ME 323

Heat Transfer

Class Project

Author & Lecturer:
Dr. Thomas A. Marino

Spring 2025

Contents

1 Project Introduction

This project provides a comprehensive analysis of heat transfer phenomena through numerical modeling. It addresses the complexities of heat transfer by systematically integrating the finite difference method for 2D conduction, boundary layer theory for convective heat transfer, and the Stefan-Boltzmann law for surface radiation effects.

The project is divided into three distinct parts.

- Part 1 focuses on solving the unsteady 2D heat conduction equation within a rectangular domain, incorporating boundary conditions and internal heat generation.
- Part 2 builds upon this by introducing convective heat transfer at a boundary, requiring the solution of boundary layer equations to determine velocity and temperature profiles and to evaluate heat flux.
- Part 3 expands the model to include surface radiation, calculating radiative heat transfer and combining it with conduction and convection to determine the net heat flux.

The successful completion of this project will yield contour plots of temperature distribution, velocity and temperature profiles, and a detailed analysis of heat transfer rates, along with discussions on grid sensitivity, convergence, and the significance of each heat transfer mode.

This project will enhance the understanding and modeling capabilities of complex heat transfer scenarios relevant to various engineering applications.

1.1 Project Objectives

- Apply the finite difference method to solve an unsteady 2D conduction problem.
- Integrate boundary layer theory to analyze convective heat transfer.
- Incorporate surface radiation to evaluate combined heat transfer effects.

1.2 Project Steps

Part 1: Finite Difference Solution to Unsteady @D Conduction

Problem Setup:

- Define a rectangular domain with fixed and unsteady boundary conditions (e.g., specified temperatures, insulated boundaries, etc.).
- Include a non-uniform internal heat generation.

Analysis:

- Discretize the domain into a grid and derive the finite difference equations using energy balance principles.
- Solve the system of equations iteratively (e.g., using Gauss-Seidel or Successive Over-Relaxation methods).

Expectations:

- Contour plots of the temperature distribution in the domain.
- Discussion of grid sensitivity and convergence.

Part 2: Boundary Layer Theory for Convection

Problem Setup:

- Add convective heat transfer at one of the boundaries (e.g., air moving past the heated surface).
- Introduce boundary layer equations (momentum and energy) to analyze the flow.

Analysis:

- Solve the boundary layer equations for velocity and temperature profiles (use numerical methods or approximations).
- Apply the appropriate convective heat transfer correlation to evaluate heat flux.

Expectations:

- Velocity and temperature profiles within the boundary layer.
- Calculated surface heat flux and comparison with the conduction-only solution.

Part 3: Surface Radiation Effects

Problem Setup:

- Include radiation exchange at the heated surface (e.g., emissivity, surrounding temperature, and view factors).

Analysis:

- Use the Stefan-Boltzmann law to calculate radiative heat transfer.
- Combine conduction, convection, and radiation heat fluxes to evaluate overall heat transfer.

Expectations:

- Net heat flux from the surface due to conduction, convection, and radiation.
- Discussion on the significance of radiation compared to other modes.

1.3 Suggested Tools

- **Mathematical Modeling Software:** MATLAB, Python (NumPy/SciPy), or equivalent.
- **Visualization:** Use tools like MATLAB's contour plots or Python's Matplotlib for graphical representation.
- **Boundary Layer Theory Resources:** Reference Schlichting's *Boundary-Layer Theory* or similar texts for boundary layer concepts.

1.4 Assessment Criteria

- **Technical Accuracy:** Correct implementation of equations and methods.
- **Numerical Results:** Convergence and sensitivity analysis for each part.
- **Integration:** Clear explanation of how conduction, convection, and radiation interact.
- **Presentation:** Professional reporting of results, including plots, tables, and discussions.

Part I

Finite Difference Solution to Unsteady 2D Conduction with Heat Generation

Problem Setup

Consider a rectangular domain ($L_x \times L_y$) with the following *example* boundary conditions:

- Left boundary ($x = 0$): Maintained at a fixed temperature (T_{left}).
- Right boundary ($x = L_x$): Insulated ($\frac{\partial T}{\partial x} = 0$).
- Top boundary ($y = L_y$): Maintained at a fixed temperature (T_{top}).
- Bottom boundary ($y = 0$): Heat Flux ($-k\frac{\partial T}{\partial n}(x_b, t) = q_b(t)$).

At least one boundary should be of type Dirichlet ($T(x_b, t) = T_b(t)$) and at least one boundary of type Neumann ($-k\frac{\partial T}{\partial n}(x_b, t) = q_b(t)$). We are interested in the temperature distribution within the domain as a function of time, starting from a given initial temperature distribution, and with the presence of a heat source.

Governing Equation

The unsteady 2D heat conduction equation *with* internal heat generation is:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{q(x, y, t)}{k}$$

where:

- ($T(x, y, t)$) is the temperature field.
- ($\alpha = \frac{k}{\rho c_p}$) is the thermal diffusivity, with (k) being the thermal conductivity, (ρ) the density, and (c_p) the specific heat capacity.
- ($qq(x, y, t)$) is the internal heat generation rate (which can be a function of position and/or time).

Discretization

The domain is discretized into a grid with spatial steps (Δx) and (Δy), and time step (Δt). The governing equation is approximated using finite difference methods:

Explicit Formulation (FTCS):

$T_{i,j}^{n+1} = T_{i,j}^{n} + \alpha \Delta t \left(\frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2} \right) + \frac{q_{i,j}^{n} \Delta t}{k}$
where ($T_{i,j}^{n}$) represents the temperature at grid point ((i,j)) and time step (n), and $q_{i,j}^{n}$ is the heat generation rate at node (i,j) at time step n.

Implicit Formulation (Crank-Nicolson):

$T_{i,j}^{n+1} = T_{i,j}^{n} + \frac{\alpha \Delta t}{2} \left(\frac{T_{i+1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i-1,j}^{n+1}}{\Delta x^2} + \frac{T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1}}{\Delta y^2} + \frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2} \right) + \frac{q_{i,j}^{n} \Delta t}{k}$
where ($T_{i,j}^{n}$) represents the temperature at grid point ((i,j)) and time step (n).

Example Solution Procedure

1. **Initialization**: Define the initial temperature distribution ($T_{i,j}^{0}$) at time ($t = 0$).
2. **Time Stepping**: For each time step (n):
   - Explicit: Calculate ($T_{i,j}^{n+1}$) directly from the known values at time step (n).
   - Implicit: Solve a system of linear equations to obtain the temperature values ($T_{i,j}^{n+1}$) at time step ($n+1$).
3. **Boundary Conditions**: Enforce the boundary conditions at each time step:
   - ($T_{0,j}^{n} = T_{\text{left}}$)
   - ($T_{M,j}^{n} = T_{M-1,j}^{n}$) (insulated right boundary)
   - ($T_{i,N}^{n} = T_{\text{top}}$)
   - ($-k \frac{\partial T_{i,0}^{n}}{\partial y} = q_{i,0}^{n}$) (heat flux)

Deliverables (but not limited to)

- **Program source code.**
- **Contour Plots**: Visualize the temperature distribution ($T(x,y,t)$) across the domain at different time instants.
- **Convergence Analysis**: Evaluate the impact of grid spacing ((Δx), (Δy)) and time step ((Δt)) on the solution's accuracy and stability. Compare the performance of explicit and implicit schemes.
- **At least one sensitivity study** (i.e. vary thermal conductivity, specific heat, boundary conditions, etc.)

Dirichlet and Neumann Boundary Conditions in Heat Transfer

In heat transfer problems, boundary conditions are crucial for defining the behavior of temperature or heat flux at the boundaries of a domain. Two commonly encountered types are **Dirichlet** and **Neumann** boundary conditions.

1. Dirichlet Boundary Condition:

- **Definition:** A Dirichlet boundary condition specifies the **temperature** at the boundary.
- **Mathematical Representation:**

$$T(x_b, t) = T_b(t)$$

  where:

  - $T(x_b, t)$ is the temperature at the boundary point x_b and time t.
  - $T_b(t)$ is the prescribed temperature at the boundary, which can be a constant or a function of time.

- **Physical Interpretation:** This condition implies that the boundary is in perfect thermal contact with a heat reservoir maintained at a fixed temperature, forcing the boundary to have that temperature.

2. Neumann Boundary Condition:

- **Definition:** A Neumann boundary condition specifies the **heat flux** at the boundary.
- **Mathematical Representation:**

$$-k\frac{\partial T}{\partial n}(x_b, t) = q_b(t)$$

  where:

  - k is the thermal conductivity of the material.
  - $\frac{\partial T}{\partial n}(x_b, t)$ is the temperature gradient at the boundary point x_b and time t in the direction normal to the boundary (n).
  - $q_b(t)$ is the prescribed heat flux at the boundary, which can be a constant or a function of time.

- **Physical Interpretation:** This condition implies that the heat flow rate across the boundary is controlled. A special case is the **adiabatic boundary condition** where $q_b(t) = 0$, indicating no heat flow across the boundary (perfect insulation).

Key Differences:

Feature	Dirichlet Boundary Condition	Neumann Boundary Condition
Specified Quantity	Temperature (T)	Heat flux (q)
Mathematical Form	$T(x_b, t) = T_b(t)$	$-k\frac{\partial T}{\partial n}(x_b, t) = q_b(t)$
Physical Meaning	Fixed temperature at the boundary	Controlled heat flow across the boundary
Example	Surface of an object in contact with a constant temperature heat source	Insulated surface of an object

In summary:

- Dirichlet conditions impose a fixed temperature at the boundary.
- Neumann conditions impose a fixed heat flux (or its derivative) at the boundary.

The choice of boundary condition depends on the specific physical situation being modeled and the available information about the system's behavior at the boundaries.

Part II
Integrated Heat Transfer Analysis: Conduction and Convection

Introduction

This project combines conduction and convection to analyze heat transfer in a rectangular domain. In Part 1, a finite difference method is used to solve the 2D steady-state conduction problem. In Part 2, boundary layer theory is applied to incorporate convective heat transfer at a boundary, enabling a more comprehensive analysis.

Part 2: Boundary Layer Theory for Convection

Problem Setup

A convective boundary condition is applied to the top boundary ($y = L_y$) of the domain where air flows past the surface. The temperature distribution in the thermal boundary layer is given by:

$$T(y) - T_s = (1 - \exp(-\mathrm{Pr} \cdot U_\infty \cdot y)) \cdot (T_\infty - T_s)$$

where:

- T_s: Surface temperature ($T_s = T_{\text{top}}$).
- T_∞: Free-stream air temperature.
- Pr: Prandtl number.
- U_∞/ν: Velocity gradient.

Heat Flux Calculation

Using Fourier's law at $y = 0$:

$$q_s = -k \frac{\partial T}{\partial y}\bigg|_{y=0}$$

The temperature gradient at $y = 0$ is derived from the given temperature profile:

$$\frac{\partial T}{\partial y}\bigg|_{y=0} = \mathrm{Pr} \cdot U_\infty \cdot (T_\infty - T_s)$$

The surface heat flux is thus:

$$q_s = -k \cdot \mathrm{Pr} \cdot U_\infty \cdot (T_\infty - T_s)$$

Integration with Part 1

The convective heat flux calculated here serves as the boundary condition for the top surface in Part 1. This integration ensures that both conduction within the domain and convection at the boundary are accounted for in the overall analysis.

Example Calculation

Given:

- $T_s = 300\,\mathrm{K}$, $T_\infty = 400\,\mathrm{K}$.
- $k = 0.0263\,\mathrm{W/m.K}$, $\mathrm{Pr} = 0.7$.
- $U_\infty/\nu = 5000\,\mathrm{m}^{-1}$.

The surface heat flux is:

$$q_s = -0.0263 \cdot 0.7 \cdot 5000 \cdot 100 = -9205\,\mathrm{W/m}^2$$

(Note: The negative sign indicates heat transfer to the surface.)

Deliverables

- Velocity and temperature profiles within the boundary layer.
- Updated temperature distribution in the domain considering convective boundary conditions.

Conclusion

This integrated approach provides a comprehensive analysis of heat transfer by combining conduction and convection. The convective boundary condition enhances the realism of the model and allows for better understanding of practical heat transfer scenarios.

Part III

Integrated Heat Transfer Analysis: Conduction, Convection, and Radiation

Introduction

This project explores conduction, convection, and radiation in a comprehensive heat transfer analysis. In Part 1, a 2D conduction problem is solved using the finite difference method. In Part 2, boundary layer theory introduces convective heat transfer at a boundary. In Part 3, surface radiation is added to account for radiative heat transfer effects. Together, these parts form a fully integrated approach to heat transfer modeling.

Integration with Part 1

The convective heat flux at the top boundary serves as the boundary condition for the finite difference solution in Part 1. This integration enables simultaneous consideration of conduction within the domain and convection at the surface.

Part 3: Surface Radiation Effects

Problem Setup

Radiative heat transfer is added to the top surface ($y = L_y$), where it exchanges energy with the surroundings. The net radiative heat flux is given by:

$$q_r = \epsilon\sigma\left(T_s^4 - T_{\text{surroundings}}^4\right)$$

where:

- ϵ: Emissivity of the surface.
- σ: Stefan-Boltzmann constant ($\sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2K^4}$).
- T_s: Surface temperature.
- $T_{\text{surroundings}}$: Temperature of the surroundings.

Net Heat Flux

The total heat flux at the top surface is the sum of convective and radiative contributions:

$$q_{\text{total}} = q_s + q_r$$

Integration with Parts 1 and 2

The radiative heat flux adds an additional boundary condition to the finite difference solution from Part 1, alongside the convective heat flux from Part 2. This integration accounts for all modes of heat transfer at the top boundary.

Example Calculation

Given:

- $\epsilon = 0.8$, $T_{\text{surroundings}} = 300\,\text{K}$.
- $T_s = 400\,\text{K}$, $k = 0.0263\,\text{W/m.K}$, $\text{Pr} = 0.7$, $U_\infty/\nu = 5000\,\text{m}^{-1}$.

The convective heat flux is:

$$q_s = -9205\,\text{W/m}^2$$

The radiative heat flux is:

$$q_r = 0.8 \cdot 5.67 \times 10^{-8} \cdot \left(400^4 - 300^4\right) = 145\,\text{W/m}^2$$

The total heat flux is:

$$q_{\text{total}} = -9205 + 145 = -9060\,\text{W/m}^2$$

(Note: Negative flux indicates heat transfer to the surface.)

Deliverables

- Updated temperature distribution considering conduction, convection, and radiation.
- Analysis of the relative contributions of radiation versus convection.

Conclusion

This integrated approach models conduction, convection, and radiation to provide a comprehensive heat transfer analysis. Students gain insight into how these modes interact in real-world scenarios.