Dynamic Behavior of a Two Degree of Freedom Mass-Spring System

SP4 – Modal Analysis

By: Abbas Hussain, Shreyes Kumar, Ali Aladhab.

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Prof. Zelalem Eshete

1. Introduction

This project involves the modal analysis of a two-degree-of-freedom mechanical system, consisting of two masses connected by three springs. The objective is to determine the system's equations of motion, natural frequencies, and mode shapes, and to analyze the response of each mass under specified initial conditions.

The system is configured with two vertically moving masses, m_1 and m_2, connected by springs with stiffness constants k_1, k_2, and k. This setup allows the system to exhibit two independent modes of vibration. By constructing free-body diagrams and deriving the equations of motion, it is possible to establish the characteristic equation, which yields the natural frequencies and associated mode shapes of the system.

The analysis considers initial conditions where m_1 is displaced vertically by 1 cm, with zero initial velocity, while m_2 starts with no initial displacement or velocity. Using these conditions, the resulting oscillatory behavior of each mass is calculated, illustrating the influence of the natural frequencies and mode shapes on the system's motion. This approach provides insights into the principles of vibrational dynamics for multi-mass systems and demonstrates the application of modal analysis in predicting system behavior.

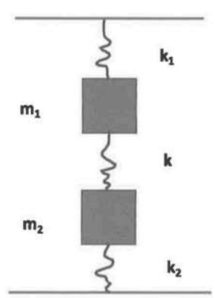


Figure 1. Mass Spring System Project is Based on

2. Discussion

2.1 Project Breakdown and Explanation

This project analyzes a two-degree-of-freedom mechanical system consisting of two masses and three springs. The objective is to determine the equations of motion, natural frequencies, and mode shapes for the system in three specific configurations, referred to as Case A, Case B, and Case C. The analysis concludes with calculating the exact mathematical solutions for each case, using initial conditions that include an initial displacement of one mass with zero initial velocities for both masses. The project is divided into three main steps as follows:

Determining Equations of Motion: The equations of motion for the system are derived using free-body diagrams and Newton's Second Law. This step involves setting up a system of differential equations that describes the motion of the masses in terms of the system's parameters, including the masses and spring constants. Each mass can move vertically, creating two degrees of freedom in the system.

Determining Natural Frequencies and Mode Shapes: By solving the characteristic equation of the system, the natural frequencies and corresponding mode shapes are obtained for each case configuration. The mode shapes describe the relative motion of the masses at each natural frequency.

Determining Mathematical Solutions: Using the initial conditions, the exact mathematical solutions for each case are derived to describe the time-dependent behavior of the masses. Each solution considers the initial displacement of m_1 by 1 cm and zero initial velocity for both masses.

2.2 Determining Equations of Motion

The equations of motion for a two-degree-of-freedom system consisting of two masses connected by three springs are derived using Newton's Second Law. The system has two masses, m_1 and m_2, connected by springs with stiffness constants k_1, k_2, and k. Each mass is capable of vertical motion, resulting in two independent degrees of freedom. The process of how to get these equations of motion are explained below.

Free-body Diagrams: The free body diagrams for mass 1 and 2 are shown below.

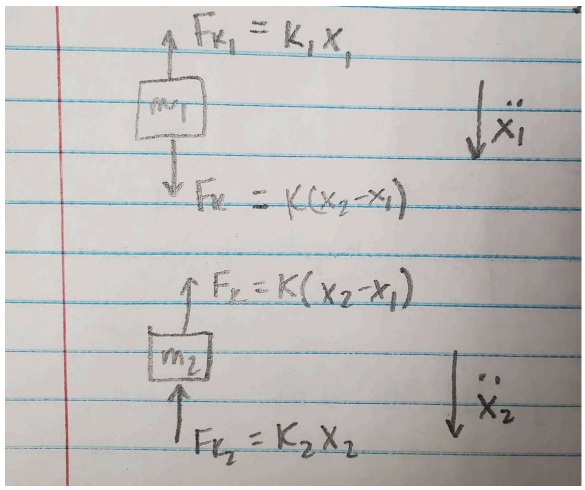


Figure 2. Free-Body Diagrams for m_1 and m_2

In constructing the free-body diagrams for each mass, the direction of acceleration for both m_1 and m_2 was set downward, aligning with the positive direction for displacements x_1 and x_2 from their equilibrium positions. This convention simplifies the equations of motion, as it ensures that the accelerations x_1'' and x_2'' correspond directly to the downward forces acting on each mass. Mass m1 is acted upon by two forces: the restoring force from spring k_1, connected to a fixed point ($F_{k1} =- k_1 x_1$), and the coupling spring k, which connects m_1 to m_2 and depends on the relative displacement $(x_1 - x_2)$, yielding $F_k = k(x_2 - x_1)$. For mass m_2, the forces include the coupling spring k, which exerts a force $F_k =- k(x_2 - x_1)$, and the restoring force from spring k_2, fixed to a stationary point, represented as $F_{k2} =- k_2 x_2$.

With these free-body diagrams and the chosen direction for acceleration, Newton's Second Law can be applied consistently to derive a system of differential equations that describes the dynamic interactions between the masses and the springs.

Equations of Motion: The equations of motion for this two-degree-of-freedom system are derived by applying Newton's Second Law to each mass. For m_1, the forces include the restoring force from spring k_1, proportional to its displacement x_1, and the coupling spring k, which depends on the relative displacement between m_1 and m_2. This yields the equation $m_1x_1'' + (k + k_1)x_1 - kx_2 = 0$. Similarly, for m_2, the forces include the restoring force from k_2, proportional to x_2, and the coupling spring k, resulting in the equation $m_2x_2'' + (k_2 + k_1)x_2 - kx_1 = 0$. These equations describe the dynamic interactions between the masses and springs and form the basis for analyzing the system's natural frequencies and mode shapes.

2.3 Determining Natural Frequencies and Mode Shapes

In order to find the natural frequencies for each case the equations of motion found above were put into matrix form. $x_1$was set to equal $x_1e^{i\omega t}$ and x_2 was set to equal $x_2e^{i\omega t}$. This was plugged into the equations of motion which resulted in the two equations $-\omega^2m_1x_1 + (k_1 + k)x_1 - kx_2 = 0$ and $-\omega^2m_2x_2 + (k_2 + k)x_2 - kx_1 = 0$. These equations were then transformed into a matrix format. To find these frequencies, the determinant of the resulting matrix was set to zero, which produced a characteristic equation. Solving this equation yielded the natural frequencies, ω_1 and ω_2, which correspond to the system's two independent modes of vibration. The final equation found from this was

$[(k_1 + k) - \omega^2m_1][(k_2 + k) - \omega^2m_2] - k^2 = 0.$

Case A: In this case $m_1 = m_2 = 1\ kg$ and $k_1 = k_2 = 10\ N/m,$ substituting these values into the characteristic equation allowed the calculation of the natural frequencies ω_1 and ω_2. It was found that $\omega_1 = \sqrt{30}\ rad/s$ and $\omega_2 = \sqrt{10}\ rad/s.$ The mode shapes were determined by substituting each frequency back into the matrix form of the equations, showing that the masses oscillate with either in-phase or out-of-phase motion due to the symmetry of the system.

Case B: In this case, $m_1 = 10\ kg$, $m_2 = 1\ kg$, and $k_1 = k_2 = 10\ N/m$. Substituting these values into the characteristic equation allowed for the calculation of the natural frequencies ω_1 and ω_2. It was found that $\omega_1 = 4.53\ rad/s$ and $\omega_2 = 1.21\ rad/s.$ The asymmetry in the mass distribution affects the mode shapes, which were determined by substituting each frequency back into the matrix form of the equations. The resulting mode shapes showed that the heavier mass m_1 exhibits smaller amplitude oscillations relative to the lighter mass m_2 in one of the modes, indicating an out-of-phase motion, while in the other mode, both masses oscillate with an in-phase motion. This configuration highlights how the mass disparity influences the relative motion and frequencies.

Case C: In this case, $m_1 = m_2 = 1\ kg$, $k_1 = k_2 = 10\ N/m$, and the coupling spring has an increased stiffness of $k\ = 100\ N/m$. Substituting these values into the characteristic equation allowed for the calculation of the natural frequencies ω_1 and ω_2. It was found that $\omega_1 = \sqrt{210}\ rad/s$ and $\omega_2 = \sqrt{10}\ rad/s.$. The significantly stiffer coupling spring affects the mode shapes, which were determined by substituting each frequency back into the matrix form of the equations. The resulting mode shapes showed that, in one mode, the masses move in-phase, oscillating together due to the symmetry of the masses and outer springs. In the other mode, the masses move out-of-phase, with one mass moving upward while the other moves downward. The increased stiffness in the coupling spring creates a higher frequency mode, emphasizing the impact of internal stiffness on the system's dynamic response.

2.4 Determining Mathematical Solutions

The mathematical solutions for each case were determined by solving the equations of motion under specific initial conditions. For each case, the system was analyzed with m_1 initially displaced by 1 cm (0.01 m) and zero initial velocities for both masses. The general solution for every case was $x_1(t) = A_1 cos(\omega_1 t) + A_2 cos(\omega_2 t)$ and $x_2(t) = B_1 cos(\omega_1 t) + B_2 cos(\omega_2 t)$.

Case A: In this case, where $m_1 = m_2 = 1\ kg$ and $k_1 = k_2 = 10\ N/m,$, the natural frequencies were found to be $\omega_1 = \sqrt{30}\ rad/s$ and $\omega_2 = \sqrt{10}\ rad/s.$ The solution for the displacement of each mass, $x_1(t)$ and $x_2(t)$ was expressed as a combination of these frequencies. After using the mode shapes and initial conditions it was found that the solution to case A was equal to:

$x_1(t) =- 0.005 cos(\sqrt{30}t) + 0.005 cos(\sqrt{10}t)$ and $x_2(t) = 0.005 cos(\sqrt{30}t) + 0.005 cos(\sqrt{10}t)$.

In this configuration, the equal mass and spring constants create a balanced system, resulting in two distinct vibrational modes that correspond to the natural frequencies ω_1 and ω_2. At the lower frequency ω_2 , the masses oscillate in-phase, moving together in the same direction, reflecting the symmetrical nature of the system. At the higher frequency ω_1 , the masses move out-of-phase, with one mass moving upward as the other moves downward. This symmetry in the mass and spring values simplifies the motion, creating clear distinctions between the in-phase and out-of-phase oscillatory modes and demonstrating how uniform system parameters lead to predictable, balanced vibrational behaviors.

Case B: In this case, where $m_1 = 10\ kg$, $m_2 = 1\ kg$, and $k_1 = k_2 = 10\ N/m$, the natural frequencies were found to be $\omega_1 = 4.53\ rad/s$ and $\omega_2 = 1.21\ rad/s.$. The solution for the displacements $x_1(t)$ and $x_2(t)$ was expressed as a combination of these frequencies. After using the mode shapes and initial conditions it was found that the solution to case B was equal to:

$x_1(t) =- 0.0003 cos(4.53t) + 0.0103 cos(1.21t)$ and

$x_2(t) = 0.00556 cos(4.53t) + 0.00557 cos(1.21t)$.

These solutions indicate that in one mode, the heavier mass m_1 has a smaller oscillation amplitude, while the lighter mass m_2 exhibits a larger amplitude. The differing frequencies and corresponding mode shapes reflect the asymmetry in mass, leading to unique amplitude and phase relationships for each mass at each frequency.

Case C: In this case , where $m_1 = m_2 = 1\ kg$, $k_1 = k_2 = 10\ N/m$, and the middle spring stiffness is significantly increased to $k = 100\ N/m$, the natural frequencies were calculated to be $\omega_1 = \sqrt{210}\ rad/s$ and $\omega_2 = \sqrt{10}\ rad/s$. The solution for the displacements $x_1(t)$ and $x_2(t)$ was expressed as a combination of these frequencies. After using the mode shapes and initial conditions it was found that the solution to case C was equal to: $x_1(t) = -\ 0.005cos(\sqrt{210}t)\ +\ 0.005cos(\sqrt{10}t)$ and $x_2(t)\ =\ 0.005cos(\sqrt{210}t)\ +\ 0.005cos(\sqrt{10}t)$. In this configuration, the increased stiffness of the middle spring introduces a higher frequency mode, represented by the higher natural frequency ω_1. The solutions show that at the lower frequency ω_2, the masses oscillate in-phase. While at the higher frequency ω_1, the masses move out-of-phase, with one mass moving upward as the other moves downward. This demonstrates the effect of a stiffer coupling spring, which creates a pronounced difference between the in-phase and out-of-phase oscillatory modes in the system.

3. Summary

This project explored the dynamic behavior of a two-degree-of-freedom mechanical system consisting of two masses connected by three springs. By deriving the equations of motion using Newton’s Second Law, the system was analyzed in three distinct configurations, each with varying mass and spring stiffness values. The natural frequencies and mode shapes were calculated for each case, providing insight into the unique vibrational modes influenced by the system's symmetry or asymmetry in mass and stiffness.

For each case, the equations of motion were transformed into matrix form to identify the natural frequencies and corresponding mode shapes. Case A, with equal masses and spring constants, exhibited symmetrical in-phase and out-of-phase oscillations. In Case B, an asymmetry in mass led to distinct amplitude and phase relationships between the masses. In Case C, a significantly stiffer coupling spring created a pronounced high-frequency mode, emphasizing the effect of internal stiffness on system response.

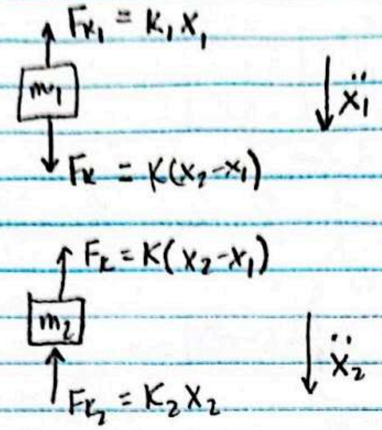
Finally, the mathematical solutions for each case were determined by applying specific initial conditions. These solutions demonstrated how the system’s natural frequencies and mode shapes govern the time-dependent behavior of each mass. The analysis highlighted how variations in mass and spring stiffness impact the system's vibrational response, providing a comprehensive understanding of the underlying dynamics of multi-mass systems with coupled oscillations. This project underscores the importance of modal analysis in predicting the behavior of mechanical systems and the effects of mass and stiffness on vibrational characteristics.

4. Appendix

Below are the hand written calculations

1

1) Draw the detailed free body diagrams of the masses of the system. Ignore gravity



2) Using the free body diagrams, determine the equations of motions for the system.

m_1:

$\Sigma F = ma$

$-k_1 x_1 + k(x_2 - x_1) = m_1 \ddot{x}_1$

$m_1 \ddot{x}_1 + k_1 x_1 + k x_1 - k x_2 = 0$

$m_1 \ddot{x}_1 + (k + k_1) x_1 - k x_2 = 0$

m_2:

$\Sigma F = ma$

$-k(x_2 - x_1) - k_2 x_2 = m_2 \ddot{x}_2$

$m_2 \ddot{x}_2 + k x_2 - k x_1 + k_2 x_2 = 0$

$m_2 \ddot{x}_2 + (k_2 + k) x_2 - k x_1 = 0$

3) For each case, what are the natural frequencies and corresponding mode shapes?

$x_1(t) = X_1 e^{i\omega t}$ $\quad$ $x_2(t) = X_2 e^{i\omega t}$

$\dot{x}_1(t) = i\omega X_1 e^{i\omega t}$ $\quad$ $\dot{x}_2(t) = i\omega X_2 e^{i\omega t}$

$\ddot{x}_1(t) = -\omega^2 X_1 e^{i\omega t}$ $\quad$ $\ddot{x}_2(t) = -\omega^2 X_2 e^{i\omega t}$

$m_1(-\omega^2 X_1 e^{i\omega t}) + (k_1 + k) X_1 e^{i\omega t} - k(X_2 e^{i\omega t}) = 0$

$-\omega^2 m_1 X_1 + (k_1 + k) X_1 - k X_2 = 0$

$m_2(-\omega^2 X_2 e^{i\omega t}) + (k_2 + k) X_2 e^{i\omega t} - k(X_1 e^{i\omega t}) = 0$

$-\omega^2 m_2 X_2 + (k_2 + k) X_2 - k X_1 = 0$

Figure 3. Handwritten Work for Step 1, 2 and 3

$$\begin{bmatrix} (k_1+k)-\omega^2 m_1 & -k \\ -k & (k_2+k)-\omega^2 m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} (k_1+k)-\omega^2 m_1 & -k \\ -k & (k_2+k)-\omega^2 m_2 \end{bmatrix} = 0$$

$$\left[(k_1+k)-\omega^2 m_1\right]\left[(k_2+k)-\omega^2 m_2\right] - \left[(-k)(-k)\right] = 0$$

$$\left[(k_1+k)-\omega^2 m_1\right]\left[(k_2+k)-\omega^2 m_2\right] - k^2 = 0$$

Case A:

$m_1 = 1\,kg$, $m_2 = 1\,kg$, $k_1 = 10\,N/m$, $k_2 = 10\,N/m$, $k = 10\,N/m$

$$\left((20)-\omega^2(1)\right)\left((20)-\omega^2(1)\right) - 100 = 0$$

$$(20-\omega^2)(20-\omega^2) - 100 = 0$$

$$(400 - 40\omega^2 + \omega^4) - 100 = 0$$

$$\omega^4 - 40\omega^2 + 300 = 0 \qquad z = \omega^2$$

$$z^2 - 40z + 300 = 0$$

$$z = \frac{40 \pm \sqrt{40^2 - 4(300)}}{2} = 20 \pm 10$$

$$z = 30, 10$$

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{30}\ \tfrac{rad}{s} \\ \omega_2 &= \sqrt{10}\ \tfrac{rad}{s} \end{aligned}}$$

$$\begin{bmatrix} 20-10 & -10 \\ -10 & 20-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$10x_1 - 10x_2 = 0$$

$$x_1 = x_2$$

$$\begin{bmatrix} 20-30 & -10 \\ -10 & 20-30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 & -10 \\ -10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-10x_1 - 10x_2 = 0$$

$$x_1 = -x_2$$

For $\omega_1 = \sqrt{30}\ \frac{rad}{s}$ For $\omega_2 = \sqrt{10}\ \frac{rad}{s}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in phase $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ out of phase

Figure 4. Handwritten Work for Step 3 Case A

Case B

$m_1 = 10\,kg,\ m_2 = 1\,kg,\ k_1 = 10\,N/m,\ k_2 = 10\,N/m,\ k = 10\,N/m$

$$((20) - \omega^2(10))((20) - \omega^2(1)) - 100 = 0$$

$$(20 - 10\omega^2)(20 - \omega^2) - 100 = 0$$

$$(400 - 20\omega^2 - 200\omega^2 + 10\omega^4) - 100 = 0$$

$$10\omega^4 - 220\omega^2 + 300 = 0$$

$$\omega^4 - 22\omega^2 + 30 = 0 \qquad \omega^2 = z$$

$$z^2 - 22z + 30 = 0$$

$$z = \frac{22 \pm \sqrt{(-22)^2 - 4(30)}}{2} = 11 \pm \frac{\sqrt{364}}{2} = 11 \pm \sqrt{91}$$

$$\omega_1 = \sqrt{11 + \sqrt{91}}$$

$$\omega_2 = \sqrt{11 - \sqrt{91}}$$

$$\boxed{\begin{aligned}\omega_1 &= 4.53\ \tfrac{rad}{s}\\ \omega_2 &= 1.21\ \tfrac{rad}{s}\end{aligned}}$$

$$\begin{bmatrix} 20 - 10(1.46) & -10 \\ -10 & 20 - 1.46 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5.4 & -10 \\ -10 & 18.54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$5.4x_1 - 10x_2 = 0 \qquad x_1 = 1.85x_2$$

$$\begin{bmatrix} 20 - (10)(20.54)\cancel{+10} & -10 \\ -10 & 20 - (20.54) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -185.4 & -10 \\ -10 & -0.54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-185.4x_1 - 10x_2 = 0$$

$$x_1 = -0.054x_2$$

For $\omega_1 = 4.53\ \frac{rad}{s}$ $\begin{bmatrix} -0.054 \\ 1 \end{bmatrix}$

For $\omega_2 = 1.21\ \frac{rad}{s}$ $\begin{bmatrix} 1.85 \\ 1 \end{bmatrix}$

Case C

$m_1 = 1\,kg,\ m_2 = 1\,kg,\ k_1 = 10\,N/m,\ k_2 = 10\,N/m,\ k = 100\,N/m$

$$((110) - \omega^2(1))((110) - \omega^2(1)) - (100)^2 = 0$$

$$(110 - \omega^2)(110 - \omega^2) - 10000 = 0$$

$$(\omega^4 - 220\omega^2 + 2100) = 0 \qquad \omega^2 = z$$

$$z^2 - 220z + 2100 = 0$$

$$= \frac{220 \pm \sqrt{(220)^2 - 4(2100)}}{2} = 110 \pm \frac{\sqrt{200}}{2} = 110 \pm 100$$

$$\boxed{\begin{aligned}\omega_1 &= \sqrt{210}\\ \omega_2 &= \sqrt{10}\end{aligned}}$$

$\omega_1 = 11.$

$\omega_2 =$

$$\begin{bmatrix} 110 - 210 & -100 \\ -100 & 110 - 210 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} -100 & -100 \\ -100 & -100 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-100x_1 - 100x_2 = 0$$

$$x_1 = -x_2$$

$$\begin{bmatrix} 110 - 10 & -100 \\ -100 & 110 - 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$100x_1 - 100x_2 = 0$$

$$x_1 = x_2$$

For $\omega_1 = \sqrt{210}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\omega_2 = \sqrt{10}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Figure 5. Handwritten Work for Step 3 Case A and B

4) Obtain the exact mathematical solution to these equations for the 3 cases defined below. In each case, the initial conditions are that m_1 is initially displaced by one centimeter vertically and that its velocity is zero. Also, m_2 has no initial displacement or velocity.

Case A:

$\omega_{1,2} = \sqrt{30}\ \frac{rad}{s},\ \sqrt{10}\ \frac{rad}{s}$

$\omega_1 : \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \omega_2 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1(0) = 0.01m$

$\dot{x}_1(0) = 0$

$x_2(0) = 0$

$\dot{x}_2(0) = 0$

$x_1(t) = A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t) \quad \omega_1 : x_1 = -x_2$

$x_2(t) = B_1\cos(\omega_1 t) + B_2\cos(\omega_2 t) \quad \omega_2 : x_1 = x_2$

$x_2 = -A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)$

$x_1(0) = A_1 + A_2 = 0.01$

$x_2(0) = -A_1 + A_2 = 0$

$(A_1 + A_2) + (-A_1 + A_2) = 0.01 + 0$

$2A_2 = 0.01$

$A_2 = 0.005$

$A_1 = 0.005$

$$\boxed{\begin{aligned} x_1(t) &= -0.005\cos(\sqrt{30}t) + 0.005\cos(\sqrt{10}t) \\ x_2(t) &= 0.005\cos(\sqrt{30}t) + 0.005\cos(\sqrt{10}t) \end{aligned}}$$

Case B:

$\omega_{1,2} = 4.53\ \frac{rad}{s},\ 1.21\ \frac{rad}{s}$

$\omega_1 : \begin{bmatrix} -0.054 \\ 1 \end{bmatrix} \quad \omega_2 : \begin{bmatrix} 1.85 \\ 1 \end{bmatrix}$

$x_1(0) = 0.01m$

$\dot{x}_1(0) = 0$

$x_2(0) = 0$

$\dot{x}_2(0) = 0$

$x_1(t) = A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t) \quad \omega_1 :$

$x_2(t) = B_1\cos(\omega_1 t) + B_2\cos(\omega_2 t)$

$x_2 = -\frac{A_1}{0.054}\cos(\omega_1 t) + \frac{A_2}{1.85}\cos(\omega_2 t)$

$x_1(0) = A_1 + A_2 = 0.01$

$x_2(0) = B_1 + B_2 = 0$

$x_2(0) = \frac{A_1}{0.054} + \frac{A_2}{1.85} = 0 \quad A_2 = 0.01 - A_1$

$x_2(0) = \frac{A_1}{0.054} + \frac{0.01 - A_1}{1.85} = 0$

$(A_1)(1.85) + (0.01 - A_1)(0.054) = 0$

$(1.85)A_1 + 0.00054 - 0.054A_1 = 0$

$A_1 = -0.0003$

$A_2 = 0.0103$

$$\boxed{\begin{aligned} x_1 &= -0.0003\cos(4.53t) + 0.0103\cos(1.21t) \\ x_2 &= -0.00556\cos(4.53t) + 0.00557\cos(1.21t) \end{aligned}}$$

Case 3:

$\omega_{1,2} = \sqrt{210}\ \frac{rad}{s},\ \sqrt{10}\ \frac{rad}{s}$

$x_1(0) = 0.01m$

$\dot{x}_1(0) = 0$

$x_2(0) = 0$

$\dot{x}_2(0) = 0$

$x_1(t) = A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t) \quad \omega_1 : x = -x_2$

$x_2(t) = B_1\cos(\omega_1 t) + B_2\cos(\omega_2 t) \quad \omega_2 : x_1 = x_2$

$x_2 = -A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t)$

$x_1(0) = A_1 + A_2 = 0.01$

$x_2(0) = -A_1 + A_2 = 0$

$(A_1 + A_2) + (-A_1 + A_2) = 0.01 + 0$

$2A_2 = 0.01$

$A_2 = 0.005$

$A_1 = 0.005$

$$\boxed{\begin{aligned} x_1(t) &= -0.005\cos(\sqrt{210}t) + 0.005\cos(\sqrt{10}t) \\ x_2(t) &= 0.005\cos(\sqrt{210}t) + 0.005\cos(\sqrt{10}t) \end{aligned}}$$

Figure 6. Handwritten Work for Case A, B and C

5. Assumptions

Linear System Behavior:

- Assumption: The springs follow Hooke's Law, and the forces are proportional to displacement. Damping, if present, is proportional to velocity.
- Reason: This linear behavior assumption simplifies the equations of motion, allowing for predictable natural frequencies and mode shapes. In reality, extreme displacements or high velocities might induce nonlinear effects, but for small oscillations, this assumption holds.

Idealized Masses (No Deformation):

- Assumption: The masses m_1 and m_2 are treated as rigid bodies with no internal deformation.
- Reason: This allows the system to be analyzed as point masses, simplifying calculations of vibrational response. In practical applications, some deformation may occur, but it is often negligible for small, controlled oscillations.

Neglecting External Forces and Friction:

- Assumption: No external forces, friction, or air resistance affects the system.
- Reason: This assumption isolates the effects of the spring forces, allowing for a clear study of natural frequencies and mode shapes. Including external forces could add complexity, making it difficult to observe the system's inherent oscillatory behavior.

Constant System Parameters:

- Assumption: The masses and spring stiffness values (i.e., ,m_1, m_2, k_1, k_2, $and\ k$) are constant.
- Reason: Assuming constant parameters ensures that the system's natural frequencies remain stable. Real systems may have slight variations due to environmental changes, but for theoretical analysis, constant parameters allow consistent results.

Two-Degree-of-Freedom System:

- Assumption: The system is treated as a two-degree-of-freedom structure, with each mass able to move vertically and independently.
- Reason: This assumption helps isolate the primary modes of vibration for analysis. Additional degrees of freedom could introduce complex interactions, which are beyond the project's scope.

Symmetric and Asymmetric Configurations:

- Assumption: Different configurations of mass and spring values are considered to be symmetric (equal values) or asymmetric (unequal values) for specific cases.
- Reason: Studying symmetric and asymmetric configurations reveals the influence of these variations on mode shapes and natural frequencies. Symmetry simplifies the equations, making it easier to identify in-phase and out-of-phase modes.