Studying and Modeling Single/Two Degree of Freedom Mass Systems

SP2 – Modeling & Analysis

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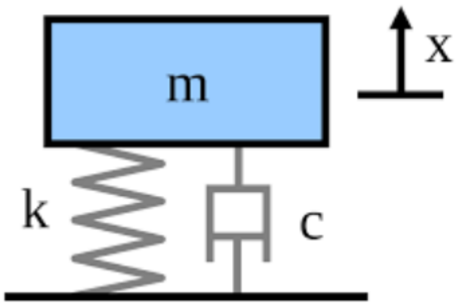
Introduction



In this project, our goal was to observe how a single/two degree of freedom mass-spring-damper system works. We had to use second-order differential equations for this system and apply different damping cases to find out the results. We were given 4 specific cases in this project and in each of these cases we had to use different damping constants. The cases were classified as undamped (c=0), underdamped (c=1), overdamped (c=8) and critically damped (c=20). With the help of MATLAB code, we were able to simulate and test our derivations and calculations to further verify our results. Furthermore, we were also required to construct a model of the mass-spring-damper system on simulink and then simulate it to further prove our previous calculations.

Discussion

1.Determining solutions for four cases

ζ=0; $0 < \zeta < 1$; ζ=1; $\zeta > 1$

To find the solutions for the differential equation of motion given above, we first assumed that the general solution of the differential equation of motion is $x(t) = e^{st}$. By substituting the solution in the equation of motion it gives us; $s^2 + 2\zeta w_n s + w_n^2)e^{st} = 0$. This is a non-zero solution. After simplifying, $s^2 + 2\zeta w_n s + w_n^2 = 0$

We used the quadratic formula to find s. Shows as; $s = \frac{-2\zeta w_n \pm \sqrt{4\zeta^2 w_n^2 - 4w_n^2}}{2}$. From the equation of motion s^2 is our A, $2\zeta w_n s$ is B, and w_n^2 is C. We get $s = w_n(- \zeta \pm \sqrt{\zeta^2 - 1}$. After conducting this information we plugged in the given four cases.

Case 1. $\zeta < 0$.
$s = \pm iw_n$
As for the general solution we got; $s = e^{iw_n t}, e^{-iw_n t}$
And as for the complete solution we got. $x = c_1 e^{iw_n t} + c_2 e^{-iw_n t}$
$x = c_3 cosw_n t + c_4 sinw_n t$. $\zeta = 0$.

Case 2 For $0 < \zeta < 1$
$s = (-\zeta \pm \sqrt{\zeta^2 - 1})w_n$
As for a general solution we conducted $s = e^{-\zeta w_n t} e^{iw_d t}, e^{\zeta iw_n t} e^{-iw_d t}$
Where $w_d = w_n \sqrt{1 - \zeta^2}$
As for the complete solution we got, $x = e^{-\zeta w_n t} [c_1 e^{iw_n t} + c_2 e^{-iw_n t}$
$x = e^{-\zeta w_n t} [cosw_d t + c_4 sinw_d t$

Case 3 For $\zeta = 1$
$\zeta = 1 \qquad s = w_n$
We got as for the general solution to be $s = c_1 + c_2 t(e^{-w_n t})$

Case 4 $\zeta > 1$
$s = (-\zeta \pm \sqrt{\zeta^2 - 1})w_n$
As for the complete solution we conducted $x = e^{-\zeta w_n t} [c_1 e^{iw_d t} + c_2 e^{-w_d t}$ as where $w_d = w_n \sqrt{\zeta^2 - 1}$.

2.Illustrating the cases using numerical values

xo=0.1m; vo = 10 m/s; m=2 kg; k=8 N/m
Cases: c=0; c=1Ns/m; c=8 Ns/m; c=20 Ns/m

To illustrate the class using numerical values we compared the mass based equation as well as the standard natural frequency based equation. $m/1 = c/2\zeta w_n = k/w_n^2$
The equations shows us that, $\zeta = c/2\sqrt{km}$ and $w_n = \sqrt{k/m}$
Applying the given into the equations;
when c=0 we get $\zeta = \frac{0}{2\sqrt{8(2)}}$=0. So when $c = 0$ $\zeta = 0$
And $w_n = \sqrt{8N/m/2kg}$ = 2rad/s. When substituting these values in the general solution that we solved on part 1 we conduct; $x = c_3 cosw_n t + c_4 sinw_n t$ we plug in the values we got for w_n. Gives us;
$x = c_3 cos(2)t + c_4 sin(2)t$. after applying the initial condition. We get; x=0.1cos2t+5sin2t.
For c=1;
$\zeta = c/2\sqrt{km}$ which gives us $1/2\sqrt{8 * 2}) = 1/8$ for $w_n = \sqrt{k/m} \rightarrow = \sqrt{8/2} = w_n = 2\ rad/s.$

$x = e^{-\zeta w_n t}[c_3 cos w_d t + c_4 sin w_d t$

$w_d = w_n\sqrt{1 - \zeta^2}$

$w_d = 2\sqrt{1 - 0.125^2}$

$w_d = 1.98 rad/s$

$x = e^{(0.25t)}[c_2 cos\, 1.98t + c_4 sin 1.98t]$

After applying the initial condition we conducted; $x = e^{(-0.25t)}[0.1 cos\, 1.98t + 5.06 sin 1.98t]$

For c= 8
We know that $\zeta = 1$ and $w_n = 2rad/s$

after substituting the values we get an equation of $x = (c_1 + c_2)e^{(-w_n t)}$ after applying the initial condition our complete solution is; $x = (0.1 + 10.2t)e^{-2t}$

When c=20
Knows as: $\zeta = 2.5,\ w_n = 2\ rad/s$

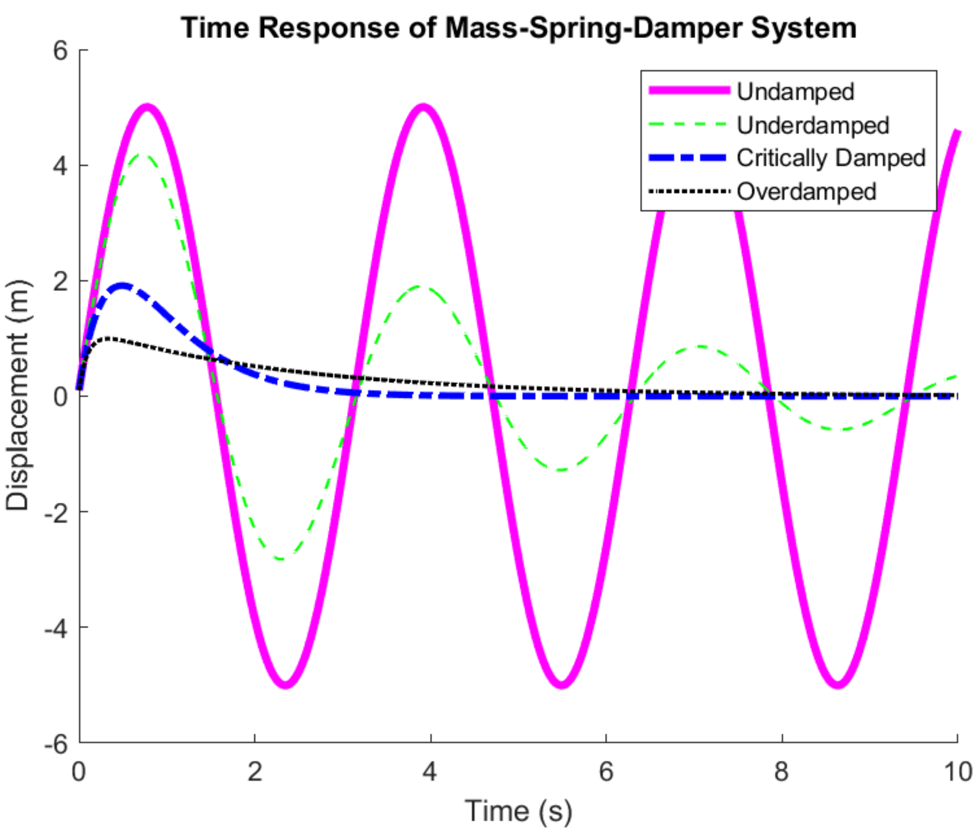
$x = e^{-5t}[c_1 e^{4.58t)} + c_2 e^{-4.58t}$

After applying the initial condition we conducted that our solution is; $x = e^{-5t}[1.2e^{4.58t} - 1.1e^{(-4.58t0}]$

3. Implementing MATLAB code and plotting the time response

After solving all the four cases and verifying our result by plugging in the given constant values, we went on to use MATLAB to implement all our theoretical work into code in order to plot the time response graph. Following is the MATLAB code we used and the graph we obtained from it:

```
m = 2;
k = 8;
x0 = 0.1;
v0 = 10;
omega_n = sqrt(k/m);
t = linspace(0, 10, 1000);
c = 0;
zeta = c / (2 * sqrt(k * m));
x_undamped = x0 * cos(omega_n * t) + (v0 / omega_n) * sin(omega_n * t);
c = 1;
zeta = c / (2 * sqrt(k * m));
omega_d = omega_n * sqrt(1 - zeta^2);
x_underdamped = exp(-zeta * omega_n * t) .* ...
    (x0 * cos(omega_d * t) + (v0 + zeta * omega_n * x0) / omega_d * sin(omega_d * t));
c = 8;
zeta = 1;
A = x0;
B = v0 + omega_n * x0;
x_critically_damped = (A + B * t) .* exp(-omega_n * t);
c = 20;
zeta = c / (2 * sqrt(k * m));
r1 = -omega_n * (zeta + sqrt(zeta^2 - 1));
r2 = -omega_n * (zeta - sqrt(zeta^2 - 1));
C1 = (v0 - r2 * x0) / (r1 - r2);
C2 = x0 - C1;
x_overdamped = C1 * exp(r1 * t) + C2 * exp(r2 * t);
figure;
hold on;
plot(t, x_undamped, 'm-', 'LineWidth', 3);
plot(t, x_underdamped, 'g--', 'LineWidth', 1);
plot(t, x_critically_damped, 'b-.', 'LineWidth', 2.5);
plot(t, x_overdamped, 'k:', 'LineWidth', 1.5);
title('Time Response of Mass-Spring-Damper System');
xlabel('Time (s)');
ylabel('Displacement (m)');
legend('Undamped', 'Underdamped', 'Critically Damped', 'Overdamped');
grid off;
hold off;
```



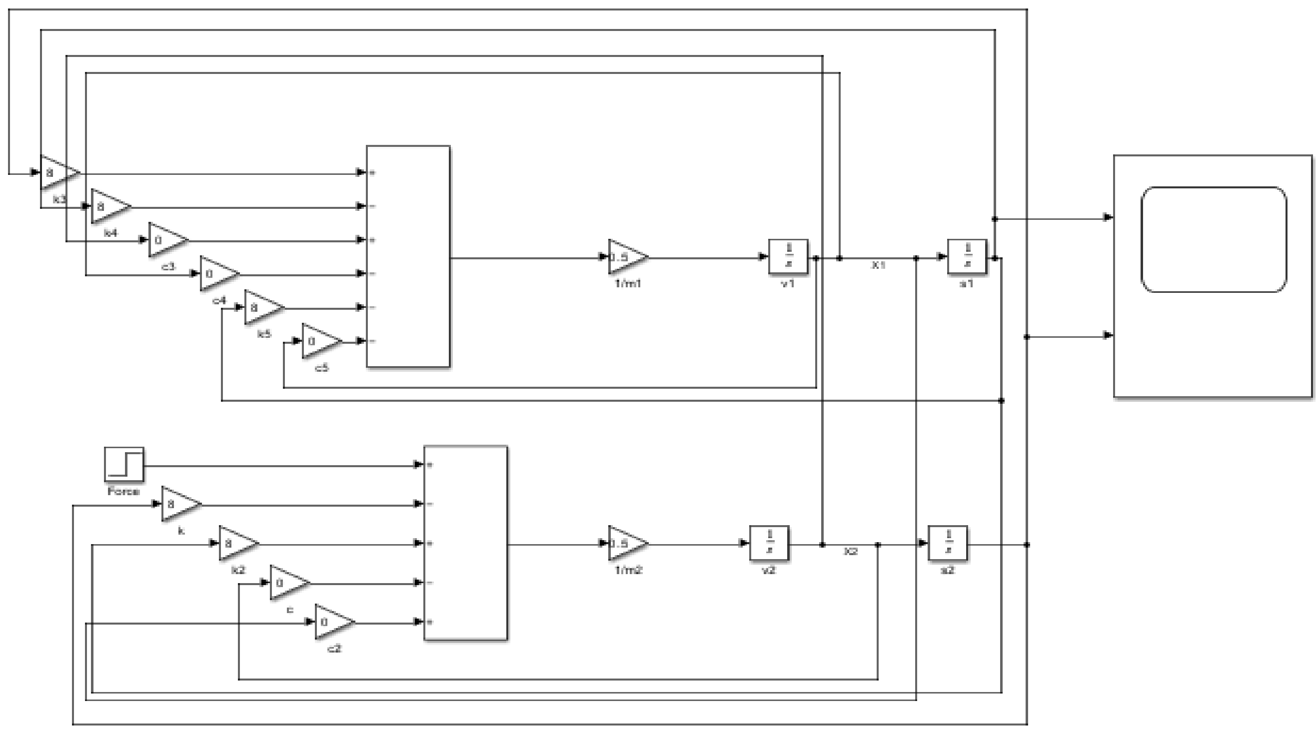
4.Simulink-Based Analysis of Two-Mass System

When constructing the two mass system in Simulink, we referred to the practice model provided to us during class that used integrator, gain and sum blocks to create a mass-spring-damper system. The only major difference between the practice model and the project model was that the project demanded a two mass system from us whereas the practice model only consisted of one mass.

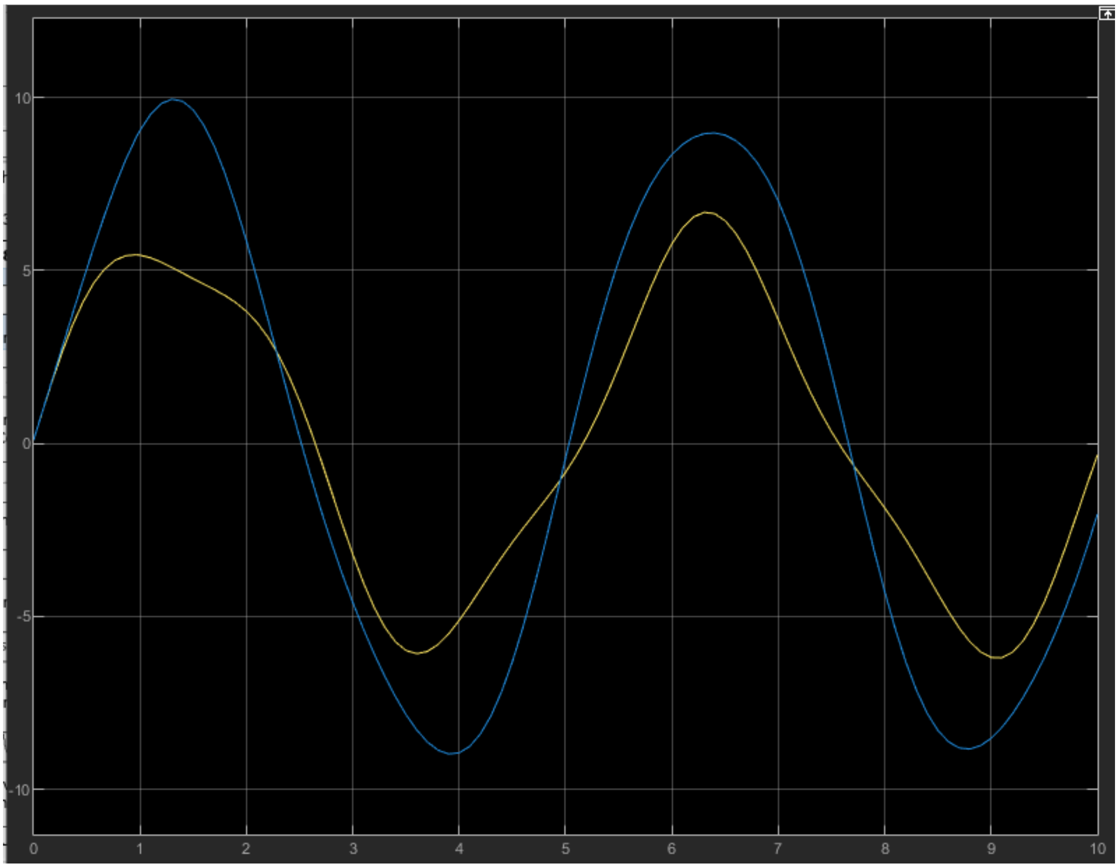
In order to make that requirement possible, we had to create two simultaneous mass systems within one model. We approached that by duplicating the one mass system we already had and then adding that on top of the existing system. The addition of a pulse function block instead of a constant block was an improvisation that we decided to go on with based on the fact that the instructions mentioned the force F(t) as a unit step function.

Apart from that, we used the gain blocks to act as the springs, dampers and masses of the system and plugged in their given constant values in order to accurately simulate the model for each damping case. The integrators helped us plug in the initial velocity and displacement of the system, which were V0=10m/s and X0=0.1m respectively. In the end we attached a scope to both of the masses to get the output of both displacements, which were denoted as X1 and X2 accordingly.

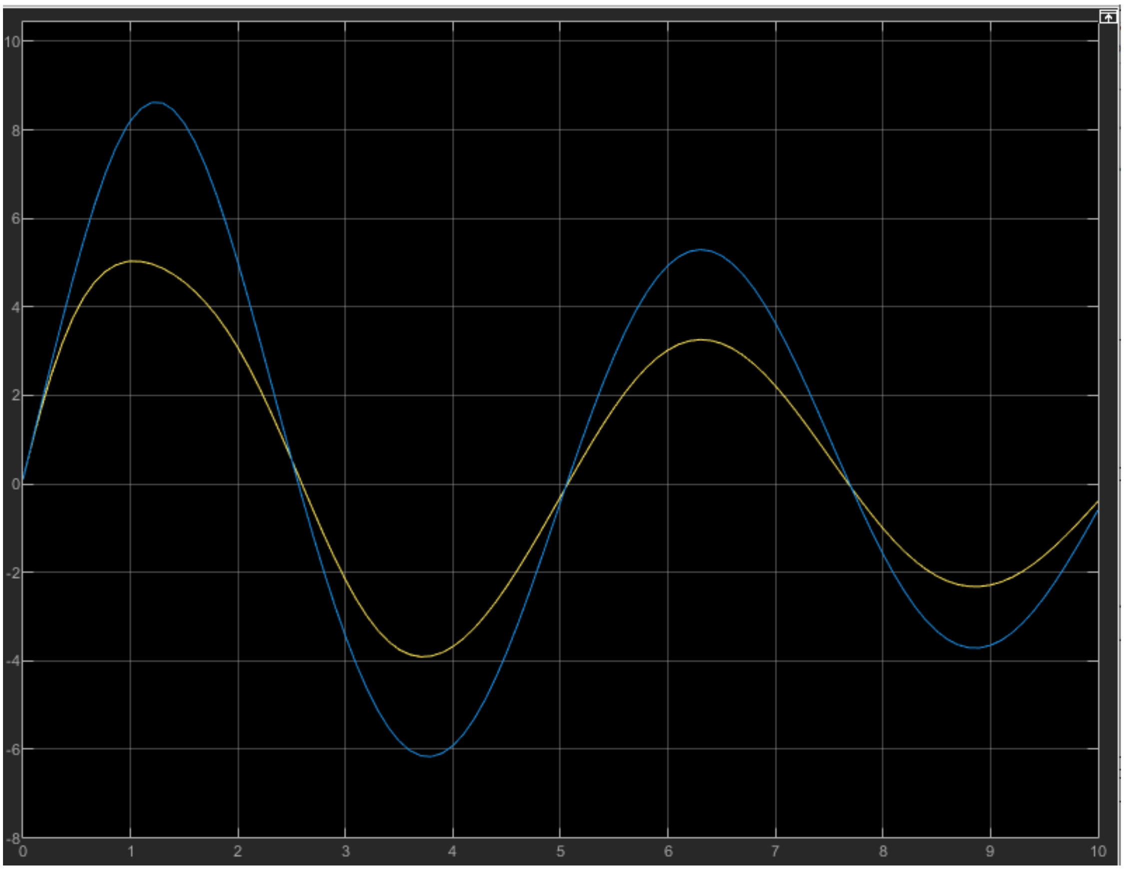
Below is the model that we constructed on Simulink along with the graphs that we obtained from the scope after simulating the model:



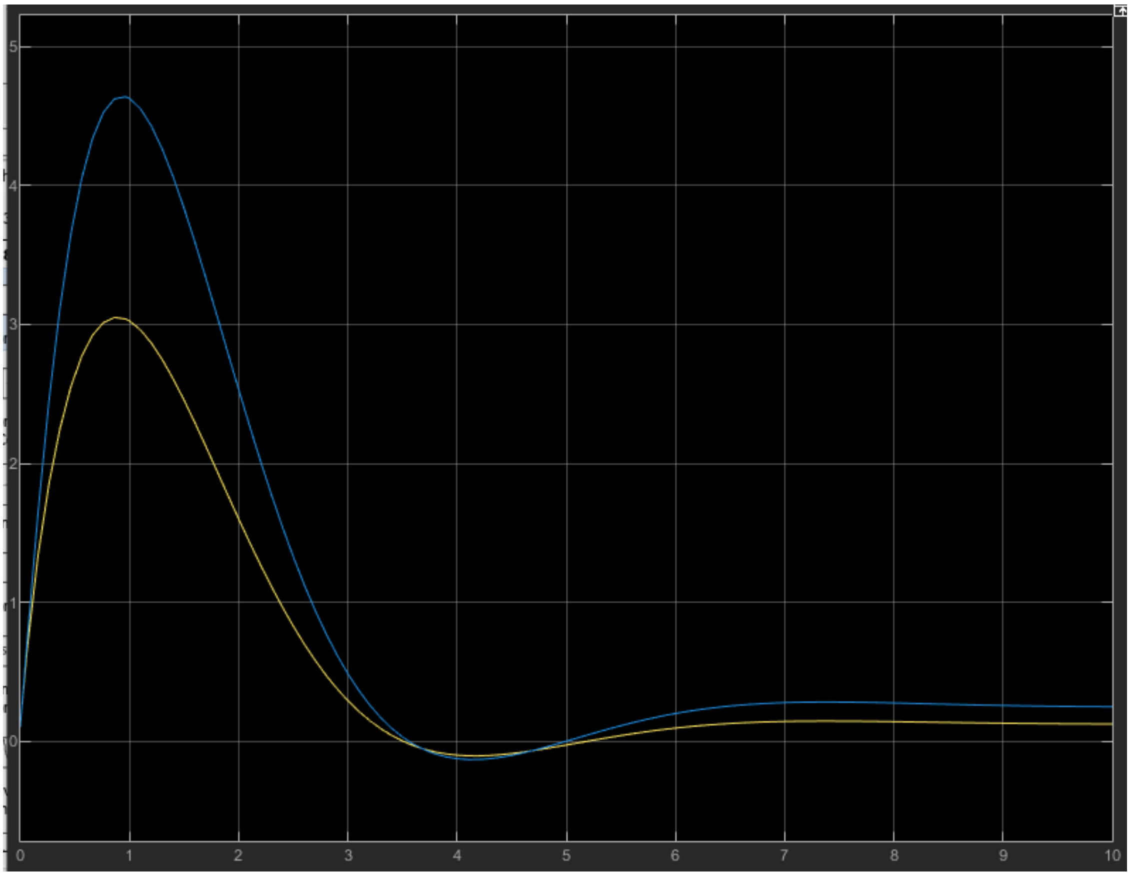
Case 1:



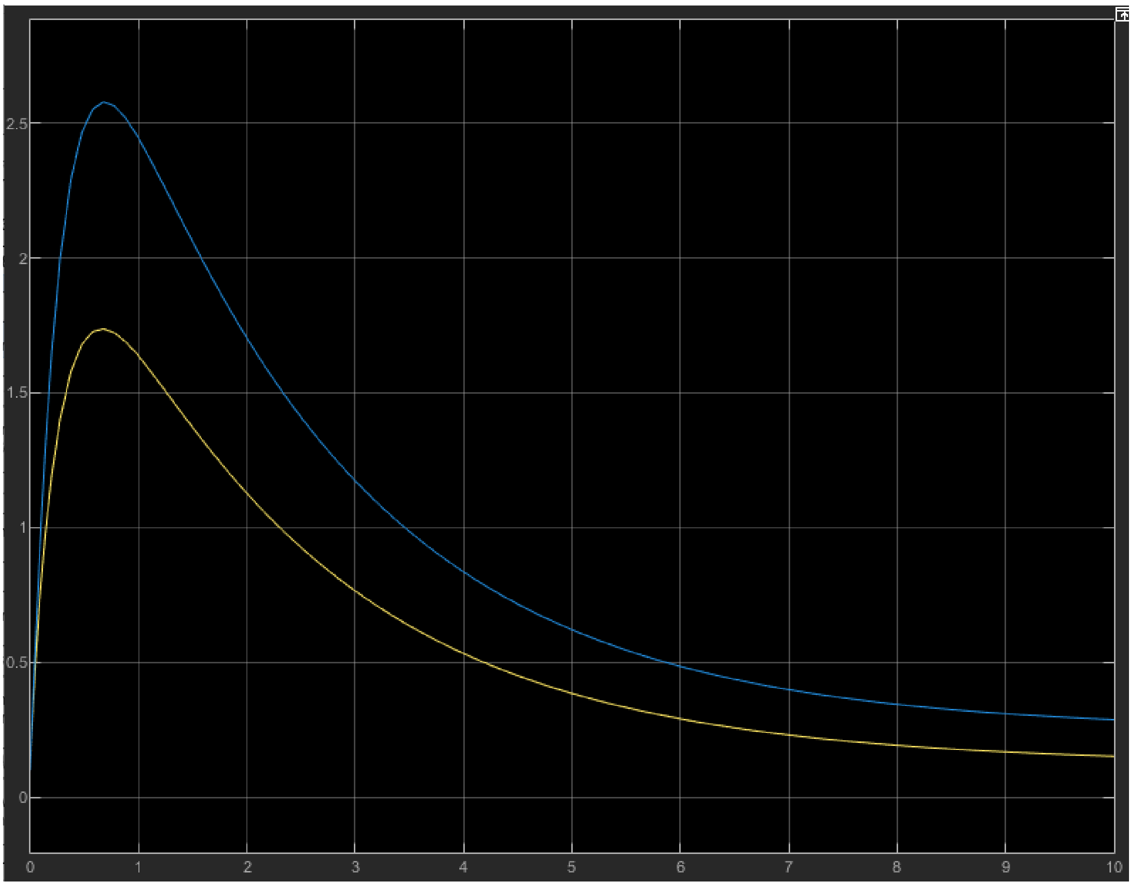
Case 2:



Case 3:



Case 4:



Summary

In conclusion, in this project we investigated the behavior of a single and two-degree-of-freedom mass-spring-damper system. The analysis involves solving second-order differential equations for different damping cases, including undamped, underdamped, critically damped, and overdamped scenarios. After deriving the general solutions for each case, numerical values were applied using specific parameters: mass ($m = 2$ kg), spring constant ($k = 8$ N/m), initial displacement ($x_0 = 0.1$ m), and initial velocity ($v_0 = 10$ m/s). MATLAB was utilized to simulate and visualize the system's time response for all four damping conditions.

Additionally, a Simulink model was constructed for a two-mass system, incorporating integrators, gain, and sum blocks to represent springs, dampers, and masses. The simulation was performed under a unit step function, and displacements of the two masses were observed. The MATLAB and Simulink simulations confirmed the theoretical calculations and provided a comprehensive understanding of the system’s dynamic behavior under different damping scenarios.

Appendix

Determin solutions for four cases given below

$m\ddot{x} + c\dot{x} + kx = 0$

$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = 0$

$x(0) = x_0$ and $\dot{x}(0) = v_0$

$\zeta = 0 \quad 0 < \zeta < 1 \quad \zeta = 1 \quad \zeta > 1$

$\mathcal{L}\dot{x} = s\,x(s) - x(0)$

$\mathcal{L}\ddot{x} = s^2 x(s) - s\,x(0) - \dot{x}(0)$

$\mathcal{L}1 = \frac{1}{s}$

$\mathcal{L}e^{-at} = \frac{1}{s+a}$

$\mathcal{L}\sin wt = \frac{w}{s^2+w^2}$

$\mathcal{L}\cos wt = \frac{s}{s^2+w^2}$

$\mathcal{L}e^{-at}\sin wt = \frac{w}{(s+a)^2+w^2}$

$\mathcal{L}e^{-at}\cos wt = \frac{s+a}{(s+a)^2+w^2}$

Formula

$(s^2 + 2\zeta w_n s + w_n^2)\,e^{st} = 0$

$s^2 + 2\zeta w_n s + w_n^2 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$A = s^2$

$B = 2\zeta w_n s$

$C = w_n^2$

$s = \frac{-2\zeta w_n \pm \sqrt{4\zeta^2 w_n^2 - 4w_n^2}}{2}$

$s = w_n(-\zeta \pm \sqrt{\zeta^2 - 1})$

$\zeta = 0$

$s = \pm i w_n$

$s = e^{(iw_n t)}, e^{-iw_n t}$

$x = c_1 e^{(iw_n t)} + c_2 e^{-iw_n t}$

$\boxed{x = c_3 \cos w_n t + c_4 \sin w_n t}$

$0 < \zeta < 1$

$s = (-\zeta \pm i\sqrt{1-\zeta^2})\,w_n$

$s = e^{(-\zeta w_n t)} e^{(i w_d t)},\ e^{(\zeta w_n t)} e^{(-i w_d t)}$

$w_d = w_n\sqrt{1-\zeta^2}$

$x = e^{(-\zeta w_n t)}\left[c_1 e^{(i w_n t)} + c_2 e^{-i w_n t}\right]$

$\boxed{x = e^{(-\zeta w_n t)}\left[c_3 \cos w_d t + c_4 \sin w_d t\right]}$

$\zeta = 1 \quad s = w_n$

$\boxed{s = (c_1 + c_2 t)\,e^{(-w_n t)}}$

$w_d = w_n\sqrt{\zeta^2 - 1}$

$\zeta > 1$

$s = (-\zeta \pm \sqrt{1-\zeta^2})\,w_n$

$\boxed{x = e^{(-\zeta w_n t)}\left[c_1 e^{(w_d t)} + c_2 e^{(-w_d t)}\right]}$

Part 1

Question 2

$\frac{m}{1} = \frac{c}{2\zeta w_n} = \frac{k}{w_n^2}$

So,

$\zeta = \frac{c}{2\sqrt{km}}, \quad w_n = \sqrt{k/m}, \quad$ when $c = 0$

$\zeta = 0, \ w_n = 2 \text{ rad/s}$

Sub values for $\zeta = 0$

$x = c_3 \cos w_n t + c_4 \sin w_n t$

$x = c_3 \cos 2t + c_4 \sin 2t$

Apply initial cond.

$x(0) = 0.1$

$c_3 = 0.1$

$\dot{x}(0) = 10$

$2c_4 = 10, \ c_4 = 5$

$\therefore \ x = 0.1 \cos 2t + 5 \sin 2t$

When $C = 20\ Ns/m$

$\zeta = 2.5$, $\omega_n = 2\ rad/s$

Sub for $\zeta > 1$

$x = e^{(-\omega_n t)}\left[C_1 e^{(\omega_d t)} + C_2 e^{(-\omega_d t)}\right]$

$\omega_d = 2\sqrt{2.5^2 - 1}$, $\omega_d = 4.58\ rad/s$

$x = e^{(-5t)}\left[C_1 e^{(4.58t)} + C_2 e^{(-4.58t)}\right]$

Apply initial Cond.

$x(0) = 0.1$

$C_1 + C_2 = 0.1$

$\dot{x}(0) = 10$

$-0.42C_1 - 9.58C_2 = 10$

$C_1 = 1.20$, $C_2 = -1.10$

$\therefore\ x = e^{(-5t)}\left[1.2e^{(4.58t)} - 1.1e^{(-4.58t)}\right]$

For $C = 1\ Ns/m$

$\zeta = 0.125$, $\omega_n = 2\ rad/s$

Sub for $0 < \zeta < 1$

$x = e^{(-\omega_n t)}\left[C_3 \cos \omega_d t + C_4 \sin \omega_d t\right]$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\omega_d = (2\ rad/s)\sqrt{1-0.125^2}$, $\omega_d = 1.98\ rad/s$

$x = e^{(-0.25t)}\left[C_3 \cos 1.98t + C_4 \sin 1.98t\right]$

Apply initial Cond.

$x(0) = 0.1$

$C_3 = 0.1$

$\dot{x}(0) = 10$

$1.98C_4 - 0.025 = 10$

$C_4 = 5.06$

$\therefore\ x = e^{(-0.25t)}\left[0.1\cos 1.98t + 5.06\sin 1.98t\right]$

When $C = 8\ Ns/m$

$\zeta = 1$, $\omega_n = 2\ rad/s$

Sub for $\zeta = 1$

$x = (C_1 + C_2 t)e^{(-\omega_n t)}$

$x = (C_1 + C_2 t)e^{(-2t)}$

Apply initial Cond.

$x(0) = 0.1$

$C_1 = 0.1$

$\dot{x}(0) = 10$

$C_2 - 0.2 = 10$, $C_2 = 10.2$

$\therefore\ x = (0.1 + 10.2t)e^{(-2t)}$

Assumptions

Linear Behavior of Springs and Dampers:

- *Assumption*: The springs and dampers follow Hooke's Law and Newton's damping law which means that the force exerted by the spring is proportional to displacement, and the damping force is proportional to velocity.
- *Reason*: This simplifies the mathematical model and allows us to use linear differential equations. Real springs and dampers may exhibit nonlinear behavior at extreme displacements or velocities, but for small oscillations, this assumption is valid.

Neglecting Friction and Air Resistance:

- *Assumption*: External forces like friction and air resistance are negligible.
- *Reason*: The focus is on understanding the system behavior due to internal damping, so excluding external forces simplifies the analysis. Friction and air resistance could introduce additional energy dissipation, but they are assumed small compared to the internal damping in the system.

Rigid Masses:

- *Assumption*: The masses in the system are considered rigid bodies with no internal deformations.
- *Reason*: This allows us to model the system as simple point masses in a two-mass-spring-damper system. In reality, objects might deform under forces, but these deformations are assumed negligible in this project.

Small Oscillations:

- *Assumption*: The system undergoes small oscillations around the equilibrium position.
- *Reason*: Small oscillations justify the use of linear equations of motion. For larger displacements, non-linear effects such as large deformations and complex damping behavior may need to be considered, but these are ignored for simplicity.