Machine Learning Prerequisites Diagnostic Test — Solutions

Part A — Calculus

Question 1

$$f(x) = 3x^2 - 5x + 7$$

(a)

$$f'(x) = 6x - 5$$

(b)

$$f'(2) = 6 \cdot 2 - 5 = 7$$

Question 2

$$f(x) = \sin(x^2)$$

Using the chain rule:

$$f'(x) = \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

Question 3

$$f(x, y) = x^2 y + 3y^2$$

(a)

$$\frac{\partial f}{\partial x} = 2xy$$

(b)

$$\frac{\partial f}{\partial y} = x^2 + 6y$$

(c)

$$\nabla f(x, y) = \begin{pmatrix} 2xy \\ x^2 + 6y \end{pmatrix}$$

Part B — Linear Algebra

Question 4

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ 10 & 20 \end{pmatrix}$$

Question 5

$$C = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix}$$

(a)

$$C^\top = \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 4 \end{pmatrix}$$

(b)

$$C \in \mathbb{R}^{2\times 3}, \quad C^\top \in \mathbb{R}^{3\times 2}$$

Question 6

$$\begin{cases} x + y = 3 \\ 2x + y = 4 \end{cases}$$

Subtracting the first equation from the second:

$$x = 1$$

Substitute into $x + y = 3$:

$$y = 2$$

$$(x, y) = (1, 2)$$

Question 7

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$$

$$= \lambda^2 - 4\lambda + 3 = 0$$

Hence:

$$\lambda_1 = 3, \quad \lambda_2 = 1$$

(b) For $\lambda = 3$:

$$(A - 3I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = 1$:

$$(A - I)v = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v = 0 \Rightarrow v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Question 8

(a) An eigenvector is a direction that is preserved (up to scaling) by a linear transformation.

(b) Eigenvalues and eigenvectors identify principal directions and variances in data (e.g. PCA).

Question 9

(a) A is invertible if and only if $\det(A) \neq 0$.

(b) Non-invertibility means the system $Ax = b$ has either no solution or infinitely many solutions.

Part C — Python Programming

Question 10

```
def sum_of_squares(n):
    s = 0
    for i in range(1, n + 1):
        s += i * i
    return s
```

Question 11

```
def sign(x):
    if x > 0:
        return "positive"
    elif x < 0:
        return "negative"
    else:
        return "zero"
```

Question 12

```
import math
y = [math.sqrt(v) for v in x]
```

Part D — NumPy

Question 13

```
np.arange(10)
```

Question 14

```
x**2
np.mean(x)
```

Question 15

```
A @ b
np.linalg.solve(A, b)
```

Question 16

(a) $X \in \mathbb{R}^{100\times 5}$

(b) $X@w \in \mathbb{R}^{100}$

(c) A linear combination of features producing one scalar output per data point

Optional Bonus

Question 17

$$f(x, y) = x^2 + y^2$$

(a)

$$\nabla f(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

(b) The gradient points in the direction of steepest increase of the function.