Problem Set 1

Methods in Computational Neuroscience 2013

August 2, 2013

Dynamics and Biophysics

1. (a) Compute the F-I curve for the leaky integrate-and-fire model:

$$C\frac{dV}{dt} = -g_L(V - V_L) + I, \tag{1}$$

with a threshold voltage V_T and reset voltage V_R such that $V_R < V_T$.

(b) Do the same for a version in which the cause of firing is synaptic:

$$C\frac{dV}{dt} = -g_L(V - V_L) + g_S(V - V_S), \tag{2}$$

where $V_S > V_T$, with $g_S > 0$ a variable parameter. Plot the curve as a function of g_S when $C = 1, V_L = -65, V_S = 0, V_T = -65$. For fun, simulate the output using Matlab, XPP, or your cell phone.

(c) Find all the bifurcations for the subthreshold dynamics of the quadratic integrate-and-fire model with adaptation as you vary I:

$$\begin{aligned} \frac{dV}{dt} &= V^2 + I - W \\ \frac{dW}{dt} &= a(bV - W). \end{aligned} \tag{3}$$

In particular, find conditions where there are 2 equilibria and no equilibria and determine the stability of the equilibria when they exist. Find the curve of the Hopf bifurcations in (I, b) space assuming $a > 0$ and the curve of saddle-nodes as well. Where do the two curves meet (Bogdanov-Takens bifurcation) as a function of a?

(d) (**Fun with planar dynamics**) In all cases below, we study $\frac{dV}{dt} = f(V, w), \frac{dw}{dt} = g(V, w)$. For all but the last part of the problem, we assume the standard potassium/voltage scenario so that $f_w < 0, g_w < 0, g_V > 0$, where f_w means the partial derivative with respect to w, etc.

i. Suppose that $f_V < 0$ at an equilibrium point. Prove that the equilibrium is stable.

ii. Now, we get a bit more specific. Suppose that $f = 0$ has the usual cubic-shaped nullclines and $g = 0$ has the usual monotonically increasing nullclines.

A. Show that if there is an equilibrium on the middle branch of the cubic, then $f_V > 0$.

B. Suppose that there is a middle branch equilibrium. Prove that the equilibrium is a saddle (node) if the slope of the w-nullcline iss less (greater) than the slope of the V-nullcline.

C. Prove that there are no limit cycles in a planar system when $f_w g_V < 0$.

2. **Single Neuron Dynamics**

The folloing is a simplified version of the Hodgkin-Huxley neuron with only two dynamical variables.

$$C\frac{dV}{dt} = -G_{Na}m_\infty^3 h(V - E_{Na}) - G_K n^4(V - E_k) - G_{leak}(V - E_{leak}) + I_{in}(t)$$
$$\tau\frac{dn}{dt} = n_\infty - n$$

where

$$m_\infty = \frac{1}{1 + \exp\left((-40 - V)/k_m\right)}$$
$$h = 0.89 - 1.1n$$
$$n_\infty = \frac{1}{1 + \exp\left((V_n - V)/k_n\right)}$$

with common parameters $C = 1\ \mu\text{F/cm}^2$, $G_K = 1.44\ \text{mS/cm}^2$, $E_{leak} = -54\ \text{mV}$, and $k_m = 7\ \text{mV}$. We have provided an XPP file for the model (`hhxpp.ode`), a MATLAB script to integrate the equations (`HH.m`), and a file to compute the phase plane in `pplane` (`hhpplane.pps`).

We want you to study the model for three different parameter sets:

i. $G_{Na} = 2\ \text{mS/cm}^2$, $G_{leak} = 0.2\ \text{mS/cm}^2$, $E_{Na} = 50\ \text{mV}$, $E_K = -77\ \text{mV}$, $V_n = -45\ \text{mV}$, $k_n = 15\ \text{mV}$, and $\tau = 10\ \text{ms}$.

ii. $G_{Na} = 2\ \text{mS/cm}^2$, $G_{leak} = 0.35\ \text{mS/cm}^2$, $E_{Na} = 65\ \text{mV}$, $E_K = -90\ \text{mV}$, $V_n = -40\ \text{mV}$, $k_n = 5\ \text{mV}$, and $\tau = 10\ \text{ms}$.

iii. $G_{Na} = 0.25\ \text{mS/cm}^2$, $G_{leak} = 0.012\ \text{mS/cm}^2$, $E_{Na} = 30\ \text{mV}$, $E_K = -77\ \text{mV}$, $V_n = -44\ \text{mV}$, $k_n = 10\ \text{mV}$, and $\tau = 50\ \text{ms}$.

For each set of parameters, we want you to:

(a) Make a phase portrait, plot the V and n nullclines, and identify the fixed points for $I_{in} = 0$. What fixed points are stable, unstable? How do you know?

(b) Study the neuron for different constant input currents I_{in} of no more than a few μA/cm^2. For each I_{in}, examine the phase portrait and simulate a short voltage trace. At some special value of I_{in}, does the neuron's behavior change drastically? What type of bifurcation is demonstrated? Identify whether the neuron is Hodgkin Type 1, 2, or 3. How can you identify the type of bifurcation and Hodgkin Type from the geometry of the nullclines?

(c) **Excitable neurons**. Study the response of the neuron to small impulses (both positive and negative). For cell (1), do this around $I_{in} = 1.3$ μA/cm^2, (2) $I_{in} = 0$, and (3) $I_{in} = 0.5$ μA/cm^2. How does the form of the impulse response relate to the bifurcation type? Is the neuron more of an integrator, resonator, or something else? Now make the pulse large enough to trigger a spike. Examine the after-polarization. Does the neuron show any after-spike facilitation? Study the response of the neuron to inhibitory inputs (with respect to the mean I_{in} above) of varying duration and amplitude. Can you get the neuron to show post-inhibitory (rebound) spiking?

Singular Value Decomposition

Recall Larry Abbott's talk and consider the following two stochastic processes driven by independent, as well as common, noise sources:

$$x_i^t = \sum_{\alpha=1}^{N} \lambda_\alpha W_{i\alpha}^x \xi_{\alpha t}^x + \sigma Z_i^x \eta_t \tag{4}$$

$$y_i^t = \sum_{\alpha=1}^{N} \lambda_\alpha W_{i\alpha}^y \xi_{\alpha t}^y + \sigma Z_i^y \eta_t \tag{5}$$

where

- x_i^t, y_i^t are the activation of the ith unit at time t for the first and second processes, respectively. Each process will have $n = 10$ units ($i = 1 \ldots 10$) and $T = 10000$ time bins ($t = 1 \ldots 10000$) .
- $W_{i\alpha}^x, W_{i\alpha}^y$ are independent orthogonal matrices that represent the feedforward basis vectors for part of the processes that is driven by independent noise.
- Z_i^x, Z_i^y are random feedforward weights for the common noise source. They are normalized such that $\sum_{i=1}^n (Z_i^x)^2 = \sum_{i=1}^n (Z_i^y)^2 = 1$.
- λ_α weights the patterns in the W's. Define $\lambda_\alpha = ae^{-\alpha/5}$, where a is chosen such that $\sum_{\alpha=1}^n \lambda_\alpha^2 = 1$.
- σ is the scale of the common input with respect to the independent input. We will choose $\sigma = 1$.
- $\xi_{\alpha t}^x, \xi_{\alpha t}^y, \eta_{\alpha t}$ are the noise sources. For all α, t they are standard normal random variables.

1. Generate the data described above. MATLAB functions you may find useful are: `randn, diag, norm, orth` (for documentation on MATLAB functions type `doc FunctionName` in the command line). After you have x, y, center the data (subtract the average unit activation).

2. Compute the cross correlation matrix (averaged across time, so it should be a 10×10 matrix): $C^n_{ij} = \sum_{t=1}^{T} x^t_i y^t_j$, and find its singular value decomposition ($C^n = U_x S U_y^T$).

   When doing so, you are recursively finding *unit* vectors u_x, u_y that maximize the expression $u_x^T C^n u_y$ (this contraction is equal to the largest singular value) and which are orthogonal to the vectors u_x, u_y chosen in the previous iterations. The vectors in every iteration are orthogonal to all of those of the previous iterations.

3. What is the dimensionality of the data as captured by SVD? Are the column(s) of U_x, U_y that correspond to the outstanding singular values a good approximation of Z^x and Z^y? Find the angles between the vectors of the reduced spaces and the Z's you used to generate the data.

4. Bonus - How do you expect the quality of the reconstruction to depend on σ? Plot the angle between the first column of U_y and Z^y as a function of σ.