Assignment 2

$$J(n) = \frac{1}{2}|e(n)|^2 = \frac{1}{2}[y_d(n) - y(n)]^2 \quad -(1)$$

$$y(n) = \sum_{K=1}^{N} w_K(n)\,\phi\{x(n), c_K, \sigma_K\} \quad -(2)$$

$$J(n) = \frac{1}{2}\left[y_d(n) - \sum_{K=1}^{N} w_K(n)\,\phi\{x(n), c_K, \sigma_K\}\right]^2 \quad -(3)$$

$$J(n) = \frac{1}{2}\left[y_d(n) - \sum_{K=1}^{N} w_K(n)\, e^{-\left[\frac{(\|x(n) - c_K(n)\|)^2}{2\sigma_K^2(n)}\right]^2}\right] \quad -(4)$$

$$w(n+1) = w(n) - \mu_w \frac{\partial}{\partial w} J(n)\Big|_{w=w(n)} \quad -(5)$$

$$\psi(n) = [\phi\{x(n), c_1, \sigma_1\}, \ldots \phi\{x(n), c_N, \sigma_N\}]^T \quad -(6)$$

a. Show that: $w(n+1) = w(n) + \mu_w e(n)\psi(n) \quad -(7)$

Comparing (5) and (7)

$$\cancel{\mu_w} e(n)\psi(n) = -\cancel{\mu_w}\frac{\partial J(n)}{\partial w}\Big|_{w=w(n)}$$

$$e(n)\psi(n) = -\frac{\partial J(n)}{\partial w}\Big|_{w=w(n)} \quad -(8)$$

$$\frac{\partial J(n)}{\partial w} = \frac{\partial J(n)}{\partial e(n)} * \frac{\partial e(n)}{\partial w(n)}$$ – from chain rule

$$J(n) = \frac{1}{2}|e(n)|^2 \quad \therefore \frac{\partial J(n)}{\partial e(n)} = e(n)$$

$$e(n) = y_d(n) - \sum_{K=1}^{N} w_K(n)\, \phi\{x(n), c_K, \sigma_K\}$$

$$\frac{\partial e(n)}{\partial w(n)} = - \sum_{K=1}^{N} \phi\{x(n), c_K, \sigma_K\}$$

$$= -[\phi\{x(n), c_1, \sigma_1\}, \cdots \phi\{x(n), c_N, \sigma_N\}]^T$$

$$\therefore \frac{\partial J(n)}{\partial w(n)} = e(n) \cdot -[\phi\{x(n), c_1, \sigma_1\}, \cdots \phi\{x(n), c_N, \sigma_N\}]^T$$

Remember (6)

$$\therefore \frac{\partial J(n)}{\partial w(n)} = -e(n)\,\varphi(n)$$

From (8)

$$e(n)\,\varphi(n) = -\frac{\partial J(n)}{\partial w(n)}$$

$$\therefore w(n+1) = w(n) + \mu_w e(n)\,\varphi(n).$$

b. $$c_K(n+1) = c_K(n) - \mu_c \frac{\partial}{\partial c_K} J(n) \Big|_{c_K = c_K(n)} \quad -(9)$$

Show that: $$c_K(n+1) = c_K(n) + \mu_c \frac{e(n)\, w_K(n)}{\sigma_K^2(n)} \phi\{x(n), c_K(n), \sigma_K\} [x(n) - c_K(n)] \quad -(10)$$

comparing (9) and (10)

$$\frac{e(n)\, w_K(n)}{\sigma_K^2(n)} \phi\{x(n), c_K(n), \sigma_K\} [x(n) - c_K(n)] = -\frac{\partial J(n)}{\partial c_K}\Big|_{c_K = c_K(n)} \quad -(12)$$

$$\frac{\partial J(n)}{\partial c_K} = \frac{\partial J(n)}{\partial e(n)} * \frac{\partial e(n)}{\partial c_K}$$

From (a), $\dfrac{\partial J(n)}{\partial e(n)} = e(n)$

From (4), $e(n) = y_d(n) - \sum_{K=1}^{N} w_K(n)\, e^{-\left(\frac{\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}\right)}$

let $z = \dfrac{-\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}$ — (11)

$$\frac{\partial e(n)}{\partial c_K} = \frac{\partial e(n)}{\partial z} * \frac{\partial z}{\partial c_K}$$

From (11), $e(n) = y_d(n) - \sum_{K=1}^{N} w_K(n)\, e^{z}$

$$\frac{\partial e(n)}{\partial z} = -\sum_{K=1}^{N} w_K(n)\, e^{z}$$

$$z = -\left(\frac{x(n)^2 - 2x(n)c_K(n) + c_K(n)^2}{2\sigma_K^2(n)}\right)$$

$$= \frac{2x(n)c_K(n) - c_K(n)^2 - x(n)^2}{2\sigma_K^2(n)}$$

$$\frac{\partial z}{\partial c_K} = \frac{2x(n) - 2c_K(n)}{2\sigma_K^2(n)} = \frac{x(n) - c_K(n)}{\sigma_K^2(n)}$$

$$\frac{\partial e(n)}{\partial c_K} = \frac{x(n) - c_K(n)}{\sigma_K^2(n)} * -\sum_{K=1}^{N} w_K(n)\, e^{z}$$

$$\therefore \frac{\partial J(n)}{\partial c_K(n)} = e(n) * \frac{x(n) - c_K(n)}{\sigma_K^2(n)} * -\sum_{K=1}^{N} w_K(n) e^{z}$$

comparing (3) and (4)

$$\phi\{x(n), c_K(n), \sigma_K(n)\} = e^{-\frac{\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}}$$

$$\frac{\partial J(n)}{\partial c_K(n)} = -\frac{e(n)\, w_K(n)}{\sigma_K^2(n)} \phi\{x(n), c_K(n), \sigma_K(n)\} (x(n) - c_K(n))$$

The above is the equivalent to (12) from earlier.

$$\therefore c_K(n+1) = c_K(n) + \mu_c \frac{e(n) w_K(n)}{\sigma_K^2(n)} \phi\{x(n), c_K(n), \sigma_K\} [x(n) - c_K(n)]$$

c. $$\sigma_K(n+1) = \sigma_K(n) - \mu_\sigma \frac{\partial}{\partial \sigma_K} J(n) \Big|_{\sigma_K = \sigma_K(n)} \quad -(13)$$

show that:

$$\sigma_K(n+1) = \sigma_K(n) + \mu_\sigma \frac{e(n)\, w_K(n)}{\sigma_K^3(n)} \phi\{x(n), c_K(n), \sigma_K\} (|x(n) - c_K(n)|)^2 \quad -(14)$$

comparing (13) and (14)

$$\frac{e(n)\, w_K(n)}{\sigma_K^3(n)} \phi\{x(n), c_K(n), \sigma_K\} (|x(n) - c_K(n)|)^2 = -\frac{\partial J(n)}{\partial \sigma_K} \quad -(15)$$

$$\frac{\partial J(n)}{\partial \sigma_K(n)} = \frac{\partial J(n)}{\partial e(n)} * \frac{\partial e(n)}{\partial \sigma_K(n)}$$

From (a), $\frac{\partial J(n)}{\partial e(n)} = e(n)$

From (4), $e(n) = y_d(n) - \sum_{K=1}^{N} w_K(n)\, e^{-\left(\frac{\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}\right)}$

Let $z(n) = \frac{-\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}$ — (16)

Using chain rule,

$$\frac{\partial e(n)}{\partial \sigma_K(n)} = \frac{\partial e(n)}{\partial z(n)} * \frac{\partial z(n)}{\partial \sigma_K(n)}$$

From (5) $\frac{\partial e(n)}{\partial z} = -\sum_{K=1}^{N} w_K(n)\, e^{z}$

$$z = -\left(\frac{[x(n) - c_K(n)]^2}{2}\right) * \sigma_K^{-2}(n)$$

$$\frac{\partial z}{\partial \sigma_K} = -\cancel{2} * \frac{(x(n) - c_K(n))^2}{\cancel{2}} * \sigma_K^{-3}(n)$$

$$= (x(n) - c_K(n))^2\, \sigma_K^{-3}(n)$$

$$\therefore \frac{\partial e(n)}{\partial \sigma_K(n)} = \frac{|x(n) - c_K(n)|^2}{\sigma_K^3(n)} * -\sum_{K=1}^{N} w_K(n)\, e^{z}$$

$$\therefore \frac{\partial J(n)}{\partial \sigma_K(n)} = e(n) * \frac{|x(n) - c_K(n)|^2}{\sigma_K^3(n)} * -\sum_{K=1}^{N} w_K(n) e^{z}$$

Remember from (3) and (4)

$$\phi\{x(n), c_K(n), \sigma_K(n)\} = e^{-\frac{\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}}$$

$$\therefore \frac{\partial J(n)}{\partial \sigma_K(n)} = -\frac{e(n) w_K(n)}{\sigma_K^3(n)} \phi\{x(n), c_K(n), \sigma_K(n)\} [|x(n) - c_K(n)|]^2$$

... proving (15)

Hence,

$$\sigma_K(n+1) = \sigma_K(n) + \mu_\sigma \frac{e(n) w_K(n)}{\sigma_K^3(n)} \phi\{x(n), c_K(n), \sigma_K(n)\} [|x(n) - c_K(n)|]^2$$