A practical theorem on gravitational wave backgrounds

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ABSTRACT

There is an extremely simple relationship between the spectrum of the gravitational wave background produced by a cosmological distribution of discrete gravitational wave sources, the total time-integrated energy spectrum of an individual source, and the present-day comoving number density of remnants. Stated in this way, the background is entirely independent of the cosmology, and only weakly dependent on the evolutionary history of the sources. This relationship allows one easily to compute the amplitude and spectrum of cosmic gravitational wave backgrounds from a broad range of astrophysical sources, and to evaluate the uncertainties therein.

Key words: gravitational waves – diffuse radiation – binaries: close – black hole physics – relativity

1 INTRODUCTION

The strongest discrete sources of gravitational waves are those which radiate large amounts of energy in a time (short compared to the age of the universe) before or after a catastrophic event. Examples include supernovae, accretion-induced collapse, black-hole formation events and merging compact binary systems, which may involve white dwarfs, neutron stars or black holes of the stellar or super-massive sort. It is becoming increasingly important to understand the 'brightness' of the night sky in gravitational waves due to the cosmic superposition of such sources. This is crucial to the design of LISA, the NASA/ESA Laser Interferometer Space Antenna, and for proposed follow-on missions at higher and lower frequencies which might search for primordial stochastic backgrounds, e.g. from inflation or phase transitions in the early universe. There have been a number of recent efforts to compute the backgrounds from particular classes of sources, by numerically integrating spectra or waveforms of sources with assumed evolutionary histories over cosmological volumes (Coward, Burman & Blair 2001; Schneider et al 2001; Ferrari, Matarrese & Schneider 1999; Rajagopal & Romani 1995).

We show here that there is an extremely simple relationship between the spectrum of the gravitational wave background produced by a cosmological distribution of such sources and the *present-day comoving number density of remnants*. The background is entirely independent of the cosmology, and in most cases is almost independent of the evolutionary history of the sources. This simple relation allows one to evaluate quickly uncertainties in estimates of gravitational wave backgrounds and to survey new backgrounds. A companion paper (Phinney 2001) in this way surveys the gravitational wave sky, and points out a number of previously unrecognized backgrounds of potential significance for LISA and future missions.

In section 2 we give a simple physical derivation of the theorem. Section 3 applies the theorem to the backgrounds produced by non-relativistic binaries in circular orbits, and compares to previous calculations of the cosmic background from double-degenerate binaries and merging super-massive black holes in galactic nuclei. The reader wanting quick numbers for other circular orbit sources should use equations 15–17. Section 4 presents a formal derivation of the theorem, and shows explicitly that it is valid for all source lifetimes, including ones much longer and much shorter than the duration of a measurement experiment.

2 THE THEOREM: PHYSICAL DERIVATION

Let f_r be the frequency of gravitational waves in a source's cosmic rest frame, and f the frequency of those waves observed today on earth, $f_r = f(1+z)$. Let the total outgoing energy emitted in gravitational waves between frequency f_r and $f_r + df_r$ be

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$$\frac{dE_{gw}}{df_r}df_r \ . \tag{1}$$

This energy, like f_r is measured in the source's cosmic rest frame, and is integrated over all solid angles and over the entire radiating lifetime of the source.

Let the number of events in unit comoving volume which occur between redshift z and z + dz be N(z)dz. Define $\Omega_{gw}(f)$ to be the present-day energy density per logarithmic frequency interval, in gravitational waves of frequency f, divided by the rest-mass energy density $\rho_c c^2$ that would be required to close the universe. Then the total present day energy density in gravitational radiation is

$$\mathcal{E}_{gw} \equiv \int_0^\infty \rho_c c^2 \Omega_{gw}(f) \, df/f \equiv \int_0^\infty \frac{\pi}{4} \frac{c^2}{G} f^2 h_c^2(f) \frac{df}{f} \,, \tag{2}$$

where $\rho_c = 3H_0^2/(8\pi G)$ is the critical density of the universe, and h_c is the characteristic amplitude of the gravitational wave spectrum over a logarithmic frequency interval $d \ln f = df/f$. h_c is related to the one-sided $(0 < f < \infty)$ spectral density $S_{h,1}$ of the gravitational wave background (cf. Thorne (1987)) by $h_c^2(f) = fS_{h,1}(f)$. If the spectral density $S_{h,2}$ is defined to be two-sided $(-\infty < f < \infty)$, cf. Ungarelli & Vecchio (2001), Cornish (2001)), the relation is $h_c^2(f) = 2fS_{h,2}(f)$. $S_h^{1/2}$ is often called the strain noise, since the mean square signal strain output of an interferometer is simply the integral over all frequencies of S_h times the response function $\mathcal{R}(f)$ of the interferometer (of order unity for wavelengths much longer than the interferometer –cf. section 3.2 of Cornish & Larson (2001)).

In any homogeneous and isotropic universe, the present-day energy density \mathcal{E}_{gw} must be equal to sum of the energy densities radiated at each redshift, divided by (1+z) to account for the redshifting of the gravitons since emission:

$$\mathcal{E}_{gw} \equiv \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} \frac{dE_{gw}}{df_r} f_r \frac{df_r}{f_r} dz \tag{3}$$

$$= \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} f_r \frac{dE_{gw}}{df_r} dz \frac{df}{f} . \tag{4}$$

Equating the two expressions in equations (2) and (4) for \mathcal{E}_{gw} frequency by frequency, we find

$$\rho_c c^2 \Omega_{gw}(f) = \frac{\pi}{4} \frac{c^2}{G} f^2 h_c^2(f) = \int_0^\infty N(z) \frac{1}{1+z} \left(f_r \frac{dE_{gw}}{df_r} \right) \Big|_{f_r = f(1+z)} dz . \tag{5}$$

This theorem is our principal result. It has the simple physical interpretation that the energy density in gravitational waves per log frequency interval is equal to the comoving number density of event remnants, times the (redshifted) energy each event produced per log frequency interval. Notice that the theorem does not depend upon the cosmological model, except for the assumption of a homogeneous and isotropic universe. Nor does it depend on any property of the sources (beaming, polarization, etc) except for their time-integrated energy spectrum, provided they are randomly oriented with respect to earth. If more than one type of source is important, the right hand side of equation 5 should be summed over the source types i, i.e.

$$N(z)\left(f_r\frac{dE_{gw}}{df_r}\right)$$
 is replaced by $\sum_i N_i(z)\left(f_r\frac{dE_{gw,i}}{df_r}\right)$. (6)

The alert reader will detect some kinship between the frequency-integrated version of this theorem (equation 4) and that of Soltan (1982) relating the contribution of quasars to the electromagnetic brightness of the night sky to the local space density of remnant super-massive black holes. This relation becomes even closer if the source we are considering is black holes or other compact objects of mass M_2 gravitationally captured in circular orbits by super-massive black holes of mass $M_1 \gg M_2$. For them $\int (dE_{gw}/df_r) df_r = M_2 c^2 e_b$, where $e_b = (e_{isco} - \int \dot{e}_H dt)$, where e_{isco} is the dimensionless binding energy per unit rest mass of the innermost stable circular orbit (0.057 for Schwarzschild black holes, 0.42 and 0.038 respectively for prograde and retrograde equatorial orbits about maximally rotating Kerr black holes), and $\int \dot{e}_H dt$ is the (small) portion of the binding energy radiated down the black hole horizon (note that this can be, and is, negative for prograde orbits around rapidly rotating black holes: the orbiting mass extracts some of their rotational energy –see table VII of Finn & Thorne (2000)). The change in the mass of the capturing black hole is $\Delta M_1 = M_2(1 - e_b)$. Thus

$$\mathcal{E}_{gw} = \int_0^\infty N(z) \frac{1}{1+z} \frac{e_b}{1-e_b} \Delta M_1 c^2 dz = \rho_{\bullet} c^2 f_m \left\langle \frac{1}{1+z} \frac{e_b}{1-e_b} \right\rangle, \tag{7}$$

where f_m is the fraction of the present-day comoving mass density ρ_{\bullet} in super-massive black holes which was grown by the capture of compact objects (as opposed, say to growth by axisymmetric accretion of gas which results in negligible gravitational radiation).

In section 4 we give a more mathematical proof of the theorem, but we first give an example of its use.

3 AN EXAMPLE: MERGING BINARIES IN CIRCULAR ORBITS

In the Newtonian limit, a circular binary of component masses M_1 and M_2 which merges due to gravitational radiation losses in less than the age of the universe has

$$\frac{dE_{gw}}{df_r} = \frac{\pi}{3} \frac{1}{G} \frac{(G\mathcal{M})^{5/3}}{(\pi f_r)^{1/3}} \text{ for } f_{\min} < f_r < f_{\max} ,$$
 (8)

where \mathcal{M} is the chirp mass, $\mathcal{M}^{5/3} = M_1 M_2 (M_1 + M_2)^{-1/3}$. Notice that to derive equation 8, one need know nothing about gravitational radiation except that it has twice the orbital frequency, and that the orbital binding energy is removed by the gravitational radiation.

The lower limit f_{\min} is set by the separation of the system at its birth (or circularisation, whichever comes first), and our derivation applies only to those systems whose initial separation is small enough that their time to merge is much shorter than the Hubble time at their birth. For example, a pair of $0.3M_{\odot}$ white dwarfs must have $f_{\min} > 10^{-4}$ Hz in order to merge through gravitational radiation in $< 10^{10}$ y. Over their lifetime, merging systems radiate a broad spectrum of frequencies, up to an upper limit f_{\max} set by the frequency at which the two bodies come into Roche lobe contact, or at which tidal dissipation begins to dominate gravitational radiation in determining the orbital evolution. For a binary whose least massive star is a white dwarf of mass $M_2 = M_{wd}M_{\odot}$, one can use the fit to white dwarf mass-radii of equation (17) of Tout et al (1997) and the approximation (good to better than 10%) that the orbital frequency $\Omega_b = \pi f_r$ of a binary is related to the equivalent volume radius of M_2 's Roche lobe R_{L2} by $\Omega_b^2 = 0.1GM_2/R_{L2}^3$, to find the maximum gravitational wave frequency emitted before the onset of mass transfer:

$$f_{\text{max}} = 0.043 \frac{M_{wd}}{[1 - (M_{wd}/1.44)^{4/3}]^{3/4}} \text{ Hz}$$
 (9)

For pairs of neutron stars, $f_{\rm max} \simeq 1.4 \times 10^3 \, {\rm Hz}$ (Uryu, Shibata & Eriguchi 2000; Shibata & Uryu 2000). For small bodies spiraling into a non-rotating black hole of mass M, $f_{\rm max} \simeq c^3/(6^{3/2}\pi GM) \simeq 4.3 \times 10^3 (M/M_{\odot})^{-1}$ (for more detail and the generalization of equation 8 to fully relativistic orbits around rotating black holes, see Phinney (2001)). Systems whose initial separation is so large that their time to merge is much longer than the Hubble time at their birth will have $f_{\rm max} \simeq f_{\rm min}$, and are neglected here.

Inserting equation 8 into equation 5 gives the gravitational wave background at $f < f_{\text{max}}$ from a population of inspiraling binaries:

$$\Omega_{gw}(f) = \frac{8\pi^{5/3}}{9} \frac{1}{c^2 H_0^2} (G\mathcal{M})^{5/3} f^{2/3} N_0 \langle (1+z)^{-1/3} \rangle \text{ and}$$
(10)

$$h_c^2(f) = \frac{4}{3\pi^{1/3}} \frac{1}{c^2} \frac{(G\mathcal{M})^{5/3}}{f^{4/3}} N_0 \langle (1+z)^{-1/3} \rangle , \qquad (11)$$

where $N_0 = \int_0^\infty N(z) dz$ is the present-day comoving number density of merged remnants, and

$$\langle (1+z)^{-1/3} \rangle = \frac{1}{N_0} \int_{z}^{z_{\text{max}}} \frac{N(z)}{(1+z)^{1/3}} dz . \tag{12}$$

 $z_{\min} = \max[0, f_{\min}/f - 1]$ and $z_{\max} = f_{\max}/f - 1$ can be set respectively to 0 and ∞ except for f just below f_{\min} or f_{\max} . In terms of the merger rate per comoving volume $\dot{N}(t_r)$, $N(z) = \dot{N}dt_r/dz$, where

$$\frac{dt_r}{dz} = \frac{1}{H_0(1+z)E(z)}, \text{ and}$$
(13)

$$E(z) = (\Omega_M (1+z)^3 + \Omega_k (1+z)^2 + \Omega_{\Lambda})^{1/2}.$$
(14)

Currently favoured values are $\Omega_M = 0.33$, $\Omega_{\Lambda} = 0.67$, $\Omega_k = 0$, $H_0 = 65 \text{ km s}^{-1}\text{Mpc}^{-1}$ (cf. Netterfield et al (2001)). The value of $\langle (1+z)^{-1/3} \rangle$ is not very sensitive to the details of N(z). For example, if we take \dot{N} increasing rapidly in the past, proportional to the cosmic star formation rate as a function of redshift given in equation (6) of Madau, Haardt & Pozzetti (2001) in a flat $\Omega_M = 1$ universe, we have $\langle (1+z)^{-1/3} \rangle = 0.74$, while a time-independent \dot{N} in a flat $\Omega_{\Lambda} = 0.67$ universe gives $\langle (1+z)^{-1/3} \rangle = 0.80$.

The one approximation made in equations 10 or 11 is that the lifetime has been assumed short compared to the expansion time of the universe –i.e. all of the inspiral occurs at the same redshift as the merger. We are thus neglecting the differential redshift of the very earliest parts of the inspiral which occur at higher redshift than the merger. But because of the $t_{mrg} \propto a^4$ dependence on initial separation a, sources generally either merge quickly or not at all, so this approximation is poor only for a small percentage of sources with finely tuned parameters.

In convenient numerical forms, equations 10 and 11 and the 1-sided strain noise $S_{h,1}^{1/2}(f > 0) = h_c/\sqrt{f}$ (the two-sided $S_{h,2}^{1/2}(-\infty < f < \infty)$ is $S_{h,1}^{1/2}/\sqrt{2}$) become

$$\Omega_{gw}(f) = 1.3 \times 10^{-17} \left(\frac{\mathcal{M}}{M_{\odot}}\right)^{5/3} \left(\frac{f}{10^{-3} \text{Hz}}\right)^{2/3} \left(\frac{N_0}{\text{Mpc}^{-3}}\right) \frac{\langle (1+z)^{-1/3} \rangle}{0.74} , \qquad (15)$$

$$h_c(f) = 3.0 \times 10^{-24} \left(\frac{\mathcal{M}}{M_{\odot}}\right)^{5/6} \left(\frac{f}{10^{-3} \text{Hz}}\right)^{-2/3} \left(\frac{N_0}{\text{Mpc}^{-3}}\right)^{1/2} \left(\frac{\langle (1+z)^{-1/3} \rangle}{0.74}\right)^{1/2} , \qquad (16)$$

$$S_{h,1}^{1/2}(f) = 1.0 \times 10^{-22} \text{Hz}^{-1/2} \left(\frac{\mathcal{M}}{M_{\odot}}\right)^{5/6} \left(\frac{f}{10^{-3} \text{Hz}}\right)^{-7/6} \left(\frac{N_0}{\text{Mpc}^{-3}}\right)^{1/2} \left(\frac{\langle (1+z)^{-1/3} \rangle}{0.74}\right)^{1/2} . \tag{17}$$

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The redshift averages in equations 15–17 are frequency-independent constants for $f_{\min}/(1+z_*) \lesssim f \lesssim f_{\max}/(1+z_*)$, where z_* is the median source redshift; outside that range the edge effects of equation 12 become important.

We now use these to estimate the cosmic gravitational wave background from merging double-degenerate binaries. For our $H_0=65~\rm km~s^{-1}Mpc^{-1}$, Fukugita, Hogan & Peebles (1998) give the mass density of stars in the universe as $\rho_*=4.4\times10^8 M_{\odot} \rm Mpc^{-3}$. After $10^{10} \rm y$ a stellar population with the initial mass function of Kroupa (2001) has 0.28 white dwarf remnants per solar mass of stellar material, giving a present-day comoving number density of $1.2\times10^8 \rm WD~Mpc^{-3}$. Of these white dwarfs, under standard assumptions of population synthesis models (equal numbers of binaries per log initial orbital separation, binary fraction 0.5, flat binary mass ratio distribution), a fraction 0.015 of the white dwarfs will undergo common envelope evolution leading to a pair of $\sim 0.3 M_{\odot}$ white dwarfs which will merge in $< 10^{10} \rm y$. Thus we estimate $N_0=1.8\times10^6 \rm Mpc^{-3}$. Inserting this and $\mathcal{M}=0.26 M_{\odot}$ into equations 15-17 and defining $f_{-3}=f/10^{-3} \rm Hz$ gives

$$\Omega_{gw}(\text{wd-wd}) = 3 \times 10^{-13} f_{-3}^{2/3} ,$$

$$h_c(\text{wd-wd}) = 1.4 \times 10^{-21} f_{-3}^{-2/3} ,$$

$$S_{h,1}^{1/2}(\text{wd-wd}) = 4 \times 10^{-20} f_{-3}^{-7/6} \text{Hz}^{-1/2} .$$
(18)

These are respectively factors of 4, 2 and 2 less than found by the very detailed calculations of Schneider et al (2001). The difference is due largely to the different choice of normalization and is representative of the true uncertainties (see Phinney (2001) for an extensive discussion): Schneider et al used the Scalo (1986) IMF, a constant Galactic supernova rate of $0.01y^{-1}$, and a binary fraction of 100 percent. If we adopted the same binary fraction, and scaled their Galactic birthrate $(0.044y^{-1})$ to that of Iben, Tutukov & Yungelson (1997) $(0.024y^{-1})$, the numbers would be in perfect (but spurious) agreement. The slight inaccuracies introduced by the theorem's neglect of systems whose merger occurs over a range of redshift (and our neglect of sources which exit common envelope evolution at frequencies much higher than 10^{-4} Hz) are small compared to the astronomical uncertainties

As a second example, we consider figure 4 of Rajagopal & Romani (1995). They computed the gravitational wave background due to mergers of super-massive black holes in galactic nuclei, at the wave frequencies accessible through pulsar timing $f \sim 0.1 - 1 \text{y}^{-1}$. Their assumptions are approximately that 0.2 of bright galaxies contained a black hole, and that each bright galaxy had 5 mergers in its lifetime, or on average one merger with another black hole. Their black hole mass distribution has a black hole space density $N_0 \sim 10^{-4} \text{Mpc}^{-3}$ and typical black hole mass of $\sim 10^{7.8} M_{\odot}$. Taking \mathcal{M} for an equal mass pair, equation 16 gives $h_c = 8 \times 10^{-17} (f/1 \text{y}^{-1})^{-2/3}$. However their mass function has $N_0 \propto M^{-0.4}$ up to $3 \times 10^9 M_{\odot}$, and those heavier black holes merging with their typical $\sim 10^{7.8} M_{\odot}$ ones have $\mathcal{M} \propto M^{0.4}$, so the larger masses make contributions to $h_c^2 \propto M^{0.27} d \ln M$. Integrating over the mass function gives a total $h_c(f) \sim 2 \times 10^{-16} (f/1 \text{y}^{-1})^{-2/3}$, in very good agreement with the $f_{S1} = 1$ curve in the simulation shown in their figure 4. Of course, beliefs and data about the numbers and mass distribution of super-massive black holes have evolved considerably since 1995, so this value is not the one preferred today (see Phinney (2001)).

It should be emphasized that in this discussion, we consider only the typical amplitude and spectrum of the backgrounds. The important issue of whether particular experiments could resolve the various backgrounds (by identifying and removing sources through optimal filtering of template waveforms, or by angular resolution, or both) is deferred to later papers.

4 THE THEOREM AGAIN: MATHEMATICAL DERIVATION

The flux of gravitational wave energy from a distant source is

$$S(t) = \frac{c^3}{16\pi G} \left(\dot{h}_+^2 + \dot{h}_\times^2 \right) . \tag{19}$$

Denote the Fourier transforms of $h_+(t)$ and $h_\times(t)$ by superscript tildes:

$$\tilde{h}_{+,\times}(f) = \int_{-\infty}^{\infty} h_{+,\times}(t)e^{-i2\pi ft} dt \text{ with inverse } h_{+,\times}(t) = \int_{-\infty}^{\infty} h_{+,\times}(f)e^{i2\pi ft} dt.$$
 (20)

Apply Parseval's relation

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{g}(f)|^2 df \tag{21}$$

to the time integral of equation 19, using the fact that the Fourier transforms of $\dot{h}_{+,\times}$ equal $i2\pi f \tilde{h}_{+,\times}(f)$:

$$\int_{-\infty}^{\infty} S(t) dt = \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} \left(\dot{h}_+^2 + \dot{h}_\times^2\right) dt$$

$$= \frac{\pi}{2} \frac{c^3}{G} \int_{0}^{\infty} f^2 \left[|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right] df , \qquad (22)$$

where in equation 22 we have made use of the fact that $h_{+,\times}$ are real, $\tilde{h}_{+,\times}^*(f) = \tilde{h}_{+,\times}(-f)$, and hence $|\tilde{h}_{+,\times}(f)|^2 = \tilde{h}_{+,\times}(f)$ $|\tilde{h}_{+,\times}(-f)|^2$. This allows us to fold the negative frequency part of the integral onto the positive frequency part. If we average over source orientations Ω_s , or equivalently, over observer positions around a given source,

$$\langle S(t)\rangle_{\Omega_s} = \frac{L_{gw}(t)}{4\pi d_L^2} \,, \tag{23}$$

where L_{gw} is the gravitational wave luminosity measured in the cosmic rest frame of the source, and d_L is the luminosity distance to the source. Since time t at redshift z=0 is related to time t_r in the source's cosmic rest frame by $dt=(1+z)dt_r$

$$\int_{-\infty}^{\infty} \langle S(t) \rangle_{\mathbf{\Omega}_s} dt = \frac{1+z}{4\pi d_L^2} \int_{-\infty}^{\infty} L_{gw}(t_r) dt_r = \frac{1+z}{4\pi d_L^2} E_{gw} , \qquad (24)$$

where E_{gw} is the rest-frame energy emitted in gravitational waves. Comparing equations 22 and 24, and using $df = (1+z)^{-1} df_r$ we get

$$\int_{-\infty}^{\infty} \langle S(t) \rangle_{\mathbf{\Omega}_s} dt = \frac{\pi}{2} \frac{c^3}{G} \frac{1}{1+z} \int_0^{\infty} f^2 \left\langle |\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right\rangle_{\mathbf{\Omega}_s} df_r$$

$$\equiv \frac{1+z}{4\pi d_L^2} \int_0^{\infty} \frac{dE_{gw}}{df_r} df_r , \qquad (25)$$

and thus identify

$$\frac{dE_{gw}}{df_r} = \frac{2\pi^2 c^3}{G} d_M^2 f^2 \left\langle |\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right\rangle_{\Omega_s} , \qquad (26)$$

where we have introduced the proper-motion distance $d_M = d_L/(1+z)$ (cf. section 5 of Hogg (2000)), which is also $1/(2\pi)$ times the proper ('comoving') circumference of the sphere about the source which passes through the earth today.

Now consider sources undergoing the catastrophic events at redshift z, at rate N per comoving volume per unit of cosmic time t_r local to the event. As seen from earth, in earth time dt, the number of events which occur in dt between redshift z

$$\frac{d\#}{dt\,dz} = \dot{N}\frac{1}{1+z}\,\frac{d\mathcal{V}_c}{dz}\,\,,\tag{27}$$

where the comoving volume element is (cf. Hogg (2000) and references therein)

$$\frac{dV_c}{dz} = 4\pi \frac{c}{H_0} d_M^2 \frac{1}{E(z)} \,,$$
(28)

where E(z) was defined in equation 14. The number of events which occur in a comoving volume between the cosmic times $t_r(z)$ and $t_r(z+dz)$ is

$$N(z) = \dot{N} \frac{dt_r}{dz} = \dot{N} \frac{1}{(1+z)H_0 E(z)} . \tag{29}$$

Thus equation 27 can be rewritten

$$\frac{d\#}{dt\,dz} = N(z)c4\pi d_M^2 \ . \tag{30}$$

The energy density in gravitational waves at z=0 is then, inserting equation 25

$$\mathcal{E}_{gw} = \int \Omega_{gw}(f) \rho_c c^2 df / f = \int_0^\infty \frac{1}{c} \left[\int_{-\infty}^\infty \langle S(t) \rangle \Omega_s dt \right] \frac{d\#}{dt \, dz} dz$$

$$= \int_0^\infty \frac{1+z}{4\pi d_L^2 c} \left[\int_0^\infty f_r \frac{dE_{gw}}{df_r} \Big|_{f_r = f(1+z)} \frac{df}{f} \right] N(z) c 4\pi d_M^2 dz$$

$$= \int_0^\infty \int_0^\infty N(z) \frac{1}{1+z} \left(f_r \frac{dE_{gw}}{df_r} \right) \Big|_{f_r = f(1+z)} dz \, \frac{df}{f} ,$$
(32)

which reproduces equation 4 and hence the statement of the theorem, equation 5.

Notice that equation 31 and the result 32 are independent of whether the timescale over which an individual source emits gravitational waves at the frequencies of interest is long or short compared to the observing time. The case when it is long compared to the observing time is appropriate e.g. for the two examples treated in section 3: double degenerate binaries in the LISA frequency band, lifetimes 10²-10⁶y, and the super-massive binaries contributing to the pulsar timing background, lifetimes 10⁴-10⁶y. In this case the number of sources contributing to the background is much larger than the number of events (binary mergers in these cases) in the observing time T, by the ratio $E_{gw}/(L_{gw}T)$. But the energy which each radiates in T is only a fraction $(L_{gw}T)/E_{gw}$ of the total given by equation 24. So the product of the number of sources and the energy

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(spectrum) radiated by each is independent of T and L_{gw} . This must be so, since \mathcal{E}_{gw} is independent of whether the observing time is 1y or 10^9 y.

The observational prospects for *removing* the background by fitting individual sources *do* depend on the observing time, however, so it is instructive to re-derive equations 10 and 11 for the case of merging binaries in circular orbits, keeping track of the number of sources contributing in each frequency interval, while explicitly showing the cancellation discussed in the previous paragraph.

For circular binaries of separation a, masses M_1 and M_2 and quadrupole gravitational wave frequency $f_r = 2/P_b$ given by

$$G(M_1 + M_2) = \pi^2 f_r^2 a^3 \,, \tag{33}$$

and the energy flux at earth averaged over source orientations and orbital phases is

$$S = \frac{\pi}{4} \frac{c^3}{G} \frac{2}{5} \left(\frac{4G^2 M_1 M_2}{c^4 a d_M} \right)^2 \left(\frac{f_r}{1+z} \right)^2 = \frac{L_{gw}}{(1+z)^2 d_M^2} . \tag{34}$$

If the rate of mergers per comoving volume is \dot{N} , then the space density N(a)da of systems with separations between a and a + da is given by the continuity equation

$$\frac{d}{da}\left(\dot{a}N(a)\right) = -\dot{N}\delta(a) , \qquad (35)$$

where we have assumed that most of the sources are born at much larger separations than those of interest. Then $N(a) = \dot{N}/(-\dot{a})$. Since

$$\frac{d}{dt}\frac{GM_1M_2}{2a} = L_{gw} , \qquad (36)$$

one finds

$$\dot{a} = -\frac{2a^2 L_{gw}}{GM_1 M_2} \text{ and } N(a) = \dot{N} \frac{GM_1 M_2}{2a^2 L_{gw}}$$
 (37)

The number density of sources $N(f_r)df_r$ radiating at frequencies between f_r and $f_r + df_r$ is related to N(a) by $-N(a)da = N(f_r)df_r$, which gives, after manipulation

$$N(f_r) = \frac{5\pi}{96} \frac{c^5}{(G\mathcal{M})^{5/3}} \frac{\dot{N}}{(\pi f_r)^{11/3}} \,. \tag{38}$$

The contribution to the specific intensity I (erg cm⁻²s⁻¹sr⁻¹Hz⁻¹) of the gravitational waves from sources between z and z + dz is

$$dI = S \cdot N(f_r) \frac{dV_c}{d\Omega dz} \frac{df_r}{dt} dz . \tag{39}$$

Integrating over all redshifts gives the total specific intensity. Into this equation, insert the right side of equation 34 for S. For $N(f_r)$ use $N(a)da/df_r$, with N(a) from equation 37; for $dV_c/dzd\Omega$ use $1/(4\pi)$ times equation 28, and recall that $df_r/df = 1+z$. After using also equation 13, one gets

$$I = \frac{1}{4\pi} \int_0^\infty c \left(\dot{N} \frac{dt}{dz} \right) \frac{GM_1 M_2}{2a^2} \frac{-da}{df_r} dz . \tag{40}$$

The quantity in parenthesis is N(z). The energy density per log (observed) frequency is, using $f = f_r/(1+z)$ in the second line,

$$\Omega_{gw}(f)\rho_{c}c^{2} = \frac{f}{c}\int I \,d\Omega = \frac{4\pi}{c}fI
= \int_{0}^{\infty} \frac{N(z)}{1+z} \frac{GM_{1}M_{2}}{2a} \left(\frac{-d\ln a}{d\ln f_{r}}\right) dz
= \frac{2}{3} \int_{0}^{\infty} \frac{N(z)}{1+z} \frac{GM_{1}M_{2}}{2} \left[\frac{\pi^{2}f_{r}^{2}}{G(M_{1}+M_{2})}\right]^{1/3}
= \frac{\pi^{2/3}}{3} \frac{(G\mathcal{M})^{5/3}}{G} f^{2/3} \int_{0}^{\infty} \frac{N(z) \,dz}{(1+z)^{1/3}} .$$
(41)

The final form reproduces equation 11, while equation 38 allows one to check the instantaneous number of sources contributing to a given frequency interval of the background.

5 CONCLUSION

We have shown that in order to estimate the contribution to the gravitational wave background from a population of sources, one need not specify a cosmology. The only work involved is some relativity (often simple): computing $f_r dEgw/df_r$ for the life-history of an individual source, and some astronomy: estimating the number of sources which have ever lived and died in a unit comoving volume of the universe, and some idea of at what redshift they did so. Insert the results in equation 5 to get the full spectrum of the background.

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