

EOSC 213: Computational Methods in Geological Engineering

Lecture 3: Separable first-order ordinary differential equation

Shadab Ahamed

Department of Earth, Ocean and Atmospheric Sciences,
The University of British Columbia

Tue, Jan 13, 2026

Course Logistics

- TA Office Hour scheduling (please sign up ASAP)
 - Link: <https://www.when2meet.com/?34223342-8MfKx>.
- **Homework 1** released
 - Submission deadline: **Fri, 16 Jan, 2026 at 11:59 pm**
 - **Before submission:** Restart jupyter kernel and run everything from the beginning in order, so the order of cell operation is visible
 - Submit your .ipynb notebook and also its exported .html version.
 - Exact **submission instructions** are available on Canvas
- Instructor Office Hour: Thu, 15 Jan, 2026, 2:00 pm - 3:00 pm (same classroom).

Where we are in the course?

- Lecture 1: Motivation, logistics, computational thinking
- Lecture 2: Python refresher (NumPy, PyTorch, plotting)
- **Lecture 3: First-order ordinary differential equations (ODEs)**
- Lecture 4: Solving first-order ODEs numerically

Goal: understand first-order ODEs as *models of change*.

What is a differential equation?

A differential equation relates:

- an unknown function (the *state*)
- its rate of change
- possibly time and parameters

General first-order ODE (model):

$$\frac{dx(t)}{dt} = f(x(t), t; \lambda)$$

- $x(t)$: state variable
- t : independent variable (often time)
- λ : parameter(s) of the model
- f : rule governing evolution

Why do we care? (Physics & Geophysics)

Differential equations appear everywhere:

- Radioactive decay

$x(t)$ = *concentration of radioactive material remaining in the sample at time t*

- Population growth

$x(t)$ = *population size at time t*

- Heat transfer

$x(t)$ = *temperature of the system at time t*

- Fluid flow

$x(t)$ = *velocity / pressure at a point or volume of fluid in a tank at time t*

- Groundwater transport

$x(t)$ = *concentration of a contaminant or tracer at time t*

They encode **physical laws** as rules of change.

Separable ordinary differential equations

A first-order ODE is called **separable** if it can be written as:

$$\frac{dx}{dt} = f(x(t), t; \lambda) = g(x; \lambda)h(t; \lambda)$$

This allows us to *separate variables*:

$$\frac{1}{g(x; \lambda)} dx = h(t; \lambda) dt$$

and then integrate both sides:

$$\int \frac{1}{g(x; \lambda)} dx = \int h(t; \lambda) dt$$

Key idea: many important physical models are separable.

Initial value problems

A first-order ODE alone does *not* specify a unique solution.

We must also specify an initial condition:

$$x(t_0) = x_0$$

This defines an **initial value problem (IVP)**.

Interpretation:

- The ODE defines possible trajectories
- The initial condition selects one

Two examples with analytic solutions

Before moving to numerical methods, we study two important cases where **closed-form (analytic) solutions** are available.

These examples help us:

- understand how ODEs encode physical laws
- see the role of initial conditions and parameters
- build intuition about their behavior, before discretization and numerical implementation

Examples we will study:

- **Exponential decay** (linear ODE)
- **Logistic growth** (nonlinear ODE)

Later, we will revisit these models using numerical methods.

Example 1: Exponential decay

Consider:

$$\frac{dx}{dt} = -\lambda x, \quad \lambda > 0,$$

where $\lambda =$ decay constant.

This models:

- radioactive decay (amount of material decays proportionally to what remains)
- cooling toward ambient temperature (Newton's law of cooling)
- damping of motion (velocity decays due to friction or drag)

Exponential decay model: analytical solution

Exponential decay model:

$$\frac{dx}{dt} = -\lambda x, \quad \lambda > 0,$$

Separate variables:

$$\frac{1}{x} dx = -\lambda dt$$

Integrate:

$$\ln |x| = -\lambda t + C_1$$

Exponentiate:

$$x(t) = Ce^{-\lambda t}, \quad \text{where } C = e^{C_1}$$

Apply initial condition $x(0) = x_0$:

$$x(t) = x_0 e^{-\lambda t}$$

Asymptotic limit as $t \rightarrow \infty$:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x_0 e^{-\lambda t} = 0$$

Initial condition matters (different solutions of the **same** model)

Consider the same linear ODE:

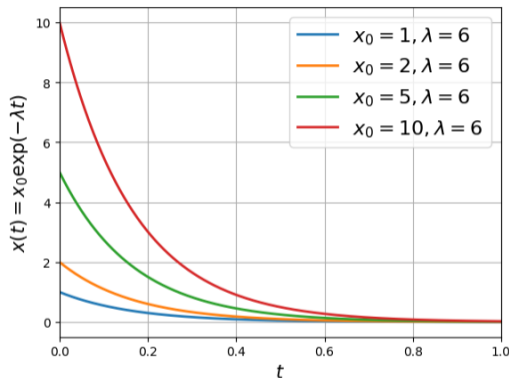
$$\frac{dx}{dt} = -\lambda x, \quad \lambda > 0$$

General solution: $x(t) = Ce^{-\lambda t}$

An **initial condition** $x(0) = x_0$ determines C :

$$x(t) = x_0 e^{-\lambda t}$$

- Same model (same λ)
- Different $x(0) \Rightarrow$ different trajectory
- Solution is **unique** once $x(0)$ is given



Model parameters matter (different models)

Consider the linear ODE:

$$\frac{dx}{dt} = -\lambda x, \quad \lambda > 0$$

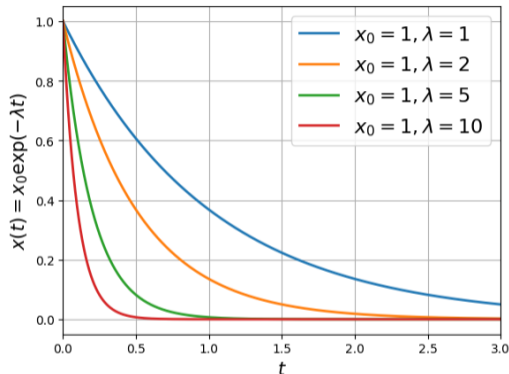
General solution: $x(t) = x_0 e^{-\lambda t}$

Two different sources of variation:

- **Initial condition** $x(0) = x_0$: selects a specific solution
- **Parameter** λ : defines a different model

Changing λ changes:

- the decay rate
- the time scale of the process
- the physical behavior of the system

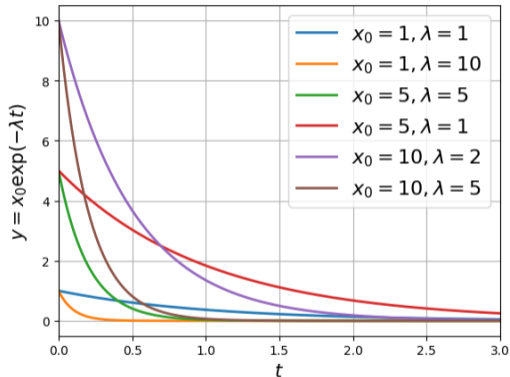


Takeaway: parameters define the model; initial conditions define the trajectory. s

Qualitative behavior: What does this solution tell us?

- Solution decays smoothly to zero
- x_0 controls the initial value
- Larger $\lambda \Rightarrow$ faster decay

Key idea: parameters control qualitative behavior.



Example 2: Logistic growth

Consider:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right), \quad r > 0, K > 0$$

where r = growth rate and K = carrying capacity.

This models:

- population growth with limited resources (carrying capacity)
- spread of an epidemic in a finite population (early growth then saturation)
- resource-limited growth in ecology / microbiology (bacteria in a dish)
- adoption / diffusion of a technology (slow \rightarrow fast \rightarrow saturation)
- recharge / recovery processes with saturation (e.g., soil moisture recovery)

Logistic growth model: analytical solution (I)

Logistic model:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right), \quad r > 0, K > 0$$

Separate variables:

$$\frac{dx}{x \left(1 - \frac{x}{K}\right)} = r dt$$

Rewrite the left-hand side using partial fractions:

$$\frac{1}{x \left(1 - \frac{x}{K}\right)} = \frac{K}{x(K - x)} = \frac{1}{x} + \frac{1}{K - x}$$

Integrate:

$$\int \left(\frac{1}{x} + \frac{1}{K - x} \right) dx = \int r dt \implies \ln|x| - \ln|K - x| = rt + C_1$$

Logistic growth model: analytical solution (II)

From integration:

$$\ln\left(\frac{x}{K-x}\right) = rt + C_1$$

Exponentiate:

$$\frac{x}{K-x} = Ce^{rt}, \quad \text{where } C = e^{C_1}$$

Solve for $x(t)$:

$$x(t) = \frac{K}{1 + Ae^{-rt}}, \quad \text{where } A = 1/C$$

Apply initial condition $x(0) = x_0$:

$$A = \frac{K - x_0}{x_0}$$

Final solution:

$$x(t) = \frac{K}{1 + \left(\frac{K-x_0}{x_0}\right) e^{-rt}}$$

Logistic growth model: analytical solution (III)

Final solution:

$$x(t) = \frac{K}{1 + \left(\frac{K-x_0}{x_0} \right) e^{-rt}}$$

Asymptotic limit as $t \rightarrow \infty$:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \frac{K}{1 + \left(\frac{K-x_0}{x_0} \right) e^{-rt}} = K$$

Interpretation:

- Early-time behavior: exponential growth
- Long-time behavior: saturation to K

Initial condition matters (different solutions of the **same** model)

Consider the logistic growth model:

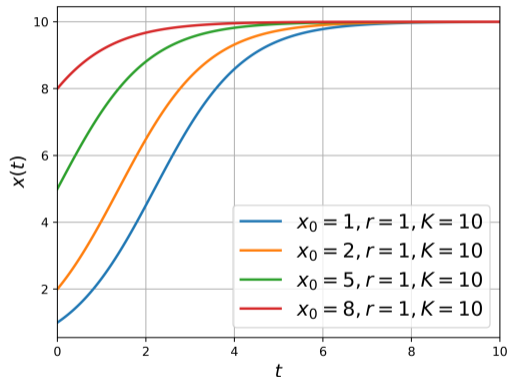
$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right), \quad r > 0, K > 0$$

General solution:

$$x(t) = \frac{K}{1 + \left(\frac{K-x_0}{x_0}\right) e^{-rt}}$$

An **initial condition** $x(0) = x_0$ selects a specific solution.

- Same model (same r , same K)
- Different $x(0) \Rightarrow$ different trajectories
- All solutions approach the same carrying capacity



Model parameters matter (different models)

General solution to the logistic model:

$$x(t) = \frac{K}{1 + \left(\frac{K-x_0}{x_0} \right) e^{-rt}}$$

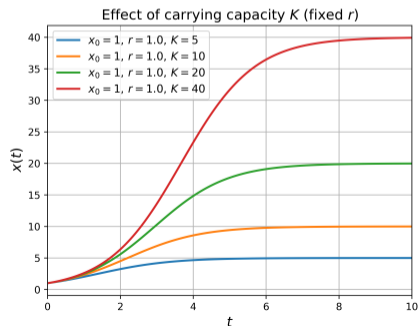
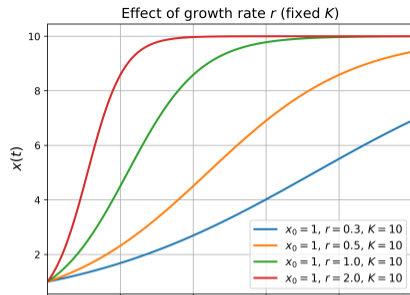
Two different sources of variation:

- **Initial condition** $x(0) = x_0$: selects a trajectory
- **Parameters** (r, K) : define the model

Changing parameters changes:

- r : growth rate (how fast the population grows)
- K : carrying capacity (long-term equilibrium)
- overall qualitative behavior

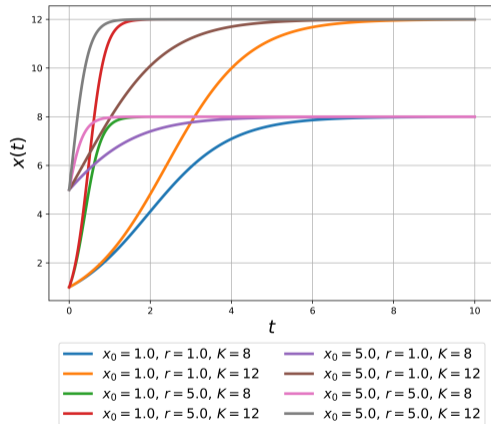
Takeaway: parameters encode physical constraints of the system.



Qualitative behavior: What does this solution tell us?

- Growth is approximately exponential at early times
- Growth slows down as $x(t)$ approaches K
- K is a stable equilibrium (carrying capacity)
- Larger $r \Rightarrow$ faster approach to equilibrium

Key idea: nonlinear terms introduce saturation and equilibria.



Integrating factor method for solving first-order linear ODEs

We consider equations of the form:

$$\frac{dx}{dt} + p(t)x = q(t)$$

This is called a **first-order linear ODE**.

Examples:

- $\frac{dx}{dt} + \lambda x = 0$ (exponential decay)
- $\frac{dx}{dt} + \lambda x = f(t)$ (forced system)

Goal: find a general analytic method to solve this equation.

Key idea: turn the left-hand side into a product

Suppose we could rewrite the equation as:

$$\frac{d}{dt}(\mu(t)x(t)) = \mu(t)q(t)$$

Using the product rule:

$$\frac{d}{dt}(\mu x) = \mu \frac{dx}{dt} + \mu'(t)x$$

We want this to match:

$$\mu \frac{dx}{dt} + \mu p(t)x$$

This suggests choosing $\mu(t)$ so that:

$$\mu'(t) = \mu(t)p(t)$$

Deriving the integrating factor

We require:

$$\frac{d\mu}{dt} = p(t) \mu$$

This is a separable ODE:

$$\frac{1}{\mu} d\mu = p(t) dt$$

Integrate:

$$\ln |\mu| = \int p(t) dt$$

Exponentiate:

$$\mu(t) = e^{\int p(t) dt}$$

This function $\mu(t)$ is called the integrating factor.

General solution using integrating factors

Multiply the original ODE by $\mu(t)$:

$$\mu \frac{dx}{dt} + \mu p(t) x = \mu q(t)$$

Left-hand side becomes:

$$\frac{d}{dt}(\mu x)$$

So we have:

$$\frac{d}{dt}(\mu x) = \mu q(t)$$

Integrate both sides:

$$\mu(t) x(t) = \int \mu(t) q(t) dt + C$$

Solve for $x(t)$:

$$x(t) = \frac{1}{\mu(t)} \left(\int \mu(t) q(t) dt + C \right)$$

Example: solve a simple ODE

Solve:

$$\frac{dx}{dt} + 2x = e^{-t}$$

Here $p(t) = 2$, so the integrating factor is:

$$\mu(t) = e^{\int 2 dt} = e^{2t}$$

Multiply the equation by e^{2t} :

$$\frac{d}{dt}(e^{2t}x) = e^t$$

Integrate:

$$e^{2t}x = \int e^t dt = e^t + C$$

Final solution:

$$x(t) = e^{-t} + Ce^{-2t}$$

Which ODEs can integrating factors solve?

We consider equations of the form:

$$\frac{dx}{dt} + p(t)x = q(t)$$

This is called a **first-order linear ODE**.

Example: exponential decay

$$\frac{dx}{dt} = -\lambda x \quad \Rightarrow \quad \frac{dx}{dt} + \lambda x = 0$$

$$\text{Here: } p(t) = \lambda, \quad q(t) = 0$$

Contrast: logistic growth

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right)$$

- contains a nonlinear term x^2
- **cannot** be written in the linear form

When no analytic solutions exist

Many realistic ODEs **cannot be solved in closed form**.

Common reasons include:

- strong nonlinearities
- time-dependent forcing
- coupling between multiple variables

Examples:

- **Nonlinear drag:**

$$\frac{dv}{dt} = -av - bv|v|$$

- **Forced systems:**

$$\frac{dx}{dt} = -x + \sin(t^2)$$

- **Coupled populations:**

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -y + xy$$

Why numerical methods are unavoidable

In practice:

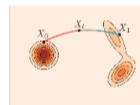
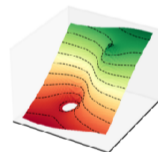
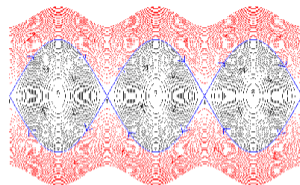
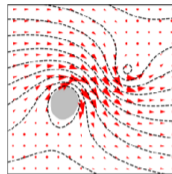
- most systems are nonlinear
- parameters vary in space and time
- models involve many coupled equations

Even when an analytic solution exists:

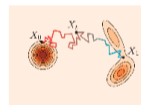
- it may be too complicated to evaluate
- it may not generalize to realistic conditions

Numerical methods let us:

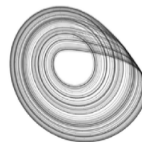
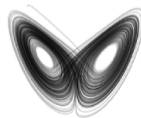
- approximate solutions robustly
- explore parameter sensitivity
- simulate real-world systems



(a) Flow



(b) Diffusion



Summary

- First-order ODEs define rules of evolution
- Initial conditions select trajectories
- Parameters shape behavior
- Some first-order ODEs admit analytic solutions
 - via **separation of variables**
 - via **integrating factors** for first-order linear ODEs
- Many real-world systems cannot be solved analytically, motivating numerical methods

Next lecture: finite difference methods, numerical time stepping, and implementation of first-order ODEs.