

# EOSC 213: Computational Methods in Geological Engineering

Lecture 11: Interpreting systems of ODEs via eigenvalues  
+ Forward Euler/ Explicit Midpoint / Backward Euler for systems

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# Course Logistics and Announcements

## Homework 02

- HW02 is posted on Canvas; submission deadline: **Thursday, 20th Feb, 2026**
  - No extensions as I want to release HW02 solutions before Midterm I
- This homework is good practice for Midterm I which scheduled on **24th Feb, 2026**
  - Instructions for Midterm I will be posted on Canvas soon
- Please follow submission instructions carefully:
  - Written solutions: **single PDF**
  - Python notebook: submit **.ipynb and .html** export. These should match EXACTLY
  - -2 if **.html** file is missing
  - **Make sure your submitted files are correct and complete**

## Recap from Lecture 10

Last lecture, we:

- wrote systems of ODEs using vectors:  $\mathbf{x}' = \mathbf{F}(\mathbf{x}, t)$
- focused on **linear systems**:  $\mathbf{x}' = A\mathbf{x}$
- used **phase portraits** ( $y(t)$  vs.  $x(t)$  plots for different initial conditions) to observe qualitative behavior of the system
- learned how to compute eigenvalues from  $\det(A - \lambda I) = 0$

**Today:** how eigenvalues predict behavior, and how numerical methods extend to systems.

# Lecture 11: What we will learn today

## ① Part I: Equilibrium points

- what are equilibrium points?
- stable, unstable equilibrium and saddle points

## ② Part II: Interpreting dynamics of the system from eigenvalues of $A$

- real positive/negative eigenvalues
- complex eigenvalues (spirals)
- purely imaginary eigenvalues (centers)

## ③ Part III: Forward Euler / Explicit Midpoint / Backward Euler for systems

- vector form of the updates
- for  $\mathbf{x}' = A\mathbf{x}$ : amplification matrices
- stability depends on eigenvalues again

## Learning goals

By the end of today, you should be able to:

- predict decay/growth/oscillation of  $\mathbf{x}' = A\mathbf{x}$  from eigenvalues of  $A$
- recognize the main eigenvalue cases (real, complex, imaginary)
- write Forward Euler / Midpoint / Backward Euler for a **system**
- for  $\mathbf{x}' = A\mathbf{x}$ , identify the **amplification matrix** of each method
- connect numerical stability to eigenvalues (same idea as scalar case)

## Practice problem: diagonalizable 3D system

Consider the linear system of ODEs

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

### Tasks:

- ① Compute the eigenvalues of  $A$ .
- ② Find one eigenvector corresponding to each eigenvalue.
- ③ Write down the general solution of the system.

*Hint: For an upper-triangular matrix, the eigenvalues are the diagonal entries.*

## Solution: Eigenvalues

We are given

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \quad \frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

Since  $A$  is upper triangular, the eigenvalues are the diagonal entries:

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 4.$$

All eigenvalues are distinct.

## Solution: Eigenvalues via characteristic equation

The eigenvalues satisfy the characteristic equation

$$\det(A - \lambda I) = 0.$$

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For

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 1 & 0 \\ 0 & 3 - \lambda & 1 \\ 0 & 0 & 4 - \lambda \end{bmatrix},$$

## Solution: Eigenvalues via characteristic equation

The eigenvalues satisfy the characteristic equation

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the determinant is

$$\det(A - \lambda I) = (2 - \lambda)(3 - \lambda)(4 - \lambda).$$

## Solution: Eigenvalues via characteristic equation

The eigenvalues satisfy the characteristic equation

$$\det(A - \lambda I) = 0.$$

For

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 1 & 0 \\ 0 & 3 - \lambda & 1 \\ 0 & 0 & 4 - \lambda \end{bmatrix},$$

the determinant is

$$\det(A - \lambda I) = (2 - \lambda)(3 - \lambda)(4 - \lambda).$$

Setting this equal to zero gives

$$(2 - \lambda)(3 - \lambda)(4 - \lambda) = 0.$$

## Eigenvalues

$$\lambda = 2, \quad 3, \quad 4.$$

## Solution: Eigenvector for eigenvalue $\lambda_1 = 2$

Computing the eigenvector for the eigenvalue  $\lambda_1 = 2$ , let us solve,

$$(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0} \quad \Rightarrow \quad (A - 2I)\mathbf{v}_1 = \mathbf{0}.$$

We compute

$$A - 2I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

## Solution: Eigenvector for eigenvalue $\lambda_1 = 2$

Computing the eigenvector for the eigenvalue  $\lambda_1 = 2$ , let us solve,

$$(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0} \Rightarrow (A - 2I)\mathbf{v}_1 = \mathbf{0}.$$

We compute

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So we solve

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the system  $x_2 = 0$ ,  $x_3 = 0$ .

So one eigenvector is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

## Solution: Eigenvector for eigenvalue $\lambda_2 = 3$

Computing the eigenvector for the eigenvalue  $\lambda_2 = 3$ , let us solve,

$$(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0} \quad \Rightarrow \quad (A - 3I)\mathbf{v}_2 = \mathbf{0}.$$

We compute

$$A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

## Solution: Eigenvector for eigenvalue $\lambda_2 = 3$

Computing the eigenvector for the eigenvalue  $\lambda_2 = 3$ , let us solve,

$$(A - \lambda_2 I)\mathbf{v}_2 = \mathbf{0} \Rightarrow (A - 3I)\mathbf{v}_2 = \mathbf{0}.$$

We compute

$$A - 3I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

So we solve

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the system  $x_2 = x_1$ ,  $x_3 = 0$ .

So one eigenvector is

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

## Solution: Eigenvector for eigenvalue $\lambda_3 = 4$

Computing the eigenvector for the eigenvalue  $\lambda_3 = 4$ , let us solve,

$$(A - \lambda_3 I)\mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad (A - 4I)\mathbf{v}_3 = \mathbf{0}.$$

We compute

$$A - 4I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

## Solution: Eigenvector for eigenvalue $\lambda_3 = 4$

Computing the eigenvector for the eigenvalue  $\lambda_3 = 4$ , let us solve,

$$(A - \lambda_3 I)\mathbf{v}_3 = \mathbf{0} \Rightarrow (A - 4I)\mathbf{v}_3 = \mathbf{0}.$$

We compute

$$A - 4I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we solve

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives the system  $x_2 = 2x_1$ ,  $x_3 = 2x_1$ .

So one eigenvector is

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

## Solution: General solution of the ODE

For a diagonalizable matrix, the solution is a linear combination of eigenmodes:

$$\mathbf{x}(t) = c_1 e^{2t} \mathbf{v}_1 + c_2 e^{3t} \mathbf{v}_2 + c_3 e^{4t} \mathbf{v}_3.$$

Explicitly,

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

### Key takeaway

Each eigenvalue contributes an independent exponential mode. The values of  $c_1$ ,  $c_2$  and  $c_3$  are obtained from the initial conditions.

# Equilibrium points for a dynamical system

We often write a system of ODEs as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t),$$

- $\mathbf{x}(t)$  is the *state* (a vector of unknowns)
- $\mathbf{f}(\mathbf{x})$  is the *vector field* (it tells us the direction of change)

# Equilibrium point (steady state)

**Definition.** A vector  $\mathbf{x}_e$  is an **equilibrium point** if,

$$\mathbf{f}(\mathbf{x}_e) = \mathbf{0}.$$

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**Definition.** A vector  $\mathbf{x}_e$  is an **equilibrium point** if,

$$\mathbf{f}(\mathbf{x}_e) = \mathbf{0}.$$

Interpretation:

- If  $\mathbf{x}(0) = \mathbf{x}_e$ , then  $\frac{d\mathbf{x}}{dt} = \mathbf{0}$  and the solution stays constant:

$$\mathbf{x}(t) \equiv \mathbf{x}_e.$$

- Equilibria are the “no net change” states of the system.

# How to judge stability in 1D?

Consider a scalar ODE

$$\frac{dx}{dt} = f(x).$$

An equilibrium  $x_e$  satisfies  $f(x_e) = 0$ .

To determine stability, ask:

- What is the sign of  $f(x)$  just left of  $x_e$ ?
- What is the sign of  $f(x)$  just right of  $x_e$ ?

This tells us whether trajectories move toward or away from  $x_e$ .

## Stable vs. unstable equilibrium (intuition)

**Stable equilibrium:** small perturbation  $\Rightarrow$  trajectories move back toward  $x_e$ .

**Unstable equilibrium:** small perturbation  $\Rightarrow$  trajectories move away from  $x_e$ .

# Stable vs. unstable equilibrium (intuition)

**Stable equilibrium:** small perturbation  $\Rightarrow$  trajectories move back toward  $x_e$ .

**Unstable equilibrium:** small perturbation  $\Rightarrow$  trajectories move away from  $x_e$ .

## Mental picture

- Bottom of a bowl: stable
- Top of a hill: unstable

## Example 1: Stable equilibrium in 1D

Consider

$$\frac{dx}{dt} = -x.$$

Equilibrium:

$$-x = 0 \Rightarrow x_e = 0.$$

Sign analysis:

- If  $x > 0$ , then  $\dot{x} < 0$  (moves left)
- If  $x < 0$ , then  $\dot{x} > 0$  (moves right)

Conclusion

All nearby trajectories move toward  $x = 0$ .

⇒ **Stable equilibrium**

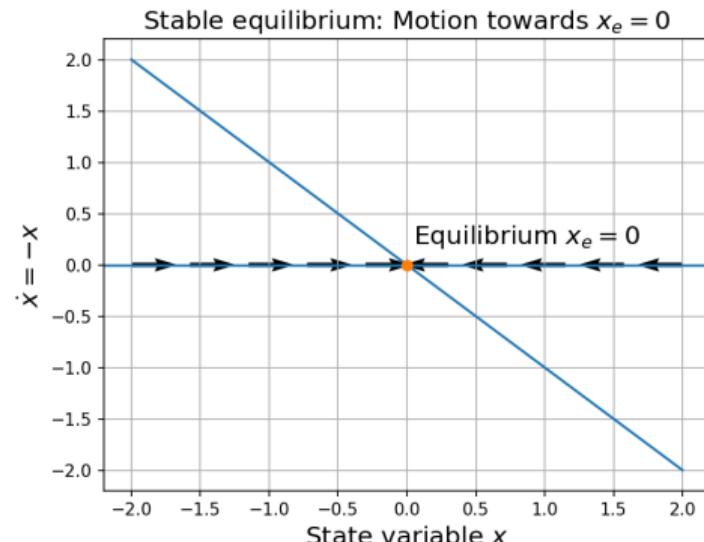


Figure: Stable equilibrium at  $x_e = 0$ . Perturbations around  $x_e = 0$  will bring the state back to  $x_e = 0$ .

## Example 2: Unstable equilibrium in 1D

Consider

$$\frac{dx}{dt} = x.$$

Equilibrium:

$$x = 0 \quad \Rightarrow \quad x_e = 0.$$

Sign analysis:

- If  $x > 0$ , then  $\dot{x} > 0$  (moves right)
- If  $x < 0$ , then  $\dot{x} < 0$  (moves left)

### Conclusion

All nearby trajectories move away from  $x = 0$ .  $\Rightarrow$  **Unstable equilibrium**

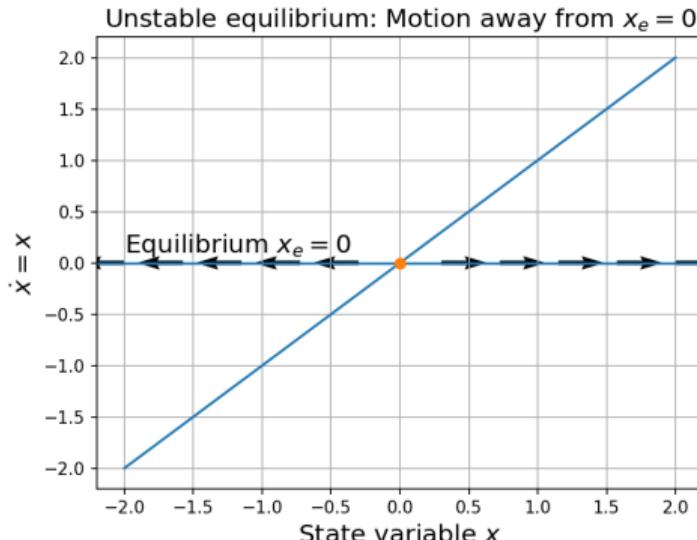


Figure: Unstable equilibrium at  $x_e = 0$ . Perturbations around  $x_e = 0$  will move the state away from  $x_e = 0$ .

## Example 3: Semi-stable equilibrium in 1D

Consider

$$\frac{dx}{dt} = x^2.$$

Equilibrium:

$$x^2 = 0 \Rightarrow x_e = 0.$$

Sign analysis:

- If  $x > 0$ , then  $\dot{x} > 0$  (moves right)
- If  $x < 0$ , then  $\dot{x} > 0$  (moves right)

### Conclusion

Trajectories approach  $x = 0$  from the left, but move away on the right.

⇒ **Semi-stable equilibrium**

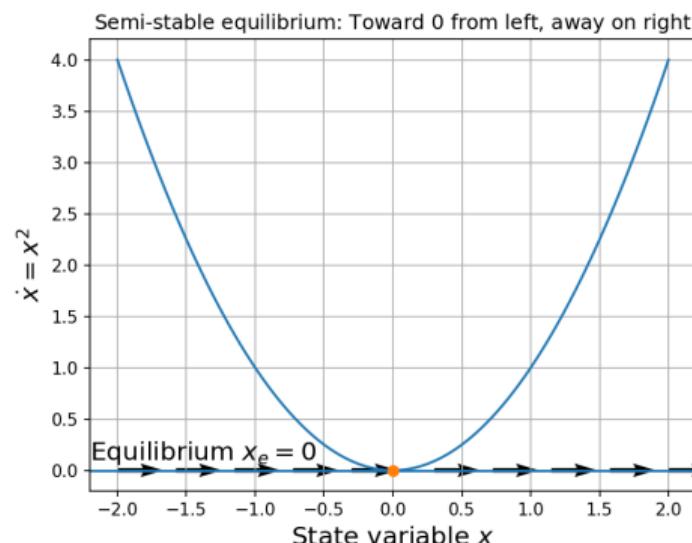


Figure: Semi-stable equilibrium at  $x_e = 0$

## Takeaway from 1D examples

- Stability in 1D can be understood purely from the sign of  $f(x)$
- Small perturbations tell the full story
- This intuition generalizes to higher dimensions using eigenvalues

*In higher dimensions, different directions behave like different 1D problems.*

# Why equilibria matter

Equilibria are the reference points for understanding dynamics:

- Many systems spend a lot of time *near* equilibria
- Long-time behavior is often about approaching or moving away from equilibria
- Stability questions are naturally posed around equilibria

## Key question

If we start near  $x_e$ , do we return towards it or move away from it?

## Example: find equilibria and their stability

Consider the scalar ODE,

$$\frac{dx}{dt} = x^2 - 1.$$

Equilibria satisfy  $\frac{dx}{dt} = 0$ , i.e.

$$x^2 - 1 = 0 \quad \Rightarrow \quad x_e = -1, 1.$$

### Conclusion

The sign of the local dynamics makes  
 $x_e = -1$  stable and  $x_e = +1$  unstable.

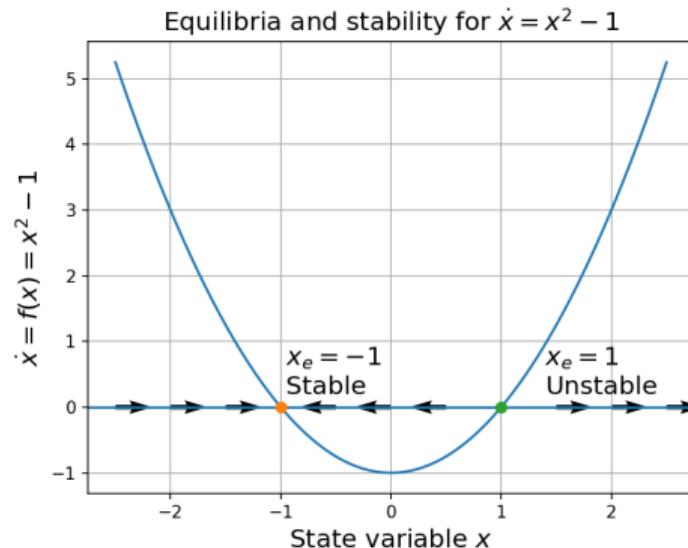


Figure: Two equilibrium points,  $x_e = -1$  (stable) and  $x_e = 1$  (unstable).

## Example 1: Stable equilibrium in 2D

Consider the linear system,

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -2y.$$

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**Behavior along each direction:**

- $x$ -direction:  $\dot{x} = -x \Rightarrow$  decay toward 0
- $y$ -direction:  $\dot{y} = -2y \Rightarrow$  faster decay toward 0

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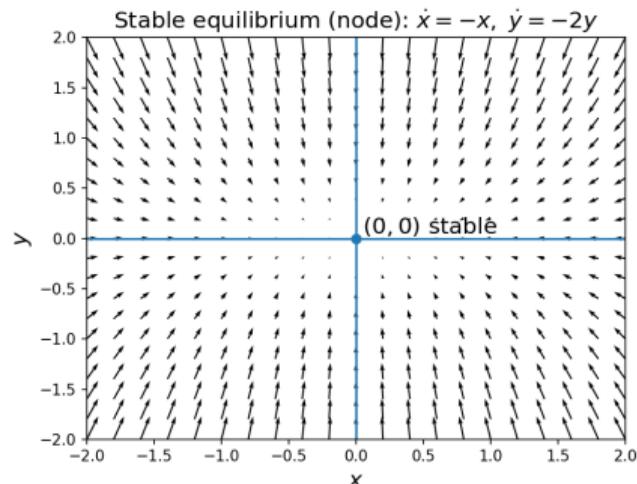
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- $y$ -direction:  $\dot{y} = -2y \Rightarrow$  faster decay toward 0

**Conclusion**

All nearby trajectories decay toward the equilibrium.

$\Rightarrow$  **Stable equilibrium (stable node)**



**Figure:** All solutions decay to  $(x_e, y_e) = (0, 0)$  making this a stable equilibrium point.

## Example 2: Unstable equilibrium in 2D

Consider the linear system,

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 2y.$$

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Consider the linear system,

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### Equilibrium:

$$x = 0, \quad y = 0 \quad \Rightarrow \quad (x_e, y_e) = (0, 0).$$

### Behavior along each direction:

- $x$ -direction:  $\dot{x} = x \Rightarrow$  growth away from 0
- $y$ -direction:  $\dot{y} = 2y \Rightarrow$  growth away from 0

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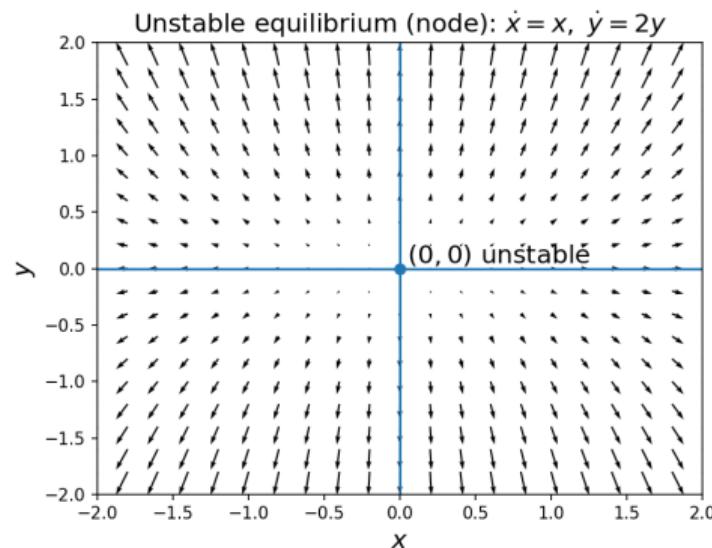
$$x = 0, \quad y = 0 \quad \Rightarrow \quad (x_e, y_e) = (0, 0).$$

**Behavior along each direction:**

- $x$ -direction:  $\dot{x} = x \Rightarrow$  growth away from 0
- $y$ -direction:  $\dot{y} = 2y \Rightarrow$  growth away from 0

**Conclusion**

All nearby trajectories move away from the equilibrium.  $\Rightarrow$  **unstable node**



**Figure:** All solutions move away from  $(x_e, y_e) = (0, 0)$  making this an unstable equilibrium point.

## Example 3: Two equilibria with different stability

Consider the nonlinear system,

$$\frac{dx}{dt} = x - x^2, \quad \frac{dy}{dt} = -y + x - x^2.$$

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### Equilibria:

$$x - x^2 = 0 \Rightarrow x = 0, 1$$

$$-y + x - x^2 = 0 \Rightarrow y = 0$$

So the equilibrium points are:  $(0, 0)$  and  $(1, 0)$ .

## Example 3: Two equilibria with different stability

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### Equilibria:

$$x - x^2 = 0 \Rightarrow x = 0, 1$$

$$-y + x - x^2 = 0 \Rightarrow y = 0$$

So the equilibrium points are:  $(0, 0)$  and  $(1, 0)$ .

### Stability:

- $(0, 0)$ : stable in one direction and unstable in another (saddle point)
- $(1, 0)$ : trajectories decay toward the equilibrium in all directions

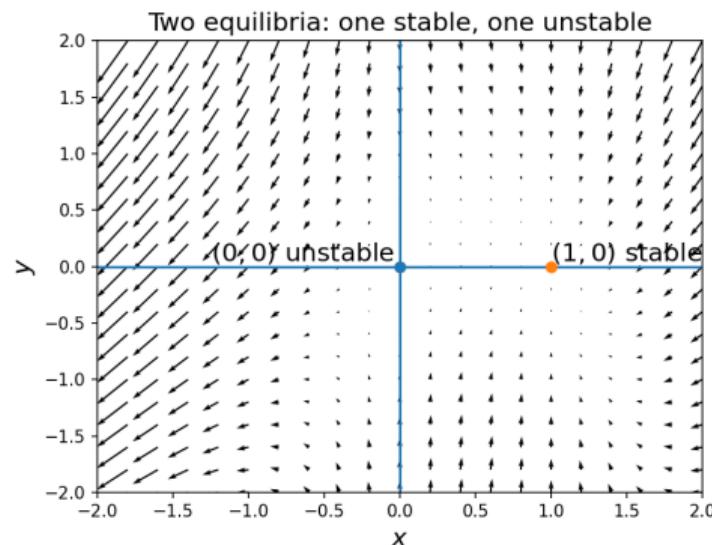


Figure: Two equilibrium points:  $(0,0)$  is unstable while  $(1,0)$  is stable.

## Example 4: Saddle point equilibrium in 2D

Consider the linear system,

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y.$$

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**Equilibrium:**  $x = 0, y = 0 \Rightarrow (x_e, y_e) = (0, 0)$ .

## Example 4: Saddle point equilibrium in 2D

Consider the linear system,

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y.$$

**Equilibrium:**  $x = 0, y = 0 \Rightarrow (x_e, y_e) = (0, 0)$ .

**Behavior along each direction:**

- $x$ -direction:  $\dot{x} = x \Rightarrow$  solutions grow away from 0  
(unstable)
- $y$ -direction:  $\dot{y} = -y \Rightarrow$  solutions decay toward 0  
(stable)

## Example 4: Saddle point equilibrium in 2D

Consider the linear system,

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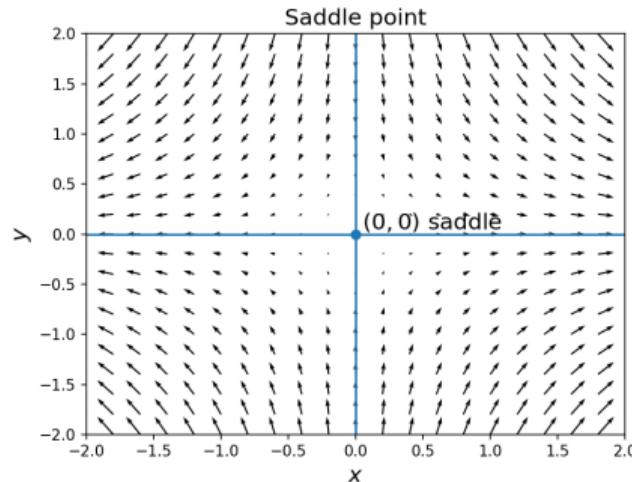
**Equilibrium:**  $x = 0, y = 0 \Rightarrow (x_e, y_e) = (0, 0)$ .

**Behavior along each direction:**

- $x$ -direction:  $\dot{x} = x \Rightarrow$  solutions grow away from 0 (unstable)
- $y$ -direction:  $\dot{y} = -y \Rightarrow$  solutions decay toward 0 (stable)

### Conclusion

The equilibrium is stable in one direction and unstable in another.



**Figure:** The equilibrium point  $(0, 0)$  is unstable along  $x$  axis and stable along  $y$  axis, making it a saddle point.

## Linear systems: equilibria

For a linear (affine) system,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b},$$

equilibria satisfy,

$$A\mathbf{x}_e + \mathbf{b} = \mathbf{0}.$$

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For a linear (affine) system,

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equilibria satisfy,

$$A\mathbf{x}_e + \mathbf{b} = \mathbf{0}.$$

- If  $\mathbf{b} = \mathbf{0}$ , then  $\mathbf{x}_e = \mathbf{0}$  is always an equilibrium.
- If  $A$  is invertible, then the equilibrium is

$$\mathbf{x}_e = -A^{-1}\mathbf{b}.$$

## Preview: from equilibria to eigenvalues

To analyze behavior *near* an equilibrium, we study how small perturbations evolve.  
This motivates linear models of the form

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u},$$

and leads directly to eigenvalues/eigenvectors as the tools for stability and “modes”.

## Takeaway

- Equilibrium point  $\mathbf{x}_e$  means  $\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$ .
- Starting at  $\mathbf{x}_e$  gives a constant solution  $\mathbf{x}(t) \equiv \mathbf{x}_e$ .
- Understanding motion *near* equilibria sets up stability and eigenvalue analysis.

## The key idea: eigenvectors are modes

For the linear system,

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

eigenpairs  $(\lambda, \mathbf{v})$  of  $A$  satisfy,

$$A\mathbf{v} = \lambda\mathbf{v}.$$

A fundamental building block (a “mode”) is,

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v}.$$

So, eigenvalues tell us:

- **how fast** a mode grows/decays (via  $\Re(\lambda)$ )
- whether it **oscillates** (via  $\Im(\lambda)$ )

## From modes to the general solution

For linear systems,

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the full solution is built by **combining modes**.

## From modes to the general solution

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the full solution is built by **combining modes**. If  $A$  has  $n$  independent eigenpairs  $(\lambda_k, \mathbf{v}_k)$ , then the solution can be written as

$$\mathbf{x}(t) = \sum_{k=1}^n c_k e^{\lambda_k t} \mathbf{v}_k,$$

where the constants  $c_k$  are determined by the initial condition  $\mathbf{x}(0)$ .

## From modes to the general solution

For linear systems,

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

the full solution is built by **combining modes**. If  $A$  has  $n$  independent eigenpairs  $(\lambda_k, \mathbf{v}_k)$ , then the solution can be written as

$$\mathbf{x}(t) = \sum_{k=1}^n c_k e^{\lambda_k t} \mathbf{v}_k,$$

where the constants  $c_k$  are determined by the initial condition  $\mathbf{x}(0)$ .

- Each term is an independent **eigenmode**
- The system's behavior is the **superposition** of all modes
- Modes with larger  $\Re(\lambda)$  dominate at long times

**Key message:** eigenvalues and eigenvectors fully determine the qualitative behavior of the system.

## Important note: when do exponential modes apply?

So far, we have described solutions of  $\mathbf{x}' = A\mathbf{x}$  using exponential modes of the form,

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v}.$$

### Important clarification:

- This description is exact when the matrix  $A$  is **diagonalizable**
- Diagonalizable means  $A$  has enough independent eigenvectors to form a basis
- In that case, the solution can be written as a combination of exponential eigenmodes

If  $A$  is **not** diagonalizable:

- solutions may involve additional polynomial factors multiplying  $e^{\lambda t}$
- the analysis becomes more technical

### For this course

You do *not* need to know these details. Just remember that superposition of exponential eigenmodes is *not* the most general solution for a general matrix  $A$ , and a full treatment is beyond the scope of EOSC 213.

## Real eigenvalues: sign matters

If  $\lambda \in \mathbb{R}$ :

$$e^{\lambda t} = \begin{cases} \text{decays to 0} & \lambda < 0, \\ \text{stays constant} & \lambda = 0, \\ \text{grows (blows up)} & \lambda > 0. \end{cases}$$

For 2D systems:

- both eigenvalues  $< 0 \Rightarrow$  stable (trajectories go to origin)
- one  $> 0$  and one  $< 0 \Rightarrow$  saddle (some directions grow, some decay)
- both  $> 0 \Rightarrow$  unstable (trajectories move away)

## Example 1: Stable node (both eigenvalues negative)

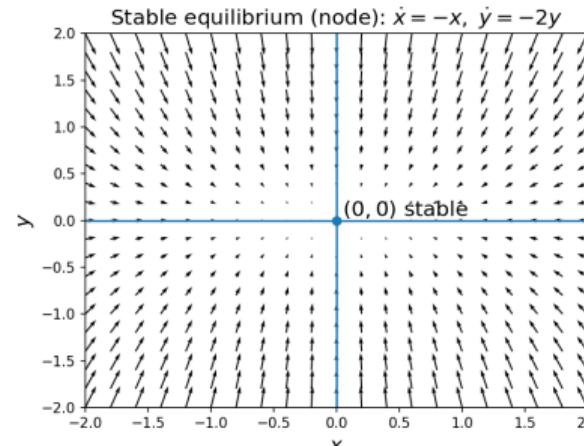
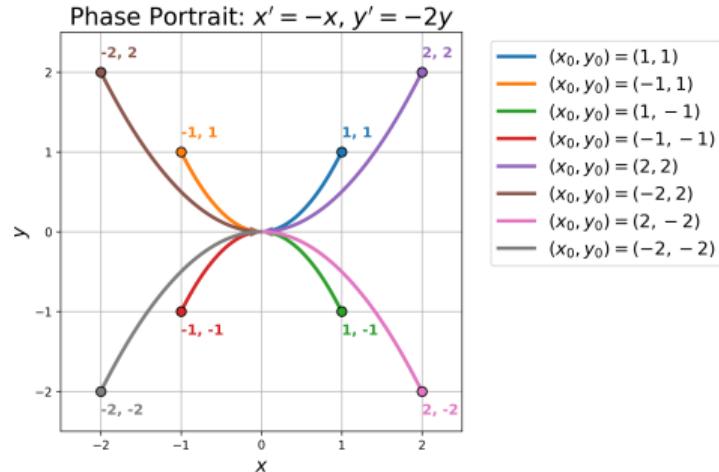
Take

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Eigenvalues:  $\lambda_1 = -1$ ,  $\lambda_2 = -2$  (both negative).

Qualitative behavior:

- all trajectories decay toward  $(0, 0)$
- $y$  decays faster than  $x$  (two time scales)
- phase portrait looks like an inward “sink”



## Example 2: Saddle (one positive, one negative)

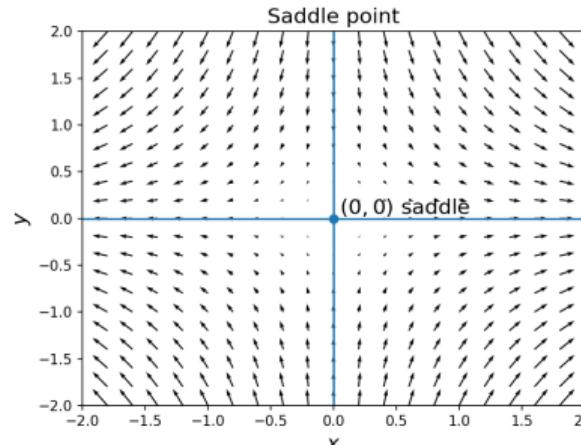
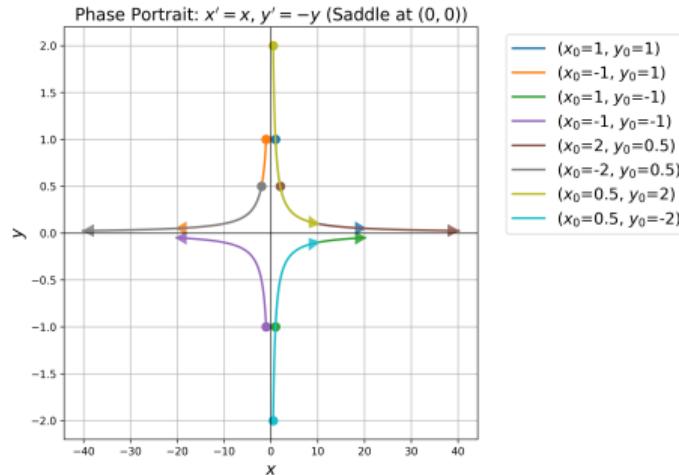
Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues:  $\lambda_1 = 1$  (growth),  $\lambda_2 = -1$  (decay).

Qualitative behavior:

- along one direction trajectories move away from origin
- along the other direction trajectories move toward origin
- phase portrait shows a **saddle** (stable manifold + unstable manifold)



## Complex eigenvalues (spirals)

If eigenvalues are complex:

$$\lambda = a \pm ib \quad (a, b \in \mathbb{R}, \ b \neq 0),$$

then

$$e^{\lambda t} = e^{at} (\cos(bt) + i \sin(bt)).$$

Interpretation:

- $\Re(\lambda) = a$  controls growth/decay of the amplitude
- $\Im(\lambda) = b$  controls oscillation frequency

So trajectories typically spiral.

## Complex eigenvalues: Spiral sink vs. spiral source

For  $\lambda = a \pm ib$ :

- $a < 0$ : amplitude decays  $\Rightarrow$  **spiral sink** (stable spiral)
- $a > 0$ : amplitude grows  $\Rightarrow$  **spiral source** (unstable spiral)
- $a = 0$ : constant amplitude  $\Rightarrow$  **center** (closed orbits)

**Takeaway:**  $\Re(\lambda)$  tells stability,  $\Im(\lambda)$  tells oscillation.

## Example 1: Spiral sink (complex eigenvalues with negative real part)

Take

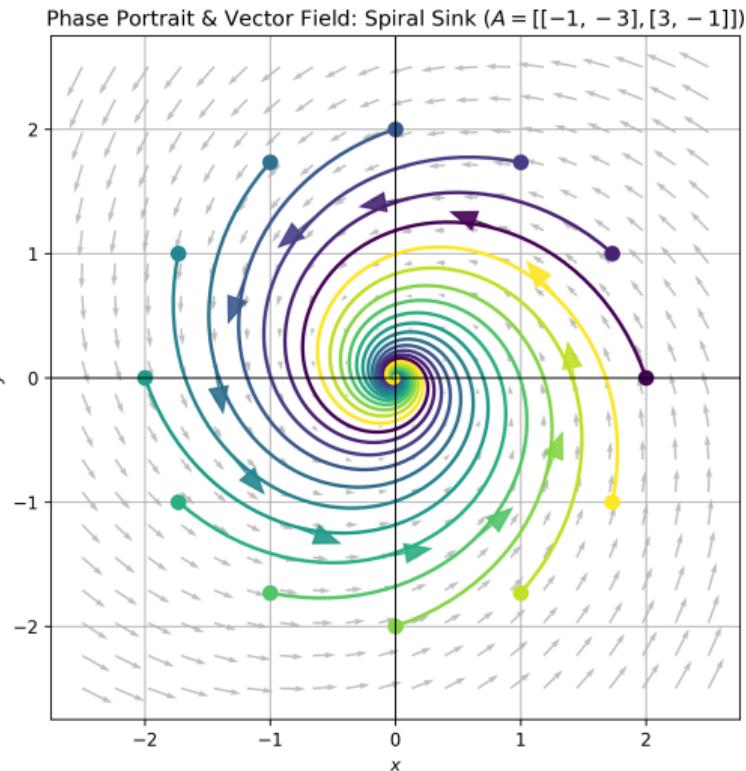
$$A = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix}$$

**Eigenvalues:**  $\lambda_{1,2} = -1 \pm 3i$

This matrix produces rotation (off-diagonal terms or imaginary part of eigenvalues) plus decay (negative diagonal terms or real part of eigenvalues).

### Qualitative behavior:

- trajectories rotate while shrinking
- phase portrait spirals into the origin



**Figure:** Spiral sink: complex eigenvalue with negative real part.

## Example 2: Spiral source (complex eigenvalues with positive real part)

Take

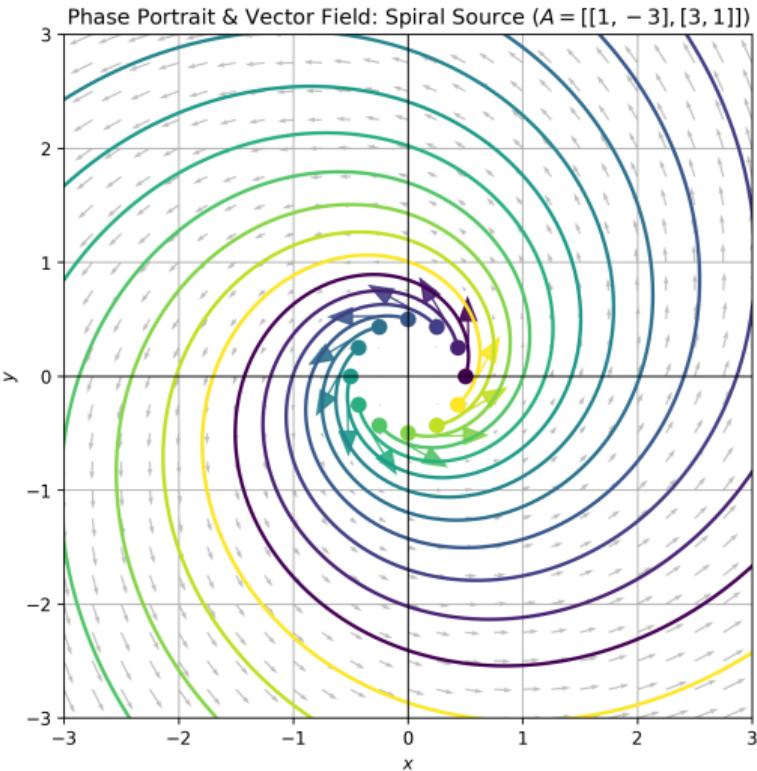
$$A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

**Eigenvalues:**  $\lambda_{1,2} = 1 \pm 3i$

This matrix produces rotation (off-diagonal terms or imaginary part of eigenvalues) plus decay (positive diagonal terms or real part of eigenvalues).

### Qualitative behavior:

- trajectories rotate and move outward
- phase portrait spirals out away from the origin towards infinity



**Figure:** Spiral source: complex eigenvalue with positive real part.

## Example 3: Closed orbits (purely imaginary eigenvalues)

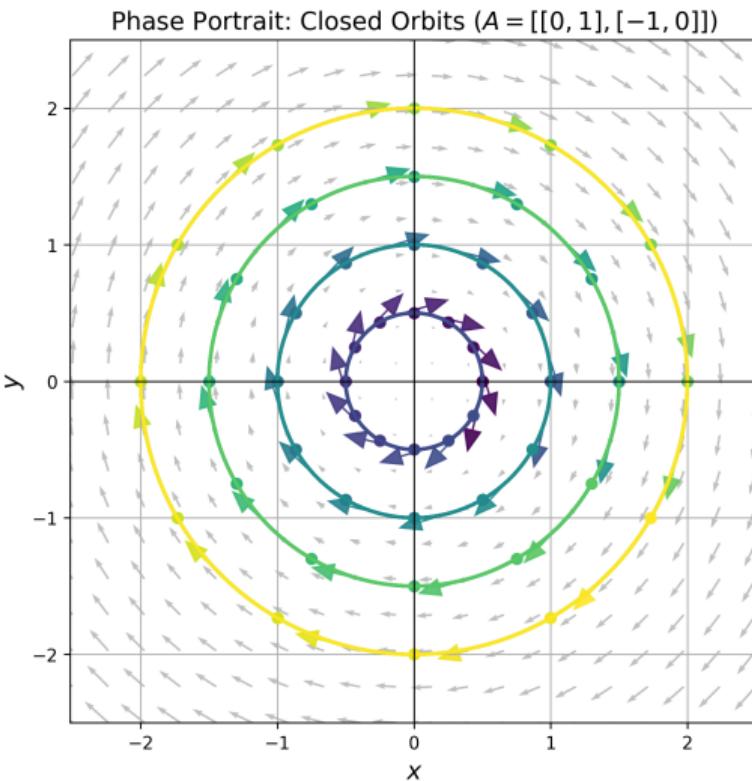
Take the matrix for the undamped oscillator,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

**Eigenvalues:**  $\lambda_{1,2} = \pm i$

**Qualitative behavior:**

- no decay, no growth
- closed orbits around the origin
- sustained oscillations (constant amplitude)



**Figure:** Closed orbits: eigenvalue are purely imaginary.

## Quick classification summary (2D)

For  $\dot{\mathbf{x}} = A\mathbf{x}$  with eigenvalues  $\lambda_1, \lambda_2$ :

- $\lambda_1, \lambda_2 < 0 \Rightarrow$  stable node (sink)
- $\lambda_1, \lambda_2 > 0 \Rightarrow$  unstable node (source)
- opposite signs  $\Rightarrow$  saddle
- $a \pm ib$  with  $a < 0 \Rightarrow$  spiral sink
- $a \pm ib$  with  $a > 0 \Rightarrow$  spiral source
- $\pm ib$  (pure imaginary)  $\Rightarrow$  center

**Eigenvalues  $\Rightarrow$  qualitative phase portrait.**

## A useful diagnostic: trace and determinant (2D)

For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{tr}(A) = a + d, \quad \det(A) = ad - bc.$$

Eigenvalues satisfy:

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0.$$

The discriminant

$$\Delta = \text{tr}(A)^2 - 4\det(A)$$

tells if eigenvalues are real ( $\Delta \geq 0$ ) or complex ( $\Delta < 0$ ).

## Systems look the same to numerical methods

For a general system:

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

numerical methods work almost exactly like the scalar case:

- replace  $x$  by  $\mathbf{x}$
- replace  $f$  by  $\mathbf{F}$
- updates become vector updates

The key difference: **vectors and (sometimes) linear solves.**

# Forward Euler for systems

Forward Euler:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \mathbf{F}(t_n, \mathbf{x}_n).$$

- explicit: easy to implement
- one function evaluation per step
- stability can still restrict  $h$

# Explicit Midpoint for systems

Explicit Midpoint:

**Half-step:**

$$\mathbf{x}_{n+\frac{1}{2}} = \mathbf{x}_n + \frac{h}{2} \mathbf{F}(t_n, \mathbf{x}_n)$$

**Full step:**

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \mathbf{F}\left(t_n + \frac{h}{2}, \mathbf{x}_{n+\frac{1}{2}}\right)$$

- second-order accurate
- still explicit
- still can have a stability limit

## Backward Euler for systems

Backward Euler:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \mathbf{F}(t_{n+1}, \mathbf{x}_{n+1}).$$

- implicit:  $\mathbf{x}_{n+1}$  appears on both sides
- requires solving an equation (often nonlinear) each step
- can be much more stable for stiff decay problems

## Special case: linear system $\dot{\mathbf{x}} = A\mathbf{x}$

Now let

$$\dot{\mathbf{x}} = A\mathbf{x}$$

with constant matrix  $A$ .

Then numerical methods become:

- matrix-vector multiplications (explicit methods)
- linear system solves (implicit methods)

This is where **amplification matrices** appear.

## Forward Euler on $\dot{\mathbf{x}} = A\mathbf{x}$

Forward Euler:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + hA\mathbf{x}_n = (I + hA)\mathbf{x}_n.$$

**Amplification matrix:**

$$G_{FE} = I + hA.$$

After  $n$  steps:

$$\mathbf{x}_n = (I + hA)^n \mathbf{x}_0.$$

## Backward Euler on $\dot{\mathbf{x}} = A\mathbf{x}$

Backward Euler:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + hA\mathbf{x}_{n+1}.$$

Rearrange:

$$(I - hA)\mathbf{x}_{n+1} = \mathbf{x}_n$$

so

$$\mathbf{x}_{n+1} = (I - hA)^{-1}\mathbf{x}_n.$$

**Amplification matrix:**

$$G_{BE} = (I - hA)^{-1}.$$

## Explicit Midpoint on $\dot{\mathbf{x}} = A\mathbf{x}$

Half-step:

$$\mathbf{x}_{n+\frac{1}{2}} = \mathbf{x}_n + \frac{h}{2}A\mathbf{x}_n = \left(I + \frac{h}{2}A\right)\mathbf{x}_n.$$

Full step:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + hA\mathbf{x}_{n+\frac{1}{2}} = \mathbf{x}_n + hA\left(I + \frac{h}{2}A\right)\mathbf{x}_n.$$

So:

$$\mathbf{x}_{n+1} = \left(I + hA + \frac{(hA)^2}{2}\right)\mathbf{x}_n.$$

**Amplification matrix:**

$$G_{EM} = I + hA + \frac{(hA)^2}{2}.$$

## Stability idea for systems (same philosophy)

For decay systems, we want the numerical method to **damp** modes.

For a matrix update:

$$\mathbf{x}_{n+1} = G\mathbf{x}_n,$$

stability is governed by eigenvalues of  $G$ .

**Rule of thumb:**

$$|\mu| < 1 \quad \text{for all eigenvalues } \mu \text{ of } G.$$

So again: **eigenvalues determine stability**.

Eigenvalues of  $G$  relate to eigenvalues of  $A$

If  $A\mathbf{v} = \lambda\mathbf{v}$ , then:

**Forward Euler:**

$$G_{FE}\mathbf{v} = (I + hA)\mathbf{v} = (1 + h\lambda)\mathbf{v} \Rightarrow \mu = 1 + h\lambda.$$

**Backward Euler:**

$$G_{BE}\mathbf{v} = (I - hA)^{-1}\mathbf{v} = \frac{1}{1 - h\lambda}\mathbf{v} \Rightarrow \mu = \frac{1}{1 - h\lambda}.$$

**Explicit Midpoint:**

$$\mu = 1 + h\lambda + \frac{(h\lambda)^2}{2}.$$

## Interpretation: stability reduces to checking each eigenvalue

For  $\dot{\mathbf{x}} = A\mathbf{x}$ , each eigenvalue  $\lambda$  of  $A$  behaves like a scalar mode.

So stability is controlled by the set  $\{h\lambda_k\}$ .

- if some  $\lambda_k$  is large negative (fast decay), explicit methods may require very small  $h$
- this is the beginning of **stiffness**
- implicit methods can remain stable for much larger  $h$

## A quick example: choosing $h$ from eigenvalues

Suppose  $A$  has eigenvalues:

$$\lambda_1 = -1, \quad \lambda_2 = -50.$$

Forward Euler stability on each mode requires roughly:

$$|1 + h\lambda_k| < 1 \Rightarrow 0 < h < \frac{2}{|\lambda_k|}.$$

So the tightest constraint comes from  $-50$ :

$$h < \frac{2}{50} = 0.04.$$

Even though one mode is “slow” ( $-1$ ), the “fast” mode forces a tiny step size.

## Computational note: what changes in code?

### Forward Euler / Midpoint:

- compute  $\mathbf{F}(t, \mathbf{x})$  and do vector arithmetic
- (for linear systems) compute  $A\mathbf{x}$  via matrix-vector product

### Backward Euler:

- solve a system each step
- linear case: solve  $(I - hA)\mathbf{x}_{n+1} = \mathbf{x}_n$
- nonlinear case: often use Newton or fixed-point iteration

## Summary: what to remember

- For  $\dot{\mathbf{x}} = A\mathbf{x}$ , eigenvalues predict behavior:
  - $\Re(\lambda) < 0$  decay (stable),  $\Re(\lambda) > 0$  growth (unstable)
  - $\Im(\lambda) \neq 0$  introduces oscillations (spirals/centers)
- Forward Euler / Midpoint / Backward Euler extend to systems by replacing  $x$  with  $\mathbf{x}$
- For linear systems, methods become  $\mathbf{x}_{n+1} = G\mathbf{x}_n$  with an amplification matrix  $G$
- Numerical stability depends on eigenvalues of  $G$ , which are functions of eigenvalues of  $A$