The Efficiency of Search Processes - A Study using Phototactic Robotics

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Aim of the project

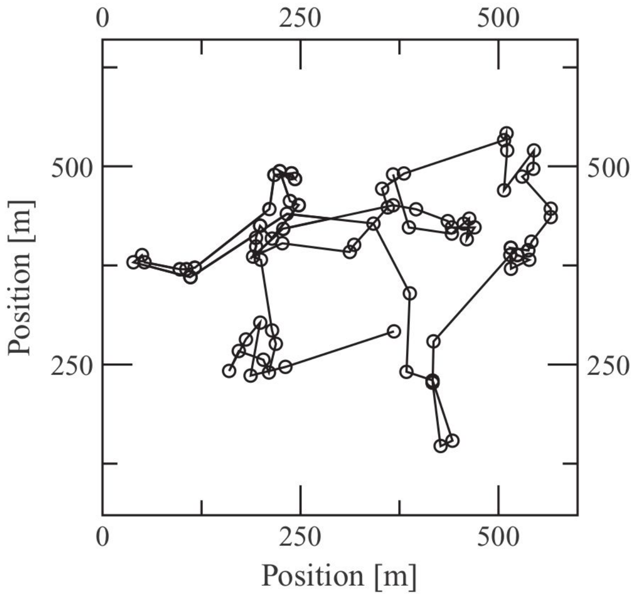
- To study different kinds of diffusive process theoretically (normal and anomalous diffusion).

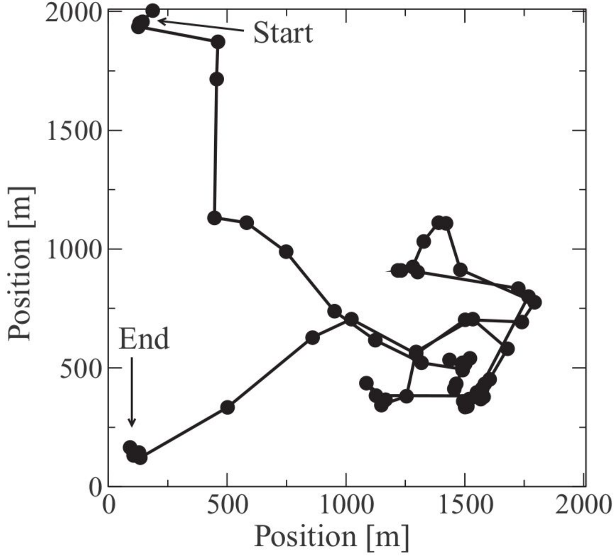
- To perform phototactic search experiments with *jump steps* drawn from various distributions.

Why study Movement and Search Processes?

- Ubiquitous in nature. Eg: animals searching for food or mates, reactants searching each other to react and form products, protein searching for target sequences on DNA, etc.
- Study of one system allows explanation of phenomena occurring in other systems (Universality).
- Closely connected to the area of random walk theory, stochastic processes, etc.

Monkey = Human?





(1) **Walk patterns of Spider Monkeys in Yucatan Peninsula:** Ramos-Fernández, G., Mateos, J. L., Miramontes, O., et al. 2004. Lévy walk patterns in the foraging movements of spider monkeys (Ateles geoffroyi), Behavioral Ecology and Sociobiology.

(2) **Lévy walk pattern for human mobility:** Rhee, I., Shin, M., Hong, S., Lee, K., and Chong, S. 2008. On the Lévy-walk nature of human mobility. In IEEE INFOCOM 2008 Proceedings. Phoenix, Arizona: Curran Associates.

Central Limit Theorem?

- Stability condition: When does the probability density function $P_N(S)$ of sum $S = X_1 + X_2 + \ldots + X_N$ of independent and identically distributed (as $p(x)$) random variables have the distribution as $p(x)$ (upto a scale-factor) ?

- The standard answer is that $p(x)$ should be a Gaussian as the sum of *N* Gaussians is again a Gaussian but with *N* times the variance of the original.

- But, Lévy proved that there exists solutions other than Gaussians as well. However, these other solutions involve random variables with infinite variances.

Skew Lévy α-stable distribution

The most general solution for the probability density function of *S* and its characteristic function $\phi(t)$, known as the skew Lévy α-stable distribution, is given by

$$P(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \exp[-itS]\phi(t)$$

$$\phi(t) = \exp\{it\nu - |ct|^{\alpha}(1 - i\beta sign(t)\Phi)\}$$

$$\Phi = \begin{cases} \tan(\alpha\pi/2), & \alpha \neq 1 \\ -\frac{2}{\pi} ln|t|, & \alpha = 1 \end{cases}$$

Where ν is the shift, *β* represents skewness and *c* is a scale. For the Lévy index $\alpha \in (0, 2]$, the distribution has a power-law tail with exponent given by $\mu = \alpha + 1$. The three special cases are:

- **Gaussian distribution:** $\alpha = 2$, $\sigma^2 = 2c^2$ (*β* irrelevant)
- **Cauchy distribution:** $\alpha = 1, \beta = 0$
- **Lévy distribution:** $\alpha = 1/2, \beta = -1$

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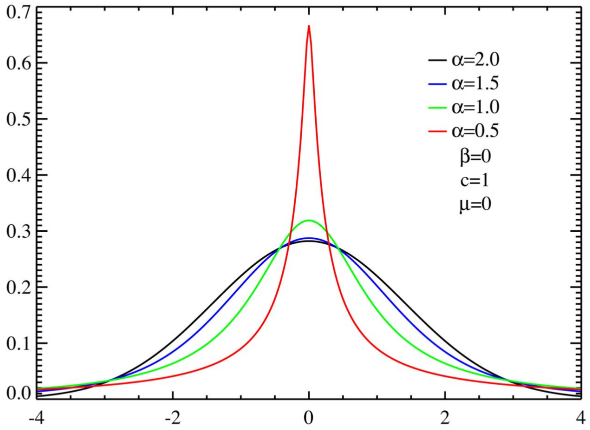

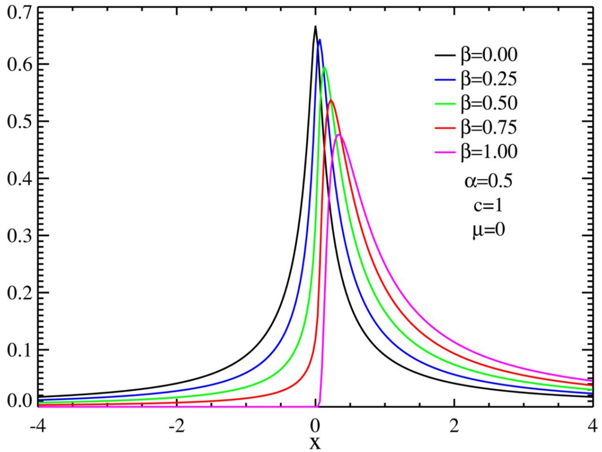

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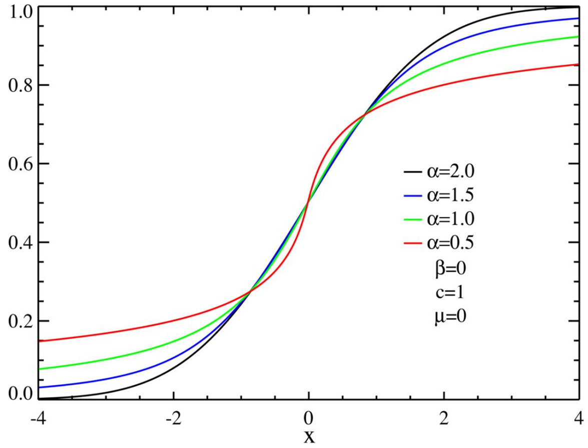
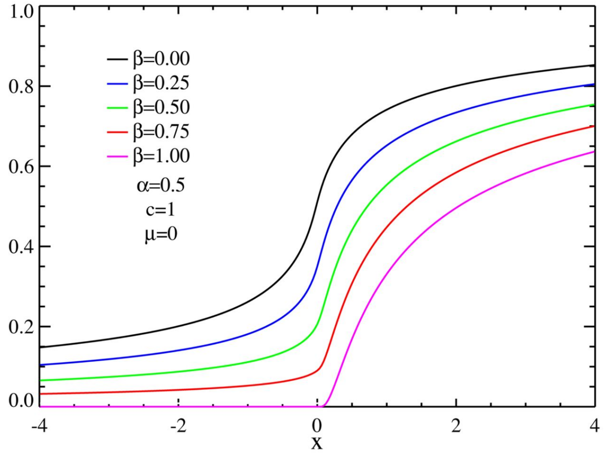




Source: https://en.wikipedia.org/wiki/Stable_distribution

Plan

1. Biological encounters as *reaction-diffusion* processes.
2. Normal diffusion/Brownian motion.
3. Anomalous diffusion.
4. Lévy flights and Lévy walks.
5. Evidence for Lévy flight foraging hypothesis.
6. Experiments.
7. Conclusion and scope.

Reaction-Diffusion Processes

$$\partial_t \mathbf{u} = D\nabla^2 \mathbf{u} + R(\mathbf{u})$$

- In 1D, $\partial_t u = D\partial_x^2 u + R(u)$, known as the Kolmogorov-Petrovsky-Piskunov equation.

- When $R(u) = 0$, Fick's Law of diffusion (pure diffusion).
- When $R(u) = ru(1-u)$, Fisher's equation.
- Pure Reaction Case: Lotka-Voltera equations (# predator = *v*, #prey = *u*)

$$\frac{\partial u}{\partial t} = au - buv$$
$$\frac{\partial v}{\partial t} = cuv - dv$$

Langevin and Fokker-Planck Equations

- Langevin Equation: $$F_{tot} = m\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} + \eta(t)$$

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = 2\lambda k_B T\delta(t - t')$$

Gaussian distributed noise term is uncorrelated in time.

- Fokker-Planck Equation: $$\frac{\partial}{\partial t}P(x,t) = D\frac{\partial^2}{\partial x^2}P(x,t)$$

With initial conditions, $t = 0, \quad P(x,0) = \delta(x)$, the solution is given by

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}}\exp\left[-\frac{x^2}{4Dt}\right]$$

Normal diffusion

- The second moment of *x* scales linearly in time. Also, the higher moments do not grow independently. All moments are finite.

$$\langle x^2 \rangle = 2Dt$$

$$\langle x^2 \rangle \sim t$$
$$\langle x^{2n} \rangle \sim t^{2n}$$
$$\langle x^{2n} \rangle \sim \langle x^2 \rangle^n$$

- The Fokker-Planck equation had a first-order differential in time and second-order in space. As a result, the mean squared displacement (second moment) scales linearly in time. With the use of fractional derivatives in time or space or both, the mean square displacement need no longer grow linearly in time. Instead, sublinear or superlinear growth are possible leading to subdiffusion and superdiffusion, respectively.

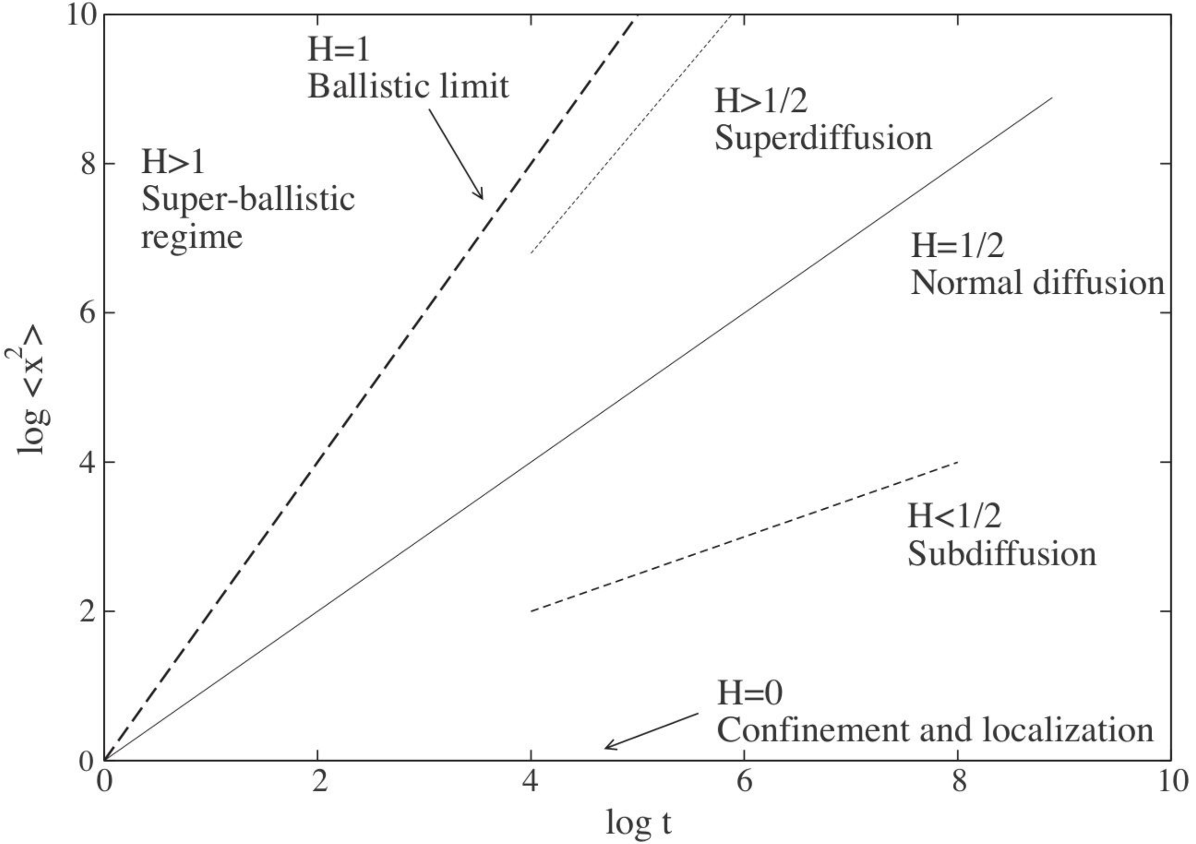
Anomalous diffusion

- One usually defines a quantity called Hurst exponent H to quantify anomalous diffusion.

$$\langle x^2 \rangle \sim t^{2H}, \qquad \langle |x|^q \rangle \sim t^{qH(q)}$$

- For superdiffusion, $H > \frac{1}{2}$, and for subdiffusion, $H < \frac{1}{2}$.
- Sometimes, second moment is not sufficient to quantify the full behaviour of the probability density functions. Hence, one also defines the generalized Hurst exponents, $H(q)$.
- When $H(q) = H$ for all q, the process is known as a monofractal walk. When $H(q)$ is different for different values of q, it is known as a multifractal walk.

- **Lévy flights and walks fall in the superdiffusive regime with $H > \frac{1}{2}$.**



H=1
Ballistic limit
H>1
Super-ballistic
regime
H>1/2
Superdiffusion
H=1/2
Normal diffusion
H<1/2
Subdiffusion
H=0
Confinement and localization
log <x²>
log t
0
2
4
6
8
10

Lévy flights and Lévy walks

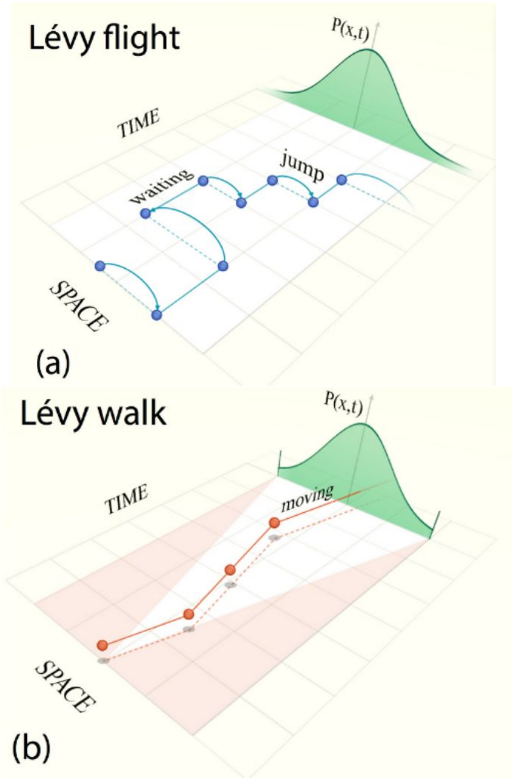
- Lévy flights are characterized by waiting times τ followed by instantaneous dislocations ℓ in space.
- For Lévy walks, the relocation happens at a constant speed v, hence the time and space are coupled, via $\ell = v\tau$.
- $\psi(\ell, \tau)$ denotes the joint probability density function for a step to take a duration τ and have jump length ℓ. Here, $\lambda(\ell)$, $\omega(\tau)$ are the marginal probability density functions. T and $\langle \ell^2 \rangle$ are the mean pausing time and second moment of jump size, respectively. $\lambda(\ell)$, $\omega(\tau)$ are distributed as power-law with exponent $\mu = \alpha + 1$ in the long-time limit.

$$\lambda(l) = \int_0^\infty \psi(l,\tau)d\tau$$

$$\omega(\tau) = \int_{-\infty}^\infty \psi(l,\tau)dl$$

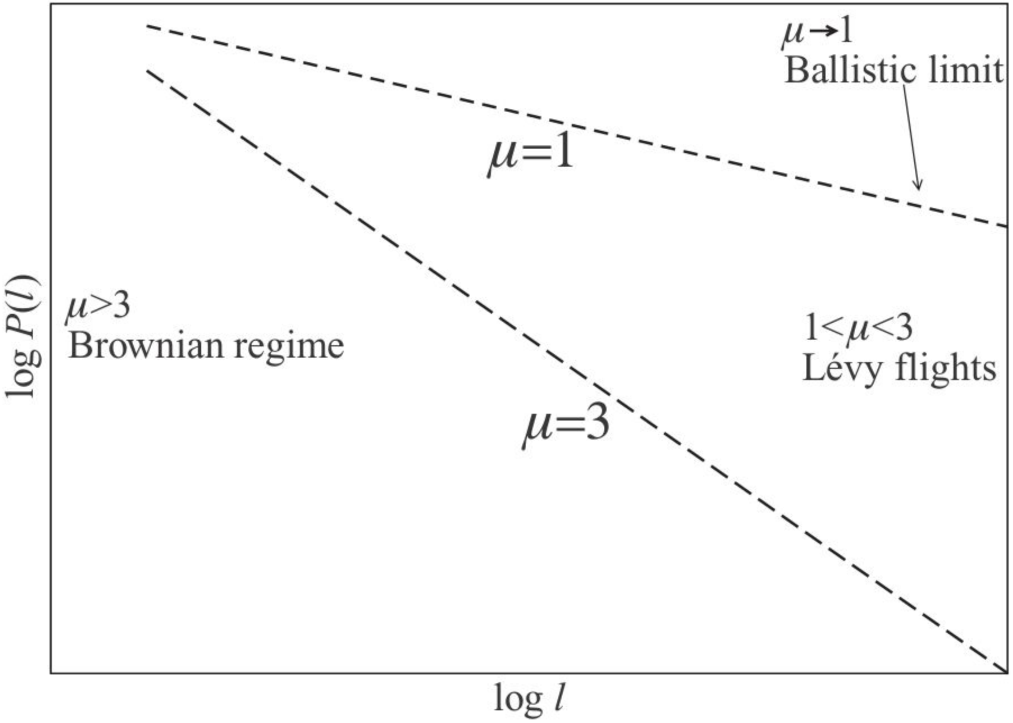
$$T = \int_0^\infty \tau\omega(\tau)d\tau$$

$$\langle l^2 \rangle = \int_{-\infty}^\infty l^2\lambda(l)dl$$



Picture Courtesy: Lévy walks, V. Zaburdaev, S. Denisov, and J. Klafter, Rev. Mod. Phys. **87**, 483

μ = α + 1



$\mu \to 1$
Ballistic limit
$\mu=1$
$\mu>3$
Brownian regime
$1<\mu<3$
Lévy flights
$\mu=3$
log $P(l)$
log l

Scaling for Lévy flights

- Lévy flights arise when the jump size distribution has a power-law tail $\lambda(\ell) \sim \ell^{-\mu}$, leading to diverging variance for $\mu < 3$, where $\mu = \alpha + 1$.
- For $\alpha < 2$, one cannot define the mean squared displacement because it diverges. One can study moments of the order less than α. Instead, we define pseudo mean square displacement with grows as $\sim t^{1/\alpha}$.

Therefore, Hurst exponent $H = \frac{1}{\alpha} = \frac{1}{\mu - 1}$.

Hence the moments satisfy, $\langle |x^q| \rangle^{1/q} \sim t^{1/\alpha}, \qquad q < \alpha$.

Scaling for Lévy walks

- Scaling behavior is more difficult to derive in this case, though it can be understood intuitively as follows:
- Since the time and space are coupled, both λ(ℓ) ~ $\ell^{-\mu}$ and ω(τ) ~ $\tau^{-\mu}$, where (μ = α + 1).

$$t \sim N, \qquad 1 < \alpha < 2$$
$$x = \sum l_i$$
$$\langle l^2 \rangle \sim \int_0^t l^2 l^{-(\alpha+1)} dl \sim t^{2-\alpha}$$

- <x^2> is proportional to <ℓ^2> because <ℓ^2> represents the characteristic scale of the jumps.

$$\langle x^2 \rangle \sim \langle l^2 \rangle t$$
$$\langle x^2 \rangle \sim t^{3-\alpha}, \qquad 1 < \alpha < 2$$

- As the result, the Hurst exponent $H = \frac{3-\alpha}{2} = \frac{4-\mu}{2}, \qquad 1 < \alpha < 2.$

Optimal *μ* for Lévy walks by Vishwanathan et. al.(1999)

- The mean free path of the searcher is $\lambda = (2r\rho)^{-1}$ where *r* is the sight radius and *ρ* is the density of the targets.
- Efficiency $\eta = \frac{1}{\langle l \rangle N}$, where <ℓ> is the mean jump size and *N* is the average number of flights between two successive targets.
- The average number of steps between two targets (in non-destructive case): $N_n \sim \left(\frac{\lambda}{r}\right)^{(\mu-1)/2}$
- The mean jump size $\langle l \rangle = \frac{\int_r^\lambda l.l^{-\mu} dl + \lambda \int_\lambda^\infty l^{-\mu} dl}{\int_r^\infty l^{-\mu} dl} = \left(\frac{\mu-1}{2-\mu}\right)\left(\frac{\lambda^{2-\mu} - r^{2-\mu}}{r^{1-\mu}}\right) + \frac{\lambda^{2-\mu}}{r^{1-\mu}}$
- When targets are plentiful, $\lambda \leq r$ and $N_n \sim 1,\quad \langle l \rangle \sim \lambda$, in which case, the efficiency is independent of *μ*.
- When targets are sparse, $\lambda \gg r$, the optimal value of the exponent, $\mu_{opt} = 2 - 1/[ln(\lambda/r)]^2 \approx 2$

Evidence from nature

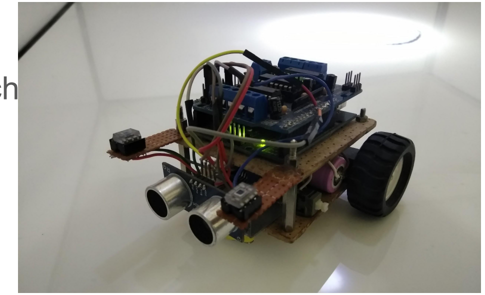
Lévy flight foraging hypothesis: since Lévy flights and walks can optimize search efficiencies, therefore natural selection should have led to adaptations for Lévy flight foraging.

Some evidences from previous works:

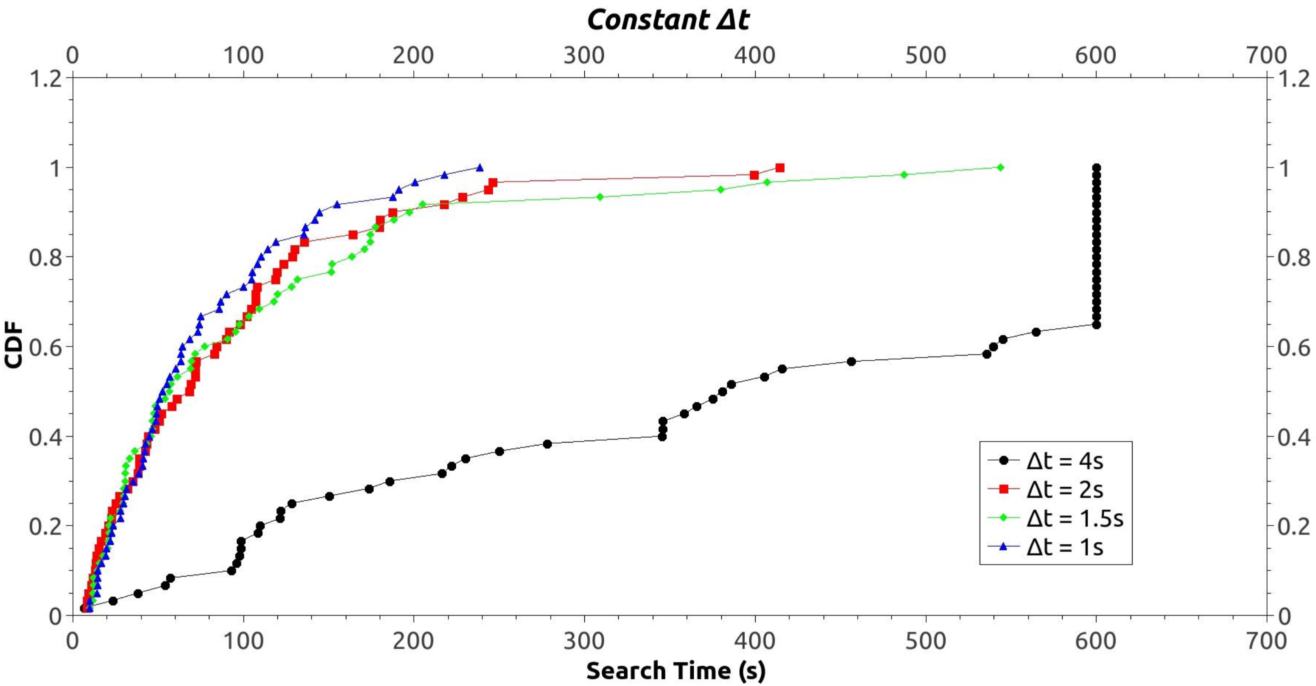
- Forcadi *et. al.* reported that fallow deer (*Dama dama*) performs Lévy walks while searching alone for food, with exponent $\mu = 2.16 \pm 0.13$ [1].
- Reynold *et al.* studied butterflies and have speculated that Lévy flight searches might underlie visually prompted mate location in these organisms [2]. In a separate study, they noted the behavior of *Angrotis segetum* moths and found complex flight patterns compatible with scale-free Lévy flight searches [3].
- Atkinson *et al.* used radio-tracking techniques employed by field ecologists to look for fractal behavior in the foraging trajectories of the African side-striped jackal. They reported evidence of superdiffusion [4] .
- Dai *et al.* studied a herd of elephants in South Africa and found evidence that they follow a stochastic process known as the Lévy-modulated correlated random walk [5].

Experiments with phototactic robot

- The robot performed a **random walk** with v = 4 cm/s.
- In one experiment, the jump steps were constant (correlated Brownian motion), in the other, they were drawn from a power-law distribution $P(\Delta t) \sim (\Delta t)^{-\mu}$ (Lévy walks with heavy tails). The pausing time τ between consecutive jumps was negligible and can be safely assumed to be zero.
- After each jump, an angle between -90º to 90º was chosen randomly with respect to the forward motion (hence subsequent steps are correlated).
- Search time (or, time for convergence) was noted.
- Light sensing was done only between two consecutive jumps, so it was possible to reach the light spot and overshoot and miss the target.
- Boundary effects cannot be ignored.

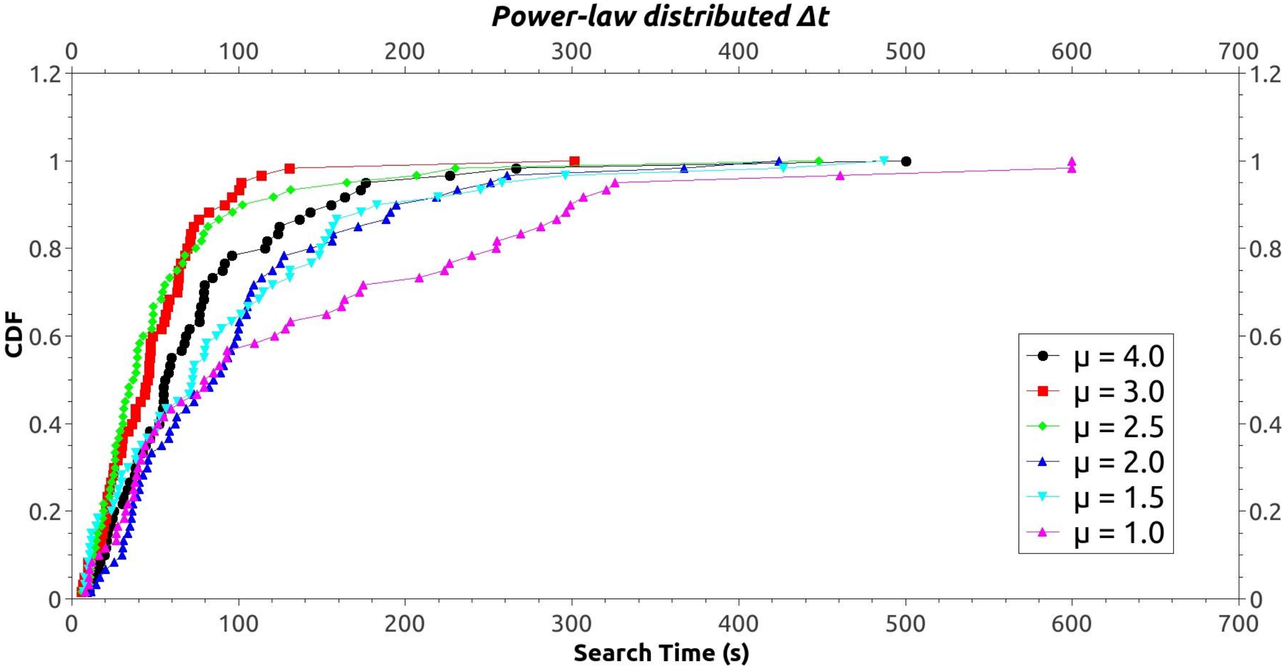


Graphs and Discussions



The search with smaller Δt is more likely to converge. With larger Δt, overshooting begins to kick in.

Graphs and Discussions



For μ > 3, the second moment becomes finite (normal diffusion) and efficiency drops.

Conclusion and scope

- Lévy walks/flights do improve search efficiency, in certain situations if not all.
- Similar experiments can be carried out with a chemotactic robot sensing CO_2, Ethanol, etc. Would require deeper thoughts because chemical species will diffuse in the air (unlike the light spot), hence sensing will be challenging.
- Applications of such robots in industries producing toxic gases; could detect leakage.
- Another application could be in designing rescue and search robots for use during natural calamities like earthquakes.
- A better understanding of search processes can help in understanding processes at the molecular/enzyme level. Lomholt *et al.* studied models of searches by proteins with targets on the DNA, combining normal diffusion and Lévy type diffusion [6].

References

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Thank you