2.1.2

Let G be a graph:

(a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.

(b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

Solution

(a)

($\Rightarrow$) Let G be a tree. Thus, G is connected (by definition), and acyclic. Since G is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

($\Leftarrow$) Let G be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in G, so G is a connected acyclic graph, or a tree.

(b)

($\Rightarrow$) Let G be a tree, and let e be an edge such that $e \notin E(G)$, and $e = uv$. Then, we create a cycle from the path $uTv + e$ — since there is only one path uTv, this means that $uTv + e$ is a unique cycle.

($\Leftarrow$) Suppose toward contradiction that adding e to the tree G yielded more than one cycle in the graph $G+e$. Then, the graph $G = G + e - e$ would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that G was a tree, we have reached a contradiction, meaning that e added exactly one cycle to the tree G.

2.1.6

Let T be a tree with average degree a. In terms of a, find $n(T)$.

Solution

$$\begin{aligned} d_{\text{avg}} &= \frac{2e(T)}{n(T)} \\ a &= \frac{2(n(T)-1)}{n(T)} \\ an &= 2n - 2 \\ (a-2)n &= -2 \\ n &= \boxed{\frac{-2}{a-2}} \end{aligned}$$

Needs to be corroborated

2.1.7

Prove that every n-vertex graph with m edges has at least $n - m + 1$ cycles.

2.1.12

Compute the diameter and radius of $K_{m,n}$.

2.1.13

Prove that every graph with diameter d has an independent set with at least $\lfloor \frac{1+d}{2} \rfloor$ vertices.