

1.3.3

Let u and v be adjacent vertices in G . Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G .

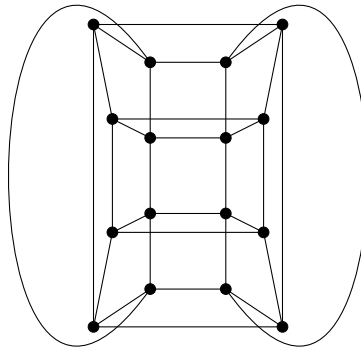
Solution

Let $u \leftrightarrow v \in G$. In order for uv to be in a triangle in G , u and v must share a common neighbor. By using the property of inclusion and exclusion, we can find the set as follows:

$$\begin{aligned} |N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &= d(u) + d(v) - n(G) \end{aligned}$$

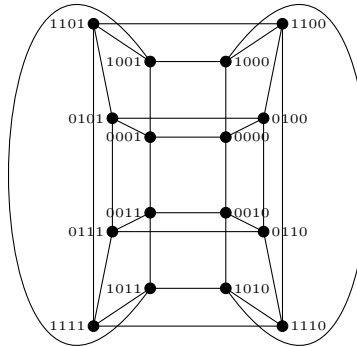
1.3.4

Prove that the graph below is isomorphic to Q_4 .



Solution

We can assign tuples to the graph as follows:



1.3.6

Given graphs G and H , determine the number of components and maximum degree in $G + H$ in terms of the parameters for G and H .

Solution

We can find the number of components in $G + H$ by summing the number of components in G and the number of components in H .

The maximum degree in $G + H$ is equal to $\max\{\Delta(G), \Delta(H)\}$.

1.3.7

Determine the maximum number of edges in a bipartite subgraph of P_n , C_n , and K_n .

Solution

For the graph P_n , we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets A and B . This is a bipartition since a path does not include any repeated vertices or edges, so A and B are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of P_n is the number of edges in P_n , which is $n - 1$.

For C_n , we have two values of the maximum number of edges in a bipartite subgraph of C_n :

- If n is even, then C_n is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of C_n is the number of edges in C_n , which is n .
- If n is odd, then C_n is not a bipartite graph. After one edge deletion, we get that $C_n - e = P_n$, which is bipartite, so the maximum number of edges in a bipartite subgraph of C_n is the number of edges in P_n , which is $n - 1$.

For the graph K_n , there are two options for the maximum number of edges in a bipartite subgraph depending on the value of n :

- If n is even, then the subgraph $K_{\frac{n}{2}, \frac{n}{2}}$ is the maximal bipartite subgraph, meaning that the number of edges is

equal to $n^2/4$. We know that $K_{\frac{n}{2}, \frac{n}{2}}$ is a subgraph of K_n because the vertex set is the same, and K_n is complete, so any subset of edges is a subset of the edge set of K_n .

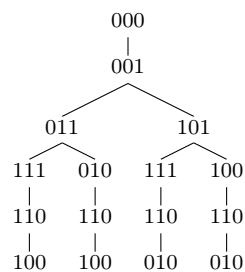
- If n is odd, then the subgraph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the maximal bipartite subgraph, because $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ and K_n is complete. Therefore, the total number of edges is $\lfloor \frac{n^2}{4} \rfloor$.

1.3.26 (a)

Count the 6-cycles in Q_3 .

Solution

Without loss of generality, we are starting from tuple $(0,0,0)$, henceforth 000. Starting from 000 going to 001. We can then map out the path as follows:



Since there are three starting options from 000, namely 001, 010, and 100, we can see there are 12 6-cycles in Q_3 .