1.3.17

Let G be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

Solution

(a)

Assume toward contradiction that deleting a vertex of degree $\Delta(G)$ increases the average degree.

$$\begin{aligned}
d'_{\text{avg}} &> d_{\text{avg}} \\
\frac{2e(G) - 2\Delta(G)}{n(G) - 1} &> \frac{2e(G)}{n(G)} \\
\frac{2e(G) - 2\Delta(G)}{2e(G)} &> \frac{n(G) - 1}{n(G)} \\
1 - \frac{\Delta(G)}{e(G)} &> 1 - \frac{1}{n(G)} \\
\frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} &> 0 \\
\frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{\text{avg}}} &> 0 \\
\frac{d_{\text{avg}} - 2\Delta(G)}{n(G)} &> 0 \\
d_{\text{avg}} - 2\Delta(G) &> 0 \\
d_{\text{avg}} &> 2\Delta(G)
\end{aligned}$$

However, we have reached a contradiction — by definition, $\Delta(G) \geq d_{\text{avg}}$, meaning that $d_{\text{avg}} \not> \Delta(G)$, let alone $2\Delta(G)$.

(b)

Deleting a vertex of the graph $K_{1,1}$ yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in $K_{1,1}$.

1.3.25

Prove that every cycle of length $2r$ in a hypercube is contained within a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r.

1.3.31

Using complete graphs and counting arguments, prove the following:

(a) $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ for $0 \leq k \leq n$.

(b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \leq \binom{n}{2}$.

Solution

(a)

K_n edge decomposition.

(b)

Cluster graphs vs. K_n.

1.3.41

Prove or disprove: if G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

Solution

Let $u, v \in V(G)$ and let $d(u) = \lceil \frac{n}{2} \rceil$. Then, u is adjacent to $\lceil \frac{n}{2} \rceil$ vertices and nonadjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Let $u \not\leftrightarrow v$.

We want to show that there exists some other vertex such that there exists a u, v path through that vertex. We know that $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$ and $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$.

Since $u \not\leftrightarrow v$, $N(u), N(v) \subseteq V(G) - \{u, v\}$. So, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq \left(\lceil \frac{n}{2} \rceil\right) + \left(\lfloor \frac{n}{2} \rfloor - 1\right) - (n-2) = 1$.

Therefore, there must be at least one element in $N(u) \cap N(v)$, meaning G is connected.