## 2.1.2

Let $G$ be a graph:

(a) Prove that $G$ is a tree if and only if $G$ is connected and every edge is a cut-edge.

(b) Prove that $G$ is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

## Solution

### (a)

($\Rightarrow$) Let $G$ be a tree. Thus, $G$ is connected (by definition), and acyclic. Since $G$ is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

($\Leftarrow$) Let $G$ be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in $G$, so $G$ is a connected acyclic graph, or a tree.

### (b)

($\Rightarrow$) Let $G$ be a tree, and let $e$ be an edge such that $e \notin E(G)$, and $e = uv$. Then, we create a cycle from the path $uTv + e$ — since there is only one path $uTv$, this means that $uTv + e$ is a unique cycle.

($\Leftarrow$) Suppose toward contradiction that adding $e$ to the tree $G$ yielded more than one cycle in the graph $G + e$. Then, the graph $G = G + e - e$ would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that $G$ was a tree, we have reached a contradiction, meaning that $e$ added exactly one cycle to the tree $G$.

## 2.1.6

Let $T$ be a tree with average degree $a$. In terms of $a$, find $n(T)$.

## Solution

$$\begin{aligned} d_{\text{avg}} &= \frac{2e(T)}{n(T)} \\ a &= \frac{2(n(T)-1)}{n(T)} \\ an &= 2n-2 \\ (a-2)n &= -2 \\ n &= \boxed{\frac{-2}{a-2}} \end{aligned}$$

Needs to be corroborated

## 2.1.7

Prove that every $n$-vertex graph with $m$ edges has at least $m - n + 1$ cycles.

**Solution**

Base Case If $m = 0$, then since this graph has zero edges, it has zero cycles, and since $0 \geq 1 - n$, we have proven the base case.

Inductive Hypothesis For an $n$-vertex graph with $0 \leq k \leq m$ vertices, then $G$ has at least $k-n+1$ cycles.

Proof If $e$ is an edge within a cycle of $G$, then $G - e$ has $k - 1$ edges, and has seen a reduction of 1 cycle, so $G - e$ has at least $(k - 1) - n + 1 = (k - n + 1) - 1$ cycles. If $e$ is not within a cycle, then $G$ has seen no reduction in cycles, but $G - e$ is predicted to have at least $(k - n + 1) - 1$ cycles, which it does by our assumption. Therefore, we have proven the inductive hypothesis for both cases.

**2.1.12**

Compute the diameter and radius of $K_{m,n}$.

**Solution**

The diameter of $K_{m,n}$ is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of $K_{m,n}$ is also 2 — the eccentricity of every vertex in $K_{m,n}$ is 2, so the radius must also be 2.

**2.1.13**

Prove that every graph with diameter $d$ has an independent set with at least $\left\lceil \frac{1+d}{2} \right\rceil$ vertices.

**Solution**

Let $G$ be a graph with diameter $d$, and let $u \in V(G)$ be a vertex with eccentricity $d$. Let $P$ be a maximal $u, v$ path of length $d$. Then, $P$ has $d + 1$ vertices. So, $P$ has a maximal independent set containing every other vertex, with total cardinality of $\left\lceil \frac{d+1}{2} \right\rceil$. Therefore, $G$ has an independent set with at least $\left\lceil \frac{d+1}{2} \right\rceil$ vertices.