

1.1**Individual****1.1.1**

Determine which complete bipartite graphs are complete graphs.

$K_{1,1}$ is the only complete bipartite graph that is complete

1.1.3

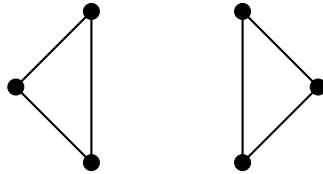
Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

$$K_{m,n} = \begin{bmatrix} a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_n \\ a_1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ a_2 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ b_1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ b_2 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

1.1.5

Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle.

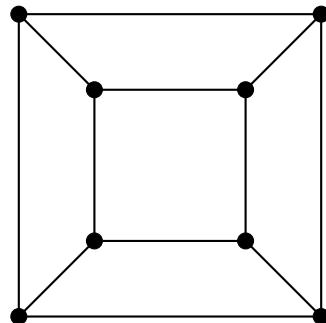
Let G be the following graph:

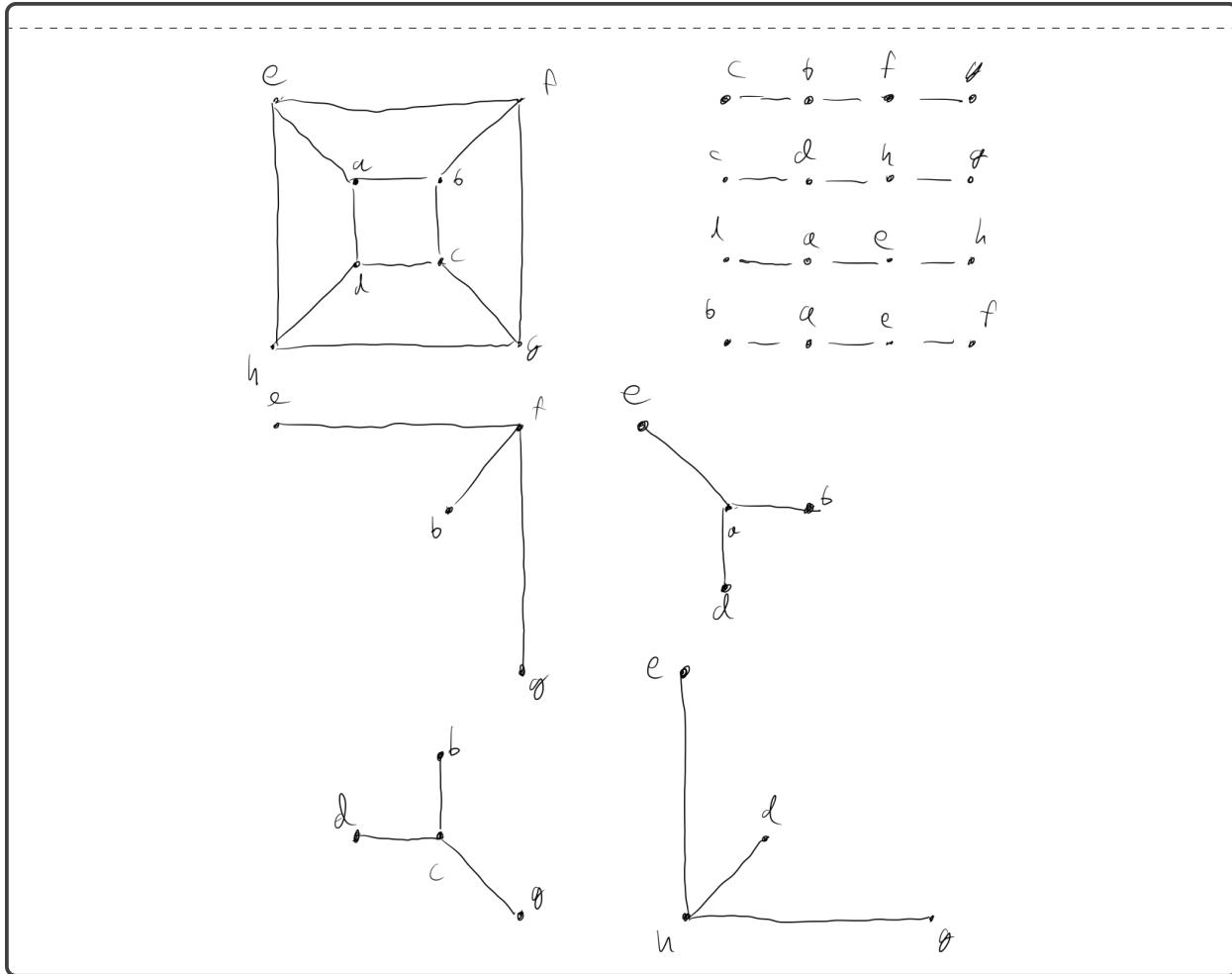


Every vertex in G has a degree 2, yet G is not a cycle.

1.1.8

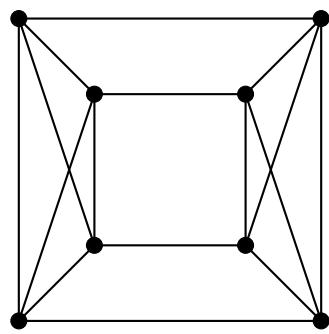
Prove that the 8 vertex graph below decomposes into copies of $K_{1,3}$ and also into copies of P_4

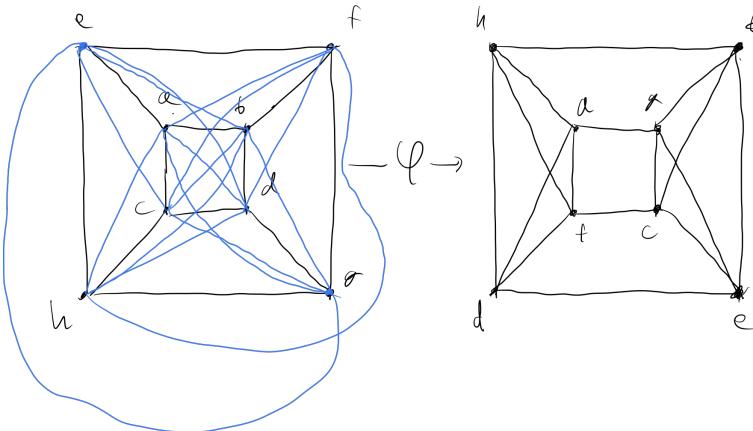




1.1.9

Prove that the graph below is isomorphic to the complement of the previous graph





1.1.10

Prove or disprove: the complement of a simple disconnected graph must be connected.

Let G be a graph that is disconnected. We want to show that $\forall x, y \in V(G), \exists xz$ path. We can split into two cases.

- Suppose $x \not\leftrightarrow y$ in G . Then, in \bar{G} , $x \leftrightarrow y$ by the definition of a graph complement.
- Suppose $x \leftrightarrow y$ in G . Then, since G is disconnected, we know that there must be some $z \in V(G)$ such that there is no xz path. Since there is no xz path, then there is no yz path. In particular, this means $x \not\leftrightarrow z$ and $y \not\leftrightarrow z$ in G . Therefore, in \bar{G} , we have that $x \leftrightarrow z$ and $y \leftrightarrow z$, meaning there is a path between x and y .

Group

1.1.13

Let G be the graph whose vertex set is the set of k -tuples with coordinates $\{0, 1\}$, with x adjacent to y if x and y differ by exactly one position. Determine whether G is bipartite.

G is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

1.1.26

Let G be a graph with girth 4 in which every vertex has degree k . Prove that G has at least $2k$ vertices. Determine all such graphs with $2k$ vertices.

Suppose G is a graph with girth 4 with every vertex of degree k . Let $v_i \in V(G)$. Then, there must be k vertices which v_i is adjacent to. However, none of these vertices can be adjacent to themselves or G would have girth 3. Thus, we can form a bipartition such that v_i is in a set of at least k vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to k vertices in

a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least $2k$ vertices.

The graphs with exactly $2k$ vertices are the $K_{n,n}$ complete bipartite graphs.

1.1.27

Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k , then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with $k^2 + 1$ vertices.

Let G be a simple graph with girth 5. Suppose that every vertex of G has degree k . Let $u \in V(G)$. Then, u has k adjacent vertices, each of which is not adjacent to each other (or else the girth of G would be 3). Let this set be N . The elements of N cannot have any other common neighbors aside from u , or else the girth of G would be 4, meaning each has $k - 1$ distinct neighbors. Therefore, the total number of vertices in our graph includes u , the elements of N that are the k distinct neighbors of u , and the $k(k-1)$ distinct vertices for each vertex in N . Therefore, our total is $1+k+k(k-1) = k^2+1$.

If there were any vertex with degree greater than k , then there would be additional vertices beyond the $k^2 + 1$ vertices necessary for a k -regular graph.

For $k = 2$, we have the graph C_5 for an example of a graph with $k^2 + 1$ vertices, and for $k = 3$ we have the Petersen graph.

1.1.30

Let G be a simple graph with adjacency matrix A and incidence matrix M . Prove that the degree of v_i is the i th diagonal entry of A^2 and MM^T . What do the entries in position (i, j) of A^2 and MM^T say about G ?

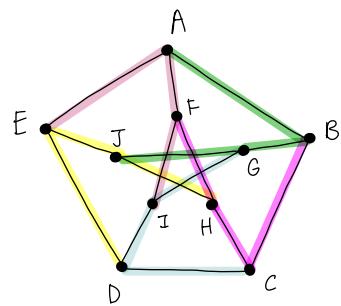
Let A be the adjacency matrix for a simple graph G . In A , every vertex's corresponding row and column are identical, meaning that the entry $A_{i,i}^2$ will be equal to $r_i c_i$ for row i and column i corresponding to v_i . Thus, $r_i c_i$ is equal to $|c_i|^2$, which is equal to the sum of the elements of c_i , which is equal to the degree of v_i .

Let M be the incidence matrix for a simple graph G . In MM^T , the diagonal element $MM_{i,i}^T$ will be equal to $r_i r_i^T$, where r_i represents the edge incidence row of v_i . This is equal to $|r_i^T|^2$, which is equal to the sum of the elements of r_i , which is equal to the number of edges incident on v_i , which is equal to the degree of v_i .

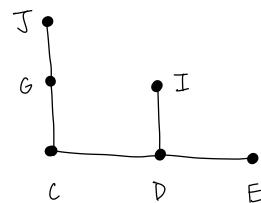
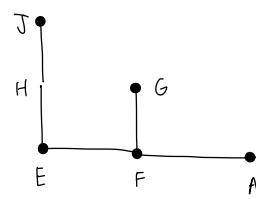
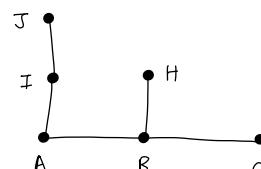
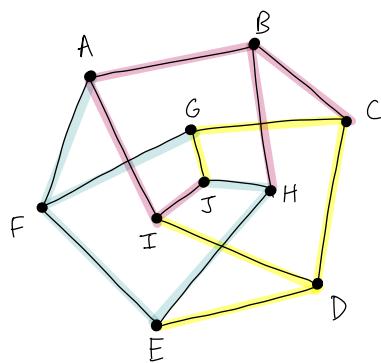
The entry in position (i, j) in both A^2 and MM^T shows whether vertices v_i and v_j are adjacent to each other.

1.1.34

Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4 .



$A - B - G - J$
 $B - C - H - F$
 $C - D - I - G$
 $D - E - J - H$
 $E - A - F - I$



1.2

Individual

1.2.1

Determine whether the following statements are true or false:

- Every disconnected graph has an isolated vertex.
 - A graph is connected if and only if some vertex is connected to all other vertices.
 - The edge set of every closed trail can be partitioned into edge sets of cycles.
 - If a maximal trail in a graph is not closed, then its endpoints have odd degree.
-
- False; we can imagine a graph with two components, each of which consists of K_3 , where there are no isolated vertices.
 - True; since $\forall u, v \in G, \exists u, v$ path by the definition of a connected graph, this means any vertex must have a path to any other vertex.
 - True; every closed trail contains within it a cycle — we can delete the edge set of this cycle, and find cycles within remaining components until we reach isolated vertices.
 - True; if there were a maximal trail with an endpoint of even degree, then we would be able to extend the trail further by re-entering the endpoint vertex.

1.2.5

Let v be a vertex of a connected simple graph G . Prove that v has a neighbor in every component of $G - v$. Explain why this allows us to conclude that no graph has a cut-vertex of degree 1.

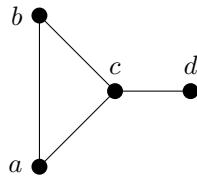
Suppose that $G - v$ is connected. Then, since G is connected, and $v \in V(G)$, it must be the case that v is connected to every component of $G - v$, meaning that it has a neighbor in every component of $G - v$ as $G - v$ is connected.

Now suppose that $G - v$ is disconnected, meaning that it has more than one component after removing v . Before, v must have been connected to every vertex in G as G was a simple connected graph, and afterwards $G - v$ is no longer connected, meaning that v is a cut-vertex. This means v must have been adjacent to a vertex in each component of $G - v$, as removing the incident edges on v along with v increased the number of components from the original 1 that was in G .

From this result, we can conclude that no cut-vertex has degree 1 as removing a vertex of degree 1 and its incident edges does not increase the number of components in G , since there is only one edge incident on a vertex of degree 1.

1.2.6

In the graph below, find all the maximal paths, maximal cliques, and maximal independent sets. Also, find all the maximum paths, cliques, and independent sets.



- The maximal paths are as follows:
 - d, c, b, a
 - d, c, a, b
 - a, b, c, d
 - b, a, c, d
 - b, c, a
 - c, b, a
 - a, c, b
- The maximal cliques are K_3 consisting of a, b, c and K_2 consisting of c, d .
- The maximal independent sets are $\{a, d\}$ and $\{b, d\}$.
- The maximum path is any of those paths listed above with length 4.
- The maximum clique is K_3 .
- The maximum independent sets are those listed above with size 2.

1.2.8

Determine the values of m and n such that $K_{m,n}$ is Eulerian.

$$m, n \in 2\mathbb{Z}^+$$

1.2.10

Prove or disprove:

- (a) Every Eulerian bipartite graph has an even number of edges.
- (b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

(a)

Let G be an Eulerian bipartite graph. Since G is Eulerian, it must contain an Eulerian cycle, meaning that as seen above, there are an even number of vertices, meaning that there are an even number of edges in G .

(b)

Let G be an Eulerian simple graph with an even number of vertices. Since G is Eulerian, this means there must be an Eulerian circuit C that traverses every edge exactly once in G . Every vertex in G must have even degree (or else we would require a backtrack in our Eulerian cycle, which is not a circuit); a simple pairing of the vertices would yield that we have $\lfloor n/2 \rfloor$ edges, and to complete the cycle we need $2(n/2) + 2k$ edges for n vertices and some integer k . Therefore, there must be an even number of edges.

Group

1.2.20

Let v be a cut-vertex of a simple graph G . Prove that $\overline{G} - v$ is connected.

Let $x, y, v \in V(\overline{G})$, where v is a cut-vertex of G .

Suppose x and y belong to distinct components of $G - v$. Then, $xy \notin E(G)$, meaning that $xy \in E(\overline{G})$, meaning there is an x, y path in \overline{G} , so there is an x, y path in $\overline{G} - v$.

Suppose x and y are in the same component of $G - v$. Since v is a cut-vertex, this means there must be at least two components in $G - v$. Let H_1 be the component that x, y are in, while $\exists w \in H_2$ is a vertex in H_2 disjoint from H_1 . Since H_1 and H_2 are disjoint, this means the components do not contain any edges between them, so $x \not\leftrightarrow w$ and $y \not\leftrightarrow w$ in $G - v$ — however, this means that $x \leftrightarrow w$ and $y \leftrightarrow w$ in \overline{G} , meaning that $\exists x, y$ path in $\overline{G} - v$.

1.2.22

Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Let G be a graph where there exists a partition of its vertices into two non-empty sets such that there is no edge with endpoints in both sets. Call these sets A and B . By our assumptions, $\forall u \in A$ and $\forall v \in B$, $\nexists e$ such that $e = uv$. Therefore, we cannot create a path between any $u \in A$ and any $v \in B$ as there is no edge to connect any element in A and any element in B . Therefore, G is disconnected.

Suppose G is a disconnected graph. Then, G contains more than one component — we can create a partition of $V(G)$ by letting H_1, H_2, \dots, H_k refer to the k components of G . Each of these components is necessarily disjoint from every other component. By taking $H = H_1 \cup H_2 \cup \dots \cup H_{k-1}$ as one set and H_k as our other set, we know that H_1, \dots, H_k are all disjoint, meaning that H and H_k are disjoint, meaning that there is no edge connecting any vertex H with any vertex in H_k , meaning we have created a partition of G such that there exists no edge between any vertex in one set and any vertex in the other set.

1.2.26

Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

Suppose G is bipartite. Then, there exists a partition of the vertices $V = X \sqcup Y$ such that X and Y are independent sets. Let H be a subgraph of G , and let $H_X = X \cap V(H)$ and $H_Y = Y \cap H$. Because H is a subgraph of G , each vertex of H must be an element of either H_X or H_Y , or that $V(H) = H_X \sqcup H_Y$. WLOG, let $|H_X| > |H_Y|$. Since $H_X \subseteq X$ and X is an independent set, H_X is an independent subset consisting of at least half of $V(H)$.

Suppose every subgraph of G has an independent set consisting of at least half of $V(H)$. We will suppose toward contradiction that G is not bipartite. Then, G must contain an odd cycle, H_1 . However, an independent set of H_1 consists of at most $\lfloor \frac{|V(H_1)|}{2} \rfloor < \frac{|V(H_1)|}{2}$, otherwise two vertices would be adjacent. Because H_1 is an independent set with less than half of the elements of $V(H)$, we have reached a contradiction. Therefore, G must be bipartite.

1.2.38

Prove that every n -vertex graph with at least n edges contains a cycle.

We proceed via induction as follows:

For the base case where $|V(G)| = 1$, we know that there is a cycle with one edge that connects back on the vertex.

For the case where $|V(G)| > 1$, if $v \in V(G)$ has degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges, so by our inductive hypothesis, we know that $G - v$ contains a cycle. Meanwhile, if $\forall v \in V(G), d(v) \geq 2$, we know by Lemma 1.2.25 that G contains a cycle.

1.3

Individual

1.3.3

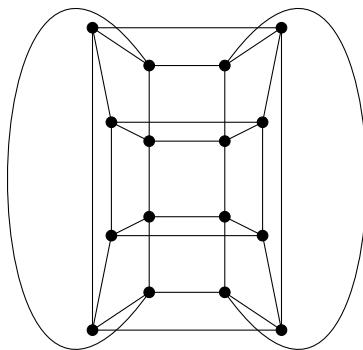
Let u and v be adjacent vertices in G . Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G .

Let $u \leftrightarrow v \in G$. In order for uv to be in a triangle in G , u and v must share a common neighbor. By using the property of inclusion and exclusion, we can find the set as follows:

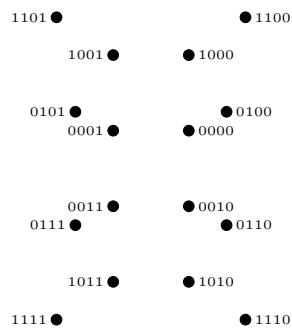
$$\begin{aligned} |N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &= d(u) + d(v) - n(G) \end{aligned}$$

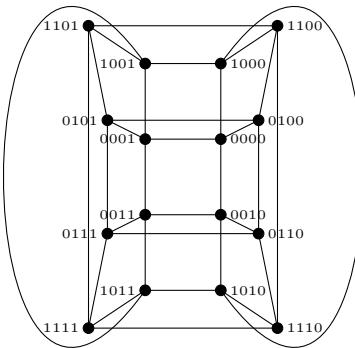
1.3.4

Prove that the graph below is isomorphic to Q_4 .



We can assign tuples to the graph as follows:





1.3.6

Given graphs G and H , determine the number of components and maximum degree in $G + H$ in terms of the parameters for G and H .

We can find the number of components in $G + H$ by summing the number of components in G and the number of components in H .

The maximum degree in $G + H$ is equal to $\max\{\Delta(G), \Delta(H)\}$.

1.3.7

Determine the maximum number of edges in a bipartite subgraph of P_n , C_n , and K_n .

For the graph P_n , we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets A and B . This is a bipartition since a path does not include any repeated vertices or edges, so A and B are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of P_n is the number of edges in P_n , which is $n - 1$.

For C_n , we have two values of the maximum number of edges in a bipartite subgraph of C_n :

- If n is even, then C_n is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of C_n is the number of edges in C_n , which is n .
- If n is odd, then C_n is not a bipartite graph. After one edge deletion, we get that $C_n - e = P_n$, which is bipartite, so the maximum number of edges in a bipartite subgraph of C_n is the number of edges in P_n , which is $n - 1$.

For the graph K_n , there are two options for the maximum number of edges in a bipartite subgraph

depending on the value of n :

- If n is even, then the subgraph $K_{\frac{n}{2}, \frac{n}{2}}$ is the maximal bipartite subgraph, meaning that the number of edges is equal to $n^2/4$. We know that $K_{\frac{n}{2}, \frac{n}{2}}$ is a subgraph of K_n because the vertex set is the same, and K_n is complete, so any subset of edges is a subset of the edge set of K_n .
- If n is odd, then the subgraph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the maximal bipartite subgraph, because $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ and K_n is complete. Therefore, the total number of edges is $\lfloor \frac{n^2}{4} \rfloor$.

1.3.26 (a)

Count the 6-cycles in Q_3 .

In order to find a 6-cycle in Q_3 , we select two vertices to delete and see if we can find a cycle from the graph $Q_3 - \{u, v\}$. Vertices are either adjacent, distance 2 (antipodal on the same face), or antipodal (distance 3) with each other.

ADJACENT: If two vertices are adjacent, we can find one cycle from the remaining vertices after deletion. There are $(8)(3)/2$ sets of adjacent vertices, for a total of 12 from this selection.

ANTIPODAL ON THE SAME FACE: If two vertices are antipodal, we cannot find a cycle from the remaining graph after deletion.

ANTIPODAL: If two vertices are antipodal, we can find one cycle from the remaining vertices after deletion. There are 4 sets of antipodal vertices.

We find a total of 16 6-cycles in Q_3 .

Group

1.3.17

Let G be a graph with at least two vertices. Prove or disprove:

- Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.
- Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

(a)

Assume toward contradiction that deleting a vertex of degree $\Delta(G)$ increases the average degree.

$$\begin{aligned}
 d'_{\text{avg}} &> d_{\text{avg}} \\
 \frac{2e(G) - 2\Delta(G)}{n(G) - 1} &> \frac{2e(G)}{n(G)} \\
 \frac{2e(G) - 2\Delta(G)}{2e(G)} &> \frac{n(G) - 1}{n(G)} \\
 1 - \frac{\Delta(G)}{e(G)} &> 1 - \frac{1}{n(G)} \\
 \frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} &> 0 \\
 \frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{\text{avg}}} &> 0 \\
 \frac{d_{\text{avg}} - 2\Delta(G)}{n(G)} &> 0 \\
 d_{\text{avg}} - 2\Delta(G) &> 0 \\
 d_{\text{avg}} &> 2\Delta(G)
 \end{aligned}$$

However, we have reached a contradiction — by definition, $\Delta(G) \geq d_{\text{avg}}$, meaning that $d_{\text{avg}} \not> \Delta(G)$, let alone $2\Delta(G)$.

(b)

Deleting a vertex of the graph $K_{1,1}$ yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in $K_{1,1}$.

1.3.25

Prove that every cycle of length $2r$ in a hypercube is contained within a subcube of dimension at most r . Can a cycle of length $2r$ be contained in a subcube of dimension less than r .

Let C be a cycle of length $2r$ in Q_n . Then, C contains $2r$ n -tuples. For every tuple in C , there exists a “switched” tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since C is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are $2r$ switches (corresponding to the $2r$ edges in C), this means there are at most r coordinates that are switched, then switched back sometime along the cycle’s path. This means the other $n - r$ coordinates are fixed, implying that $C \subseteq Q_r$, the r -dimensional subcube of Q_k .

There is a cycle of length 8 in Q_3 , defined as follows: $000 \rightarrow 001 \rightarrow 011 \rightarrow 010 \rightarrow 110 \rightarrow 111 \rightarrow 101 \rightarrow 100 \rightarrow 000$.

1.3.31

Using complete graphs and counting arguments, prove the following:

$$(a) \binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2} \text{ for } 0 \leq k \leq n.$$

$$(b) \text{ If } \sum n_i = n, \text{ then } \sum \binom{n_i}{2} \leq \binom{n}{2}.$$

(a)

We can consider a decomposition of the edges of K_n into the edge set of K_k and K_{n-k} , and some connector edges.

The edge set of K_n has cardinality $\binom{n}{2}$, the edge set of K_k has cardinality $\binom{k}{2}$, and the edge set of K_{n-k} has cardinality $\binom{n-k}{2}$. In order for this set of edges to be a full decomposition, we need a graph that connects all the vertices in K_k with all the vertices in K_{n-k} , which takes $k(n-k)$ edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph G , where $|V(G)| = n$ with maximal clique components H_1, \dots, H_k . Each of these components has $e(H_i) = \binom{|V(H_i)|}{2}$, with total $\sum_{i=1}^k \binom{|V(H_i)|}{2}$. In comparison, if we consider $e(K_G)$, where K_G is defined as the complete graph on the vertices of G , then that value is $\binom{n}{2}$, and $n = \sum_{i=1}^k |V(H_i)|$. Therefore, the size of the edge set of G is less than or equal to the sum of the sizes of the edge sets of maximal clique components H_i (because the maximal clique components of G could just be G itself).

1.3.41

Prove or disprove: if G is an n -vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

Let $u, v \in V(G)$ and let $d(u) = \lceil \frac{n}{2} \rceil$. Then, u is adjacent to $\lceil \frac{n}{2} \rceil$ vertices and nonadjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Let $u \not\sim v$.

We want to show that there exists some other vertex such that there exists a u, v path through that vertex. We know that $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$ and $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$.

Since $u \not\sim v$, $N(u), N(v) \subseteq V(G) - \{u, v\}$. So, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (\lceil \frac{n}{2} \rceil) + (\lfloor \frac{n}{2} \rfloor - 1) - (n - 2) = 1$.

Therefore, there must be at least one element in $N(u) \cap N(v)$, meaning G is connected.