

1.3.17

Let G be a graph with at least two vertices. Prove or disprove:

- (a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.
- (b) Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

Solution

(a)

Assume toward contradiction that deleting a vertex of degree $\Delta(G)$ increases the average degree.

$$\begin{aligned}
 d'_{\text{avg}} &> d_{\text{avg}} \\
 \frac{2e(G) - 2\Delta(G)}{n(G) - 1} &> \frac{2e(G)}{n(G)} \\
 \frac{2e(G) - 2\Delta(G)}{2e(G)} &> \frac{n(G) - 1}{n(G)} \\
 1 - \frac{\Delta(G)}{e(G)} &> 1 - \frac{1}{n(G)} \\
 \frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} &> 0 \\
 \frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{\text{avg}}} &> 0 \\
 \frac{d_{\text{avg}} - 2\Delta(G)}{n(G)} &> 0 \\
 d_{\text{avg}} - 2\Delta(G) &> 0 \\
 d_{\text{avg}} &> 2\Delta(G)
 \end{aligned}$$

However, we have reached a contradiction — by definition, $\Delta(G) \geq d_{\text{avg}}$, meaning that $d_{\text{avg}} \not> \Delta(G)$, let alone $2\Delta(G)$.

(b)

Deleting a vertex of the graph $K_{1,1}$ yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in $K_{1,1}$.

1.3.25

Prove that every cycle of length $2r$ in a hypercube is contained within a subcube of dimension at most r . Can a cycle of length $2r$ be contained in a subcube of dimension less than r .

Solution

Let C be a cycle of length $2r$ in Q_n . Then, C contains $2r$ n -tuples. For every tuple in C , there exists a “switched” tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since C is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are $2r$ switches (corresponding to the $2r$ vertices), this means

there are r coordinates that are switched, then switched back sometime along the cycle's path. This means the other $n - r$ coordinates are fixed, implying that $C \subseteq Q_r$, the r -dimensional subcube of Q_k .

There is a cycle of length 8 in Q_3 , defined as follows: $000 \rightarrow 001 \rightarrow 011 \rightarrow 010 \rightarrow 110 \rightarrow 111 \rightarrow 101 \rightarrow 100 \rightarrow 000$.

1.3.31

Using complete graphs and counting arguments, prove the following:

$$(a) \binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2} \text{ for } 0 \leq k \leq n.$$

$$(b) \text{ If } \sum n_i = n, \text{ then } \sum \binom{n_i}{2} \leq \binom{n}{2}.$$

Solution

(a)

We can consider a partition of the edges of K_n into the edge set of K_k and K_{n-k} , and some connector edges.

The edge set of K_n has cardinality $\binom{n}{2}$, the edge set of K_k has cardinality $\binom{k}{2}$, and the edge set of K_{n-k} has cardinality $\binom{n-k}{2}$. In order for this set of edges to be a full partition, we need to connect all the vertices in K_k and all the edges in K_{n-k} , which takes $k(n-k)$ edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph G , where $|V(G)| = n$ with maximal clique components H_1, \dots, H_k . Each of these components has $e(H_i) = \binom{|V(H_i)|}{2}$, with total $\sum_{i=1}^k \binom{|V(H_i)|}{2}$. In comparison, if we consider $e(K_G)$, where K_G is defined as the complete graph on the vertices of G , then that value is $\binom{n}{2}$, and $n = \sum_{i=1}^k |V(H_i)|$. Therefore, the size of the edge set of G is less than or equal to the sum of the sizes of the edge sets of maximal clique components H_i (because the maximal clique components of G could just be G itself).

1.3.41

Prove or disprove: if G is an n -vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

Solution

Let $u, v \in V(G)$ and let $d(u) = \lceil \frac{n}{2} \rceil$. Then, u is adjacent to $\lceil \frac{n}{2} \rceil$ vertices and nonadjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Let $u \not\leftrightarrow v$.

We want to show that there exists some other vertex such that there exists a u, v path through that vertex. We know that $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$ and $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$.

Since $u \not\leftrightarrow v$, $N(u), N(v) \subseteq V(G) - \{u, v\}$. So, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (\lceil \frac{n}{2} \rceil) + (\lfloor \frac{n}{2} \rfloor - 1) - (n - 2) = 1$.

Therefore, there must be at least one element in $N(u) \cap N(v)$, meaning G is connected.