

1.3.17

Let  $G$  be a graph with at least two vertices. Prove or disprove:

- (a) Deleting a vertex of degree  $\Delta(G)$  cannot increase the average degree.
- (b) Deleting a vertex of degree  $\delta(G)$  cannot decrease the average degree.

Solution

(a)

Assume toward contradiction that deleting a vertex of degree  $\Delta(G)$  increases the average degree.

$$\begin{aligned}
 d'_{\text{avg}} &> d_{\text{avg}} \\
 \frac{2e(G) - 2\Delta(G)}{n(G) - 1} &> \frac{2e(G)}{n(G)} \\
 \frac{2e(G) - 2\Delta(G)}{2e(G)} &> \frac{n(G) - 1}{n(G)} \\
 1 - \frac{\Delta(G)}{e(G)} &> 1 - \frac{1}{n(G)} \\
 \frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} &> 0 \\
 \frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{\text{avg}}} &> 0 \\
 \frac{d_{\text{avg}} - 2\Delta(G)}{n(G)} &> 0 \\
 d_{\text{avg}} - 2\Delta(G) &> 0 \\
 d_{\text{avg}} &> 2\Delta(G)
 \end{aligned}$$

However, we have reached a contradiction — by definition,  $\Delta(G) \geq d_{\text{avg}}$ , meaning that  $d_{\text{avg}} \not> \Delta(G)$ , let alone  $2\Delta(G)$ .

(b)

Deleting a vertex of the graph  $K_{1,1}$  yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in  $K_{1,1}$ .

1.3.25

Prove that every cycle of length  $2r$  in a hypercube is contained within a subcube of dimension at most  $r$ . Can a cycle of length  $2r$  be contained in a subcube of dimension less than  $r$ .

Solution

Let  $C$  be a cycle of length  $2r$  in  $Q_n$ . Then,  $C$  contains  $2r$   $n$ -tuples. For every tuple in  $C$ , there exists a “switched” tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since  $C$  is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are  $2r$  switches (corresponding to the  $2r$  vertices), this means

there are  $r$  coordinates that are switched, then switched back sometime along the cycle's path. This means the other  $n - r$  coordinates are fixed, implying that  $C \subseteq Q_r$ , the  $r$ -dimensional subcube of  $Q_k$ .

There is a cycle of length 8 in  $Q_3$ , defined as follows:  $000 \rightarrow 001 \rightarrow 011 \rightarrow 010 \rightarrow 110 \rightarrow 111 \rightarrow 101 \rightarrow 100 \rightarrow 000$ .

## 1.3.31

Using complete graphs and counting arguments, prove the following:

(a)  $\binom{n}{2} = \binom{k}{2} + k(n - k) + \binom{n - k}{2}$  for  $0 \leq k \leq n$ .

(b) If  $\sum n_i = n$ , then  $\sum \binom{n_i}{2} \leq \binom{n}{2}$ .

## Solution

(a)

We can consider a partition of the edges of  $K_n$  into the edge set of  $K_k$  and  $K_{n-k}$ , and some connector edges.

The edge set of  $K_n$  has cardinality  $\binom{n}{2}$ , the edge set of  $K_k$  has cardinality  $\binom{k}{2}$ , and the edge set of  $K_{n-k}$  has cardinality  $\binom{n-k}{2}$ . In order for this set of edges to be a full partition, we need to connect all the vertices in  $K_k$  and all the edges in  $K_{n-k}$ , which takes  $k(n - k)$  edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph  $G$ , where  $|V(G)| = n$  with maximal clique components  $H_1, \dots, H_k$ . Each of these components has  $e(H_i) = \binom{|V(H_i)|}{2}$ , with total  $\sum_{i=1}^k \binom{|V(H_i)|}{2}$ . In comparison, if we consider  $e(K_G)$ , where  $K_G$  is defined as the complete graph on the vertices of  $G$ , then that value is  $\binom{n}{2}$ , and  $n = \sum_{i=1}^k |V(H_i)|$ . Therefore, the size of the edge set of  $G$  is less than or equal to the sum of the sizes of the edge sets of maximal clique components  $H_i$  (because the maximal clique components of  $G$  could just be  $G$  itself).

## 1.3.41

Prove or disprove: if  $G$  is an  $n$ -vertex simple graph with maximum degree  $\lceil n/2 \rceil$  and minimum degree  $\lfloor n/2 \rfloor - 1$ , then  $G$  is connected.

## Solution

Let  $u, v \in V(G)$  and let  $d(u) = \lceil \frac{n}{2} \rceil$ . Then,  $u$  is adjacent to  $\lceil \frac{n}{2} \rceil$  vertices and nonadjacent to  $\lfloor \frac{n}{2} \rfloor$  vertices. Let  $u \not\sim v$ .

We want to show that there exists some other vertex such that there exists a  $u, v$  path through that vertex. We know that  $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$  and  $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$ .

Since  $u \not\sim v$ ,  $N(u), N(v) \subseteq V(G) - \{u, v\}$ . So,  $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (\lceil \frac{n}{2} \rceil) + (\lfloor \frac{n}{2} \rfloor - 1) - (n - 2) = 1$ .

Therefore, there must be at least one element in  $N(u) \cap N(v)$ , meaning  $G$  is connected.