

## 1.1

## Individual

## 1.1.1

Determine which complete bipartite graphs are complete graphs.

$K_{1,1}$  is the only complete bipartite graph that is complete

## 1.1.3

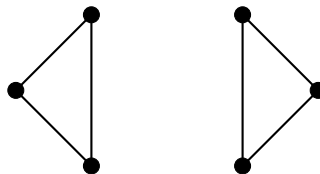
Using rectangular blocks whose entries are all equal, write down an adjacency matrix for  $K_{m,n}$ .

$$K_{m,n} = \begin{matrix} & \begin{matrix} a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_n \end{matrix} \\ \begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix} & \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{matrix}$$

## 1.1.5

Prove or disprove: If every vertex of a simple graph  $G$  has degree 2, then  $G$  is a cycle.

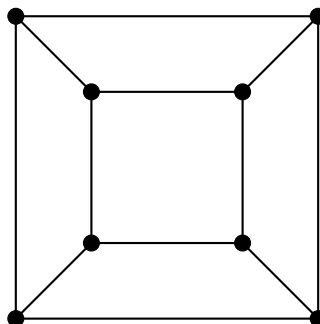
Let  $G$  be the following graph:

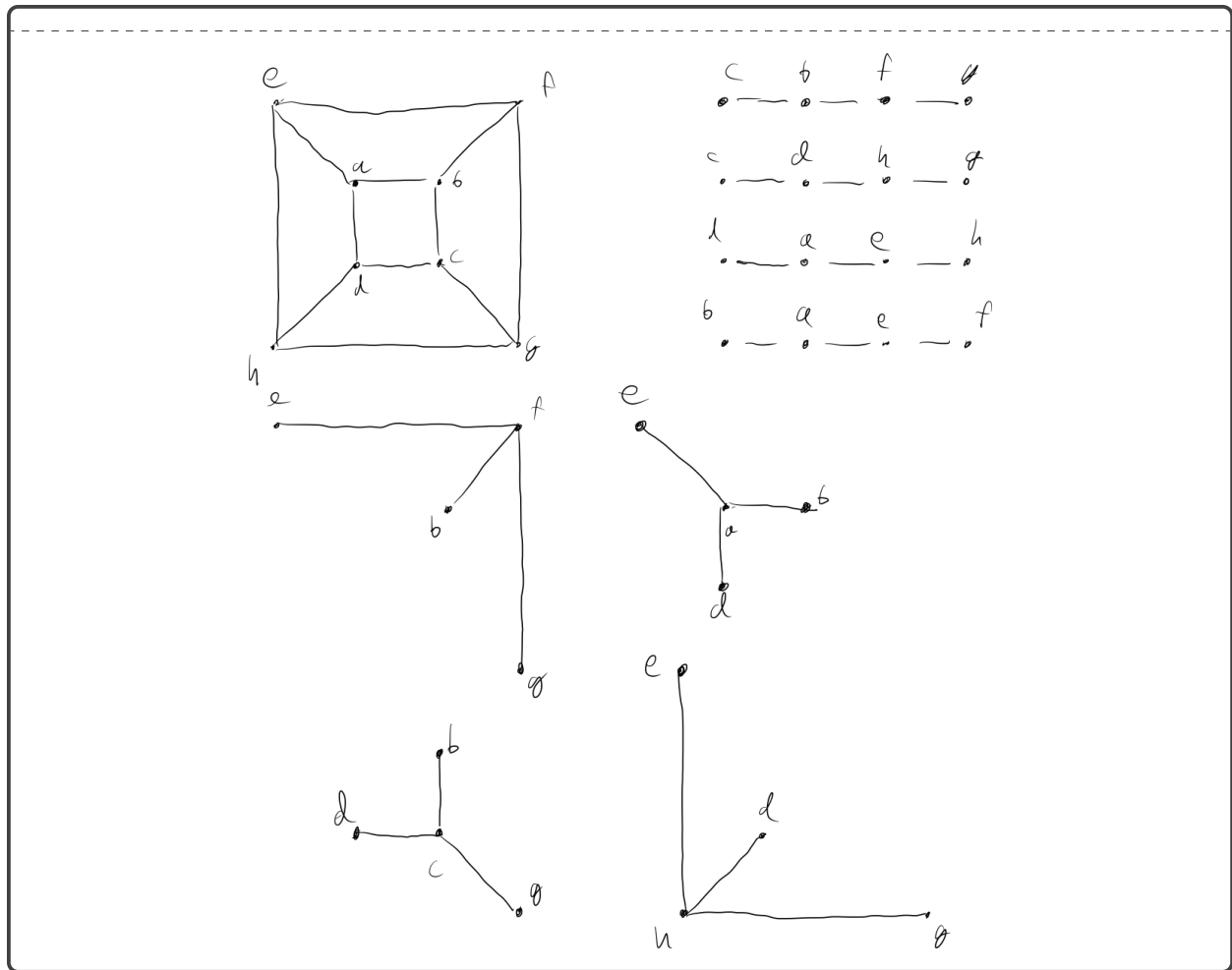


Every vertex in  $G$  has a degree 2, yet  $G$  is not a cycle.

## 1.1.8

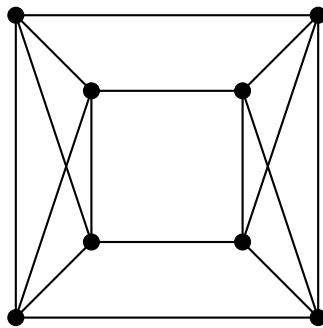
Prove that the 8 vertex graph below decomposes into copies of  $K_{1,3}$  and also into copies of  $P_4$

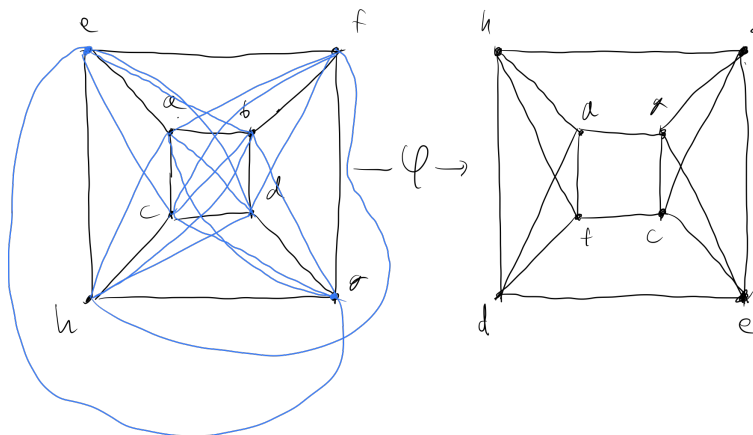




1.1.9

Prove that the graph below is isomorphic to the complement of the previous graph





1.1.10

Prove or disprove: the complement of a simple disconnected graph must be connected.

Let  $G$  be a graph that is disconnected. We want to show that  $\forall x, y \in V(G), \exists xz$  path. We can split into two cases.

- Suppose  $x \not\leftrightarrow y$  in  $G$ . Then, in  $\overline{G}$ ,  $x \leftrightarrow y$  by the definition of a graph complement.
- Suppose  $x \leftrightarrow y$  in  $G$ . Then, since  $G$  is disconnected, we know that there must be some  $z \in V(G)$  such that there is no  $xz$  path. Since there is no  $xz$  path, then there is no  $yz$  path. In particular, this means  $x \not\leftrightarrow z$  and  $y \not\leftrightarrow z$  in  $G$ . Therefore, in  $\overline{G}$ , we have that  $x \leftrightarrow z$  and  $y \leftrightarrow z$ , meaning there is a path between  $x$  and  $y$ .

## Group

1.1.13

Let  $G$  be the graph whose vertex set is the set of  $k$ -tuples with coordinates  $\{0, 1\}$ , with  $x$  adjacent to  $y$  if  $x$  and  $y$  differ by exactly one position. Determine whether  $G$  is bipartite.

$G$  is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

1.1.26

Let  $G$  be a graph with girth 4 in which every vertex has degree  $k$ . Prove that  $G$  has at least  $2k$  vertices. Determine all such graphs with  $2k$  vertices.

Suppose  $G$  is a graph with girth 4 with every vertex of degree  $k$ . Let  $v_i \in V(G)$ . Then, there must be  $k$  vertices which  $v_i$  is adjacent to. However, none of these vertices can be adjacent to themselves or  $G$  would have girth 3. Thus, we can form a bipartition such that  $v_i$  is in a set of at least  $k$  vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to  $k$  vertices in a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least  $2k$  vertices.

The graphs with exactly  $2k$  vertices are the  $K_{n,n}$  complete bipartite graphs.

1.1.27

Let  $G$  be a graph with girth 5. Prove that if every vertex of  $G$  has degree at least  $k$ , then  $G$  has at least  $k^2 + 1$  vertices. For  $k = 2$  and  $k = 3$ , find one such graph with  $k^2 + 1$  vertices.

Let  $G$  be a simple graph with girth 5. Suppose that every vertex of  $G$  has degree  $k$ . Let  $u \in V(G)$ . Then,  $u$  has  $k$  adjacent vertices, each of which is not adjacent to each other (or else the girth of  $G$  would be 3). Let this set be  $N$ . The elements of  $N$  cannot have any other common neighbors aside from  $u$ , or else the girth of  $G$  would be 4, meaning each has  $k - 1$  distinct neighbors. Therefore, the total number of vertices in our graph includes  $u$ , the elements of  $N$  that are the  $k$  distinct neighbors of  $u$ , and the  $k(k - 1)$  distinct vertices for each vertex in  $N$ . Therefore, our total is  $1 + k + k(k - 1) = k^2 + 1$ .

If there were any vertex with degree greater than  $k$ , then there would be additional vertices beyond the  $k^2 + 1$  vertices necessary for a  $k$ -regular graph.

For  $k = 2$ , we have the graph  $C_5$  for an example of a graph with  $k^2 + 1$  vertices, and for  $k = 3$  we have the Petersen graph.

1.1.30

Let  $G$  be a simple graph with adjacency matrix  $A$  and incidence matrix  $M$ . Prove that the degree of  $v_i$  is the  $i$ th diagonal entry of  $A^2$  and  $MM^T$ . What do the entries in position  $(i, j)$  of  $A^2$  and  $MM^T$  say about  $G$ ?

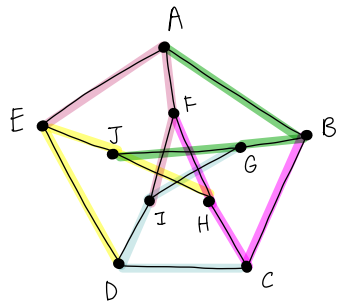
Let  $A$  be the adjacency matrix for a simple graph  $G$ . In  $A$ , every vertex's corresponding row and column are identical, meaning that the entry  $A_{i,i}^2$  will be equal to  $r_i c_i$  for row  $i$  and column  $i$  corresponding to  $v_i$ . Thus,  $r_i c_i$  is equal to  $|c_i|^2$ , which is equal to the sum of the elements of  $c_i$ , which is equal to the degree of  $v_i$ .

Let  $M$  be the incidence matrix for a simple graph  $G$ . In  $MM^T$ , the diagonal element  $MM_{i,i}^T$  will be equal to  $r_i r_i^T$ , where  $r_i$  represents the edge incidence row of  $v_i$ . This is equal to  $|r_i^T|^2$ , which is equal to the sum of the elements of  $r_i$ , which is equal to the number of edges incident on  $v_i$ , which is equal to the degree of  $v_i$ .

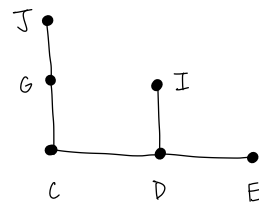
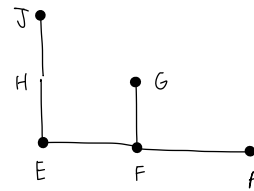
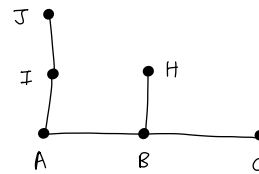
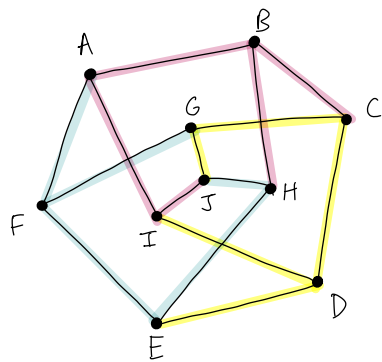
The entry in position  $(i, j)$  in both  $A^2$  and  $MM^T$  shows whether vertices  $v_i$  and  $v_j$  are adjacent to each other.

1.1.34

Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of  $P_4$ .



$A-B-G-J$   
 $B-C-H-F$   
 $C-D-I-G$   
 $D-E-J-H$   
 $E-A-F-I$



## 1.2

## Individual

## 1.2.1

Determine whether the following statements are true or false:

- Every disconnected graph has an isolated vertex.
  - A graph is connected if and only if some vertex is connected to all other vertices.
  - The edge set of every closed trail can be partitioned into edge sets of cycles.
  - If a maximal trail in a graph is not closed, then its endpoints have odd degree.
- 
- False; we can imagine a graph with two components, each of which consists of  $K_3$ , where there are no isolated vertices.
  - True; since  $\forall u, v \in G, \exists u, v$  path by the definition of a connected graph, this means any vertex must have a path to any other vertex.
  - True; every closed trail contains within it a cycle — we can delete the edge set of this cycle, and find cycles within remaining components until we reach isolated vertices.
  - True; if there were a maximal trail with an endpoint of even degree, then we would be able to extend the trail further by re-entering the endpoint vertex.

## 1.2.5

Let  $v$  be a vertex of a connected simple graph  $G$ . Prove that  $v$  has a neighbor in every component of  $G - v$ . Explain why this allows us to conclude that no graph has a cut-vertex of degree 1.

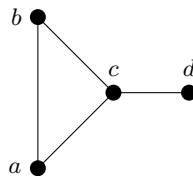
Suppose that  $G - v$  is connected. Then, since  $G$  is connected, and  $v \in V(G)$ , it must be the case that  $v$  is connected to every component of  $G - v$ , meaning that it has a neighbor in every component of  $G - v$  as  $G - v$  is connected.

Now suppose that  $G - v$  is disconnected, meaning that it has more than one component after removing  $v$ . Before,  $v$  must have been connected to every vertex in  $G$  as  $G$  was a simple connected graph, and afterwards  $G - v$  is no longer connected, meaning that  $v$  is a cut-vertex. This means  $v$  must have been adjacent to a vertex in each component of  $G - v$ , as removing the incident edges on  $v$  along with  $v$  increased the number of components from the original 1 that was in  $G$ .

From this result, we can conclude that no cut-vertex has degree 1 as removing a vertex of degree 1 and its incident edges does not increase the number of components in  $G$ , since there is only one edge incident on a vertex of degree 1.

## 1.2.6

In the graph below, find all the maximal paths, maximal cliques, and maximal independent sets. Also, find all the maximum paths, cliques, and independent sets.



- The maximal paths are as follows:
  - $d, c, b, a$
  - $d, c, a, b$
  - $a, b, c, d$
  - $b, a, c, d$
  - $b, c, a$
  - $c, b, a$
  - $a, c, b$
- The maximal cliques are  $K_3$  consisting of  $a, b, c$  and  $K_2$  consisting of  $c, d$ .
- The maximal independent sets are  $\{a, d\}$  and  $\{b, d\}$ .
- The maximum path is any of those paths listed above with length 4.
- The maximum clique is  $K_3$ .
- The maximum independent sets are those listed above with size 2.

## 1.2.8

Determine the values of  $m$  and  $n$  such that  $K_{m,n}$  is Eulerian.

$$m, n \in 2\mathbb{Z}^+$$

## 1.2.10

Prove or disprove:

- (a) Every Eulerian bipartite graph has an even number of edges.
- (b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

(a)

Let  $G$  be an Eulerian bipartite graph. Since  $G$  is Eulerian, it must contain an Eulerian cycle, meaning that as seen above, there are an even number of vertices, meaning that there are an even number of edges in  $G$ .

(b)

Let  $G$  be an Eulerian simple graph with an even number of vertices. Since  $G$  is Eulerian, this means there must be an Eulerian circuit  $C$  that traverses every edge exactly once in  $G$ . Every vertex in  $G$  must have even degree (or else we would require a backtrack in our Eulerian cycle, which is not a circuit); a simple pairing of the vertices would yield that we have  $\lfloor n/2 \rfloor$  edges, and to complete the cycle we need  $2(n/2) + 2k$  edges for  $n$  vertices and some integer  $k$ . Therefore, there must be an even number of edges.

## Group

1.2.20

Let  $v$  be a cut-vertex of a simple graph  $G$ . Prove that  $\overline{G} - v$  is connected.

Let  $x, y, v \in V(\overline{G})$ , where  $v$  is a cut-vertex of  $G$ .

Suppose  $x$  and  $y$  belong to distinct components of  $G - v$ . Then,  $xy \notin E(G)$ , meaning that  $xy \in E(\overline{G})$ , meaning there is an  $x, y$  path in  $\overline{G}$ , so there is an  $x, y$  path in  $\overline{G} - v$ .

Suppose  $x$  and  $y$  are in the same component of  $G - v$ . Since  $v$  is a cut-vertex, this means there must be at least two components in  $G - v$ . Let  $H_1$  be the component that  $x, y$  are in, while  $\exists w \in H_2$  is a vertex in  $H_2$  disjoint from  $H_1$ . Since  $H_1$  and  $H_2$  are disjoint, this means the components do not contain any edges between them, so  $x \not\leftrightarrow w$  and  $y \not\leftrightarrow w$  in  $G - v$  — however, this means that  $x \leftrightarrow w$  and  $y \leftrightarrow w$  in  $\overline{G}$ , meaning that  $\exists x, y$  path in  $\overline{G} - v$ .

1.2.22

Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Let  $G$  be a graph where there exists a partition of its vertices into two non-empty sets such that there is no edge with endpoints in both sets. Call these sets  $A$  and  $B$ . By our assumptions,  $\forall u \in A$  and  $\forall v \in B$ ,  $\nexists e$  such that  $e = uv$ . Therefore, we cannot create a path between any  $u \in A$  and any  $v \in B$  as there is no edge to connect any element in  $A$  and any element in  $B$ . Therefore,  $G$  is disconnected.

Suppose  $G$  is a disconnected graph. Then,  $G$  contains more than one component — we can create a partition of  $V(G)$  by letting  $H_1, H_2, \dots, H_k$  refer to the  $k$  components of  $G$ . Each of these components is necessarily disjoint from every other component. By taking  $H = H_1 \cup H_2 \cup \dots \cup H_{k-1}$  as one set and  $H_k$  as our other set, we know that  $H_1, \dots, H_k$  are all disjoint, meaning that  $H$  and  $H_k$  are disjoint, meaning that there is no edge connecting any vertex  $H$  with any vertex in  $H_k$ , meaning we have created a partition of  $G$  such that there exists no edge between any vertex in one set and any vertex in the other set.

1.2.26

Prove that a graph  $G$  is bipartite if and only if every subgraph  $H$  of  $G$  has an independent set consisting of at least half of  $V(H)$ .

Suppose  $G$  is bipartite. Then, there exists a partition of the vertices  $V = X \sqcup Y$  such that  $X$  and  $Y$  are independent sets. Let  $H$  be a subgraph of  $G$ , and let  $H_X = X \cap V(H)$  and  $H_Y = Y \cap H$ . Because  $H$  is a subgraph of  $G$ , each vertex of  $H$  must be an element of either  $H_X$  or  $H_Y$ , or that  $V(H) = H_X \sqcup H_Y$ . WLOG, let  $|H_X| \geq |H_Y|$ . Since  $H_X \subseteq X$  and  $X$  is an independent set,  $H_X$  is an independent subset consisting of at least half of  $V(H)$ .

Suppose every subgraph of  $G$  has an independent set consisting of at least half of  $V(H)$ . We will suppose toward contradiction that  $G$  is not bipartite. Then,  $G$  must contain an odd cycle,  $H_1$ . However, an independent set of  $H_1$  consists of at most  $\left\lfloor \frac{|V(H_1)|}{2} \right\rfloor < \frac{|V(H_1)|}{2}$ , otherwise two vertices would be adjacent. Because  $H_1$  is an independent set with less than half of the elements of  $V(H)$ , we have reached a contradiction. Therefore,  $G$  must be bipartite.

1.2.38

Prove that every  $n$ -vertex graph with at least  $n$  edges contains a cycle.



We proceed via induction as follows:

For the base case where  $|V(G)| = 1$ , we know that there is a cycle with one edge that connects back on the vertex.

For the case where  $|V(G)| > 1$ , if  $v \in V(G)$  has degree at most 1, then  $G - v$  has  $n - 1$  vertices and at least  $n - 1$  edges, so by our inductive hypothesis, we know that  $G - v$  contains a cycle. Meanwhile, if  $\forall v \in V(G), d(v) \geq 2$ , we know by Lemma 1.2.25 that  $G$  contains a cycle.

### 1.3

#### Individual

##### 1.3.3

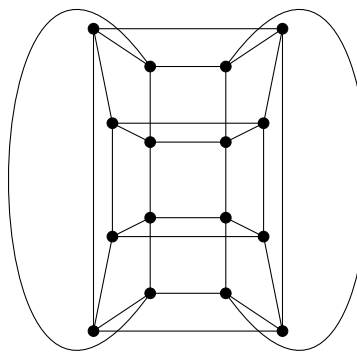
Let  $u$  and  $v$  be adjacent vertices in  $G$ . Prove that  $uv$  belongs to at least  $d(u) + d(v) - n(G)$  triangles in  $G$ .

Let  $u \leftrightarrow v \in G$ . In order for  $uv$  to be in a triangle in  $G$ ,  $u$  and  $v$  must share a common neighbor. By using the property of inclusion and exclusion, we can find the set as follows:

$$\begin{aligned} |N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &\geq d(u) + d(v) - n(G) \end{aligned}$$

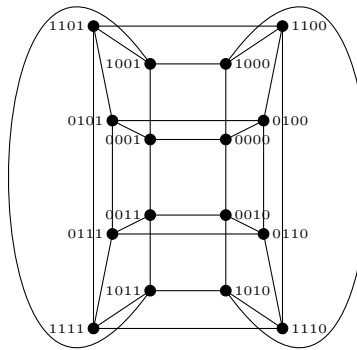
##### 1.3.4

Prove that the graph below is isomorphic to  $Q_4$ .



We can assign tuples to the graph as follows:

1101 ●	● 1100
1001 ●	● 1000
0101 ●	● 0100
0001 ●	● 0000
0011 ●	● 0010
0111 ●	● 0110
1011 ●	● 1010
1111 ●	● 1110



### 1.3.6

Given graphs  $G$  and  $H$ , determine the number of components and maximum degree in  $G + H$  in terms of the parameters for  $G$  and  $H$ .

We can find the number of components in  $G + H$  by summing the number of components in  $G$  and the number of components in  $H$ .

The maximum degree in  $G + H$  is equal to  $\max\{\Delta(G), \Delta(H)\}$ .

### 1.3.7

Determine the maximum number of edges in a bipartite subgraph of  $P_n$ ,  $C_n$ , and  $K_n$ .

For the graph  $P_n$ , we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets  $A$  and  $B$ . This is a bipartition since a path does not include any repeated vertices or edges, so  $A$  and  $B$  are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of  $P_n$  is the number of edges in  $P_n$ , which is  $n - 1$ .

For  $C_n$ , we have two values of the maximum number of edges in a bipartite subgraph of  $C_n$ :

- If  $n$  is even, then  $C_n$  is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of  $C_n$  is the number of edges in  $C_n$ , which is  $n$ .
- If  $n$  is odd, then  $C_n$  is not a bipartite graph. After one edge deletion, we get that  $C_n - e = P_n$ , which is bipartite, so the maximum number of edges in a bipartite subgraph of  $C_n$  is the number of edges in  $P_n$ , which is  $n - 1$ .

For the graph  $K_n$ , there are two options for the maximum number of edges in a bipartite subgraph depending on the value of  $n$ :

- If  $n$  is even, then the subgraph  $K_{\frac{n}{2}, \frac{n}{2}}$  is the maximal bipartite subgraph, meaning that the number of edges is equal to  $n^2/4$ . We know that  $K_{\frac{n}{2}, \frac{n}{2}}$  is a subgraph of  $K_n$  because the vertex set is the same, and  $K_n$  is complete, so any subset of edges is a subset of the edge set of  $K_n$ .
- If  $n$  is odd, then the subgraph  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$  is the maximal bipartite subgraph, because  $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$  and  $K_n$  is complete. Therefore, the total number of edges is  $\lfloor \frac{n^2}{4} \rfloor$ .

#### 1.3.26 (a)

Count the 6-cycles in  $Q_3$ .

In order to find a 6-cycle in  $Q_3$ , we select two vertices to delete and see if we can find a cycle from the graph  $Q_3 - \{u, v\}$ . Vertices are either adjacent, distance 2 (antipodal on the same face), or antipodal (distance 3) with each other.

ADJACENT: If two vertices are adjacent, we can find one cycle from the remaining vertices after deletion. There are  $(8)(3)/2$  sets of adjacent vertices, for a total of 12 from this selection.

ANTIPODAL ON THE SAME FACE: If two vertices are antipodal, we cannot find a cycle from the remaining graph after deletion.

ANTIPODAL: If two vertices are antipodal, we can find one cycle from the remaining vertices after deletion. There are 4 sets of antipodal vertices.

We find a total of 16 6-cycles in  $Q_3$ .

### Group

#### 1.3.17

Let  $G$  be a graph with at least two vertices. Prove or disprove:

- Deleting a vertex of degree  $\Delta(G)$  cannot increase the average degree.
- Deleting a vertex of degree  $\delta(G)$  cannot decrease the average degree.

(a)

Assume toward contradiction that deleting a vertex of degree  $\Delta(G)$  increases the average degree.

$$\begin{aligned}
 d'_{\text{avg}} &> d_{\text{avg}} \\
 \frac{2e(G) - 2\Delta(G)}{n(G) - 1} &> \frac{2e(G)}{n(G)} \\
 \frac{2e(G) - 2\Delta(G)}{2e(G)} &> \frac{n(G) - 1}{n(G)} \\
 1 - \frac{\Delta(G)}{e(G)} &> 1 - \frac{1}{n(G)} \\
 \frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} &> 0 \\
 \frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{\text{avg}}} &> 0 \\
 \frac{d_{\text{avg}} - 2\Delta(G)}{n(G)} &> 0 \\
 d_{\text{avg}} - 2\Delta(G) &> 0 \\
 d_{\text{avg}} &> 2\Delta(G)
 \end{aligned}$$

However, we have reached a contradiction — by definition,  $\Delta(G) \geq d_{\text{avg}}$ , meaning that  $d_{\text{avg}} \not> \Delta(G)$ , let alone  $2\Delta(G)$ .

(b)

Deleting a vertex of the graph  $K_{1,1}$  yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in  $K_{1,1}$ .

1.3.20

Count the cycles  $n$ -cycles in  $K_n$  and the  $2n$ -cycles in  $K_{n,n}$ .

To count the cycles in  $K_n$ , we start at a vertex  $v$  and choose an edge out of  $n - 1$  options. After choosing the edge, there are  $n - 2$  options remaining that do not backtrack to  $v$ , and so on and so forth. Therefore, there are  $(n - 1)!$  cycles in  $K_n$ .

Let  $v \in A$ , where  $A$  and  $B$  are the order  $n$  sets that partition  $G$ . Then, there are  $n$  possible vertices in  $B$  which can be the second element in our cycle — afterwards, there are  $n - 1$  elements in  $A$ , and after that there are  $n - 1$  elements in  $B$ , and so on and so forth. Therefore, there are  $n!(n - 1)!$  possible options for cycles in  $K_{n,n}$ .

1.3.25

Prove that every cycle of length  $2r$  in a hypercube is contained within a subcube of dimension at most  $r$ . Can a cycle of length  $2r$  be contained in a subcube of dimension less than  $r$ .

Let  $C$  be a cycle of length  $2r$  in  $Q_n$ . Then,  $C$  contains  $2r$   $n$ -tuples. For every tuple in  $C$ , there exists a “switched” tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since  $C$  is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are  $2r$  switches (corresponding to the  $2r$  edges in  $C$ ), this means there are at most  $r$  coordinates that are switched, then switched back sometime along the cycle’s path. This means the other

$n - r$  coordinates are fixed, implying that  $C \subseteq Q_r$ , the  $r$ -dimensional subcube of  $Q_k$ .

There is a cycle of length 8 in  $Q_3$ , defined as follows:  $000 \rightarrow 001 \rightarrow 011 \rightarrow 010 \rightarrow 110 \rightarrow 111 \rightarrow 101 \rightarrow 100 \rightarrow 000$ .

## 1.3.31

Using complete graphs and counting arguments, prove the following:

- (a)  $\binom{n}{2} = \binom{n}{k} + k(n - k) + \binom{n-k}{2}$  for  $0 \leq k \leq n$ .  
 (b) If  $\sum n_i = n$ , then  $\sum \binom{n_i}{2} \leq \binom{n}{2}$ .

(a)

We can consider a decomposition of the edges of  $K_n$  into the edge set of  $K_k$  and  $K_{n-k}$ , and some connector edges.

The edge set of  $K_n$  has cardinality  $\binom{n}{2}$ , the edge set of  $K_k$  has cardinality  $\binom{k}{2}$ , and the edge set of  $K_{n-k}$  has cardinality  $\binom{n-k}{2}$ . In order for this set of edges to be a full decomposition, we need a graph that connects all the vertices in  $K_k$  with all the vertices in  $K_{n-k}$ , which takes  $k(n - k)$  edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph  $G$ , where  $|V(G)| = n$  with maximal clique components  $H_1, \dots, H_k$ . Each of these components has  $e(H_i) = \binom{V(H_i)}{2}$ , with total  $\sum_{i=1}^k \binom{V(H_i)}{2}$ . In comparison, if we consider  $e(K_G)$ , where  $K_G$  is defined as the complete graph on the vertices of  $G$ , then that value is  $\binom{n}{2}$ , and  $n = \sum_{i=1}^k |V(H_i)|$ . Therefore, the size of the edge set of  $G$  is less than or equal to the sum of the sizes of the edge sets of maximal clique components  $H_i$  (because the maximal clique components of  $G$  could just be  $G$  itself).

## 1.3.41

Prove or disprove: if  $G$  is an  $n$ -vertex simple graph with maximum degree  $\lceil n/2 \rceil$  and minimum degree  $\lfloor n/2 \rfloor - 1$ , then  $G$  is connected.

Let  $u, v \in V(G)$  and let  $d(u) = \lceil \frac{n}{2} \rceil$ . Then,  $u$  is adjacent to  $\lceil \frac{n}{2} \rceil$  vertices and nonadjacent to  $\lfloor \frac{n}{2} \rfloor$  vertices. Let  $u \not\sim v$ .

We want to show that there exists some other vertex such that there exists a  $u, v$  path through that vertex. We know that  $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$  and  $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$ .

Since  $u \not\sim v$ ,  $N(u), N(v) \subseteq V(G) - \{u, v\}$ . So,  $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (\lceil \frac{n}{2} \rceil) + (\lfloor \frac{n}{2} \rfloor - 1) - (n - 2) = 1$ .

Therefore, there must be at least one element in  $N(u) \cap N(v)$ , meaning  $G$  is connected.

## 2.1

## Individual

## 2.1.2

Let  $G$  be a graph:

- (a) Prove that  $G$  is a tree if and only if  $G$  is connected and every edge is a cut-edge.
- (b) Prove that  $G$  is a tree if and only if adding any edge with endpoints in  $V(G)$  creates exactly one cycle.

(a)

( $\Rightarrow$ ) Let  $G$  be a tree. Thus,  $G$  is connected (by definition), and acyclic. Since  $G$  is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

( $\Leftarrow$ ) Let  $G$  be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in  $G$ , so  $G$  is a connected acyclic graph, or a tree.

(b)

( $\Rightarrow$ ) Let  $G$  be a tree, and let  $e$  be an edge such that  $e \notin E(G)$ , and  $e = uv$ . Then, we create a cycle from the path  $uTv + e$  — since there is only one path  $uTv$ , this means that  $uTv + e$  is a unique cycle.

( $\Leftarrow$ ) Suppose toward contradiction that adding  $e$  to the tree  $G$  yielded more than one cycle in the graph  $G + e$ . Then, the graph  $G = G + e - e$  would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that  $G$  was a tree, we have reached a contradiction, meaning that  $e$  added exactly one cycle to the tree  $G$ .

## 2.1.6

Let  $T$  be a tree with average degree  $a$ . In terms of  $a$ , find  $n(T)$ .

$$\begin{aligned}
 d_{\text{avg}} &= \frac{2e(T)}{n(T)} \\
 a &= \frac{2(n(T) - 1)}{n(T)} \\
 an &= 2n - 2 \\
 (a - 2)n &= -2 \\
 n &= \boxed{\frac{2}{2 - a}}
 \end{aligned}$$

## 2.1.7

Prove that every  $n$ -vertex graph with  $m$  edges has at least  $m - n + 1$  cycles.

**BASE CASE** If  $m = 0$ , then since this graph has zero edges, it has zero cycles, and since  $0 \geq 1 - n$ , we have proven the base case.

**INDUCTIVE HYPOTHESIS** For an  $n$ -vertex graph with  $0 \leq k \leq m$  vertices, then  $G$  has at least  $k - n + 1$  cycles.

**PROOF** If  $e$  is an edge within a cycle of  $G$ , then  $G - e$  has  $k - 1$  edges, and has seen a reduction of 1 cycle, so

$G - e$  has at least  $(k - 1) - n + 1 = (k - n + 1) - 1$  cycles. If  $e$  is not within a cycle, then  $G$  has seen no reduction in cycles, but  $G - e$  is predicted to have at least  $(k - n + 1) - 1$  cycles, which it does by our assumption. Therefore, we have proven the inductive hypothesis for both cases.

## 2.1.12

Compute the diameter and radius of  $K_{m,n}$ .

The diameter of  $K_{m,n}$  is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of  $K_{m,n}$  is also 2 — the eccentricity of every vertex in  $K_{m,n}$  is 2, so the radius must also be 2.

## 2.1.13

Prove that every graph with diameter  $d$  has an independent set with at least  $\lceil \frac{1+d}{2} \rceil$  vertices.

Let  $G$  be a graph with diameter  $d$ , and let  $u \in V(G)$  be a vertex with eccentricity  $d$ . Let  $P$  be a maximal  $u, v$  path of length  $d$ . Then,  $P$  has  $d + 1$  vertices. So,  $P$  has a maximal independent set containing every other vertex, with total cardinality of  $\lceil \frac{d+1}{2} \rceil$ . Therefore,  $G$  has an independent set with at least  $\lceil \frac{d+1}{2} \rceil$  vertices.