2.1.2

Let G be a graph:

(a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.

(b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

(a)

($\Rightarrow$) Let G be a tree. Thus, G is connected (by definition), and acyclic. Since G is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

($\Leftarrow$) Let G be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in G, so G is a connected acyclic graph, or a tree.

(b)

($\Rightarrow$) Let G be a tree, and let e be an edge such that $e \notin E(G)$, and $e = uv$. Then, we create a cycle from the path $uTv + e$ — since there is only one path uTv, this means that $uTv + e$ is a unique cycle.

($\Leftarrow$) Suppose toward contradiction that adding e to the tree G yielded more than one cycle in the graph $G + e$. Then, the graph $G = G + e - e$ would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that G was a tree, we have reached a contradiction, meaning that e added exactly one cycle to the tree G.

2.1.6

Let T be a tree with average degree a. In terms of a, find $n(T)$.

$$
\begin{aligned}
d_{\text{avg}} &= \frac{2e(T)}{n(T)} \\
a &= \frac{2(n(T)-1)}{n(T)} \\
an &= 2n - 2 \\
(a-2)n &= -2 \\
n &= \boxed{\frac{2}{2-a}}
\end{aligned}
$$

2.1.7

Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

Base Case If $m = 0$, then since this graph has zero edges, it has zero cycles, and since $0 \geq 1 - n$, we have proven the base case.

Inductive Hypothesis For an n-vertex graph with $0 \leq k \leq m$ vertices, then G has at least $k-n+1$ cycles.

Proof If e is an edge within a cycle of G, then $G-e$ has $k-1$ edges, and has seen a reduction of 1 cycle, so $G-e$ has at least $(k-1)-n+1=(k-n+1)-1$ cycles. If e is not within a cycle, then G has seen no reduction in cycles, but $G-e$ is predicted to have at least $(k-n+1)-1$ cycles, which it does by our assumption. Therefore, we have proven the inductive hypothesis for both cases.

2.1.12

Compute the diameter and radius of $K_{m,n}$.

The diameter of $K_{m,n}$ is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of $K_{m,n}$ is also 2 — the eccentricity of every vertex in $K_{m,n}$ is 2, so the radius must also be 2.

2.1.13

Prove that every graph with diameter d has an independent set with at least $\left\lceil \frac{1+d}{2} \right\rceil$ vertices.

Let G be a graph with diameter d, and let $u \in V(G)$ be a vertex with eccentricity d. Let P be a maximal u,v path of length d. Then, P has $d+1$ vertices. So, P has a maximal independent set containing every other vertex, with total cardinality of $\left\lceil \frac{d+1}{2} \right\rceil$. Therefore, G has an independent set with at least $\left\lceil \frac{d+1}{2} \right\rceil$ vertices.