1.3.3

Let u and v be adjacent vertices in G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

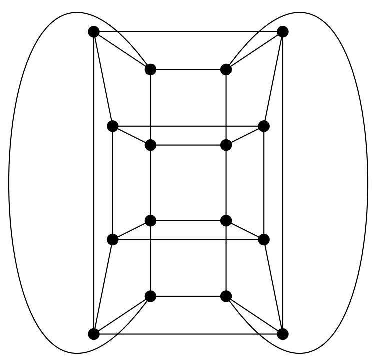
Solution

Let $u \leftrightarrow v \in G$. In order for uv to be in a triangle in G, u and v must share a common neighbor. By using the property of inclusion and exclusion, we can find the set as follows:

$$\begin{aligned}
|N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\
|N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\
&= d(u) + d(v) - n(G)
\end{aligned}$$

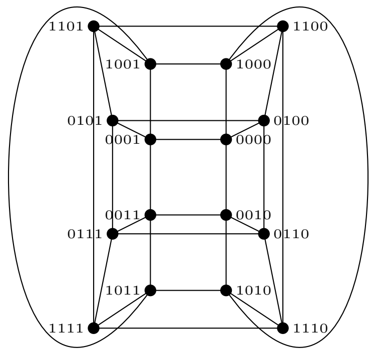
1.3.4

Prove that the graph below is isomorphic to Q_4.



Solution

We can assign tuples to the graph as follows:



1.3.6

Given graphs G and H, determine the number of components and maximum degree in $G+H$ in terms of the parameters for G and H.

Solution

We can find the number of components in $G+H$ by summing the number of components in G and the number of components in H.

The maximum degree in $G+H$ is equal to $\max\{\Delta(G), \Delta(H)\}$.

1.3.7

Determine the maximum number of edges in a bipartite subgraph of P_n, C_n, and K_n.

Solution

For the graph P_n, we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets A and B. This is a bipartition since a path does not include any repeated vertices or edges, so A and B are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of P_n is the number of edges in P_n, which is $n-1$.

For C_n, we have two values of the maximum number of edges in a bipartite subgraph of C_n:

- If n is even, then C_n is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of C_n is the number of edges in C_n, which is n.
- If n is odd, then C_n is not a bipartite graph. After one edge deletion, we get that $C_n - e = P_n$,

which is bipartite, so the maximum number of edges in a bipartite subgraph of C_n is the number of edges in P_n, which is $n-1$.

For the graph K_n, there are two options for the maximum number of edges in a bipartite subgraph depending on the value of n:

- If n is even, then the subgraph $K_{\frac{n}{2},\frac{n}{2}}$ is the maximal bipartite subgraph, meaning that the number of edges is equal to $n^2/4$. We know that $K_{\frac{n}{2},\frac{n}{2}}$ is a subgraph of K_n because the vertex set is the same, and K_n is complete, so any subset of edges is a subset of the edge set of K_n.
- If n is odd, then the subgraph $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ is the maximal bipartite subgraph, because $\lfloor\frac{n}{2}\rfloor+\lceil\frac{n}{2}\rceil = n$ and K_n is complete. Therefore, the total number of edges is $\lfloor\frac{n^2}{4}\rfloor$.

1.3.26 (a)

Count the 6-cycles in Q_3.

Solution

In order to find a 6-cycle in Q_3, we select two vertices to delete and see if we can find a cycle from the graph $Q_3-\{u,v\}$. Vertices are either adjacent, distance 2 (antipodal on the same face), or antipodal (distance 3) with each other.

Adjacent If two vertices are adjacent, we can find one cycle from the remaining vertices after deletion. There are $(12)(3)/2$ sets of adjacent vertices, for a total of 18 from this selection.

Antipodal on the same face If two vertices are antipodal, we cannot find a cycle from the remaining graph after deletion.

Antipodal If two vertices are antipodal, we can find one cycle from the remaining vertices after deletion. There are 4 sets of antipodal vertices.

We find a total of 22 6-cycles in Q_3.