

2.1.22

Let T be an n -vertex tree with one vertex of each degree $2 \leq i \leq k$; the remaining $n - k + 1$ vertices are leaves. Determine n in terms of k .

We will find the number of vertices in T by finding the number of edges in T and adding 1. For $2 \leq i \leq k$ corresponding to each of the non-leaf vertices, summation yields $\frac{k(k+1)}{2} - 1$ edges. However, this scheme double-counts each edge, so we have to subtract the $k - 2$ edges connecting the $k - 1$ non-leaf vertices, yielding $\frac{k(k+1)}{2} - k + 1$ edges. Finally, because T is a tree, we get that T has $\frac{k(k+1)}{2} - k + 2$ vertices.

2.1.27

Let d_1, \dots, d_n be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees d_1, \dots, d_n if and only if $\sum d_i = 2n - 2$.

- (\Rightarrow) Suppose that for some tree T , d_1, \dots, d_n are the degrees of the vertices of the tree. Since T is a tree, this means $e(G) = n - 1$, and $\sum d_i = 2e(G)$, meaning $\sum d_i = 2(n - 1) = 2n - 2$.
- (\Leftarrow) Suppose that $\sum d_i = 2n - 2$ for d_1, \dots, d_n corresponding to the degrees of the vertices in G . By a previous result, we know that $\sum d_i = 2e(G)$, meaning that $\sum d_i = 2(n - 1)$, implying that $e(G) = n - 1$. We can find a tree G with $n - 1$ edges by letting G be connected with $n - 1$ edges.

2.1.33

Let G be a connected n -vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

- (\Rightarrow) Let G be a connected n -vertex graph with exactly one cycle. If we delete an edge from this cycle, then $G - e$ is acyclic, as well as connected (since e is not a cut-edge), so $G - e$ has $n - 1$ edges. Adding back e , we get that G has n edges.
- (\Leftarrow) Let G be a connected n -vertex graph with n edges. Then, G contains a spanning tree that contains all n vertices. Therefore, $T \subseteq G$ contains $n - 1$ edges. By adding another edge, we get that $e(G) = e(T) + 1 = n - 1 + 1$. Thus, G has exactly one cycle.

2.1.34

Let T be a tree with k edges, and let G be a n -vertex simple graph with more than $n(k - 1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n \geq k$.