1.1

Individual

1.1.1

Determine which complete bipartite graphs are complete graphs.

Solution

$K_{1,1}$ is the only complete bipartite graph that is complete

1.1.3

Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

Solution

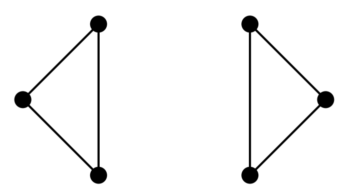
$$K_{m,n} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \begin{array}{c} \begin{array}{cccccccc} a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_n \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

1.1.5

Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle.

Solution

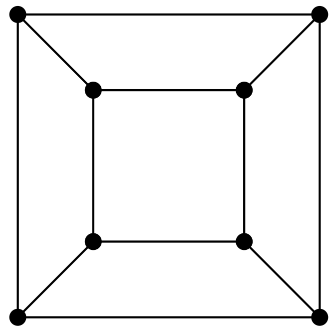
Let G be the following graph:



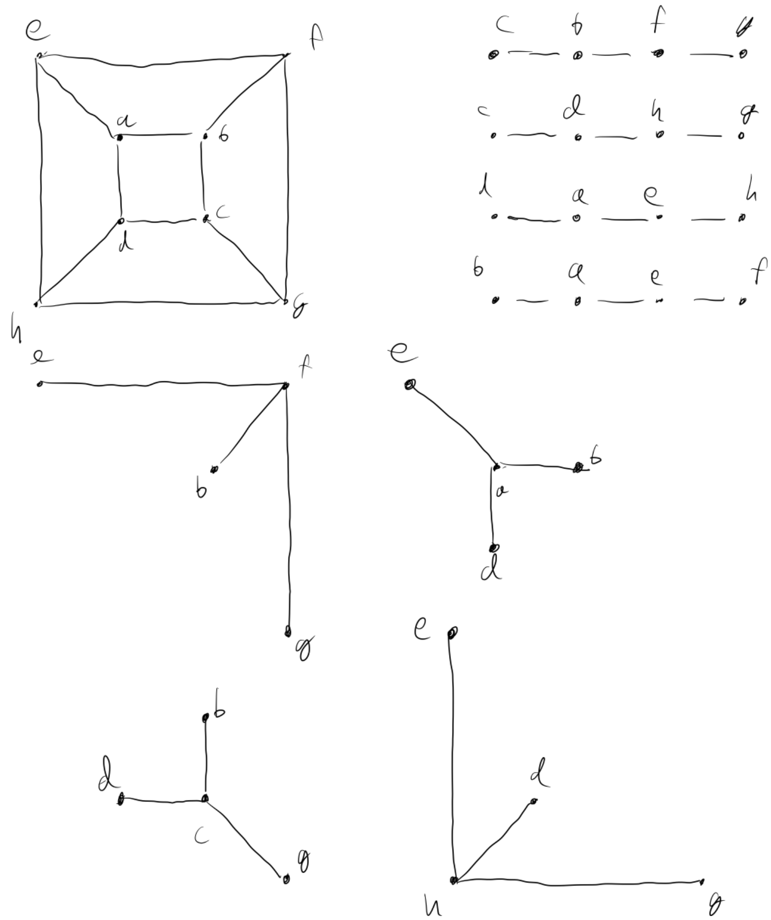
Every vertex in G has a degree 2, yet G is not a cycle.

1.1.8

Prove that the 8 vertex graph below decomposes into copies of $K_{1,3}$ and also into copies of P_4

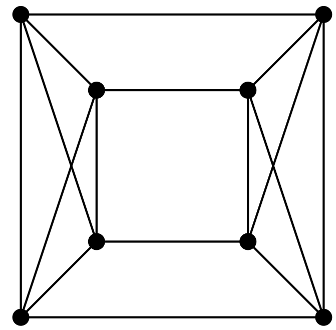


Solution

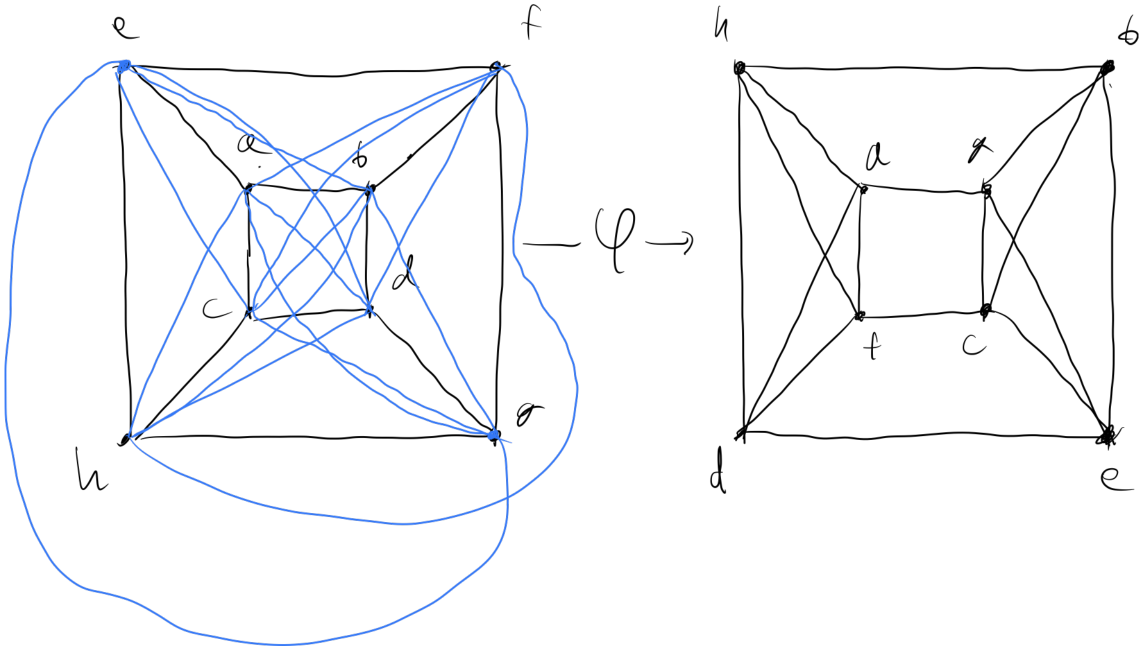


1.1.9

Prove that the graph below is isomorphic to the complement of the previous graph



Solution



1.1.10

Prove or disprove: the complement of a simple disconnected graph must be connected.

Solution

Let G be a graph that is disconnected. We want to show that $\forall x, y \in V(G), \exists xz$ path. We can split into two cases.

- Suppose $x \not\leftrightarrow y$ in G. Then, in $\overline{G}$, $x \leftrightarrow y$ by the definition of a graph complement.
- Suppose $x \leftrightarrow y$ in G. Then, since G is disconnected, we know that there must be some $z \in V(G)$ such that there is no xz path. Since there is no xz path, then there is no yz path. In particular, this means $x \not\leftrightarrow z$ and $y \not\leftrightarrow z$ in G. Therefore, in $\overline{G}$, we have that $x \leftrightarrow z$ and $y \leftrightarrow z$, meaning there is a path between x and y.

Group

1.1.13

Let G be the graph whose vertex set is the set of k-tuples with coordinates $\{0, 1\}$, with x adjacent to y if x and y differ by exactly one position. Determine whether G is bipartite.

Solution

G is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

1.1.26

Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least $2k$ vertices. Determine all such graphs with $2k$ vertices.

Solution

Suppose G is a graph with girth 4 with every vertex of degree k. Let $v_i \in V(G)$. Then, there must be k vertices which v_i is adjacent to. However, none of these vertices can be adjacent to themselves or G would have girth 3. Thus, we can form a bipartition such that v_i is in a set of at least k vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to k vertices in a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least $2k$ vertices.

The graphs with exactly $2k$ vertices are the $K_{n,n}$ complete bipartite graphs.

1.1.27

Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with $k^2 + 1$ vertices.

Solution

Let G be a simple graph with girth 5. Suppose that every vertex of G has degree k. Let $u \in V(G)$. Then, u has k adjacent vertices, each of which is not adjacent to each other (or else the girth of G would be 3). Let this set be N. The elements of N cannot have any other common neighbors aside from u, or else the girth of G would be 4, meaning each has $k - 1$ distinct neighbors. Therefore, the total number of vertices in our graph includes u, the elements of N that are the k distinct neighbors of u, and the $k(k-1)$ distinct vertices for each vertex in N. Therefore, our total is $1 + k + k(k-1) = k^2 + 1$.

If there were any vertex with degree greater than k, then there would be additional vertices beyond the $k^2 + 1$ vertices necessary for a k-regular graph.

For $k = 2$, we have the graph C_5 for an example of a graph with $k^2 + 1$ vertices, and for $k = 3$ we have the Petersen graph.

1.1.30

Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry of A^2 and MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

Solution

Let A be the adjacency matrix for a simple graph G. In A, every vertex's corresponding row and column are identical, meaning that the entry $A^2_{i,i}$ will be equal to $r_i c_i$ for row i and column i corresponding to v_i. Thus, $r_i c_i$ is equal to $|c_i|^2$, which is equal to the sum of the elements of c_i, which is equal to the degree of v_i.

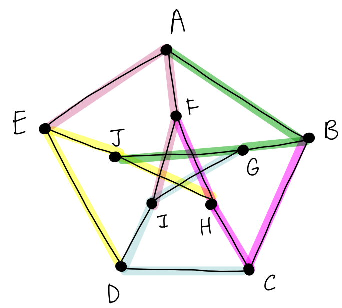
Let M be the incidence matrix for a simple graph G. In MM^T, the diagonal element $MM^T_{i,i}$ will be equal to $r_i r_i^T$, where r_i represents the edge incidence row of v_i. This is equal to $\left|r_i^T\right|^2$, which is equal to the sum of the elements of r_i, which is equal to the number of edges incident on v_i, which is equal to the degree of v_i.

The entry in position (i, j) in both A^2 and MM^T shows whether vertices v_i and v_j are adjacent to each other.

1.1.34

Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4.

Solution



A – B – G – J
B – C – H – F
C – D – I – G
D – E – J – H
E – A – F – I