# 1.1

## Individual

**1.1.1**

Determine which complete bipartite graphs are complete graphs.

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$K_{1,1}$ is the only complete bipartite graph that is complete

**1.1.3**

Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

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$$K_{m,n} = \begin{array}{cc} & \begin{array}{cccccccc} a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_n \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{array} & \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

**1.1.5**

Prove or disprove: If every vertex of a simple graph $G$ has degree 2, then $G$ is a cycle.

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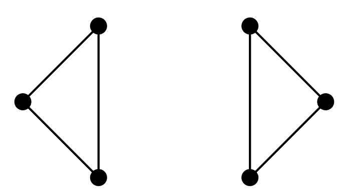
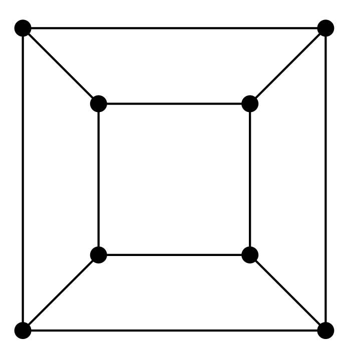
Let $G$ be the following graph:

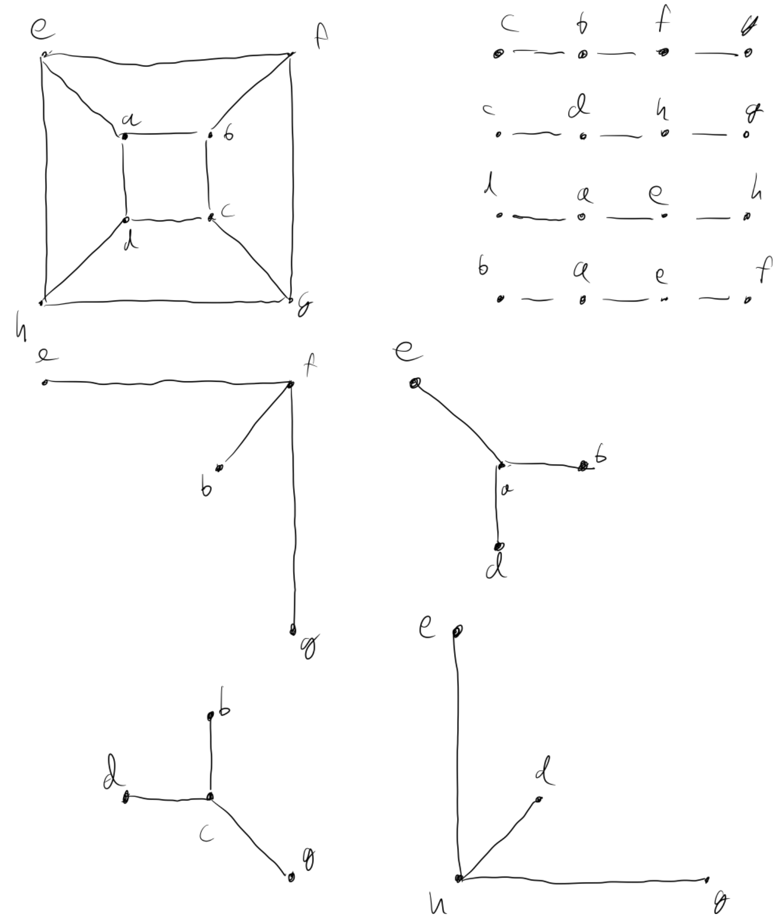


Every vertex in $G$ has a degree 2, yet $G$ is not a cycle.

**1.1.8**

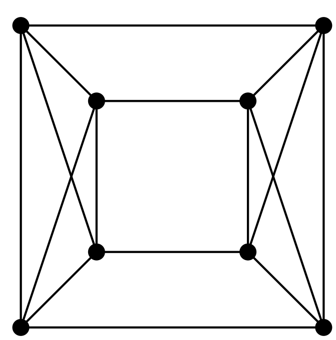
Prove that the 8 vertex graph below decomposes into copies of $K_{1,3}$ and also into copies of $P_4$

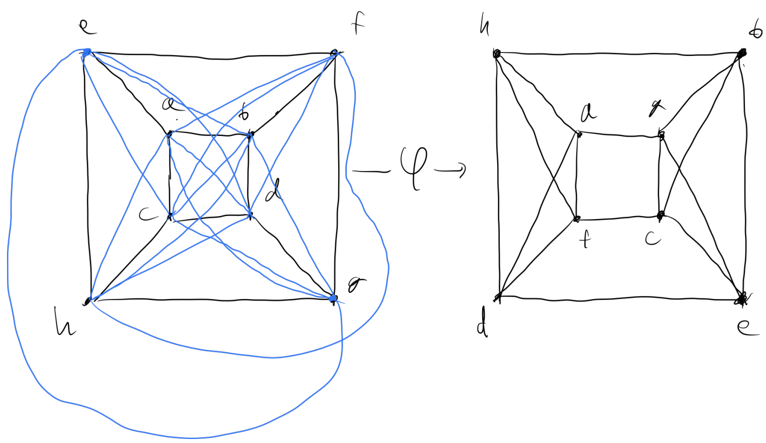




## 1.1.9

Prove that the graph below is isomorphic to the complement of the previous graph






**1.1.10**

Prove or disprove: the complement of a simple disconnected graph must be connected.

Let $G$ be a graph that is disconnected. We want to show that $\forall x, y \in V(G), \exists xz$ path. We can split into two cases.

- Suppose $x \not\leftrightarrow y$ in $G$. Then, in $\overline{G}$, $x \leftrightarrow y$ by the definition of a graph complement.
- Suppose $x \leftrightarrow y$ in $G$. Then, since $G$ is disconnected, we know that there must be some $z \in V(G)$ such that there is no $xz$ path. Since there is no $xz$ path, then there is no $yz$ path. In particular, this means $x \not\leftrightarrow z$ and $y \not\leftrightarrow z$ in $G$. Therefore, in $\overline{G}$, we have that $x \leftrightarrow z$ and $y \leftrightarrow z$, meaning there is a path between $x$ and $y$.

## Group

**1.1.13**

Let $G$ be the graph whose vertex set is the set of $k$-tuples with coordinates $\{0, 1\}$, with $x$ adjacent to $y$ if $x$ and $y$ differ by exactly one position. Determine whether $G$ is bipartite.

$G$ is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

**1.1.26**

Let $G$ be a graph with girth 4 in which every vertex has degree $k$. Prove that $G$ has at least $2k$ vertices. Determine all such graphs with $2k$ vertices.

Suppose $G$ is a graph with girth 4 with every vertex of degree $k$. Let $v_i \in V(G)$. Then, there must be $k$ vertices which $v_i$ is adjacent to. However, none of these vertices can be adjacent to themselves or $G$ would have girth 3. Thus, we can form a bipartition such that $v_i$ is in a set of at least $k$ vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to $k$ vertices in a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least $2k$ vertices.

The graphs with exactly $2k$ vertices are the $K_{n,n}$ complete bipartite graphs.

### 1.1.27

Let $G$ be a graph with girth 5. Prove that if every vertex of $G$ has degree at least $k$, then $G$ has at least $k^2+1$ vertices. For $k=2$ and $k=3$, find one such graph with $k^2+1$ vertices.

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Let $G$ be a simple graph with girth 5. Suppose that every vertex of $G$ has degree $k$. Let $u \in V(G)$. Then, $u$ has $k$ adjacent vertices, each of which is not adjacent to each other (or else the girth of $G$ would be 3). Let this set be $N$. The elements of $N$ cannot have any other common neighbors aside from $u$, or else the girth of $G$ would be 4, meaning each has $k-1$ distinct neighbors. Therefore, the total number of vertices in our graph includes $u$, the elements of $N$ that are the $k$ distinct neighbors of $u$, and the $k(k-1)$ distinct vertices for each vertex in $N$. Therefore, our total is $1+k+k(k-1)=k^2+1$.

If there were any vertex with degree greater than $k$, then there would be additional vertices beyond the $k^2+1$ vertices necessary for a $k$-regular graph.

For $k=2$, we have the graph $C_5$ for an example of a graph with $k^2+1$ vertices, and for $k=3$ we have the Petersen graph.

### 1.1.30

Let $G$ be a simple graph with adjacency matrix $A$ and incidence matrix $M$. Prove that the degree of $v_i$ is the $i$th diagonal entry of $A^2$ and $MM^T$. What do the entries in position $(i,j)$ of $A^2$ and $MM^T$ say about $G$?

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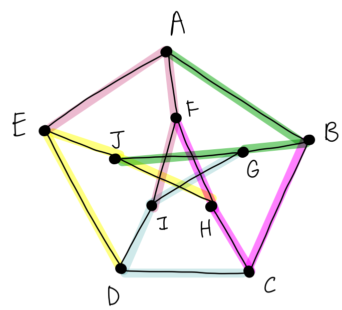
Let $A$ be the adjacency matrix for a simple graph $G$. In $A$, every vertex's corresponding row and column are identical, meaning that the entry $A^2_{i,i}$ will be equal to $r_ic_i$ for row $i$ and column $i$ corresponding to $v_i$. Thus, $r_ic_i$ is equal to $|c_i|^2$, which is equal to the sum of the elements of $c_i$, which is equal to the degree of $v_i$.

Let $M$ be the incidence matrix for a simple graph $G$. In $MM^T$, the diagonal element $MM^T_{i,i}$ will be equal to $r_ir_i^T$, where $r_i$ represents the edge incidence row of $v_i$. This is equal to $\left|r_i^T\right|^2$, which is equal to the sum of the elements of $r_i$, which is equal to the number of edges incident on $v_i$, which is equal to the degree of $v_i$.

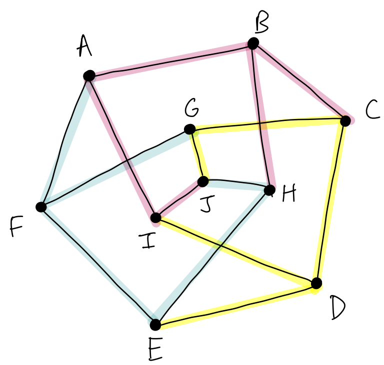
The entry in position $(i,j)$ in both $A^2$ and $MM^T$ shows whether vertices $v_i$ and $v_j$ are adjacent to each other.

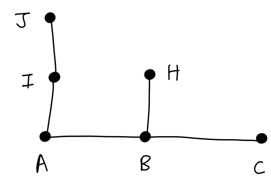
### 1.1.34

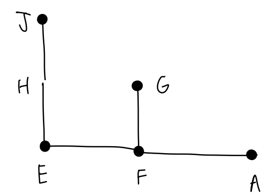
Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of $P_4$.

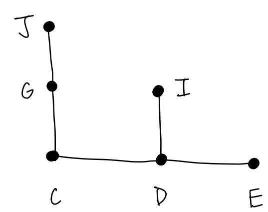



A – B – G – J
B – C – H – F
C – D – I – G
D – E – J – H
E – A – F – I



## 1.2

### Individual

**1.2.1**

Determine whether the following statements are true or false:

- Every disconnected graph has an isolated vertex.
- A graph is connected if and only if some vertex is connected to all other vertices.
- The edge set of every closed trail can be partitioned into edge sets of cycles.
- If a maximal trail in a graph is not closed, then its endpoints have odd degree.

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- False; we can imagine a graph with two components, each of which consists of $K_3$, where there are no isolated vertices.
- True; since $\forall u, v \in G, \exists u, v$ path by the definition of a connected graph, this means any vertex must have a path to any other vertex.
- True; every closed trail contains within it a cycle — we can delete the edge set of this cycle, and find cycles within remaining components until we reach isolated vertices.
- True; if there were a maximal trail with an endpoint of even degree, then we would be able to extend the trail further by re-entering the endpoint vertex.

**1.2.5**

Let $v$ be a vertex of a connected simple graph $G$. Prove that $v$ has a neighbor in every component of $G - v$. Explain why this allows us to conclude that no graph has a cut-vertex of degree 1.

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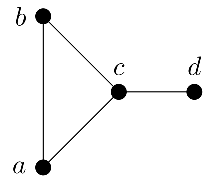
Suppose that $G - v$ is connected. Then, since $G$ is connected, and $v \in V(G)$, it must be the case that $v$ is connected to every component of $G - v$, meaning that it has a neighbor in every component of $G - v$ as $G - v$ is connected.

Now suppose that $G - v$ is disconnected, meaning that it has more than one component after removing $v$. Before, $v$ must have been connected to every vertex in $G$ as $G$ was a simple connected graph, and afterwards $G - v$ is no longer connected, meaning that $v$ is a cut-vertex. This means $v$ must have been adjacent to a vertex in each component of $G - v$, as removing the incident edges on $v$ along with $v$ increased the number of components from the original 1 that was in $G$.

From this result, we can conclude that no cut-vertex has degree 1 as removing a vertex of degree 1 and its incident edges does not increase the number of components in $G$, since there is only one edge incident on a vertex of degree 1.

**1.2.6**

In the graph below, find all the maximal paths, maximal cliques, and maximal independent sets. Also, find all the maximum paths, cliques, and independent sets.




- The maximal paths are as follows:
  - $d, c, b, a$
  - $d, c, a, b$
  - $a, b, c, d$
  - $b, a, c, d$
  - $b, c, a$
  - $c, b, a$
  - $a, c, b$
- The maximal cliques are $K_3$ consisting of $a, b, c$ and $K_2$ consisting of $c, d$.
- The maximal independent sets are $\{a, d\}$ and $\{b, d\}$.
- The maximum path is any of those paths listed above with length 4.
- The maximum clique is $K_3$.
- The maximum independent sets are those listed above with size 2.

### 1.2.8

Determine the values of $m$ and $n$ such that $K_{m,n}$ is Eulerian.

$$m, n \in 2\mathbb{Z}^+$$

### 1.2.10

Prove or disprove:

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

#### (a)

Let $G$ be an Eulerian bipartite graph. Since $G$ is Eulerian, it must contain an Eulerian cycle, meaning that as seen above, there are an even number of vertices, meaning that there are an even number of edges in $G$.

(b)

Let $G$ be an Eulerian simple graph with an even number of vertices. Since $G$ is Eulerian, this means there must be an Eulerian circuit $C$ that traverses every edge exactly once in $G$. Every vertex in $G$ must have even degree (or else we would require a backtrack in our Eulerian cycle, which is not a circuit); a simple pairing of the vertices would yield that we have $\lfloor n/2 \rfloor$ edges, and to complete the cycle we need $2(n/2)+2k$ edges for $n$ vertices and some integer $k$. Therefore, there must be an even number of edges.

## Group

1.2.20

Let $v$ be a cut-vertex of a simple graph $G$. Prove that $\overline{G} - v$ is connected.

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Let $x, y, v \in V(\overline{G})$, where $v$ is a cut-vertex of $G$.

Suppose $x$ and $y$ belong to distinct components of $G-v$. Then, $xy \notin E(G)$, meaning that $xy \in E(\overline{G})$, meaning there is an $x, y$ path in $\overline{G}$, so there is an $x, y$ path in $\overline{G} - v$.

Suppose $x$ and $y$ are in the same component of $G - v$. Since $v$ is a cut-vertex, this means there must be at least two components in $G - v$. Let $H_1$ be the component that $x, y$ are in, while $\exists w \in H_2$ is a vertex in $H_2$ disjoint from $H_1$. Since $H_1$ and $H_2$ are disjoint, this means the components do not contain any edges between them, so $x \not\leftrightarrow w$ and $y \not\leftrightarrow w$ in $G - v$ — however, this means that $x \leftrightarrow w$ and $y \leftrightarrow w$ in $\overline{G}$, meaning that $\exists x, y$ path in $\overline{G} - v$.

1.2.22

Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

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Let $G$ be a graph where there exists a partition of its vertices into two non-empty sets such that there is no edge with endpoints in both sets. Call these sets $A$ and $B$. By our assumptions, $\forall u \in A$ and $\forall v \in B$, $\nexists e$ such that $e = uv$. Therefore, we cannot create a path between any $u \in A$ and any $v \in B$ as there is no edge to connect any element in $A$ and any element in $B$. Therefore, $G$ is disconnected.

Suppose $G$ is a disconnected graph. Then, $G$ contains more than one component — we can create a partition of $V(G)$ by letting $H_1, H_2, \ldots, H_k$ refer to the $k$ components of $G$. Each of these components is necessarily disjoint from every other component. By taking $H = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ as one set and $H_k$ as our other set, we know that $H_1, \ldots, H_k$ are all disjoint, meaning that $H$ and $H_k$ are disjoint, meaning that there is no edge connecting any vertex $H$ with any vertex in $H_k$, meaning we have created a partition of $G$ such that there exists no edge between any vertex in one set and any vertex in the other set.

1.2.26

Prove that a graph $G$ is bipartite if and only if every subgraph $H$ of $G$ has an independent set consisting of at least half of $V(H)$.

Suppose $G$ is bipartite. Then, there exists a partition of the vertices $V = X \sqcup Y$ such that $X$ and $Y$ are independent sets. Let $H$ be a subgraph of $G$, and let $H_X = X \cap V(H)$ and $H_Y = Y \cap H$. Because $H$ is a subgraph of $G$, each vertex of $H$ must be an element of either $H_X$ or $H_Y$, or that $V(H) = H_X \sqcup H_Y$. WLOG, let $|H_X| > |H_Y|$. Since $H_X \subseteq X$ and $X$ is an independent set, $H_X$ is an independent subset consisting of at least half of $V(H)$.

Suppose every subgraph of $G$ has an independent set consisting of at least half of $V(H)$. We will suppose toward contradiction that $G$ is not bipartite. Then, $G$ must contain an odd cycle, $H_1$. However, an independent set of $H_1$ consists of at most $\lfloor \frac{|V(H_1)|}{2} \rfloor < \frac{|V(H_1)|}{2}$, otherwise two vertices would be adjacent. Because $H_1$ is an independent set with less than half of the elements of $V(H)$, we have reached a contradiction. Therefore, $G$ must be bipartite.

1.2.38

Prove that every $n$-vertex graph with at least $n$ edges contains a cycle.

We proceed via induction as follows:

For the base case where $|V(G)| = 1$, we know that there is a cycle with one edge that connects back on the vertex.

For the case where $|V(G)| > 1$, if $v \in V(G)$ has degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n-1$ edges, so by our inductive hypothesis, we know that $G-v$ contains a cycle. Meanwhile, if $\forall v \in V(G), d(v) \geq 2$, we know by Lemma 1.2.25 that $G$ contains a cycle.