2.1.22

Let T be an n-vertex tree with one vertex of each degree $2 \leq i \leq k$; the remaining $n-k+1$ vertices are leaves. Determine n in terms of k.

We will find the number of vertices in T by finding the number of edges in T and adding 1. For $2 \leq i \leq k$ corresponding to each of the non-leaf vertices, summation yields $\frac{k(k+1)}{2} - 1$ edges. However, this scheme double-counts each edge, so we have to subtract the $k-2$ edges connecting the $k-1$ non-leaf vertices, yielding $\frac{k(k+1)}{2} - k + 1$ edges. Finally, because T is a tree, we get that T has $\frac{k(k+1)}{2} - k + 2$ vertices.

2.1.27

Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \dots, d_n$ if and only if $\sum d_i = 2n - 2$.

($\Rightarrow$) Suppose that for some tree T, $d_1, \dots, d_n$ are the degrees of the vertices of the tree. Since T is a tree, this means $e(G) = n - 1$, and $\sum d_i = 2e(G)$, meaning $\sum d_i = 2(n-1) = 2n - 2$.

($\Leftarrow$) Suppose that $\sum d_i = 2n - 2$ for $d_1, \dots, d_n$ corresponding to the degrees of the vertices in G. By a previous result, we know that $\sum d_i = 2e(G)$, meaning that $\sum d_i = 2(n-1)$, implying that $e(G) = n - 1$. We can find a tree G with $n-1$ edges by letting G be connected with that many edges.

2.1.33

Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

2.1.34

Let T be a tree with k edges, and let G be a n-vertex simple graph with more than $n(k-1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n \geq k$.