1.1

Individual

1.1.1

Determine which complete bipartite graphs are complete graphs.

- -

$K_{1,1}$ is the only complete bipartite graph that is complete

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1.1.3

Using rectangular blocks whose entries are all equal, write down an adjacency matrix for $K_{m,n}$.

- -

$$K_{m,n} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_m \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \begin{array}{cccccccc} a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_n \\ \left[0 \right. & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & \left. 0 \right] \end{array}$$

1.1.5

Prove or disprove: If every vertex of a simple graph G has degree 2, then G is a cycle.

- -
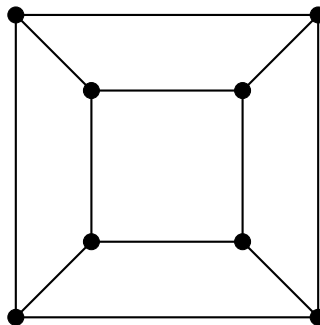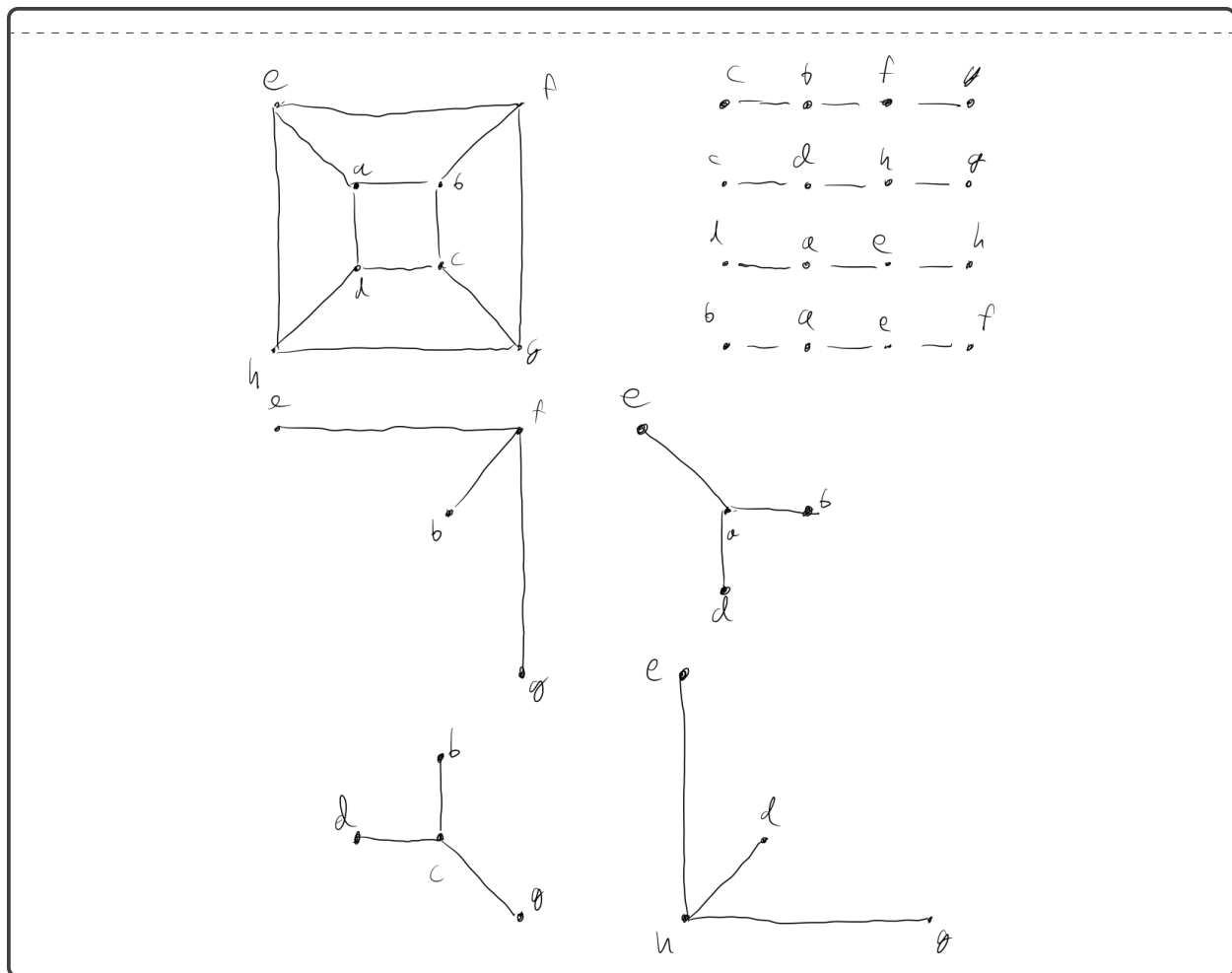
Let G be the following graph:

Every vertex in G has a degree 2, yet G is not a cycle.
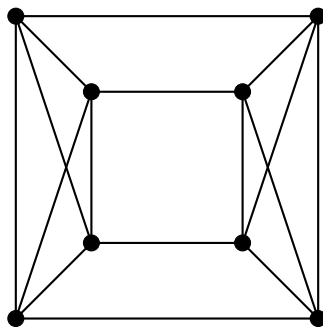

1.1.8

Prove that the 8 vertex graph below decomposes into copies of $K_{1,3}$ and also into copies of P_4
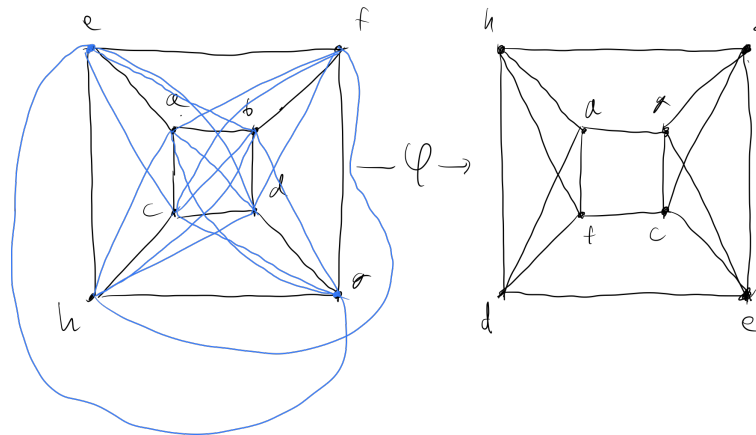
1.1.9

Prove that the graph below is isomorphic to the complement of the previous graph

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1.1.10

Prove or disprove: the complement of a simple disconnected graph must be connected.

- -

Let G be a graph that is disconnected. We want to show that $\forall x, y \in V(G), \exists xz$ path. We can split into two cases.

- Suppose $x \not\leftrightarrow y$ in G. Then, in $\overline{G}$, $x \leftrightarrow y$ by the definition of a graph complement.

- Suppose $x \leftrightarrow y$ in G. Then, since G is disconnected, we know that there must be some $z \in V(G)$ such that there is no xz path. Since there is no xz path, then there is no yz path. In particular, this means $x \not\leftrightarrow z$ and $y \not\leftrightarrow z$ in G. Therefore, in $\overline{G}$, we have that $x \leftrightarrow z$ and $y \leftrightarrow z$, meaning there is a path between x and y.

Group

1.1.13

Let G be the graph whose vertex set is the set of k-tuples with coordinates $\{0, 1\}$, with x adjacent to y if x and y differ by exactly one position. Determine whether G is bipartite.

- -

G is bipartite — we can find a bipartition by separating the set into a set of tuples which differ by an even number of positions and a set of tuples which differ by an odd number of positions. Since odd numbers differ from each other by at least 2 places, and even numbers differ from each other by at least 2 places, we know that each subset of tuples is not adjacent to each other, but is adjacent to the other set.

1.1.26

Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least 2k vertices. Determine all such graphs with 2k vertices.

- -

Suppose G is a graph with girth 4 with every vertex of degree k. Let $v_i \in V(G)$. Then, there must be k vertices which v_i is adjacent to. However, none of these vertices can be adjacent to themselves or G would have girth 3. Thus, we can form a bipartition such that v_i is in a set of at least k vertices such that each vertex is not adjacent to itself, and each vertex in this set is adjacent to k vertices in a disjoint set where each vertex in this set is not adjacent to any other vertex in this set. Therefore, there are at least 2k vertices.

The graphs with exactly 2k vertices are the $K_{n,n}$ complete bipartite graphs.

1.1.27

Let G be a graph with girth 5. Prove that if every vertex of G has degree at least k, then G has at least $k^2 + 1$ vertices. For $k = 2$ and $k = 3$, find one such graph with $k^2 + 1$ vertices.

- -

Let G be a simple graph with girth 5. Suppose that every vertex of G has degree k. Let $u \in V(G)$. Then, u has k adjacent vertices, each of which is not adjacent to each other (or else the girth of G would be 3). Let this set be N. The elements of N cannot have any other common neighbors aside from u, or else the girth of G would be 4, meaning each has $k - 1$ distinct neighbors. Therefore, the total number of vertices in our graph includes u, the elements of N that are the k distinct neighbors of u, and the $k(k - 1)$ distinct vertices for each vertex in N. Therefore, our total is $1 + k + k(k - 1) = k^2 + 1$.

If there were any vertex with degree greater than k, then there would be additional vertices beyond the $k^2 + 1$ vertices necessary for a k-regular graph.

For $k = 2$, we have the graph C_5 for an example of a graph with $k^2 + 1$ vertices, and for $k = 3$ we have the Petersen graph.

1.1.30

Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry of A^2 and MM^T. What do the entries in position (i, j) of A^2 and MM^T say about G?

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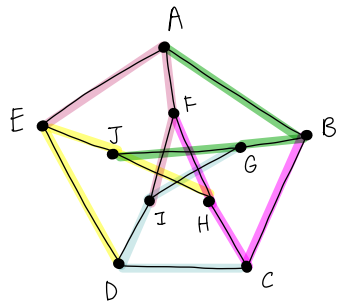
Let A be the adjacency matrix for a simple graph G. In A, every vertex's corresponding row and column are identical, meaning that the entry $A^2_{i,i}$ will be equal to $r_i c_i$ for row i and column i corresponding to v_i. Thus, $r_i c_i$ is equal to $|c_i|^2$, which is equal to the sum of the elements of c_i, which is equal to the degree of v_i.

Let M be the incidence matrix for a simple graph G. In MM^T, the diagonal element $MM^\mathsf{T}_{i,i}$ will be equal to $r_i r_i^\mathsf{T}$, where r_i represents the edge incidence row of v_i. This is equal to $\left|r_i^\mathsf{T}\right|^2$, which is equal to the sum of the elements of r_i, which is equal to the number of edges incident on v_i, which is equal to the degree of v_i.
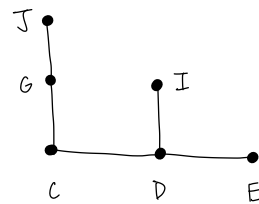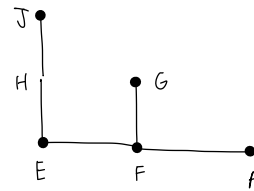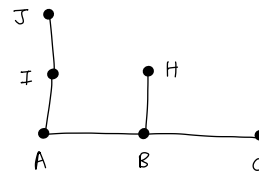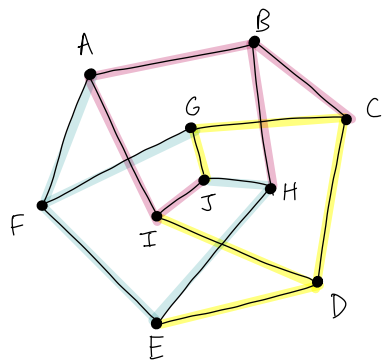
The entry in position (i, j) in both A^2 and MM^T shows whether vertices v_i and v_j are adjacent to each other.

1.1.34

Decompose the Petersen graph into three connected subgraphs that are pairwise isomorphic. Also decompose it into copies of P_4.

A — B — G — J
B — C — H — F
C — D — I — G
D — E — J — H
E — A — F — I

1.2

Individual

1.2.1

Determine whether the following statements are true or false:

- Every disconnected graph has an isolated vertex.

- A graph is connected if and only if some vertex is connected to all other vertices.

- The edge set of every closed trail can be partitioned into edge sets of cycles.

- If a maximal trail in a graph is not closed, then its endpoints have odd degree.

- -

- False; we can imagine a graph with two components, each of which consists of K_3, where there are no isolated vertices.

- True; since $\forall u, v \in G, \exists u, v$ path by the definition of a connected graph, this means any vertex must have a path to any other vertex.

- True; every closed trail contains within it a cycle — we can delete the edge set of this cycle, and find cycles within remaining components until we reach isolated vertices.

- True; if there were a maximal trail with an endpoint of even degree, then we would be able to extend the trail further by re-entering the endpoint vertex.

1.2.5

Let v be a vertex of a connected simple graph G. Prove that v has a neighbor in every component of $G - v$. Explain why this allows us to conclude that no graph has a cut-vertex of degree 1.

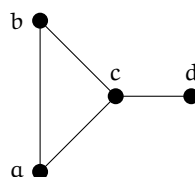- -

Suppose that $G - v$ is connected. Then, since G is connected, and $v \in V(G)$, it must be the case that v is connected to every component of $G - v$, meaning that it has a neighbor in every component of $G - v$ as $G - v$ is connected.

Now suppose that $G - v$ is disconnected, meaning that it has more than one component after removing v. Before, v must have been connected to every vertex in G as G was a simple connected graph, and afterwards $G - v$ is no longer connected, meaning that v is a cut-vertex. This means v must have been adjacent to a vertex in each component of $G - v$, as removing the incident edges on v along with v increased the number of components from the original 1 that was in G.

From this result, we can conclude that no cut-vertex has degree 1 as removing a vertex of degree 1 and its incident edges does not increase the number of components in G, since there is only one edge incident on a vertex of degree 1.

1.2.6

In the graph below, find all the maximal paths, maximal cliques, and maximal independent sets. Also, find all the maximum paths, cliques, and independent sets.

- The maximal paths are as follows:

  - d, c, b, a
  - d, c, a, b
  - a, b, c, d
  - b, a, c, d
  - b, c, a
  - c, b, a
  - a, c, b

- The maximal cliques are K_3 consisting of a, b, c and K_2 consisting of c, d.

- The maximal independent sets are $\{a, d\}$ and $\{b, d\}$.

- The maximum path is any of those paths listed above with length 4.

- The maximum clique is K_3.

- The maximum independent sets are those listed above with size 2.

1.2.8

Determine the values of m and n such that $K_{m,n}$ is Eulerian.

$$m, n \in 2\mathbb{Z}^+$$

1.2.10

Prove or disprove:

(a) Every Eulerian bipartite graph has an even number of edges.

(b) Every Eulerian simple graph with an even number of vertices has an even number of edges.

(a)

Let G be an Eulerian bipartite graph. Since G is Eulerian, it must contain an Eulerian cycle, meaning that as seen above, there are an even number of vertices, meaning that there are an even number of edges in G.

(b)

Let G be an Eulerian simple graph with an even number of vertices. Since G is Eulerian, this means there must be an Eulerian circuit C that traverses every edge exactly once in G. Every vertex in G must have even degree (or else we would require a backtrack in our Eulerian cycle, which is not a circuit); a simple pairing of the vertices would yield that we have $\lfloor n/2 \rfloor$ edges, and to complete the cycle we need $2(n/2) + 2k$ edges for n vertices and some integer k. Therefore, there must be an even number of edges.

Group

1.2.20

Let v be a cut-vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

Let $x, y, v \in V(\overline{G})$, where v is a cut-vertex of G.

Suppose x and y belong to distinct components of $G - v$. Then, $xy \notin E(G)$, meaning that $xy \in E(\overline{G})$, meaning there is an x, y path in $\overline{G}$, so there is an x, y path in $\overline{G} - v$.

Suppose x and y are in the same component of $G - v$. Since v is a cut-vertex, this means there must be at least two components in $G - v$. Let H_1 be the component that x, y are in, while $\exists w \in H_2$ is a vertex in H_2 disjoint from H_1. Since H_1 and H_2 are disjoint, this means the components do not contain any edges between them, so $x \nleftrightarrow w$ and $y \nleftrightarrow w$ in $G - v$ — however, this means that $x \leftrightarrow w$ and $y \leftrightarrow w$ in $\overline{G}$, meaning $\exists x, y$ path in $\overline{G} - v$.

1.2.22

Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

Let G be a graph where there exists a partition of its vertices into two non-empty sets such that there is no edge with endpoints in both sets. Call these sets A and B. By our assumptions, $\forall u \in A$ and $\forall v \in B$, $\nexists e$ such that $e = uv$. Therefore, we cannot create a path between any $u \in A$ and any $v \in B$ as there is no edge to connect any element in A and any element in B. Therefore, G is disconnected.

Suppose G is a disconnected graph. Then, G contains more than one component — we can create a partition of $V(G)$ by letting $H_1, H_2, \ldots, H_k$ refer to the k components of G. Each of these components is necessarily disjoint from every other component. By taking $H = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ as one set and H_k as our other set, we know that $H_1, \ldots, H_k$ are all disjoint, meaning that H and H_k are disjoint, meaning that there is no edge connecting any vertex H with any vertex in H_k, meaning we have created a partition of G such that there exists no edge between any vertex in one set and any vertex in the other set.

1.2.26

Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$.

Suppose G is bipartite. Then, there exists a partition of the vertices $V = X \sqcup Y$ such that X and Y are independent sets. Let H be a subgraph of G, and let $H_X = X \cap V(H)$ and $H_Y = Y \cap H$. Because H is a subgraph of G, each vertex of H must be an element of either H_X or H_Y, or that $V(H) = H_X \sqcup H_Y$. WLOG, let $|H_X| > |H_Y|$. Since $H_X \subseteq X$ and X is an independent set, H_X is an independent subset consisting of at least half of $V(H)$.

Suppose every subgraph of G has an independent set consisting of at least half of $V(H)$. We will suppose toward contradiction that G is not bipartite. Then, G must contain an odd cycle, H_1. However, an independent set of H_1 consists of at most $\left\lfloor \frac{|V(H_1)|}{2} \right\rfloor < \frac{|V(H_1)|}{2}$, otherwise two vertices would be adjacent. Because H_1 is an independent set with less than half of the elements of $V(H)$, we have reached a contradiction. Therefore, G must be bipartite.

1.2.38

Prove that every n-vertex graph with at least n edges contains a cycle.

We proceed via induction as follows:

For the base case where $|V(G)| = 1$, we know that there is a cycle with one edge that connects back on the vertex.

For the case where $|V(G)| > 1$, if $v \in V(G)$ has degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges, so by our inductive hypothesis, we know that $G - v$ contains a cycle. Meanwhile, if $\forall v \in V(G), d(v) \geq 2$, we know by Lemma 1.2.25 that G contains a cycle.
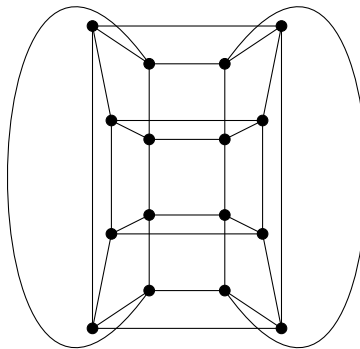
1.3

Individual

1.3.3

Let u and v be adjacent vertices in G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

Let $u \leftrightarrow v \in G$. In order for uv to be in a triangle in G, u and v must share a common neighbor. By using the principle of inclusion and exclusion, we can find the set as follows:

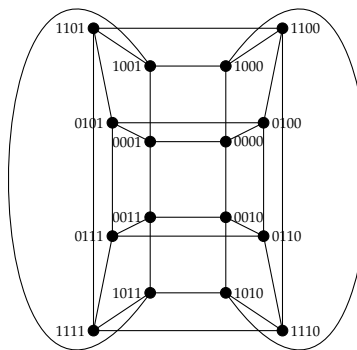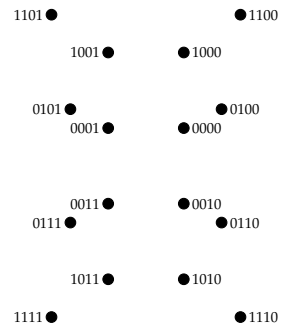$$|N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)|$$
$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)|$$
$$\geq d(u) + d(v) - n(G)$$

1.3.4

Prove that the graph below is isomorphic to Q_4.

We can assign tuples to the graph as follows:

1.3.6

Given graphs G and H, determine the number of components and maximum degree in G + H in terms of the parameters for G and H.

- -

We can find the number of components in G + H by summing the number of components in G and the number of components in H.

The maximum degree in G + H is equal to $\max\{\Delta(G), \Delta(H)\}$.

1.3.7

Determine the maximum number of edges in a bipartite subgraph of P_n, C_n, and K_n.

- -

For the graph P_n, we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets A and B. This is a bipartition since a path does not include any repeated vertices or edges, so A and B are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of P_n is the number of

edges in P_n, which is $n - 1$.

For C_n, we have two values of the maximum number of edges in a bipartite subgraph of C_n:

- If n is even, then C_n is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of C_n is the number of edges in C_n, which is n.

- If n is odd, then C_n is not a bipartite graph. After one edge deletion, we get that $C_n - e = P_n$, which is bipartite, so the maximum number of edges in a bipartite subgraph of C_n is the number of edges in P_n, which is $n - 1$.

For the graph K_n, there are two options for the maximum number of edges in a bipartite subgraph depending on the value of n:

- If n is even, then the subgraph $K_{\frac{n}{2}, \frac{n}{2}}$ is the maximal bipartite subgraph, meaning that the number of edges is equal to $n^2/4$. We know that $K_{\frac{n}{2}, \frac{n}{2}}$ is a subgraph of K_n because the vertex set is the same, and K_n is complete, so any subset of edges is a subset of the edge set of K_n.

- If n is odd, then the subgraph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the maximal bipartite subgraph, because $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ and K_n is complete. Therefore, the total number of edges is $\left\lfloor \frac{n^2}{4} \right\rfloor$.

1.3.26 (a)

Count the 6-cycles in Q_3.

- -

In order to find a 6-cycle in Q_3, we select two vertices to delete and see if we can find a cycle from the graph $Q_3 - \{u, v\}$. Vertices are either adjacent, distance 2 (antipodal on the same face), or antipodal (distance 3) with each other.

ADJACENT: If two vertices are adjacent, we can find one cycle from the remaining vertices after deletion. There are $(8)(3)/2$ sets of adjacent vertices, for a total of 12 from this selection.

ANTIPODAL ON THE SAME FACE: If two vertices are antipodal, we cannot find a cycle from the remaining graph after deletion.

ANTIPODAL: If two vertices are antipodal, we can find one cycle from the remaining vertices after deletion. There are 4 sets of antipodal vertices.

We find a total of 16 6-cycles in Q_3.

Group

1.3.17

Let G be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot decrease the average degree.

(a)

Assume toward contradiction that deleting a vertex of degree $\Delta(G)$ increases the average degree.

$$d'_{avg} > d_{avg}$$

$$\frac{2e(G) - 2\Delta(G)}{n(G) - 1} > \frac{2e(G)}{n(G)}$$

$$\frac{2e(G) - 2\Delta(G)}{2e(G)} > \frac{n(G) - 1}{n(G)}$$

$$1 - \frac{\Delta(G)}{e(G)} > 1 - \frac{1}{n(G)}$$

$$\frac{1}{n(G)} - \frac{\Delta(G)}{e(G)} > 0$$

$$\frac{1}{n(G)} - \frac{2\Delta(G)}{n(G)d_{avg}} > 0$$

$$\frac{d_{avg} - 2\Delta(G)}{n(G)} > 0$$

$$d_{avg} - 2\Delta(G) > 0$$

$$d_{avg} > 2\Delta(G)$$

However, we have reached a contradiction — by definition, $\Delta(G) \geq d_{avg}$, meaning that $d_{avg} \not> \Delta(G)$, let alone $2\Delta(G)$.

(b)

Deleting a vertex of the graph $K_{1,1}$ yields a graph with one vertex of degree zero, which is lower than the average degree of 1 in $K_{1,1}$.

1.3.20

Count the cycles n-cycles in K_n and the $2n$-cycles in $K_{n,n}$.

To count the cycles in K_n, we start at a vertex v and choose an edge out of $n - 1$ options. After choosing the edge, there are $n - 2$ options remaining that do not backtrack to v, and so on and so forth. Therefore, there are $(n - 1)!$ cycles in K_n.

Let $v \in A$, where A and B are the order n sets that partition G. Then, there are n possible vertices in B which can be the second element in our cycle — afterwards, there are $n - 1$ elements in A, and after that there are $n - 1$ elements in B, and so on and so forth. Therefore, there are $n!(n - 1)!$ possible options for cycles in $K_{n,n}$.

1.3.25

Prove that every cycle of length $2r$ in a hypercube is contained within a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r.

Let C be a cycle of length $2r$ in Q_n. Then, C contains $2r$ n-tuples. For every tuple in C, there exists a "switched" tuple where every coordinate is equal to its other, corresponding coordinate, except for one. Since C is a cycle, every coordinate that is switched must be returned to its original state at the end of the cycle — since there are $2r$ switches (corresponding to the $2r$ edges in C), this means there are at most r coordinates that are switched, then switched back sometime along the cycle's path. This means the other $n - r$ coordinates are fixed, implying

that $C \subseteq Q_r$, the r-dimensional subcube of Q_k.

There is a cycle of length 8 in Q_3, defined as follows: $000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000$.

1.3.31

Using complete graphs and counting arguments, prove the following:

(a) $\binom{n}{2} = \binom{n}{k} + k(n-k) + \binom{n-k}{2}$ for $0 \leq k \leq n$.

(b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \leq \binom{n}{2}$.

(a)

We can consider a decomposition of the edges of K_n into the edge set of K_k and K_{n-k}, and some connector edges.

The edge set of K_n has cardinality $\binom{n}{2}$, the edge set of K_k has cardinality $\binom{k}{2}$, and the edge set of K_{n-k} has cardinality $\binom{n-k}{2}$. In order for this set of edges to be a full decomposition, we need a graph that connects all the vertices in K_k with all the vertices in K_{n-k}, which takes $k(n-k)$ edges. Therefore, we have shown the following result:

$$\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$$

(b)

Consider the graph G, where $|V(G)| = n$ with maximal clique components $H_1, \ldots, H_k$. Each of these components has $e(H_i) = \binom{V(H_i)}{2}$, with total $\sum_{i=1}^{k} \binom{V(H_i)}{2}$. In comparison, if we consider $e(K_G)$, where K_G is defined as the complete graph on the vertices of G, then that value is $\binom{n}{2}$, and $n = \sum_{i=1}^{k} |V(H_i)|$. Therefore, the size of the edge set of G is less than or equal to the sum of the sizes of the edge sets of maximal clique components H_i (because the maximal clique components of G could just be G itself).

1.3.41

Prove or disprove: if G is an n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$, then G is connected.

Let $u, v \in V(G)$ and let $d(u) = \lceil \frac{n}{2} \rceil$. Then, u is adjacent to $\lceil \frac{n}{2} \rceil$ vertices and nonadjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Let $u \not\leftrightarrow v$.

We want to show that there exists some other vertex such that there exists a u, v path through that vertex. We know that $|N(u)| = d(u) = \lceil \frac{n}{2} \rceil$ and $|N(v)| = d(v) \geq \delta(G) = \lfloor \frac{n}{2} \rfloor - 1$.

Since $u \not\leftrightarrow v$, $N(u), N(v) \subseteq V(G) - \{u, v\}$. So, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (\lceil \frac{n}{2} \rceil) + (\lfloor \frac{n}{2} \rfloor - 1) - (n-2) = 1$.

Therefore, there must be at least one element in $N(u) \cap N(v)$, meaning G is connected.

2.1

Individual

2.1.2

Let G be a graph:

    (a) Prove that G is a tree if and only if G is connected and every edge is a cut-edge.

    (b) Prove that G is a tree if and only if adding any edge with endpoints in $V(G)$ creates exactly one cycle.

(a)

($\Rightarrow$) Let G be a tree. Thus, G is connected (by definition), and acyclic. Since G is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.

($\Leftarrow$) Let G be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in G, so G is a connected acyclic graph, or a tree.

(b)

($\Rightarrow$) Let G be a tree, and let e be an edge such that $e \notin E(G)$, and $e = uv$. Then, we create a cycle from the path $uTv + e$ — since there is only one path uTv, this means that $uTv + e$ is a unique cycle.

($\Leftarrow$) Suppose toward contradiction that adding e to the tree G yielded more than one cycle in the graph $G + e$. Then, the graph $G = G + e - e$ would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that G was a tree, we have reached a contradiction, meaning that e added exactly one cycle to the tree G.

2.1.6

Let T be a tree with average degree a. In terms of a, find $n(T)$.

$$d_{avg} = \frac{2e(T)}{n(T)}$$
$$a = \frac{2(n(T) - 1)}{n(T)}$$
$$an = 2n - 2$$
$$(a - 2)n = -2$$
$$n = \boxed{\frac{2}{2 - a}}$$

2.1.7

Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles.

BASE CASE If $m = 0$, then since this graph has zero edges, it has zero cycles, and since $0 \geq 1 - n$, we have proven the base case.

INDUCTIVE HYPOTHESIS For an n-vertex graph with $0 \leq k \leq m$ vertices, then G has at least $k - n + 1$ cycles.

PROOF If e is an edge within a cycle of G, then $G - e$ has $k - 1$ edges, and has seen a reduction of 1 cycle, so $G - e$ has at least $(k - 1) - n + 1 = (k - n + 1) - 1$ cycles. If e is not within a cycle, then G has seen no reduction

in cycles, but $G - e$ is predicted to have at least $(k - n + 1) - 1$ cycles, which it does by our assumption. Therefore, we have proven the inductive hypothesis for both cases.

2.1.12

Compute the diameter and radius of $K_{m,n}$.

- -

The diameter of $K_{m,n}$ is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of $K_{m,n}$ is also 2 — the eccentricity of every vertex in $K_{m,n}$ is 2, so the radius must also be 2.

2.1.13

Prove that every graph with diameter d has an independent set with at least $\left\lceil \frac{1+d}{2} \right\rceil$ vertices.

- -

Let G be a graph with diameter d, and let $u \in V(G)$ be a vertex with eccentricity d. Let P be a maximal u, v path of length d. Then, P has $d + 1$ vertices. So, P has a maximal independent set containing every other vertex, with total cardinality of $\left\lceil \frac{d+1}{2} \right\rceil$. Therefore, G has an independent set with at least $\left\lceil \frac{d+1}{2} \right\rceil$ vertices.

Group

2.1.22

Let T be an n-vertex tree with one vertex of each degree $2 \leq i \leq k$; the remaining $n - k + 1$ vertices are leaves. Determine n in terms of k.

- -

We will find the number of vertices in T by finding the number of edges in T and adding 1. For $2 \leq i \leq k$ corresponding to each of the non-leaf vertices, summation yields $\frac{k(k+1)}{2} - 1$ edges. However, this scheme double-counts each edge, so we have to subtract the $k - 2$ edges connecting the $k - 1$ non-leaf vertices, yielding $\frac{k(k+1)}{2} - k + 1$ edges. Finally, because T is a tree, we get that T has $\frac{k(k+1)}{2} - k + 2$ vertices.

2.1.27

Let $d_1, \ldots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

- -

($\Rightarrow$) Suppose that for some tree T, $d_1, \ldots, d_n$ are the degrees of the vertices of the tree. Since T is a tree, this means $e(G) = n - 1$, and $\sum d_i = 2e(G)$, meaning $\sum d_i = 2(n - 1) = 2n - 2$.

($\Leftarrow$) Suppose that $\sum d_i = 2n - 2$ for $d_1, \ldots, d_n$ corresponding to the degrees of the vertices in G. By a previous result, we know that $\sum d_i = 2e(G)$, meaning that $\sum d_i = 2(n - 1)$, implying that $e(G) = n - 1$. We can find a tree G with $n - 1$ edges by letting G be connected with $n - 1$ edges.

2.1.29

Every tree is bipartite. Prove that every tree has a leaf in its larger partite set (or in both sets if the partite sets have equal size).

- -

Let $T = X \sqcup Y$, and suppose without loss of generality that $|X| \geq |Y|$, and suppose that X has no vertices of degree 1 within it. Then, every vertex in X has degree at least 2, meaning that the total number of edges in T is at least $2n(X)$. However, since $n(X) \geq \frac{n(T)}{2}$, this means the number of edges in T is at least $n(T)$, which would

contradict our assumption that T is a tree.

2.1.33

Let G be a connected n-vertex graph. Prove that G has exactly one cycle if and only if G has exactly n edges.

- -

($\Rightarrow$) Let G be a connected n-vertex graph with exactly one cycle. If we delete an edge from this cycle, then $G - e$ is acyclic, as well as connected (since e is not a cut-edge), so $G - e$ has $n - 1$ edges. Adding back e, we get that G has n edges.

($\Leftarrow$) Let G be a connected n-vertex graph with n edges. Then, G contains a spanning tree that contains all n vertices. Therefore, $T \subseteq G$ contains $n - 1$ edges. By adding another edge, we get that $e(G) = e(T) + 1 = n - 1 + 1$. Thus, G has exactly one cycle.

2.1.34

Let T be a tree with k edges, and let G be a n-vertex simple graph with more than $n(k - 1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n > k$.

- -

We will use induction to prove that $T \subseteq G$ as follows:

BASE CASE Suppose $n = k + 1$. Then, we can find the following:

$$e(G) > (k + 1)(k - 1) - \binom{k}{2}$$

$$e(G) > (k^2 - 1) - \frac{k(k - 1)}{2}$$

$$e(G) > \frac{k^2 - 1}{2} + \frac{k^2 - 1 - (k^2 - k)}{2}$$

$$e(G) > \frac{k^2 + k}{2} - 1$$

$$e(G) > \frac{k(k + 1)}{2} - 1$$

This means $e(G) = \frac{k(k+1)}{2}$ in the base case, meaning G is the complete graph on $k + 1$ vertices, where $\delta(G) = k$. By Theorem 2.1.8, we know that $T \subseteq G$.

INDUCTIVE HYPOTHESIS If $n > k + 1$, $e(G) > n(k - 1) - \binom{k}{2}$, then either $\delta(G) \geq k$ or, if $\delta(G) < k$, then $e(G - x) > (n - 1)(k - 1) - \binom{k}{2}$ for $\delta(G) = d(x)$.

PROOF If $\delta(G) \geq k$, then we know by Theorem 2.1.8 that $T \subseteq G$. Otherwise, suppose $\delta(G) < k$, and let $d(x) = \delta(G)$. Let $G' = G - x$.

$$e(G') = e(G) - \delta(G)$$

$$e(G') \geq e(G) - (k - 1)$$

$$e(G') > n(k - 1) - \binom{k}{2} - (k - 1)$$

$$e(G') > (n - 1)(k - 1) - \binom{k}{2}$$

Therefore, the inductive hypothesis is proven.

2.1.35

Let T be a tree. Prove that the vertices of T all have odd degree if and only if for all $e \in E(T)$, both components of $T - e$ are of odd order.

($\Rightarrow$) Let T be a tree. We will suppose toward contradiction with two cases:

CASE 1 Suppose $T - e$ has exactly one component of odd order. Then, T has odd order, meaning that by a previous result, we know that there must exist at least one vertex of even degree in T, otherwise $\sum d(v)$ would be odd.

CASE 2 Suppose $T - e$ has two components of even order. Let X and Y be the two components of $T - e$.

SUBCASE 2.1 Suppose every vertex in X is of odd degree. Then, $T = T - e + e$ would increase the degree of a vertex in X by 1, making that particular vertex have even degree.

SUBCASE 2.2 Suppose there is exactly one vertex in X of even degree. This would mean X has an odd number of vertices of odd degree, which we have previously shown is not possible.

SUBCASE 2.3 Suppose there is more than one vertex in X of even degree. Then, T would contain at least one vertex of even degree (as reconnecting the edge would only increase the degree of one vertex).