

## 1.3.3

Let  $u$  and  $v$  be adjacent vertices in  $G$ . Prove that  $uv$  belongs to at least  $d(u) + d(v) - n(G)$  triangles in  $G$ .

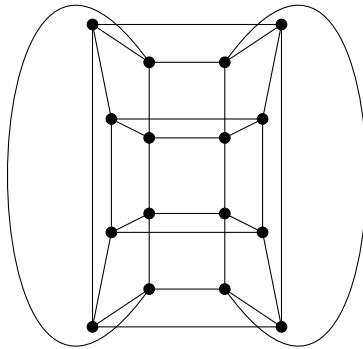
## Solution

Let  $u \leftrightarrow v \in G$ . In order for  $uv$  to be in a triangle in  $G$ ,  $u$  and  $v$  must share a common neighbor. By using the property of inclusion and exclusion, we can find the set as follows:

$$\begin{aligned} |N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &= d(u) + d(v) - n(G) \end{aligned}$$

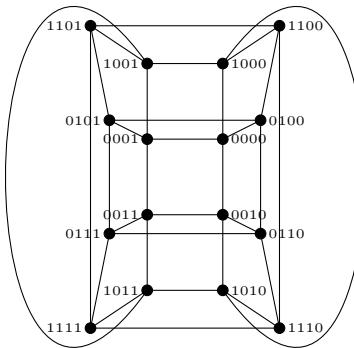
## 1.3.4

Prove that the graph below is isomorphic to  $Q_4$ .



## Solution

We can assign tuples to the graph as follows:



## 1.3.6

Given graphs  $G$  and  $H$ , determine the number of components and maximum degree in  $G + H$  in terms of the parameters for  $G$  and  $H$ .

## Solution

We can find the number of components in  $G + H$  by summing the number of components in  $G$  and the number of components in  $H$ .

The maximum degree in  $G + H$  is equal to  $\max\{\Delta(G), \Delta(H)\}$ .

## 1.3.7

Determine the maximum number of edges in a bipartite subgraph of  $P_n$ ,  $C_n$ , and  $K_n$ .

## Solution

For the graph  $P_n$ , we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets  $A$  and  $B$ . This is a bipartition since a path does not include any repeated vertices or edges, so  $A$  and  $B$  are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of  $P_n$  is the number of edges in  $P_n$ , which is  $n - 1$ .

For  $C_n$ , we have two values of the maximum number of edges in a bipartite subgraph of  $C_n$ :

- If  $n$  is even, then  $C_n$  is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of  $C_n$  is the number of edges in  $C_n$ , which is  $n$ .
- If  $n$  is odd, then  $C_n$  is not a bipartite graph. After one edge deletion, we get that  $C_n - e = P_n$ , which is bipartite, so the maximum number of edges in a bipartite subgraph of  $C_n$  is the number of edges in  $P_n$ , which is  $n - 1$ .

For the graph  $K_n$ , there are two options for the maximum number of edges in a bipartite subgraph depending on the value of  $n$ :

- If  $n$  is even, then the subgraph  $K_{\frac{n}{2}, \frac{n}{2}}$  is the maximal bipartite subgraph, meaning that the number of edges is

equal to  $n^2/4$ . We know that  $K_{\frac{n}{2}, \frac{n}{2}}$  is a subgraph of  $K_n$  because the vertex set is the same, and  $K_n$  is complete, so any subset of edges is a subset of the edge set of  $K_n$ .

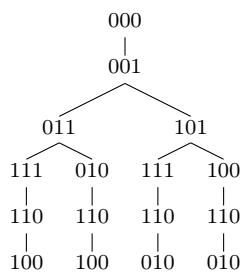
- If  $n$  is odd, then the subgraph  $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$  is the maximal bipartite subgraph, because  $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$  and  $K_n$  is complete. Therefore, the total number of edges is  $\lfloor \frac{n^2}{4} \rfloor$ .

## 1.3.26 (a)

Count the 6-cycles in  $Q_3$ .

## Solution

Without loss of generality, we are starting from tuple  $(0, 0, 0)$ , henceforth 000. Starting from 000 going to 001. We can then map out the path as follows:



Since there are three starting options from 000, namely 001, 010, and 100, we can see there are 12 6-cycles in  $Q_3$ .