

## 2.1.2

Let  $G$  be a graph:

- (a) Prove that  $G$  is a tree if and only if  $G$  is connected and every edge is a cut-edge.
- (b) Prove that  $G$  is a tree if and only if adding any edge with endpoints in  $V(G)$  creates exactly one cycle.

(a)

- $\Rightarrow$  Let  $G$  be a tree. Thus,  $G$  is connected (by definition), and acyclic. Since  $G$  is acyclic, this means that there are no edges within cycles, so by definition, every edge is a cut-edge.
- $\Leftarrow$  Let  $G$  be a connected graph such that every edge is a cut-edge. Since there are no non-cut-edge edges, this means there are no cycles in  $G$ , so  $G$  is a connected acyclic graph, or a tree.

(b)

- $\Rightarrow$  Let  $G$  be a tree, and let  $e$  be an edge such that  $e \notin E(G)$ , and  $e = uv$ . Then, we create a cycle from the path  $uTv + e$  — since there is only one path  $uTv$ , this means that  $uTv + e$  is a unique cycle.
- $\Leftarrow$  Suppose toward contradiction that adding  $e$  to the tree  $G$  yielded more than one cycle in the graph  $G + e$ . Then, the graph  $G = G + e - e$  would have at least one cycle, as we deleted an edge from one cycle in a graph with more than one cycle. However, since we assumed that  $G$  was a tree, we have reached a contradiction, meaning that  $e$  added exactly one cycle to the tree  $G$ .

## 2.1.6

Let  $T$  be a tree with average degree  $a$ . In terms of  $a$ , find  $n(T)$ .

$$d_{\text{avg}} = \frac{2e(T)}{n(T)}$$

$$a = \frac{2(n(T) - 1)}{n(T)}$$

$$an = 2n - 2$$

$$(a - 2)n = -2$$

$$n = \boxed{\frac{2}{2-a}}$$

## 2.1.7

Prove that every  $n$ -vertex graph with  $m$  edges has at least  $m - n + 1$  cycles.

**BASE CASE** If  $m = 0$ , then since this graph has zero edges, it has zero cycles, and since  $0 \geq 1 - n$ , we have proven the base case.

**INDUCTIVE HYPOTHESIS** For an  $n$ -vertex graph with  $0 \leq k \leq m$  vertices, then  $G$  has at least  $k - n + 1$  cycles.

**PROOF** If  $e$  is an edge within a cycle of  $G$ , then  $G - e$  has  $k - 1$  edges, and has seen a reduction of 1 cycle, so  $G - e$  has at least  $(k - 1) - n + 1 = (k - n + 1) - 1$  cycles. If  $e$  is not within a cycle, then  $G$  has seen no reduction in cycles, but  $G - e$  is predicted to have at least  $(k - n + 1) - 1$  cycles, which it does by our assumption. Therefore, we have proven the inductive hypothesis for both cases.

### 2.1.12

Compute the diameter and radius of  $K_{m,n}$ .

The diameter of  $K_{m,n}$  is equal to 2 — for vertices in the same independent set, it requires two edges to traverse between them.

The radius of  $K_{m,n}$  is also 2 — the eccentricity of every vertex in  $K_{m,n}$  is 2, so the radius must also be 2.

### 2.1.13

Prove that every graph with diameter  $d$  has an independent set with at least  $\lceil \frac{1+d}{2} \rceil$  vertices.

Let  $G$  be a graph with diameter  $d$ , and let  $u \in V(G)$  be a vertex with eccentricity  $d$ . Let  $P$  be a maximal  $u, v$  path of length  $d$ . Then,  $P$  has  $d + 1$  vertices. So,  $P$  has a maximal independent set containing every other vertex, with total cardinality of  $\lceil \frac{d+1}{2} \rceil$ . Therefore,  $G$  has an independent set with at least  $\lceil \frac{d+1}{2} \rceil$  vertices.