1.3.3

Let u and v be adjacent vertices in G. Prove that uv belongs to at least $d(u) + d(v) - n(G)$ triangles in G.

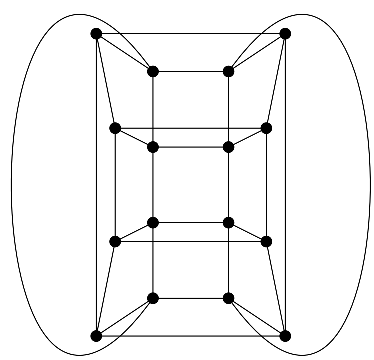
Solution

Let $u \leftrightarrow v \in G$. In order for uv to be in a triangle in G, u and v must share a common neighbor. By using the property of inclusion and exclusion, we can find the set as follows:

$$\begin{aligned}|N(u) \cup N(v)| &= |N(u)| + |N(v)| - |N(u) \cap N(v)| \\ |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &= d(u) + d(v) - n(G)\end{aligned}$$

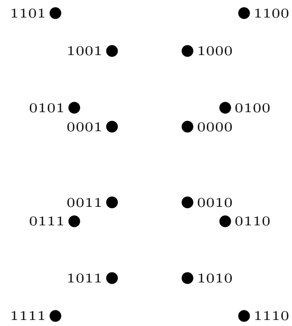
1.3.4

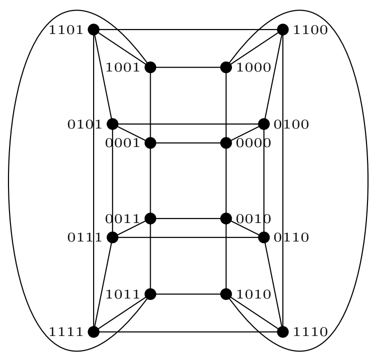
Prove that the graph below is isomorphic to Q_4.



Solution

We can assign tuples to the graph as follows:





1.3.6

Given graphs G and H, determine the number of components and maximum degree in $G + H$ in terms of the parameters for G and H.

Solution

We can find the number of components in $G + H$ by summing the number of components in G and the number of components in H.

The maximum degree in $G + H$ is equal to $\max\{\Delta(G), \Delta(H)\}$.

1.3.7

Determine the maximum number of edges in a bipartite subgraph of P_n, C_n, and K_n.

Solution

For the graph P_n, we will create a bipartition by starting at an endpoint of the path and alternating vertices in the sets A and B. This is a bipartition since a path does not include any repeated vertices or edges, so A and B are independent sets. Therefore, the maximum number of edges in a bipartite subgraph of P_n is the number of edges in P_n, which is $n-1$.

For C_n, we have two values of the maximum number of edges in a bipartite subgraph of C_n:

- If n is even, then C_n is a bipartite graph already, meaning that the maximum number of edges in a bipartite subgraph of C_n is the number of edges in C_n, which is n.
- If n is odd, then C_n is not a bipartite graph. After one edge deletion, we get that $C_n - e = P_n$, which is bipartite, so the maximum number of edges in a bipartite subgraph of C_n is the number of edges in P_n, which is $n-1$.

For the graph K_n, there are two options for the maximum number of edges in a bipartite subgraph depending on the value of n:

- If n is even, then the subgraph $K_{\frac{n}{2},\frac{n}{2}}$ is the maximal bipartite subgraph, meaning that the number of edges is equal to $n^2/4$. We know that $K_{\frac{n}{2},\frac{n}{2}}$ is a subgraph of K_n because the vertex set is the same, and K_n is complete, so any subset of edges is a subset of the edge set of K_n.
- If n is odd, then the subgraph $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ is the maximal bipartite subgraph, because $\lfloor\frac{n}{2}\rfloor + \lceil\frac{n}{2}\rceil = n$ and K_n is complete. Therefore, the total number of edges is $\lfloor\frac{n^2}{4}\rfloor$.

1.3.26 (a)

Count the 6-cycles in Q_3.

Solution

In order to find a 6-cycle in Q_3, we select two vertices to delete and see if we can find a cycle from the graph $Q_3 - \{u, v\}$. Vertices are either adjacent, distance 2 (antipodal on the same face), or antipodal (distance 3) with each other.

ADJACENT: If two vertices are adjacent, we can find one cycle from the remaining vertices after deletion. There are (8)(3)/2 sets of adjacent vertices, for a total of 12 from this selection.

ANTIPODAL ON THE SAME FACE: If two vertices are antipodal, we cannot find a cycle from the remaining graph after deletion.

ANTIPODAL: If two vertices are antipodal, we can find one cycle from the remaining vertices after deletion. There are 4 sets of antipodal vertices.

We find a total of 16 6-cycles in Q_3.