## 2.1.22

Let $T$ be an $n$-vertex tree with one vertex of each degree $2 \leq i \leq k$; the remaining $n-k+1$ vertices are leaves. Determine $n$ in terms of $k$.

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We will find the number of vertices in $T$ by finding the number of edges in $T$ and adding 1. For $2 \leq i \leq k$ corresponding to each of the non-leaf vertices, summation yields $\frac{k(k+1)}{2} - 1$ edges. However, this scheme double-counts each edge, so we have to subtract the $k-2$ edges connecting the $k-1$ non-leaf vertices, yielding $\frac{k(k+1)}{2} - k + 1$ edges. Finally, because $T$ is a tree, we get that $T$ has $\frac{k(k+1)}{2} - k + 2$ vertices.

## 2.1.27

Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \dots, d_n$ if and only if $\sum d_i = 2n-2$.

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($\Rightarrow$) Suppose that for some tree $T$, $d_1, \dots, d_n$ are the degrees of the vertices of the tree. Since $T$ is a tree, this means $e(G) = n-1$, and $\sum d_i = 2e(G)$, meaning $\sum d_i = 2(n-1) = 2n-2$.

($\Leftarrow$) Suppose that $\sum d_i = 2n-2$ for $d_1, \dots, d_n$ corresponding to the degrees of the vertices in $G$. By a previous result, we know that $\sum d_i = 2e(G)$, meaning that $\sum d_i = 2(n-1)$, implying that $e(G) = n-1$. We can find a tree $G$ with $n-1$ edges by letting $G$ be connected with $n-1$ edges.

## 2.1.33

Let $G$ be a connected $n$-vertex graph. Prove that $G$ has exactly one cycle if and only if $G$ has exactly $n$ edges.

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($\Rightarrow$) Let $G$ be a connected $n$-vertex graph with exactly one cycle. If we delete an edge from this cycle, then $G-e$ is acyclic, as well as connected (since $e$ is not a cut-edge), so $G-e$ has $n-1$ edges. Adding back $e$, we get that $G$ has $n$ edges.

($\Leftarrow$) Let $G$ be a connected $n$-vertex graph with $n$ edges. Then, $G$ contains a spanning tree that contains all $n$ vertices. Therefore, $T \subseteq G$ contains $n-1$ edges. By adding another edge, we get that $e(G) = e(T) + 1 = n - 1 + 1$. Thus, $G$ has exactly one cycle.

## 2.1.34

Let $T$ be a tree with $k$ edges, and let $G$ be a $n$-vertex simple graph with more than $n(k-1) - \binom{k}{2}$ edges. Use Proposition 2.1.8 to prove that $T \subseteq G$ if $n > k$.

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We will use induction to prove that $T \subseteq G$ as follows:

Base Case Suppose $n = k+1$. Then, we can find the following:

$$\begin{aligned}
e(G) &> (k+1)(k-1) - \binom{k}{2}\\
e(G) &> (k^2-1) - \frac{k(k-1)}{2}\\
e(G) &> \frac{k^2-1}{2} + \frac{k^2-1-(k^2-k)}{2}\\
e(G) &> \frac{k^2+k}{2} - 1\\
e(G) &> \frac{k(k+1)}{2} - 1
\end{aligned}$$

This means $e(G) = \frac{k(k+1)}{2}$ in the base case, so $G$ is the complete graph on $k+1$ vertices, so $\delta(G) = k$, meaning $T \subseteq G$.

Inductive Hypothesis If $n > k+1$, $e(G) > n(k-1) - \binom{k}{2}$, and $\delta(G) < k$, then we have that $e(G-x) > (n-1)(k-1) - \binom{k}{2}$, where $d(x) = \delta(G)$.

Proof Let $G' = G - x$, where $\delta(G) \leq k-1$ and $d(x) = \delta(G)$. Then, we have the following:

$$\begin{aligned}
e(G') &= e(G) - \delta(G)\\
e(G') &\geq e(G) - (k-1)\\
e(G') &> n(k-1) - \binom{k}{2} - (k-1)\\
e(G') &> (n-1)(k-1) - \binom{k}{2}
\end{aligned}$$

Therefore, the inductive hypothesis is proven, and we have shown the formula to be correct.