Functions of One Random Variable.

We'll begin our exploration of the distribution of function of random variables, by focusing on simple functions of one random variable. For example, if X is a continuous random variable, and we take a function of X, say:

$$Y = u(X).$$

then Y is also a continuous random variable that has its own probability distribution.

We'll learn how to find the probability density function of Y, using two different techniques, namely, the distribution function technique and the change-of-variable technique.

Upon completion of this lesson, one should be able to:

- To learn how to use the distribution function technique to find the probability function $Y = u(X)$, a one-to-one transformation of a random variable X.
- To learn how to ~~change~~ use the change-of-variable technique to find

the probability distribution of $Y = u(X)$ a one-to-one transformation of a random variable X.

- To learn how to use the change of variable technique to find the probability distribution of $Y = u(X)$ a two-to-one transformation of a random variable X.

Distribution Function Technique

$$Z = \frac{X - \mu}{\sigma}$$

follows a standard normal distribution when X is normally distributed with mean μ & standard deviation σ.

~~And we used the distribution~~

Use distribution function technique to find the p.d.f of the random variable $Y = u(X)$ by:

1. first, finding the cumulative distribution function:

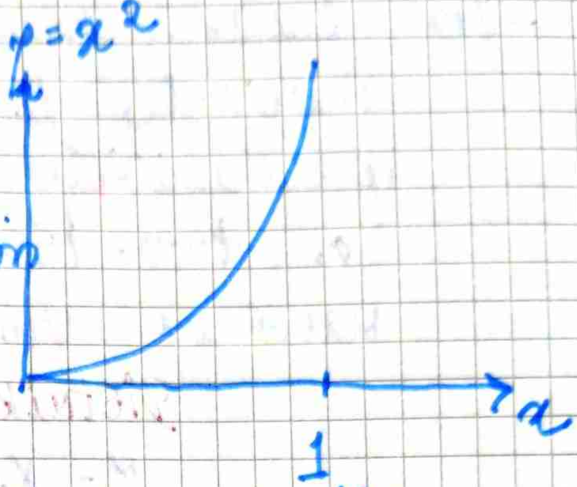
$$F_Y(y) = P(Y \leq y)$$

2. Then, differentiating distribution function $F(y)$ to get the probability density function $f(y)$. That is

$$f_Y(y) = F_Y'(y).$$

Example : Let X be a continuous random variable with the following probability density function

$$f(x) = 3x^2$$



for $0<x<1$. What is the probability density function $Y = X^2$?

Solution : If one looks close at the graph of the function of $Y = X^2$ one might note that (1) the function is an increasing function of x (2) $0 \le Y \le 1$.

That noted, let's now use the distribution function technique to find the p.d.f of Y.

First, we find the cumulative distribution function of Y

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le y^{1/2}) = F_X(y^{1/2})$$

$$= \int_0^{y^{1/2}} 3t^2\,dt \Rightarrow \left[t^3\right]_{t=0}^{t=y^{1/2}} \Rightarrow y^{3/2},\ 0<y<1$$

Having shown that the cumulative distribution function of Y is

$$F_Y(y) = y^{\frac{3}{2}}$$

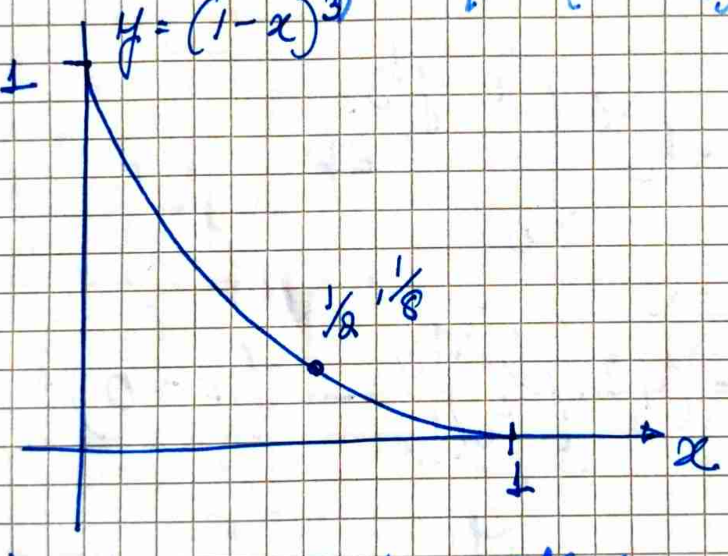
for $0 \le y < 1$, we now just need to differentiate $F(y)$ to get the probability density function $f(y)$.

$$f_Y(y) = F_Y'(y) = \frac{3}{2} y^{1/2} \quad \text{for } 0 < y < 1.$$

example: Let X be a continuous random variable with the following probability density function:

$$f(x) = 3(1-x)^2 \quad \text{for } 0 < x < 1.$$

What is the probability density function of $Y = (1-X)^3$?



solution: If we look at the graph above $Y = (1-X)^3$, we might note that the function is a decreasing function of X, $0 < y < 1$. Let's use the distribution function technique to find the p.d.f of Y.

1. first, we find the cumulative distribution function of Y.

$F_Y(y) = P(Y \le y) = P((1-x^3) \le y)$

$= P(1-x \le y^{1/3})$

$= P(-x \le y^{1/2} - 1)$

$\Rightarrow P(X \ge 1 - y^{1/3})$

$\Rightarrow 1 - P_X(1-y^{1/3})$

$\Rightarrow 1 - \int_0^{1-y^{1/3}} 3(1-t)^2 dt$

$\Rightarrow 1 + (1-t)^3 \Big]_0^{1-y^{1/3}} \Rightarrow$

$\Rightarrow 1 + \left[[1-(1-y^{1/3})]^3 - (1-0)^3 \right]$

$\Rightarrow 1 + [+y - 1]$

$= y$

having shown that the cumulative distribution function of Y is

$F_Y(y) = y$, for $0 \le y \le 1$.

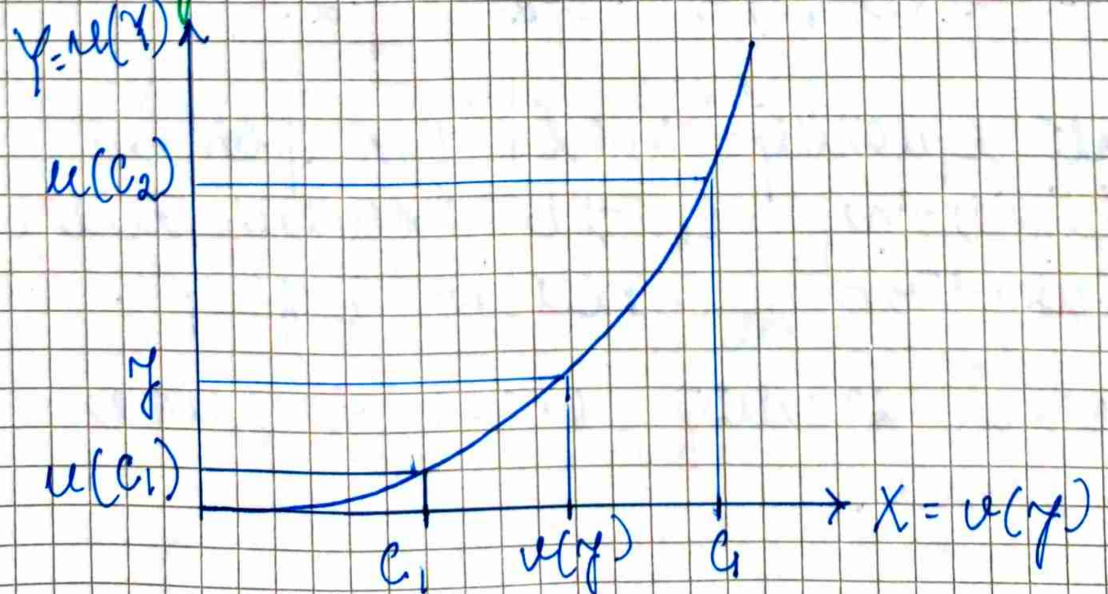
We now just need to differentiate $F(y)$ to get the probability density function $f(y)$. Doing so, we get

$$f_Y(y) = F_Y'(y) = 1 \quad \text{for } 0<y<1.$$

Change-of-Variable Technique.

In the previous section, we used the distribution function technique in two different examples. In the first example, the transformation of X involve an increasing function while in the second example, the transformation of X involved a decreasing function. Now, we will generalize what we did for an increasing function & then for a decreasing function. The generalizations lead to what is called the change-of-variable technique.

Generalization for an increasing funcⁿ



The curve, represents the continuous and increasing function $Y = u(X)$. If you put an x-value, such as c_1 and c_2 into the function $Y = u(x)$, you get a value, such as $u(c_1)$ & $u(c_2)$.

But because the function is continuous and increasing, an inverse function $X = v(y)$ exists. In that case, if you put into the function $X = v(y)$ you get an x-value such as $v(y)$.

Deriving, distribution of Y:

$$F_Y(y) \stackrel{1}{=} P(Y \le y) =$$

$$\stackrel{2}{=} P(u(X) \le y)$$

$$\stackrel{3}{=} P(X \le v(y))$$

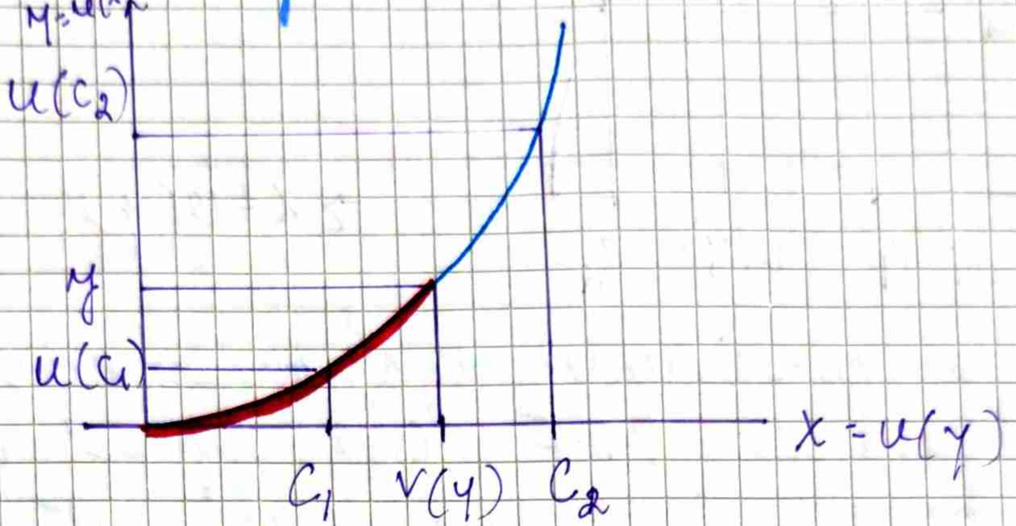
$$\stackrel{4}{=} \int_{c_1}^{v(y)} f(x)\,dx.$$

for $d_1 = u(c_1) < y < u(c_2) = d_2$.

The first equality holds for from the definition of the cumulative distribution function of Y.

The second equality holds because $Y = u(X)$.

The third equality holds because, as shown in red, on the following graph for the portion for which $u(x) \leq y$, it is also true $X \leq v(y)$.



And the last equality holds from the definition of probability for a continuous random variable X.

Now, we just have to take the derivative of $F_Y(y)$, the cumulative distribution function of Y, to get $f_Y(y)$, the probability density function of Y.

Fundamental Theorem of Calculus

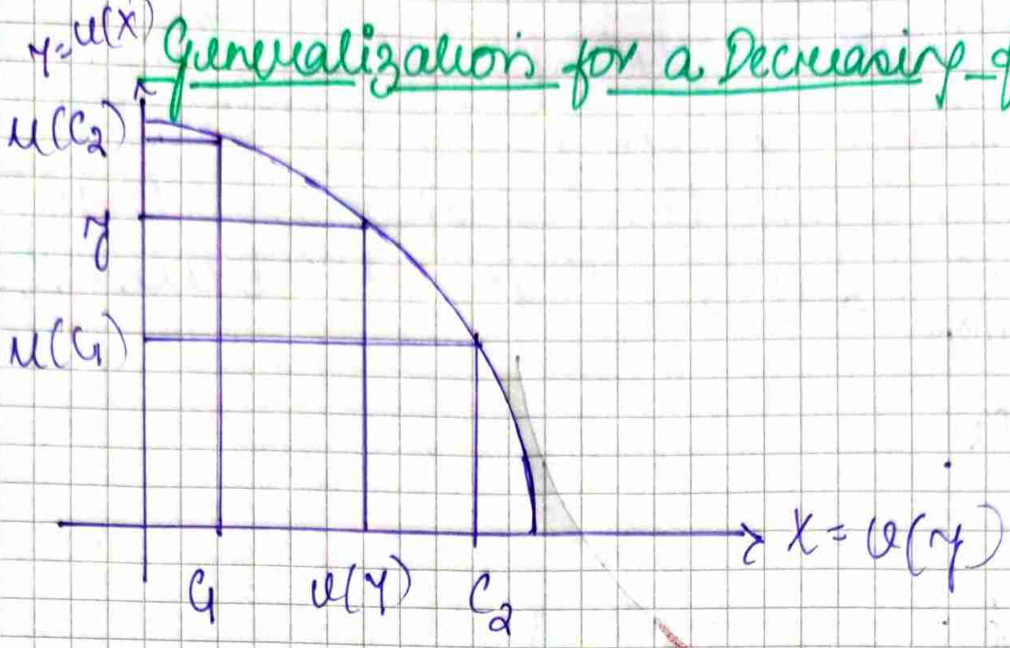
$$A(x+h) - A(x) \approx f(x) \cdot h$$

The fundamental theorem of calculus, in conjuction with the chain rule

$$f_Y(y) = F_Y'(y) = f_X(v(y))\, v'(y)$$

for $d_1 = u(c_1) < y < u(c_2) = d_2$.

Generalization for a Decreasing function



Let X be a continuous random variable with a generic P.D.F $f(x)$ defined over the support $c_1 \le x \le c_2$. And, let $Y = u(X)$ be a continuous decreasing function of X with inverse function $X = v(Y)$.

The curve, represents the continuous and decreasing function $y = u(x)$. Again, if you put an x-value, such as c_1 & c_2, into the function $Y = u(X)$, you get a y-value $u(c_1)$ & $u(c_2)$. But, because the function is continuous & decreasing an inverse function $X = v(Y)$ exists.

Distribution of Y is then:

$$F_Y(y) = P(Y \le y) = P(u(X) \le y)$$

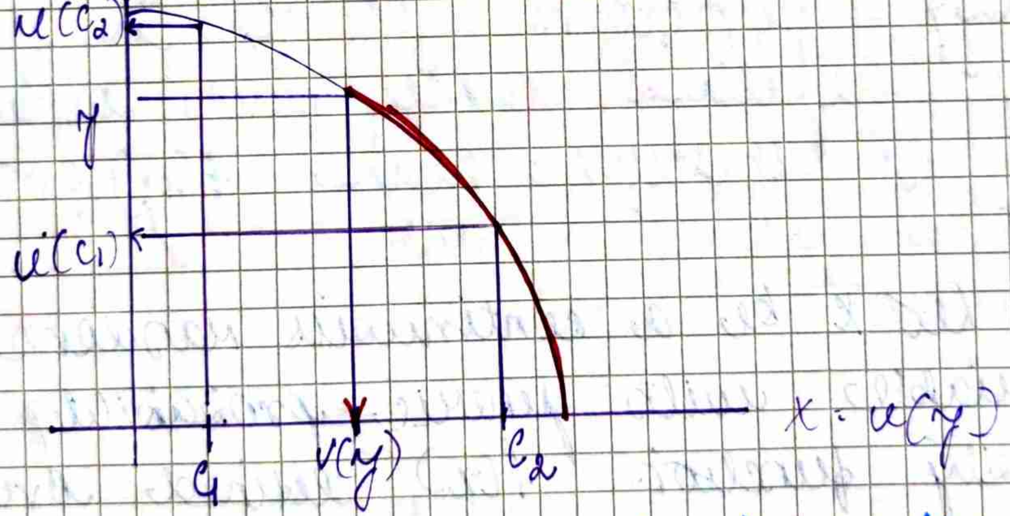
$$= P(X \ge v(y))$$

$$= P(X \geq v(y))$$

$$= 1 - P(X \leq v(y))$$

$$= 1 - \int_{c_1}^{v(y)} f(x)\,dx. \quad \text{for } d_2 = u(c_2) < y < u(c_1) = d_1.$$

The first equality holds from the definition of the cumulative distribution function of Y. The second equality holds because $Y = u(X)$.
The third equality holds, because as shown in red on the following graph for the portion of the function for which $u(X) \leq y$, it is also true that $X \geq v(y)$.



The fourth equality holds from the rule of complementary events. And, the last equality holds from the definition of probability for a continuous random variable. Now we just have to take the derivative of

$F_Y(y)$.

The cumulative distribution function of Y, to get $f_Y(y)$, the probability density function of Y.

Again, the fundamental theorem of calculus in conjunction with the chain rule, tells us the derivative is:

$$f_Y(y) = F_Y'(y) = -f_X(v(y))\, v'(y).$$

for $d_2 = u(c_2) < y < u(c_1) = d_1$.

The p.d.f $f(y)$ is negative, but note that the derivative of $v(y)$ is negative, as $X = v(Y)$ is a decreasing function of Y. Therefore, the two negatives cancel each other out, & therefore make $f(y)$ positive.

Definition: Let X be a continuous random variable with generic probability density function $f(x)$ defined over the support $c_1 < x < c_2$.

And, let $Y = u(X)$ be an invertible function of X with inverse function $X = v(Y)$. Then, using the <u>change-of-variable technique</u>

The probability density function of Y is:

$$f_Y(y) = f_X(v(y)) \times |v'(y)|$$

defined over the support $u(c_1) < y < u(c_2)$.

example

Let's return to our example in which X is a continuous random variable with the following probability density function:

$$f(x) = 3x^2 \quad \text{for } 0 < x < 1.$$

Use the change-of-variable to find the probability density function of $Y = X^2$.

Soln: Note that the function

$$Y = X^2$$

defined over the interval $0 < x < 1$ is an invertible function. The inverse function is:

$$x = v(y) = \sqrt{y} = y^{\frac{1}{2}}$$

for $0 < y < 1$. Now take the derivative of $v(y)$, we get

$$v'(y) = \frac{1}{2} y^{-\frac{1}{2}}$$

Therefore, the change-of-variable technique:

$$f_Y(y) = f_X(v(y)) \times |v'(y)|$$

$$\Rightarrow 3\left[y^{\frac{1}{2}}\right]^2 \cdot \frac{1}{2} y^{-\frac{1}{2}}$$

$$\Rightarrow \frac{3}{2} y^{\frac{1}{2}}. \qquad 0 < y < 1.$$

example: Let's return to our example in which X is a continuous random variable with the following probability density function:

$$f(x) = 3(1-x)^2 \quad \text{for } 0 < x < 1.$$

Use the change-of-variable technique to find the probability density function $Y = (1-X)^3$.

soln: Note that the function:

$$Y = (1-X)^3$$

defined on the interval $0 < x < 1$ is an invertible function. The inverse function is:

$$x = v(y) = 1 - y^{\frac{1}{3}}. \quad \text{for } 0 < y < 1.$$

(That range is because when $x = 0, y = 1$. & when $x = 1, y = 0$).

Now, taking the derivative of $v(y)$ we get:

$$v'(y) = \frac{1}{3} y^{-2/3}.$$

Therefore, the change-of-variable technique

$$f_Y(y) = f_X(v(y)) \times |v'(y)|$$

$$= 3\left[1-(1-y^{1/3})\right]^{2} \times \frac{1}{3} y^{-2/3}$$

$$\Rightarrow \cancel{y}\, 3 \cdot y^{2/3} \cdot \frac{1}{3} y^{-2/3} = 1.$$

$$f_Y(y) = 1. \quad \text{for} \quad 0 < y < 1.$$

Transformations of Two Random Variables

In this lesson, we consider the situation where we have two random variables and we are interested in the joint distribution of two new random variables which are a transformation of the original one.

Such a transformation is called a Bivariate transformation.

We use a generalization of the change of variables techniques which we learnt in previous section.

Recall, that for the univariate (one random variable) situation. Given X with the pdf $f(x)$ and the transformation $Y = u(X)$ with the single-valued inverse $X = v(Y)$, then the PDF of Y is given by

$$g(y) = |v'(y)| \, f[v(y)]$$

Now suppose (X_1, X_2) has joint density $f(x_1, x_2)$ and support S_x.

Let (Y_1, Y_2) be some function of (X_1, X_2) defined by $Y_1 = u_1(X_1, X_2)$ and

$$Y_2 = u_2(X_1, X_2)$$

with the single valued inverse given by

$$X_1 = v_1(Y_1, Y_2)$$

$$X_2 = v_2(Y_1, Y_2).$$

Let S_Y be the support of Y_1, Y_2.

Then, we usually find S_Y by considering the image of S_x under transformation (Y_1, Y_2). Say given $x_1, x_2 \in S_X$, we can find $(y_1, y_2) \in S_Y$ by

$$~~x_1 \in~~ \quad x_1 = v_1(y_1, y_2), \quad x_2 = v_2(y_1, y_2)$$

The joint PDF of Y_1 & Y_2 is

$$\boxed{g(y_1, y_2) = |J|\, f\left[u_1(y_1, y_2), u_2(y_1, y_2)\right]}.$$

In the above expression $|J|$ refers to the absolute value of the Jacobian J. The Jacobian J is given by:

$$\begin{vmatrix} \dfrac{\partial u_1(y_1, y_2)}{\partial y_1} & \dfrac{\partial u_1(y_1, y_2)}{\partial y_2} \\ \dfrac{\partial u_2(y_1, y_2)}{\partial y_1} & \dfrac{\partial u_2(y_1, y_2)}{\partial y_2} \end{vmatrix}$$

i.e it is the determinant of the matrix

$$\begin{pmatrix} \partial u_1(y_1, y_2)/\partial y_1 & \partial u_1(y_1, y_2)/\partial y_2 \\ \partial u_2(y_1, y_2)/\partial y_1 & \partial u_2(y_1, y_2)/\partial y_2 \end{pmatrix}$$

example: Suppose X_1 & X_2 are independent exponential random variables with parameter $\alpha = 1$, so that

$$f_{X_1}(x_1) = e^{-x_1} \quad 0 < x_1 < \alpha$$

$$f_{X_2}(x_2) = e^{-x_2} \quad 0 < x_2 < \alpha.$$

The joint PDF is given by

$$f(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2) = e^{-x_1 - x_2}$$

$$0 < x_1 < \alpha, \quad 0 < x_2 < \alpha.$$

Consider the transformation

$$Y_1 = X_1 - X_2, \quad Y_2 = X_1 + X_2.$$

- We wish to find the joint distribution of Y_1 & Y_2.

We have:

$$x_1 - x_2 + x_1 + x_2 = Y_1 + Y_2$$

$$2x_1 = Y_1 + Y_2$$

$$x_1 = \frac{y_1 + y_2}{2}$$

$$x_1 - x_2 - x_1 - x_2 = y_1 - y_2$$

$$-2x_2 = y_1 - y_2$$

$$x_2 = \frac{y_2 - y_1}{2}$$

or

$$v_1(y_1, y_2) = \frac{y_1 + y_2}{2}, \quad v_2(y_1, y_2) = \frac{y_2 - y_1}{2}$$

The Jacobian $|J|$ is:

$$\begin{vmatrix} \frac{\partial\left(\frac{y_1+y_2}{2}\right)}{\partial y_1} & \frac{\partial\left(\frac{y_1+y_2}{2}\right)}{\partial y_2} \\ \frac{\partial\left(\frac{y_2-y_1}{2}\right)}{\partial y_1} & \frac{\partial (y_2-y_1)/2}{\partial y_2} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} \Rightarrow \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$g(y_1, y_2) = e^{-v_1(y_1, y_2) - v_2(y_1, y_2)} |J|$$

$$= \frac{1}{2} e^{-\left[\frac{y_1+y_2}{2}\right] - \left[\frac{y_2-y_1}{2}\right]}$$

$$= \frac{1}{2} e^{-\left[\frac{y_1+y_2+y_2-y_1}{2}\right]}$$

$$\Rightarrow \frac{1}{2} e^{-\frac{2y_2}{2}}$$

$$= \frac{1}{2} e^{-y_2}$$

Now, we determine the support of (Y_1, Y_2). Since $0 \le x_1 < \infty$, $0 \le x_2 < \infty$.

we have $0 \le \frac{y_1 + y_2}{2} < \infty$, $0 \le \frac{y_2 - y_1}{2} < \infty$

or $0 \le y_1 + y_2 < \infty$, $0 \le y_2 - y_1 < \infty$.

This may be rewritten as

$-\underline{\underline{y_2}} < y_1 < y_2$, $0 \le y_2 < \infty$

$\cancel{0 \le y_1 + y_2 - y_2}$

Using, the joint PDF, we may find the Marginal PDF of Y_2 as

$$g(y_2) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_1$$

$$\Rightarrow \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2}\, dy_1$$

$$= \frac{1}{2} \left[e^{-y_2}\, y_1 \right]_{y_1 = -y_2}^{y_1 = y_2}$$

$$= \frac{1}{2} e^{-y_2} \left[y_2 + y_2 \right]$$

$$= y_2 e^{-y_2} \quad 0 < y_2 < \infty.$$