Conditional Distributions

Source

Here , we will investigate how one of the random variables, **say Y , behaves given that another random variable, say X , has already behaved in a certain way**.

For example , we might want to know the probability that Y, the number of car accidents in July on a particular curve in the road, equals 2 given that X, the number of cars in June caught speeding on the curve is more than 50. Ie. **Conditional probability distribution** of a random variable Y given another random variable X.

What is a conditional Distribution?

Let us start our investigation of conditional distributions by using an example to help enlighted us about the **distinction between a joint(bivariate) probability distribution and a conditional probability distribution**

Example

A safety Offier for an auto insurance company in Connecticut was interested in learning how the extent of an individual's injury in an automobile accident relates to the type of safety restraint the individual was wearing at the time of the accident. As a result , the Safety Officer used statewide ambulance and police records to compile the following two-way table of joint probabilities.

f (x,y)	Type of Restraint *(Y)*			
Extent of Injury *(X)*	None (0)	Belt Only (1)	Belt and Harness (2)	$f_X(x)$
None (0)	0.065	0.075	0.06	0.20
Minor (1)	0.175	0.16	0.115	0.45
Major (2)	0.135	0.10	0.065	0.30
Death (3)	0.025	0.015	0.01	0.05
$f_Y(y)$	0.40	0.35	0.25	1.00

The Safety Officer created the random variable X, the extent of injury, by arbitrarily assigning values $0, 1, 2$ and 3 to each of the possible outcome. Similarly, the Safety Officer created the random variable Y, the type of restrain , by arbitrarily assigning values $0, 1$ and 2 to each possible outcomes.

Among other things, the Safety Officer was interested in answering the following questions:

- What is the probability that a randomly selected person in an automobile accident was wearing a seat belt and has only a minor injury ?
- If a randomly selected person wears no restrains, what is the probability of death?
- If a randomly selected person sustains no injury, what is the probability the person was wearing a belt and harness?

Definition

A **joint(Bivariate) probability distribution** describes the probability that a randomly selected person from the *population* has the *two characteristics* of interest.

Solution to the 1^{st} question

Let

A = the event that a randomly selected person in a car accident has a minor injury.

B = the event that the randomly selected person was wearing only a seat belt.

From the table we get

$$P(A \text{ and } B) = P(X = 1, Y = 1) = f(1, 1) = 0.16$$

That is, there is a 16% chance that a randomly selected person in an accident is wearing a seat belt and has only a minor injury.

Definition

Conditional Probability Distribution

A **conditional probability distribution** is a probability distribution for a sub-population. That is, a conditional probability distribution describes the probability that a randomly selected person from a sub-population has the *one characteristic of interest*

Solution to the 2^{nd} question

Here, we need to use the definition of conditional probability to calculate the desired probability.

Dissecting the Safety Officer's questions into two parts by identifying the subpopulation and the characteristic of interest. The **subpopulation** is the population of people wearing no restrains , and the **characteristic of interest** is death. Then, using the definition of conditional probability, we determine that the desrired probability is:

$$\begin{aligned} P(D|NR) &= \frac{P(D \bigcap NR)}{P(NR)} \\ &= \frac{P(X=3, Y=0)}{P(Y=0)} \\ &= \frac{f(3,0)}{f_Y(0)} \\ &= \frac{0.025}{0.40} \\ &= 0.0625 \end{aligned}$$

That is, there is 6.25% change of death of a random selected person in an automobile accident, if the person wears no restraint.

Here, we are simply trying to get the feel of how a conditional probability distribution describes the probability that a randomly selected person from a *sub-population* has the *one characteristic of interest*.

Definition

Conditional probability mass function of X

The conditional probability mass function of X, given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} \text{ provided } f_Y(y) > 0$$

Similarly,

The conditional probability mass function of Y, given that $X = x$, is defined by

$$g(x|y) = \frac{f(x,y)}{f_X(x)} \text{ provided } f_X(x) > 0$$

Example

Let X be a discrete random variable with support $S_1 = \{0, 1\}$ and let Y be a discrete random variable with support $S_2 = \{0, 1, 2\}$. Suppose, in tabular form, that X and Y have the following **joint probability distribution** $f(x, y)$:

f (x,y)		Y		$f_X(x)$
X	0	1	2	
0	1/8	2/8	1/8	4/8
1	2/8	1/8	1/8	4/8
$f_Y(y)$	3/8	3/8	2/8	1

What is the conditional distribution of X given Y? That is what is $g(x|y)$?

Solution : Using the formula $g(x|y) = \frac{f(x,y)}{f_Y(y)}$, the conditional distribution of X given Y is :

$g(x\|y)$	Y		
X	0	1	2
0	1/3	2/3	1/2
1	2/3	1/3	1/2
$f_Y(y)$	1	1	1

For example :

$g(0|0) = \frac{f(0,0)}{f_Y(0)} = \frac{1/8}{3/8} = \frac{1}{3}$

What is the conditional distribution of Y given X ? That is , what is $h(y|x)$

Solution

Using the formual $h(x|y) = \frac{f(x,y)}{f_X(x)}$

$h(y\|x)$		Y			
		0	1	2	
X	0	1/4	2/4	1/4	1
	1	2/4	1/4	1/4	1

For example , for $x = 0$ and $y = 0$

$h(0|0) = \frac{f(0,0)}{f_X(0)} = \frac{1/8}{4/8} = \frac{1}{4}$

So, we've used the definition to find the conditional distribution of X given Y, as well as the conditional distribution of Y given X. We should now have enough experience with conditional distributions to believe that the following statements are true:

1. Conditional distributions are valid probability mass functions in their own right. That is, the conditional probabilities are between 0 and 1 , inclusive

$$0 \leq g(x|y) \leq 1 \text{ and } 0 \leq (y|x) \leq 1$$

For each sub-population, the conditional probabilities sum to 1:

$$\sum_x g(x|y) = 1 \text{ and } \sum_y h(y|x) = 1$$

2. In general, the conditional distributions of X given Y does not equal the conditional distribution of Y given X. That is

$$g(x|y) \neq h(y|x)$$

Conditional Means and Variances

Definition

Suppose X and Y are discrete random variables. Then, the **conditional mean of Y given** $X = x$ is defined as :

$$\mu_{Y|X} = E[Y|x] = \sum_y yh(y|x)$$

And, the **conditional mean of X given** $Y = y$ is defined as

$$\mu_{X|Y} = E[X|y] = \sum_x xg(x|h)$$

The **conditional variance of Y given** $X = x$ is

$$\sigma^2_{Y|x} = E\{[Y - \mu_{Y|x}]^2|x\} = \sum_y [y - \mu_{Y|x}]h(y|x)$$

or, alternatively , using the usual shortcut

$$\sigma^2_{Y|x} = E[Y^2|x] - \mu^2_{Y|x} = \left[\sum_y y^2 h(y|x)\right] - \mu^2_{Y|x}$$

The **conditional variance of X given** $Y = y$ is

$$\sigma^2_{X|y} = E\{[X - \mu_{X|y}]^2|y\} = \sum_x [x - \mu_{X|y}]g(x|y)$$

or, alternatively , using the usual shortcut

$$\sigma^2_{X|y} = E[X^2|y] - \mu^2_{X|y} = \left[\sum_x X^2 g(x|y)\right] - \mu^2_{X|y}$$

Example

let X be a discrete random variable with support $S_1 = \{0, 1\}$ and let Y be a discrete random variable with support $S_2 = \{0, 1, 2\}$. Suppose, in tabular form, that X and Y have the following joint probability distribution $f(x, y)$:

f (x,y) X	Y 0	1	2	$f_X(x)$
0	$^1/_8$	$^2/_8$	$^1/_8$	$^4/_8$
1	$^2/_8$	$^1/_8$	$^1/_8$	$^4/_8$
$f_Y(y)$	$^3/_8$	$^3/_8$	$^2/_8$	1

What is the conditional mean of Y given $X = x$?

Solution

We previously determined that the conditional distribution of Y given X is

$h(y\|x)$		Y 0	1	2	
X	0	$^1/_4$	$^2/_4$	$^1/_4$	1
	1	$^2/_4$	$^1/_4$	$^1/_4$	1

$$\mu_{Y|0} = E[Y|0] = \sum_y yh(y|x) = 0(\tfrac{1}{4}) + 1(\tfrac{2}{4}) + 2(\tfrac{1}{4}) = 1$$
$$\mu_{Y|1} = E[Y|1] = \sum_y yh(y|x) = 0(\tfrac{2}{4}) + 1(\tfrac{1}{4}) + 2(\tfrac{1}{4}) = \tfrac{3}{4}$$

Note that the conditional mean of $Y|X = x$ depends on x , and depends on x alone. One might want to think about these conditional means in terms of sub-populations again. The mean of Y is likely to depend on the sub-population. The mean of Y is 1 for the $X = 0$ sub-population ,and the mean of Y is $\frac{3}{4}$ for the $X = 1$ sub-population.

Rather than calculating the average weight of an adult , for example one would probably want to calculate the average weight for the sub-population of females and the average weight for the sub-population of males, because the average weight no doubt depends on the sub-population.

What is the conditional variance of Y given $X = 0$

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\begin{aligned}
\sigma^2_{Y|0} &= E\{[Y - \mu_{Y|0}]^2 |x\} \\
&= E\{[Y - 1]^2 |0\} = \sum_y (y-1)^2 h(y|x) \\
&= (0-1)^2(\frac{1}{4}) + (1-1)^2(\frac{2}{4}) + (2-1)^2(\frac{1}{4}) \\
&= (\frac{1}{4}) + 0 + (\frac{1}{4}) \\
&= \frac{2}{4}
\end{aligned}
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