Effecient Algorithms for Present and Future Values

The PV of the cash flows $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \ldots + \frac{C_n}{(1+y)^n}$$

C_t are the casdh flows, y is the interest rate, and n is the term of the investment. we can easily verif that the variable d is equal to $(1+y)^i$ at the beginning of the for loop. As a result, the variable x becomes the partial sum $\sum_{t=1}^{i} C_t(1+y)^{-t}$ at the end of each loop.

$\text{Algorithm for evaluating present value:}$

$\text{input}: y, n, C_t(1 \leq t \leq n);$
real: x, d;
$x := 0;$
$d := 1 + y$;
for $(i = 1 \text{ to } n)$
{
 $x := x + (C_i/d)$;
 $d := d \times (1+y)$
}
return x;

Double-click (or enter) to edit

```
!pyppeteer-install
```

```
/bin/bash: pyppeteer-install: command not found
```

```
def PV(y,n,C):
  x = 0
  d = 1+y
  for i in range(0,n):
    x = x + (C/d)
    d = d * (1+y)
  return x
```

Conversion between Compounding Methods

We can compare interest rates with different compounding methods by converting one into the other. Suppose that r_1 is the annual rate with continuous compounding and r_2 is the equivalent rate compounded m times per annum. Then $[1 + (r_2/m)]^m = e^{r_1}$

Therefore,

$$r_1 = mln(1 + \frac{r_2}{m})$$
$$r_2 = m(e^{r_1/m} - 1)$$

Example : Consider an interest rate of 10 with quarterly compounding. The equivalent rate with continuous compounding is $4 \times ln(1 + \frac{0.1}{4}) = 0.09877$ or 9.877%

Simple Compounding

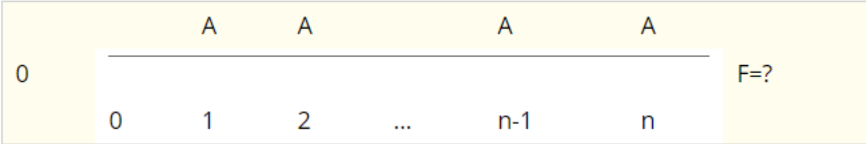
Besides periodic compounding and continuous compounding (hence compound interest), there is a diffferent scheme for computing interest called **simple compounding** (hence simple interest). Under this scheme, interest is computed on the original principal. Suppose that P dollars is borrwed at an annual rate of r. The simple interest each year is Pr.

Annuity

An annuity is a series of equal payments made at equal intervals of time. Financial activities like installment payments, monthly rentals, life-insurance premiums, monthly retirement benefits, are familiar examples of annuity.

Annuity can be certain or uncertain. In $annuity\ certain$, the specific amount of payments are set to being and end at a specific length of time. A good example of annuity certain is the monthly payments of a car loan where the amount and number of payments are known. In $annuity\ uncertain$, the annuitant may be paid according to certain event. Example of annuity uncertain is life and accident insurance. In this example, the start of payment is known known and the amount of payment is dependent to which event.

Think of an example , where we deposit A dollars every year in an imaginary bank account that gives i percent interest and we can repeat this for n years. We want to know how much we will have at the end of year nth.



Calculate future value of each single investment and then the cumulative future worth of these equal investments.

Future value of first investment occured at time period $1: A(1+i)^{n-1}$
And similarly, Future value of second intestment occured at time period $2: A(1+i)^{n-2}$
Future value of third intestment occured at time period $3: A(1+i)^{n-3}$
Future value of last intestment occured at time $n: A(1+i)^{n-n} = A$

So, the summation of all future values is

$$F = A(1+i)^{n-1} + A(1+i)^{n-2} + A(1+i)^{n-3} + \ldots + A \tag{1}$$

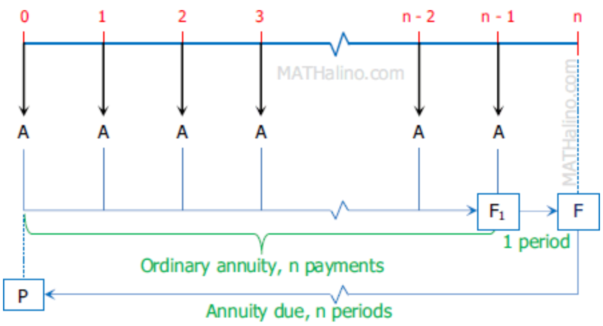
By multiplying both sides of $(i+1)$, we will have

$$F(1+i) = A(1+i)^{n} + A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots + A(1+i) \tag{2}$$

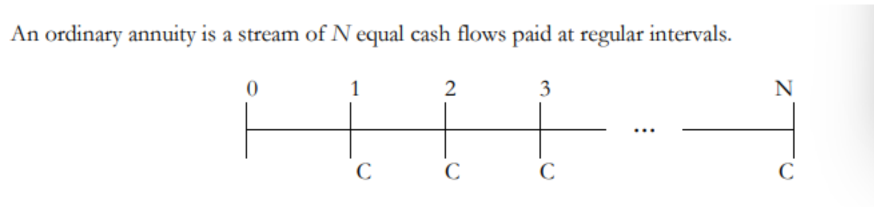
By subtracting first equation from second one, we will have

$$F(1+i) - F = A(1+i)^{n} + A(1+i)^{n-1} + A(1+i)^{n-2} + \ldots + A(1+i) - [A(1+i)^{n-1} + A(1+i)^{n-2} + A(1+i)^{n-3} + \ldots + A]$$
$$F(1+i) - F = A(i+1)^{n} - A$$
$$F = A[(1+i)^{n} - 1]/i$$

Therefore, equation (3) can determine the future value of uniform series of equal investments as $F = A[(i+1)^n - 1]/i$ The factor $\frac{(1+i)^n - 1}{i}$ is called *equal-payment-series* compound-amount factor.



Cash flow Diagram of Annuity Due



Annuity Due

In annuity due, the equal payments are made at the beginning of each compounding period starting from the first period. As seem from the above diagram.

$$F = \frac{A[(1+i)^n - 1]}{i}(1+i)$$

If m payments of A dollars each are recevied per year , then

$$A\frac{[(1+\frac{r}{m})^{mn} - 1]}{\frac{r}{m}}, A\frac{[(1+\frac{r}{m})^{mn} - 1]}{\frac{r}{m}}(1+\frac{r}{m}) \tag{3.6}$$

The PV of or general annuity

$$PV = \sum_{i=1}^{nm} C(1+\frac{r}{m})^{-i} = C\frac{1-(1+\frac{r}{m})^{-nm}}{\frac{r}{m}}$$

An annuity that lasts forever is called a **perpetual annuity**. We canderive its PV from the equation (3.6) by letting n go to infinity:

$PV = \frac{mC}{r}$

This formula is useful for valuing perpetual fixed-coupon debts. For example consider a financial instrument promising to pay $100 once a year forever. If the interest rate is 10%. $PV = 100 * (1/0.1) = 1000$ dollars

Exercise: Derive the PV formula for the general annuity due.

Solution : Present values of future cash flows

$$
\begin{aligned}
PV &= \frac{C}{(1+r)^1} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^{n-3}} + \ldots + \frac{C}{(1+r)^n} \\
&= C[\frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3}, \ldots + \frac{1}{(1+r)^n}] \\
&= C\frac{\frac{1}{(1+r)}[1-(1/(1+r))^n]}{1-(1/1+r)} \\
&\text{Sum of a geometric Sequence} \\
&= C\frac{1-(1+r)^{-n}}{1+r-1} \\
&= C\frac{1-(1+r)^{-n}}{r}
\end{aligned}
$$

Yields

Yields and return are both measurements of an investment's performance.

What is yield ?

Yield refers to how much income an investment generates, separate from the principal. It's commonly used to refer to interest payments an investor recevies on a bond or divident payments on a stock. For example, bond A has a $\$1000$ face value and pays a semiannual coupon of $\$10$. Over one year, bond A yields $\$20$, or 2%. This is known as the cost yield because it' based on the cost or value of the bond.

However, most people buy bonds on the secondar market and not directly from the issuer, meaning they more or less than face value. Considering the same bond A for $\$900$, the $\$20$ coupon paments based on the current $\$900$ price would be a yield of 2.2% This is known as the **current yield** because it's based on the **current price of the bond.**

From book

The term **yield** denotes the return of investment and has many variants. The **nominal yield** is the **coupon rate** of the bond. In the *Wall Street Journal*, for instanc, a corporate bond issued at $AT\&T$ is quoted as follows:

Company	*Cur Yld.*	*Vol.*	*Close*	*Net chg.*
ATT85/831	8.1	162	1061/2	−3/8

This bond matures in the year 2031 and has a nominal ield of $8\frac{5}{8}\%$, which is part of the identificaition of the bond. The **current yield** is the annual coupon interest divided by the market price.

In the preceding case, the annual interest is $8\frac{5}{8} \times 1000/100 = 86.25$, assuming a par value of $\$1000$. The closing price is $106\frac{1}{2} \times 1000/100 = 1065\$$. Therefore, $86.25/1065 \approx 8.1\%$ is the current yield at market closing. The preceding two yield measure are of little use in comparing returns.

Interest Rate of Return

For the rest of this section , the yield we are concerned with, unless stated otherwise, is the **internal rate of return (IRR). The IRR is the interest rate that equates an investment's PV** with its price P:

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_n}{(1+y)^n} \tag{3.11}$$

The right hand side of equation (3.11) is the PV of the cash flows $C_1, C_2, \ldots, C_n$ discounted at the IRR.

```
# Calculate the IRR
def irr(C,r,n,f):

  ############################################
  # C : Series of payment amount made at equal intervals
  # r : Internal Rate of return
  # n : Number of years (time period)
  # f : frequency of payments within a year
  ############################################

  d = 1+(r/f)
```

```
total_period = n*f
print("total period",total_period)
PV = 0 # initialized the PV value
for i in range(1,total_period+1):
  PV +=C/(d)**i
  #print(C/(d)**i)
return PV
```

A bank lent a borrower $\$260,000$ for 15 years to purchase a house.This 15-year mortgage has a monthly payment of $\$2000$. The annual yield is 4.583%

```
# A bank lent a borrower $260,000 for 15 years to purchase a house.
# This 15-year mortgage has a monthly payment of $2000. The annual yield is
# 4.583%
C = 2000
r = 0.04583
n = 15
f = 12
irr(C,r,n,f)
```

```
total period 180
259996.19610819177
```

A financial instrument promises to pay $\$1000$ for the next 3 and sells for $\$2500$. Its yield is 9.7%, which can be verfied as follows. With 0.097 as the discounting rate

```
# A bank lent a borrower $260,000 for 15 years to purchase a house.
# This 15-year mortgage has a monthly payment of $2000. The annual yield is
# 4.583%
C = 1000
r = 0.097
n = 3
f = 1
irr(C,r,n,f)
```

```
total period 3
2500.0453114933393
```

A financial instrument can be bought for $\$1000$ and the investor will end up with $\$2000$ 5 years from now. Calculate the yield

$$1000 = 2000 \times (1+y)^{-5}$$
$$(1+y)^{-5} = \frac{1000}{2000}$$
$$1+y = (\frac{1000}{2000})^{\frac{1}{5}}$$
$$y = (\frac{1000}{2000})^{-\frac{1}{5}} - 1$$
$$1.1486 - 1$$
$$14.86$$

Given the cash flow $C_1, C_2, \ldots, C_n$, its FV is

$$FV = \sum_{t=1}^{n} C_t(1+y)^{n-t} \tag{3.12}$$

By equation (3.11), the yield y makes the preceding FV equal to $P(1+y)^n$. Hence, in principal, multiple cash flows can be reduced to a single cash flow $P(1+y)^n$ at maturity.

Deriving the relationship between FV and PV

Let $n = 3$
Calculating FV using (3.12)

$$FV = C(1+y)^{n-1} + C(1+y)^{n-2} + \ldots + C(1+y)^{n-n}$$
$$FV = C(1+y)^2 + C(1+y)^1 + C(1+y)^0$$
$$FV = C(1+y)^2 + C(1+y) + C$$

Calculating PV using (3.11)

$$PV = \frac{C}{(1+y)} + \frac{C}{(1+y)^2} + \frac{C}{(1+y)^3}$$

Multiplying both sides of the above equation with $(1+y)^3$ we get

$$PV(1+y)^3 = \frac{C(1+y)^3}{(1+y)} + \frac{C(1+y)^3}{(1+y)^2} + \frac{C(1+y)^3}{(1+y)^3}$$

$$PV(1+y)^3 = C + C(1+y) + C(1+y)^2$$

Thr R.H.S of the above equation is nothing by the value of FV from equation (3.12)

Therefore,

$PV(1+y)^n = FV$

note: Substituting n back in place of 3. **The equations (3.11) and (3.12) mean the same thing because both implicitly assume that all cash flows are reinvested at the same rate as the IRR y.**

A general yield measure would be : Calculate the FV and then find the yield that equates it with the PV. This is the **holding period return(HPR)** methodology. With the HPR it is no longer mandatory that all cash flows be reinvested at the same rate. Instead, explicit assumptions about the reinvestment rates must be made for the cash flows. Suppose that the reinvestment rate has been determined to be r_e. Then the FV is

$$FV = \sum_{t=1}^{n} C_t(1+r_e)^{n-t}$$

We then solve for the **holding period yield** y such that $FV = P(1+y)^n$.

Net Present Value

Consider an investment that has the cash flows $C_1, C_2, \ldots, C_n$ and is selling for P. For an investor who believes that this security should have a return rate of y^*, then the **net present value (NPV)** is

$$\sum_{t=1}^{n} \frac{C_t}{(1+y^*)^t} - P$$

The IRR is thus the return rate that nullifies the NPV. In general, the NPV is the difference between the PV's of cash inflows and cash outflow. Businesses are often assumed to maximize their assets NPV.

The management is presented with the following proposals:

		Net Cash Flow at end of		
Proposal	*Investment Now*	*Year 1*	*Year 2*	*Year 3*
A	9,500	4,500	2,000	6,000
B	6,000	2,500	1,000	5,000

It believes that the company can earn 15% effective on projects of this kind. The NPV for Proposal A is

It believes that the company can earn 15% effective on products of this kind.

The NPV for Proposal A is

$$\frac{4500}{1.15} + \frac{2000}{1.15^2} + \frac{6000}{1.15^3} - 9500 = -129.57$$

The NPV for Proposal B is

$$\frac{2500}{1.15} + \frac{1000}{1.15^2} + \frac{5000}{1.15^3} - 6000 = 217.64$$

Proposal A is therefore dropped in favor of Proposal B.

Q. Repeat the calculation for above example for an expected return of 4%.

The NPV for Proposal A is

$$\frac{4500}{1.04} + \frac{2000}{1.04^2} + \frac{6000}{1.04^3} - 9500 = 2010$$

The NPV for Proposal B is

$$\frac{2500}{1.04} + \frac{1000}{1.04^2} + \frac{5000}{1.04^3} - 6000 = 1773$$

Proposal B is therefore dropped in favor of Proposal A.

Numerical Methods for Finding Yields

Coputing the yield amount to solving $f(y) = 0$ for $y \geq -1$, where

$$f(y) = \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P \tag{3.13}$$

and P is the market price. The function $f(y)$ is monotonic in y if the C_ts are all positive.

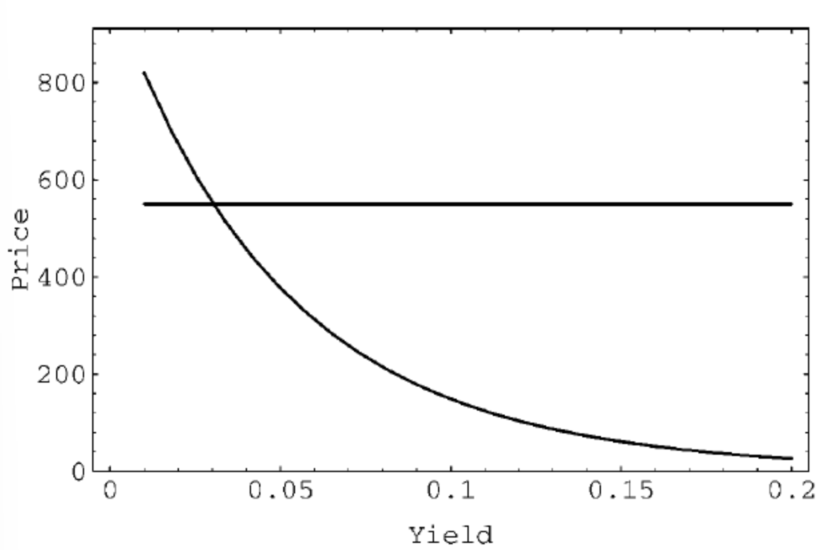


Figure 3.3: Computing yields. The current market price is represented by the horizontal line, and the PV of the future cash flow is represented by the downward-sloping curve. The desired yield is the value on the *x* axis at which the two curves intersect.

The Bisection Method

The bisection method for solving equations

One of the simplest and failure -free methods to solve equations such as $(3.13$ and for well-behaved function is the bisection method. Start with two numbers a and b, written as $\xi \in [a, b]$. If we evaluated f at the midpoint $c \equiv (a+b)/2$, then

- $f(c) = 0$
- $f(a)f(c) < 0$
- $f(c)f(b) < 0$

In the first case we are done, in the second case we continue the preocess with the new bracket $[a, c]$ and in the third case we continue with $[c, b]$. Note that the bracket is halved in the latter two cases. After n steps, we will have confined ξ within a bracket of length $(b-a)/2^n$.

The bisection method. The number ϵ is an upper bound on the absolute error of the returned value C: $|\xi - C| \leq \epsilon$
The initial bracket $[a, b]$ gurantees the existence of a root with $f(a)f(b) < 0$ condition.
$"$ if $[f(c) = 0]$ $"$ may be replaced with testing if $|f(c)|$ is a very small number.

$\text{input } \epsilon, a \text{ and } b (b > a \text{ and } f(a)f(b) < 0);$

$\text{real length } c;$

$\text{length} := b - a$

$\text{while [length } > \epsilon]\{$

$$\begin{aligned} &c := (b+a)/2; \\ &\text{if } [f(c) = 0] \text{ return } c; \\ &\text{else if} [f(a)f(c) < 0] b := c; \\ &\text{else } a := c; \end{aligned}$$

$\}$
$\text{return } c;$

Solving the above NPV problem using the Bisection Method.

```
def bisection_method(a,b,eps,f_x):

  len = b - a
  function_values = []
  while len > eps:
    c = (b+a)/2
    print("Value of root c and function f_x at c",c,f_x(c))
```

```
    function_values.append(f_x(c))
    if f_x(c) == 0:
      return c
    elif f_x(a)*f_x(c)<0:
      b = c
    else:
      a = c
    len = abs(b - a)
  return c,function_values
```

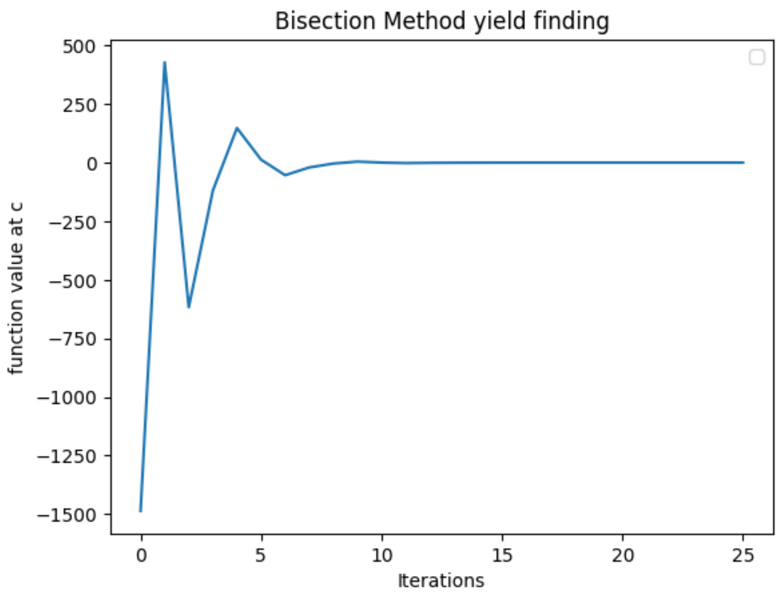
```
f_x = lambda y : 4500/(1+y) + 2000/(1+y)**2 + 6000/(1+y)**3 - 9500
a = -0.01
b = 0.5
eps = 1e-8
c,ouput_vals= bisection_method(a,b,eps,f_x)
```

```
Value of root c and function f_x at c 0.245 -1486.0795094055047
Value of root c and function f_x at c 0.1175 427.78304623197255
Value of root c and function f_x at c 0.18125 -616.9411261793884
Value of root c and function f_x at c 0.14937499999999998 -119.35962857230697
Value of root c and function f_x at c 0.1334375 147.61215681328758
Value of root c and function f_x at c 0.14140624999999998 12.529453167006068
Value of root c and function f_x at c 0.145390625 -53.80788626016147
Value of root c and function f_x at c 0.14339843749999998 -20.73820673631053
Value of root c and function f_x at c 0.14240234375 -4.129224014161082
Value of root c and function f_x at c 0.141904296875 4.193890254351572
Value of root c and function f_x at c 0.1421533203125 0.030778606351304916
Value of root c and function f_x at c 0.14227783203125 -2.049611136708336
Value of root c and function f_x at c 0.142215576171875 -1.009513397824776
Value of root c and function f_x at c 0.1421844482421875 -0.48939168195283855
Value of root c and function f_x at c 0.14216888427734375 -0.2293126097356435
Value of root c and function f_x at c 0.14216110229492188 -0.09926851972340955
Value of root c and function f_x at c 0.14215721130371095 -0.03424533620091097
Value of root c and function f_x at c 0.14215526580810547 -0.0017334598051093053
Value of root c and function f_x at c 0.14215429306030275 0.014522549554385478
Value of root c and function f_x at c 0.14215477943420413 0.006394538944732631
Value of root c and function f_x at c 0.14215502262115481 0.002330538089154288
Value of root c and function f_x at c 0.14215514421463016 0.0002985387691296637
Value of root c and function f_x at c 0.1421552050113678 -0.0007174606071203016
Value of root c and function f_x at c 0.14215517461299898 -0.00020946094082319178
Value of root c and function f_x at c 0.14215515941381457 4.4538908696267754e-05
Value of root c and function f_x at c 0.14215516701340677 -8.246101606346201e-05
```

```
import matplotlib.pyplot as plt
import numpy as np
import warnings

warnings.filterwarnings("ignore")
x = np.arange(len(ouput_vals))
plt.plot(x,ouput_vals)
plt.title("Bisection Method yield finding")
plt.xlabel("Iterations")
plt.ylabel("function value at c")
plt.legend()
```

```
WARNING:matplotlib.legend:No artists with labels found to put in legend.  Note that artists whose label
<matplotlib.legend.Legend at 0x7f863c7b07c0>
```



The Newton-Raphson method for solving equations

Algorithm

$$
\begin{aligned}
&\text{input} : \epsilon, x_{initial};\\
&\text{real} : x_{new}, x_{old};\\
&x_{old} := x_{initial};\\
&x_{new} := \infty;\\
&\qquad\qquad\text{while } [|x_{new} - x_{old} > \epsilon]\\
&\qquad\qquad\qquad x_{new} = x_{old} - f(x_{old})/f'(x_{old})\\
&\text{return } x_{new};
\end{aligned}
$$

Q. Exercise : Let $f(x) = x^3 - x^2$ and start with the guess $x_0 = 2.0$ to the equation $f(x) = 0$. Iterate the Newton-Raphson method five times.

```
def newton_raphson(f_x,f_dx,x_0,eps):

  x_old = x_0
  x_new = 100000000
  for i in range(0,5):
    x_new = x_old - f_x(x_old)/f_dx(x_old)
    x_old = x_new
    print("iteration i",i,":",x_new)


```

```
f_x = lambda x : x**3 - x**2
f_dx = lambda  x: 3*x**2 - 2*x
x_0  =2
eps = 1e-8
newton_raphson(f_x,f_dx,x_0,eps)
```

```
iteration i 0 : 1.5
iteration i 1 : 1.2
iteration i 2 : 1.05
iteration i 3 : 1.0043478260869565
iteration i 4 : 1.0000373203955963
```

Bonds

A bond is a contract between the issuer(borrrower) and the bondholder(lender). The issuer promises to pay the bondholder interest, if any, and principal on the remaining balance. Bonds usually refer to long-term debt. A bond has a **pay value**. The **redemption date** or **maturity date** specifies the date on which the loan will be repaid.

A bond pays interest at the coupon rate on its par value at regular time intervals until the maturity date. The payment is usually made semiannually in the United States.

The **redemption value** is the amount to be paid at a redemption date. A bons is **redeemed at par** if the redemption value is the same as the par value. Redemption date and maturity date may differ.

There are several ways to **redeem** or **retire** a bond. A bond is redeemed at maturity if the principal is repaid at maturity. Most corporate bonds are **callable**, meaning that the issuer can retire some or all of the bonds before the stated maturity, usually at a price above the par value.

Because this provision gives the issuer the advantage of calling a bond when the prevailing interest rate is much lower than the coupon rate, the bondholders usually demand a premium. A callable bond may also have **call protection** so that it is not callable for the first few years.

Refunding involves using the proceeds from the issuance of new bonds to retire old ones. A corporation may deposit money into a **sinking fund** and use the funds to buy back some or all of the bonds.

Convertible bonds can be converted into the issuer's common stock.

Tresury securities with maturities of 1 year or less are discount securities: the $T - bills$. Tresury securities with original maturities between 2 and 10 years are called **Tresury notes (T-notes)** . Those with maturities greater than ten years are called **Tresury bonds(T-bonds)**. Both T-notes and T-bonds are coupon securities, paying interest every 6 months.

What is a Bond Quote ?

A bond quote is the last price at which a bond traded, expressed as a percentage of par value and converted to a point scal.e

Par value is generally set at 100, representing 100% of a bond's face value of $\$1000$/ For example, if a corporate bond is quoted at 99 , that means it is trading at 99% of face value. In this case, the caost to buy each bond is $\$990$.

How a Bond Quote Works

Price quotes for bonds are represented by a percentage of the bond's par value, which is converted to a numeric value, then multiplied by 10, in order to determine the cost per bond. Bond quotes can also be expressed as fractions.

For example, corporate bonds are quotes $\frac{1}{8}$ increments, while government bonds are quoted in increments of $\frac{1}{32}$. Therefore, a bond quote of $99\ 1/4$ represents 99.25% of par.

Valuation

Let us begin with **pure discount bonds**, also known as **zero-coupon bonds** or simply **zeros**. They promise a single payment in the future and are sold at a discount from the par value. The price of a zero-coupon bond thay pays F dollars in n periods is $F/(1+r)^n$, where r is the interest rate per period.

Example : Suppose the interest rate is 8% compounded semiannually. A zero-coupon bond that pays that par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value and will be quoted at 20.83. If the interest rate is 9% instead, the same bond will be priced at only $17.19.If the bond matures in 10 years instead of 20, its price would be 45.64 with an 8% interest rate. Clearly both the maturity and the market interest rate have a profound impact on price.

A **level-coupon bond** pays interest based on the coupon rate and the par value which is paid at maturity. IF F denotes the par value and C denotes the coupon, then the cash flow is shown as

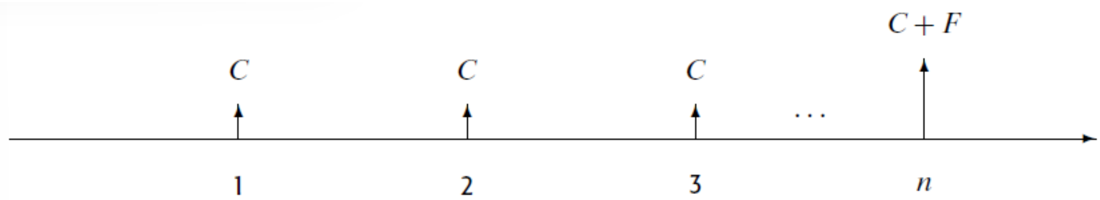


Figure 3.7: Cash flow of level-coupon bond.

$$PV = \sum_{i=1}^{n} \frac{C}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^n}$$
$$= C\frac{1-(1+\frac{r}{m})^{-n}}{\frac{r}{m}} + \frac{F}{(1+\frac{r}{m})^n} \qquad (3.18)$$

where n is the number of cash flows, m is the number of payments per year, and r is the annual interest rate compounded m times per annum. Note that $C = Fc/m$ where c is the annual coupon rate.

Bond Yield Rate vs. Coupon Rate

A bond's coupon rate is the rate of interest it pays annually, while its yield is the rate of return it generates. A bond's coupon rate is expressed as a percentage of its par value. The par value is simply the face value of the bond or the value of the bond as stated by the issuing entity.

Thus,a $\$1000$ bond with a coupon rate of 6% pays $\$60$ in interest annually and a $\$2000$ bond with a coupon rate 6% pays $\$120$ in interest annually.

Example : Consid a 20-year 9% bond with coupon paid semiannually. This means that a payment of $1000 * 0.09/2 = 45$ dollars will be made every 6 months until maturity, and $\$1000$ will be paid at maturity. Its price can be computed from above equation. with $n = 2 \times 20, r = 0.08, m = 2, F = 1$and $C = 1 \times 0.09/2$ .The result is $1.09896 or 109.896\%$ of par value.

The **yield to maturity** of a level-coupon bond is its IRR when the bond is held to maturity. In other words, it is the r that satisfies $eq(3.18)$ with the PV being the bond price. For example , for an investor with a 15% BEY to maturity, a 10-year bond with a coupon rate of 10% paid semiannually should sell for percent of par.

$$5 \times \frac{1-[1+(0.15/2)]^{-2\times10}}{0.15/2} + \frac{100}{[1+(0.15/2)]^{2\times10}} = 74.5138$$

For a callable bond,

The **yield to stated maturity** measure its yield to maturity as if it were not callable.

The **yield to call** is the yield to maturity satisfied by $EQ(3.18)$, with n denoting the number of remaining coupon payments until the first call date and F replaced with the **call price**, the price at which the bond will be called.

The related **yield to par call** assumes the call price is the par value.

The **yield to effective maturity** replaces n with the **effective maturity date**, the redemption date when the bond is called.

The **yield to worst** is the minimum of the yields to call under all possible call dates.

Price Behaviors

The price of a bond goes in the opposite direction from that of interest rate movements: Bond prices fall when interest rates rise, and vice versa. This is because the PV decreases as interest rate increase.

Equation (3.18) can be used to show that a level-coupon bond will be selling **at a premium**(above its par value) when its coupon rate is above the market interest rate,

at par(at its par value) when its coupon rate is equal to the market interest rate, and

at a discount(below its par value) when its coupon rate is below the market interest rate.

3.5 Bonds

Figure 3.8: Price/yield relations. A 15-year 9% coupon bond is assumed.

Yield (%)	*Price (% of par)*
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

The table above shows the relation between the price of a bond and the required yield. Bonds selling at par are called **par bonds**.

The price/yield relation has a convex shape. Convexity is attractive for bondholders because the price decrease per percent rate increase is smaller than the price increase per percent rate decrease.

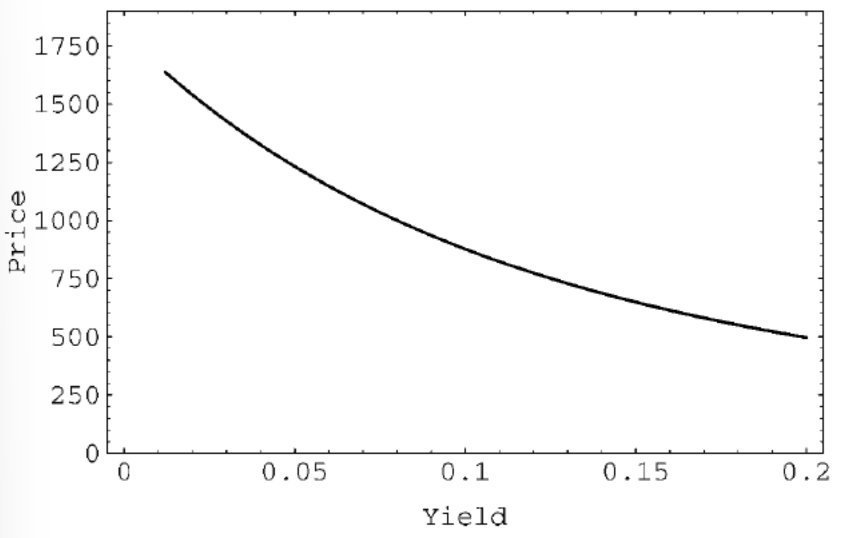


Figure 3.9: Price vs. yield. Plotted is a bond that pays 8% interest on a par value of $1,000, compounded annually. The term is 10 years.

As the maturity date draws near, a bond selling at a discount will see its price move up towards par, a bond selling at par will see its price remain at par, and a bond selling at a premium will see its price move down toward par.

Besides the two reasons cited for causing bond prices to change (interest rate movements and a nonpar bond moving toward maturity), other reasons include changes in the yield spread to T-Bonds for non T-Bonds, changes in perceived credit quality of the issuer, and changes in the value of the embedded option.

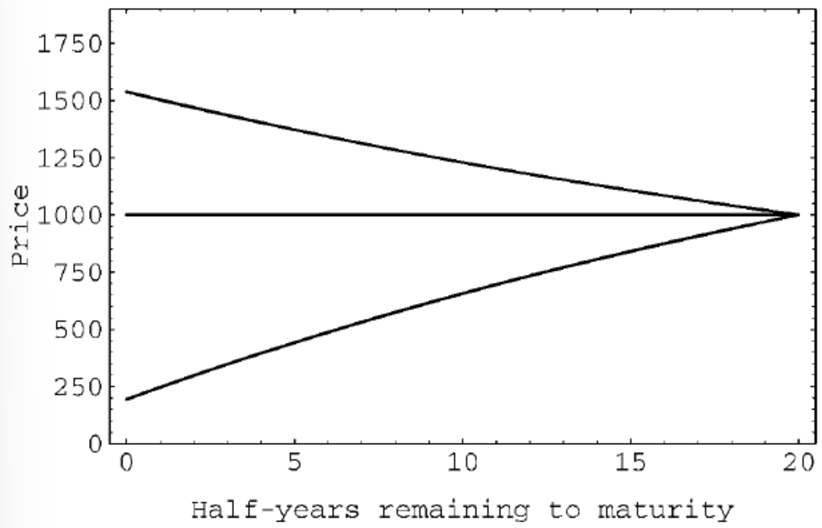


Figure 3.10: Relations between price and time to maturity. Plotted are three curves for bonds, from top to bottom, selling at a premium, at par, and at a discount, with coupon rates of 12%, 6%, and 2%, respectively. The coupons are paid semiannually. The par value is $1,000, and the required yield is 6%. The term is 10 years (the x axis is measured in half-years).

Exercise: Prove that a level-coupon bond will be sold at par if its coupon is the same as the market interest rate.

Solution : let C = rF , as $(C = \frac{cF}{r})$

$$\begin{aligned} PV &= C \times \frac{1-(1+r)^{-n}}{r} + \frac{F}{(1+r)^n} \\ &= (rF) * \frac{1-(1+r)^{-n}}{r} + \frac{F}{(1+r)^n} \\ &= F * (1-(1+r)^{-n}) + F(1+r)^{-n} \\ &= F - F(1+r)^{-n} + F(1+r)^{-n} \\ &= F \end{aligned}$$

Day Count Conventions

Handling the issue of dating correctly is critical to any financial software. In the so-called actual/actual day count convention, the first "actual" refers to the actual number of days in a month, and the second refers to the actual number of days in a coupon period.

For example, for coupon-bearing Tresury securities, the number of days between June $17, 1992$ and October $1, 1992,$ is 106: 13 days in June, 31 days in July, 31 days in August, 30 days in September and 1 day in October.

A convention popular with corporate the municipal bonds and agency securities is $30/360$. Here each month is assumed to have 30 days and each year 360 days. The number of days between June 17 1992 and October $1, 1992$ is now 104 : 13 days in June, 30 days in July, August, September and 1 day in October.

In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ and to date $D_2 \equiv (y_2, m_2, d_2)$ under the $30/360$ convention can be computed by

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1)$$

where
y_i denotes the years
m_i denote the months
d_i denote the days

Accrued Interest

Up to now, we have assumed that the next coupon payment date is exactly one period (6 months for bonds, for instance) from now. In reality , the settlement date may fall on any day between two coupon payment dates and yield measures have to be adjusted accordingly. Let

$$\omega \equiv \frac{\text{number of days between the settlement and the next coupon date}}{\text{number of days in the coupon period}}$$

the day count is based on the convention applicable to the security in question. The price is now calculated by

$$PV = \sum_{i=0}^{n-1} \frac{C}{(1+\frac{r}{m})^i} + \frac{C}{(1+\frac{r}{m})^\omega} + \frac{F}{(1+\frac{r}{m})^{n-1}} + \frac{F}{(1+\frac{r}{m})^\omega}$$
$$= \sum_{i=0}^{n-1} \frac{C}{(1+\frac{r}{m})^{\omega+i}} + \frac{F}{(1+\frac{r}{m})^{\omega+n-1}} \tag{3.20}$$

where n is the number of remaining coupon payments. This price is called the **full price,dirty price or invoice price**. Equation (3.20) reduces to (3.18) then $\omega = 1$.

As the issuer of the bond will not send the next coupon to the seller after the transaction, the buyer has to pay the sellter part of the coupon during the time the bond was owned by the seller. The convention is that the buyer pays the quoted price plus the **accrued interest** calculated by :

$$C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}}$$
$$= C \times (1-\omega)$$

The yield to maturity is the r satisfying equation (3.20) when the PV is the invoice price, the sum of the quoted price and the accrued interest. As the quoted price in United States does not include the accrued interest, it is also called the **clean price** or **flat price**.

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