Bond Duration and Convexity

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Bond duration and convexity are measures of the sensitivity of bond price to interest rate (i.e yield) changes. Bond price is a function of time (i) and discount rate (y). The equation for bond price at time zero is the discounted value of expected future cash flows. The bond price equation in mathematical term is

$$P_0 = \sum_{i=1}^{n} [C_i \times (1 + y)^i] \tag{1}$$

Provided that estimates future cash flow does not change, bond price will change with the passage of time (i) and with change in yield (y). Using Taylor Series Approximation , the change in bond price , $\triangle P$, over some small interval , $\triangle i$, is given by the following partial differential equation (PDE)

$$\triangle P = \frac{\delta P}{\delta i} \triangle i + \frac{\delta P}{\delta y} \delta y + \frac{1}{2} \frac{\delta^2 P}{\delta y^2} (\triangle y)^2 \tag{2}$$

The first terms in the PDE is $\frac{\delta P}{\delta i} \triangle i$, which is the first derivative of the price equation with respect to time (i). This partial derivative measures the change in price that occurs with the passage of time while holding yield (y) constant. We are not interested in holding yield constant. We want to hold time constant and measure the change in price due to an instantaneous change in yield. For our purposes we ignore this derivative.

The second term in the PDE is $\frac{\delta P}{\delta y} \delta y$ which is the first derivative of the price equation with respect to yield (y). This partial derivative measures the first order change in price that occurs when yield changes while holding time (i) constant. We are interested in this derivative because this is what we want to know .. how will bond price change with an instantaneous change in yield? This derivative is related to a bond's duration and is linear measure of how bond price changes in response to interest rate changes. This derivative is always negative (Bond price increases with a decrease in interest rates. Bond price decreases with an increase in interest rates).

The third term in the PDE is $\frac{1}{2} \frac{\delta^2 P}{\delta y^2} (\triangle y)^2$, which is the second derivative of the price equation with respect to yield (y) and measure how bond duration changes as interest rates change. A non-callable bond has positive convexity. Positive convexity means that a 100 bases point increase in rates will decrease bond price by less than what a 100 bases point decrease in rates will increase bond price. Convexity is a good thing that fixed income investors will pay for.

We now rewrite equation (2) by dividing both sides by bond price such that the new PDE describes the percentage change in bond price. The new PDE becomes

$$\frac{\triangle P}{P} = \frac{1}{P}\frac{\delta P}{\delta i}\triangle i + \frac{1}{P}\frac{\delta P}{\delta y}\delta y + \frac{1}{2}\frac{1}{P}\frac{\delta^2 P}{\delta y^2}(\triangle y)^2 \tag{3}$$

We can rewrite equation (1) by defining $\theta = (1+y)$. The bond price equation becomes

$$P_0 = \sum_{i=1}^{n} C_i \times \theta^{-i} \tag{4}$$

The first and second derivatives (respectively) of equation (4) with respect to θ is

$$\begin{aligned} \tfrac{\delta P}{\delta \theta} &= \textstyle\sum_{i=1}^{n} C_i(-i)\theta^{-i-1} \\ &= -\theta^{-1}\textstyle\sum iC_i\theta^{-i} \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\delta^2 P}{\delta \theta^2} &= -(-1)\theta^{-2}\sum_{i=1}^{n} iC_i\theta^{-i} + (-\theta^{-1})\sum(-i^2)C_i(\theta^{-i-1}) \\ &= \theta^{-2}\sum iC_i\theta^{-i} + \theta^{-2}\sum i^2C_i\theta^{-i} \\ &= \theta^{-2}\sum_{i=1}^{n}(i+i^2)C_i\theta^{-i} \end{aligned} \tag{6}$$

We will define $\frac{1}{P}\frac{\delta P}{\delta y}\delta y$ as bond duration and $\frac{1}{P}\frac{\delta^2 P}{\delta y^2}$ as bond convexity. After making these substitutions into equation (3)and then multiply by bond price , we get

$$\triangle P = P \times Duration \times \triangle\theta + \frac{1}{2} \times P \times Convexity \times (\triangle\theta)^2 \tag{7}$$

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