Calculus Notes

The Limit Of A Variable

An infinitely Large Variable

In this section we shall consider ordered variables that vary in a special way defined as follows : "the variable approaches limit".

Definition 1: A constant number a is said to be the *limit* of a variable x , if for every preassigned arbitrarily small positive number ϵ it is possible to indicate a value of the variable x such that all subsequent values of the variable will satisfy the inequality

$$|x - a| < \epsilon$$

If the number a is the limit of the variable x, one says that x approaches the limit a; in symbols we have

$$x \to a \; or \; lim\, x = a.$$

In geometric terms, limit may be defined as follows.

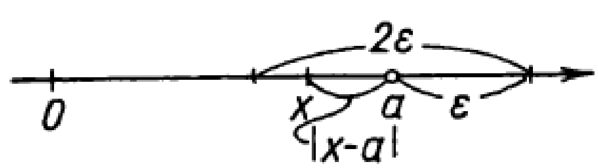

Fig. 28.

The constant number a is the limit of the variable x if for any preassigned arbitrarily small neighbourhood with centre in the point a and with radius ϵ there is a value of x such tat all points corresponding to subsequent values of the variable will be within this neighbourhood.

Let us consider several cases of variables approaching limits.

Example 1. The variable x takes on successive values:

$$x_1 = 2; x_2 = 1 + \frac{1}{2};\; x_3 = 1 + \frac{1}{3},\; \ldots, x_n = 1 + \frac{1}{n}$$

We shall prove that this variable has unity as its limit . We have

$$\}x_n - 1| = |\left(1 + \frac{1}{n}\right) - 1| = \frac{1}{n}$$

For any ϵ , all subsequent values of the variable begin with n , where $\frac{1}{n} < \epsilon$ or $n > \frac{1}{\epsilon}$ will satisfy the inequality $|x_n - 1| < \epsilon$ and the proof is complete.

It will be noted that the variable decreases as it approaches the limit.

Example 2: The variable x takes on successive values:

$x_1 = 1 - \frac{1}{2}$; $x_2 = 1 + \frac{1}{2^2}$, $x_3 = 1 - \frac{1}{2^3}$;

$x_4 = 1 + \frac{1}{2^4}$, $\ldots x_n = 1 + (-1)^n \frac{1}{2^n}$

The variable has a limit of unity

$$|x_n - 1| = |\left(1 + (-1)^n \frac{1}{2^n}\right) - 1| = \frac{1}{2^n}$$

For any ϵ , beginning with n , which satisfies the relation

$$\frac{1}{2^n} < \epsilon$$

from which it follows that

$$\begin{aligned} 2^n &> \frac{1}{\epsilon} \\ n \, log \, 2 &> log \, \frac{1}{\epsilon} \\ n &> \frac{log \, \frac{1}{\epsilon}}{log \, 2} \end{aligned}$$

all subsequent values of x will satisfy the relation.

$$|x_n - 1| < \epsilon$$

It will be noted here that the values of the variables are greater than or less than the limit, and the variable approaches its limit by "oscillating about it".

Note : One should not think that every variable has a limit. Let the variable x take on the following successive values:

$$x_1 = \frac{1}{2}, x_2 = 1 - \frac{1}{4}, x_3 = \frac{1}{8}, x_4 = 1 - \frac{1}{16}, \ldots, x_{2k} = 1 - \frac{1}{2^{2k}}, x_{2k+1} = \frac{1}{2^{2k+1}}$$

For k sufficiently large, the value x_{2k} and all subsequent values with event labels will differ from unity by as small a number , while the next value x_{2k+1} and all subsequent values of x with odd labels will differ from zero by as small a number . Consequently, the variable x does not approach a limit.

<u>The Limit Of A Function</u>

In this section we shall consider cases of the variation of a function when the argument x approaches a certain limit a or infinity.

Definition 1: Let the function $y = f(x)$ be defined in a certain neighbourhood of the point a or at certain points of this neighbourhood. The *function* $y = f(x)$ *approaches the limit* $b(y \to b)$ as x approaches $a(x \to a)$, if for every positive number ϵ, no matter how small, it is possible to indicate a positive number δ such that for all x, different from a and satisfying the inequality

$$|x - a| < \delta$$

we have the inequality

$$|y - b| < \epsilon$$

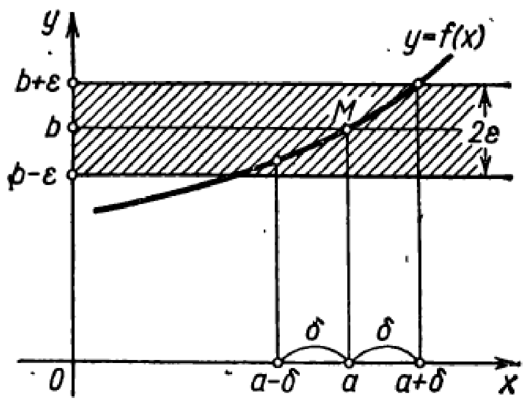

Fig. 31.

If b is the *limit of the function f(x) as* $x \to a$,we write.

$$\lim_{x \to a} f(x) = b$$

or $f(x) \to b$ as $x \to a$.
Since from the inequality $|x - a| < \delta$ there follows the inequality $|f(x) - b| < \epsilon$, this means that for all points x that are not more distant from the point a than δ, the points M of the graph of the function $y = f(x)$ lie within a band of width 2ϵ bounded by the lined $y = b - \epsilon$ and $y = b + \epsilon$

Note 1 : We may define the limit of the function $f(x)$ as $x \to a$ as follows.

Let a variable x assume values such (that is, ordered in such fashion) that if

$$|x^{*} - a| > |x^{**} - a|$$

Then x^{**} is the subsequent value and x^{*} is the preceding value ; but if

$$|\overline{x}^{*} - a| = |\overline{x}^{**} - a|,$$

then $\overline{x}^{**}$ is the subsequent value and $\overline{x}^{*}$ is the preceding value.

In other words, of two points on a number scale, the subsequent one is that which is closer to the point a at equal distances, the subsequent one is that which is to the right of the point a .
Let a variable quantity x ordered in this fashion approach the limit $a\left[x \to a \; or \lim x = a\right]$

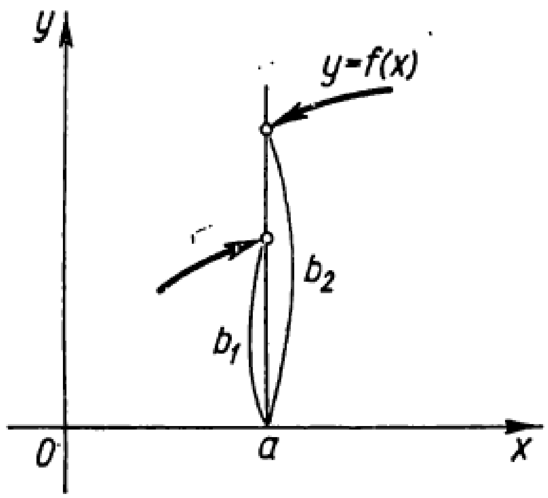

Fig. 32.

Note 2: If $f(x)$ approaches the limit b_1 as x approaches a certain number $a,$ so that x takes on only values less than $a,$ we write $\lim\limits_{x \to a-0} f(x) = b_1$ and call b_1 the limit of the function $f(x)$ on the left of the point $a.$ If x takes on only values greater than $a,$ we write $\lim\limits_{x \to a+0} f(x) = b_2$ and call b_2 the limit of the function on the right of the point $a.$
It can be be proved that if the limit on the right and the limit on the left exist and are equal, that is $b_1 = b_2 = b$ then b will be the limit in the sense of the foregoing definition of a limit at the point $a.$ And conversely , if there exists a limit b of a function at the point a, then there exist limits of the function at the point a both on the right and on the left and they are equal.

Continuity Of Function

Let the function $y = f(x)$ be defined for some value x_0 and in some neighbourhood with cetnre at x_0. Let $y_0 = f(x_0)$.

If x receives some positive or negative increment Δx and assumes the value $x = x_0 + \Delta x$, then the function y too will receive an increment $\Delta y.$ The new increased value of the function will be

$y_0 + \Delta y = f(x_0 + \Delta x)$

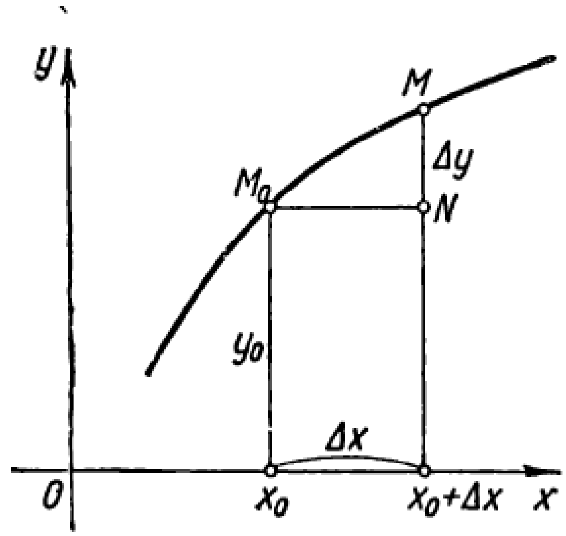

Fig. 47.

The new increased value of the function will be $y_0 + \Delta y \;=\; f(x_0 + \Delta x)$.
The increment of the function Δy will be expressed by the formula

$$\Delta y \;=\; f(x_0 + \Delta x) \;-f(x_0)$$

Definition 1 : The function $y = f(x)$ is called *continuous for the value* $x = x_0$ (or at point $x_0)$ if it is defined in some neighbourhood of the point x_0 (obviously, at the point x_0 as well) and if

$$\lim_{\Delta x \to 0} \Delta y \;=\; 0 \tag{1}$$

or which is the same thing

$$\lim_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] = 0 \tag{2}$$

In descriptive geometrical terms, the continuity of a function at a given point signifies that the difference of the ordinates of the graph of the function $y = f(x)$ at the point $x_0 + \Delta x$ and x_0 will, in absolute magnitude, be arbitrarily small, provided $|\Delta x|$ is sufficiently small.

Example 1 : We shall prove that the function $y = x^2$ is continuous at an arbitrary point x_0. Indeed.

$$y_0 \;=\; x_0^2,\; y_0 + \Delta y \;=\; (x_0 + \Delta x)^2 \;= x_0^2 + 2x\Delta x + \Delta x^2$$

$$\Delta y = x_0^2 + 2x\Delta x + \Delta x^2 - y_0 = x_0^2 + 2x\Delta x + \Delta x^2 - x_0^2 = 2x\Delta x + \Delta x^2,$$

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \left(2x_0 \Delta x + \Delta x^2\right) = 2x_0 \lim_{\Delta x \to 0} \Delta x + \Delta x \lim_{\Delta x \to 0} \Delta x = 0$$

for any way that Δx may approach zero.

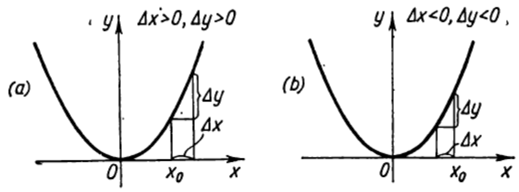

Fig. 48.

Theorem : *Every elementary function is continuous at each point at which it is defined.* The condition of continuity (2) may be written thus:

$$\lim_{\Delta x \to 0} f(x_0 + \Delta x) = f(x_0)$$

or

$$\lim_{x \to x_0} f(x) = f(x_0),$$

but

$$x_0 = \lim_{x \to x_0} x.$$

Consequently,

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right)$$

In other words, in order to find the limit of a continuous function as $x \to x_0$ it is sufficient to substitute into the expression of the function the value of the argument , x_0 in place of the argument x.

Definition 2 : If the function $y = f(x)$ is continuous at each point of a certain interval (a, b), where $a < b$, then it is said that the *function is continuous in this interval.*

If the function is also defined for $x = a$ and $\lim\limits_{x \to a+0} f(x) = f(a)$, it is said that $f(x)$ at the point $x = a$ *is continuous on the right.*

If $\lim\limits_{x \to b-0} f(x) = f(b)$, it is said that the function $f(x)$ *is continuous on the left* of the point $x = b$.

If the function $f(x)$ is continuous at each point of the interval (a, b) and is continuous at the end points of the interval, on the right and left, respectively, it is said that the function $f(x)$ is *continuous over the closed interval* $[a, b]$.

Definition 3 : If the function $f(x)$ is such that there exist finite limits $\lim\limits_{x \to x_0+0} f(x) = f(x_0 + 0)$ and $\lim\limits_{x \to x_0-0} f(x) = f(x_0 - 00,$ but either $\lim\limits_{x \to x_0+0} f(x) \neq \lim\limits_{x \to x_0-0} f(x)$ or the value of the function $f(x)$ at $x = x_0$ is not defined , then $x = x_0$ is called a *point of discontinuity of the first kind.*

Certain Properties of Continuous Functions

In this section we shall consider a number of properties of functions that are continuous on an interval. These properties will be stated in the form of theorems given without proof.

Theorem 1 : *If a function* $y = f(x)$ *is continuous on some interval* $[a, b]$ $(a \leq x \leq b)$, *there will be , on this interval at least one point* $x = x_1$ *such that the value of the function at point will satisfy the relation*

$$f(x_1) \geq f(x),$$

where x *is any other point of the interval, and there will be at least one point* x_2 such that the value of the function at this point will satisfy the relation

$$f(x_2) \leq f(x).$$

We shall call the value of the function $f(x_1)$ the *greatest value* of the function $y = f(x)$ on the interval $[a, b]$ and the value of the function $f(x_2)$ the *smallest (least) value* of the function on the
interval $[a, b]$.

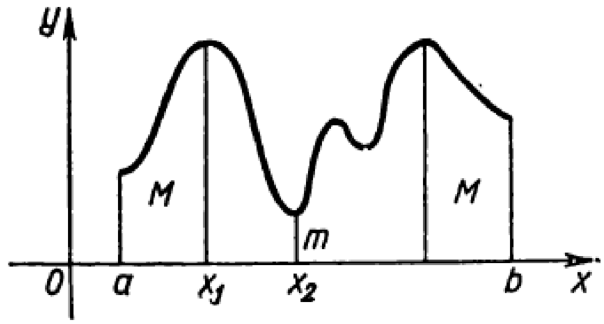

Fig. 51.

A function continuous on the interval $a \leq x \leq b$ attains on this interval (at least once) a greatest value M and a smallest value m.

Theorem 2 : *Let the function* $y = f(x)$ *be continuous on the interval* $[a, b]$ *and at the end point of this interval let it take on values of different sign; then between the points* a *and* b *there will be at least one point* $x = c$, at which the function becomes zero:

$$f(c) = 0, a < c < b.$$

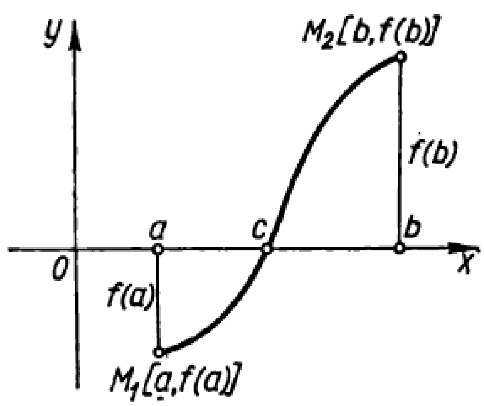

Fig. 52.

This theorem has a simple geometrical meaning. The graph of a continuous function $y = f(x)$ joining point $M_1[a, f(a)]$ and $M_2[b, f(b)]$ where $f(a) < 0$ and $f(b) > 0$ or

$f(a) > 0$ *and* $f(b) < 0$ cuts the x-axis at least at one point.

Theorem 3 : *Let the function* $y = f(x)$ *be defined and continuous in the interval* $[a, b]$. *If at the end points of this interval the function takes on unequal value* $f(a) = A$ *and* $f(b) = B$. *then no matter what the number* μ *between number* A *and* B, there will be a point $x = c$ *between a and b such that* $f(c) = \mu$.

In the given case, any straight line $y = \mu$ cuts the graph of the function $y = f(x)$.

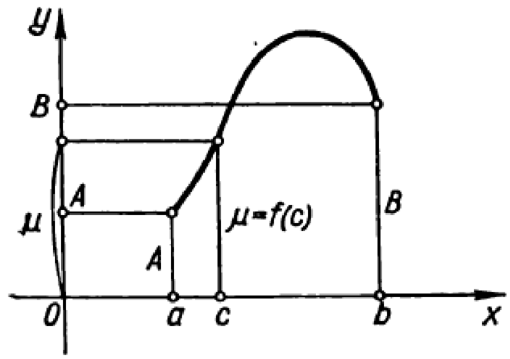

Fig. 54.

Corollary of Theorem 3 : *If a function* $y = f(x)$ *is continuous in some interval and takes on a greater value and a least value, then in this interval it takes on, at least once, any value lying between the greatest and the least values.*

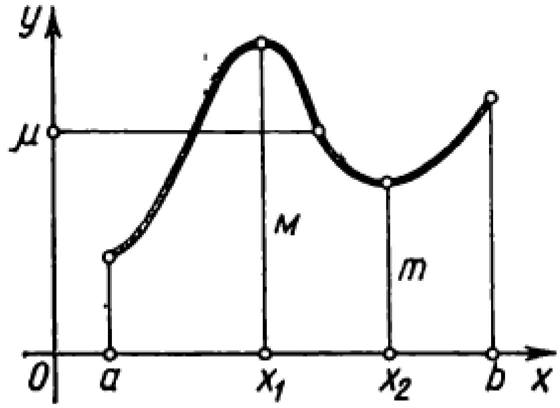

Fig. 55.

Indeed, let $f(x_1) = M$ and $f(x_2) = m$. Consider the interval $[x_1, x_2]$. By theorem 3, in this interval the function $y = f(x)$ takes on any value μ lying between M and m. But the interval $[x_1, x_2]$ lies inside the interval under consideration in which the function $f(x)$ is defined.

Derivative and Differential

Velocity Motion

Let us consider the rectilinear motion of some solid, say a stone thrown vertically upwards. Idealising the situation and disregarding dimensions and shapes, we shall always represent such a body in the form of a moving point M. The distance s of the moving point reckoned from some initial position M_0 will depend on the time t; in other words, s will be a function of time t:

$$s = f(t) \tag{3}$$

Fig. 56.

At some instant of time $t,$ let the moving point M be at a distance s from the initial point $M_0,$ and at some later instance $t + \Delta t$ let the point be at $M,$ a distance $s + \Delta s$ from the initial position as seen in the figure above.

Thus, during the interval of time Δt the distance s changed by the quantity Δs. In such cases, one says that during the time Δt the quantity s received an increment $\Delta s.$

Let us consider the ratio $\dfrac{\Delta s}{\Delta t}$; it gives us the average velocity of motion of the point during the time $\Delta t.$

$$v_{av} = \frac{\Delta s}{\Delta t} \tag{4}$$

The average velocity cannot in all cases give an exact picture of the rate of translation of the point M at time $t.$ If , for example the body moved very fact at the beginning of the interval Δt and very slow at the end , the average velocity obviously cannot reflect these peculiarities in the motion of the point and give us a correct idea of the true velocity of motion at time $t.$

In order to express more precisely this true velocity in terms of the average velocity, one has to take a small interval of time $\Delta t.$ The most complete description of the rate of motion of the point at time t is given by the limit which the average velocity approaches as $\Delta t \to 0.$

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{5}$$

Thus, the *rate (velocity) of motion at a given instant* is the limit of the ratio of increment of path Δs to increment of time Δt, as the time increment approaches zero.

$$\begin{aligned} \Delta s &= f(t+\Delta t) - f(t), \\ v &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \end{aligned} \tag{6}$$

Example : Find the velocity of uniformly accelerated motion at an arbitrary time t and at $t = 2$ sec if the relation of the path traversed to the time is expressed by the formula

$$s = \frac{1}{2} g t^2$$

Solution : At time t we have $s = \frac{1}{2} g t^2$; at time $t + \Delta t$ we get

$$s + \Delta s = \frac{1}{2} g (t+\Delta t)^2 = \frac{1}{2} g \left(t^2 + 2t\Delta t + \Delta t^2\right)$$

We find Δs

$$\Delta s = \frac{1}{2} g \left(t^2 + 2t\Delta t + \Delta t^2\right) - \frac{1}{2} g t^2 = g t \Delta t + \frac{1}{2} t \Delta t^2.$$

We form the ratio $\frac{\Delta s}{\Delta t}$:

$$\frac{\Delta s}{\Delta t} = \frac{g t \Delta t + \frac{1}{2} t \Delta t^2}{\Delta t} = g t + \frac{1}{2} t \Delta t$$

by definition we have

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \left(g t + \frac{1}{2} g \Delta t \right) = g t$$

Thus, the velocity at an arbitrary time t is $v = g t$.
At $t = 2$, we have $(v)_{t=2} = 9.8 * 2 = 19.6\ m/sec$.

Definition of Derivative

Let there be a function

$$y = f(x) \tag{7}$$

defined in a certain interval. The function $y = f(x)$ has a definite value for each value of the argument x in this interval.

Let the argument x receive a certain increment Δx. Then the function y will receive a certain increment Δy. Thus, with the value of the argument x we will have $y + \Delta y = f(x + \Delta x)$, with the value of the argument $x + \Delta x$ we will have $y + \Delta y = f(x + \Delta x)$.

Let us find the increment of the function Δy :

$$\Delta y = f(x + \Delta x) - f(x) \tag{8}$$

Forming the ratio of the increment of the function to the increment of the argument, we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{9}$$

We then find the limit of this ratio as $\Delta x \to 0$. If the limit exists, it is called the **derivative** of the given function $f(x)$ and is denoted by $f'(x)$. Thus, by definition,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

or

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{10}$$

Consequently, the *derivative* of a given function $y = f(x)$ with respect to the argument x is the limit of the ratio of the increment of the function Δy to the increment of the argument Δx, when that latter approaches zero in arbitrary fashion.

The operation of finding the derivative of a function $f(x)$ is called *differentiation of the function* .

Example : $y = \frac{1}{x};\ find\ y'$

Solution : Reasoning , we get

$$
\begin{array}{rccc}
 & y & = & \frac{1}{x} \\
\Longrightarrow & y+\Delta y & = & \frac{1}{x+\Delta x} \\
\Longrightarrow & \Delta y & = & \frac{1}{x+\Delta x} - y \\
 & & = & \frac{1}{x+\Delta x} - \frac{1}{x} \\
 & & = & \frac{x-x-\Delta x}{x(x+\Delta x)} \\
 & & = & -\frac{\Delta x}{x(x+\Delta x)} \\
\Longrightarrow & \frac{\Delta y}{\Delta x} & = & -\frac{1}{x(x+\Delta x)} \\
y' = & \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} & = & \lim_{\Delta x \to 0} -\frac{1}{x(x+\Delta x)} \\
 & & = & -\frac{1}{x^2}
\end{array}
$$

Geometric Meaning of the Derivative

We take a curve with a fixed point M_0 on it. Taking a point M_1 on the curve we draw the secant M_0M_1. If the point M_1 approaches the point M_0 without limit, the secant M_0M_1 will occupy various position M_0M_1', M_0M_1'' and so on.

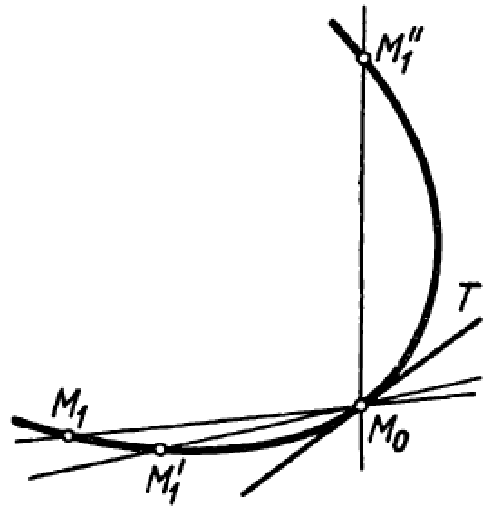

Fig. 57.

If, in the limitless approach of the point M_1(along the curve) to the point M_0 **from either side,**
the secant tends to occupy the position of a definite straight line M_0T, this line is called the **tangent** to the curve at the point M_0.

Let us consider the function $f(x)$ and the corresponding curve in the below figure.

$$y = f(x)$$

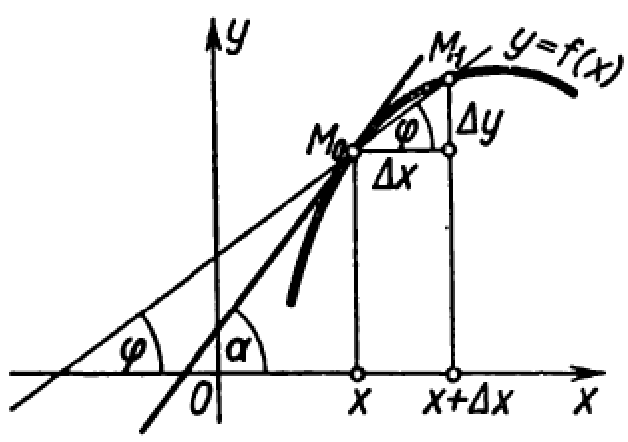

Fig. 58.

At a certain value of x the function has the value $y = f(x)$. Corresponding to these values of x and y on the curve we have the point $M_0(x, y)$.

let us increases the argument x by Δx. Corresponding to the new value of the argument, $x + \Delta x$, we have an increased value of the function $y + \Delta y = f(x + \Delta x)$.

Another corresponding point on the curve will be $M_1(x + \Delta x, y + \Delta y)$.

Draw the secant M_0M_1 and denote by φ the angle formed by the secant and the positive direction of the x-axis. Form the ration $\frac{\Delta y}{\Delta x}$. From the figure 58, it follows immediately,

$$\frac{\Delta y}{\Delta x} = \tan\varphi \tag{11}$$

Now if Δx approaches zero, the point M_1 will move along the curve always approaching M_0. The secant M_0M_1 will turn about M_0 and the angle φ will change in Δx.

If $\Delta x \to 0$ the angle φ approaches a certain limit α, the straight line passing through M_0 and forming an angle α with the positive direction of the abscissa axis will be the sought-for line tangent. It is easy to find its slope:

$$\tan\alpha = \lim_{\Delta x \to 0} \tan\varphi \;=\; \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

Hence,

$$f'(x) \;=\; \tan\alpha \tag{12}$$

which means that the values of the derivative $f'(x)$, for a given value of the argument x, is equal to the tangent of the angle formed with the positive direction of the x-axis by the line tangent to the graph of the function $f(x)$ at the corresponding point $M_0(x, y)$.