Putnam Mathematical Competition Analysis

This document provides a detailed examination of each problem from the 85th William Lowell Putnam Mathematical Competition.

# Problem A1

Problem Statement:  
Determine all positive integers n for which there exist positive integers a, b, and c satisfying  
[  
2a^n + 3b^n = 4c^n.  
]  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem A2

Problem Statement:  
For which real polynomials p is there a real polynomial q such that  
[  
p(p(x)) - x = (p(x) - x)^2 q(x)  
]  
for all real x?  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem A3

Problem Statement:  
Let S be the set of bijections  
[  
T colon 1,2,3 times 1,2,dots,2024 to 1,2,dots,6072  
]  
such that T(1,j) < T(2,j) < T(3,j) for all j in 1,2,dots,2024 and T(i,j) < T(i,j+1) for all i in 1,2,3 and j in 1,2,dots,2023. Do there exist a and c in 1,2,3 and b and d in 1,2,dots,2024 such that the fraction of elements T in S for which T(a,b) < T(c,d) is at least 1/3 and at most 2/3?  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem A4

Problem Statement:  
Find all primes p > 5 for which there exists an integer a and an integer r satisfying 1 leq r leq p-1 with the following property: the sequence 1,a,a^2,dots,a^p-5 can be rearranged to form a sequence b\_0,b\_1,b\_2,dots,b\_p-5 such that b\_n-b\_n-1-r is divisible by p for 1 leq n leq p-5.  
  
Key Components:  
- This is a computational/analytical problem  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem A5

Problem Statement:  
Consider a circle Omega with radius 9 and center at the origin (0,0), and a disc Delta with radius 1 and center at (r,0), where 0 leq r leq 8. Two points P and Q are chosen independently and uniformly at random on Omega. Which value(s) of r minimize the probability that the chord overlinePQ intersects Delta?  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem A6

Problem Statement:  
Let c\_0,c\_1,c\_2,dots be the sequence defined so that   
[  
frac1-3x-sqrt1-14x+9x^24 = sum\_k=0^infty c\_k x^k  
]  
for sufficiently small x. For a positive integer n, let A be the n-by-n matrix with i,j-entry c\_i+j-1 for i and j in 1,dots,n. Find the determinant of A.  
  
Key Components:  
- This is a computational/analytical problem  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem B1

Problem Statement:  
Let n and k be positive integers. The square in the ith row and jth column of an n-by-n grid contains the number i+j-k. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly 1,2,dots,n?  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem B2

Problem Statement:  
Two convex quadrilaterals are called emphpartners if they have three vertices in common and they can be labeled ABCD and ABCE so that E is the reflection of D across the perpendicular bisector of the diagonal overlineAC. Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent?  
[A diagram has been omitted.]  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem B3

Problem Statement:  
Let r\_n be the nth smallest positive solution to tan x = x, where the argument of tangent is in radians. Prove that  
[  
0 < r\_n+1 - r\_n - pi < frac1(n^2+n)pi  
]  
for n geq 1.  
  
Key Components:  
- This is a proof-based problem  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem B4

Problem Statement:  
Let n be a positive integer. Set a\_n,0 = 1. For k geq 0, choose an integer m\_n,k uniformly at random from the set 1,dots,n, and let  
[  
a\_n,k+1 = begincases a\_n,k + 1, & mboxif m\_n,k > a\_n,k;   
a\_n,k, & mboxif m\_n,k = a\_n,k;   
a\_n,k-1, & mboxif m\_n,k < a\_n,k.  
endcases  
]  
Let E(n) be the expected value of a\_n,n. Determine lim\_nto infty E(n)/n.  
  
Key Components:  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem B5

Problem Statement:  
Let k and m be positive integers. For a positive integer n, let f(n) be the number of integer sequences x\_1,dots,x\_k,y\_1,dots,y\_m,z satisfying 1 leq x\_1 leq cdots leq x\_k leq z leq n and 1 leq y\_1 leq cdots leq y\_m leq z leq n. Show that f(n) can be expressed as a polynomial in n with nonnegative coefficients.  
  
Key Components:  
- This requires mathematical demonstration  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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# Problem B6

Problem Statement:  
For a real number a, let F\_a(x) = sum\_n geq 1 n^a e^2n x^n^2 for 0 leq x < 1.  
Find a real number c such that  
beginalign\*  
& lim\_x to 1^- F\_a(x) e^-1/(1-x) = 0 qquad mboxfor all a < c, and   
& lim\_x to 1^- F\_a(x) e^-1/(1-x) = infty qquad mboxfor all a > c.  
endalign\*  
  
Key Components:  
- This is a computational/analytical problem  
  
Approach Strategy:  
1. First, understand the given conditions  
2. Break down the problem into smaller parts  
3. Consider relevant theorems and techniques

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