Weekly 3 Problems

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1. (a) By considering the function $f(x) = \arctan(x)$, show that the set of real numbers has the same cardinality as the interval $(-\pi/2, \pi/2)$.

Solution: We know that f is continuous over $\mathbb{R}$ with codomain in $(-\pi/2, \pi/2)$. f^{-1} is also continuous over $(-\pi/2, \pi/2)$. We can create a bijection from $(-\pi/2, \pi/2)$ to $\mathbb{R}$ with f.

$$\begin{aligned}\forall x, y \in \mathbb{R}, f(x) = f(y) &\implies \arctan(x) = \arctan(y) \\ &\implies \tan\arctan(x) = \tan\arctan(y) \\ &\implies x = y\end{aligned}$$

f is injective.

$$\forall x \in (-\pi/2, \pi/2), \exists y \in \mathbb{R} \ni (y = \tan(x))$$

f in surjective. We have proven that $f : \mathbb{R} \to (-\pi/2, \pi/2)$ is a bijection. Therefore, the cardinality of $\mathbb{R}$ is equal to the cardinality of $(-\pi/2, \pi/2)$.

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(b) By considering the function $f(x) = \dfrac{x}{1+|x|}$, show that the set of real numbers has the same cardinality as the interval $(-1, 1)$.

Solution: We know that f is continuous over $\mathbb{R}$ with codomain in $(-1, 1)$. f^{-1} is also continuous over $(-1, 1)$. We can create a bijection from $(-1, 1)$ to $\mathbb{R}$ with f.

$$f(x) = \begin{cases} \frac{x}{1+x} & x \geq 0 \\ \frac{x}{1-x} & x < 0 \end{cases} \iff f^{-1}(x) = \begin{cases} \frac{x}{1-x} & x \geq 0 \\ \frac{x}{1+x} & x < 0 \end{cases}$$

$$\therefore f(x) = \frac{x}{1+|x|} \iff f^{-1}(x) = \frac{x}{1-|x|}$$

$$
\begin{aligned}
\forall x, y \in \mathbb{R}, f(x) = f(y) &\implies \frac{x}{1+|x|} = \frac{y}{1+|y|} \\
&\implies f^{-1}\left(\frac{x}{1+|x|}\right) = f^{-1}\left(\frac{y}{1+|y|}\right) \\
&\implies \frac{\frac{x}{1+|x|}}{1-|\frac{x}{1+|x|}|} = \frac{\frac{y}{1+|y|}}{1-|\frac{y}{1+|y|}|} \\
&\implies \frac{\frac{x}{1+|x|}}{1-\frac{|x|}{|1+|x||}} = \frac{\frac{y}{1+|y|}}{1-\frac{|y|}{|1+|y||}} \\
&\implies \frac{\frac{x}{1+|x|}}{1-\frac{|x|}{1+|x|}} = \frac{\frac{y}{1+|y|}}{1-\frac{|y|}{1+|y|}} \\
&\implies x = y
\end{aligned}
$$

f is injective.

$$
\forall x \in (-1,1), \exists y \ni \left(y = \frac{x}{1-|x|}\right)
$$

f is surjective. We have proven that $f : \mathbb{R} \to (-1,1)$ is a bijection. Therefore, the cardinality of $\mathbb{R}$ is equal to the cardinality of $(-1,1)$.

$\blacksquare$

(c) Show that two non-empty open intervals have the same cardinality.

Solution: Suppose two non-empty intervals (a,b) and (c,d). Let $f(x) = \dfrac{x-a}{b-a}$ and $g(x) = \dfrac{x-c}{d-c}$. Then $f(x)$ and $g(x)$ are both continuous functions on (a,b) and (c,d) respectively. Since $f(x)$ and $g(x)$ are continuous functions, they have the same inverse functions. Therefore, $f^{-1}(x)$ and $g^{-1}(x)$ are both continuous functions on $(0,1)$.

$$
f^{-1}(x) = (b-a)x + a \qquad g^{-1}(x) = (d-c)x + c
$$

We can create a bijection between from (a,b) to (c,d). Lets say we have a element $\omega \in (a,b)$. $f(\omega) \in (0,1)$. We can put this value in $g^{-1}(x)$ to get a value that is in (c,d).

$$
\omega \in (a,b) \implies g^{-1}(f(\omega)) \in (c,d)
$$

We can do the same thing in the other direction. Lets say we have a element $\varpi \in (c,d)$. $g(\varpi) \in (0,1)$. We can put this value in $f^{-1}(x)$ to get a value that is in (a,b).

$$
\varpi \in (c,d) \implies f^{-1}(g(\varpi)) \in (a,b)
$$

Therefore, we have a bijection between (a, b) and (c, d). Since the two intervals are non-empty, they have the same cardinality.

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(d) Show that set of real number has the same cardinality as any non-empty open interval.

Solution: Suppose we have a non-empty open interval (a, b). Let $f(x) = \dfrac{x-a}{b-a}$. Then $f(x)$ is a continuous function on (a, b). Since $f(x)$ is a continuous function, it has an inverse function. Therefore, $f^{-1}(x)$ is a continuous function on $(0, 1)$.

$$f^{-1}(x) = (b-a)x + a$$

Now $g : (0, 1) \to \mathbb{R}$, defined as,

$$g(x) = \tan\left(\frac{2x-1}{2}\pi\right)$$

Then $g(x)$ is a continuous function on $(0, 1)$. Since $g(x)$ is a continuous function, it has an inverse function. Therefore, $g^{-1}(x)$ is a continuous function on $\mathbb{R}$.
Lets say we have a element $\omega \in (a, b)$. $f(\omega) \in (0, 1)$. We can put this value in $g(x)$ to get a value that is in $\mathbb{R}$.

$$\omega \in (a, b) \implies g(f(\omega)) \in \mathbb{R}$$

We can do the same thing in the other direction. Lets say we have a element $\varpi \in \mathbb{R}$. $g(\varpi) \in (0, 1)$. We can put this value in $f^{-1}(x)$ to get a value that is in (a, b).

$$\varpi \in \mathbb{R} \implies f^{-1}(g^{-1}(\varpi)) \in (a, b)$$

Therefore, we have a bijection between (a, b) and $\mathbb{R}$. Since the two intervals are non-empty, they have the same cardinality.

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2. (a) Let I be the set of decimal of the form $0.d_1d_2\cdots$. Construct a one to one function from I to $I \times I$.

Solution:
Let f be a function that takes some a decimal number a of form $0.d_1d_2d_3\cdots$ and map it to the ordered pair $(0.d_1d_3d_5\cdots, 0.d_2d_4d_6\cdots)$. Then f is a one to one function from I to $I \times I$.
We will prove f to be injective with the following steps.

$$\begin{aligned}
&\forall a, b \in I, f(a) = f(b) \\
&\quad\implies (0.a_1a_3\cdots, 0.a_2a_4\cdots) = (0.b_1b_3\cdots, 0.b_2b_4\cdots) \\
&\quad\implies \forall i > 0, a_i = b_i \\
&\quad\implies a = b
\end{aligned}$$

(b) Find either an onto function from I to $I \times I$ or a one-to-one function for $I \times I$ to I.

Solution:
Let f be a function that takes some ordered pair (a, b) and map it to the decimal number $0.a_1a_2\cdots$. Then f is a onto function from I to $I \times I$.
We will prove f to be surjective with the following steps.

$$\begin{aligned}
&\forall (a, b) \in I \times I, \exists c \in I, f(c) = (a, b) \\
&\quad\implies f(0.c_1c_2c_3c_4\cdots) = (0.a_1a_2\cdots, 0.b_1b_2\cdots) \\
&\quad\implies \forall i > 0, c_{2i-1} = a_i \wedge c_{2i} = b_i \\
&\quad\implies c = 0.a_1b_1a_2b_2\cdots
\end{aligned}$$

(c) Do I and $I \times I$ have the same cardinality?

Solution:
We shown that our choice of function f is injective as well as surjective, that mean f is injective. Therefore, I and $I \times I$ have the same cardinality.

3. (a) Let $f : A \to B$ be one-to-one and onto. Define $g : B \to A$ by saying $a = g(b) \iff b = f(a)$. Show that g is a function and that it is one-to-one and onto.

Solution:
We will first show g is injective.

$$\begin{aligned}\forall b_1, b_2 \in B, & g(b_1) = g(b_2) \\ & \implies f(g(b_1)) = f(g(b_2)) \\ & \implies b_1 = b_2\end{aligned}$$

We will now show g is surjective.

$$\begin{aligned}\forall a \in A, & \exists b \in B \ni g(b) = a \\ & \implies f(a) = f(g(b)) \\ & \implies b = f(a)\end{aligned}$$

Therefore, g is a function and that it is one-to-one (injective) and onto (surjective).

(b) Show that, if $a \in A$, then $g(f(a)) = a$, if $b \in B$, then $f(g(b)) = b$. Explain why g is called the g **inverse function** of f.

Solution:
We will show $\forall a \in A$, $g(f(a)) = a$.

$$\begin{aligned}\forall a \in A, & f(a) = b \\ & \iff g(f(a)) = g(b) \\ & \iff g(f(a)) = a\end{aligned}$$

We will show $\forall b \in B$, $f(g(b)) = b$.

$$\begin{aligned}\forall b \in B, & g(b) = a \\ & \iff f(g(b)) = f(a) \\ & \iff f(g(b)) = b\end{aligned}$$

Above results shows that,

$$(\forall a \in A, g(f(a)) = a) \land (\forall b \in B, f(g(b)) = b) \iff (f = g^{-1}) \land (g = f^{-1})$$

That's why g is called the inverse function of f.

$\blacksquare$

(c) Examine your proof of (a) and carefully pick out which properties of f lead to which properties of g.

Solution:
The properties are that,

1. f is bijective.
2. g was defined as $a = g(b) \iff b = f(a)$

(d) If A is finite and $f : A \to B$ is one-to-one, show that the number of elements of $f(A) = \{y \in B : \exists x \in A \ni (y = f(x))\}$ is same as the number of elements of A.

Solution:
We can prove the argument with,

$$\forall y \in B, \exists x \in A \ni y = f(x)$$

For every element in codomain we have some element in domain. Due to the definition of function the element in domain will be unique. Therefore, the number of elements of $f(A)$ is same as the number of elements of A.

$$\therefore |A| = |B|$$

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(e) Show that it is impossible to have a one-to-one correpondance between a finite set and one of its proper subsets.

Solution:
We will prove this contradiction with the following steps. Suppose two sets A and B such that $A \subset B$ and a bijective function $f : A \to B$. f will map elements from domain to themselves in codomain.

$$\forall a \in A, \exists b \in B \ni b = f(a) = a$$
$$A \subset B \implies \exists b' \in B \ni b' \notin A$$

f is not mapping any element to b', therefore f is not a bijective function. Contradiction. Therefore, it is impossible to have a one-to-one correpondance between a finite set and one of its proper subsets.

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4. A relation on a set X is a set of ordered pairs of elements of X. A relation is often denoted by a symbol like $\sim$, and we write "$x \sim y$" (and say "x is related to y") to indicate that (x, y) is an element of the relation. The relation is called an equivalence relation on X if it has the properties:

   1. $x \sim x$ for all $x \in X$
   2. if $x \sim y$ then $y \sim x$
   3. if $x \sim y$ and $y \sim z$ then $x \sim z$

   (a) Show that $=$ and $\leq$ are equivalence relations on the real numbers, but that $<$ is not.

Solution: For $=$,

$$\begin{aligned} \forall x \in X, & x = x \\ x = y & \implies y = x \\ (x = y) \wedge (y = z) & \implies (x = z) \end{aligned}$$

We can say that every number is equal to itself. If a number is equal to the other number then the other number is equal to the first number. If one number is equal to another number and that another number is equal to some other number, then the one number is equal to some other number. Hence $=$ is an equivalence relation over set of numbers.
For $\leq$,

$$\begin{aligned} \forall x \in X, & x \leq x \\ x \leq y & \implies y \leq x \\ (x \leq y) \wedge (y \leq z) & \implies (x \leq z) \end{aligned}$$

The reflexivity and transivity is trivial and true. We have to discuss over symmertic property. This can be divided into two cases because $x \leq y$ is equivalent to $(x = y) \vee (x < y)$. If the number x and y are equal then we have proved that $=$ relation is symmertic, otherwise x can be less than y and in this case $y \leq x$ is false. and we know that if our premise is true and conclusion is false then the implication is false. $\therefore \leq$ is not symmertic over set of numbers. Hence $\leq$ is not an equivalence relation over set of numbers.
For $<$, we only need to discuss the reflexivity of numbers on $<$.

$$\forall x \in X, x < x$$

This can not be true because no number can be less than to itself. $\therefore <$ is not an equivalence relation over set of numbers.

$\blacksquare$

(b) Is $\subseteq$ an equivalence relation on sets?

Solution: Let A, B and C be any set. It is for sure that any set is an improper subset of itself, which makes it the subeset of itself.

$$\therefore A \subseteq A$$

If $A \subseteq B$, this is equivalent of saying $(A \subset B) \vee (A = B)$. If $A \subseteqq B$ (improper subeset), which means they are equal then they are subeset of each other means $A \subseteq B$ and $B \subseteq A$. If $A \subset B$, then $B \not\subset A$, but if the premise is true and conclusion is false then the implication is false.

$$A \subseteq B \nRightarrow B \subseteq A$$

Hence, $\subseteq$ is not an equivalent relation over collection of sets.

$\blacksquare$

(c) Is ... is related to ... an equivalence relation on the set of people?

Solution: There are different type of relations among humans; like biological relation, social relation and ethnic relations etc. To prove the statement I would assume that we are talking about the biological relations.

Then If two people let say p_1 and p_2 are a member of set $\mathbb{P}$ that contains all people in the world. If p_1 is the father of p_2, then p_2 is related to p_1, but only a person can be related to his ancestors, ancestors can not be related to the person. Therefore, p_1 is not related to p_2. Hence the relation is not symmertic, therefore it is not equivalent relation.

(d) Is $\cdots$ is acquainted with $\cdots$ an equivalence relation on the set of people?

Solution:

If two people p_1 and p_2 are acquainted with each other, then they are related to each other. If p_1 is acquainted with p_2, then p_2 is acquainted with p_1. If p_1 is acquainted with p_2 and p_2 is acquainted with p_3, then p_1 is acquainted with p_3. Also, p_1 is acquainted with itself. Hence, the relation is equivalent relation.

(e) Let X be a set and an equivalence relation on X. For any element a of X, let $X_a = \{x \in X : x \sim a\}$. Show that:

1. $X_a \neq \emptyset$ for all a,
2. if $X_a \cap X_b \neq \emptyset$, then $X_a = X_b$,
3. $X = \bigcup\limits_a X_a$

Solution:

1. X_a can not be empty because for equivalence relation to hold it must be reflexive. Therefore,
$$\{a\} \subseteq X_a \implies X_a \neq \emptyset$$

2. Let ω be any element from $X_a \cap X_b$,
$$\begin{aligned} &\omega \in X_a \cap X_b \\ \implies &(\omega \in X_a) \wedge (\omega \in X_b) \\ \implies &(\omega \sim a) \wedge (\omega \sim b) \\ \implies &(a \sim \omega) \wedge (\omega \sim b) \\ \implies &a \sim b \\ \implies &b \in X_a \end{aligned}$$

   b will also be in X_a and due to transivity all the element that are related to b would also be in X_a. This means all elements of X_b are in X_b and vice versa.
$$X_a = X_b$$
   We can also state this as,
$$X_a \cap X_b \neq \emptyset \implies a \sim b$$

3. The contrapostive of the implication we had while proving (2) is,
$$a \nsim b \implies X_a \cap X_b = \emptyset$$
   This means if a is not related to b they do not have any element in common. This shows that the relation $\sim$ partitions the set. And if take union of all those partitions we would have the orginal set.
$$X = \bigcup_{a \in X} X_a$$

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