Differential & Integral Calculus

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Function, Domain & Range

1. a) Define function with example. Describe the various types of function. (Au-25)
   b) Test whether; the following functions are even or odd:
   i. f(x) = $1 - \frac{1}{x^2}$
   ii. f(x) = $x + \frac{1}{x}$
   iii. f(x) = $\ln(x + \sqrt{1 + x^2})$
2. a) Define domain & range of a function with example.
   b) Find the domain & range of the following function:
   i. f(x) = x+3
   ii. f(x) = x^2+x+2
   iii. f(x) = $\sqrt{x+3}$
   iv. f(x) = $\sqrt{16 - x^2}$
   v. $f(x) = \sqrt{x^2 - 9}$ (Au-25)
   vi. f(x) = $\sqrt{\frac{1-x}{x}}$ (Sp-25)
   vii. $f(x) = \frac{x+1}{x-1}$
   viii. f(x) = $\frac{x-3}{2x+1}$
   ix. f(x) = $\frac{x}{|x|}$
   x. f(x) = $|x| - x$
   xi. f(x) = $|x+1| + |x-1|$
   xii. f(x) = $|x| + |x-1|$
   xiii. f(x) = $|x+2| + |x-2|$
   xiv. f(x) = $|x+3| + |x-3|$. (Sp-24)
   xv. f(x) = $|x+1| + |x| + |x-1|$
   xvi. $f(x) = \begin{cases} 3, & -3 \le x < -1 \\ -6x-3, & -1 \le x \le 0 \\ 3x-3, & 0 < x \le 1 \end{cases}$
3. Draw the graph of the following function:
   i. f(x) = $\sqrt{x}$
   ii. f(x) = $\frac{1}{x}$
   iii. f(x) = $-x^2 + 1$
   iv. f(x) = $3 - 2x - x^2$
   v. $f(x) = \begin{cases} x^2+1, & x < 0 \\ x, & 0 \le x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$
   vi. $f(x) = \sin x, \quad \cos x, \quad \tan x$
   vii. $f(x) = a^x, a > 1$
   viii. $f(x) = |x| + |x-1|$

ix. $f(x) = \begin{cases} 3 + 2x, & \frac{-3}{2} < x < 0 \\ 3 - 2x, & 0 \le x < \frac{3}{2} \\ -3 - 2x, & x \ge \frac{3}{2} \end{cases}$

Limit

1. Define limit of a function.
2. a) If $f(x) = \begin{cases} 1 + 2x, & \frac{-1}{2} \le x < 0 \\ 1 - 2x, & 0 \le x < \frac{1}{2} \\ 2x - 1, & x > \frac{1}{2} \end{cases}$

   Does limit $\lim_{x \to 0} f(x)$ and $\lim_{x \to \frac{1}{2}} f(x)$ exists?

   b) If $f(x) = \begin{cases} x^2, & x < 1 \\ 3, & x = 1 \\ x^2 + 1, & x > 1 \end{cases}$

   Does limit $\lim_{x \to 1} f(x)$ exists?

   c) If $f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$

   Does limit $\lim_{x \to 0} f(x)$ exists?

   d) Let, $f(x) = \frac{3x + |x|}{7x - 5|x|}$ does limit $\lim_{x \to 0} f(x)$ exists?

Continuity

1. Define continuity of a function.
2. Test the continuity of the following function:

   $f(x) = \begin{cases} 5, & 0 < x < 1 \\ 10, & 1 \le x \le 2 \\ 15, & 2 < x \le 3 \end{cases}$ at x = 2
3. Test the continuity of the following function:

   $f(x) = \begin{cases} \frac{1 - \cos x}{x^2}; & x \ne 0 \\ 1; & x = 0 \end{cases}$ at $x = 0$ (Au-25)
4. Test the continuity of the following function at x = 0 and $x = \frac{3}{2}$

   $f(x) = \begin{cases} 3 + 2x, & \frac{-3}{2} \le x < 0 \\ 3 - 2x, & 0 \le x < \frac{3}{2} \\ -3 - 2x, & x \ge \frac{3}{2} \end{cases}$ (Sp-24)
5. Test the continuity at x = 0 and x = 1

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x, & 0 \le x \le 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

6. Test the continuity at x = 0,1,2.

$$f(x) = \begin{cases} -x^2, & x \le 0 \\ 5x - 4, & 0 < x \le 1 \\ 4x^2 - 3x, & 1 < x < 2 \\ 3x + 4, & x \ge 2 \end{cases}$$

7. Test the continuity at x = 0

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

8. Test the continuity at x = 0

$$f(x) = \begin{cases} -cosx, & x \ne 0 \\ cosx, & x = 0 \end{cases}$$

9. Test the continuity at x = 0

$$f(x) = \begin{cases} \frac{\log_e(1+x)}{\tan^{-1} x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

10. Test the continuity at x = 1,2

$$f(x) = \begin{cases} \log x, & 0 < x \le 1 \\ 0, & 1 < x < 2 \\ 1 + x^2, & x \ge 2 \end{cases}$$

11. Test the continuity of the function, $f(x) = |x| + |x + 1|$ at x = 1

Differentiability

1. Define differentiability of a function.
2. Test whether the function $f(x) = x^2 + 5$ is differentiable at the point x = 1.
3. A function f(x) is defined as;

$$f(x) = \begin{cases} 5x + 4, & 0 < x \le 1 \\ 4x^2 - 3x, & 1 < x < 2 \end{cases}$$

   i) Test the continuity at x = 1
   ii) Test whether the above function is differentiable at the point x = 1.

4. Test the differentiability of the following function at x = 0

$$f(x)=\begin{cases}3+2x, & \frac{-3}{2}<x<0\\ 3-2x, & 0\le x<\frac{3}{2}\\ -3-2x, & x\ge\frac{3}{2}\end{cases}\qquad \text{(Sp-24)}$$

5. Test the differentiability of the function at $x=\frac{\pi}{2}$

$$f(x)=\begin{cases}1, & x<0\\ 1+sinx, & 0\le x<\frac{\pi}{2}\\ 2+(x-\frac{\pi}{2})^2, & x\ge\frac{\pi}{2}\end{cases}\qquad \text{(Sp-24)}$$

6. Test the differentiability of the function at $x=2$

$$f(x)=\begin{cases}3x+2; & x>2\\ 12-x^2; & x\le 2\end{cases}\qquad \text{(Sp-25)}$$

7. Test the differentiability of the function at $x=2$

$$f(x)=\begin{cases}x; & 0<x<1\\ 2-x; & 1\le x\le 2\\ x-\frac{1}{2}x^2; & x>2\end{cases}\qquad \text{(Sp-25)}$$

Indeterminate Form

1. $\lim\limits_{x\to0}\frac{xe^x-\log(1+x)}{x^2}$
2. $\lim\limits_{x\to0}\frac{e^x-e^{-x}+2sinx-4x}{x^5}$
3. $\lim\limits_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$
4. $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}$
5. $\lim\limits_{x\to3}\frac{x^2-9}{x-3}$
6. $\lim\limits_{x\to0}\frac{\sqrt{1+x}-1}{x}$
7. $\lim\limits_{x\to a}\frac{x^{\frac{7}{2}}-a^{\frac{7}{2}}}{\sqrt{x}-\sqrt{a}}$
8. $\lim\limits_{x\to0}\frac{1-\cos x}{x^2}$
9. $\lim\limits_{x\to0}x^2\log x^2$
10. $\lim\limits_{x\to0}\frac{\tan x-\sin x}{x^3}$. (Sp-24)
11. Evaluate the limit using L. Hospital's rule $\lim\limits_{x\to0}(\frac{1}{x})^{\tan x}$; (Sp-25)
12. Evaluate the limit using L. Hospital's rule $\lim\limits_{x\to\infty}(e^x+1)^{\frac{-2}{x}}$. (Au-25)

Differentiation

1. Describe the physical meaning of derivative of a function.
2. The radius of a circle increases at a rate of 3 cm/sec. Find the rate of change of the area when i) the radius is 5 cm. ii) the area is 4π cm^2 (Sp-24)
3. By definition (First principal method), find the differential co-efficient of the following function:
   i. $f(x) = x^n, a^x, \ln x\ (Au-25), \log_a x$
   ii. $f(x) = e^x (Sp-25),\ e^{mx}, e^{\sqrt{x}}$
   iii. $f(x) = sinx\ (Sp-24), cosx, tanx, cotx, secx, cosecx$
   iv. $f(x) = \sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \cot^{-1} x, \sec^{-1} x, \csc^{-1} x$
4. Differentiate the following functions or derive the first derivatives from the following functions:
   i. $y = \sec\tan^{-1} x$
   ii. $y = \log\sec(ax+b)^3$ (Sp-24)
   iii. $y = \ln(x^2+2)$
   iv. $y = 5x^{\frac{3}{5}} + tan^2(2x)$
   v. $y = \log_a x + \log_x a$
   vi. $y = (\ln\sin x)^2$
   vii. $y = \frac{x^n}{\log_a x}$
   viii. $y = \tan^{-1}\frac{a+bx}{b-ax}$
   ix. $y = \sqrt[3]{x^2+\sqrt{x}}$
   x. $y = 10^{\ln\sin x}$
   xi. $y = e^{\sin x} - (\sin x^2)^2$
   xii. $y = x^{x^x}, x^{e^x}, (\sin x)^{\ln x}, x^{\cos^{-1} x}$
   xiii. $y = sin^2(\ln x^2)$
   xiv. $y = (\sec\tan^{-1} x)(x^5 + x^{\frac{1}{3}})^{\frac{5}{3}}$
   xv. $y = \sin(2\tan^{-1}\sqrt{\frac{1-x}{1+x}})$
   xvi. $y = \sin^{-1}(\frac{x+\sqrt{1-x^2}}{\sqrt{2}})$
5. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \cdots \dots\dots\dots \infty}}}$ $then\ find\ \frac{dy}{dx}$.
6. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \dots\dots\dots \infty}}}$; then show that, $y_1 = \frac{\cos x}{2y-1}$ (Sp-25)
7. If $y = cosx^{cosx^{cosx^{\cdot^{\cdot^{\cdot^{\infty}}}}}}$ then find $\frac{dy}{dx}$ (Au-25)
8. Differentiate $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\sec^{-1}\frac{1}{2x^2-1}$ (Au-25)
9. Differentiate $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$. (Sp-24, Sp-25))

10. If $C = 1 + r\cos\theta + \frac{r^2\cos 2\theta}{2!} + \frac{r^3\cos 3\theta}{3!} + \cdots \dots \dots \dots$

$S = r\sin\theta + \frac{r^2\sin 2\theta}{2!} + \frac{r^3\sin 3\theta}{3!} + \cdots \dots \dots \dots$

Prove that, $C\frac{dC}{dr} + S\frac{dS}{dr} = (C^2 + S^2)\cos\theta$

11. The radius of a circle increases at rate of 3 cm/sec. Find the rate of change of the area when i) the radius is 5 cm ii) the area is $4\pi cm^2$. (Sp-24)

Successive Differentiation

1. Find the n'th derivative of the following functions;
   i. $y = \ln x,\ \frac{1}{x},\ a^x,\ e^{mx},\ x^m,\ \sin x,\ \cos x,\ \ln(ax+b),\ \sin(ax+b),\ \cos(ax+b)$
2. State and prove Leibnitz's theorem.
3. If $y = x^n \log x$; prove that, $y_{n+1} = \frac{n!}{x}$. (Sp-24)
4. If $y = \tan^{-1} x$ then show that, $(1+x^2)y_{n+1} + 2nxy_n + n^2y_{n-1} - ny_{n-1} = 0$; (Sp-25)
5. If $y = e^{a\cos^{-1}x}\ or\ y = e^{a\sin^{-1}x}$ then show that, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$
6. If $x = \sin(\frac{1}{m}\ln y)\ then\ show\ that, (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ (Au-25)
7. $If\ \ln y = \tan^{-1} x,\ then\ show\ that,\ (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$
8. $If\ y = a\sin^{-1}x + b\cos^{-1}x\ then\ show\ that,\ (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$
9. If $y = f(x) = \sin^{-1} x$ then show that,
   i. $(1-x^2)y_2 - xy_1 = 0$
   ii. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$
   iii. $(y_n)_0 = (n-2)^2.(n-4)^2 \dots \dots \dots \dots 3^2.1^2$ for n is odd number.
   iv. $(y_n)_0 = 0$ for n is even number.

General Theorem & Expansion

Rolle's Theorem

1. State and prove **Rolle's** theorem.
2. Verify Rolle's theorem, $f(x) = 2x^3 + x^2 - 4x - 2$
3. Verify the Rolle's theorem for the function $f(x) = x^2 - 3x + 2$. in the interval (1, 2). (Au-25)
4. Suppose we are asked to determine whether Rolle's theorem can be applied to $f(x) = x^4 - 2x^2$ on the closed interval [-2, 2]. And if so, find all values of c in the interval that satisfies the theorem. (Sp-24)
5. Verify the Rolle's theorem for $f(x) = 2x^3 + x^2 - 4x - 2$ over $[-\sqrt{2}, \sqrt{2}]$; (Sp-25)

Mean Value Theorem

6. State and prove **Mean Value** theorem.
7. Verify the mean value theorem for the function, $f(x) = 3 + 2x - x^2$ in the interval (0, 1); (Sp-25)
8. Verify the mean value theorem for the function, $f(x) = 3x^3 + 7x^2 - 11x - 15 = 0$
9. Verify the mean value theorem for the function, $f(x) = \frac{x^3}{4} + 1$ over the interval [0, 2]. (Au-25)
10. Verify the mean value theorem for $f(x) = 2x^3 - 8x + 1\ when\ a = 1\ and\ b = 3.$ (Sp-24)
11. Verify the mean value theorem for the function, $f(x) = \ x^2 - 5x + 7$ in the interval $1 \leq x \leq 3.$

Taylor's Series:

1. State Taylor's series.
2. Expand $\frac{1}{x}$ in ascending power of $(x - 2)$ by Toylor's series. (Au-25)

Maclaurin's Series

12. State and prove **Maclaurian's** theorem.
13. Obtain the Maclaurian's series generated by the function, $f(x) = \log(1 + x)\ (Sp - 24, Au - 25), e^x, e^{mx}, sinx, cosx\ (Sp - 25), \ln x\ etc.$

<u>Recommendation Book:</u>

1. A text book on differential calculus - P. K.Bhattacharjee
2. Differential calculus – P.N. Chatterjee