Dynamics of a Rigid Body

Rigid body:

Definition: A rigid body is defined as that body which does not undergo any change in shape and volume when external forces are applied on it.

When forces are applied on a rigid body, the distance between any two particles of the body will remain unchanged, however larger force may cause any deformation in the object. **In actual practice, nobody is perfectly rigid. For practical purpose solid bodies are taken as rigid bodies.**

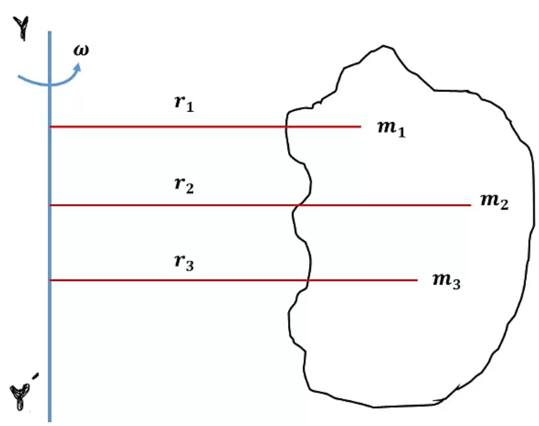
Moment of Inertia:

"We know that a body maintains the current state of motion unless acted upon by an external force." The measure of the inertia in the linear motion is the mass of the system and its angular counterpart is the so-called moment of inertia.

Definition: The moment of inertia of a body about an axis is the product of its mass and the square of its distance from that axis.

Mathematical interpretation:

The moment of inertia of a body is not only related to its mass but also the distribution of the mass throughout the body. Let us consider a body of mass M and any axis YY'. Imagine the body to be composed of a large number of particles of masses m_1, m_2, m_3, etc. at distance r_1, r_2, r_3…etc. from the axis YY' (as mentioned in the following figure).



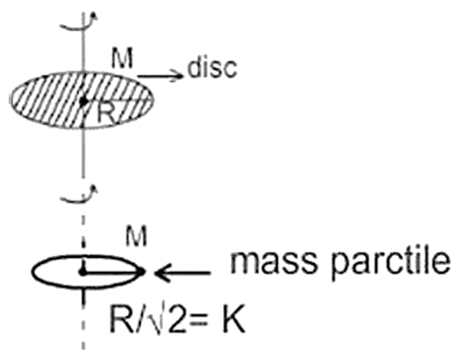
Then, the moment of inertia of the particle m_1 about YY' is $m_1 r_1^2$, that of the particle m_2 is $m_2 r_2^2$ and so on. Therefore, the moment of inertia, I of the whole body, about the axis YY' is equal to the sum of $m_1 r_1^2$, $m_2 r_2^2$, $m_3 r_3^2$..etc. Thus,

$$I = m_1 r_1^{2} + m_2 r_2^{2} + m_3 r_3^{3} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{I = \sum_{i=1}^{n} m_i r_i^2}$$

Radius of Gyration:

Definition: The radius of gyration of the body about an axis may be defined as the distance of a mass point from the same axis, whose mass is equal to the mass of the whole body and whose moment of Inertia is equal to the moment of inertia of the body, if rotated about the same axis. The example is given below:



So for disc and particle,

$$I = \frac{MR^2}{2} \text{..........}(1)$$

$$I = MK^2 \text{...........}(2)$$

Comparing (1)&(2)

$$K = \frac{R}{\sqrt{2}}.$$

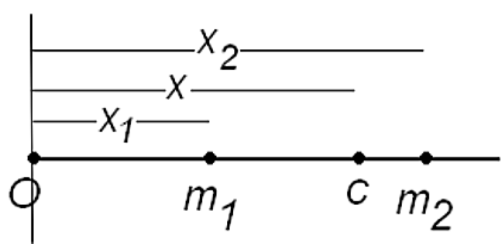
Here K is the radius of Gyration.

Center of Mass:

The terms "center of mass" and "center of gravity" are used synonymously in a uniform gravity field to represent the unique point in an object or system that can be used to describe the system's response to external forces and torques.

Definition: The concept of the center of mass is that of an average of the masses factored by their distances from a reference point. Or, the center of mass of a body or system is the point at which its entire mass may be considered to be concentrated for analyzing motion and the effect of external forces.

- It represents the average position of all the mass in a body or system.
- All external forces effectively act at this point.
- The motion of the whole body can be studied by analyzing the motion of its center of mass.
- If no external force acts, the center of mass remains at rest or moves with constant velocity.



As shown in the figure, consider two particles having masses m_1 and m_2 lying on the X-axis at distances x_1 and x_2, respectively, from the origin (O). The center of mass of this system is that point whose distance from the origin O is given by,

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Here, x is the mass-weighted average position of x_1 and x_2. The center of mass of the two particles of equal mass lies at the center (on the line joining the two particles between the two particles).

If the moments (mass × distance) about the center of mass are equal and opposite, then the system balances there.

From this, we can rearrange: $m_1 r_1 = m_2 r_2$………………. (1)

Or, $m_1 = m_2 \frac{r_2}{r_1}$…………………(2)

The center of mass lies on the line connecting the two masses.

Consider a set of n particles whose masses are m_1 m_2, m_3, … m_n and whose vector relative to an origin O are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, …….. $\mathbf{r}_n$ respectively

The centre of mass of this set of particles is defined as the point with position vector $\mathbf{r}_{CM}$

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M}$$

Here M is the total mass of the body.

Rotational Kinetic Energy:

Let us consider a rigid body rotates with angular velocity ω about an axis XY. Let the body contain `n` number of particles and their mass are $m_{1,} m_{2,}\ m_3$ … … … . m_n respectively and distance from the rotational axis $r_{1,} r_{2,}\ r_3$ … … … . r_n respectively.

Hence, if the tangential velocity of the particles is $v_1, v_2, v_3, \ldots \ldots \ldots . . v_n$, When the kinetic energy of the particle of mass m_1, is

$$k_1 = \frac{1}{2} m_1 v_1^2.$$

And we know, for the first particle, $v_1 = r_1\omega$ [ω is constant for any particle]

Similarly, the kinetic energy of the particle of mass m_2 is

$$k_2 = \frac{1}{2} m_2 \omega^2 r_2^2 \dots\dots\dots\dots\dots\dots.(2)$$

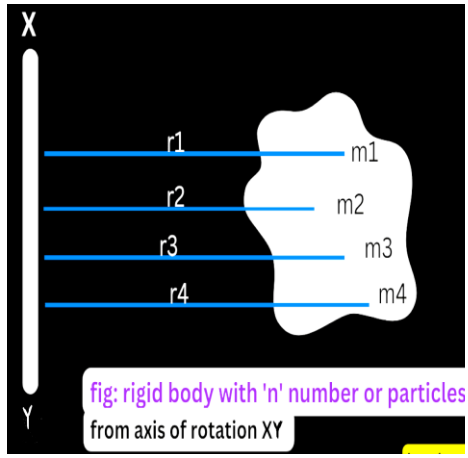
Similarly,

$$k_3 = \frac{1}{2} m_3 \omega^2 r_3^2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$k_n = \frac{1}{2} m_n \omega^2 r_n^2.$$



Hence, the kinetic energy of the rigid body is equal to the sum of the kinetic energy of all the particles.,

$$k = k_1 + k_2 + k_3 + \dots\dots\dots\dots\dots.. + k_n.$$

$$= \frac{1}{2} m_1 \omega^2 r_1^2 + \frac{1}{2} m_2 \omega^2 r_2^2 + \dots\dots\dots. + \frac{1}{2} m_n \omega^2 r_n^2.$$

$$= \frac{1}{2} \omega^2 (m_1 r_1^2 + +m_2 r_2^2 + \dots\dots\dots + +m_n r_n^2)$$

$$= \frac{1}{2} \omega^2 \sum_{i=1}^{n} m_i r_i^2$$

$$= \frac{1}{2} \omega^2 I \dots\dots\dots. \because I = \sum_{i=1}^{n} m_i r_i^2$$

$$\therefore k = \frac{1}{2} \omega^2 I.$$

Therefore, the kinetic energy of rotation

$= \frac{1}{2} \times$(moment of inertia) $\times$ $(angular\ velocity)^2$

Theorem of Perpendicular Axes:

(a) For a plane Lamina:

The theorem of perpendicular axes states that the moment of inertia of a plane laminar body about an axis perpendicular to the plane is equal to the sum of the moment of inertia about two mutually perpendicular axes in the plane of the lamina, such that the three mutually perpendicular axes have a common point of intersection.

Let us consider a plane lamina having the axes OX and OY in the plane of the lamina. The axis OZ passes through O and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles, each of mass m. Let a particle of mass m be at P with coordinates (x,y) and situated at a distance r from the point of intersection of the axes.

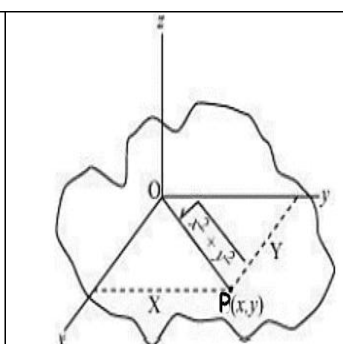


Fig.-01: Theorem of Perpendicular Axes for plane laminar.

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$$\therefore\ r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

The moment of inertia of the particle P about the axis

$$OZ = mr^2$$

The moment of inertia of the whole lamina about the axis OZ is given by

$$I_Z = \sum mr^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

The moment of inertia of the whole lamina about the axis OX is given by

$$I_X = \sum my^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Similarly,

$$I_Y = \sum mx^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Therefore from equation (2)

$$I_Z = \sum mr^2 = \sum m(x^2 + y^2)$$

$$I_Z = \sum mx^2 + \sum my^2 = I_Y + I_X$$

$$\therefore\ \boldsymbol{I_Z = I_Y + I_X}$$

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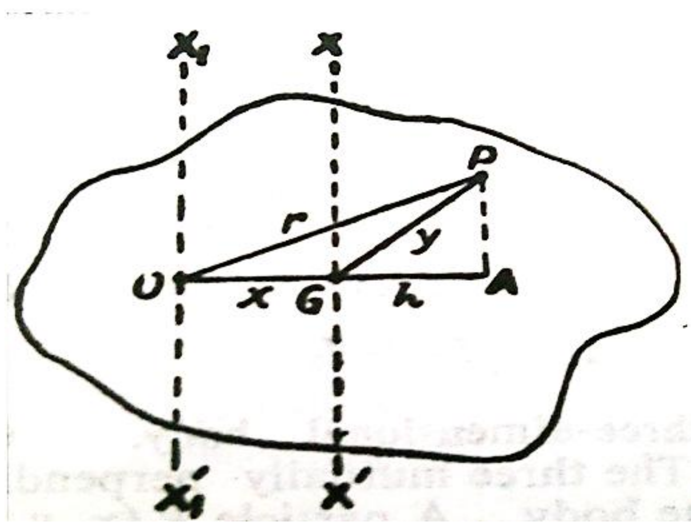
Theorem of Parallel Axes:

The theorem of parallel axes states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through the center of gravity and the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.



Let us consider a plane laminar body having its centre of gravity at G. The axis $X\acute{X}$ passes through the centre of gravity and its perpendicular to the plane of the laminar. The axis $X_1X_1{}'$ passes through the point O and is parallel to the axis $X\acute{X}$. The distance between the two parallel axes is x, which is shown in Fig.-01.

Let the lamina be divided into large number of particles each of mass m. The moment of inertia of the particle of mass m at P about the axis $X_1X_1{}'$ is equal to mr^2.The moment of inertia of the whole lamina about the axis X_1X_1' is given by

$$I_o = \sum mr^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

In the ΔOPA

$$OP^2 = (OA)^2 + (AP)^2$$
$$r^2 = (x+h)^2 + (AP)^2$$
$$r^2 = x^2 + 2xh + h^2 + (AP)^2$$
$$r^2 = x^2 + 2xh + h^2 + y^2 - h^2$$
$$r^2 = x^2 + y^2 + 2xh$$

Putting the above value in equation (1)

$$I_o = \sum m(x^2 + y^2 + 2xh)$$
$$I_o = \sum mx^2 + \sum my^2 + \sum m(2xh)$$
$$I_o = Mx^2 + I_G + 2x\sum mh$$

Here $\sum my^2 = I_G$ and $\sum mh = 0$

This is because the body balances about centre of mass at G. Therefore, the algebraic some of moments of all the particles about the centre of gravity, i.e.,

$$\sum mgh = 0$$

As g is constant

$$\sum mh = 0$$

Hence, we can write

$$\boldsymbol{I_o = I_G + Mx^2}$$

Conservation Theorem of Energy:

Statement: Energy may be transformed from one kind to another, but it cannot be created or destroyed; the total energy is constant. This is the principle of the conservation of energy.

This theorem can be expressed as

$$\Delta K + \Delta U + Q + (Change\ in\ other\ form\ of\ energy) = 0$$

Here, ΔK is the change in kinetic energy.

ΔU is the change in potential energy.

Q is the heat produced due to friction.

Conservation Theorem of Momentum:

Statement:

If the vector sum of the external forces on a system is zero, the total momentum of the system is constant.

Let us consider an idealized system consisting of two particles that interact with each other but not with anything else. Each particle exerts a force on the other; according to Newton's third law, the two forces are always equal in magnitude and opposite in direction. Hence the impulses that act on the two particles will be equal and opposite, and the changes in momentum of the two particles will be equal and opposite.

Let $\vec{F}_{\text{B on A}}$ be the force exerted by particle B on particle A and $\vec{F}_{\text{A on B}}$ be the force exerted by A on B.

Therefore, the rate of change of momentum of the two particles is

$$\vec{F}_{B\ on\ A} = \frac{d\vec{P}_A}{dt}, \qquad \vec{F}_{A\ on\ B} = \frac{d\vec{P}_B}{dt} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where, $\vec{P}_A$ = momentum of particle A.

$\vec{P}_B$ = momentum of particle B.

Since $\vec{F}_{B\ on\ A}$ and $\vec{F}_{A\ on\ B}$ are equal and opposite, then

$$\vec{F}_{B\ on\ A} = -\vec{F}_{A\ on\ B}$$

$$\Rightarrow \vec{F}_{B\ on\ A} + \vec{F}_{A\ on\ B} = 0 \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

From equation (1) and (2) , we get

$$\frac{d\vec{P}_A}{dt} + \frac{d\vec{P}_B}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}(\vec{P}_A + \vec{P}_B) = 0$$

$$\Rightarrow \frac{d\vec{P}}{dt} = 0 \qquad [\vec{P} = Total\ momentum = \vec{P}_A + \vec{P}_B$$

$$\therefore \vec{P} = Constant.$$

Collision:

In a collision, a relatively large force acts on each colliding particle for a relatively short time. The basic idea of a "collision" is that the motion of the colliding particles changes rather abruptly and that we can make a relatively clean separation of times that are "before the collision" and those that are "after the collision".

There are two types of collision: (i) Elastic Collision, (ii) Inelastic Collision

Elastic Collision: A collision in which the kinetic energy of the system is the same after the collision as before.

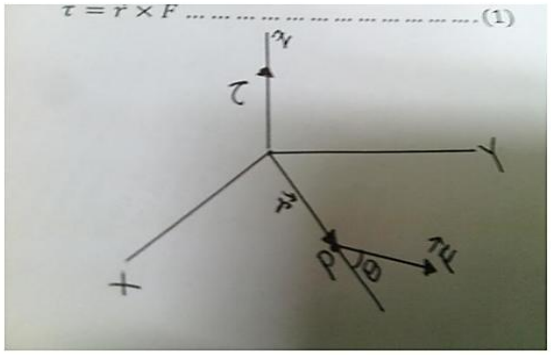
Inelastic Collision: A collision in which the total kinetic energy after the collision is less than that before the collision is called an inelastic collision.

Torque:

Torque is defined as the tendency of a force to rotate an object about an axis.

If a force $\vec{F}$ act on a single particle at a point P whose position with respect to the origin O of the inertial reference frame is given by the displacement vector $\overrightarrow{r}$, the torque τ acting on the particle with respect to the origin O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$



Torque is a vector quantity. Its magnitude is given by

$$\tau = rF\ sin\theta$$

where θ is angle between $\vec{r}$ and $\vec{F}$; its direction is normal to the plane formed by $\vec{r}$ and $\vec{F}$.

Flywheel:

A flywheel is a heavy metal disc with its mass concentrated mostly in its rim. The wheel is capable of rotation about a horizontal or a vertical axis. A thick rod, which is called the axle, passes through the center of gravity of the wheel, which rotates about the rod as the axis. The rod and the wheel are rigidly connected. The wheel is on a supported horizontal axis (Fig. 01). One end of a flexible cord is fixed to a small peg on the axle. The other end of the cord, which is wrapped around the axle, carries a mass M. The length of the cord is such that it becomes detached from the axle when the mass strikes the ground.