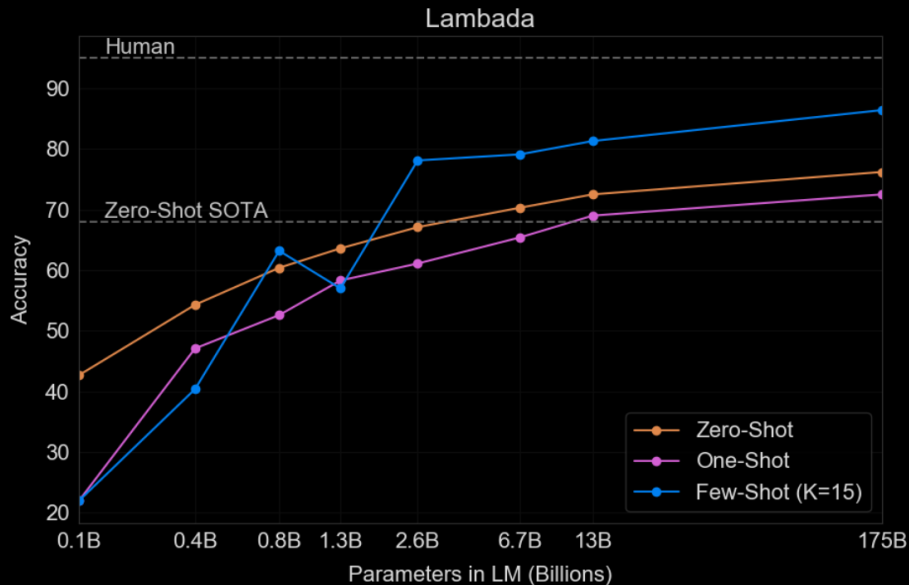
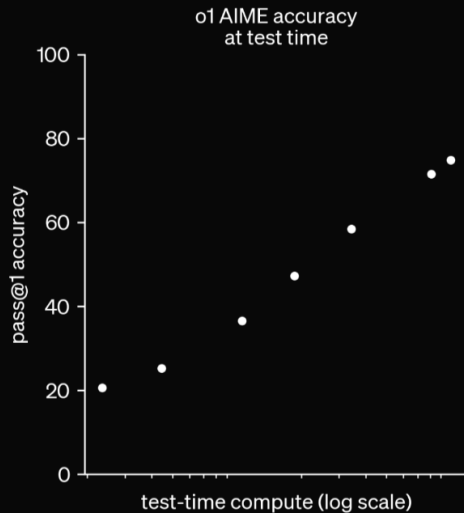
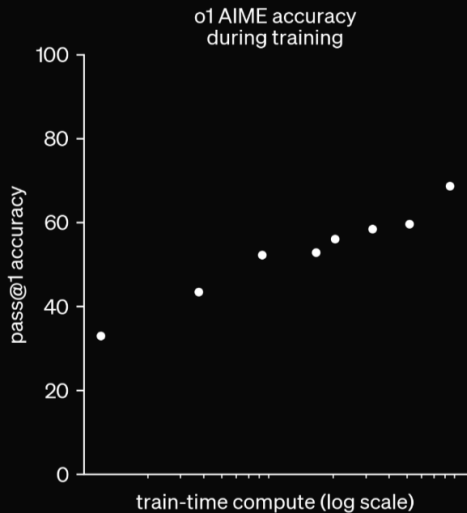


# **Speculations on Test-Time Scaling**

Sasha Rush Daniel Ritter

Cornell





For any finite set  $X$ , let  $|X|$  denote the number of elements in  $X$ . Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs  $(A, B)$  such that  $A$  and  $B$  are subsets of  $\{1, 2, 3, \dots, n\}$  with  $|A| = |B|$ . For example,  $S_2 = 4$  because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\}$$

giving  $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$ . Let  $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. Find the remainder when  $p + q$  is divided by 1000.

# The Bitter Lesson



The bitter lesson is based on the historical observations that 1) AI researchers have often tried to build knowledge into their agents, 2) this always helps in the short term, and is personally satisfying to the researcher, but 3) in the long run it plateaus and even inhibits further progress, and 4) breakthrough progress eventually arrives by an opposing approach based on scaling computation by **search and learning**.

# Importance of Search



The most important [lesson] is that I and other researchers simply didn't know how much of a difference scaling up search would make. If I had seen those scaling results at the start of my PhD, I would have shifted to researching search algorithms for poker much sooner and we probably would have gotten superhuman poker bots much sooner.

# Sources

- Survey of the public literature
- Synthesis of discussions with expert
- Rumors from social media

Thanks to Lewis Tunstall, Edward Beeching, Aviral Kumar, Charlie Snell, Michael Hassid, Yoav Artzi, Risab Agarwal, Kanishk Gandhi, Wenting Zhao, Yuntian Deng, Nathan Lambert, Noah Goodman

# Outline

Introduction

The Clues

Technical Background

The Suspects

What do we do now?



## o1 Description



Our large-scale **reinforcement learning algorithm** teaches the model how to think productively using its **chain of thought** in a highly **data-efficient** training process.

# Implication

- RL; Signal from verifiable problems
- CoT; Test-time occurs in token stream
- Data Efficient; Bounded set of problems

# Current Assumptions

- Single final language model
- Not following from expert examples
- Behavior is “learned”.

## Review: Chain of Thought

- In the process of answering a question, the model generates intermediate steps.
- These steps provide an scratchpad for hard technical problems.
- A limited number of examples provided.

# o1 Chain of Thought



o1 learns to hone its chain of thought and refine the strategies it uses. It learns to recognize and **correct its mistakes**. It learns to **break down tricky steps** into simpler ones. It learns to try a **different approach** when the current one isn't working.

## o1 CoT: Outlining

### *Implementation Outline:*

- 1. Capture input string as argument.*
- 2. Remove any spaces (if any).*
- 3. Parse input string to extract numbers as arrays.*
  - Since the input is in the format '[1,2],[3,4]', we can:*
  - Remove outer brackets if necessary.*
  - Split the string by '],' to get each row.*
  - For each row, remove '[' and ']', then split by ',' to get elements.*
- 4. Build a 2D array in bash (arrays containing arrays).*

## o1 CoT: Planning

*First, the cipher seems connected to the plaintext.*

*Given the time constraints, perhaps the easiest way is to try to see patterns.*

*Option 1: Try to find mapping from letters to letters.*

*Do any letters match?*

*First, let's write down the ciphertext and plaintext letters on top of each other.*

## o1 CoT: Backtracking

Similarly,  $m(x)^* (-x^2) = (-x^{2n+2} + m^{2n-2}x^{2n} + \text{lower terms})m(x)^* (-x^2) = (-x^{2n+2} + m^{2n-2}x^{2n} + \text{lower terms})$ .

Wait, actually, this may not help us directly without specific terms. An alternative is to consider the known polynomials.

So  $m(x) = k \dots$



## o1 CoT: Self-Evaluation

*Let's analyze each option.*

*Option A: "because appetite regulation is a field of staggering complexity."*

*Is that a good explanation? Hmm.*

*Option B: "because researchers seldom ask the right questions."*

*Does this make sense with the main clause?*

# Summary

- CoT provides test-time scaling
- CoT looks like search / planning in a classical sense
- RL needed to induce this behavior

# Outline

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# Technical Background

- Formalize sampling of latent reasoning
- Techniques from combinatorial sampling
- No learning yet.

Question: 4 baskets. 3 have 9 apples, 15 oranges, 14 bananas each. 4th has 2 less of each. Total fruits?

Let's solve step-by-step:

Fruits in one of first 3 baskets:  $9 + 15 + 14 = 38$

Total in first 3 baskets:  $38 * 3 = 114$

4th basket:  $(9-2) + (15-2) + (14-2) = 32$

Total fruits:  $114 + 32 = 146$

Answer: 146 fruits

# Stepwise CoT Sampling

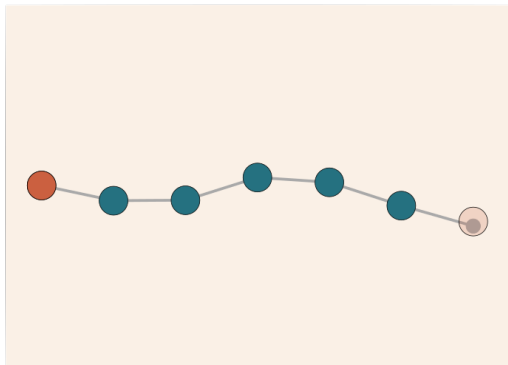
- $x$ ; problem specification
- $z_{1:T} \in \mathcal{S}^T$ ; chain of thought (CoT) steps
- $y \in \mathcal{Y}$ ; final answer

$$p(y|x) = \mathbb{E}_z p(y|x, z)$$

# Warm-up: Ancestral Sampling

$$z_{1:T} \sim p(\cdot | x)$$

$$y \sim p(\cdot | x, z_{1:T})$$



$T$  is the amount of test-time compute

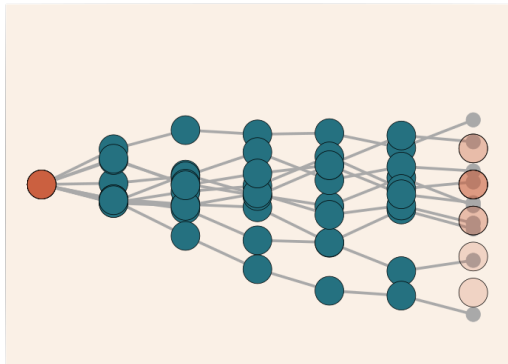
# Warm-up: Monte-Carlo (Self-Consistency)

For  $N$  samples,

$$z_{1:T} \sim p(\cdot | x)$$

$$y^n \sim p(\cdot | x, z_{1:T})$$

Pick majority choice  $y^n$



# Assumption: Verifier

$$\text{Ver}_x : \mathcal{Y} \rightarrow \{0, 1\}$$

Common datasets:

- Regular expression for math [Cobbe et al., 2021]
- Unit test for code [?]
- Test questions for science [Hendrycks et al., 2021a]



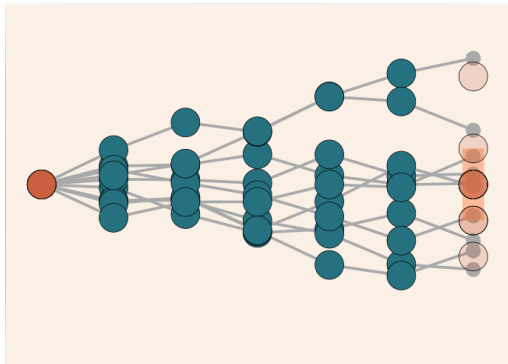
# Rejection Sampling / Best-of-N

For  $n = 1$  to  $N$ :

$$z^n \sim p(z|x)$$

$$y^n \sim p(y|x, z^n)$$

Verified set  $\{y^n : \text{Ver}_x(y^n)\}$

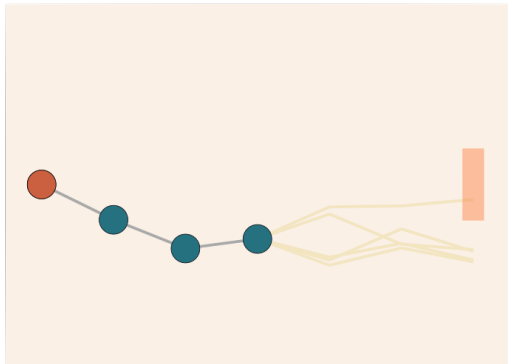


# Monte-Carlo Roll-Outs

Given partial CoT  $z_{1:t}$ , expected value,

$$\mathbb{E}_{y \sim p(\cdot|x)} \text{Ver}(y)$$

Monte Carlo for this expectation.



# Goal: Learning with Latent CoTs

Maximum likelihood;

$$\max_{\theta} \sum \log p(y|x; \theta) = \\ \sum \log \mathbb{E}_z p(y|x, z; \theta)$$

Classic combinatorial  
expectation

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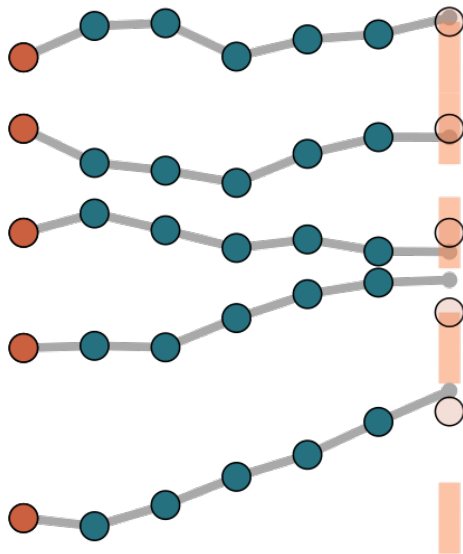
What do we do now?

# The Suspects

- Guess + Check
- Guided Search
- AlphaZero
- Learn to Search

# Suspect 1: Guess + Check

- 1) Sample  $N$  CoTs
- 2) Check if successful
- 3) Train on good ones



# Framework: Rejection Sampling EM

$$\max_{\theta} \sum \log E_{z \sim p(z|x; \theta)} p(y|x, z)$$

- E-Step: For  $n = 1$  to  $N$ :

$$z^n \sim p(\cdot|x)$$

$$y^n \sim p(\cdot|x, z^n)$$

Keep verified set  $\mathcal{Z} = \{z^n : \text{Ver}(y^n)\}$

- M-Step: Fit  $\theta' \leftarrow \arg \max_{\theta} \sum_{z \in \mathcal{Z}} \log p(z|x; \theta)$

# Variants

- Best-of-N Training [Cobbe et al., 2021]
- STaR [Zelikman et al., 2022]
- ReST [Gulcehre et al., 2023]
- ReST-EM [Singh et al., 2023]
- Filtered Rejection Sampling [Nakano et al., 2021]



# Reinforcement Learning?

- Batched  $\rightarrow$  Compute trajectories first, then train
- Online  $\rightarrow$  Update after each example
- Specific algorithm choice (PPO, etc)

# Empirical Results

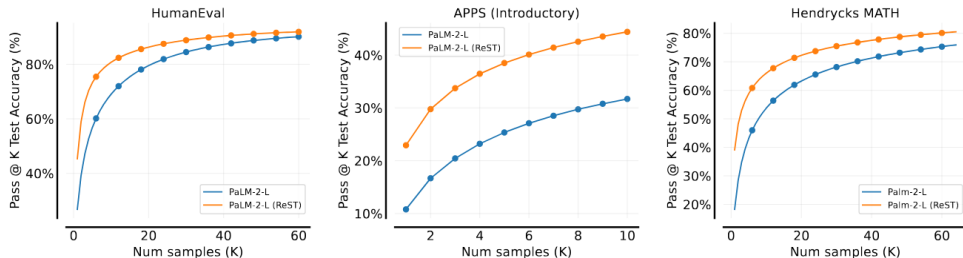


Figure 5 | **Pass@K results** for PaLM-2-L pretrained model as well as model fine-tuned with ReST<sup>EM</sup>. For a fixed number of samples K, fine-tuning with ReST<sup>EM</sup> substantially improves Pass@K performance. We set temperature to 1.0 and use nucleus sampling with  $p = 0.95$ .

# Is this o1?

## Pro

- ✓ Extremely simple and scalable
- ✓ Positive results in past work

# Is this o1?

## Pro

- ✓ Extremely simple and scalable
- ✓ Positive results in past work

- × No evidence this learns to correct, plan
- × Computationally inefficient search

## Suspect 2: Guided Search

- 1) During CoT sampling, use a heuristic to improve trajectories
- 2) Check if final versions are successful
- 3) Train on good ones

# Framework: Beam Search with Guide

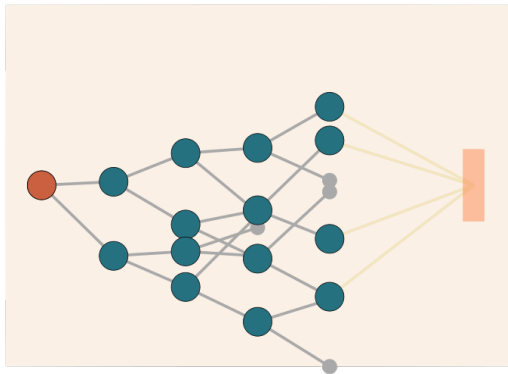
$r : \mathcal{S}^t \rightarrow \mathbb{R}$ ; Guide function

For each step  $t$

1. Sample next step,

$$z_t \sim p(\cdot | \mathbf{x}, z_{1:t-1}^n)$$

2. Keep the top  $N$  samples, ordered by  $r(z_t)$



# Guide Variants

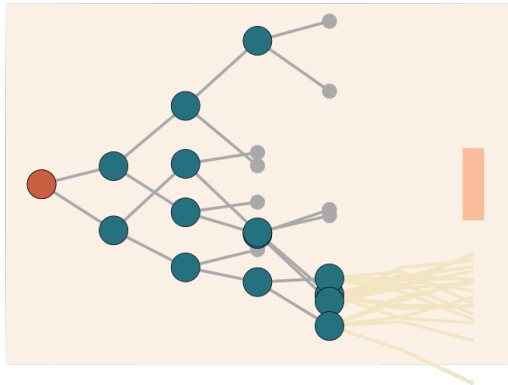
- Monte Carlo Roll-outs
- Learned Value Function (PRM)
- Interleaved Value Function

# Beam Search with Roll-Outs

For a  $z_t$ , sample answers

$$y^n \sim p(\cdot | x, z_{1:t-1})$$

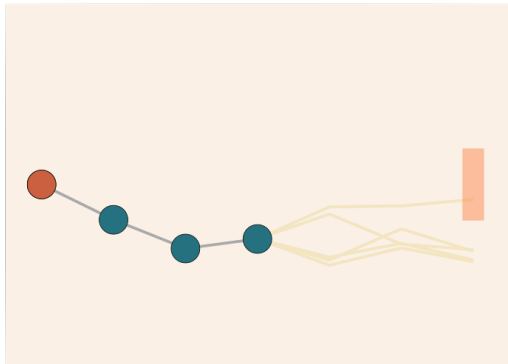
$$r_{MC}(z_t) = \frac{1}{N} \sum_{i=1}^N \text{Ver}(y^i)$$





# Learned Value Function

- Rollouts are costly, so instead learn a model  $r_\psi(z_t)$  to approximate rollouts
- Use  $r_{MC}$  to determine labels to train  $r_\psi$



# Interleaved Value Functions

- Combine  $r_{MC}$  and  $r_{\psi}$
- $r_{Inter}(z_t) = (1 - \alpha)r_{MC}(z_t) + \alpha r_{\psi}(z_t)$

# Variants

- Search Heuristic
- Value Function
- PRM; Process Reward Model  
[Uesato et al., 2022, Lightman et al., 2023]
- PAV; Process Advantage Verifier [Setlur et al., 2024]

# Test-time Guides Outperform Self-consistency

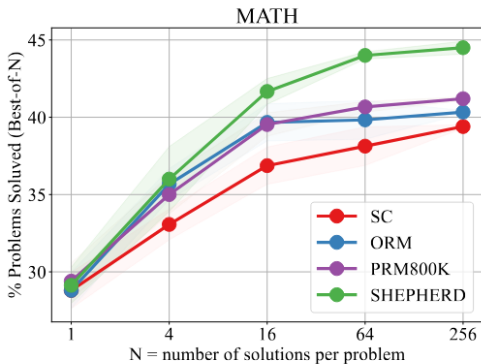
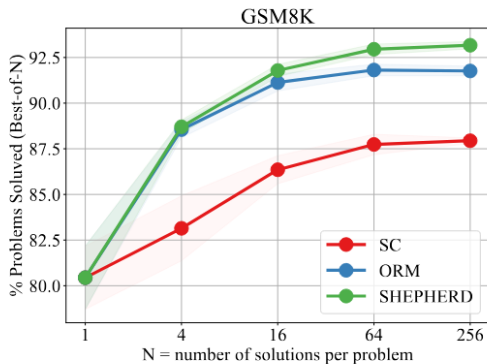


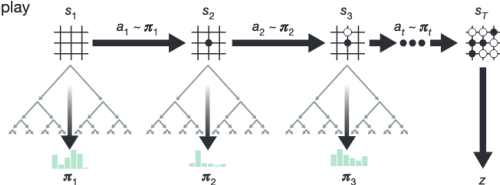
Figure 3: Performance of LLaMA2-70B using different verification strategies across different numbers of solution candidates on GSM8K and MATH.

## Is this o1?

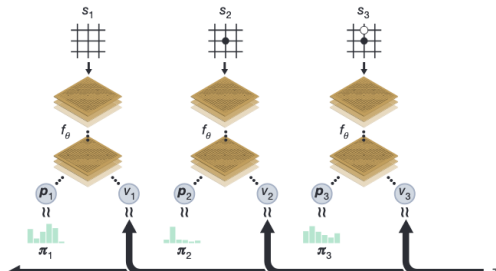
- ✓ RS needs to be more efficient.
- ✓ Learned rewards are effective
- ✗ o1 is a single test-time model
- ✗ Not clear if this is enough for planning.

# Reminder: AlphaZero

**a** Self-play



**b** Neural network training



- Canonical example of self-learning
- Scaling model without data

## Suspect 3: AlphaZero

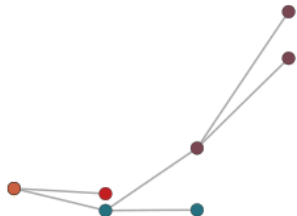
- 1) Self-play using guided-search with exploration
- 2) Label final outcomes of self-play games
- 3) Train guide and generator

# Framework: Expert Iteration

- Iterative algorithm combining learned model + expert search with a verifier.
- Generate samples using  $p(\textcolor{brown}{y}, \textcolor{teal}{z}|\textcolor{brown}{x})$ , reward model  $r(\textcolor{teal}{z}_t)$ , and search algorithm (e.g. beam search)
- Label samples using  $x(\textcolor{brown}{y})$
- Train  $p(\textcolor{brown}{y}, \textcolor{teal}{z}|\textcolor{brown}{x})$ ,  $r(\textcolor{teal}{z}_t)$  on the labeled samples, and repeat



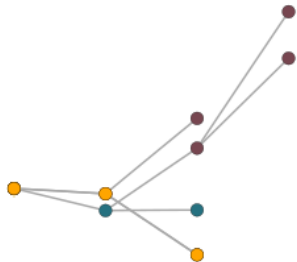
# Review: MCTS



# Review: MCTS



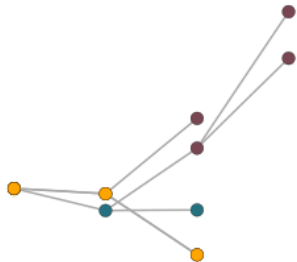
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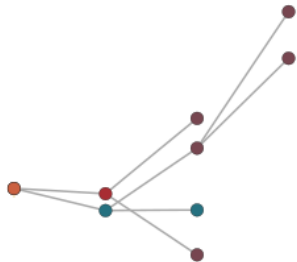
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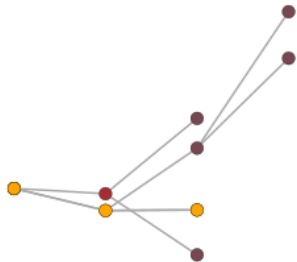
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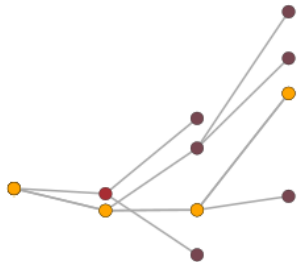
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## Review: MCTS

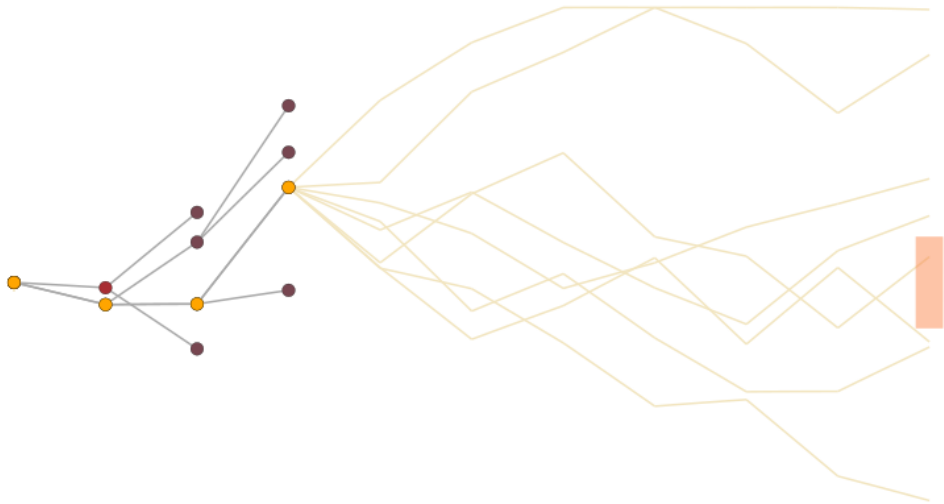


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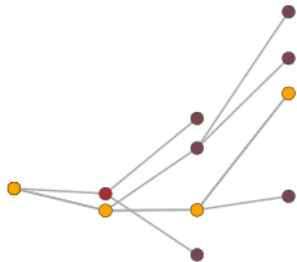




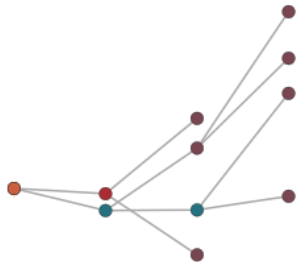
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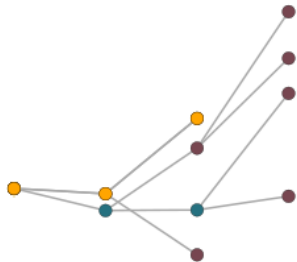
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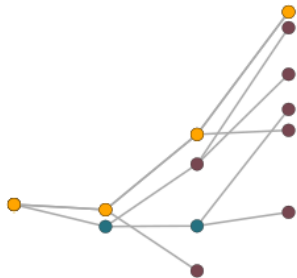
# Review: MCTS



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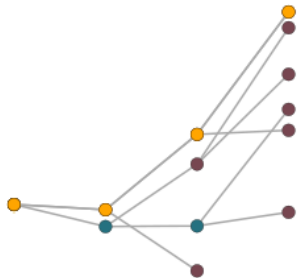
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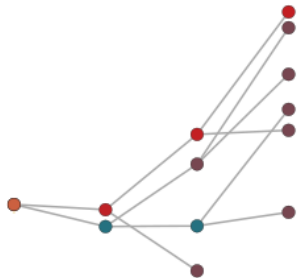
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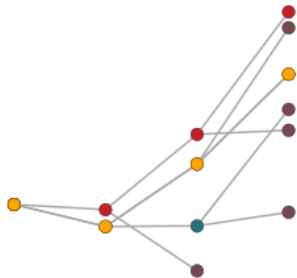


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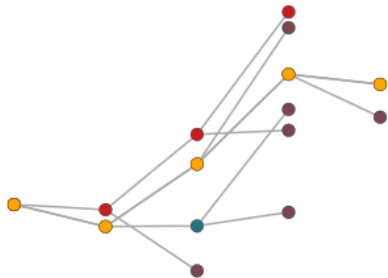




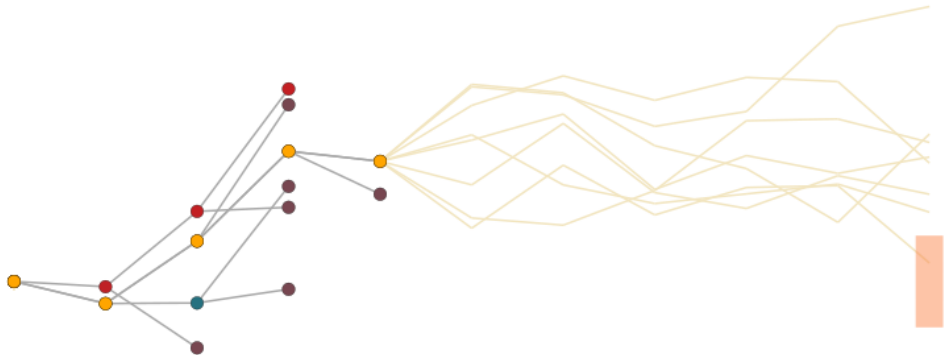
## Review: MCTS



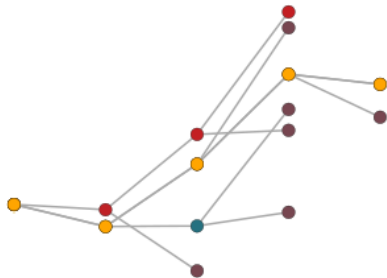
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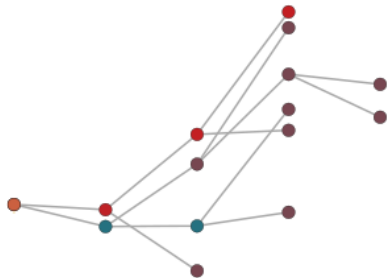
# Review: MCTS



# Review: MCTS



# Review: MCTS



# UCB for Language

- Selection: Walk down tree to leaf  $z_{t-1}$
- Expand: Sample  $N$  next steps  $z_t^n$ , pick one at random
- Rollouts: Sample steps  $z_{t+1} \dots z_T$
- Backprop: Update nodes counts  $z_{1:t}$  based on results

## Compared with Guided Search

- MCTS-UCB explores States
- 
-

# Empirical Results

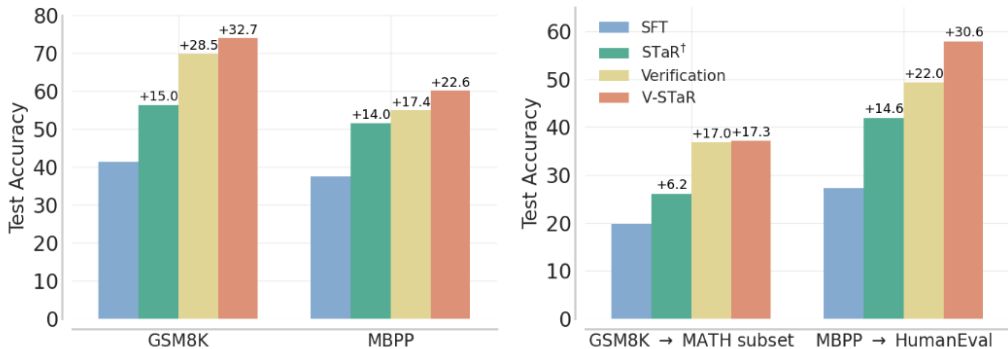


Figure 8: Test accuracy of 13B V-STaR compared to baselines. We report Best-of-64 for verification-based methods and Pass@1 for others. **(Left)** Test accuracy for training tasks. **(Right)** Transfer evaluation of GSM8K and MBPP trained models on MATH subset and HumanEval respectively.



# Is this o1?

✓ Major demonstrated RL result

✓ Scales to more train-time search

✗ Costly to maintain open states

✗ More complex algorithmically

## Suspect 4: Learning to Correct

- Sample  $N$  Successful CoTs
- Edit to inject incorrect expansions before correct ones.
- Train on correcting trajectories

# Self-Correction

- Argument: Training on  $x, z_1^*, y$  is too easy.
- Train instead on  $x, z', z_1^*, y$
- Model should learn to self-correct

# Score

- Positive rewards

# Challenge: Collapse

- Model may learn to just ignore negative
- 
-

# Generalized: Stream of Search

- Find  $z_{1:T}^*$  as optimal length CoT
- Find  $z'_{1:T'}$  with  $T' > T$  through backtracking tree search
- Train model on  $z'_{1:T'}$

# From Tree to Stream



# Empirical Results

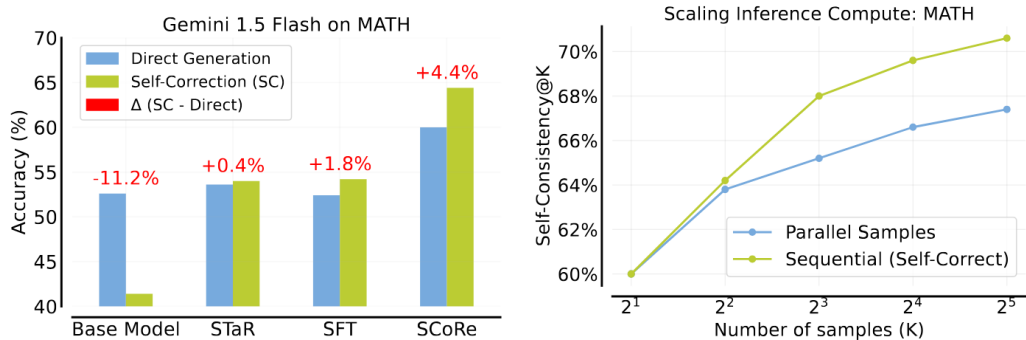


Figure 1 | **Left:** *SCoRe* achieves state-of-the-art self-correction performance on MATH; **Right:** *SCoRe* inference-time scaling: spending samples on *sequential* self-correction becomes more effective than only on *parallel* direct samples (Section 6.2).



# Empirical Results

Table 2 | **Performance of SCoRe on MATH.** SCoRe not only attains a higher accuracy at both attempts, but also provides the most positive self-correction performance  $\Delta(\mathbf{t1}, \mathbf{t2})$ .

Approach	Acc.@t1	Acc.@t2	$\Delta(\mathbf{t1}, \mathbf{t2})$	$\Delta^{i \rightarrow c}(\mathbf{t1}, \mathbf{t2})$	$\Delta^{c \rightarrow i}(\mathbf{t1}, \mathbf{t2})$
Base model	52.6%	41.4%	-11.2%	4.6%	15.8%
Self-Refine (Madaan et al., 2023)	52.8%	51.8%	-1.0%	3.2%	4.2%
STaR w/ $\mathcal{D}_{\text{StaR}}^+$ (Zelikman et al., 2022)	53.6%	54.0%	0.4%	2.6%	2.2%
Pair-SFT w/ $\mathcal{D}_{\text{SFT}}$ (Welleck et al., 2023)	52.4%	54.2%	1.8%	5.4%	3.6%
<b>SCoRe (Ours)</b>	<b>60.0%</b>	<b>64.4%</b>	<b>4.4%</b>	<b>5.8%</b>	<b>1.4%</b>

## Is this o1?

✓ Learns to correct and plan

× Complex training process

✓ Single test-time model

× Limited empirical evidence

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# Replication

**Does it need to be the same?**



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