Speculations on Test-Time Scaling

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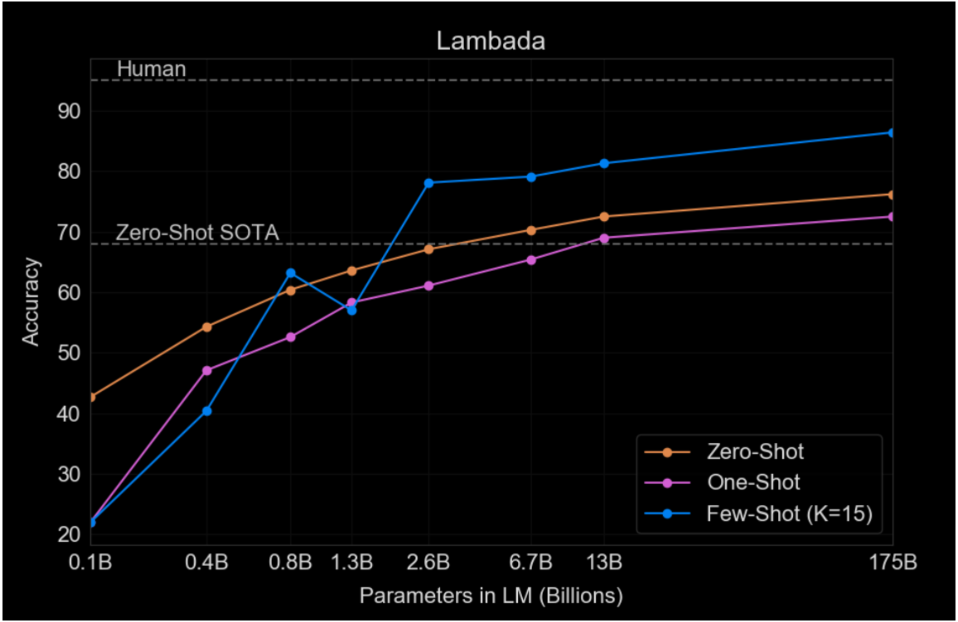
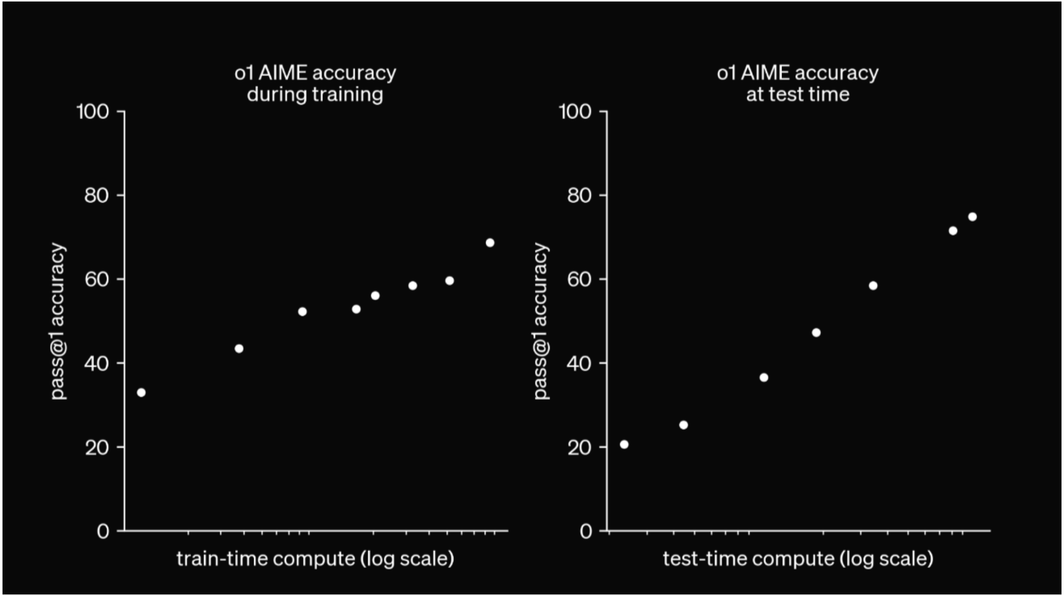
Cornell


Lambada
Human
Zero-Shot SOTA
Accuracy
90
80
70
60
50
40
30
20
0.1B
0.4B
0.8B
1.3B
2.6B
6.7B
13B
175B
Parameters in LM (Billions)
Zero-Shot
One-Shot
Few-Shot (K=15)

[?]


o1 AIME accuracy
during training
100
80
60
40
20
0
pass@1 accuracy
train-time compute (log scale)
o1 AIME accuracy
at test time
100
80
60
40
20
0
pass@1 accuracy
test-time compute (log scale)

[?]

AIME

[?]

For any finite set X, let $|X|$ denote the number of elements in X. Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \cdots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\}$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

The Bitter Lesson

[?]

The bitter lesson is based on the historical observations that 1) AI researchers have often tried to build knowledge into their agents, 2) this always helps in the short term, and is personally satisfying to the researcher, but 3) in the long run it plateaus and even inhibits further progress, and 4) breakthrough progress eventually arrives by an opposing approach based on scaling computation by **search and learning**.

Importance of Search

https://x.com/polynoamial/status/1840822629625688469

The most important [lesson] is that I and other researchers simply didn't know how much of a difference scaling up search would make. If I had seen those scaling results at the start of my PhD, I would have shifted to researching search algorithms for poker much sooner and we probably would have gotten superhuman poker bots much sooner.

Overview

- Survey of the public literature
- Synthesis of discussions with expert
- Rumors

Thanks to Lewis Tunstall, Edward Beeching, Aviral Kumar, Charlie Snell, Michael Hassid, Yoav Artzi, Risab Agarwal, Kanishk Gandhi, Wenting Zhao, Yuntian Deng, Nathan Lambert, Noah Goodman

Outline

Introduction

The Clues

Technical Background

The Suspects

What do we do now?

o1 Description

[?]

Our large-scale **reinforcement learning algorithm** teaches the model how to think productively using its **chain of thought** in a highly **data-efficient** training process.

Implication

- RL; Signal from verifiable problems
- CoT; Test-time occurs in token stream
- Data Efficient; Bounded set of problems

Current Assumptions

- Single final language model
- Not following from expert examples
- Behavior is “learned”.

Review: Chain of Thought

[?, ?]

- In the process of answering a question, the model generates intermediate steps.
- These steps provide an scratchpad for hard technical problems.
- A limited number of examples provided.

[?]

o1 Chain of Thought

o1 learns to hone its chain of thought and refine the strategies it uses. It learns to recognize and **correct its mistakes**. It learns to **break down tricky steps** into simpler ones. It learns to try a **different approach** when the current one isn't working.

o1 CoT: Outlining

Implementation Outline:
1. Capture input string as argument.
2. Remove any spaces (if any).
3. Parse the input string to extract numbers into arrays.
- Since the input is in the format ’[1,2],[3,4],[5,6]’, we can:
- Remove outer brackets if necessary.
- Split the string by ’],’ to get each row.
- For each row, remove ’[’ and ’]’, then split by ’,’ to get elements.
4. Build a 2D array in bash (arrays containing arrays).

o1 CoT: Planning

[?]

First, the cipher seems connected to the plaintext.
Given the time constraints, perhaps the easiest way is to try to see patterns.
Option 1: Try to find mapping from letters to letters.
Do any letters match?
First, let’s write down the ciphertext and plaintext letters on top of each other.

o1 CoT: Backtracking

Similarly, m(x)⋆(−x 2) = (−x2n + 2 + m2n−2x2n + lower terms)m(x) * (−x 2) = (−x 2n + 2 + m 2n−2 x 2n + lower terms). Wait, actually, this may not help us directly without specific terms.An alternative is to consider the known polynomials. So m(x) = k ...

o1 CoT: Self-Evaluation

Let's analyze each option.
Option A: "because appetite regulation is a field of staggering complexity."
Is that a good explanation? Hmm.
Option B: "because researchers seldom ask the right questions."
Does this make sense with the main clause?

Summary

- CoT provides test-time scaling
- CoT looks like search / planning in a classical sense
- RL needed to induce this behavior

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Technical Background

[?]

- Formalize sampling of latent reasoning
- Techniques from combinatorial sampling
- No learning yet.

Question: 4 baskets. 3 have 9 apples, 15 oranges, 14 bananas each. 4th has 2 less of each. Total fruits?
Let's solve step-by-step:
Fruits in one of first 3 baskets: 9 + 15 + 14 = 38
Total in first 3 baskets: 38 * 3 = 114
4th basket: (9-2) + (15-2) + (14-2) = 32
Total fruits: 114 + 32 = 146

Answer: 146 fruits

Stepwise CoT Sampling

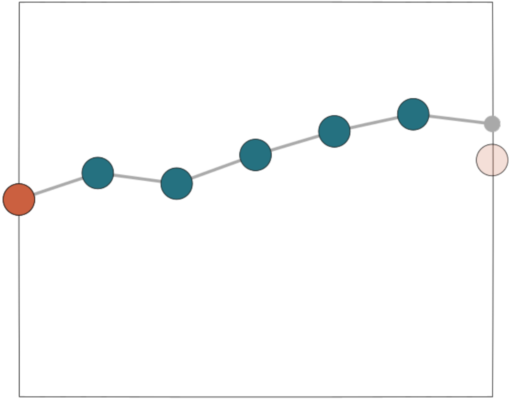
- x; problem specification
- $z_{1:T} \in \mathcal{S}^T$; chain of thought (CoT) steps
- $y \in \mathcal{Y}$; final answer

$$p(y|x) = \mathbb{E}_z p(y|x, z)$$

Warm-up: Ancestral Sampling

[?]

$$z_{1:T} \sim p(\cdot|x)$$
$$y \sim p(\cdot|x, z_{1:T})$$



T is the amount of test-time compute

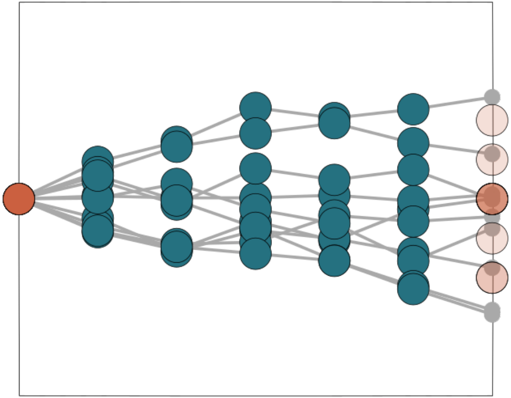
Warm-up: Monte-Carlo (Self-Consistency)

[?]

For N samples,

$$z_{1:T} \sim p(\cdot|x)$$
$$y^n \sim p(\cdot|x, z_{1:T})$$

Pick majority choice y^n



[Uesato et al., 2022]

Assumption: Verifier

$$\mathsf{Ver}_x : \mathcal{Y} \rightarrow \{0, 1\}$$

Examples:

- Regular expression for math
- Unit test for code
- Test questions for science

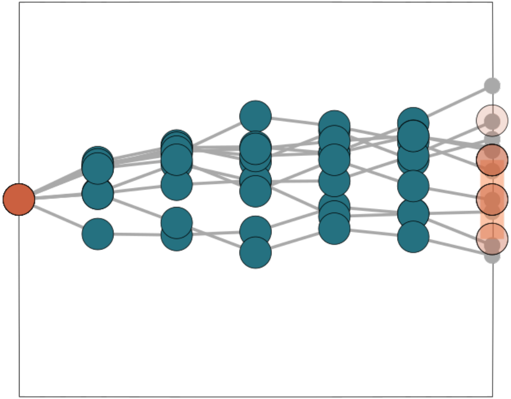
[?, Lightman et al., 2023]

Warm up: Rejection Sampling / Best-of-N

For $n = 1$ to N:

$$z^n \sim p(z|x)$$
$$y^n \sim p(y|x, z^n)$$

Verified set $\{y^n : \text{Ver}(y^n)\}$

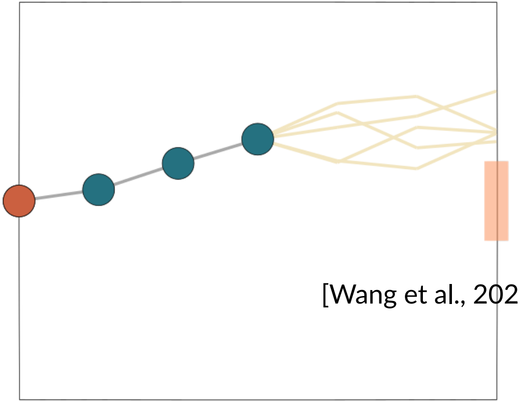


Warm up: Monte-Carlo Roll-Outs

Given partial CoT $z_{1:t}$, expected value,

$$\mathbb{E}_{y \sim p(y|z,x), z_{t:T}} \text{Ver}(y)$$

Monte Carlo for this expectation.



[Wang et al., 2023]

Goal: Learning with Latent CoTs

Maximum likelihood;

$$\max_{\theta} \sum \log p(y|x;\theta) =$$

$$\sum \log \mathbb{E}_z p(y|x, z; \theta)$$

Classic combinatorial expectation

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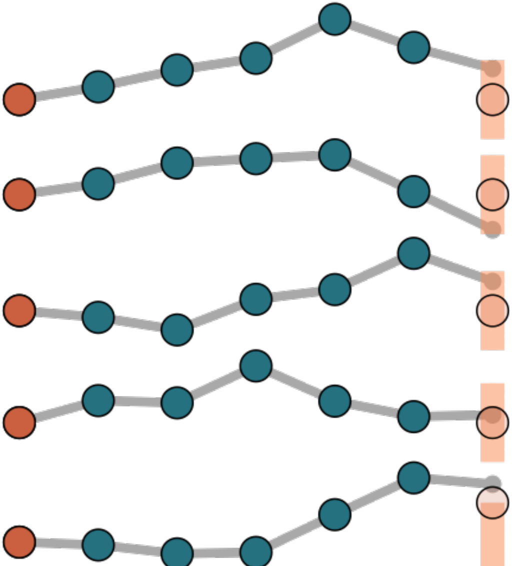
What do we do now?

The Suspects

- Guess + Check
- Guided Search
- AlphaZero
- Learn to Search

Suspect 1: Guess + Check

- 1) Sample N CoTs
- 2) Check if successful
- 3) Train on good ones



Formalization: Rejection Sampling EM

$$\max_{\theta} \sum \log E_{z \sim p(z|x;\theta)} p(y|x, z)$$

- E-Step: For $n = 1$ to N:

$$z^n \sim p(\cdot|x)$$
$$y^n \sim p(\cdot|x, z^n)$$

  Keep verified set $\mathcal{Z} = \{z^n : \text{Ver}(y^n)\}$

[?, ?]

- M-Step: Fit $\theta' \leftarrow \arg\max_{\theta} \sum_{z \in \mathcal{Z}} \log p(z|x; \theta)$

[Zelikman et al., 2022, Gulcehre et al., 2023,
Singh et al., 2023]

G+C Variants

- Best-of-N Training
- STaR
- ReST
- ReST-EM
- Filtered Rejection Sampling

G+C Variants

- Batched -> Compute trajectories first, then train with behavioral cloning

- Online -> Use policy gradient-like steps to update after each example

G+C Empirical Results

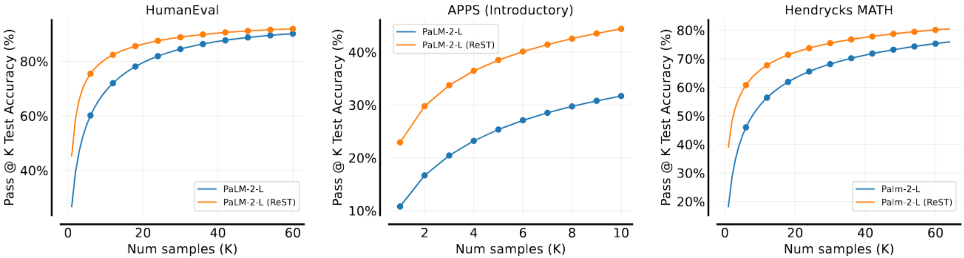


Figure 5 | **Pass@K results** for PaLM-2-L pretrained model as well as model fine-tuned with ReST^{EM}. For a fixed number of samples K, fine-tuning with ReST^{EM} substantially improves Pass@K performance. We set temperature to 1.0 and use nucleus sampling with $p = 0.95$.

G+C Why might this be right?

- Extremely simple and scalable
- Good baseline in past work

- No evidence this learns to correct, plan
- Well-explored in literature with marginal gains

More Structure?

- Rejection sampling may be really inefficient.
- Particularly on hard problems, may get no signal

Suspect 2: Guided Search

- During CoT sampling, use a heuristic to improve trajectories

- Check if final versions are successful

- Train on good ones

GS: Beam Search with Guide

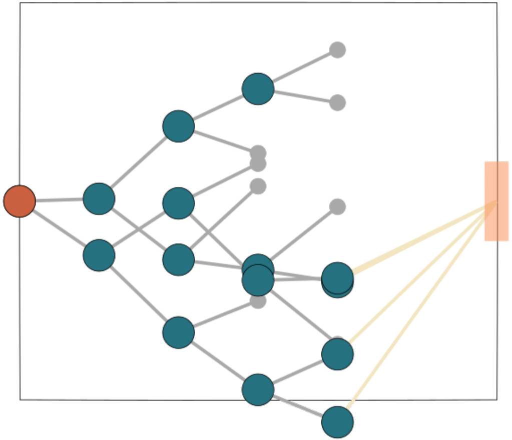
$r : \mathcal{S}^t \to \mathbb{R}$; Guide function

For each step t

1. Sample next step,

$$z_t \sim p(\cdot | x, z^i_{1:t-1})$$

2. Keep the top N samples, ordered by $r(z_t)$



What to use as Guide?

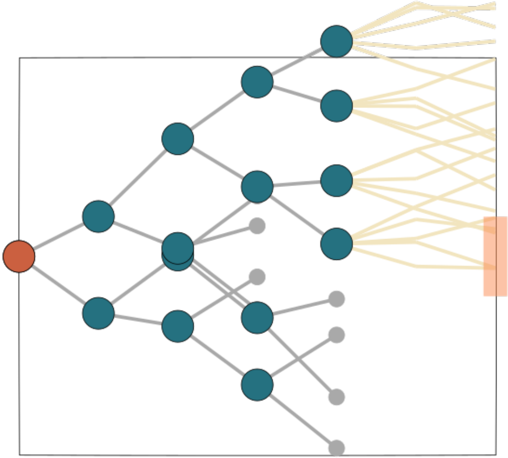
- Monte Carlo Roll-outs
- Learned Value Function
- Interleaved Value Function

Beam Search with Roll-Outs

For a z_t, sample answers

$$y^n \sim p(\cdot | x, z_{1:t-1})$$

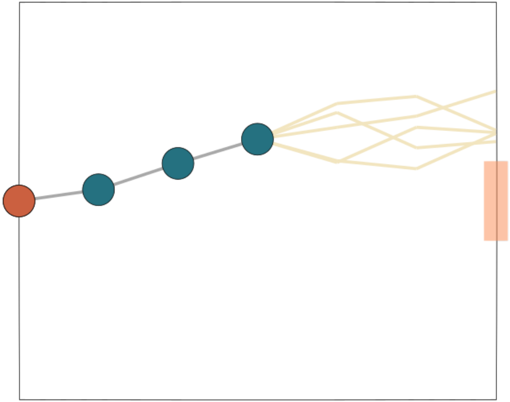
$$r_{MC}(z_t) = \frac{1}{N} \sum_{i=1}^{n} \text{Ver}(y^i)$$



Amortized Roll-Outs

- Rollouts are costly, so instead learn a model $r_\psi(z_t)$ to approximate rollouts

- Use r_{MC} to determine labels to train r_ψ



Interleaved

What about test time?

- Learned rewards can improve test-time without verifier.
-

Snell et al., 2024]

Terminology

- Value
- PRM
- PAV
- Math Shepard.
- snell.

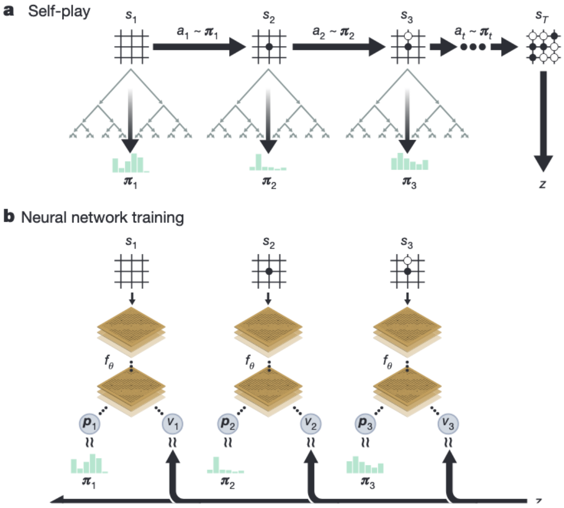
Why might this be right?

- OpenAI is exploring
- Makes RS more efficient.
- Learned rewards are effective
- Assumption: o1 is a single test-time model (although could train or distill-in)
- Not clear if it learns planning.

More Structure

- Improving search seems critical.

Reminder: AlphaZero



Suspect 3: AlphaZero

- Self-play using guided-search with exploration
- Label final outcomes of self-play games
- Train guide and generator

Formalized: Expert Iteration

[Anthony et al., 2017]

- Iterative algorithm combining learned model + expert search with a verifier.

- Generate samples using $p(y, z|x)$, reward model $r(z_t)$, and search algorithm (e.g. beam search)

- Label samples using $Ver(y)$

- Train $p(y, z|x)$, $r(z_t)$ on the labeled samples, and repeat

MCTS exploration

UCB for Language

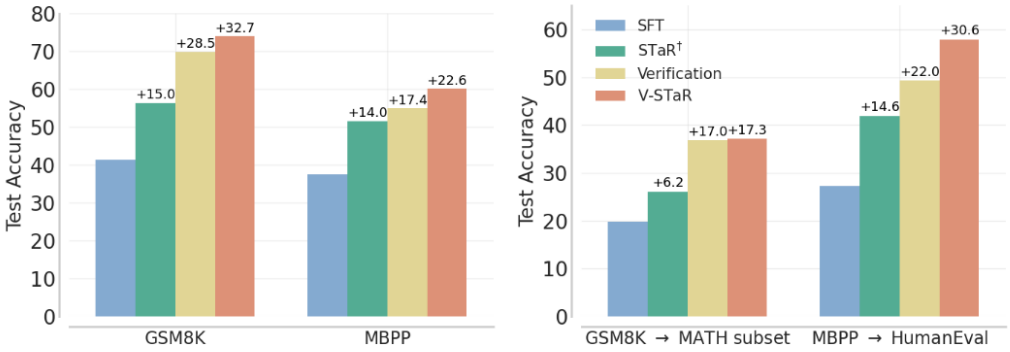
- Selection: Walk down tree to leaf z_{t-1}
- Expand: Sample K next steps z_t^i, pick one at random
- Rollouts: Sample $z_{t+1} \dots z_T$
- Backprop: Update nodes counts $z_{1:t}$ based on results

Compared with Search

- System builds in exploration
- Scales to more train-time search

- Costly to maintain open states
- More complex algorithmically

Empirical Results



[?]

Figure 8: Test accuracy of 13B V-STaR compared to baselines. We report Best-of-64 for verification-based methods and Pass@1 for others. **(Left)** Test accuracy for training tasks. **(Right)** Transfer evaluation of GSM8K and MBPP trained models on MATH subset and HumanEval respectively.

Why might this be right?

- Major demonstrated RL result
-
-
-

More Structure

- Can we force the model to search?

Suspect 4: Learning to Correct

- Sample N Successful CoTs
- Edit to inject incorrect expansions before correct ones.
- Train on correcting trajectories

Self-Correction

[Gandhi et al., 2024]

- Argument: Training on x, z_1^*, y is too easy.
- Train instead on x, z', z_1^*, y
- Model should learn to self-correct

Score

- Positive rewards

Challenge: Collapse

- Model may learn to just ignore negative
-
-

Generalized: Stream of Search

- Find $z^*_{1:T}$ as optimal length CoT
- Find $z'_{1:T'}$ with $T' > T$ through backtracking tree search
- Train model on $z'_{1:T'}$

From Tree to Stream

Empirical Results

Score results

Why might this be right?

Less Structure?

- Maybe this is all too much...
- Could this be done without a verifier?

Outline

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What do we do now?

Replication

Does it need to be the same?

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