# Multi-Objective Optimization

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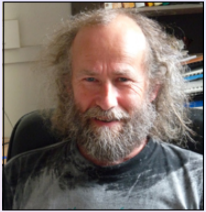
**LECTURE 4**

# Test Problems



Some examples of test suites that have been proposed in the specialized literature to evaluate single-objective evolutionary algorithms are the following:

- The 5 test problems from De Jong [1975] for unconstrained optimization.
- The 12 test problems from Michalewicz & Schoenauer [1996] for constrained optimization.
- The 62 test problems from Schwefel [1995] for evaluating evolution strategies.

# Test Problems

- The test problems proposed by Whitley et al. [1996] and Goldberg [1989].
- The test problems from Yao & Liu [1996; 1997] used to assess performance of evolutionary programming and evolution strategies.
- The deceptive problems from Goldberg and Mühlenbein.
- The 8 test problemas from Digalakis & Margaritis [2000].
- The multimodal test problems from Levy [1981], the test problems from Corana [1987], the test problems from Freudenstein-Roth and the test problems from Goldstein-Price [1981].
- Ackley's function and Wirestrass' function [Bäck et al., 1997].

# Test Problems

A set of test problems to evaluate the performance of a MOEA should include the following features (both in genotypic and in phenotypic space):

- Continuous vs. discontinuous vs. discrete
- Differentiable vs. non-differentiable
- Convex vs. concave
- Modality (unimodal, multi-modal)
- Numerical vs. alphanumeric
- Quadratic vs. nonquadratic
- Type of constraints (equalities, inequalities, linear, nonlinear)
- Low vs. high dimensionality (genotype, phenotype)
- Deceptive vs. nondeceptive
- Biased vs. unbiased portions of the true Pareto front

# Test Problems

Test problems should range in difficulty from “easy” to “hard” as well as attempt to represent generic real-world situations.

Dynamically changing environments can include “moving cones” [Morrison & de Jong, 1999] with movement ranging from predictable to chaotic to non-stationary and deceptive.

R.W. Morrison and K.A. de Jong, “**A Test Problem Generator for Non-Stationary Environments**”, in *1999 IEEE Congress on Evolutionary Computation*, pp. 2047–2053, IEEE Press, Washington, D.C., USA, 1999.

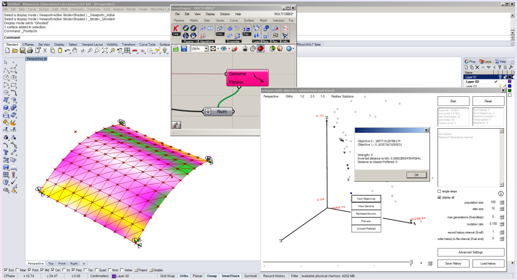
# Test Problems

One should also consider the following guidelines suggested by Whitley et al. [1996] in developing generic test suites:

- Some test suite problems are *resistant* to simple search strategies.
- Test suites contain nonlinear, unseparable & unsymmetric problems.
- Test suites contain scalable problems.
- Some test suite problems have scalable evaluation cost.
- Test problems have a canonical representation (ease of use).

D. Whitley, K. Mathias, S. Rana and J. Dzubera, "**Evaluating Evolutionary Algorithms**", *Artificial Intelligence*, **85**:245–276, 1996.

# Test Problems



Ideally, test problems used to evaluate a MOEA should contain features and difficulties similar to those found in the real-world problem(s) that we aim to solve.

However, the specialized literature presents a wide number of “artificial” test problems that emphasize certain aspects that are indeed difficult for most MOEAs, but that don’t necessarily represent the difficulties found in real-world problems.

## Unconstrained Problems

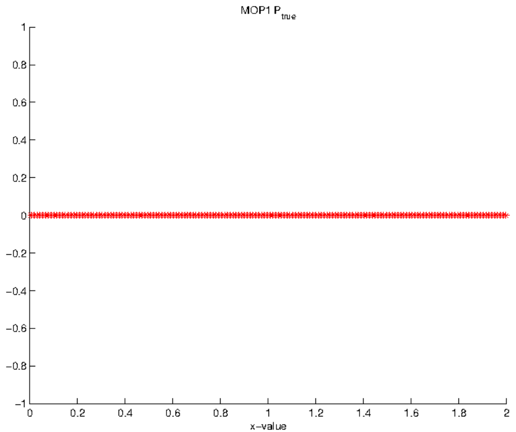
**MOP 1**: This is the first test problem used by David Schaffer. Historically, it has a very high relevance, because it was the first test problem proposed to evaluate the performance of a MOEA. However, this problem is so simple that its Pareto front can be obtained in an analytic form. $PF_{true}$ is convex and the problem has a single decision variable.

$F = (f_1(x), f_2(x))$, where

$$\begin{aligned} f_1(x) &= x^2, \\ f_2(x) &= (x-2)^2 \end{aligned}$$

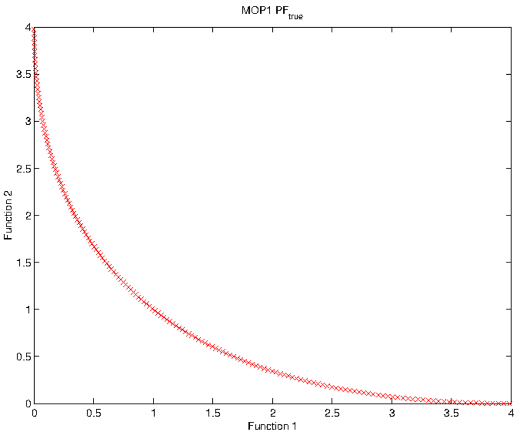
where: $-10^5 \leq x \leq 10^5$

# Test Problems




## Unconstrained Problems

$P_{true}$ of **MOP 1**




## Unconstrained Problems

$PF_{true}$ of **MOP 1**

## Unconstrained Problems

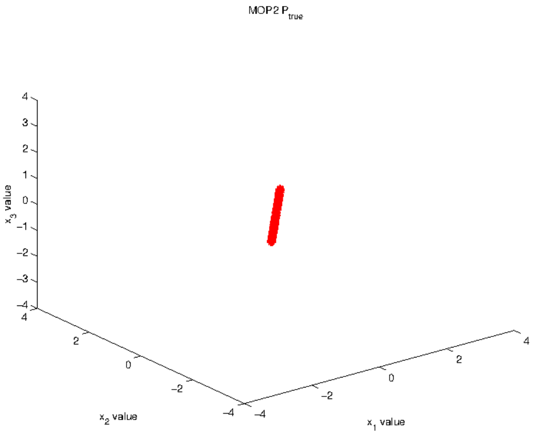
**MOP 2**: This is the second test problem proposed by Fonseca. It is scalable. It is possible to add decision variables to this test problem without changing the shape of $PF_{true}$ (the Pareto front is concave in this case).

$F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= 1 - \exp(-\sum_{i=1}^{n}(x_i - \frac{1}{\sqrt{n}})^2), \\ f_2(\vec{x}) &= 1 - \exp(-\sum_{i+1}^{n}(x_i + \frac{1}{\sqrt{n}})^2) \end{aligned}$$

where: $-4 \leq x_i \leq 4;\ i = 1, 2, 3$

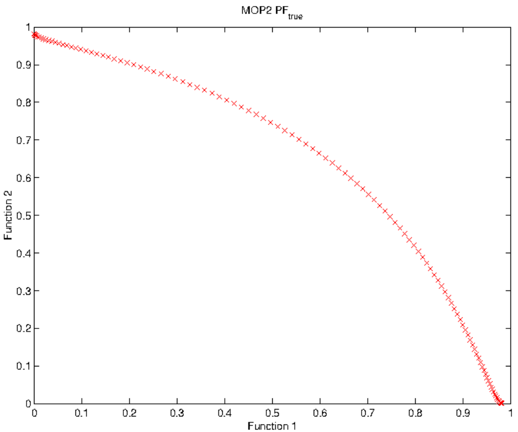
# Test Problems




## Unconstrained Problems

$P_{true}$ of **MOP 2**

# Test Problems




## Unconstrained Problems

$PF_{true}$ of **MOP 2**

## Unconstrained Problems

**MOP 3**: Proposed by Carlo Poloni. Both $P_{true}$ and $PF_{true}$ are disconnected.

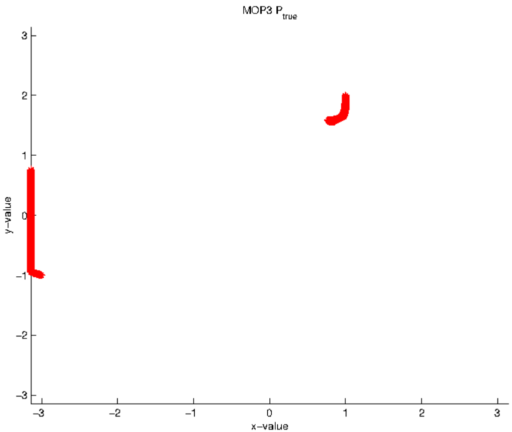
Maximize $F = (f_1(x,y), f_2(x,y))$, where

$$\begin{aligned} f_1(x,y) &= -[1 + (A_1 - B_1)^2 + (A_2 - B_2)^2], \\ f_2(x,y) &= -[(x+3)^2 + (y+1)^2] \end{aligned}$$

where: $-3.1416 \leq x, y \leq 3.1416$,

$$\begin{aligned} A_1 &= 0.5 \sin 1 - 2 \cos 1 + \sin 2 - 1.5 \cos 2, \\ A_2 &= 1.5 \sin 1 - \cos 1 + 2 \sin 2 - 0.5 \cos 2, \\ B_1 &= 0.5 \sin x - 2 \cos x + \sin y - 1.5 \cos y, \\ B_2 &= 1.5 \sin x - \cos x + 2 \sin y - 0.5 \cos y \end{aligned}$$

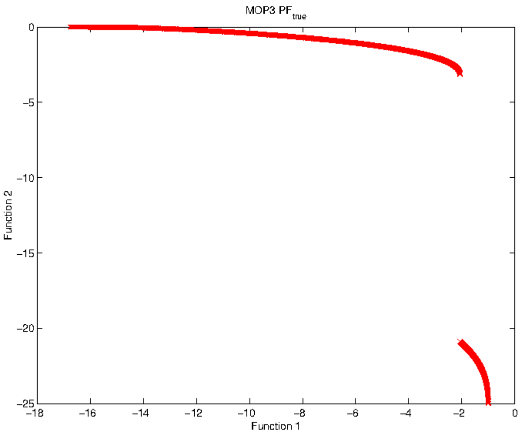
# Test Problems




## Unconstrained Problems

$P_{true}$ of **MOP 3**

# Test Problems




## Unconstrained Problems

$PF_{true}$ of **MOP 3**

## Unconstrained Problems

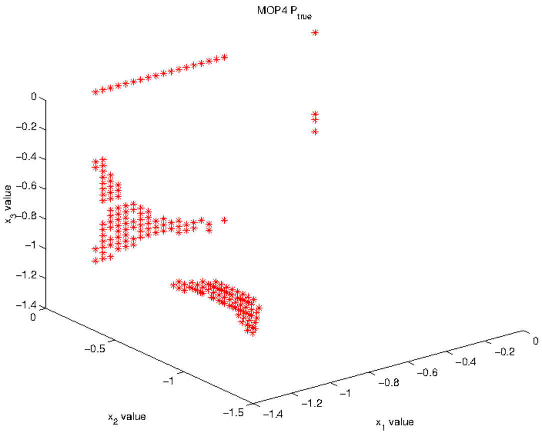
**MOP 4**: Proposed by Kursawe. There are disconnected and asymmetrical portions in $P_{true}$. $PF_{true}$ consists of 3 disconnected curves. It allows the use of an arbitrary number of decision variables, although scaling this test problem changes the shape of $PF_{true}$.

$F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= \sum_{i=1}^{n-1}(-10e^{(-0.2)*\sqrt{x_i^2+x_{i+1}^2}}), \\ f_2(\vec{x}) &= \sum_{i=1}^{n}(|x_i|^a + 5\sin(x_i)^b) \end{aligned}$$

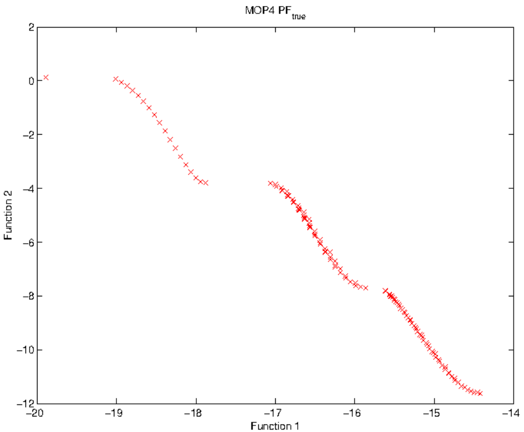
where: $-5 \leq x_i \leq 5$; $i = 1, 2, 3$; $a = 0.8$, $b = 3$

# Test Problems




## Unconstrained Problems

$P_{true}$ of **MOP 4**




## Unconstrained Problems

$PF_{true}$ of **MOP 4**

## Unconstrained Problems

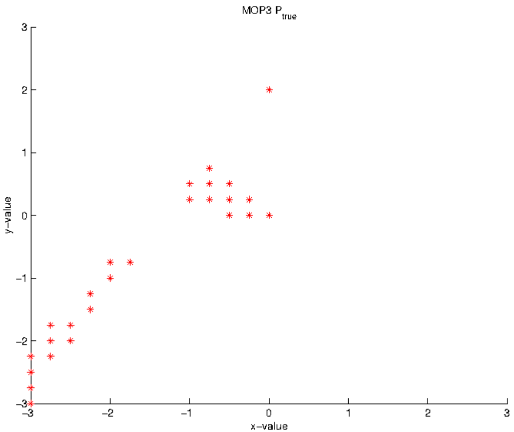
**MOP 5**: Proposed by Viennet. It has disconnected regions in $P_{true}$. $PF_{true}$ is a three-dimensional curve.

$F = (f_1(x, y), f_2(x, y), f_3(x, y))$, where

$$\begin{aligned}
f_1(x, y) &= 0.5 * (x^2 + y^2) + \sin(x^2 + y^2), \\
f_2(x, y) &= \frac{(3x - 2y + 4)^2}{8} + \frac{(x - y + 1)^2}{27} + 15, \\
f_3(x, y) &= \frac{1}{(x^2 + y^2 + 1)} - 1.1e^{(-x^2 - y^2)}
\end{aligned}$$

where: $-30 \leq x, y \leq 30$

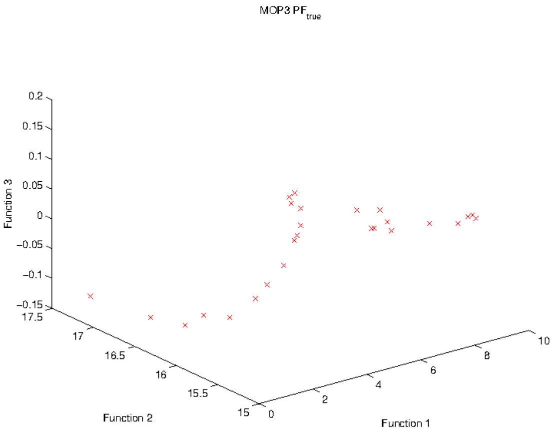
# Test Problems




## Unconstrained Problems

$P_{true}$ of **MOP 5**

# Test Problems




## Unconstrained Problems

*PF$_{true}$* of **MOP 5**

## Unconstrained Problems

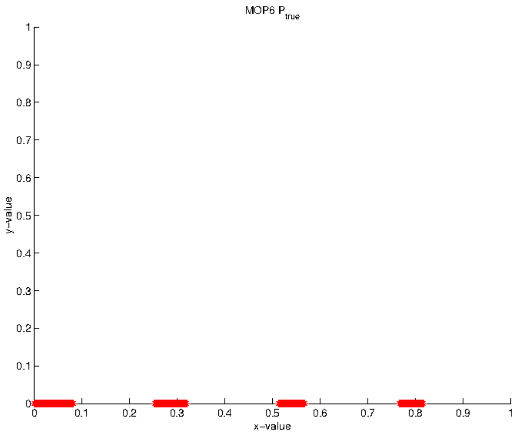
**MOP 6**: Proposed by Deb. Both $P_{true}$ and $PF_{true}$ are disconnected.

$F = (f_1(x,y), f_2(x,y))$, where

$$\begin{aligned} f_1(x,y) &= x, \\ f_2(x,y) &= (1+10y) * \\ &\quad [1 - (\frac{x}{1+10y})^{\alpha} - \frac{x}{1+10y}\sin(2\pi q x)] \end{aligned}$$

where: $0 \leq x, y \leq 1$,

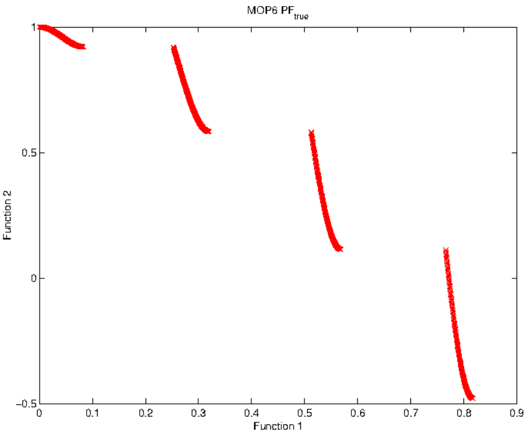
$$\begin{aligned} q &= 4, \\ \alpha &= 2 \end{aligned}$$




## Unconstrained Problems

$P_{true}$ of **MOP 6**

# Test Problems




## Unconstrained Problems

$PF_{true}$ of **MOP 6**

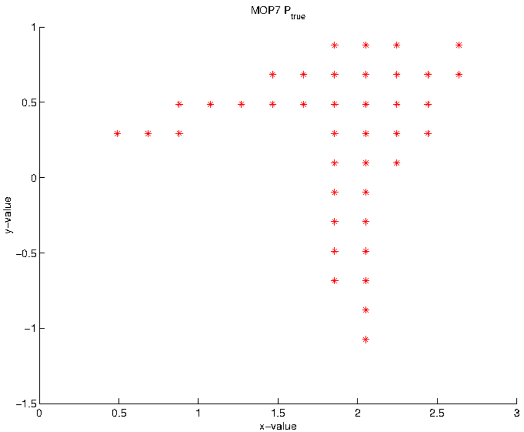
## Unconstrained Problems

**MOP 7**: Proposed by Viennet. $P_{true}$ is connected and $PF_{true}$ is a surface. This problem is relatively easy to solve by any MOEA.

$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where

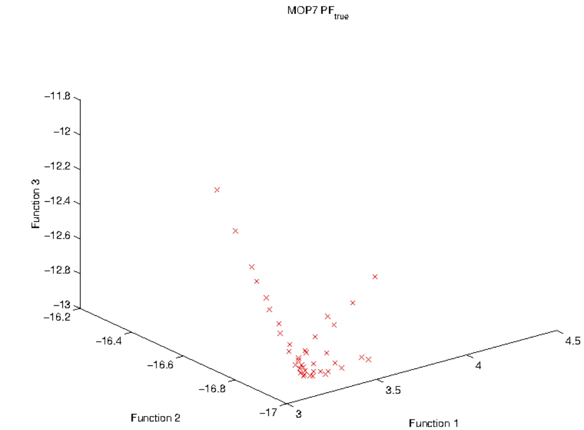
$$
\begin{aligned}
f_1(x,y) &= \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3, \\
f_2(x,y) &= \frac{(x+y-3)^2}{36} + \frac{(-x+y+2)^2}{8} - 17, \\
f_3(x,y) &= \frac{(x+2y-1)^2}{175} + \frac{(2y-x)^2}{17} - 13
\end{aligned}
$$

where: $-400 \leq x, y \leq 400$




## Unconstrained Problems

$P_{true}$ of **MOP 7**




## Unconstrained Problems

$PF_{true}$ of **MOP 7**

## Constrained Problems

Historically, constraints have been handled in MOEAs through the use of penalty functions [Richardson et al., 1989].

However, many other methods to handle constraints are currently available, although few of them have been specifically designed for MOEAs.

Jon T. Richardson, Mark R. Palmer, Gunar Liepins, and Mike Hilliard, “**Some Guidelines for Genetic Algorithms with Penalty Functions**”, in J. David Schaffer (Ed), *Proceedings of the Third International Conference on Genetic Algorithms*, pp. 191–197, Morgan Kaufmann Publishers, San Mateo, California, USA, 1989.

B. Y. Qu and P. N. Suganthan, “**Constrained Multi-objective Optimization Algorithm with an Ensemble of Constraint Handling Methods**, *Engineering Optimization*, Vol. 43, No. 4, pp. 403–416, 2011.

### Constrained Problems

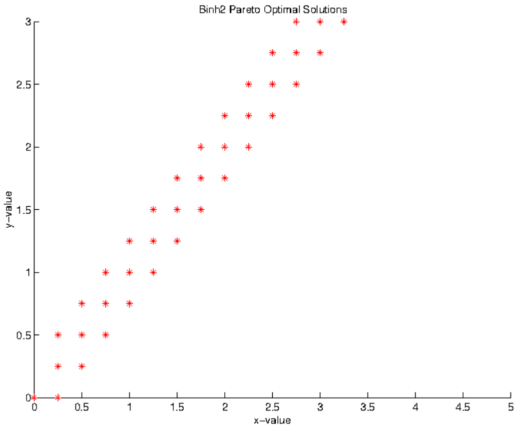
**MOP-C1**: Proposed by Binh. In this case, $P_{true}$ is an area and $PF_{true}$ is a single convex curve.

$F = (f_1(x,y), f_2(x,y))$, where

$$\begin{aligned} f_1(x,y) &= 4x^2 + 4y^2, \\ f_2(x,y) &= (x-5)^2 + (y-5)^2 \end{aligned}$$

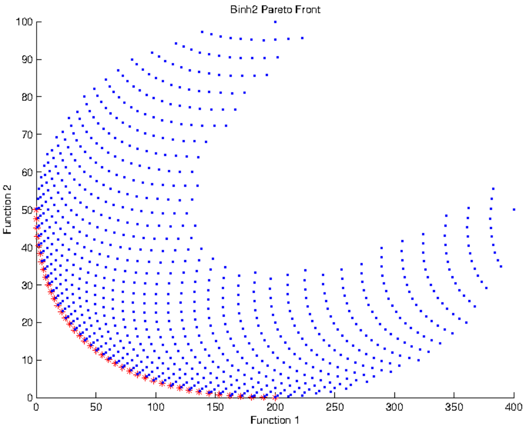
where:

$$0 \leq x \leq 5, \; 0 \leq y \leq 3$$
$$0 \geq (x-5)^2 + y^2 - 25,$$
$$0 \geq -(x-8)^2 - (y+3)^2 + 7.7$$




## Constrained Problems

$P_{true}$ of **MOP-C1**




## Constrained Problems

$PF_{true}$ of **MOP-C1**

## Constrained Problems

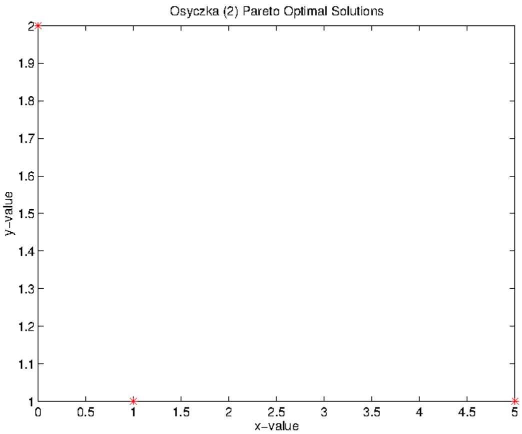
**MOP-C2**: Proposed by Osyczka. Both $P_{true}$ and $PF_{true}$ are disconnected.

$$
\begin{aligned}
f_1(\vec{x}) &= -(25(x_1-2)^2 + (x_2-2)^2 + (x_3-1)^2 \\
&\quad + (x_4-4)^2 + (x_5-1)^2, \\
f_2(\vec{x}) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2
\end{aligned}
$$

$0 \le x_1, x_2, x_6 \le 10$, $1 \le x_3, x_5 \le 5$, $0 \le x_4 \le 6$,

$$
\begin{aligned}
0 &\le x_1 + x_2 - 2, \\
0 &\le 6 - x_1 - x_2, \\
0 &\le 2 - x_2 + x_1, \\
0 &\le 2 - x_1 + 3x_2, \\
0 &\le 4 - (x_3-3)^2 - x_4 \\
0 &\le (x_5-3)^2 + x_6 - 4
\end{aligned}
$$

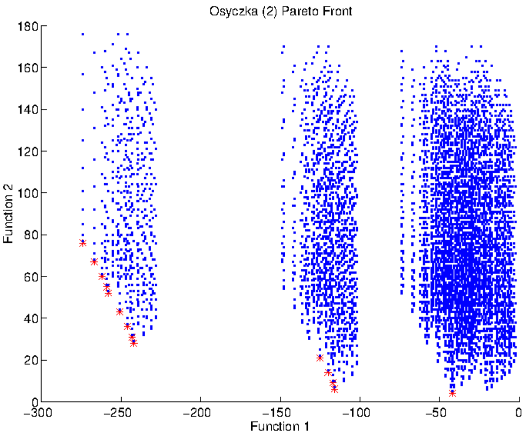
# Test Problems




## Constrained Problems

$P_{true}$ of **MOP-C2**

# Test Problems




## Constrained Problems

*$PF_{true}$* of **MOP-C2**

## Constrained Problems

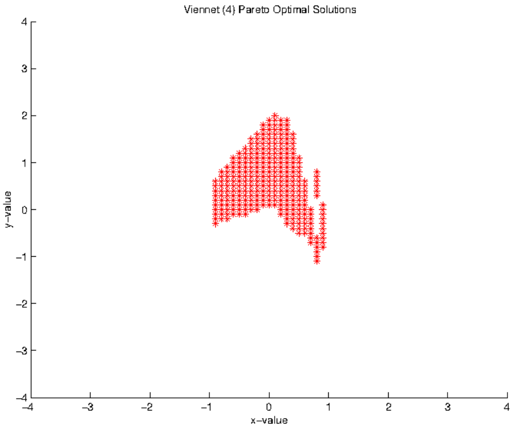
**MOP-C3**: Proposed by Viennet. $P_{true}$ is connected but it's asymmetrical. $PF_{true}$ is a 3D curve.

$$
\begin{aligned}
f_1(x,y) &= \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3, \\
f_2(x,y) &= \frac{(x+y-3)^2}{175} + \frac{(2y-x)^2}{17} - 13, \\
f_3(x,y) &= \frac{(3x-2y+4)^2}{8} + \frac{(x-y+1)^2}{27} + 15
\end{aligned}
$$

$-4 \leq x, y \leq 4,$

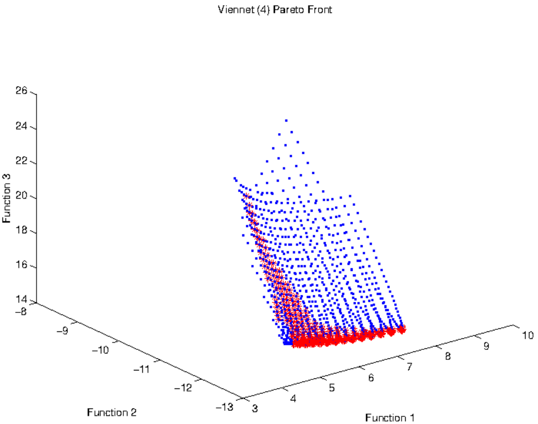
$$
\begin{aligned}
y &< -4x + 4, \\
x &> -1, \\
y &> x - 2
\end{aligned}
$$

# Test Problems




## Constrained Problems

$P_{true}$ of **MOP-C3**




## Constrained Problems

$PF_{true}$ of **MOP-C3**

## Constrained Problems

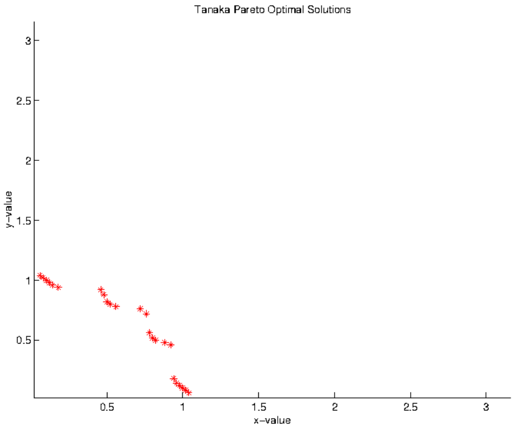
**MOP-C4**: Proposed by Tanaka. $P_{true}$ is connected, but $PF_{true}$ is disconnected.

$$\begin{aligned} f_1(x,y) &= x, \\ f_2(x,y) &= y \end{aligned}$$

$0 < x, y \leq \pi,$

$$\begin{aligned} 0 &\geq -(x^2) - (y^2) \\ &\quad +1 + \\ &\quad (a \cos \\ &\quad (b \arctan(x/y))) \\ a &= 0.1 \\ b &= 16 \end{aligned}$$

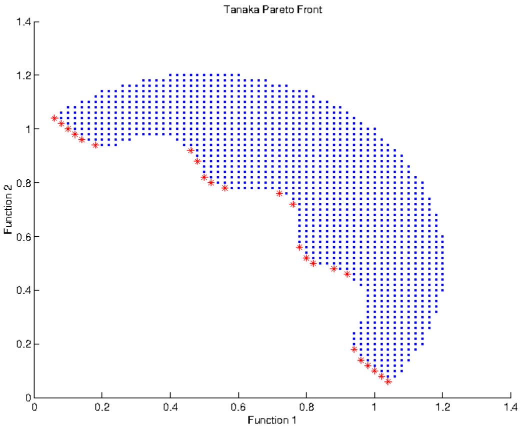
# Test Problems




## Constrained Problems

$P_{true}$ of **MOP-C4**

# Test Problems




## Constrained Problems

*PF*$_{true}$ of **MOP-C4**

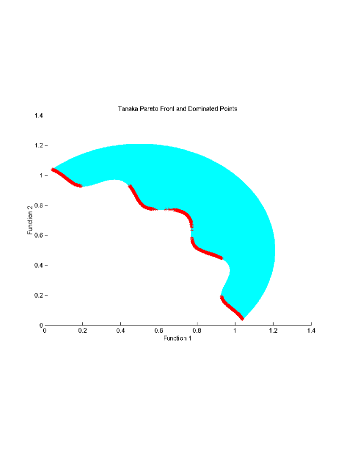
# Test Problems

## Constrained Problems

Considering specific variations of these two MOP-C4 parameters along with an *absolute* operator on the last term of the constraint, can result in the following general landscapes:

- Standard Tanaka phenotype with $a = .1$ and $b = 16$ (shown in the previous slide).
- Smaller continuous regions with $a = .1$, $b = 32$.
- Increased distance between regions using the absolute value on the last term of the constraint, and $a = .1$, $b = 16$.
- Increased distance between regions using the absolute values on the last term of the constraint, and $a = .1$, $b = 32$.
- Deeper periodic regions using the absolute value on the last term of the constraint, and $a = .1(x^2 + y^2 + 5xy)$, $b = 32$.
- Non-periodic regions on front using the absolute value on the last term of the constraint, and $a = .1(x^2 + y^2 + 5xy)$, $b = 8(x^2 + y^2)$.

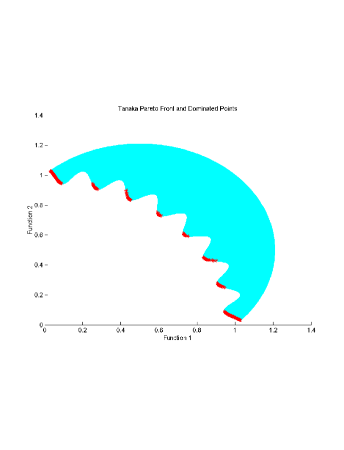
# Test Problems




## Constrained Problems

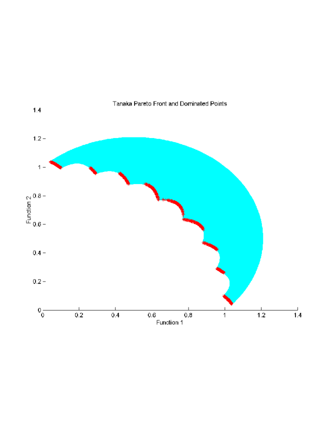
MOP-C4 (Tanaka). With $a = .1, b = 16$ we obtain the original shape of $PF_{true}$.

# Test Problems




## Constrained Problems

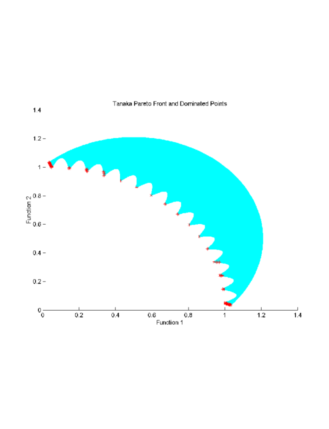
MOP-C4 (Tanaka). With $a = .1, b = 32$ we have smaller contiguous regions of $PF_{true}$.




## Constrained Problems

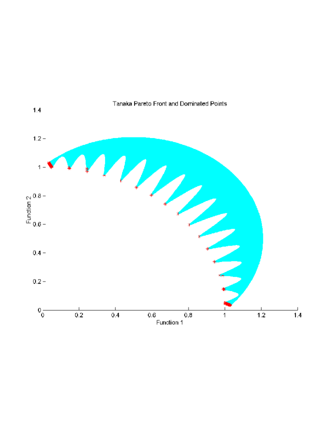
MOP-C4 (Tanaka). With $a = .1, b = 16$ and using the absolute value on the last term of the constraint, the distance between the regions of $PF_{true}$ is increased.

# Test Problems




## Constrained Problems

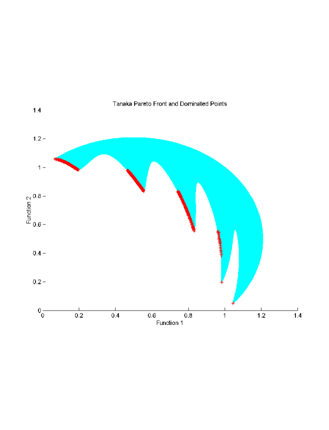
MOP-C4 (Tanaka). With $a = .1, b = 32$ and using the absolute values on the last term of the constraint, there is a bigger increase in the distance between the regions of $PF_{true}$.




## Constrained Problems

MOP-C4 (Tanaka). With $a = .1(x^2 + y^2 + 5xy)$, $b = 32$, and the absolute values of the last term of the constraint, we have deeper periodic regions of $PF_{true}$.

# Test Problems




## Constrained Problems

MOP-C4 (Tanaka). With $a = .1(x^2 + y^2 + 5xy)$, $b = 8(x^2 + y^2)$, we have non-periodic regions of $PF_{true}$.

# Test Problems

## Constrained Problems

By selecting values for the two parameters $(a, b)$ different landscape structures evolve. $P_{true}$ of course is the same by function definition. Although the central Pareto curve in the original Tanaka problem appears not to be continuous due to numerical accuracy, it is continuous in reality. The two internal sections of this curve are very difficult to find numerically because of the near horizontal or vertical slope at the associated points, respectively.

In general, $(a, b)$ controls the length of the continuous region on the Pareto front. As this region is decreased, a MOEA finds fewer points on $PF_{true}$ due to the discretization of $\vec{x}$; i.e., a more difficult problem.

## Constrained Problems

By increasing the value of $a$, the length of the "cuts" become deeper requiring the search to proceed along a narrower corridor, again more difficult to solve. One can also move away from the periodic nature of the disconnected $PF_{true}$ regions by changing $b$ from the initial value of 16 (more optimal solutions in one direction or the other). Finding all closely packed $PF_{true}$ regions then becomes difficult for the MOEA.

As reflected in the figures, such parameter variations can cause MOEA searches to become more difficult as the size of feasible regions are decreased. Just by changing a few parameters very different phenotype landscapes result.

## Test Problems Generators

MOP test functions can also be generated by using the single-objective functions. A methodology for constructing MOPs exhibiting desired characteristics has been proposed by Deb [1999].

Kalyanmoy Deb, "**Multi-Objective Genetic Algorithms: Problem Difficulties and Construction of Test Problems**", *Evolutionary Computation*, **7**(3):205-230, Fall 1999.

He points out that when computationally derived a non-uniform distribution of vectors may exist in some Pareto front. He limits his initial test construction efforts to unconstrained MOPs of only two functions; his construction methodology then places restrictions on the two component functions so that resultant MOPs exhibit desired properties. To accomplish this he defines various generic bi-objective optimization problems, such as the example of the next slide.

# Test Problems

## Test Problems Generators

Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= f(x_1, \ldots, x_m), \\ f_2(\vec{x}) &= g(x_{m+1}, \ldots, x_N)\, h(f(x_1, \ldots, x_m), g(x_{m+1}, \ldots, x_N)) \end{aligned} \tag{1}$$

where function $f_1$ is a function of $(m < N)$ decision variables and $f_2$ a function of all $N$ decision variables.

The function $g$ is one of $(N - m)$ decision variables which are not included in function $f$.

The function $h$ is directly a function of $f$ and $g$ function values. The $f$ and $g$ functions are also restricted to positive values in the search space, i.e., $f > 0$ and $g > 0$.

## Test Problems Generators

Deb lists five functions each for possible $f$ and $g$ instantiation, and four for $h$. These functions may then be "mixed and matched" to create MOPs with desired characteristics.

He states these functions have the following general effect:

- *f* – This function controls vector representation uniformity along the Pareto front.
- *g* – This function controls the resulting MOP's characteristics – whether it is multifrontal or has an isolated optimum.
- *h* – This function controls the resulting Pareto front's characteristics (e.g., convex, disconnected, etc.)

These functions respectively influence search along and towards the Pareto front, and the shape of a Pareto front in $\mathbb{R}^2$. Deb implies that a MOEA has difficulty finding $PF_{true}$ because it gets "trapped" in a local Pareto front.

## Test Problems Generators

**MOP-G1**: This is an example of the test problems generated with Deb's methodology. In this case, $PF_{true}$ is convex.

$$
\begin{aligned}
f_1(x_1) &= x_1, \\
f_2(\vec{x}) &= g(1 - \sqrt{(f_1/g)}) \\
g(\vec{x}) &= 1 + 9\sum_{i=2}^{m} x_i/(m-1)
\end{aligned}
$$

$m = 30; 0 \leq x_i \leq 1$

# Test Problems

## Test Problems Generators

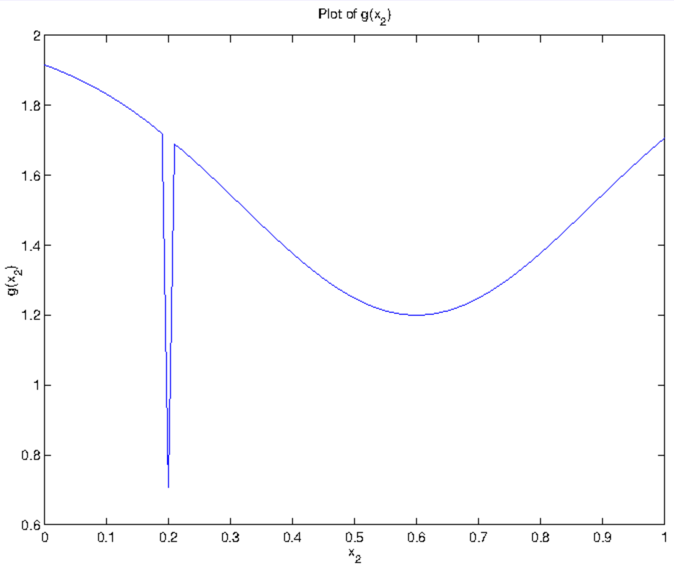
Deb's methodology is, with no doubt, an important contribution to the automatic generation of multi-objective test problems. However, it is not free of problems. Let's consider the following example:

Minimize $F = (f_1(x_1, x_2), f_2(x_1, x_2))$, where

$$\begin{aligned} f_1(x_1, x_2) &= x_1, \\ f_2(x_1, x_2) &= \frac{2.0 - \exp\{-(\frac{x_2 - 0.2}{0.004})^2\} - 0.8\exp\{-(\frac{x_2 - 0.6}{0.4})^2\}}{x_1}. \end{aligned} \tag{2}$$

In this case, $f_2$ can also be represented as $\frac{g(x_2)}{x_1}$. Thus, $g(x_2)$ is the bimodal function shown in the next slide. That figure shows the optima located at $g(0.6) \approx 1.2$ and $g(0.2) \approx 0.7057$.

# Test Problems



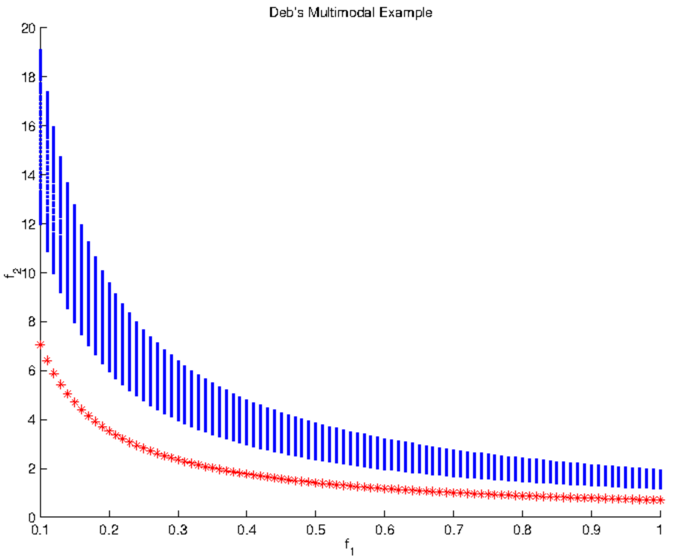
### Test Problems Generators

The figure of the next slide shows the Pareto fronts (proposed by Deb) corresponding to this test problem.

The lower portion of the upper vectorial band is denominated $PF_{local}$ by Deb, and the lower band is $PF_{true}$.

The solutions corresponding to $P_{local}$ are $\{(x_1, x_2) \mid x_2 \approx 0.6\}$ and those corresponding to $P_{true}$ are $\{(x_1, x_2) \mid x_2 \approx 0.2\}$.

# Test Problems



# Test Problems

## Test Problems Generators

Deb indicates that there will be difficulties to find $PF_{true}$ for this problem because MOEAs will tend to get trapped in $PF_{local}$.

However, this is not a phenotypic effect, but more a problem due to the discretization of the genotypic search space. In this problem, the difficulty relies not only on the existence of $PF_{local}$, but rather on the number of discrete points close to the global optimum of $g(x_2)$.

This problem really shows deception rather than multi-frontality, because the search space was uniformly discretized.

# Test Problems

## Test Problems Generators

For constrained test MOPs, Deb [2001] suggests extending his methodology in the following way:

$$
\begin{aligned}
f_1(\vec{x}) &= x_1 \\
f_2(\vec{x}) &= g(\vec{x}) \exp(-f_1(\vec{x})/g(\vec{x}))
\end{aligned}
$$

subject to:

$$
c_j(x) = f_2(\vec{x}) - a_j \exp(-b_j f_1(\vec{x})) \geq 0, \; j = 1, 2, \ldots J \tag{3}
$$

There are $J$ inequalities, each of which has 2 parameters $(a_j, b_j)$, which makes that part of feasible region of the original (unconstrained) problem is now infeasible.

## Test Problems Generators

Kalyanmoy Deb, Amrit Pratap and T. Meyarivan, "**Constrained Test Problems for Multi-objective Evolutionary Optimization**", in Eckart Zitzler et al. (Eds.), *First International Conference on Evolutionary Multi-Criterion Optimization*, pp. 284–298. Springer-Verlag. Lecture Notes in Computer Science No. 1993, 2001

An example of this methodology is the following:
Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= x_1 \\ f_2(\vec{x}) &= (1 + x_2)/x_1 \\ 0.1 \leq x_1 \leq 1.0 & \\ 0.0 \leq x_2 \leq 5.0 & \end{aligned}$$

subject to:

$$\begin{aligned} c_1(\vec{x}) &= x_2 + 9x_1 \geq 6 \\ c_2(\vec{x}) &= -x_2 + 9x_1 \geq 1 \end{aligned} \quad (4)$$

## Test Problems Generators

In fact, the next generic form is suggested:
Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= x_1 \\ f_2(\vec{x}) &= g(\vec{x})(1 - f_1(\vec{x})/g(\vec{x}) \end{aligned}$$

subject to:

$$\begin{aligned} c_j(\vec{x}) &= \cos(\theta)(f_2(\vec{x}) - e) - \sin(\theta)f_1(\vec{x})) \geq \\ & a|\sin(b\pi(\sin)\theta)(f_2(\vec{x}) - e) + \cos(\theta)f_1(\vec{x}))^c)|^d, \\ & j = 1, 2, ...J \end{aligned} \tag{5}$$

## Test Problems Generators

With 6 parameters ($\theta$, $a$, $a$, $c$, $d$, $e$), $x_1$ is restricted to the range [0,1] and $g(\vec{x})$ determines the bounds of the other decision variables.

Selecting values for the 6 parameters, we can generate different fitness landscapes.

It is worth noting that $d$ controls the length of the continuous region of the Pareto front. As we decrease this region, a MOEA will tend to find less points of $PF_{true}$ because of the discretization of $\vec{x}$.

## Test Problems Generators

If we increase the value of $a$, the length of the "cuts" becomes more profound, which requires the search to proceed through a narrowed corridor. Evidently, this makes more difficult the search.

We can also depart from the periodic disconnected regions of $PF_{true}$ by changing $c$ from its initial value of 1.

$\theta$ and $e$ control the slope and the change of direction of $PF_{true}$, respectively.

## Zitzler-Deb-Thiele (ZDT) Test Problems

Each of the test problems shown next is structured in the same way and it consists of 3 functions $f_1$, $g$, $h$:

$$\begin{aligned} \textit{Minimize}: F(\vec{x}) &= (f_1, f_2), \\ \textit{subject to}: f_2(\vec{x}) &= g(x_2, \ldots, x_m)h(f_1(x_1), g(x_2, \ldots, x_m)), \\ \textit{where}: \vec{x} &= (x_1, \ldots, x_M). \end{aligned} \tag{6}$$

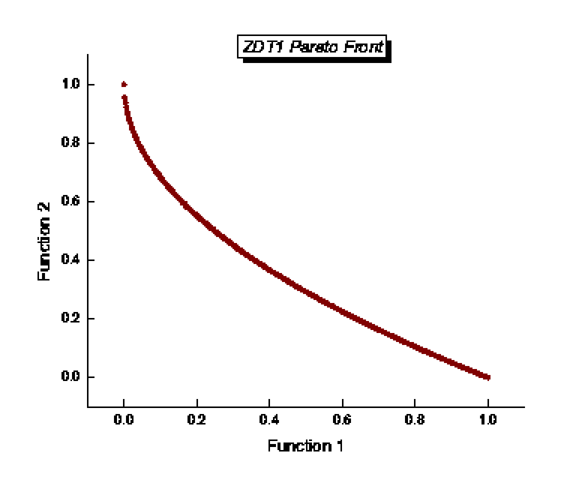
$f_1$ is a function of only the first decision variable, $g$ is a function of the $m-1$ remaining decision variables, and the parameters of $h$ are the values of $f_1$ and $g$.

## Zitzler-Deb-Thiele (ZDT) Test Problems

Eckart Zitzler, Kalyanmoy Deb and Lothar Thiele, "**Comparison of Multiobjective Evolutionary Algorithms: Empirical Result**", *Evolutionary Computation*, **8**(2):173-195, Summer 2000.

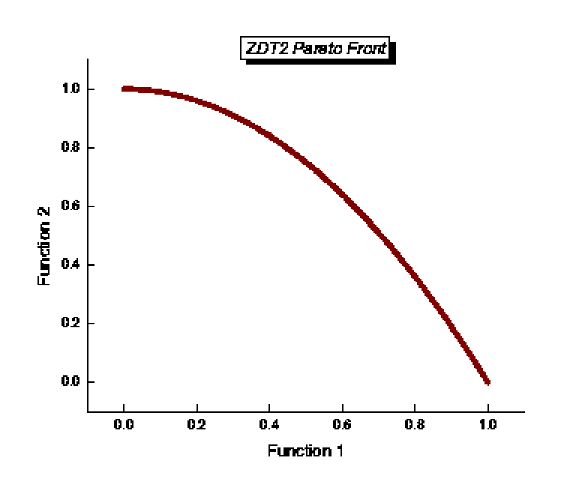
The test problems differ in these 3 functions and in the number of decision variables $m$, as well as in the values that the decision variables can take. These problems have been heavily used to validated MOEAs in the specialized literature.

# Test Problems



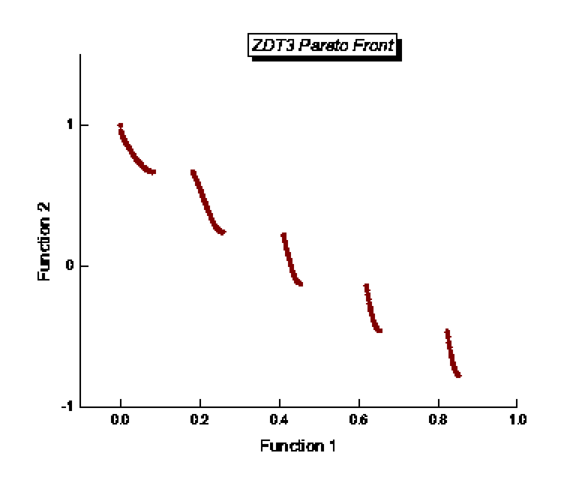

## ZDT Test Problems

**ZDT1**



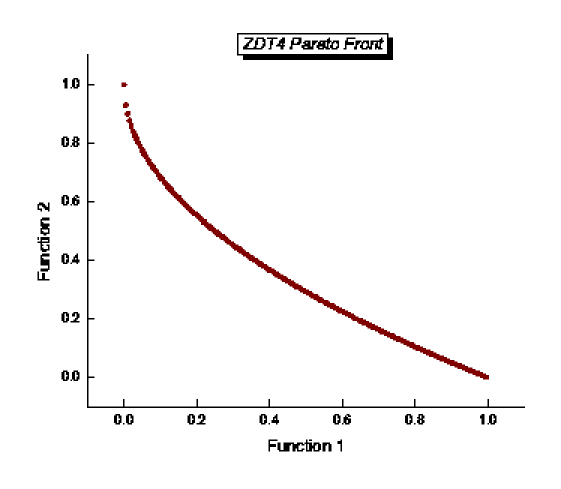

## ZDT Test Problems

**ZDT2**



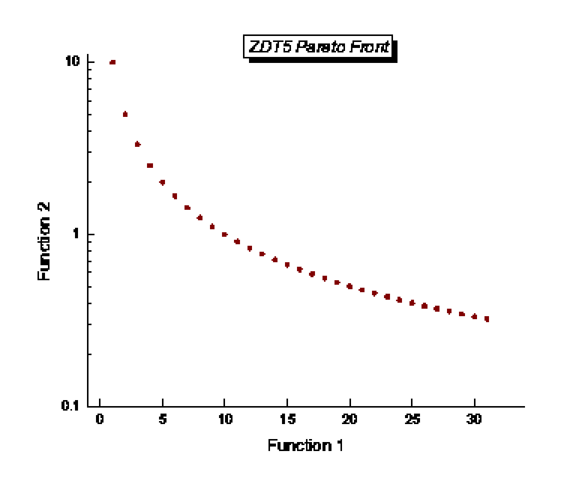

## ZDT Test Problems

**ZDT3**



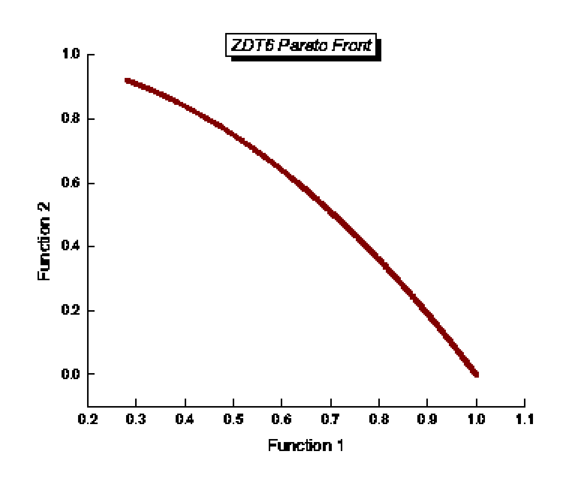

## ZDT Test Problems

**ZDT4**




## ZDT Test Problems

### ZDT5




## ZDT Test Problems

**ZDT6**

# Test Problems

## Deb-Thiele-Laumanns-Zitzler (DTLZ) Test Problems

Another desirable feature of a test problem is that it can scale up to any number of dimensions.

Since the mapping between the genotypic and the phenotypic space can be considerably nonlinear, we can exploit this property to generate test problems with a high degree of difficulty.

Deb et al. [2002,2005] proposed the so-called Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite in which the problems are scalable to a number of objectives defined by the user. This test suite has also been very popular in the specialized literature.

Kalyanmoy Deb, Lothar Thiele, Marco Laumanns and Eckart Zitzler, "**Scalable Test Problems for Evolutionary Multiobjective Optimization**", in Ajith Abraham, Lakhmi Jain and Robert Goldberg (editors), *Evolutionary Multiobjective Optimization. Theoretical Advances and Applications*, pp. 105–145, Springer, USA, 2005.

## DTLZ1

$PF_{true}$ is linear, separable and multimodal.

Minimize:

$$f_1(x) = \frac{1}{2} x_1 x_2 \ldots x_{M-1}(1 + g(x_M)), \tag{7}$$

$$f_2(x) = \frac{1}{2} x_1 x_2 \ldots (1 - x_{M-1})(1 + g(x_M)), \tag{8}$$

$$\vdots \qquad \vdots \tag{9}$$

$$f_{M-1}(x) = \frac{1}{2} x_1 (1 - x_2)(1 + g(x_M)), \tag{10}$$

$$f_M(x) = \frac{1}{2}(1 - x_1)(1 + g(x_M)), \tag{11}$$

$$\text{subject to } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n \tag{12}$$

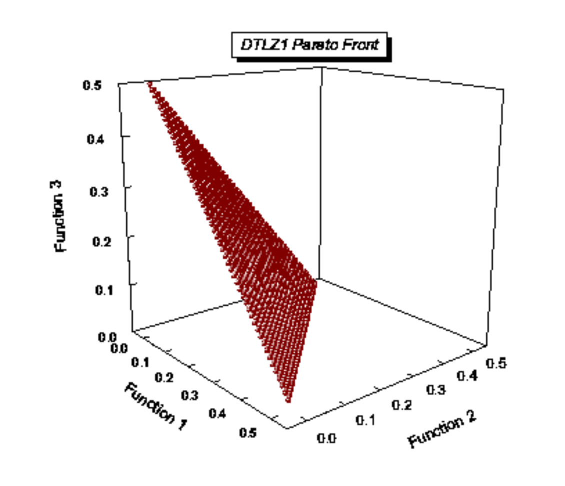
$$\text{where: } g(x_M) = 100 \left[ |x_M| + \sum_{x_i \in x_M} (x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5)) \right] \tag{13}$$

## DTLZ1

It is normally adopted with $M = 3$. The Pareto optimal set is located at $x_M^* = 0$ and the values of the objective functions at the linear hyperplane $\sum_{m=1}^{M} = 0.5$.

The search space contains $(11^k - 1)$ local Pareto fronts ($k$ is a value defined by the user, such that the number of decision variables is: $n = M + k - 1$. It is common to adopt $k = 5$).

# Test Problems




## DTLZ Test Problems

**DTLZ1**

## DTLZ2

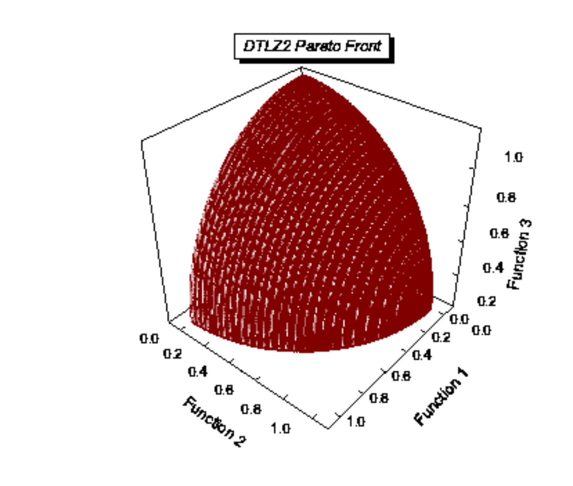
Minimize:

$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\cos(x_{M-2}\pi/2)\cos(x_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\cos(x_{M-2}\pi/2)\sin(x_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\sin(x_{M-2}\pi/2),\\
\vdots &\qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(x_1\pi/2)\sin(x_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(x_1\pi/2).\\
\text{subject to: } & 0 \le x_i \le 1 \quad \forall \quad i = 1, 2, ..., n\\
\text{where: } & g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2
\end{aligned}
$$

## DTLZ2

The Pareto optimal set is located at: $x_i = 0.5$ for every $x_i \in \mathbf{x}_M$ and all the objective functions have to satisfy: $\sum_{i=1}^{M}(f_i)^2 = 1$. It is suggested to use $k = |\mathbf{x}_M| = 10$.

The total number of decision variables is: $n = M + k - 1$.




## DTLZ Test Problems

**DTLZ2**

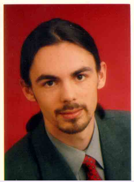
## DTLZ3

Minimize:

$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\cos(x_{M-2}\pi/2)\cos(x_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\cos(x_{M-2}\pi/2)\sin(x_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\sin(x_{M-2}\pi/2),\\
\vdots &\qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(x_1\pi/2)\sin(x_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(x_1\pi/2).
\end{aligned}
$$

subject to: $0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$

where: $g(x_M) = 100[|x_M| + \sum_{x_i \in x_M} (x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))]$

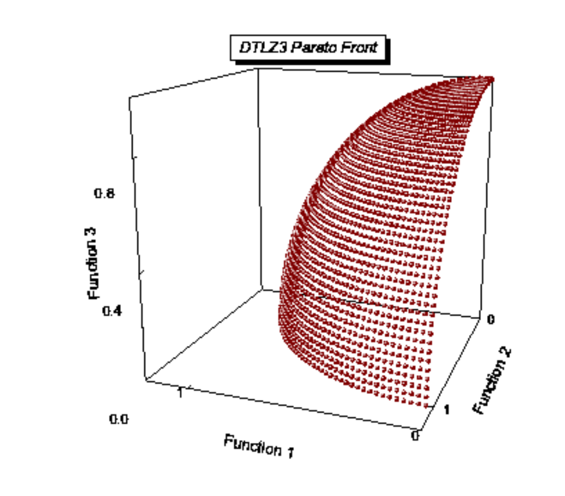


## DTLZ3

It is suggested that $k = |x_M| = 10$. There is a total of $n = M + k - 1$ decision variables.

The function $g$ described before, introduces ($3k$ - 1) false Pareto fronts. All of these false Pareto fronts are parallel to the global Pareto front and, therefore, a MOEA can get easily trapped in one of them before converging to the Pareto optimal front which is located at $g^* = 0$.

The true Pareto front corresponds to $x_M = (0.5, \ldots, 0.5)^T$.




## DTLZ Test Problems

**DTLZ3**

## DTLZ4

Minimize:

$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\cos(x_2^{\pi}\pi/2)\ldots\cos(x_{M-2}^{\pi}\pi/2)\cos(x_{M-1}^{\pi}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\cos(x_2^{\pi}\pi/2)\ldots\cos(x_{M-2}^{\pi}\pi/2)\sin(x_{M-1}^{\pi}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\cos(x_2^{\pi}\pi/2)\ldots\sin(x_{M-2}^{\pi}\pi/2),\\
&\vdots \qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\sin(x_2^{\pi}\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(x_1^{\pi}\pi/2).
\end{aligned}
$$

$$\text{subject to: } 0 \le x_i \le 1 \quad \forall \quad i = 1, 2, \ldots, n$$

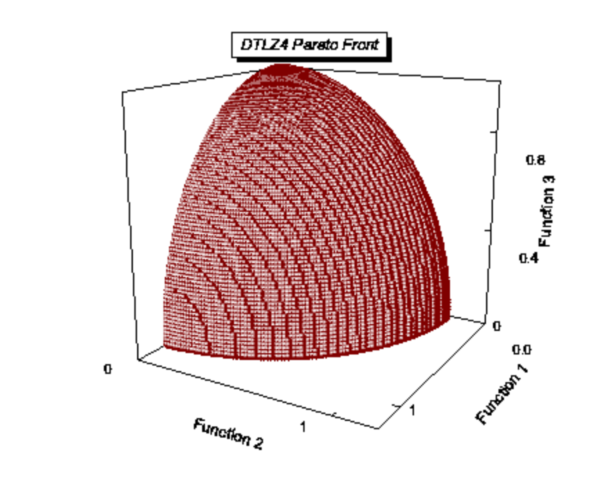
$$\text{where: } g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2$$

## DTLZ4

It is suggested to use $\alpha = 100$ in this case. Again, all the decision variables $x_1$ to $x_{M-1}$ are varied in the range $(0 : 1)$.

It is also suggested to use $k = 10$. There are $n = M + k - 1$ decision variables in this problem.

In this case, there is dense set of solutions close to the plane $f_M - f_1$.




## DTLZ Test Problems

**DTLZ4**

## DTLZ5

Minimize:

$$\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\dots\cos(\theta_{M-2}\pi/2)\cos(\theta_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\dots\cos(\theta_{M-2}\pi/2)\sin(\theta_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\dots\sin(\theta_{M-2}\pi/2),\\
&\vdots \qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\sin(\theta_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(\theta_1\pi/2).
\end{aligned}$$

subject to: $0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$

where: $\theta_i = \frac{\pi}{4(1+g(x_M))}(1+2g(x_M)x_i)$, for $i = 2, 3, \dots, (M-1)$

$$g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2$$

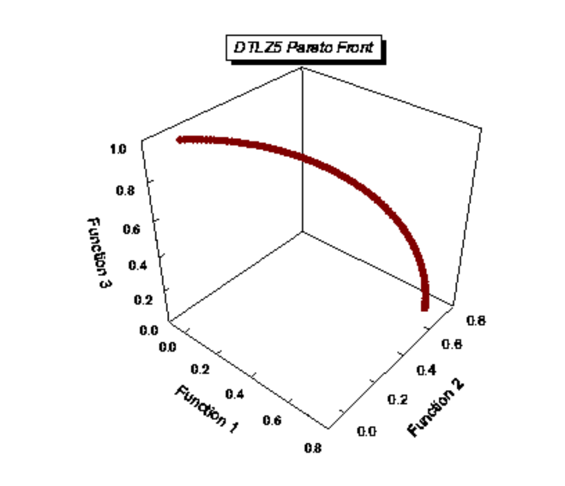
## DTLZ5

It is suggested to use function $g$ with $k = |\mathbf{x}_M| = 10$. Also, there are $n = M + k - 1$ decision variables and the Pareto optimal set corresponds to $x_i = 0.5$ for every $x_i \in \mathbf{x}_M$ and every objective function must satisfy: $\sum_{i=1}^{M} (f_i)^2 = 1$.

This problem evaluates the capability of a MOEA to converge to a curve.

It is suggested to use ($M \in [5, 10]$).

# Test Problems




## DTLZ Test Problems

**DTLZ5**

## DTLZ6

Minimize:

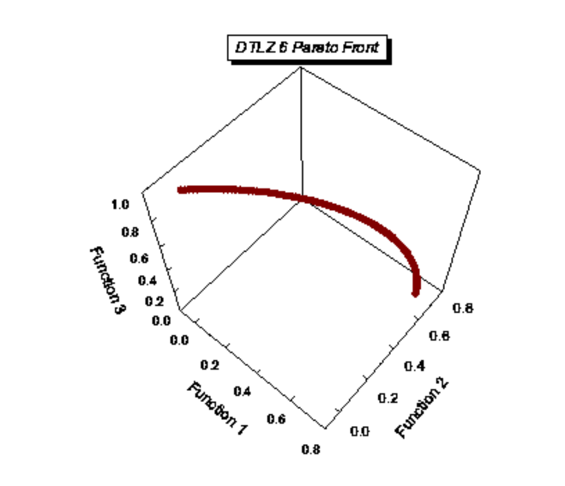
$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\ldots\cos(\theta_{M-2}\pi/2)\cos(\theta_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\ldots\cos(\theta_{M-2}\pi/2)\sin(\theta_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\ldots\sin(\theta_{M-2}\pi/2),\\
\vdots &\qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\sin(\theta_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(\theta_1\pi/2).\\
\text{subject to: } & 0 \le x_i \le 1 \quad \forall \quad i = 1, 2, ..., n\\
\text{where: } \theta_i &= \frac{\pi}{4(1+g(x_M))}(1+2g(x_M)x_i), \forall i = 2, 3, \ldots, (M-1)\\
g(x_M) &= \sum_{x_i \in X_M} (x_i)^{0.1}
\end{aligned}
$$

**DTLZ6**

The Pareto optimal set is located at $x_i = 0$ for every $x_i \in x_M$.

The size of the vector $x_M$ is chosen as 10 and the total number of decision variables is identical to the one used for DTLZ5.

# Test Problems




## DTLZ Test Problems

**DTLZ6**

## DTLZ7

Minimize:

$$
\begin{aligned}
f_1(x) &= x_1, \\
f_2(x) &= x_2, \\
\vdots \quad & \quad \vdots \\
f_{M-1}(x) &= x_{M-1} \\
f_M(x) &= (1 + g(x_M)) \cdot h(f_1, f_2, \ldots, f_{M-1} g(x)) \\
\text{subject to: } 0 \leq x_i \leq 1 \quad \forall \quad i &= 1, 2, ..., n \\
\text{where: } g(x) &= 1 + \frac{9}{|x_M|} \sum_{x_i \in x_M} x_i, \\
h(f_1, f_2, \ldots, f_{M-1}, g) &= M - \sum_{i=1}^{M-1} \left( \frac{f_i}{1 + g(x)} (1 + sin(3\pi f_i)) \right)
\end{aligned}
$$

# Test Problems

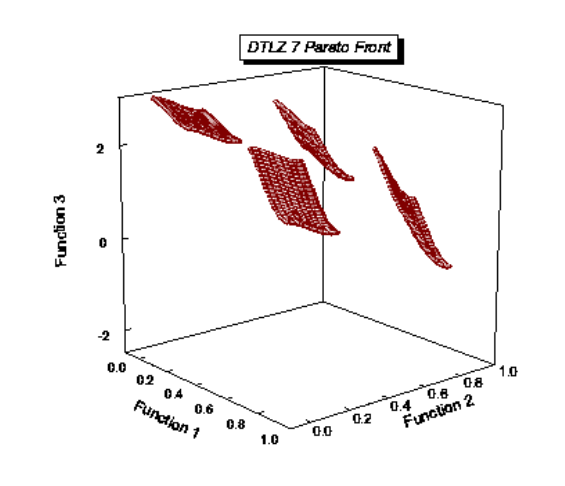

## DTLZ7

This problem has $2M - 1$ disconnected Pareto optimal regions.

$g$ requires $k = |x_M j|$ decision variables and the total number of decision variables is $n = M + k - 1$. It is suggested to use $k = 20$.

The Pareto optimal set corresponds to $x_M = 0$.

This problem aims to test the ability of a MOEA to maintain, simultaneously, solutions at different regions of the search space.




## DTLZ Test Problems

**DTLZ7**

# Test Problems

## DTLZ8

Minimize:

$$f_j(x) = \frac{1}{\lfloor n/M \rfloor} \sum_{\lfloor i=(j-1)\frac{n}{M} \rfloor}^{\lfloor j\frac{n}{M} \rfloor} (x_i), \forall j = 1, 2, \ldots, M,$$

$$\text{subject to: } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } g_j(x) = f_M(x) + 4f_j(x) - 1 \geq 0, \forall j = 1, 2, \ldots, (M-1)$$

$$g_M(x) = 2f_M(x) + min_{i,j=1,i\neq j}^{M-1} [f_i(x) + f_j(x)] - 1 \geq 0,$$

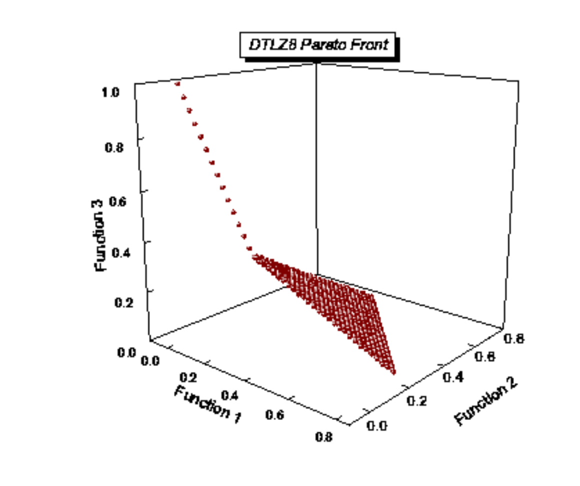
# Test Problems

## DTLZ8

The number of decision variables must be larger than the number of objectives $n > M$. It is suggested to use $n = 10M$.

This problem has $M$ constraints. The true Pareto front is a combination of a straight line and a hyperplane.

The straight line is the intersection of the first $(M - 1)$ constraints (with $f_1 = f_2 = \ldots = f_M - 1$ and the hyperplane is represented through constraint $g_M$).




## DTLZ Test Problems

**DTLZ8**

## DTLZ9

Minimize:

$$f_j(x) = \frac{1}{\lfloor n/M \rfloor} \sum_{\lfloor i=(j-1)\frac{n}{M} \rfloor}^{\lfloor j\frac{n}{M} \rfloor} \left( x_i^{0.1} \right), \forall j = 1, 2, \ldots, M,$$

$$\text{subject to: } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } g_j(x) = f_M^2(x) + f_j^2(x) - 1 \geq 0, \forall j = 1, 2, \ldots, (M-1)$$

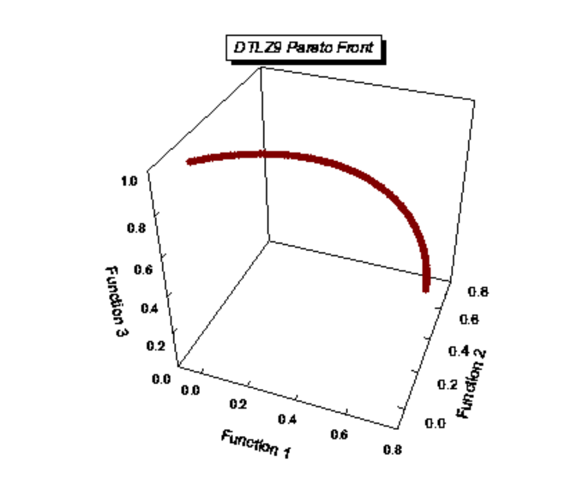
## DTLZ9

The number of decision variables must be larger than the number of objectives. It is suggested to use: $n = 10M$.

The true Pareto front is a curve with $f_1 = f_2 = \ldots = f_M - 1$, similar to the Pareto front of DTLZ5. However, in this case, the density of solutions decreases as we approach the Pareto optimal region.

The Pareto front is at the intersection of all the $(M - 1)$ constraints, which can cause difficulties to a MOEA.

# Test Problems




## DTLZ Test Problems

**DTLZ9**

## Okabe's Test Problems

Tatsuya Okabe et. al [2004] proposed a methodology to generate multi-objective test problems based on a mapping of probability density functions from decision variable space to objective function space. They also provide two examples of this methodology.

The basic idea is to depart from an initial space (called $S^2$) between decision variable space and objective function space and from there, they build both spaces by applying appropriate functions to $S^2$. For this sake, the authors proposed to use the inverse of the generation operation (i.e., deformation, rotation and translation).

Tatsuya Okabe, Yaochu Jin, Markus Olhofer and Bernhard Sendhoff, "**On Test Functions for Evolutionary Multi-objective Optimization**", in Xin Yao et al. (editors), *Parallel Problem Solving from Nature - PPSN VIII*, Springer-Verlag, Lecture Notes in Computer Science, Vol. 3242, pp. 792–802, Birmingham, UK, September 2004.

## Okabe's Test Problems

**OKA1:**

Minimize:

$$
\begin{aligned}
f_1 &= x_1', \\
f_2 &= \sqrt{2\pi} - \sqrt{|x_1'|} + 2|x_2' - 3\cos(x_1') - 3|^{\frac{1}{2}}, \\
&\text{where:} \\
x_1' &= \cos(\pi/12)x_1 - \sin(\pi/12)x_2, \\
x_2' &= \sin(\pi/12)x_1 + \cos(\pi/12)x_2, \\
&\text{subject to:} \\
x_1 &\in [6\sin(\pi/12), 6\sin(\pi/12) + 2\pi\cos(\pi/12)], \\
x_2 &\in [-2\pi\sin(\pi/12), 6\cos(\pi/12)],
\end{aligned}
\tag{14}
$$

# Test Problems

## Okabe's Test Problems

The Pareto optimal set is located at: $x'_2 = 3\cos(x'_1 + 3)$ and $x'_1 \in [0, 2\pi]$.

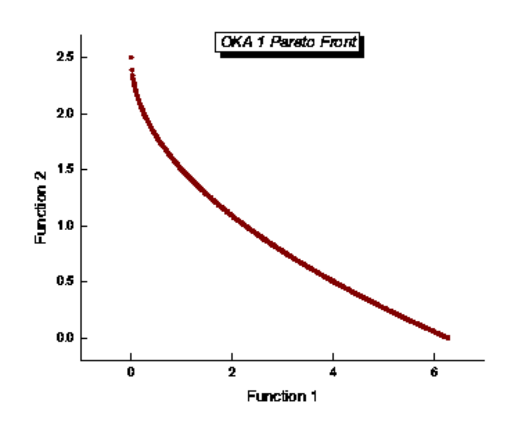
The Pareto front is located at: $f_2 = \sqrt{(2\pi)} - \sqrt{f_1}$ and $f_1 \in [-\pi, \pi]$.

The Distribution indicator is:

$$D_{x \to f} = \frac{3}{2}|x'_2 - 3\cos(x'_1) - 3|^{\frac{2}{3}} \tag{15}$$

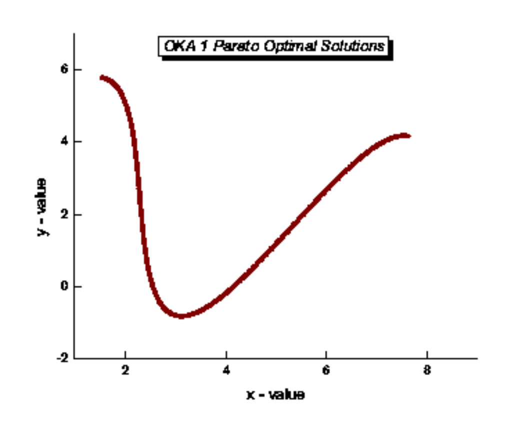
The Distribution indicator measures the amount of distortion that the probability density suffers in decision variable space under the mapping from decision variable space to objective function space.

# Test Problems




## Okabe's Test Problems

**Pareto front of OKA1**




## Okabe's Test Problems

**Pareto optimal set of OKA1**

## Okabe's Test Problems

**OKA2:**

Minimize:

$$
\begin{aligned}
f_1 &= x_1, \\
f_2 = 1 - \frac{1}{4\pi^2}(x_1+\pi)^2 + |x_2 - 5\cos(x_1)|^{\frac{1}{3}} + |\tfrac{x}{3} - 5\sin(x_1)|^{\frac{1}{3}}&, \\
\text{subject to:}& \\
x_1 &\in [-\pi, \pi], \\
x_2, x_3 &\in [-5, 5]
\end{aligned}
$$

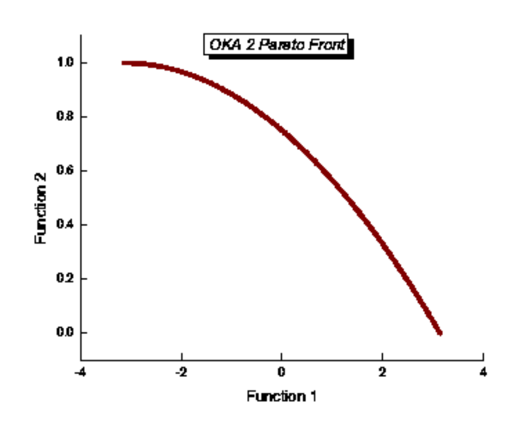
## Okabe's Test Problems

The Pareto optimal set is located at: $(x_1, x_2, x_3) = (x_1, 5\cos(x_1), 5\sin(x_1))$ and $x_1 \in [-\pi, \pi]$.

The true Pareto front is located at: $f_2 = 1 - \frac{1}{4\pi^2}(f_1 + \pi)^2$ and $f_1 \in [-\pi, \pi]$.

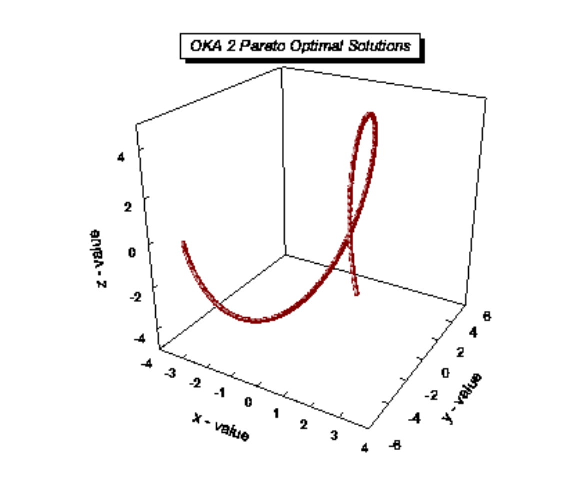
The Distribution indicator is: $D_{x \to f} = 9|x_2 - 5\cos(x_1)|^{\frac{2}{3}}|x_3 - 5\sin(x_1)|^{\frac{2}{3}}$.

# Test Problems




## Okabe's Test Problems

**Pareto front of OKA2**




## Okabe's Test Problems

**Pareto optimal set of OKA2**

## WFG Test Problems

Huband et al. [2006] proposed a methodology to design test problems which are quite challenging for MOEAs. The set that they used to exemplify their methodology is known as the Walking-Fish-Group (WFG) test suite.

In the next slides, we show the shapes for the objective functions and the transformation functions.

Simon Huband, Phil Hingston, Luigi Barone and Lyndon While, "**A Review of Multiobjective Test Problems and a Scalable Test Problem Toolkit**", *IEEE Transactions on Evolutionary Computation*, Vol. 10, No. 5, pp. 477–506, October 2006.

# WFG Test Problems (Shapes for the objective functions)

$$
\begin{aligned}
\text{linear}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1} x_i \\
\text{linear}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m} x_i\right)(1-x_{M-m+1}) \\
\text{linear}_M(x_1,\ldots,x_{M-1}) &= 1-x_1
\end{aligned}
$$

$$
\begin{aligned}
\text{convex}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1}(1-\cos(x_i\pi/2)) \\
\text{convex}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m}(1-\cos(x_i\pi/2))\right)(1-\sin(x_{M-m+1}\pi/2)) \\
\text{convex}_M(x_1,\ldots,x_{M-1}) &= 1-\sin(x_1\pi/2)
\end{aligned}
$$

# WFG Test Problems (Shapes for the objective functions)

$$\begin{aligned}
\text{concave}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1}\sin(x_i\pi/2)) \\
\text{concave}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m}\sin(x_i\pi/2)\right)\cos(x_{M-m+1}\pi/2) \\
\text{concave}_M(x_1,\ldots,x_{M-1}) &= \cos(x_1\pi/2)
\end{aligned}$$

$$\text{mixed}_M(x_1,\ldots,x_{M-1}) = \left(1 - x_1 - \frac{\cos(2A\pi x_1 + \pi/2)}{2A\pi}\right)^{\alpha}$$

$$\text{disc}_M(x_1,\ldots,x_{M-1}) = 1 - x_1^{\alpha}\cos^2(Ax_1^{\beta}\pi)$$

# WFG Test Problems (Transformation functions)

$$\text{b\_poly}(y, \alpha) = y^{\alpha}$$

$$\text{b\_flat}(y, A, B, C) = A + \min(0, \lfloor y - B \rfloor) \frac{A(B - y)}{B} - \min(0, \lfloor C - y \rfloor) \frac{(1 - A)(y - C)}{1 - C}$$

$$\text{b\_param}(y, u(\vec{y}'), A, B, C) = y^{B+(C-B)\left(A-(1-2u(\vec{y}'))\left|\left\lfloor 0.5-u(\vec{y}')\right\rfloor+A\right|\right)}$$

$$\text{s\_linear}(y, A) = \frac{|y - A|}{|\lfloor A - y \rfloor + A|}$$

$$\text{s\_decept}(y, A, B, C) = 1 + (|y - A| - B) \left( \frac{\lfloor y - A + B \rfloor \left(1 - C + \frac{A-B}{B}\right)}{A - B} + \frac{\lfloor A + B - y \rfloor \left(1 - C + \frac{1-A-B}{B}\right)}{1 - A - B} + \frac{1}{B} \right)$$

$$\text{s\_multi}(y, A, B, C) = \frac{1 + \cos\left((4A + 2)\pi \left(0.5 - \frac{|y-C|}{2(\lfloor C-y \rfloor + C)}\right)\right) + 4B \left(\frac{|y-C|}{2(\lfloor C-y \rfloor + C)}\right)^2}{b + 2}$$

$$\text{r\_sum}(\vec{y}, \vec{w}) = \frac{\sum_{i=1}^{|\vec{y}|} w_1 y_i}{\sum_{i=1}^{|\vec{y}|} w_i}$$

$$\text{r\_nonsep}(\vec{y}, A) = \frac{\sum_{j=1}^{|\vec{y}|} \left(y_j + \sum_{k=0}^{A-2} \left|y_j - y_{1+(j+k) mod |\vec{y}|}\right|\right)}{\frac{|\vec{y}|}{A} \left\lceil \frac{A}{2} \right\rceil \left(1 + 2A - 2 \left\lceil \frac{A}{2} \right\rceil\right)}$$

**WFG1:**
Minimize

$$
\begin{aligned}
f_{m=1:M-1}(\vec{x}) &= x_M + S_m \mathrm{convex}_m(x_1, \ldots, x_{M-1}) \\
f_M(\vec{x}) &= x_M + S_M \mathrm{mixed}_M(x_1, \ldots, x_{M-1})
\end{aligned}
$$

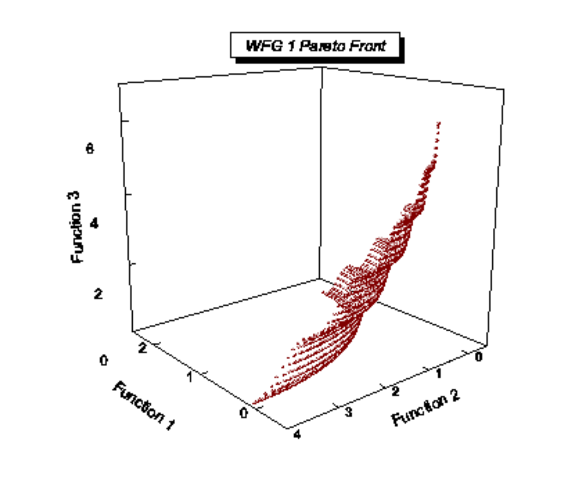
where

$$
\begin{aligned}
y_{i=1:M-1} &= \mathrm{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [2((i-1)k/(M-1)+1), \ldots, 2ik/(M-1)]) \\
y_M &= \mathrm{r\_sum}([y'_{k+1}, \ldots, y'_n], [2(k+1), \ldots, 2n]) \\
y'_{i=1:n} &= \mathrm{b\_poly}(y''_i, 0.02) \\
y''_{i=1:k} &= y'''_i \\
y''_{i=k+1:n} &= \mathrm{b\_flat}(y'''_i, 0.8, 0.75, 0.85) \\
y'''_{i=1:k} &= z_{i,[0,1]} \\
y'''_{i=k+1:n} &= \mathrm{s\_linear}(z_{i,[0,1]}, 0.35)
\end{aligned}
$$

# Test Problems

For all problems:
The decision vector is $z = [z_1, \ldots, z_k, z_{k+1}, \ldots, z_n]$ where $0 \leq z_i \leq z_{i,\max}$.

$$\begin{aligned}
z_{i=1:n,\max} &= 2i \\
z_{i=1:n,[0,1]} &= \frac{z_i}{z_{i,\max}} \\
x_{i=1:M-1} &= \max(y_M, A_i)(y_i - 0.5) + 0.5 \\
x_M &= y_M \\
S_{m=1:M} &= 2m \\
A_1 &= 1 \\
A_{2:M-1} &= \begin{cases} 0, & \text{for WFG3} \\ 1, & \text{otherwise} \end{cases}
\end{aligned}$$




## WFG Test Problems

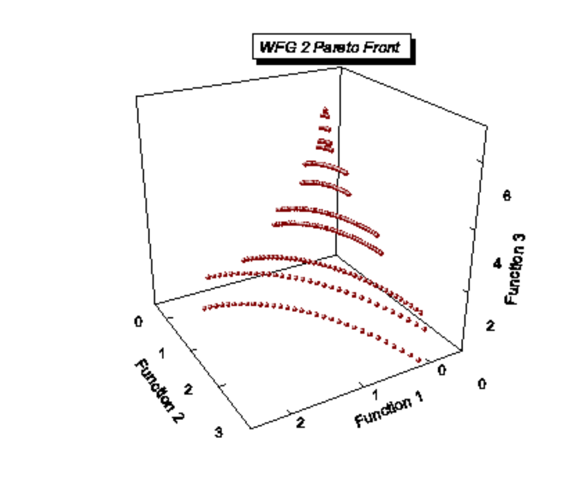
**Pareto front of WFG1**

**WFG2:**
Minimize

$$
\begin{aligned}
f_{m=1:M-1}(\vec{x}) &= x_M + S_m \mathrm{convex}_m(x_1, \ldots, x_{M-1}) \\
f_M(\vec{x}) &= x_M + S_M \mathrm{disc}_M(x_1, \ldots, x_{M-1})
\end{aligned}
$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \mathrm{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \mathrm{r\_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:k+l/2} &= \mathrm{r\_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\
y''_{i=1:k} &= z_{i,[0,1]} \\
y''_{i=k+1:n} &= \mathrm{s\_linear}(z_{i,[0,1]}, 0.35)
\end{aligned}
$$




## WFG Test Problems

**Pareto front of WFG2**

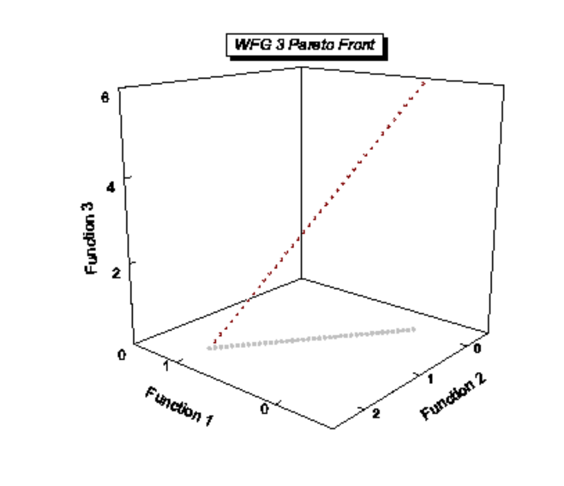
**WFG3:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{linear}_m(x_1, \ldots, x_{M-1})$$

where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:k+l/2} &= \text{r\_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\
y''_{i=1:k} &= z_{i,[0,1]} \\
y''_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35)
\end{aligned}
$$

# Test Problems




## WFG Test Problems

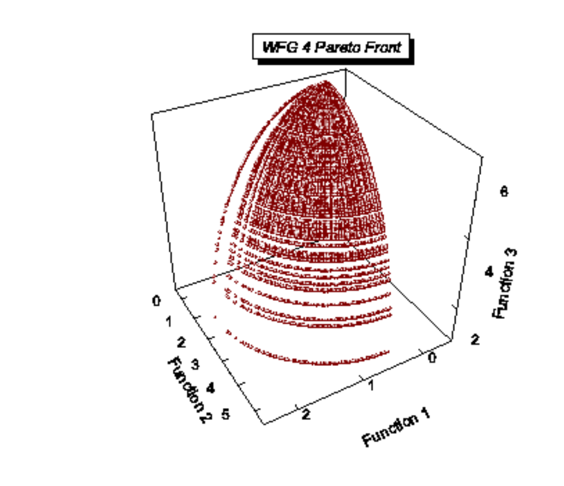
**Pareto front of WFG3**

**WFG4:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\ y'_{i=1:n} &= \text{s\_multi}(z_{i,[0,1]}, 30, 10, 0.35) \end{aligned}$$




## WFG Test Problems

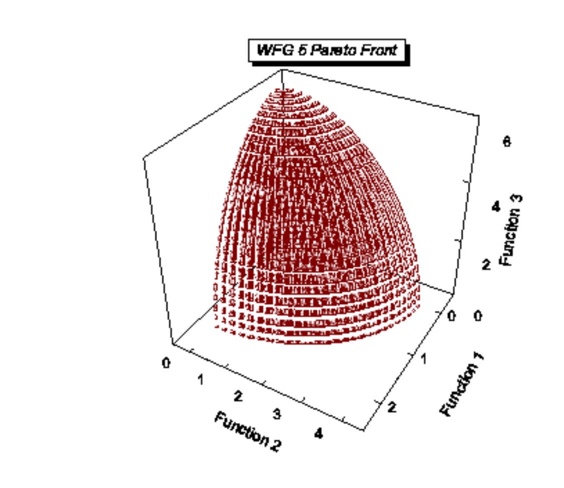
**Pareto front of WFG4**

**WFG5:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\ y'_{i=1:n} &= \text{s\_decept}(z_{i,[0,1]}, 0.35, 0.001, 0.05) \end{aligned}$$




## WFG Test Problems

**Pareto front of WFG5**

# Test Problems

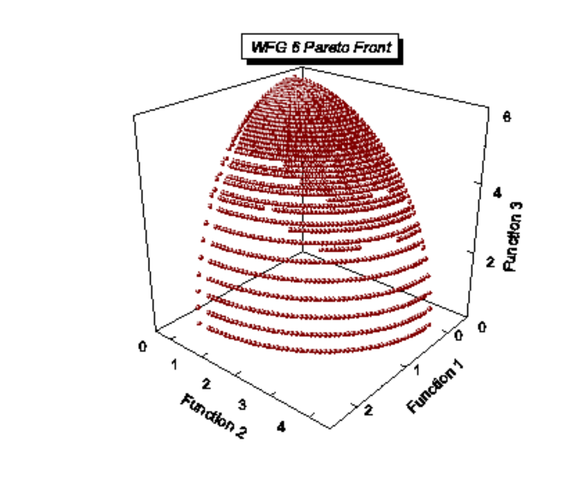
**WFG6:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\ y_M &= \text{r\_nonsep}([y'_{k+1}, \ldots, y'_n], l) \\ y'_{i=1:k} &= z_{i,[0,1]} \\ y'_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35) \end{aligned}$$

# Test Problems




## WFG Test Problems

**Pareto front of WFG6**

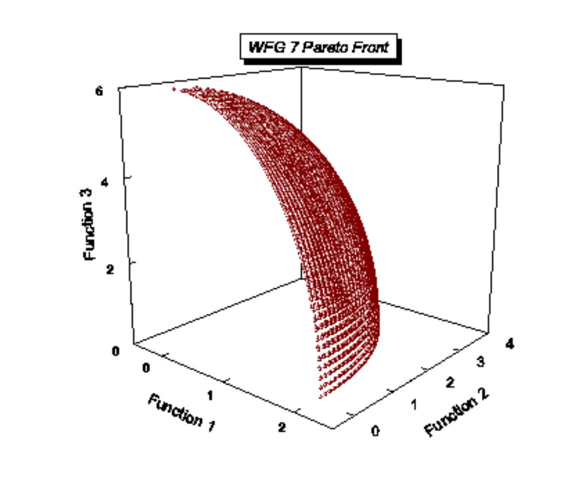
**WFG7:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \text{r_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:n} &= \text{s_linear}(y''_i, 0.35) \\
y''_{i=1:k} &= \text{b_param}(z_{i,[0,1]}, \text{r_sum}([z_{i+1,[0,1]}, \ldots, z_{n,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_{i=k+1:n} &= z_{i,[0,1]}
\end{aligned}$$

# Test Problems




## WFG Test Problems

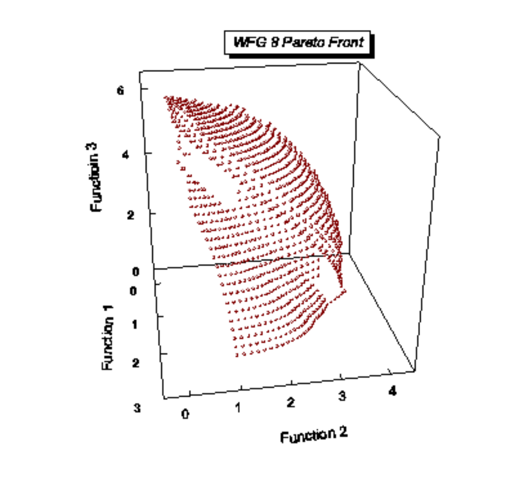
**Pareto front of WFG7**

**WFG8:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \text{r_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:n} &= \text{s_linear}(y''_i, 0.35) \\
y''_{i=1:k} &= z_{i,[0,1]} \\
y''_{i=k+1:n} &= \text{b_param}(z_{i,[0,1]}, \text{r_sum}([z_{1,[0,1]}, \ldots, z_{i-1,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50)
\end{aligned}$$




## WFG Test Problems

**Pareto front of WFG8**

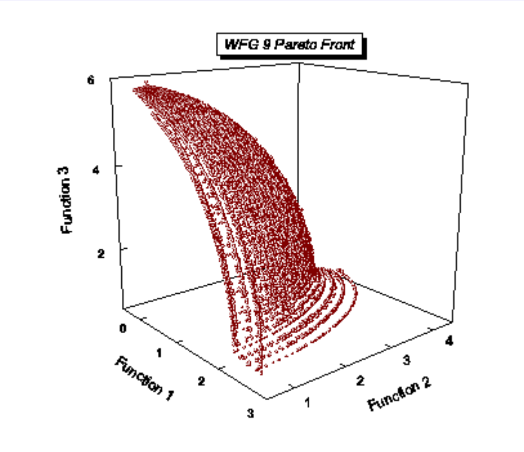
**WFG9:**
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \mathrm{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \mathrm{r_nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\
y_M &= \mathrm{r_nonsep}([y'_{k+1}, \ldots, y'_n], l) \\
y'_{i=1:k} &= \mathrm{s_decept}(y''_i, 0.35, 0.001, 0.05) \\
y'_{i=k+1:n} &= \mathrm{s_multi}(y''_i, 30, 95, 0.35) \\
y''_{i=1:n-1} &= \mathrm{b_param}(z_{i,[0,1]}, \mathrm{r_sum}([z_{i+1,[0,1]}, \ldots, z_{n,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_n &= z_{n,[0,1]}
\end{aligned}$$

# Test Problems




## WFG Test Problems

**Pareto front of WFG9**

# Test Problems

There are other multi-objective test problems available to assess performance of a MOEA.

Peng Cheng, "**A Tunable Constrained Test Problems Generator for Multi-objective Optimization**", in *Proceedings of the 2008 Second International Conference on Genetic and Evolutionary Computing (WGEC'2008)*, pp. 96–100, IEEE Computer Society, Washington, DC, USA, September 2008, ISBN 978-0-7695-3334-6.

Michael T.M. Emmerich and André H. Deutz, "**Test Problems Based on Lamé Superspheres**", in Shigeru Obayashi et al. (Eds), *Evolutionary Multi-Criterion Optimization, 4th International Conference, EMO 2007*, pp. 922–936, Springer. Lecture Notes in Computer Science Vol. 4403, Matshushima, Japan, March 2007.

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# Test Problems

## *NP*-Complete Multi-Objective Problems

| ***NP*-Complete Problem** | **Example** |
|---|---|
| Traveling Salesperson | Min energy, time, and/or distance; Max expansion |
| Coloring | Min number of colors, number of each color |
| Set/Vertex Covering | Min total cost, over-covering |
| Maximum Independent Set (Clique) | Max set size; Min geometry |

## *NP*-Complete Multi-Objective Problems

| ***NP*-Complete Problem** | **Example** |
|---|---|
| Vehicle Routing | Min time, energy, and/or geometry |
| Scheduling | Min time, deadlines, wait time, resource use |
| Layout | Min space, overlap, costs |
| *NPC*-Problem Combinations | Vehicle scheduling and routing |
| 0/1 Knapsacks - Bin Packing | Max profit; Min weight |
| Minimum Spanning Trees | tuple weighted edges; minimum weighting |