Non-Dominated Sorting
(Multi-Dimensional Sorting)

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4 April 2019
IIT Patna

Outline

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Motivation

Ticket	Travel Time (in Hrs.)	Ticket Cost (In Thousand)
Tkt_1	1	1
Tkt_2	1	2
Tkt_3	3	1
Tkt_4	2	3
Tkt_5	4	2

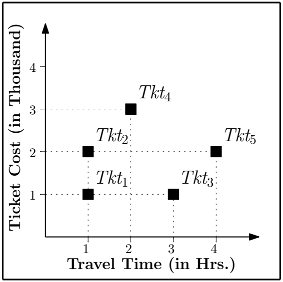


Figure 1 : Ticket Comparison

Outline

Solution Representation

Solution Representation

A solution '*sol*' in M-dimensional space is represented as

$$sol = \{f_1(sol), f_2(sol), \ldots, f_M(sol)\} \tag{1}$$

where $f_m(sol), 1 \leq m \leq M$ is the value of solution '*sol*' in m^{th} dimension.

Representation of 5 solutions

$sol_1 = \{1, 1\}$
$sol_2 = \{1, 2\}$
$sol_3 = \{3, 1\}$
$sol_4 = \{2, 3\}$
$sol_5 = \{4, 2\}$

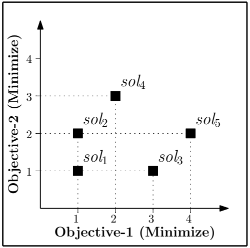


Figure 2 : Solutions in 2-dimensional space.

Outline

Dominance Relationship

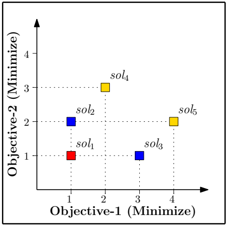
Definition: Dominance Relationship (for minimization problem)

A solution $sol_i = \{f_1(sol_i), f_2(sol_i), \ldots, f_M(sol_i)\}$ dominates another solution $sol_j = \{f_1(sol_j), f_2(sol_j), \ldots, f_M(sol_j)\}$ denoted as $sol_i \prec sol_j$ *iff*

1. $f_m(sol_i) \leq f_m(sol_j) \quad \forall m \in \{1, 2, \ldots, M\}$
2. $f_m(sol_i) < f_m(sol_j) \quad \exists m \in \{1, 2, \ldots, M\}$

sol_i and sol_j are non-dominated represented as $sol_i \preceq sol_j$ *iff* neither $sol_i \prec sol_j$ nor $sol_j \prec sol_i$

In the Figure



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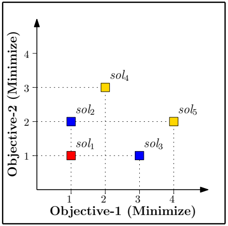
$sol_1 \prec \{sol_2, sol_3, sol_4, sol_5\}$



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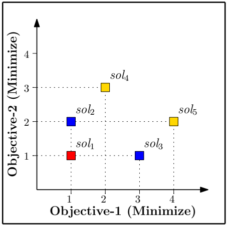

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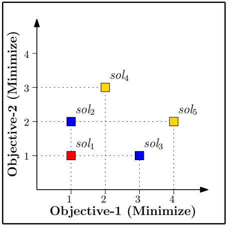

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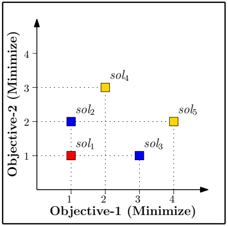

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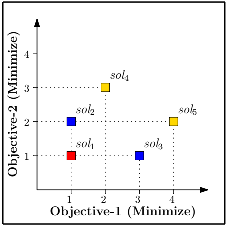
$sol_3 \preceq sol_4$



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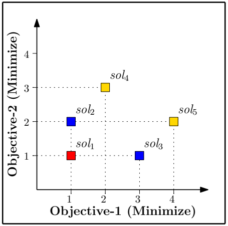

Questions?

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Q1

$sol_i = \{1, 1\} \quad sol_j = \{1, 5\}$

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Relationship: $sol_i \prec sol_j$

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Q2

$sol_i = \{1, 2, 1\} \quad sol_j = \{1, 1, 1\}$

Questions?

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$sol_i = \{1, 1\} \quad sol_j = \{1, 5\}$
Relationship: $sol_i \prec sol_j$

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Q2

$sol_i = \{1, 2, 1\} \quad sol_j = \{1, 1, 1\}$

Relationship: $sol_j \prec sol_i$

Q3

$sol_i = \{1, 2, 1\} \quad sol_j = \{1, 1, 2\}$

Questions?

Q1

$sol_i = \{1, 1\}$ $\quad$ $sol_j = \{1, 5\}$
Relationship: $sol_i \prec sol_j$

Q2

$sol_i = \{1, 2, 1\}$ $\quad$ $sol_j = \{1, 1, 1\}$
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Relationship: $sol_i \prec sol_j$

Q2

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Relationship: $sol_j \prec sol_i$

Q3

$sol_i = \{1, 2, 1\} \qquad sol_j = \{1, 1, 2\}$
Relationship: $sol_i \preceq sol_j$

Q4

$sol_i = \{4, 2, 1, 3\} \qquad sol_j = \{1, 1, 2, 1\}$

Questions?

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$sol_i = \{1, 1\} \quad sol_j = \{1, 5\}$
Relationship: $sol_i \prec sol_j$

Q2

$sol_i = \{1, 2, 1\} \quad sol_j = \{1, 1, 1\}$
Relationship: $sol_j \prec sol_i$

Q3

$sol_i = \{1, 2, 1\} \quad sol_j = \{1, 1, 2\}$
Relationship: $sol_i \preceq sol_j$

Q4

$sol_i = \{4, 2, 1, 3\} \quad sol_j = \{1, 1, 2, 1\}$
Relationship: $sol_j \preceq sol_i$

Outline

Non-Domination Sorting

Definition: Non-Dominated Sorting [1]

Given a set of N solutions $\{sol_1, sol_2, \ldots, sol_N\}$ in an M-dimensional space. Non-Dominated Sorting divides these solutions in $K(1 \leq K \leq N)$ different fronts $\{F_1, F_2, \ldots, F_K\}$ which are arranged in decreasing order of their dominance. The division of the solutions in fronts is such that

1. $\forall sol_i, sol_j \in F_k\colon\ sol_i \preceq sol_j \qquad 1 \leq k \leq K$
2. $\forall sol \in F_k,\ \exists sol' \in F_{k-1}\colon\ sol' \prec sol \qquad 2 \leq k \leq K$

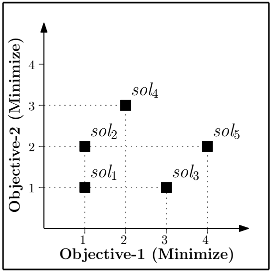


Figure 3 : Solutions

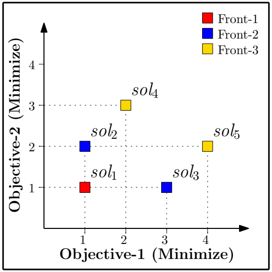


Figure 4 : Non-Dominated Fronts

Different Approaches

1. Naive Approach [4]
2. Deductive Sort [3]

Outline

1. Non-Dominated Sorting
   - Motivation
   - Solution Representation
   - Dominance Relationship
   - Problem Definition
2. Approaches
   - Naive Approach
   - Efficient Non-Dominated Sort (ENS)
3. Conclusions & Future Work

Naive Approach: Basic Idea

1. All the solutions belonging to a particular front are obtained together.
   - F_1 is obtained
   - F_2 is obtained

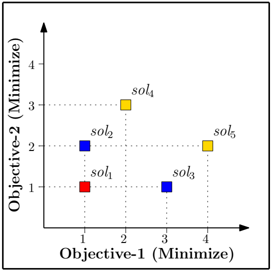
     $\vdots$
   - F_K is obtained
2. Each solution is compared with all other solutions.
3. The solutions which are not dominated by any other solution are assigned to the current front.

Domination Count

Definition: Domination Count

Domination count of a solution '*sol*' in population $\mathbb{P}$ is the number of solutions in $\mathbb{P}$ which dominates solution '*sol*'.

In the Figure



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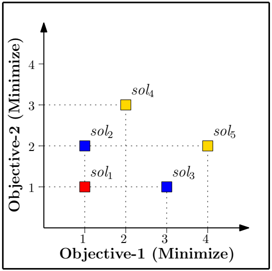
Domination Count of $sol_1 = 0$



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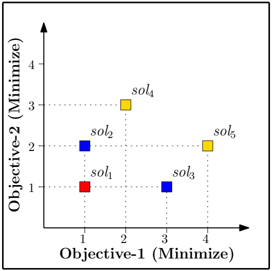

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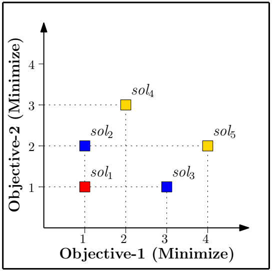

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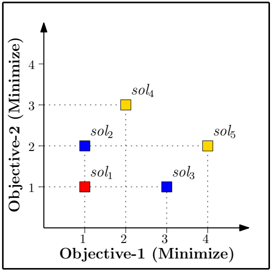
Domination Count of $sol_1 = 0$
Domination Count of $sol_2 = 1$
Domination Count of $sol_3 = 1$
Domination Count of $sol_4 = 2$



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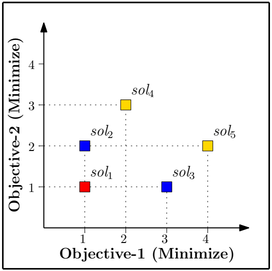
Domination Count of $sol_4 = 2$
Domination Count of $sol_5 = 3$



Working Example

1. For solution sol_1
   - $sol_1 \prec \{sol_2, sol_3, sol_4, sol_5\}$
2. For solution sol_2
   - $sol_2 \prec \{sol_4, sol_5\}$
   - $sol_2 \preceq \{sol_3\}$
   - $sol_2 \succ \{sol_1\}$
3. For solution sol_3
   - $sol_3 \prec \{sol_5\}$
   - $sol_3 \preceq \{sol_2, sol_4\}$
   - $sol_3 \succ \{sol_1\}$
4. For solution sol_4
   - $sol_4 \preceq \{sol_3, sol_5\}$
   - $sol_4 \succ \{sol_1, sol_2\}$
5. For solution sol_5
   - $sol_5 \preceq \{sol_4\}$
   - $sol_5 \succ \{sol_1, sol_2, sol_3\}$

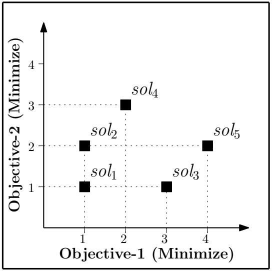


Figure 5 : Solutions

$n_{sol_1} = 0 \quad n_{sol_2} = 1$

$n_{sol_3} = 1 \quad n_{sol_4} = 2$

$n_{sol_5} = 3$

Working Example ...

1. For solution sol_2
   - $sol_2 \prec \{sol_4, sol_5\}$
   - $sol_2 \preceq \{sol_3\}$
2. For solution sol_3
   - $sol_3 \prec \{sol_5\}$
   - $sol_3 \preceq \{sol_2, sol_4\}$
3. For solution sol_4
   - $sol_4 \preceq \{sol_3, sol_5\}$
   - $sol_4 \succ \{sol_2\}$
4. For solution sol_5
   - $sol_5 \preceq \{sol_4\}$
   - $sol_5 \succ \{sol_2, sol_3\}$

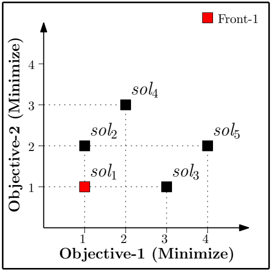


Figure 6 : Solutions

$n_{sol_2} = 0 \quad n_{sol_3} = 0$

$n_{sol_4} = 1 \quad n_{sol_5} = 2$

Working Example . . .

1. For solution sol_4
   - $sol_4 \preceq \{sol_5\}$
2. For solution sol_5
   - $sol_5 \preceq \{sol_4\}$

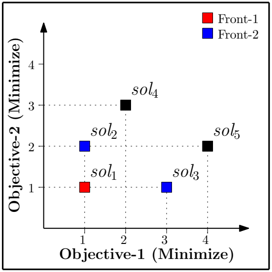


Figure 7 : Solutions

$n_{sol_4} = 0 \quad n_{sol_5} = 0$

Working Example . . .

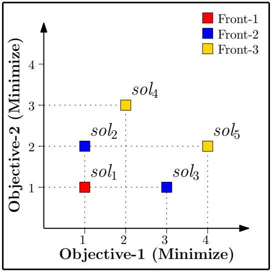


Figure 8 : Non-Dominated Fronts

Complexity Analysis

Space Complexity

- Domination count of each of the solutions is stored.
- Initially, the number of solutions is N.
- Space complexity $= \mathcal{O}(N)$

Time Complexity

- Each solution is compared with all other solutions.
  - Time required $= \mathcal{O}(MN^2)$
- This process may be repeated –
  - **Minimum 1 time:** When N solutions are in single fronts.
  - **Maximum N times:** When N solutions are in N different fronts.
- Time complexity
  - Best Case $= \mathcal{O}(MN^2)$
  - Worst Case $= \mathcal{O}(MN^3)$

Complexity Analysis

Space Complexity

- Domination count of each of the solutions is stored.
  - Storage requirement $= \mathcal{O}(N)$
- The set of solutions dominated by all the solutions is stored.
  - Storage requirement $= \mathcal{O}(N^2)$
- Space complexity $= \mathcal{O}(N^2)$

Time Complexity

- Each solution is compared with all other solutions exactly once.
  - Time required $= \mathcal{O}(MN^2)$
- The set of dominated solutions are traversed once and domination count value is reduced.
  - Time required –
    - When N solutions are in single fronts: $\mathcal{O}(N)$
    - When N solutions are in N different fronts: $\mathcal{O}(N^2)$
- Total time complexity $= \mathcal{O}(MN^2)$

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1. Non-Dominated Sorting
   - Motivation
   - Solution Representation
   - Dominance Relationship
   - Problem Definition

2. Approaches
   - Naive Approach
   - Efficient Non-Dominated Sort (ENS)

3. Conclusions & Future Work

Efficient Non-Dominated Sort: Basic Idea

- Various approaches usually compare a solution with all other solutions in the population before assigning it to a front.
- ENS compares the solutions with only those solutions that have already been assigned to a front.

Efficient Non-Dominated Sort

First Phase: Pre-Sorting

The solutions are sorted in ascending order based on the objective [2].
Advantage: When two solutions sol_i and $sol_j, i > j, 1 \leq i, j \leq N$ are compared, only two possibilities

- sol_i is non-dominated with sol_j
- sol_i is dominated by sol_j.

Second Phase: Assignment

Sorted solutions are assigned to their respective front.

Working Example

Solution	Objectives
sol_1	1,1
sol_2	1,2
sol_3	3,1
sol_4	2,3
sol_5	4,2

(a)

Solution	Objectives
sol_1	1,1
sol_2	1,2
sol_4	2,3
sol_3	3,1
sol_5	4,2

(b)

Table 1 : (a). A set of 5 solutions where two objectives are associate with each solution. (b). Solutions in sorted order based on objectives.

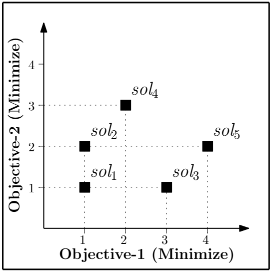

Figure 9 : Solutions

Working Example . . .

- $F_1 = \{sol_1\}$

Solution	Objectives
sol_2	1,2
sol_4	2,3
sol_3	3,1
sol_5	4,2

Table 2 : Un-assigned Solutions

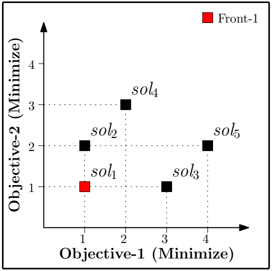

Figure 10 : Solutions

Working Example ...

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$

Solution	Objectives
sol_4	2,3
sol_3	3,1
sol_5	4,2

Table 3 : Un-assigned Solutions

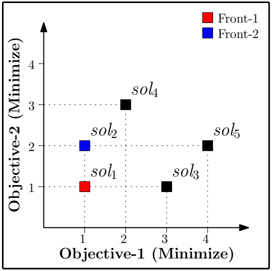

Figure 11 : Solutions

Working Example ...

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_4\}$

Solution	Objectives
sol_3	3,1
sol_5	4,2

Table 4 : Un-assigned Solutions

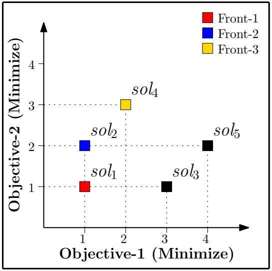

Figure 12 : Solutions

Working Example . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2, sol_3\}$
- $F_3 = \{sol_4\}$

Solution	Objectives
sol_5	4,2

Table 5 : Un-assigned Solutions

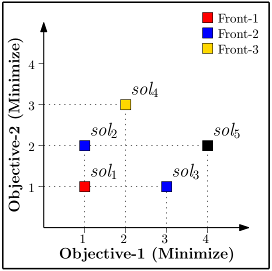

Figure 13 : Solutions

Working Example ...

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2, sol_3\}$
- $F_3 = \{sol_4, sol_5\}$

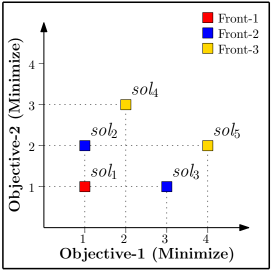

Figure 14 : Non-Dominated Fronts

Searching Techniques

To assign solution to a front, two search techniques can be used –

- Sequential search
- Binary search

Sequential

To obtain the position of un-assigned solution, sequential search is used in the sorted set of fronts.

Binary

To obtain the position of un-assigned solution, binary search is used in the sorted set of fronts.

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2, sol_3\}$
- $F_3 = \{sol_4\}$

Solution	Objectives
sol_5	4,2

Table 6 : Un-assigned Solutions

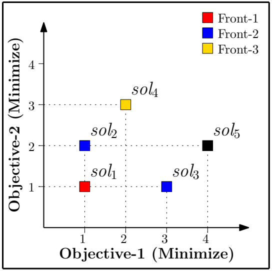

Figure 15 : Solutions

Sequential Search

Solution sol_5 is compared with the solutions of F_1, F_2, F_3 in a sequential manner. Order of comparison –

- F_1
- F_2
- F_3

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2, sol_3\}$
- $F_3 = \{sol_4\}$

Solution	Objectives
sol_5	4,2

Table 7 : Un-assigned Solutions

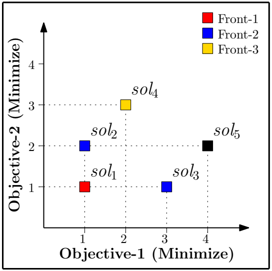

Figure 16 : Solutions

Binary Search

Solution sol_5 is first compared with F_2 and then depending on the dominance relationship of sol_5 with the solutions of F_2, sol_5 is compared with either F_1 or F_3. Order of comparison –

- F_2
- F_3

Efficient Non-Dominated Sort

Based on sequential search and binary search technique, there are two approaches derived from ENS:

- ENS-SS: Efficient Non-Dominated Sort Based on Sequential Search
- ENS-BS: Efficient Non-Dominated Sort Based on Binary Search

Complexity Analysis: First Phase

- Solutions are sorted based on the objective.
- Heap sort is used: $\mathcal{O}(N \log N)$
- While comparing two solutions minimum 1 and maximum M objective may be considered.
- Time Complexity:
  - Best Case: $\mathcal{O}(N \log N)$
  - Worst Case: $\mathcal{O}(MN \log N)$

Complexity Analysis: Second Phase

- A solution sol_i belongs to front F_k, iff
  - **Condition 1:** sol_i is dominated by at-least one solution belonging to fronts $F_1, F_2, \ldots, F_{k-1}$.
  - **Condition 2:** sol_i is non-dominated with all the previous solutions of front F_k.
- Both these conditions are considered to obtain the time complexity.

Complexity Analysis of ENS-SS: Worst case

All the solutions are in single front

- **Condition 1:** Not Applicable
- **Condition 2:**

$$\text{No. of dominance comparisons} = 0 + 1 + 2 + \ldots + (N-1) = \sum_{i=1}^{N}(i-1) = \frac{1}{2}N(N-1) \qquad (2)$$

All the solutions are in different fronts

- **Condition 1:**

$$\text{No. of dominance comparisons} = 0 + 1 + 2 + \ldots + (N-1) = \sum_{k=1}^{N}(k-1) = \frac{1}{2}N(N-1) \qquad (3)$$

- **Condition 2:** Not Applicable

Complexity Analysis of ENS-SS: Best case

N solutions are equally divided into $\sqrt{N}$ fronts[1]

- **Condition 1:**

$$\text{No. of dominance comparisons} = \sum_{k=1}^{\sqrt{N}} (k-1)\sqrt{N} = \frac{1}{2}N(\sqrt{N}-1) \tag{4}$$

- **Condition 2:**

$$\begin{aligned}\text{No. of dominance comparisons} &= \sum_{k=1}^{\sqrt{N}} \left[0+1+2+\ldots+(\sqrt{N}-1)\right] \\ &= \frac{1}{2}N(\sqrt{N}-1) \end{aligned} \tag{5}$$

$$\begin{aligned}\text{Total no. of dominance comparisons} &= \frac{1}{2}N(\sqrt{N}-1) + \frac{1}{2}N(\sqrt{N}-1) \\ &= N(\sqrt{N}-1) \end{aligned} \tag{6}$$

[1] Every solution in k^{th} front is dominated by all the solutions in $k-1^{th}$ front.

Complexity Analysis of ENS-BS

Worst Case: All the solutions are in single front

- **Condition 1:** Not Applicable
- **Condition 2:**

$$\begin{aligned} \text{No. of dominance comparisons} &= 0 + 1 + 2 + \ldots + (N-1) \\ &= \frac{1}{2}N(N-1) \end{aligned} \tag{7}$$

Best Case: All the solutions are in different fronts

- **Condition 1:**

$$\begin{aligned} \text{No. of dominance comparisons} &= \sum_{k=1}^{N} \lceil \log k \rceil = N \log N - (N-1) \\ &= \mathcal{O}(N \log N) \end{aligned} \tag{8}$$

- **Condition 2:** Not Applicable

Complexity of ENS-SS and ENS-BS: Summary

ENS-SS

- Worst Case: $\mathcal{O}(MN^2)$
- Best Case: $\mathcal{O}(MN\sqrt{N})$

ENS-BS

- Worst Case: $\mathcal{O}(MN^2)$
- Best Case: $\mathcal{O}(MN \log N)$

Working Example in Different Scenarios

- N solutions in single front
- N solutions in N fronts

Working Example: N Solutions in Single Front

Solution	Objectives
sol_1	1,8
sol_2	2,7
sol_3	3,6
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

(a)

Solution	Objectives
sol_1	1,8
sol_2	2,7
sol_3	3,6
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

(b)

Table 8 : (a). A set of 8 solutions where two objectives are associate with each solution. (b). Solutions in sorted order based on objectives.

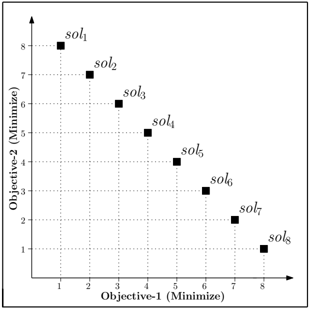

Figure 17 : Solutions

Working Example: N Solutions in Single Front ...

- $F_1 = \{sol_1\}$

Solution	Objectives
sol_2	2,7
sol_3	3,6
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 9 : Un-assigned Solutions

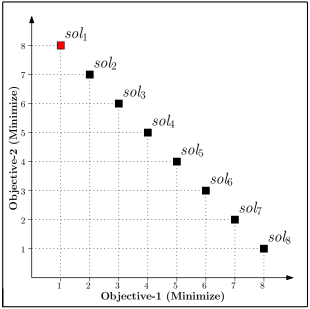

Figure 18 : Solutions

Working Example: N Solutions in Single Front ...

- $F_1 = \{sol_1, sol_2\}$

Solution	Objectives
sol_3	3,6
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 10 : Un-assigned Solutions

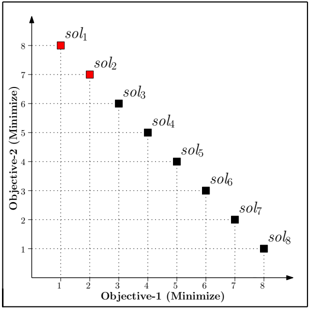

Figure 19 : Solutions

Working Example: N Solutions in Single Front . . .

- $F_1 = \{sol_1, sol_2, sol_3\}$

Solution	Objectives
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 11 : Un-assigned Solutions

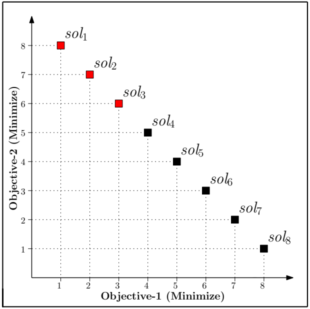

Figure 20 : Solutions

Working Example: N Solutions in Single Front . . .

- $F_1 = \{sol_1, sol_2, sol_3, sol_4\}$

Solution	Objectives
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 12 : Un-assigned Solutions

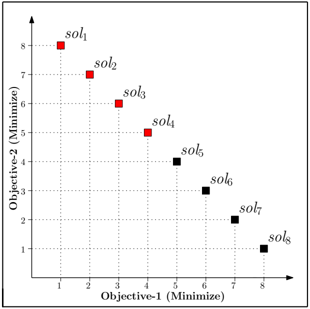

Figure 21 : Solutions

Working Example: N Solutions in Single Front ...

- $F_1 = \{sol_1, sol_2, sol_3, sol_4, sol_5\}$

Solution	Objectives
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 13 : Un-assigned Solutions

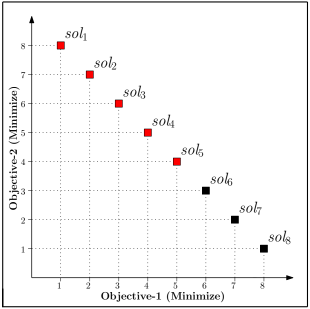

Figure 22 : Solutions

Working Example: N Solutions in Single Front ...

- $F_1 = \{sol_1, sol_2, sol_3, sol_4, sol_5, sol_6\}$

Solution	Objectives
sol_7	7,2
sol_8	8,1

Table 14 : Un-assigned Solutions

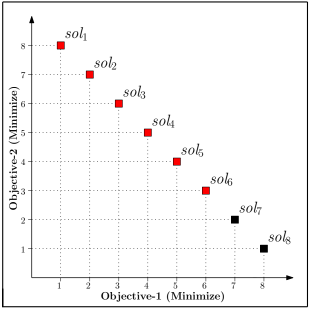

Figure 23 : Solutions

Working Example: N Solutions in Single Front . . .

- $F_1 = \{sol_1, sol_2, sol_3, sol_4, sol_5, sol_6, sol_7\}$

Solution	Objectives
sol_8	8,1

Table 15 : Un-assigned Solutions

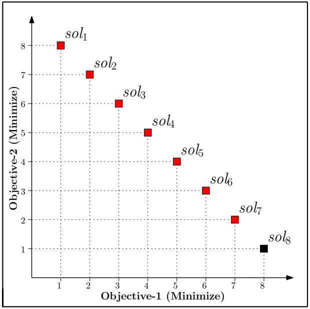

Figure 24 : Solutions

Working Example: N Solutions in Single Front ...

- $F_1 = \{sol_1, sol_2, sol_3, \ldots, sol_7, sol_8\}$

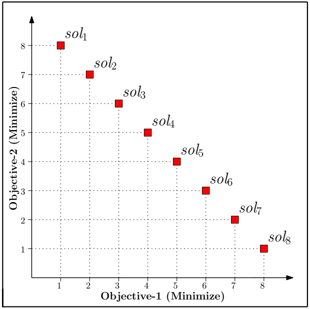

Figure 25 : Non-Dominated Fronts

Working Example: N Solutions in N Fronts

Solution	Objectives
sol_1	1,1
sol_2	2,2
sol_3	3,3
sol_4	4,4
sol_5	5,5
sol_6	6,6
sol_7	7,7
sol_8	8,8

(a)

Solution	Objectives
sol_1	1,1
sol_2	2,2
sol_3	3,3
sol_4	4,4
sol_5	5,5
sol_6	6,6
sol_7	7,7
sol_8	8,8

(b)

Table 16 : (a). A set of 8 solutions where two objectives are associate with each solution. (b). Solutions in sorted order based on objectives.

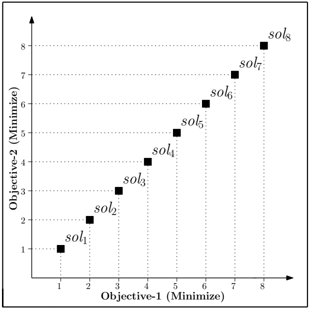

Figure 26 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$

Solution	Objectives
sol_2	2,7
sol_3	3,6
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 17 : Un-assigned Solutions

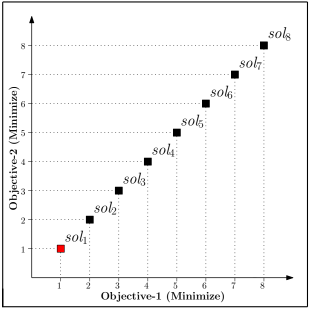

Figure 27 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$

Solution	Objectives
sol_3	3,6
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 18 : Un-assigned Solutions

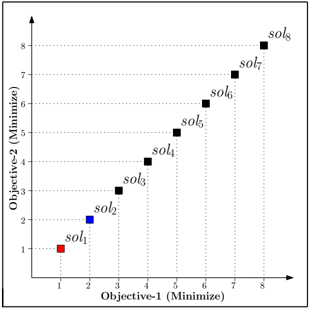

Figure 28 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_3\}$

Solution	Objectives
sol_4	4,5
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 19 : Un-assigned Solutions

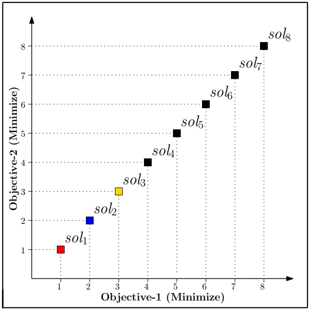

Figure 29 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_3\}$
- $F_4 = \{sol_4\}$

Solution	Objectives
sol_5	5,4
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 20 : Un-assigned Solutions

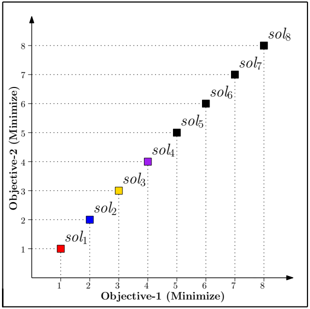

Figure 30 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_3\}$
- $F_4 = \{sol_4\}$
- $F_5 = \{sol_5\}$

Solution	Objectives
sol_6	6,3
sol_7	7,2
sol_8	8,1

Table 21 : Un-assigned Solutions

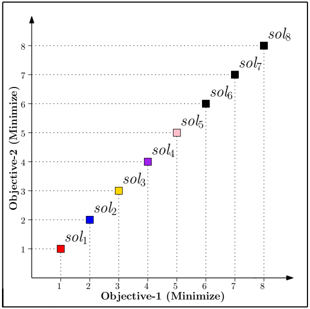

Figure 31 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_3\}$
- $F_4 = \{sol_4\}$
- $F_5 = \{sol_5\}$
- $F_6 = \{sol_6\}$

Solution	Objectives
sol_7	7,2
sol_8	8,1

Table 22 : Un-assigned Solutions

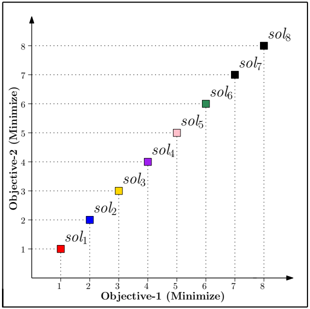

Figure 32 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_3\}$
- $F_4 = \{sol_4\}$
- $F_5 = \{sol_5\}$
- $F_6 = \{sol_6\}$
- $F_7 = \{sol_7\}$

Solution	Objectives
sol_8	8,1

Table 23 : Un-assigned Solutions

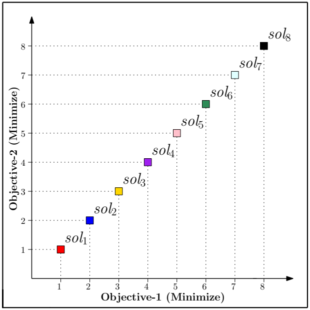

Figure 33 : Solutions

Working Example: N Solutions in N Fronts . . .

- $F_1 = \{sol_1\}$
- $F_2 = \{sol_2\}$
- $F_3 = \{sol_3\}$
- $F_4 = \{sol_4\}$
- $F_5 = \{sol_5\}$
- $F_6 = \{sol_6\}$
- $F_7 = \{sol_7\}$
- $F_8 = \{sol_8\}$

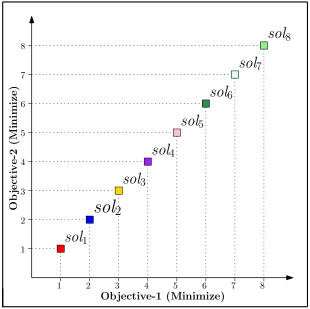

Figure 34 : Non-Dominated Fronts

Conclusions & Future Work

Conclusions

- Discussed non-dominated sorting problem and various approaches to solve it.

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Future Research Direction

- Parallel non-dominated sorting approaches.

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- Parallel non-dominated sorting approaches.
- Develop an efficient approach specially for small number of fronts.

Conclusions & Future Work

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- Discussed non-dominated sorting problem and various approaches to solve it.

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- Parallel non-dominated sorting approaches.
- Develop an efficient approach specially for small number of fronts.
- Efficient handling of duplicate solutions.

Conclusions & Future Work

Conclusions

- Discussed non-dominated sorting problem and various approaches to solve it.

Future Research Direction

- Parallel non-dominated sorting approaches.
- Develop an efficient approach specially for small number of fronts.
- Efficient handling of duplicate solutions.
- Efficiently find the dominance relationship between the solutions.

Conclusions & Future Work

Conclusions

- Discussed non-dominated sorting problem and various approaches to solve it.

Future Research Direction

- Parallel non-dominated sorting approaches.
- Develop an efficient approach specially for small number of fronts.
- Efficient handling of duplicate solutions.
- Efficiently find the dominance relationship between the solutions.
- Develop approximate non-dominated sorting approaches.

Conclusions & Future Work

Conclusions

- Discussed non-dominated sorting problem and various approaches to solve it.

Future Research Direction

- Parallel non-dominated sorting approaches.
- Develop an efficient approach specially for small number of fronts.
- Efficient handling of duplicate solutions.
- Efficiently find the dominance relationship between the solutions.
- Develop approximate non-dominated sorting approaches.
- Find the lower bound on the time complexity of non-dominated sorting.

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Thank you!