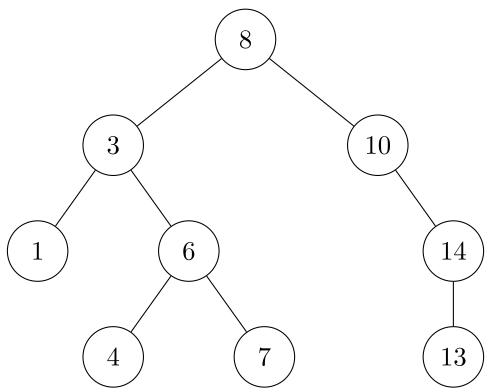
Binary Search Trees

A **Binary Search Tree (BST)** is a binary tree where each node contains a key such that:

$$\texttt{left subtree keys} < \texttt{node key} < \texttt{right subtree keys}$$

This ordering enables efficient searching, insertion, and deletion operations.

1 Example BST Structure



This BST contains keys ordered such that for every node:

$$\texttt{left subtree keys} < \texttt{node key} < \texttt{right subtree keys}$$

2 Use Cases

- Symbol tables in compilers
- Maintaining sorted datasets
- Searching and dynamic set operations

3 Time Complexity

Operation	**Average Case**	**Worst Case (Unbalanced)**
Search	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$
Insert	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$
Delete	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$

3.1 Terminology

- **BST:** A binary tree with ordered keys
- **Leaf:** A node with no children

- **Internal node:** Has at least one child
- **Height:** Number of edges on the longest downward path
- **Balanced Tree:** Height is approximately $\log n$

4 Why BST Operations Are $\mathcal{O}(\log n)$

In a **balanced** BST, each level of the tree halves the search space. With n nodes, the maximum height is approximately $\log_2 n$.

- At each comparison, we eliminate half the remaining nodes.
- This is analogous to binary search on a sorted array.

Hence, insertion, search, and deletion all take $\mathcal{O}(\log n)$ time on average for balanced trees.

4.1 C++ Implementation of a BST

```cpp
#include <iostream>
using namespace std;

struct BSTNode {
  int key;
  BSTNode* left;
  BSTNode* right;

  BSTNode(int val) : key(val), left(nullptr), right(nullptr) {}
};

class BST {
public:
  BSTNode* root;

  BST() : root(nullptr) {}

  BSTNode* insert(BSTNode* node, int key) {
    if (!node) return new BSTNode(key);
    if (key < node->key)
      node->left = insert(node->left, key);
    else if (key > node->key)
      node->right = insert(node->right, key);
    return node;
  }

  bool search(BSTNode* node, int key) const {
    if (!node) return false;
    if (key == node->key) return true;
    if (key < node->key)
      return search(node->left, key);
    else
      return search(node->right, key);
  }

  void insert(int key) {
    root = insert(root, key);
  }

  bool search(int key) const {
    return search(root, key);
  }
};
```

4.2 Tree Traversals

Traversal orders commonly used with BSTs:

Inorder Traversal (Left → Root → Right): Produces sorted order

```
void inorder(BSTNode* node) {
  if (!node) return;
  inorder(node->left);
  cout << node->key << " ";
  inorder(node->right);
}
```

Preorder Traversal (Root → Left → Right):

```
void preorder(BSTNode* node) {
  if (!node) return;
  cout << node->key << " ";
  preorder(node->left);
  preorder(node->right);
}
```

Postorder Traversal (Left → Right → Root):

```
void postorder(BSTNode* node) {
  if (!node) return;
  postorder(node->left);
  postorder(node->right);
  cout << node->key << " ";
}
```

5 Deletion in BSTs

To delete a node, we consider three cases:

- **Case 1: Leaf Node (0 children)** Simply remove the node.
- **Case 2: One Child** Replace the node with its only child.
- **Case 3: Two Children** Replace the node's value with either:
  - The **inorder successor** (smallest value in right subtree)
  - The **inorder predecessor** (largest value in left subtree)

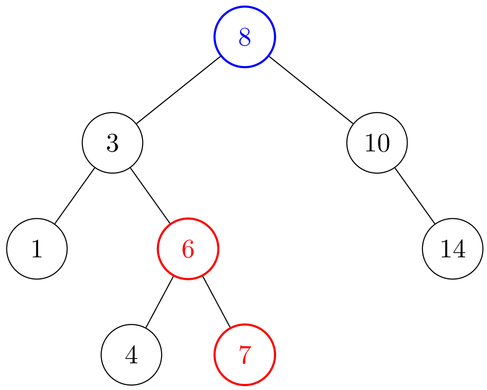
  Then delete the successor or predecessor node.

Finding the Inorder Successor

To find the inorder successor of a node in a BST, we consider two cases:

- If the node has a right subtree: successor is the leftmost node in the right subtree
- If the node has no right subtree: successor is the lowest ancestor for which the node lies in its left subtree

5.1 Visualizing Inorder Successor Cases



Red	Target Node
Blue	Inorder Successor

Case 1: Node **6** has a right child (7). Its inorder successor is the leftmost node in its right subtree: **7**.
Case 2: Node **7** has no right subtree. To find its inorder successor:

- We move up the tree from node 7 to its parent (node 6), but since 7 is a **right child**, we continue upward.
- We then move from 6 to its parent (node 3), where 6 is a **right child** again, so we continue.
- We then move from 3 to node 8, where 3 is a **left child**.
- Hence, node **8** is the lowest ancestor where node 7 (through its lineage) lies in the left subtree.

Therefore, the inorder successor of node **7** is **8**.

C++ Implementation:

Deletion Using `inorderSuccessor`

```
// Full-tree inorderSuccessor version used here
BSTNode* deleteNode(BSTNode* root, int key) {
  if (!root) return nullptr;

  if (key < root->key)
    root->left = deleteNode(root->left, key);
  else if (key > root->key)
    root->right = deleteNode(root->right, key);
  else {
    // Case 1 and 2: Node with 0 or 1 child
    if (!root->left) {
      BSTNode* temp = root->right;
      delete root;
      return temp;
    } else if (!root->right) {
      BSTNode* temp = root->left;
      delete root;
      return temp;
    }

    // Case 3: Node with 2 children
    BSTNode* succ = inorderSuccessor(root, root); // full-tree root and current node
    root->key = succ->key;
    root->right = deleteNode(root->right, succ->key);
  }
  return root;
}
```

Deletion Function in C++

```
BSTNode* deleteNode(BSTNode* root, int key) {
  if (!root) return nullptr;

  if (key < root->key)
    root->left = deleteNode(root->left, key);
  else if (key > root->key)
    root->right = deleteNode(root->right, key);
  else {
    // Case 1 and 2: Node with 0 or 1 child
    if (!root->left) {
      BSTNode* temp = root->right;
      delete root;
      return temp;
    }
    else if (!root->right) {
      BSTNode* temp = root->left;
      delete root;
      return temp;
    }

    // Case 3: Node with 2 children
    BSTNode* succ = findMin(root->right);
    root->key = succ->key;
    root->right = deleteNode(root->right, succ->key);
  }
  return root;
}
```

Note: The deletion algorithm maintains the BST invariant by rearranging nodes appropriately. If balance is not preserved, performance can degrade to $\mathcal{O}(n)$.

5.2 Degenerate Cases and Balancing

A poorly structured BST can degrade to a linked list, resulting in $\mathcal{O}(n)$ time operations. To prevent this:

- Use self-balancing trees like AVL or Red-Black Trees
- Randomize insertions or rebalance periodically

Balanced BSTs ensure height $\mathcal{O}(\log n)$, which guarantees logarithmic performance for search-related operations.

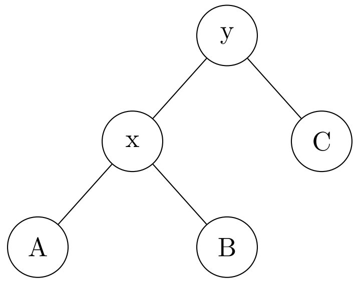
6 Tree Rotations

Tree rotations are fundamental operations used in self-balancing binary search trees (e.g., AVL and Red-Black Trees) to restore height balance after insertions or deletions. A rotation is a local restructuring operation that preserves the BST ordering.

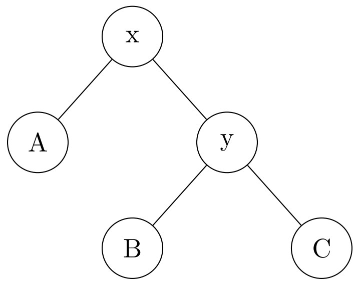
6.1 Right Rotation (`rotateRight`)

Right rotation is applied at a node y with a left child x to reduce the height of the left subtree.

Before:



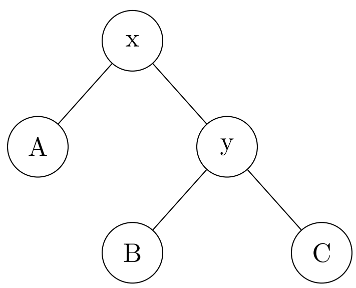
After `rotateRight(y)`:



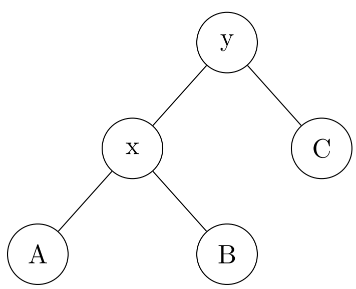
6.2 Left Rotation (`rotateLeft`)

Left rotation is applied at a node x with a right child y to reduce the height of the right subtree.

Before:



After `rotateLeft(x)`:



6.3 Preserving the BST Property During Rotations

Tree rotations maintain the binary search tree property:

$$\texttt{left subtree} < \texttt{node} < \texttt{right subtree}$$

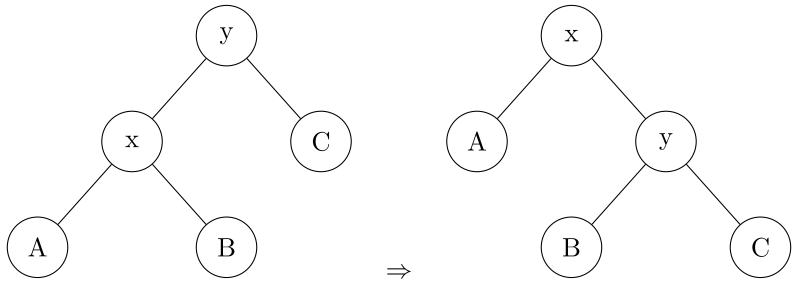
Consider the right rotation at node y (where x is its left child):

Before rotation:

- Subtree A: All keys < x.key

- Subtree B: x.key < keys < y.key
- Subtree C: All keys > y.key

Right Rotation:



After rotation:

- Subtree A remains in the left of x
- Subtree B becomes left of y, and since B's keys were > x.key and < y.key, they remain valid
- Subtree C remains to the right of y

All relative key orderings are preserved, satisfying:

$$\mathtt{A} < x < \mathtt{B} < y < \mathtt{C}$$

Both left and right rotations preserve in-order traversal and relative ordering of keys. Hence, the BST property remains intact after any single rotation.

7 Practice problem

Leetcode Problem 108: *Convert Sorted Array to Binary Search Tree*
`https://leetcode.com/problems/convert-sorted-array-to-binary-search-tree/description/`