Scale Inconsistency and the Boundary Problem

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Abstract

The method of pairwise comparisons, first described by Marquis de Condorcet in 1785, was identified in 1972 by Thomas Saaty's development of the Analytic Hierarchy Process. Since then, major research has been devoted to the critical analysis of the AHP, from various perspectives. In most cases, particular attention is paid to inconsistencies produced by the decision makers' preferences.

Yet, in the literature, there are no references concerning inconsistencies derived directly by the ratio scales. It is well known that in some cases, the restrictions of pairwise comparisons are bound to cause inconsistencies occurring solely by the scale itself, even if the decision maker is consistent. For example, in Saaty's 1 – 9 scale, if a is considered to be 5 times more important than b and b is considered to be 6 times more important than c, then c is correctly judged to be 30 times more important than a. However, this is not acceptable as it is outside the bounds of the scale, which gives that c is 9 times more important than a. This is known as the boundary problem.

The goal of this proposal is to research how priorities are derived from ratio scales and which properties must be satisfied for a consistent preference relation. Thereafter, the derived axiomatic assumptions remove inconsistencies coming from the decision maker, reviewing the current ratio-scaling approaches and associated problems. Secondary research shows how contradictions are exhibited as far as the sensitivity of the ratio scales are concerned and discuss ways to explain these. A "soft" operational research direction is followed with a positivism methodology in order to reach conclusions based on an ideal of objectivity and independence around the results.

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Introduction

Today Borda's and Condorcet's contribution (**See Appendix A**) to Decision Theory (DT) is widely accepted and acknowledged. Further development in DT can be attributed to Daniel Bernoulli with his Utility Theory (Bernoulli 1954). Perhaps the most critical contribution comes from the axiomatic work of Neumann and Morgenstern (Neumann, Morgenstern 1944) which was the basis for further developments in DT. Nevertheless, throughout the years, different traditions and schools emerged which brought some dichotomies in DT (Brunelli 2011).

The first of these dichotomies is between the stated preferences which are expressed by the decision maker and the revealed preferences which are based on the behavior of each decision maker (Gilboa, Maccheroni, Marinacci,Schmeidler 2010). This research is based on stated preferences assuming that each decision maker participating in a group can express their preferences on all pair of alternatives.

A second dichotomy in DT occurs between the normative approach which concerns how decisions are made and the descriptive approach (Missier, Mantyla, Bruine 2010) which is concerned with how decisions are actually made. The normative approach is been used, with the assumption of ideal conditions, whereas each decision maker has all the information they need to make rational (consistent) preferences.

A third dichotomy in DT appears in the need of generalization of those of decisions under uncertainty, called robust decisions in some cases, and dynamic decisions in another. Decisions under uncertainty have been studied extensively (Keeney, Raiffa 1976). For the purpose of the research proposal the assumption is that there exists no uncertainty.

Yet, the most notable dichotomy takes place in Decision Analysis (DA) which is the core of DT dealing with the mathematical methods of decision. In this case, two main streams have emerged, particularly since 1970 when research and development in multicriteria was first introduced (Roy 1991). These are the American stream involved with Multi Criteria Decision Making Systems (MCDMS) and the European stream involved with Multi Criteria Aiding Systems (MCDAS) (Salomon, Montevechi 2001). MCDAS use the Aggregation–Disaggregation Approach. In the disaggregation phase a preference model is constructed on the decision makers' judgment policy which is based on a limited set of reference actions that are well known to them. The information obtained in the aggregation phase thence constructs a value (utility) function, similarly as in MCDMS. Initial development of such system was proposed with POLICY DSS (Hammond 1978). Other methodological frameworks include UTA (Jacquet-Lagreze, Siskos 1982), PREFCALC (Jacquet-Lagreze 1984), MINORA (Siskos, Spiridakos, Yannacopoulos 1993), MIIDAS (Siskos, Spiridakos, Yannacopoulos 1999). These trends are discussed (Siskos, Grigoroudis 2010). MCDMS Systems that use the Outranking Relation Approach include the ELECTRE (Leyva-López, González 2003) and the PROMETHE (Brans,Vincke 1985, Macharis, Brans, Mareschal 1998, Macharis, Springael, De Brucker, Verbeke 2004). MCDMS based on a Value System Approach includes systems such as the AHP (Saaty 2001), MAUT (Keeney and Raiffa 1976) and MACBETH (Belton, Steward 2002). Zeleny's (1982) concepts of objective attribute and criterion is particularly interesting and broad. Referring to the simpler scheme proposed by Keeney and Raiffa and Saaty (1990), the AHP (**See Appendix B**) is utilized as it aims to construct a value function on a cardinal scale. The decision maker expresses not only their preference but also the degree of preference in order to derive rankings or priority (weight) vector using the Eigenvector Social Choice Function.

Research Methodology

Primary research and perspective can thus (References in Appendix A and B), define the direction of the research methodology. Interest of the proposal is revolved only in the inconsistencies produced directly as a result of the ratio scale and not with the decision makers' inconsistency. To do that, the ordinal scale is researched and the probabilistic properties of non transitivity is observed, which increase markedly as the number of decision elements are increased relative to the number of decision makers, actually making the decision (**See Appendix A**). Further, the conditions that need to be satisfied so as to achieve consensus in group decision is also observed. Therefore, in order to observe inconsistency coming solely from the ratio scale and to eliminate inconsistencies produced by the decision maker the following series of assumptions are necessary.

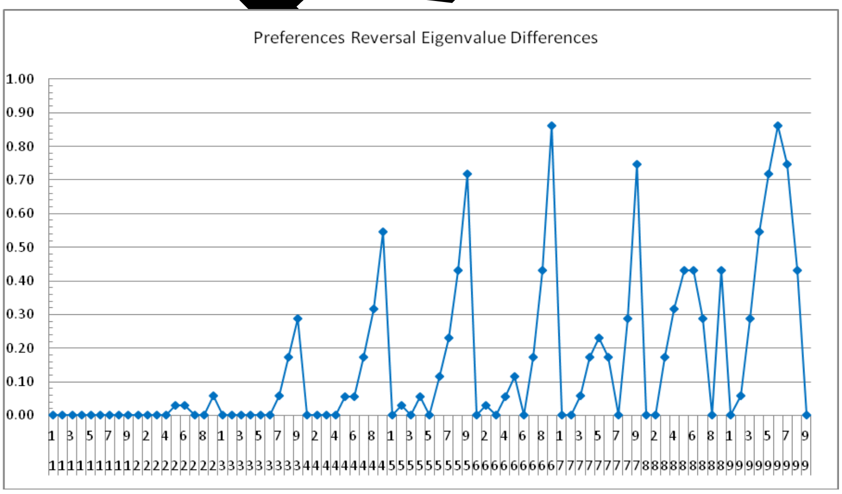
The first assumption is that the stated preferences are to be used, so that each decision maker participating in a group decision expresses their preferences on all pairwise comparisons of the decision elements.

Next, the normative approach is followed, which means that each decision maker makes consistent preferences. This also assumes that there is certainly no uncertainty involved. Finally, as the AHP methodology (**See Appendix B**) is considered, this means using a value systems approach based on a cardinal scale, in this case Saaty's fundamental scale, presumably a priority vector can only exist if and only if the preference relation is consistent.

Based on the above assumptions, preliminary observation below, shows how inconsistencies are produced on the 1-9 linear scale. Generally, decision makers can only be consistent as long as they exhibit weak preferences in their pairwise comparisons. When strong preferences are activated, then the phenomenon known as the boundary problem comes into effect. This means, that even if the decision makers wanted to provide consistent preferences, they cannot, as it exceeds the scale that is actually available to them. Thus, if decision makers can be consistent in the boundaries of the scale they are using, over these boundaries inconsistency comes only from the scale. For example, if decision makers prefer a 6 times more than b and b 5 times more than c then c should be preferred 30 times more than a, to be consistent. Because the scale can only allow that c is preferred 9 times more than a, then using the eigenvalue there is a 14.63 % inconsistency produced by the scale. Computational analysis of the 3 decision elements a, b and c, shows the following boundary relationship for all possible preference comparisons as depicted on Graph 1.

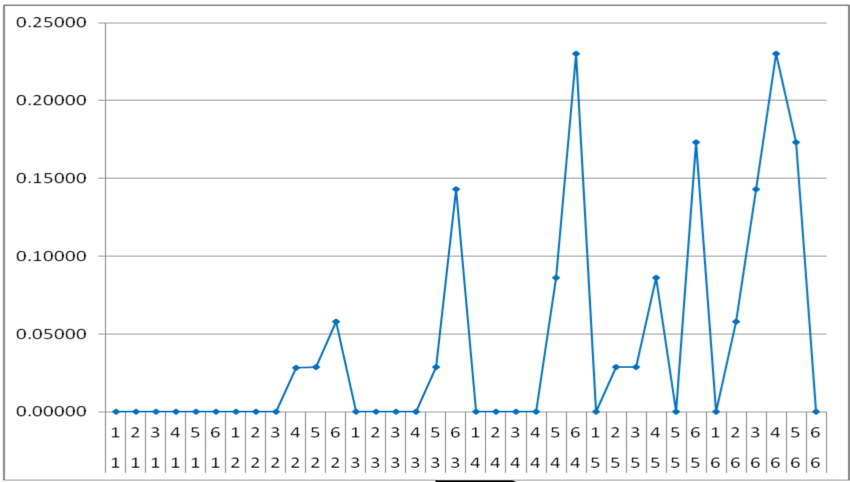
Graph 1: Inconsistencies % of c to a for pairwise comparisons of a to b and b to c.

It is quite interesting to see that if absolute preferences are used, that is, a is preferable 9 times more than b and b is preferable 9 time more than c then inconsistency is just 54.08 %. This could be an ordinal scale representation of a ***P*** b, b ***P*** to c then c ***P*** a. The question is; is this inconsistency inherited in the ordinal scale as well? Also, 48 out of the 81 (59.23%) comparisons have less than 10% inconsistency and that 23 out of the 81 (28.40%) are consistent.
Now, reversing original preference comparisons of a to b to b to c and b to c to a to b, there appears to be a similar relation. Normally, it is expected that their eigenvalues to be the same. Surprisingly, they are not. There exists noticeably a small discrepancy taking their absolute differences; these discrepancies are observed which are depicted on Graph 2.



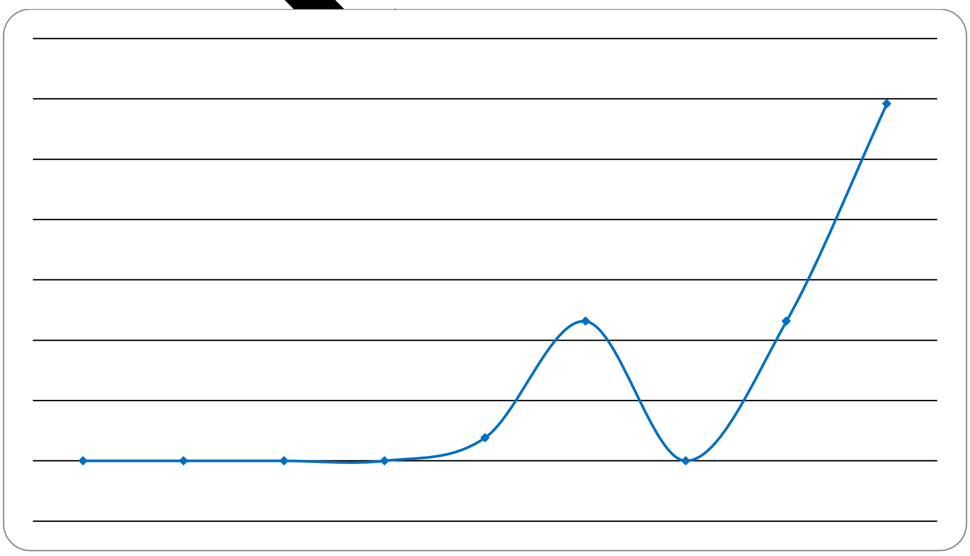
Graph 2: Eigenvalues Differences when we reverse of a to b to b to c and b to c to a to b

Observation shows, is that all consistent pairs have no absolute differences in the eigenvalues. For inconsistent pairs, the double pairs, that is, 4x4, 5x5, 6x6, 7x7, 8x8 and 9x9 indeed have no absolute diferrences too. Furthermore, the inconsistent pairs 2x7, 7x2, 2x8, 8x2, 3x4, 4x3, 3x5, 5x3, 3x6, 6x3 also have no absolute differences. The rest of the inconsistent pairs have absolute eigenvalue differences having the same values as pairs ranging from 0.029 – 0.862.
Could this mean that this arises due to the sensitivity on the intervals of the scale? To validate this, changing the interval scale from 1-9 to 1-6, the following discrepancies as are obtained as depicted on Graph 3.



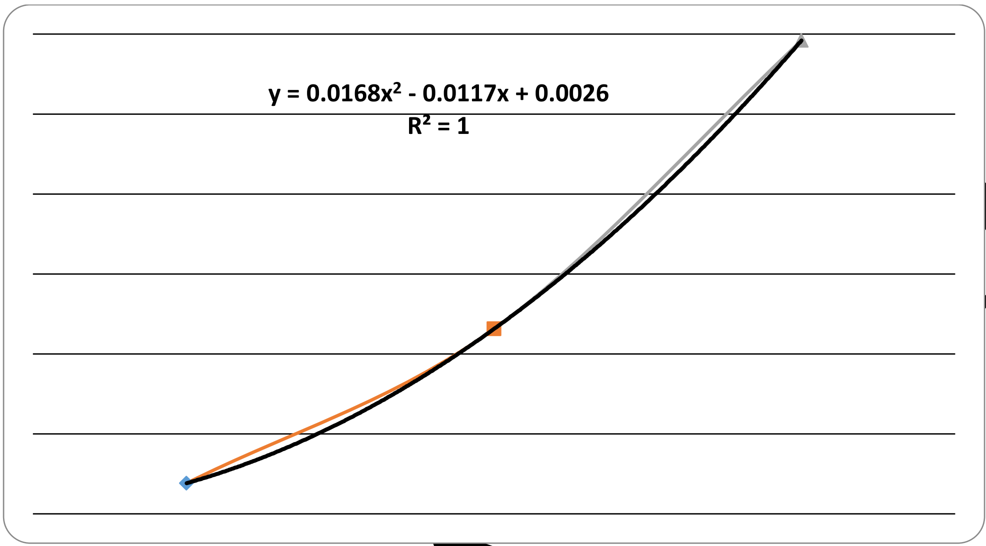
Graph 3: Eigenvalues Differences with a [illegible]

What is immediately apparent is that these differences drop markedly. Thus, a 1 -3 interval scale has no eigenvalue differences at all, which implies that this scale is now robust. Ploting the inconsistencies of the 1 – 3 scale, the following boundary relationship for all possible preference comparisons is depicted on Graph 4.



Graph 4: % Inconsistencies of c using a 1-3 scale.

As pairs of 2x3 and 3x2 gives the same inconsistency, with no eigenvalue differences, then a plot of a relationship can be seen on Graph 5. It appers to be a second order polynomial.



Graph 5: Inconsistencies relation for a 1 – 3 scale.

Conclusions

The research methodology based on some axiomatic assumptions allows to separate inconsistencies that occur only from the ratio scale and those that can occur from the decision makers. Using Saaty's linear scale, inconsistencies coming from the scale itself is studied and the boundary problem is observed. For a simple system of 3 decision elements it appears that when pairwise comparisons are reversed it is expected to obtain the same eigenvalues, nevertheless, this is not so. As far as it could be understood, it can be assumed that this discrepancy occurs due to the sensitivity (Triantaphyllou, Lootsma, Pardalos, Mann 1994) on the intervals of the scale. In order to validate the phenomenon, a different scale, namely a 1 – 6 scale can be used and observe that the eigenvalue differences are reduced markedly both in magnitude and frequency. When a 1 – 3 scale is used, the eigenvalue differences are eliminated, suggesting it is a robust scale compared either to the 1 – 6 or the 1 – 9 scale (Fullop, Koczkodaj, Szarek 2010). Returning to the inconsistencies produced with the 1 – 3 scale it can be seen that they are much less compared to Saaty's and that it exhibits a simple second order polynomial relationship. At first sight, it seems perhaps, that the scale choice might be directly related to the number of decision elements. For example, for 3 decision elements a scale of 1 – 3, for 4 decision elements a scale of 1 – 4 and so forth. Further research could be able to validate this. Nevertheless, if consistency is satisfied, then the choice of intervals for the ratio scale pays no role. In this respect Saaty's scale is as good as others (Dong, Xu, Li, 2007).

As far as the boundary problem is concerned, it seems that whatever scale may used, it will always be present. Perhaps the assymptotic scale (Donegan, Dodd, McMaster 1992) provides some solution, but pairwise comparison of strong preferences will still be difficult for decision makers, as any numbers at the extreme ends will be extremely large. What is needed presumably, is a new dynamic scale that accomodates strong preferences. Perhaps, a scale which relates each time that the third pairwise comparison will change according to the previous two comparisons. That could mean to derive a scale which will change depending on the decision makers first two pairwise comparisons. For example, if decision makers, using the 1 – 9 scale, prefer a 6 times more than b and b 5 times more than c then the new scale could be automatically created which would be a 1 – 30 linear scale. If on the other hand, decision makers prefers a 9 times more than b and b 5 times more than c then the new scale could be automatically created which would be a 1 – 45 linear scale, as shown in Figure 1; and so forth. In this way we overcome the boundary problem and can thus allow the decision makers to give consistent pairwise comparisons.

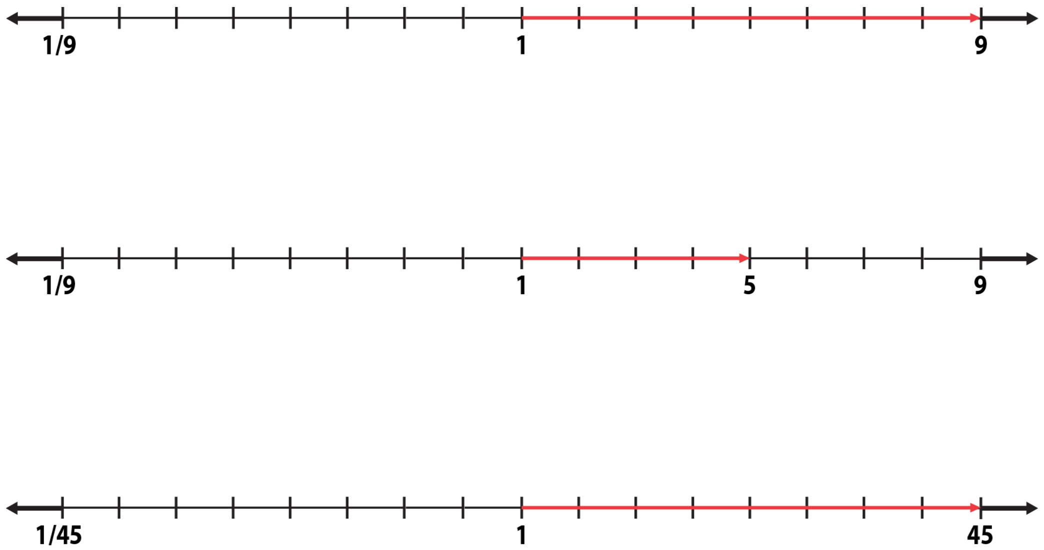


Figure 1: A Dynamic Pairwise Comparison Matrix

Although it may seem that the boundary problem is solved, it becomes immediately apparent that no matter what the new dynamic scale will be produced, its' intervals need to remain the same (Wang, Yang, Xu 2005), as in the previous two pairwise comparisons, so as to produce no difference in sensitivity and to make it easier for the decision makers to compare. Also, the scale needs to be extended, because if leaving the scale of the third pairwise comparison as it is, the decision maker will always be consistent, by just always choosing the maximum on the dynamic pairwise comparison on the scale. Finally, and most importantly, the dynamic scale need not be be

transparent to the user so as to not allow the mathematics to determine the decision makers' preferences.
As far as the intervals are concerned, no matter what the new magnitude of the scale will be, current interval will just be divisibles of 9. Extending the scale can be achieved by squaring the overall scale, that is, it becomes a 1 - 81 intervals scale. As for the transparency of the scale, the decision makers will still see a scale which is similar to the 1 – 9 interval scale, it is just the values that we will use will be different in order to obtain consistency. A simple example in Figure 2 demonstrates our principles.

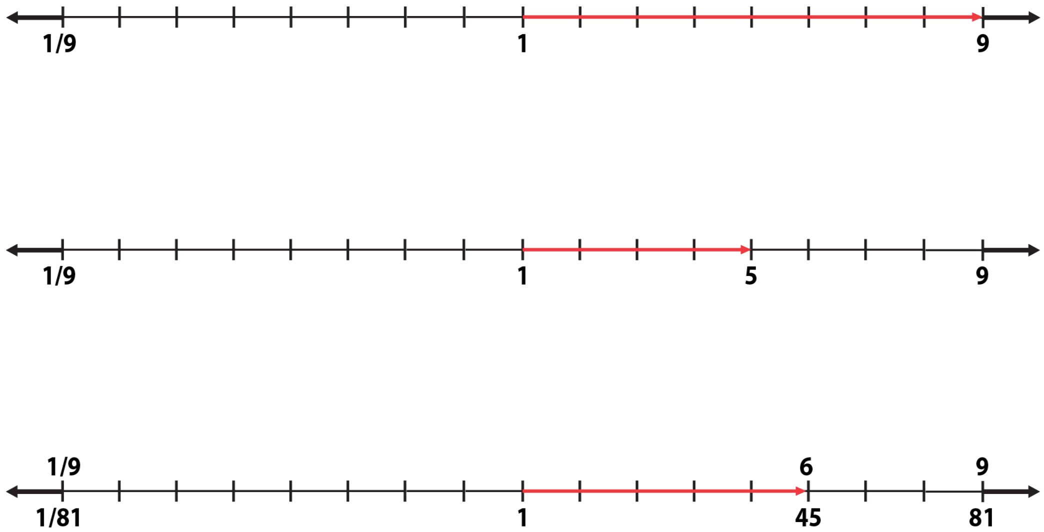


Figure 2: A Modified Dynamic Pairwise Comparison Matrix

Therefore, by solving the scale intervals, extension and transparency issues, a scale can be derived that overcomes the boundary problem, which can give consistent results and thus measure inconsistencies coming only from the decision makers. The example above shows that consistency from the scale is achieved when c is 45 times more preferable that a, or 6 times as compared to the original one. In other words, as long as there are weak preferences and the boundary phenomenon does not come into effect, our dynamic scale remains as is, a 1 – 9 scale. When strong preferences are required and the boundary phenomenon appears, then the dynamic scale could change to the 1 – 81 intervals scale, informing the decision makers of strong preferences and adjust their judgements accordingly.
It is worth noting that the boundary problem occurs only when strong preferences occur in the same direction, that is, when a **P** b and b **P** c or b **P** a and c **P** b. When this is reversed, that is, strong preferences of a **P** b and c **P** b or b **P** a and b **P** c, then the boundary effect is not observed there and Saaty's scale still gives consistent results. Perhaps further research on the additive scale (Guh, Po, Lou 2009) and comparisons (Choo 2008) with the multiplicative scale that is currently utilized may shed more light on their appropriatness.
It is deemed that further research on the inconsistencies and sensitivities derived directly from the ratio scale would be able to provide improvements on the value system approach in decision making, especially in group decision making.

Appendix A: The Condorcet Effect

The preferential voting method first introduced by Chevalier Jean-Charles de Borda (Borda 1784) in 1770 proposed that the social choice or the aggregated preorder to be obtained by summing the all the points assigned for each alternative with the most points, second alternative with one point less and so forth. In general, group decision making is understood to be a reduction of decision maker preferences among a set of alternatives to a single collective preference or group choice. In 1785 Marquis de Condorcet (Condorcet 1785) discovers the paradox of voting, the fact that social choice processes based on the principle of the majority rule can give rise to non-transitive (cyclical) rankings among the alternatives.

Assume n decision makers and 2 alternatives a and b, we symbolize the i_{th} decision maker prefers a to b by a $\boldsymbol{P_i}$ b, a is at least as good as b by a $\boldsymbol{R_i}$ b and a is indifferent to b by a $\boldsymbol{I_i}$ b. Arrows theorem on the method of majority states that a group decision function is the method of simple majority decision in which a $\boldsymbol{R}$ b holds if and only if (iff) the number of decision makers is such that a $\boldsymbol{R_i}$ b is at least as great as the number of decision maker such that b $\boldsymbol{R_i}$ a (Hwang, Lin 1987). Hence,

$a\,\boldsymbol{P}\,b \quad \#(i{:}\ a\,\boldsymbol{P_i}\,b) > \#(i{:}\ b\,\boldsymbol{P_i}\,a) \qquad i\ \varepsilon\ n$

$a\,\boldsymbol{R}\,b \quad \#(i{:}\ a\,\boldsymbol{R_i}\,b) > \#(i{:}\ b\,\boldsymbol{R_i}\,a) \qquad i\ \varepsilon\ n$

$a\,\boldsymbol{I}\,b \quad \#(i{:}\ a\,\boldsymbol{I_i}\,b) > \#(i{:}\ b\,\boldsymbol{I_i}\,a) \qquad i\ \varepsilon\ n$

which defines the necessary conditions for simple majority decisions such that $\boldsymbol{P}$, $\boldsymbol{R}$ and $\underline{\boldsymbol{I}}$ represent ordinal (binary) relation of strict simple majority, weak simple majority and tie.

Suppose that n decision makers which has a consistent set of preferences for 3 alternatives a, b and c, then 6 possible opinion categories are produced and the Condorcet effect comes into action,

$a\,\boldsymbol{P}\,b\,\boldsymbol{P}\,c$ . . .#(1)
$a\,\boldsymbol{P}\,c\,\boldsymbol{P}\,b$ . . .#(2)
$c\,\boldsymbol{P}\,a\,\boldsymbol{P}\,b$ . . .#(3)
$c\,\boldsymbol{P}\,b\,\boldsymbol{P}\,a$ . . .#(4)
$b\,\boldsymbol{P}\,c\,\boldsymbol{P}\,a$ . . .#(5)
$b\,\boldsymbol{P}\,a\,\boldsymbol{P}\,c$ . . .#(6)

Now, let n_i be the number of decision makers in each of the six categories, where i = 1,2,3,…,6. The collective preference is obtained using the Condorcet principle, comparing a to b, the six categories are regrouped into 2 classes,

$\#(i{:}\ a\,\boldsymbol{P_i}\,b) = \#(1)$ and $\#(2)$ and $\#(3) = n_1 + n_2 + n_3$
$\#(i{:}\ b\,\boldsymbol{P_i}\,a) = \#(4)$ and $\#(5)$ and $\#(6) = n_4 + n_5 + n_6$

Similarly for the other alternatives,

$(i{:}\ a\,\boldsymbol{P_i}\,c) = \#(1)$ and $\#(2)$ and $\#(6) = n_1 + n_2 + n_6$
$(i{:}\ c\,\boldsymbol{P_i}\,a) = \#(3)$ and $\#(4)$ and $\#(5) = n_3 + n_4 + n_5$
$(i{:}\ b\,\boldsymbol{P_i}\,c) = \#(1)$ and $\#(5)$ and $\#(6) = n_1 + n_5 + n_6$
$(i{:}\ c\,\boldsymbol{P_i}\,b) = \#(2)$ and $\#(3)$ and $\#(4) = n_2 + n_3 + n_4$

Inconsistency will occur if a ***P*** b, b ***P*** c and c ***P*** a or b ***P*** a, c ***P*** b and a ***P*** c. The first case occurs when the three inequalities happen simultaneously,

$n_1 + n_2 + n_3 > n_4 + n_5 + n_6$
$n_3 + n_4 + n_5 > n_1 + n_2 + n_6$
$n_1 + n_5 + n_6 > n_2 + n_3 + n_4$

Conditions of the second case can be written by reversing the inequalities above. Choosing,
$n_1 = n_3 = n_5 = 1$
$n_2 = n_4 = n_6 = 0$

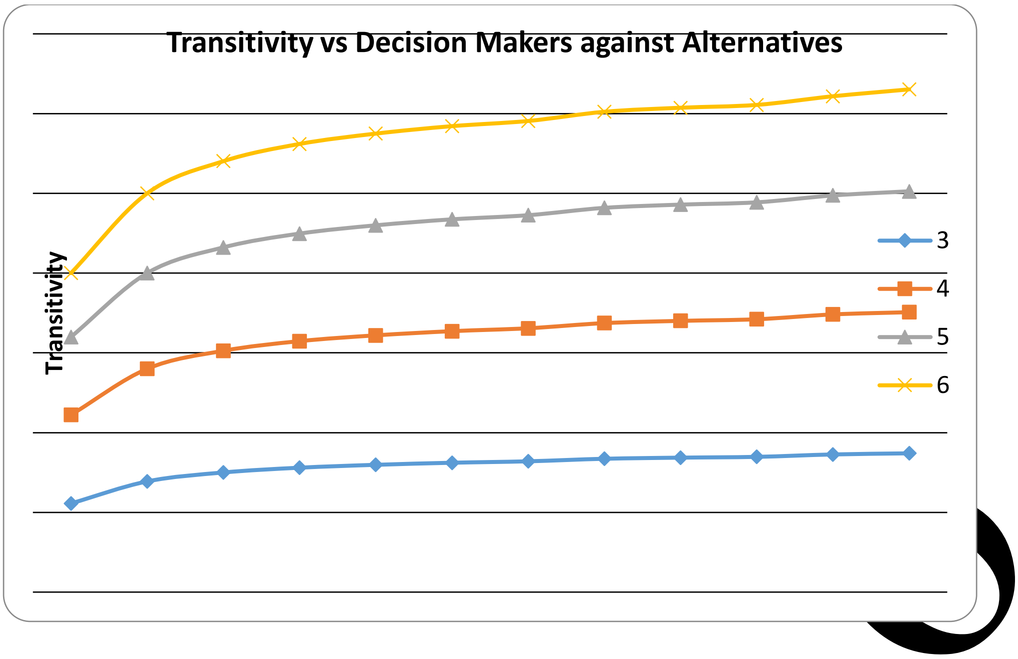
The 3 inequality conditions are satisfied. This is a collective preference of 3 decision makers who hold opinions (1), (3) and (5). The first of the 3 decision makers can hold any of the 6 opinions as shown below. Likewise for the second and third, so the result is 6 x 6 x 6 = 216 possibilities. Among these possibilities there are 12 which can give rise to the Condorcet effect, a little less than 6 %. These 12 possibilities are,

(1) (3) (5)
(1) (5) (3)
(3) (1) (5)
(3) (5) (1)
(5) (1) (3)
(5) (3) (1)

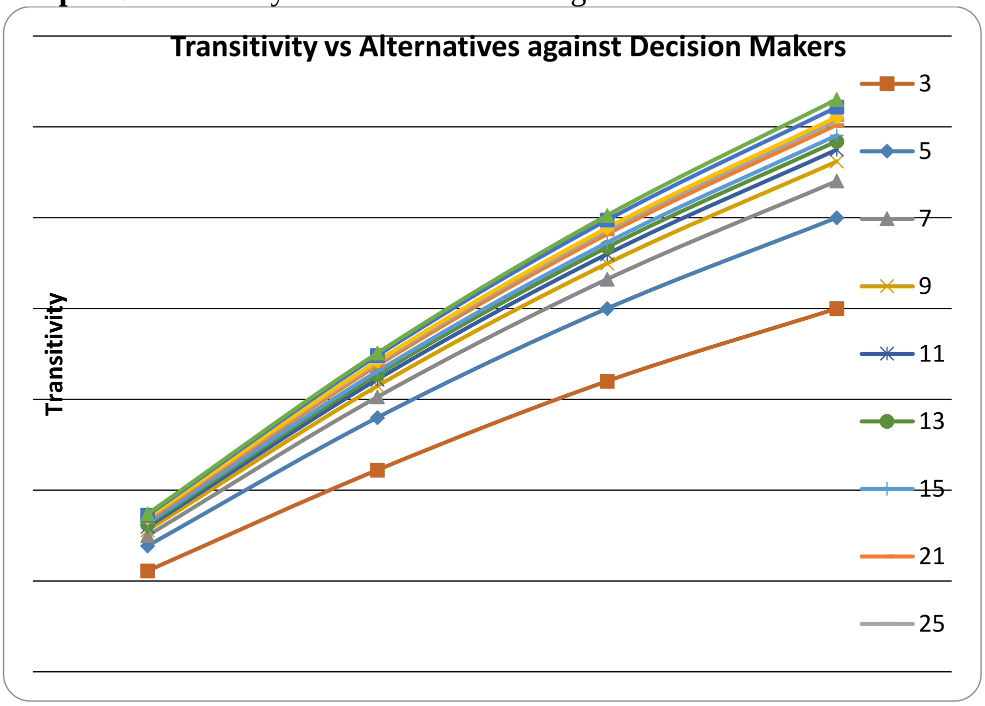
and 6 analogous arrangements for (2) (4) (6). Computational analysis below shows that the Condorcet effect represents only a small fraction of the possibilities.

	Decision Makers											
Alternatives	**3**	**5**	**7**	**[illegible]**	**11**	**13**	**15**	**21**	**25**	**29**	**59**	**∞**
3	0.0556	0.0694	0.0750	0.0780	0.0798	0.0811	0.8200	0.0836	0.0843	0.0848	0.0863	0.0870
4	0.1111	0.1400	[illegible]	[illegible]	0.1610	0.1636	0.1654	0.1686	0.1701	0.1711	0.1741	0.1755
5	0.1600	0.2000	[illegible]	0.2248	0.2300	0.2337	0.2363	0.2409	0.2429	0.2444	0.2487	0.2513
6	0.2000	[illegible]	0.2702	[illegible]	0.2875	0.2921	0.2954	0.3012	0.3037	0.3055	0.3109	0.3152

We observe that as the number of alternatives are increased the probabilities of non-transitive majorities increase towards 1 with little sensitivity to the number of decision makers.



Graph 6: Transitivity vs Decision Makers against Alternatives



Graph 7: Transitivity vs Alternatives Makers against Decision Makers

The paradox of voting has been formalized and studied by Arrow K. J. (1951) and procedures to handle this paradox has been one of the most popular preoccupations of Social Science Theory to define such functions to solve for the Condorcet effect.
A Social Choice Function can be considered as an aggregation procedure based on preferential relations. It is a mapping which assigns a non empty subset of the potential feasible subset to each ordered pair consisting of a potential feasible subset of alternatives and a profile of decision makers' preferences.

The Condorcet principle is defined as follows; for any non empty finite sets of A alternatives, n decision makers give each one's preference order of the alternatives, so a profile on A is any n-tuple oh linear orders of A. For any situation and alternatives a, b ε A, we let #(i: a $\boldsymbol{P_i}$ b) be the number of decision makers that have a preferred to b. Hence,

#(i: a $\boldsymbol{P_i}$ b) + #(i: b $\boldsymbol{P_i}$ a) = n when a ≠ b

then, the simple majority win relation on A is defined by:

a ***P*** b iff #(i: a $\boldsymbol{P_i}$ b) > #(i: b $\boldsymbol{P_i}$ a)

for all situations alternative a will be the winner whenever x ε A and a ***P*** b for all b ε A\{a}, that is, alternative a is excluded from A. The Condorcet function is given by:

$$f_c(a) = \min_{b \in A \setminus \{a\}} \#\,(i : aP_i b)$$

The properties of ordinary relation ***R*** over a set A is defined as:

1. ***R*** is **reflexive** over A iff for all a belonging to A, a ***R*** a, that is, a ***R*** a, $\forall a \in A$
2. ***R*** is **irreflexive** over A iff for all a belonging to A not a ***R*** a, that is, ≈(a ***R*** a), $\forall a \in A$
3. ***R*** is **connected** (complete) over A iff for all a and b (a≠b) belonging to A, a ***R*** b or b ***R*** a, that is, [a ***R*** b **U** b ***R*** a], $\forall a,b \in A$
4. ***R*** is **symmetric** over A iff for all a and b belonging to A, a ***R*** b implies b ***R*** a, that is, [a ***R*** b → b ***R*** a], $\forall a,b \in A$
5. ***R*** is **asymmetric** over A iff for all a and b belonging to A, a ***R*** b implies not b ***R*** a, that is, [a ***R*** b → ≈(b ***R*** a)], $\forall a,b \in A$
6. ***R*** is **antisymmetric** over A iff for all a and b belonging to A, a ***R*** b and b ***R*** a implies a is the same as b, that is, [a ***R*** b **and** b ***R*** a]→(a = b), $\forall a,b \in A$
7. ***R*** is **transitive** over A iff for all a, b and c belonging to A, a ***R*** b and b ***R*** a implies a ***R*** c, that is, [a ***R*** b and b ***R*** c]→a ***R*** c, $\forall a,b,c \in A$
8. ***R*** is **negatively transitive** over A iff for all a, b and c belonging to A not a ***R*** b and not b ***R*** a and not b R c implies not a ***R*** c, that is, [≈a ***R*** b and ≈b ***R*** c]→ ≈a ***R*** c, $\forall a,b,c \in A$

The Reflexive property means that the alternative is as good as itself, the relation is connected if whenever alternative a is not the winner then alternative b will be the winner. It is transitive if alternative a is as good as alternative b, which is itself as good as alternative c, implies that alternative a is at least as good as alternative b. Symmetry means that every alternative has an equal chance to be selected.
To define the properties of group decision rule, we assume that there are n decision makers for all a and b with each decision maker we associate a variable D, that takes the values -1, 0, 1 according whether the decision maker i prefers b to a, b P a, is

indifferent to a and b, a I b, or prefers a to b, a P b. For the set of all Decision maker preferences we write D = $\{-1, 0, 1\}^n$. In other words, an element D ε D where D = $\{D_1, D_2, D_3,\ldots,D_n\}$, then a Social Choice Function is defined to be a function F(D) = $f(D_1, D_2, D_3,\ldots,D_n)$ for all D ε D, that is, F: $\{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$. These properties are:

1. **Decisiveness:** F: D$\rightarrow$$\{-1, 0, 1\}$ id decisive iff D ≠ 0 $\rightarrow$ F(D) ≠ $\underline{0}$. It is weakly decisive if {D: F(D) = 0} ={$\underline{0}$} and strongly decisive if {D: F(D) = 0} = 0.
2. **Neutrality:** (Duality) $f(-D_1, -D_2, -D_3,\ldots,-D_n) = -f(D_1, D_2, D_3,\ldots,D_n)$
3. **Anonymity:** (Equality) $f(D_1, D_2, D_3,\ldots,D_n) = f(D_{\sigma(1)}, D_{\sigma(2)}, D_{\sigma(3)},\ldots,D_{\sigma(n)})$ if σ is a permutation on (1, 2, 3,…, n)
4. **Monotonicity:** (Positive Responsiveness) If D ≥ D' then F(D) ≥ F(D')
5. **Unanimity:** (Weak Pareto Criterion) f(1,…,1) = 1 or f(-1,…,-1) = -1
6. **Homogeneity:** F(mD) = F(D) for all positive integers of m
7. **Weak Pareto Optimality:** If D_i = 1 for all i, then F(D) = 1, or if D_i = -1 for all i, then F(D) = -1
8. **Strong Pareto Optimality:** If Di = {1, 0} for all i and D, k = 1 for some k, the F(D) = 1 and if D_i = 0 for all i, then F(D) = 0

Decisiveness means that a social choice function is such that each decision maker's preferences lead to a defined and unique solution. Neutrality prevents a build in favoritism for either alternative, since the social choice function will be reversed if the decision makers reverse their preferences, that is, each alternative is treated equally. Anonymity means that the system gives equal rights to the decision makers. Monotonicity assures that if an decision maker moves alternative a upwards in their ranking, leaving the relative standing of others unchanged, then alternative a will stand at least as well relative to the other alternatives as before. Unanimity prescribes that a wins when everyone prefers a to b and that b wins when everyone prefers b to a. Homogeneity prescribes that if an decision maker is indifferent between a and b they are replaced by two other decision makers with the same preference as the original except that one prefers a to b and the other b to a. The Pareto Optimality means that if every decision maker thinks that a is better than a, or at least as good as, then so does society.

Some other Social Choice Functions include:

1. Borda's
2. Copeland's
3. Nanson's
4. Dogson's
5. Kemeny's
6. Cook and Seiford's
7. Fishburn's
8. Eigenvector
9. Bernado's Assignment

Appendix B: The AHP Background

Saaty's AHP approach to decision making is a MCDMS dating back to 1972. In 1977 he paves the way for successive development (Saaty 1972, 1977). It copes with both the tangible (objective) and intangible (subjective) measurement of decision elements composed of criteria and alternatives arranged in a hierarchy. A defined hierarchy is called a Hierarchon (Saaty, Forman 2003). If tangible comparisons are needed a scoring scale can be used, while for intangible comparisons the fundamental (1-9) scale (Saaty 2001, 2006) seems more appropriate in order to derive priorities of these decision elements. Pairwise Comparisons (Saaty 2008), the essence of AHP is first used by Yokohama in (1921) and Thurstone in (1927). The hierarchic arrangement of a problem was first proposed by J. Miller in (1966). Finally the 1-9 scale was initially inspired by Fechner in (1860) and developed by Miller in (1956) and Stevens in (1957). Whether MCDMS or MCDAS methods are used, in effect all use a 4 step procedure (Ishizaka 2011) namely, Problem Modeling (Hierarchon Design), Weights Valuation (pairwise comparisons, scale choice), weights aggregation (priority eigenvector, consistency eigenvalue) and sensitivity analysis.

Problem Modeling

The major feature of AHP and of decision making itself is the structuring of the hierarchy into interrelated decision elements as shown in Figure 3. The Decision Analyst develops with clarity, a hierarchical representation – a Hierarchon (Saaty, Forman 2003), of an otherwise complicated problem, with the cooperation of the Decision Makers, in order to obtain mathematically the priorities of the alternatives using a cardinal scale of the agreed criteria approach. This means that the criteria and alternatives must be independent entities to represent the decision problem as thoroughly as possible; otherwise the clarity with which a problem is perceived can be reduced.

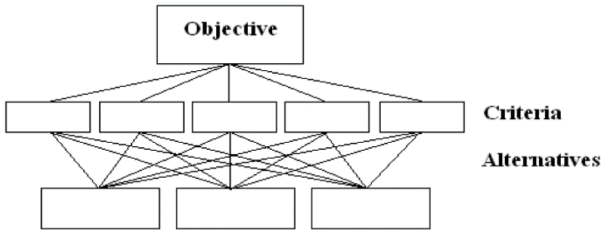


Figure 3: Three Level Hierarchon

However if criteria weights are altered, then possibly there would be observed Rank Reversal, which brings up the main criticism of AHP (Johnson 1979, Robins 1999, Finan, Hurley 2002). If criteria are deleted then it is called "wash criteria" and if added it is called "irrelevant criteria" (Saaty, Vargas, Whitetaker 2009), in either case, Saaty argues that adding or deleting irrelevant criteria cannot be used to make important decisions as it treats the weights of the criteria not as representative of their importance but as scaling constants like in MAUT (Keeney and Raiffa 1976). Similarly, it is noted that when alternatives are added or deleted which are compared on several criteria can lead to Rank Reversal (Belton, Gear 1983, Belton, Goodwin 1996, Belton, Stewart 2002). In this case, the Ideal Mode of AHP is used to avoid the aforementioned phenomenon (Saaty, Vargas 1993, Millet, Saaty 2000). Yet, some argue that the ideal mode which is used to avoid rank reversal is still not well understood (Robins 1999). On the other hand, Foreman argues that this is the strength of the method (Forman, Selly 2001).

Weight Valuation

This step requires the decision maker to make pairwise comparisons of all the decision elements. Saaty's fundamental (linear) scale of real numbers from 1 to 9 is used to systematically assign preferences in terms of cardinal scale judgments as down in (Table 2.)

Intensity of importance	Definition	Explanation
1	**Equal importance**	**Two elements contribute equally**
2	Weak	
3	**Moderate importance**	**Experience and judgment slightly favor one element over another**
4	Moderate plus	
5	**Strong importance**	**An element is strongly favored over another**
6	Strong plus	
7	**Very strong or demonstrated importance**	**An element is strongly dominant**
8	Very, very strong	
9	**Extreme importance**	**The evidence favored one element over another is of the higher order of affirmation**
Reciprocals of above	**If element *i* has one of the above nonzero numbers assigned to it when compared with activity j, the *j* has the reciprocal value when compared to *i***	**A reasonable assumption**
Rationales	**Ratio arising from the scale**	**If consistency were to be forced by obtaining n numerical values to span the matrix**

Table 2: The Fundamental Scale

Although pairwise comparisons implement an easy to use verbal mathematical terminology, there seems to be no agreement in the scientific community on how to aggregate paired ratios and convert them into weights (Chang, Lei, Jung, Lin R., Lin J., Yu, Chuang 2008, Hovanov, Kolari, Sokolov 2008, Bernasconi, Choirat, Seri 2010, Dijkstra 2010). In addition the numbers of comparisons that need to made makes the method extremely time consuming as such; no more than 9 criteria should be included which may also cause rank reversal. Furthermore, as Pairwise comparison is, in the end, are in the prerogative of the user and not the result of calculations, practitioners in the field of AHP warn that the need to be consistent should not be used to drive judgments, otherwise inappropriate weights will result (Zahedi 1986).

Main criticisms concerning the scale, involve verbal judgments themselves which naturally impose a quantitative scale (Elliot 2010). That is, if a is weakly preferred to b, implies that a is 3 times more important than 3. Sometimes the same scaling of the decision elements may be incompatible (Watson, Freeling 1982). Also, the concept of zero value which is not possible in the scale has drawn long debates (Barzilai 1997), as well as to proposing alternatives to the nine point scale to represent real world problems has stirred some contradiction among scholars (Ma, Zheng 1991, Lootsma 1993, Salo, Hamaleinen 1997), yet none seem to assert any improvements over the 1 -9 linear scale (Triantaphyllou, Lootsma, Pardalos, Mann 1994).

Several other scales have been proposed as seen in Table 3 (Saaty 1977, Harker, Vargas 1987, Lootsma 1989, Ma, Zheng 1991, Dodd, Donegan 1995, Salo, Hamaleinen 1997, Ishizaka 2010). Although each method claims to be in favor of Saaty's 1-9 scale, nonetheless the choice of the best possible scale still remains an issue of debate. Some argue that the choice depends on the person and the decision problem (Harker, Vargas, 1987, Pöyhönen 1997). Most notably, Donegan, Dodd and McMaster (1992) have proposed an asymptotic scale avoiding the boundary problem, e.g. if decision-makers enters *a* is 3 time more preferable than b and *b* is 4 times more preferable than c, they forced to an intransitive relation because the upper limit of the scale is 9 and they cannot enter *c* is 12 times more preferable than a. On this observation we pursue our research to solve the boundary problem.

	Scale Type	Scholar	Year	Definition	Parameters
1	***Linear***	Saaty	1977	$c = a \cdot x$	a>0; x = {1, 2, 3,...,9}
2	***Power***	Harker and Vargas	1987	$c = x^a$	a>1; x = {1, 2, 3,...,9}
3	***Root Square***	Harker and Vargas	1987	$c = \sqrt[a]{x}$	a>1; x = {1, 2, 3,...,9}
4	***Geometric***	Lootsma	1989	$c = a^{x-1}$	a>1; x = {1, 2, 3,...,9} or x = {1, 1.5,...,4} or other
5	***Inverse Linear***	Ma and Zheng	1991	$c = 9/(10-x)$	x = {1, 2, 3,...,9}
6	***Assymptotical***	Dodd and Donegan	1995	$c = \tanh^{-1}\left(\frac{\sqrt{3}(x-1)}{14}\right)$	x = {1, 2, 3,...,9}
7	***Balanced***	Salo and Hamaleinen	1997	$c = w/(1-w)$	w ={0.5, 0.55, 0.6,...,0.9}
8	***Logarithmic***	Ishizaka, Balkenborg and Kaplan	2010	$c = \log_a(x+(a-1))$	a>1; x = {1, 2, 3,...,9}

Table 3: Other proposed scales for pairwise comparisons.

Weights Aggregation

Using the Pairwise comparisons of the previous step, an eigenvector is used to determine the relative priorities of the decision elements (Saaty 1990, Saaty, Millet 1998, Saaty 200[illegible]). In addition, the eigenvalue is used to calculate consistency. Assuming we are given n alternatives, $A_1, A_2, A_3, \ldots, A_n$, with weights $w_1, w_2, w_3, \ldots, w_n$ respectively, the matrix will contain the ratio of the weights for each alternative with respect to all others:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \ldots & n \end{array} \\ \begin{bmatrix} w_1/w_1 & w_1/w_2 & w_1/w_3 & \ldots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & w_2/w_3 & \ldots & w_2/w_n \\ w_3/w_1 & w_3/w_2 & w_3/w_3 & \ldots & w_3/w_n \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ w_n/w_1 & w_n/w_2 & w_n/w_3 & \ldots & w_n/w_n \end{bmatrix} \end{array}$$

This leads to the eigenvalue equation:

$$\begin{bmatrix} w_1/w_1 & w_1/w_2 & w_1/w_3 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & w_2/w_3 & \cdots & w_2/w_n \\ w_3/w_1 & w_3/w_2 & w_3/w_3 & \cdots & w_3/w_n \\ . & . & . & & \\ . & . & . & & \\ . & . & . & & \\ w_n/w_1 & w_n/w_2 & w_n/w_3 & \cdots & w_n/w_n \end{bmatrix} * \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ . \\ . \\ . \\ w_n \end{bmatrix} = n * \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ . \\ . \\ . \\ w_n \end{bmatrix}$$

A•W = n•W

The solution of the above is called the principal eigenvector of A, which is the normalization of any column of A, where:

$$\sum_{i=1}^{n} w_i = 1$$

Where,

A: vector comparison matrix

W: $(w_1, w_2, w_3, \ldots, w_n)^T$. The right eigenvector of matrix A

n: scalar eigenvalue (principal maximum of matrix A)

The matrix **A = aij, aij = wi / wj = i,j = 1, 2, 3,…,n** satisfies the Reciprocal property if a is twice as much preferable than b, b is twice more preferable than c, then a is four times more preferable than c (Hamaleinen, Salo 1997a, Hamaleinen, Poyhonen, Salo 1997b, Hamaleinen, Laininen 1999).

$$\mathbf{a_{j,i} = 1 / a_{i,j}}$$

is Consistent provided the following condition is met:

$$a_{j,k} = a_{i,k} / a_{i,j},\ i, j, k = 1, 2, 3, \ldots, n$$

and Transitive provided the following condition is met:

$$\mathbf{a_{i,j} = a_{i,j+1} \bullet a_{i+1,j+2} \bullet, \ldots, a_{j-1,j}}$$

Figure 4: Phases of comparisons

In decision making if wi / wj are not precise values, then matrix A contains Inconsistencies. This leads to the eigenvalue in the form:

$$\bar{A} * \hat{W} = \lambda_{max} * \hat{W}$$

Where:

$\bar{A}$ Vector comparison matrix

$\hat{W}$ Right eigenvector of vector comparison matrix

λ_{max} Largest eigenvalue of vector comparison matrix

Inconsistency is given by $\boldsymbol{\lambda_{max} - n}$. This leads to the Consistency Index (CI):

CI = (λmax - n) / (n - 1)

And the Consistency Ratio (CR) as:

CR = (CI / RI)*100

Where, Random Index (RI) is the average consistency index of 100 randomly generated (inconsistent) Pairwise comparisons matrices. These values have been tabulated for different values of n, as shown on Table 4.

n	3	4	5	6	7	8	9
RI(n)	0.58	0.90	1.12	1.24	1.32	1.41	1.45

Table 4: Alternatives vs. Consistency Index

The main criticisms concerning the Eigenvalue/Eigenvector method is that despite being descriptive, the AHP is based on a normative interpretation of the human experience and as such the mathematics itself are made to reflect human thinking process (Holder 1990, Dyer 1990). Further, the eigenvector method for obtaining weight is usually not transparent to most decision makers.

As far as consistency is concerned, main criticism of the method stems from the fact that psychological evidence indicates that humans cannot be consistent beyond 5-9 comparisons (Zahedi 1986). AHP can only be used for 4 or fewer comparisons if confidence is needed in the results. Saaty (1977) used the perturbation theory to justify the use of the principal eigenvector as the desired priorities vector. However, Johnson (1979) showed a rank reversal problem or scale inversion with the eigenvalue method. This right and left arises only for inconsistent matrices with a dimension higher than three (Saaty, Vargas 1984a, 1984b). In order to avoid this problem, Crawford and Williams (1985) have adopted another approach in minimizing the multiplicative error. Other methods have been proposed, each one based either on the idea of the distance minimization (like the geometric mean) or the arithmetic mean of rows. Cho and Wedley (2004) have enumerated 18 different methods. However, deriving weights from interval comparison matrices, especially from inconsistency interval comparison matrices, still remains a research topic that needs to be further studied (Wang, Yang, Xu 2005). Various methods have been proposed to improve consistency (Ishizaka, Lusti 2004). Compared to optimization or perturbation methods of Saaty other new methods have appeared to improve consistency through linearization (Cavalo 2010, Benitez, Ga[illegible] 2011). Indeed, modification of the pairwise comparison matrix can also improve consistency (Bozoki, Fullop, Koczkodaj 2011, Bozoki, Fullop, Poesz 2011). Alternative non multiplicative scales and consistency measures in the AHP have been studied in depth using the AHP Manager Software package (Kusherbaeva, Sushkov, Tarmazyan 2011). It is of interest, that Sarfaraz and Maleki (2011) observe that the AHP and REMBRANDT MCDMS possess intransitivity whereas it should not and that further research is required. Non numerical rankings (Zhai 2010) and consistency of these (Janicki, Zhai 2011) have also stirred particular attention in a way of thinking of the relationship of ordinal scale compared to the various cardinal ones.

Koczkodaj (1993) defined a new consistency and compared with Saaty's (Bozooki 2007) refers to the complexity of determining the consistency index. The consistency index has been criticized by (Costa, Vansnick 2008) as it allows erroneous judgments in the pairwise comparison matrices.

Sensitivity Analysis

Sensitivity analysis may be applied on the comparisons, weights and priorities. Local priorities sensitivity has also been studied (Chen, Kocaoglu, 2008). Triantaphyllou and Sánchez (1997) have defined a sensitivity coefficient of the weights and local priorities. It is calculated with the minimum change of the current weight/local priority such as the ranking of two alternatives is reversed.

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