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# Matrix 5.5.32

ai25btech11015 – M Sai Rithik

## Problem

If

$$A = \begin{pmatrix} 1 & \cot x \\ -\cot x & 1 \end{pmatrix}, \tag{1}$$

show that

$$A^\top A^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}. \tag{2}$$

## Solution

First compute the determinant of $A$.

$$\det A = 1 \cdot 1 - (\cot x)(-\cot x) = 1 + \cot^2 x = \csc^2 x. \tag{3}$$

Hence the inverse of $A$ is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1 & -\cot x \\ \cot x & 1 \end{pmatrix} = \sin^2 x \begin{pmatrix} 1 & -\cot x \\ \cot x & 1 \end{pmatrix}. \tag{4}$$

The transpose of $A$ is

$$A^\top = \begin{pmatrix} 1 & -\cot x \\ \cot x & 1 \end{pmatrix}. \tag{5}$$

Form the product $A^\top A^{-1}$. Using $t = \cot x$ for brevity,

$$A^\top A^{-1} = \sin^2 x \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}. \tag{6}$$

Compute the inside product:

$$\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 - t^2 & -2t \\ 2t & 1 - t^2 \end{pmatrix}. \tag{7}$$

Now substitute $t = \cot x = \dfrac{\cos x}{\sin x}$ and multiply by $\sin^2 x$:

$$\sin^2 x\,(1 - t^2) = \sin^2 x \left(1 - \frac{\cos^2 x}{\sin^2 x}\right) = \sin^2 x \left(\frac{\sin^2 x - \cos^2 x}{\sin^2 x}\right) = \sin^2 x - \cos^2 x = -\cos 2x, \tag{8}$$

$$\sin^2 x\,(-2t) = -2 \sin^2 x \cdot \frac{\cos x}{\sin x} = -2 \sin x \cos x = -\sin 2x, \tag{9}$$

$$\sin^2 x\,(2t) = 2 \sin^2 x \cdot \frac{\cos x}{\sin x} = 2 \sin x \cos x = \sin 2x. \tag{10}$$

Therefore

$$A^\top A^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}, \tag{11}$$

as required.