frame=single, breaklines=true, columns=fullflexible

Matrix 5.5.32

ai25btech11015 – M Sai Rithik

Problem

If

$$A = \begin{pmatrix} 1 & \cot x \\ -\cot x & 1 \end{pmatrix},$$

show that

$$A^{\top} A^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}.$$

Step 1: Determinant and Inverse

$$\det A = 1 + \cot^2 x = \csc^2 x.$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 1 & -\cot x \\ \cot x & 1 \end{pmatrix} = \sin^2 x \begin{pmatrix} 1 & -\cot x \\ \cot x & 1 \end{pmatrix}.$$

Step 2: Transpose and Product

Transpose:

$$A^{\top} = \begin{pmatrix} 1 & -\cot x \\ \cot x & 1 \end{pmatrix}.$$

Form the product:

$$A^{\top} A^{-1} = \sin^2 x \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}, \quad t = \cot x.$$

Step 3: Simplification

Multiply matrices:

$$\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 - t^2 & -2t \\ 2t & 1 - t^2 \end{pmatrix}.$$

Now substitute $t = \frac{\cos x}{\sin x}$ and simplify:

$\sin^2 x(1-t^2) = -\cos 2x, \quad \sin^2 x(-2t) = -\sin 2x, \quad \sin^2 x(2t) = \sin 2x.$

Final Result

$$A^\top A^{-1} = \begin{pmatrix} -\cos 2x & -\sin 2x \\ \sin 2x & -\cos 2x \end{pmatrix}.$$