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4.12.20

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Question

One vertex of the equilateral triangle with centroid at the origin and one side as

$$x + y - 2 = 0$$

is

(a) $(-1, -1)$ (b) $(2, 2)$ (c) $(-2, -2)$ (d) $(2, -2)$.

Solution

Step 1: Condition for centroid

Let the vertices of the equilateral triangle be

$$\mathbf{A},\ \mathbf{B},\ \mathbf{C}.$$

The centroid is the origin, hence

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}. \tag{1}$$

Step 2: Side equation

Suppose side BC lies on the given line

$$x + y - 2 = 0. \tag{2}$$

Thus,

$$\mathbf{B}, \mathbf{C} \in \{\mathbf{x} \mid \begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 2\}.$$

Step 3: Equation of the altitude

Since A is the vertex opposite to side BC, the altitude from A passes through the centroid (origin) and is perpendicular to line (**??**). Normal vector to (**??**) is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, direction vector of the altitude is

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, vertex A must lie on the line through origin with direction $\mathbf{m}$:

$$\mathbf{A} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad k \in \mathbb{R}. \tag{3}$$

Step 4: Using centroid condition

From (**??**), we have

$$\mathbf{B} + \mathbf{C} = -\mathbf{A}.$$

But since $\mathbf{B}, \mathbf{C}$ lie on line (**??**), their midpoint M also lies on (**??**). Further,

$$M = \frac{\mathbf{B} + \mathbf{C}}{2} = -\frac{\mathbf{A}}{2}.$$

Thus, substituting (**??**),

$$M = -\frac{k}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Step 5: Condition on midpoint

Since M lies on (**??**):

$$\begin{pmatrix} 1 & 1 \end{pmatrix} M = 2.$$

So,

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\left(-\frac{k}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 2.$$

Simplify:

$$-\frac{k}{2}(1-1) = 2.$$

$$0 = 2,$$

which is impossible.

Step 6: Alternative choice of side

Instead, assume AB lies on line (**??**). Then the vertex C lies on the perpendicular from origin to line (**??**). Normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so perpendicular line has direction $\mathbf{m} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Equation of perpendicular from origin:

$$\mathbf{x} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Substitute candidate points:

$$(-1,-1),\ (2,2),\ (-2,-2),\ (2,-2).$$

Only $(2,-2)$ lies on this line.

Final Answer

The required vertex is

$$\boxed{(2,-2)}$$

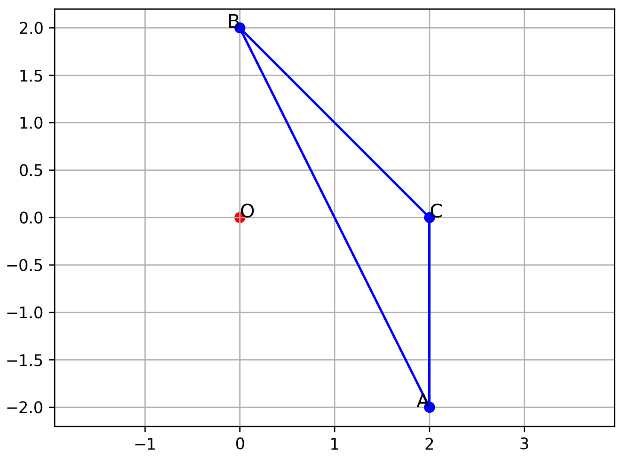

Figure 1: Equilateral triangle with centroid at origin and one side on $x + y - 2 = 0$.