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Matrix 4.8.11

ai25btech11015 – M Sai Rithik

Question

Find the vector equation of the plane determined by the points

$$A(3,-1,2), \quad B(5,2,4), \quad C(-1,-1,6),$$

and hence find the distance of this plane from the origin.

Solution

Step 1: Position vectors and edge vectors

$$\mathbf{A} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}. \tag{1}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \tag{2}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix}. \tag{3}$$

Step 2: Normal vector via cross product

Using the determinant/submatrix definition of the cross product,

$$(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} \begin{vmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 4 \end{pmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} & \begin{pmatrix} -4 \\ 4 \end{pmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} & \begin{pmatrix} -4 \\ 0 \end{pmatrix} \end{vmatrix} \end{pmatrix}. \tag{4}$$

Now evaluate the 2×2 determinants:

$$(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) = \begin{pmatrix} (3)(4) - (2)(0) \\ (2)(4) - (2)(-4) \\ (2)(0) - (3)(-4) \end{pmatrix}. \tag{5}$$

$$(\mathbf{B}-\mathbf{A})\times(\mathbf{C}-\mathbf{A}) = \begin{pmatrix}12\\16\\12\end{pmatrix}. \tag{6}$$

We may simplify (factor out 4):

$$\mathbf{N}_0 \;=\; \begin{pmatrix}3\\-4\\3\end{pmatrix}\times(-1) \quad \text{or} \quad \mathbf{N} = \begin{pmatrix}3\\-4\\3\end{pmatrix}, \tag{7}$$

(we choose $\mathbf{N} = (3,-4,3)^\top$, sign is arbitrary: any nonzero scalar multiple is a valid normal).

Step 3: Finding plane Equation

Compute the RHS constant d for the plane $\mathbf{N}^\top x = d$ using point A:

$$d \;=\; \mathbf{N}^\top\mathbf{A} = \begin{pmatrix}3 & -4 & 3\end{pmatrix}\begin{pmatrix}3\\-1\\2\end{pmatrix} = 3\cdot 3+(-4)\cdot(-1)+3\cdot 2. \tag{8}$$

$$d = 9+4+6 = 19. \tag{9}$$

Thus the plane equation in standard form is

$$\boxed{\mathbf{N}^\top x = 19 \quad \text{with} \quad \mathbf{N} = \begin{pmatrix}3\\-4\\3\end{pmatrix}}. \tag{10}$$

Scale the normal so that the right-hand side becomes 1. Define

$$\widetilde{\mathbf{N}} \;=\; \frac{\mathbf{N}}{19} \;=\; \begin{pmatrix}\frac{3}{19}\\-\frac{4}{19}\\\frac{3}{19}\end{pmatrix}. \tag{11}$$

Then the plane can be written as

$$\boxed{\widetilde{\mathbf{N}}^\top x = 1}, \tag{12}$$

since $\widetilde{\mathbf{N}}^\top x = (\mathbf{N}^\top x)/19$ and $\mathbf{N}^\top\mathbf{A} = 19$.

Step 5: Distance of the plane from the origin

For a plane written as $\widetilde{\mathbf{N}}^\top x = 1$, the perpendicular distance D from the origin to the plane equals

$$D \;=\; \frac{|1|}{\|\widetilde{\mathbf{N}}\|}. \tag{13}$$

Compute $\|\mathbf{N}\|$ first:

$$\|\mathbf{N}\| = \sqrt{3^2 + (-4)^2 + 3^2} = \sqrt{9 + 16 + 9} = \sqrt{34}. \tag{14}$$

Since $\widetilde{\mathbf{N}} = \mathbf{N}/19$, we have $\|\widetilde{\mathbf{N}}\| = \|\mathbf{N}\|/19$. Hence

$$D = \frac{1}{\|\widetilde{\mathbf{N}}\|} = \frac{19}{\|\mathbf{N}\|}. \tag{15}$$

$$\boxed{D = \frac{19}{\sqrt{34}} = \frac{19\sqrt{34}}{34} \approx 3.260}. \tag{16}$$

Final Answer

$$\boxed{\widetilde{\mathbf{N}}^\top x = 1, \quad \widetilde{\mathbf{N}} = \begin{pmatrix} \frac{3}{19} \\ -\frac{4}{19} \\ \frac{3}{19} \end{pmatrix},} \tag{17}$$

$$\boxed{\text{Distance from origin to plane} = \frac{19}{\sqrt{34}} \approx 3.260.} \tag{18}$$

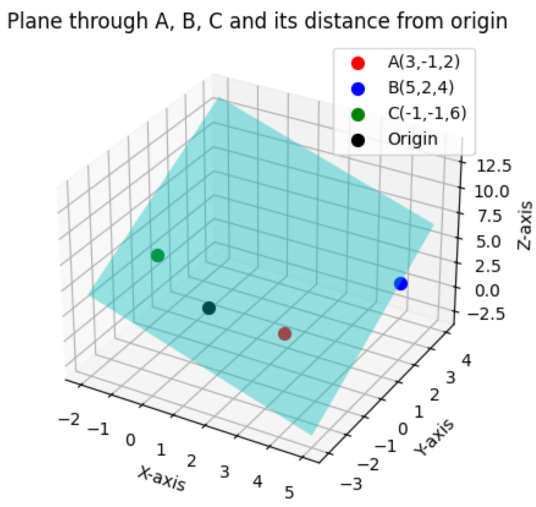

Figure 1: