AI25BTECH11034 - SUJAL CHAUHAN

4.8.30

Question:

Find the equation of a line passing through the point (2,3,2) and parallel to the line $\mathbf{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance beteween these two lines.

Theory:

Consider two parallel lines in 3D:

$$\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}, \quad \lambda \in \mathbb{R}, \tag{1}$$

$$\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}, \quad \mu \in \mathbb{R}, \tag{2}$$

where $\mathbf{a}_1, \mathbf{a}_2$ are points on the respective lines and $\mathbf{b}$ is the common direction vector.

Shortest distance between them can be given by :

$$d^2 = (\mathbf{a_2} - \mathbf{a_1})^\top (\mathbf{a_2} - \mathbf{a_1}) - \left(\frac{(\mathbf{a_2} - \mathbf{a_1})^\top \mathbf{b}}{\|\mathbf{b}\|} \right)^2 \tag{3}$$

Now we can calculate d.

Solution:

The direction vector of the given parallel lines is

$$\mathbf{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}. \tag{4}$$

The first line is given by

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}, \quad \mu \in \mathbb{R}. \tag{5}$$

The second line is

$$\mathbf{r}_2 = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}, \quad \lambda \in \mathbb{R}. \tag{6}$$

Now, we compute the difference between the position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{a}_2 - \mathbf{a}_1 \quad = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix}. \tag{7}$$

Next, we calculate the required components for the distance formula. The squared magnitude of $(\mathbf{a}_2 - \mathbf{a}_1)$ is:

$$\begin{aligned} (\mathbf{a}_2 - \mathbf{a}_1)^\top (\mathbf{a}_2 - \mathbf{a}_1) &= \|\mathbf{a}_2 - \mathbf{a}_1\|^2 && (8) \\ &= (-4)^2 + (0)^2 + (-2)^2 && (9) \\ &= 16 + 0 + 4 = 20. && (10) \end{aligned}$$

$$\begin{aligned} (\mathbf{a}_2 - \mathbf{a}_1)^\top \mathbf{b} &= \begin{pmatrix} -4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} && (11) \\ &= (-4)(2) + (0)(-3) + (-2)(6) && (12) \\ &= -8 + 0 - 12 = -20. && (13) \end{aligned}$$

The magnitude of the direction vector $\mathbf{b}$ is:

$$\begin{aligned} \|\mathbf{b}\| &= \sqrt{2^2 + (-3)^2 + 6^2} && (14) \\ &= \sqrt{4 + 9 + 36} = \sqrt{49} = 7. && (15) \end{aligned}$$

Substituting these values into the formula for the squared distance:

$$\begin{aligned} d^2 \quad &= (\mathbf{a_2} - \mathbf{a_1})^\top (\mathbf{a_2} - \mathbf{a_1}) \quad - \left(\frac{(\mathbf{a_2} - \mathbf{a_1})^\top \mathbf{b}}{\|\mathbf{b}\|} \right)^2 && (16) \\ &= 20 - \left(\frac{-20}{7} \right)^2 && (17) \\ &= 20 - \frac{400}{49} && (18) \\ &= \frac{20 \times 49 - 400}{49} && (19) \\ &= \frac{980 - 400}{49} && (20) \\ &= \frac{580}{49}. && (21) \end{aligned}$$

Finally, the shortest distance d is the square root of this value:

$$d = \sqrt{\frac{580}{49}} = \frac{\sqrt{580}}{7} = \frac{\sqrt{4 \times 145}}{7} = \frac{2\sqrt{145}}{7}. \tag{22}$$

Thus, the distance between the two parallel lines is

$$\boxed{d = \frac{2\sqrt{145}}{7}}.$$

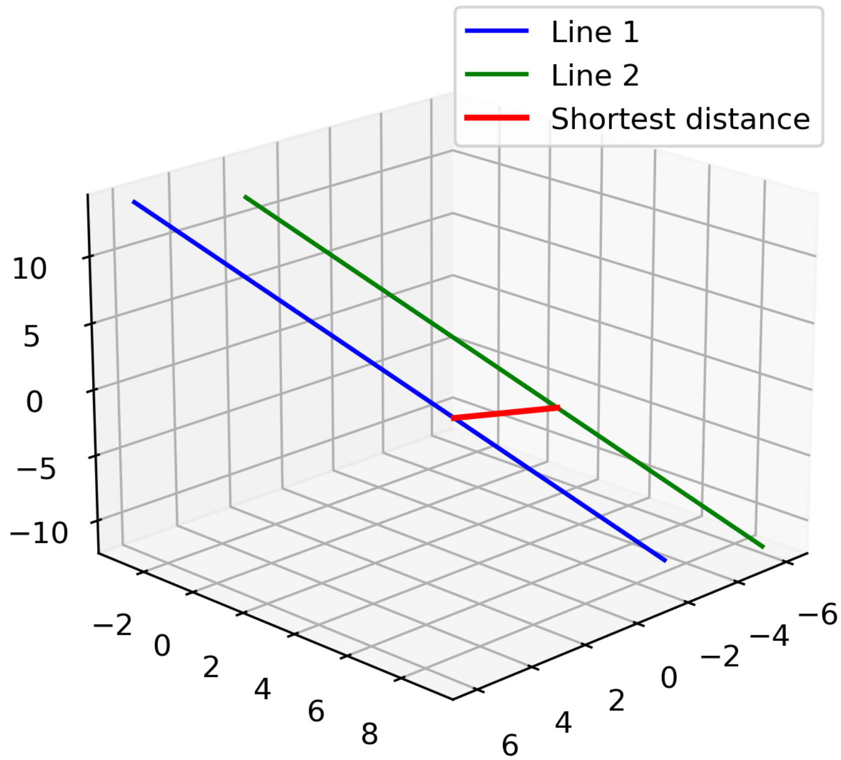

Figure 1: Shortest distance between two parallel lines