Global Curve Fitting of Frequency Response Measurements using the Rational Fraction Polynomial Method

by
Mark H. Richardson and David L. Formenti
Structural Measurement Systems
San Jose, California

ABSTRACT

The latest generation of FFT Analyzers contain still more and better features for excitation, measurement and recording of frequency response functions (FRF's) from mechanical structures. As measurement quality continues to improve, a larger variety of curve fitting methods are being developed to handle a set of FRF measurements in a global fashion. These approaches can potentially yield more consistent modal parameter values than curve fitting individual measurements independently.

In this paper, a new formulation of the Rational Fraction Polynomial method is given which can globally curve fit a set of FRF measurements. The pros and cons of this approach are discussed, and an example is included to compare the results of this method with a local curve fitting method.

INTRODUCTION

Physically speaking, a mode of vibration of a structure is characterized by a so called "natural" or "resonant' frequency at which the structure's predominant motion is a well defined waveform, called the "mode shape". A mode is the manifestation of energy which is trapped within the boundaries of the structure, and cannot readily escape.

When a structure is excited, its linear response can be shown to be a function of the combined motions of its modes of vibration. That is, the overall motion can be represented as a linear combination of the motions of each of the modes. Likewise, when the excitation source is removed from the structure, the trapped energy within it will slowly decay out until it no longer vibrates. The rate at which energy decays out of the structure is controlled by the amount of damping in the structure. Damping is also a modal property, each mode having a certain value of damping associated with it. That is, the motion comprised of heavily damped modes will decay out more quickly than that part of the motion comprised of more lightly damped modes.

Modes of vibration can be observed in practically any vibrating structure. When we measure the vibration of a structure and decompose the vibration signal into its frequency spectrum, the modes of vibration are evidenced by peaks in the spectrum. (Other peaks may be present in the spectrum due to large cyclical excitation forces). The modal peaks, however, will appear in practically any measurement made from any point on the structure.

In summary, then, each mode is a global property of the structure. Each mode is defined by a natural (or modal) frequency, a value of (modal) damping, and a mode shape.

Since modes are properties of the structure, itself, and are independent of the type of excitation force used to excite it, they should be identified from measurements which are also independent of the type of excitation. The Frequency Response Function (FRF) is such a measurement for linear systems.

The FRF is essentially a "normalized" measure of structural response. That is, it is the ratio of a response spectrum divided by the spectrum of the excitation which causes the response. Hence, the FRF is a measure of the dynamic properties between two degrees-of-freedom (DOF's) of a structure; the excitation point (and direction) and the response point (and direction). Again, the modes of the structure are indicated by the peaks in the measurement, with at least one mode defined by each peak.

Figure 1 shows in simplified form how the first three modes of a beam are identified from a set of FRF measurements made from the beam. The figure shows the imaginary part of each of the FRF measurements which were made between some (arbitrary) excitation point, and each of the

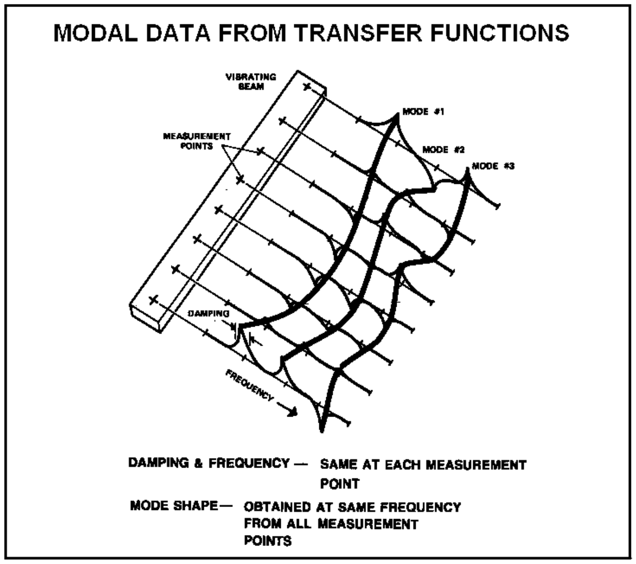

FIGURE 1

response points marked with X's. In this case, responses were measured only in the vertical direction and with a transducer that measured either displacement or acceleration. Alternatively, the FRF's could have been measured by mounting a single response transducer in one (arbitrary) location and exciting the beam at each X, in the vertical direction.

The figure shows a modal peak at the same frequency in each measurement, indicating the global nature of modal frequency. The "width" of the modal peak for each mode should also be the same in each measurement, again indicating the global nature of modal damping. Lastly, the mode shape which is defined by assembling the modal peak values from all the measurements, is global in the sense that it is defined for the entire expanse of the structure.

Mathematically speaking, modes of vibration are defined by certain parameters of a linear dynamic model for a structure. The dynamic properties of a structure can be written either as a set of differential equations in the time domain, or as a set of equations containing transfer functions in the Laplace (frequency) domain.

These equivalent models are shown in Figures 2 and 3. Regardless of which model is used, it can be shown that either model can be written in terms of the same parameters (frequencies, damping, and mode shapes) that describe the modes of vibration.

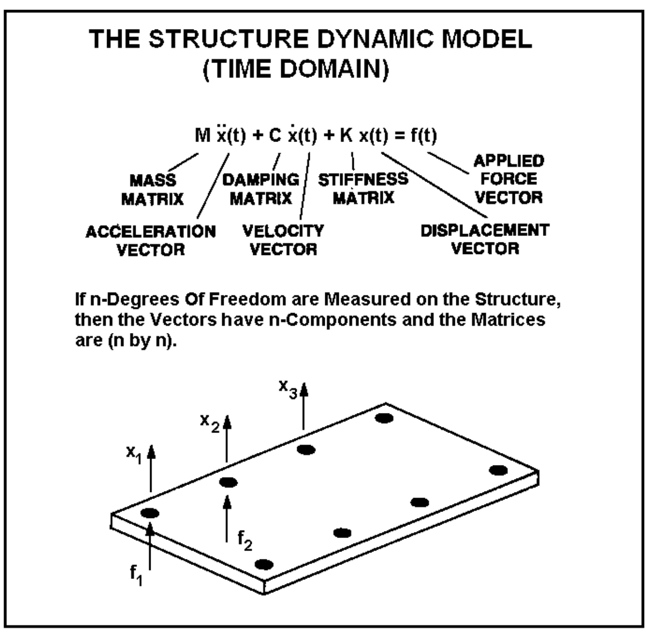

FIGURE 2

CURVE FITTING FRF's

Curve fitting, or Parameter Estimation, is a numerical process that is typically used to represent a set of experimentally measured data points by some assumed analytical function. The results of this curve fitting process are the coefficients, or parameters, that are used in defining the analytical function. With regard to the Frequency Response Function, the parameters that are calculated are its so-called modal parameters (i.e. modal frequency, damping, and residue). The curve fitting process can also be thought of as a data compression process since a large number of experimental values (the FRF measurements) can be represented by a much smaller number of modal parameters.

Various forms of the transfer function dynamic model are used to curve fit FRF measurements. The transfer function model is, in effect, evaluated along the frequency axis (i.e. **$s=j\omega$**) during the curve fitting process. The entire transfer function model is shown in Figure 3 , and it is well known [2] from examination of this model that curve fitting of one row or one column of FRF's is sufficient to identify the modal properties of the structure. However, selection of the correct row or column may be very important, depending on the modes of interest, the geometry of the structure, etc.

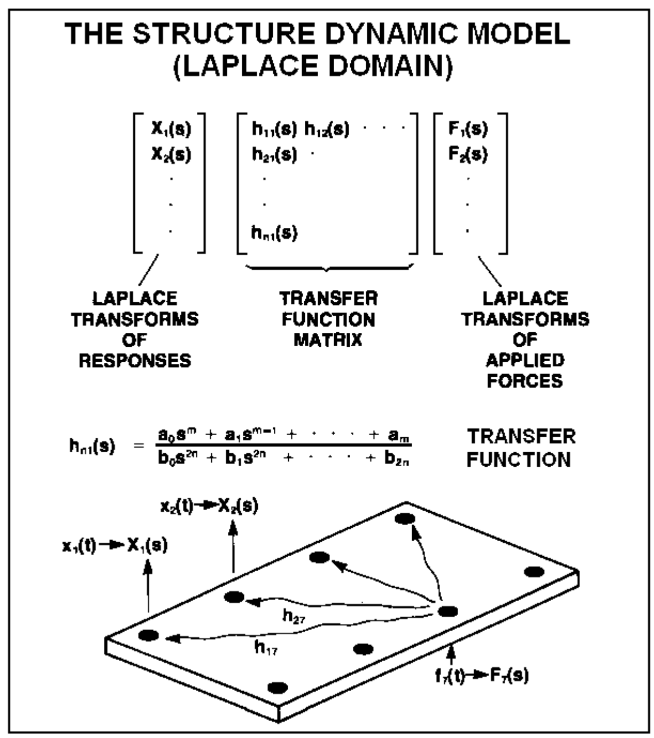

FIGURE 3

Nevertheless, once a set of FRF measurements has been made on a structure, whether they comprise one row or column, or several, the most commonly used method of curve fitting FRF's is to fit them one at a time using one of the analytical forms of the FRF shown in Figure 4.

The most commonly used form is the Partial Fraction Form. Most SDOF (or single mode) methods, and various iterative MOOF (multiple mode) methods, are based on this form of the model.

RATIONAL FRACTION POLYNOMIAL (RFP) METHOD

In a previous paper [1], a curve fitting method based on the Rational Fraction Polynomial form of the FRF was introduced. This MDOF method fits the analytical expression (1) to an FRF measurement in a least-squared error sense, and in the process, the coefficients of the numerator and denominator polynomials are identified.

Analytical Forms of the Frequency Response Function

Rational Fraction Form

$$H(\omega) = \left. \frac{\sum_{k=0}^{m} a_k s^k}{\sum_{k=0}^{n} b_k s^k} \right|_{s=j\omega} \qquad (1)$$

Partial Fraction Form

$$H(\omega) = \sum_{k=1}^{n/2} \left[\frac{r_k}{s - p_k} + \frac{r_k^*}{s - p_k^*} \right]_{s=j\omega} \qquad (2)$$

$$p_k = -\sigma_k + j\omega_k = k^{th} \text{ pole}$$

$$r_k = \text{residue for } k^{th} \text{ pole}$$

FIGURE 4

Once these coefficients are known, it is a straightforward matter to obtain the poles, the zeroes, and the modal properties (poles and residues) of the FRF. This method has been implemented in a variety of commercially available modal software packages and has been used successfully on a large variety of FRF measurements.

As pointed out in [1], not only can the RFP method handle noisy measurements (because of its least-squared error formulation) but it offers a unique way of handling the residual effects of out-of-band modes.

Most other MDOF methods require that additional "computational" modes be used in order to compensate for the residual effects of out-of-band modes in the curve fitting frequency band. With some methods, these computational modes can often cause the parameter estimates of the modes of interest to be in large error if the “right" number of computational modes is not used. Choosing the “right" number of computational modes can be a trial and error process.

The RFP method, on the other hand, allows the use of additional numerator polynomial terms as a means of compensating for the effects of out-of-band modes. As shown in [1], the use of these extra terms still permits the accurate estimation of the modal parameters of interest, and is, in general, a more foolproof means of compensation than the use of computational modes.

GLOBAL CURVE FITTING

Most curve fitting is done today in a “local" sense. That is, each measurement is individually curve fit, and the modal frequency, damping, and complex residue are estimated for each mode in the measurement. Hence, four parameters are estimated for each mode, (counting the complex residue as two parameters), and if an MDOF curve fitter is used to estimate the parameters of, for example, five modes, then a total of twenty unknown parameters must be simultaneously identified during the curve fitting process. With such a large number of unknowns, significant errors can occur in the parameter estimates. Many times the accuracy of the modal parameters is sacrificed during the curve fitting process to yield a good looking (re-synthesized) curve fit function. In other words, the values of the estimated parameters can trade off errors. This is especially true for the modal damping and residue estimates.

Accurate damping and residue estimates are, in general, more difficult to obtain than accurate frequency estimates. Damping is the most difficult parameter to estimate accurately from FRF measurements, and the residue is often tightly coupled to damping. That is, if damping is in large error, the residue estimate will be in large error even though the curve fitting function closely matches the measurement data.

One approach that can reduce errors is to divide the curve fitting process up into two steps; (1) estimate the frequency and damping parameters and (2) the residue (or mode shape) parameters. This process of using the measurement data to obtain frequency and damping estimates first, and then with known frequency and damping values to obtain mode shape estimates by a second estimation process is called Global Curve Fitting.

One advantage of Global Curve Fitting is that more accurate frequency and damping estimates can potentially be obtained by processing all of the measurements, than can be obtained from curve fitting a single measurement. Another advantage is that because damping is already known and fixed as a result of the first step, the residues are, in general, more accurately estimated during the second step.

GLOBAL FREQUENCY AND DAMPING

Reference [1] contains a special formulation of the RFP method which obtains global frequency and damping estimates from a set of FRF measurements. The Complex Exponential (or Prony) method [4] can also be formulated in a similar manner to obtain global estimates from a set of impulse response functions. (These functions would be obtained by taking the inverse FFT of each FRF measurement).

Another very straightforward method of obtaining global frequency and damping is to average together the magnitudes of all of the FRF measurements. The resulting magnitude function will then contain the resonance peaks of all of the modes in the measurement set, and this function can be curve fit to obtain frequency and damping estimates.

Probably the simplest method, though, is to select some measurements where each particular mode has a large response and then use the frequency and damping estimates from these measurements as the best approximations of the global estimates.

GLOBAL MODE SHAPE

Once the modal frequencies and damping are known, the mode shapes (which are eigenvectors) can be obtained by solving an eigenvalue/eigenvector set of equations, (3). This type of an approach generally involves the manipulation of large matrices, however, and hence requires a relatively large computer; one larger than is found in most laboratory testing systems.

The global curve fitting method introduced in this paper is much simpler and easier to implement on a small computer than an eigenvalue/eigenvector solution approach. The approach discussed here is "global" in the sense that global frequency and damping estimates are used, but is "local" in the sense that the FRF measurements are processed one at a time in order to obtain modal residue estimates. These residue estimates are then assembled from all the various measurements to obtain the mode shapes.

THE GLOBAL RFP METHOD TO ESTIMATE MODE SHAPES

The RFP method can be reformulated to take advantage of the global nature of modal frequency and damping. If the Rational Fraction Form of the FRF shown in Figure 4 is used for curve fitting, and the modal frequencies and damping are already known, the denominator polynomial (also called the characteristic polynomial) is therefore known. Hence, the only unknowns are the coefficients of the numerator polynomial. Once these coefficients are determined, then the residues, and hence the mode shapes, can be computed.

Global FRP Using Ordinary Polynomials

FRF in Terms of polynomial coefficients, a_k

$$\mathbf{h_i = \sum_{k=0}^{m} t_{i,k} a_k} \qquad \mathbf{i=1,\ldots,L} \qquad \mathbf{(3)}$$

where

$$\mathbf{t_{i.k}} = \frac{(\mathbf{j}\omega_i)^k}{\sum_{k=0}^{n} \mathbf{b_k}(\mathbf{j}\omega_i)^k}$$

$$= \frac{(\mathbf{j}\omega_i)^k}{\sum_{k=1}^{\mathbf{modes}} (\omega_k{}^2 - \omega_i{}^2 + \mathbf{j}2\sigma_k\omega_i)} \qquad \mathbf{(4)}$$

In matrix form

$$\begin{Bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h_n} \end{Bmatrix} = \begin{bmatrix} \mathbf{t}_{1,0} & & \mathbf{t_{1,m}} \\ & & \\ & & \\ \mathbf{t_{L,0}} & & \mathbf{t_{L,m}} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a_m} \end{Bmatrix}$$

or

$$\mathbf{\{H\} = [T]\{A\}} \qquad \mathbf{(5)}$$

Least Squared Error Equations

$$\mathbf{[T^*]^t[T]\{A\} = Re([T^*]^t\{Y\})} \qquad \mathbf{(6)}$$

$\mathbf{\{Y\}} =$ L-vector of measurement data.

FIGURE 5

Global FRP Using Orthogonal Polynomials

FRF in terms of polynomial coefficients, c_k

$$\mathbf{h_i} = \sum_{k=0}^{m} \mathbf{z_{i,k} c_k} \qquad \mathbf{i=1,\ldots,L} \qquad \mathbf{(7)}$$

where

$$\mathbf{z_{i,k}} = \frac{\phi_{i,k}}{\sum_{k=1}^{modes} (\omega_k{}^2 - \omega_i{}^2 + j2\sigma_k\omega_i)} = \frac{\phi_{i,k}}{\mathbf{g_i}} \qquad \mathbf{(8)}$$

$$\sum_{i=1}^{L} \frac{\phi_{i,k}\phi_{i,j}}{|\mathbf{g_i}|^2} = \begin{cases} .5 & k = j \\ 0 & k \neq j \end{cases} \qquad \mathbf{(9)}$$

In matrix form

$$\begin{Bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \vdots \\ \mathbf{h_n} \end{Bmatrix} = \begin{bmatrix} \mathbf{z}_{1,0} & \cdots & \mathbf{z_{1,m}} \\ \vdots & & \vdots \\ \mathbf{z_{L,0}} & \cdots & \mathbf{z_{L,m}} \end{bmatrix} \begin{Bmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c_m} \end{Bmatrix}$$

or

$$\{\mathbf{H}\} = [\mathbf{Z}]\{\mathbf{C}\} \qquad \mathbf{(10)}$$

Least Squared Error Equations

$$\{\mathbf{C}\} = \mathbf{Re}([\mathbf{Z}^*]^t\{\mathbf{Y}\}) \qquad \mathbf{(11)}$$

$\{\mathbf{Y}\} =$ L-vector of measurement data.

FIGURE 6

In equation (3) in Figure 5, the FRF ($\mathbf{h_i}$) is written for L data points ($\mathbf{i = 1 .... L}$). This expression of the FRF model is written in a form which isolates the unknown parameters ($\mathbf{a_k, k=0,...,m}$). The least squared error curve fitting problem can then be formulated in a manner similar to [1], and the solution equations (6) shown in Figure 5 will result.

Notice that only the right hand side of equation (6) is directly dependent on the FRF measurement data. The coefficient matrix on the left-hand side only depends on the frequency range and the number of data points used for curve fitting.

These simultaneous linear equations can be solved using a standard equation solver, but our experience has been that numerical problems occur frequently in attempting to solve them in this form. Again, as with our previous implementation of the RFP method, the left-hand side matrix in equation (6) becomes ill-conditioned due to the polynomial functions, and hence the equation solutions can contain significant errors.

Figure 6, on the other hand, contains a reformulation of the RFP method using orthogonal polynomials. Not only does this formulation eliminate the numerical problems of equation (6), but the solution equations (11) become greatly simplified and don't require a simultaneous linear equation solution at all. Notice from equation (9) that the numerator orthogonal polynomials are generated using the denominator terms ($\mathbf{g_i, i = 1,...L}$) as a weighting function. Once the matrix [Z] is computed, it can be saved and used to obtain a new set of polynomial coefficients {**C**} for each new vector of measurement data {**Y**}. Of course, the coefficients {**C**} are for the orthogonal polynomials, and the procedure described in [1] must be used to recover the coefficients {**A**} of the ordinary polynomial coefficients. The residues are then recovered by a partial fraction expansion process. These final calculations are very straightforward, though.

TEST CASE WITH HEAVY MODAL COUPLING

The Global RFP method was compared with the Local RFP method by curve fitting five measurements with three heavily coupled modes. Plots of the log magnitudes of these five measurements are shown in Figure 8 . These FRF's were synthesized from known modal parameters (listed in Figure 7), and then random noise was added to them to simulate more realistic measurements. Synthesized measurements were used so that the curve fitting results could be compared with the correct modal parameter values.

The measurements were synthesized for a frequency range of 0 Hz to 100 Hz, and the curve fitting was done between 40 Hz and 65 Hz.

Notice that measurement No.1 contains a zero (0) residue for mode No.2, indicating that this measurement was taken at a node point of Mode No. 2. Likewise, measurement No.3 was taken at a node point of mode No. 1. Measurement No.4 (or No. 5) could be a driving point measurement since all of the residues have the same sign.

MODAL DATA USED TO SYNTHESIZE MEASUREMENTS

MODE NO.:	**1**	**2**	**3**
FREQUENCY (Hz):	**50**	**52**	**55**
DAMPING (%):3	**2.5**	**3**	
	-----	**RESIDUES**	**-----**
MEASMT. NO.1:	**1**	**0**	**-1**
MEASMT. NO.2:	**.5**	**-.25**	**-.6**
MEASMT. NO.3:	**0**	**-.5**	**.6**
MEASMT. NO.4:	**-.5**	**-.75**	**-.3**
MEASMT. NO.5:	**-1**	**-1**	**-.8**

FIGURE 7

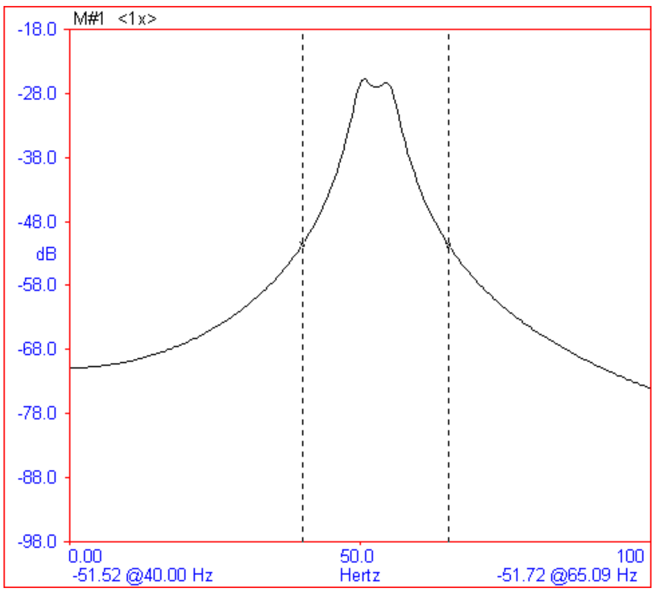

FIGURE 8.a

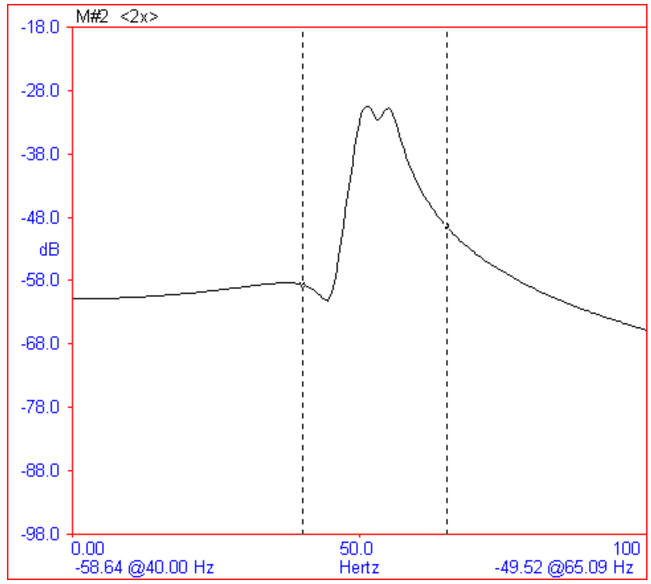

FIGURE 8.b

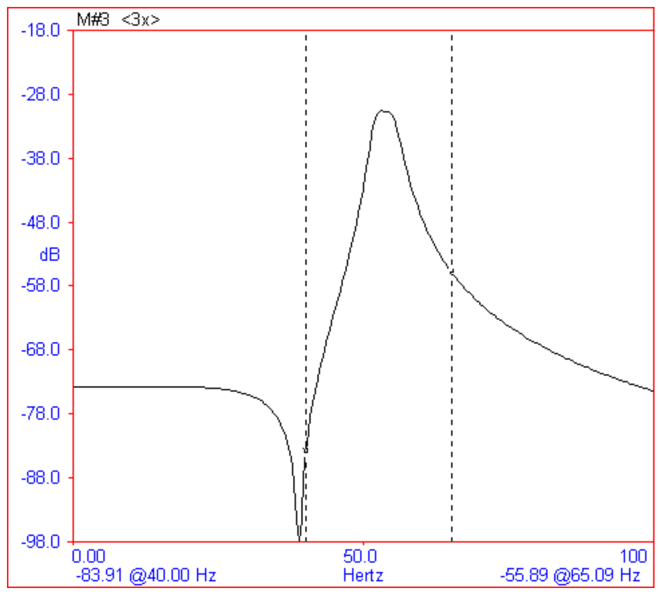

FIGURE 8.c

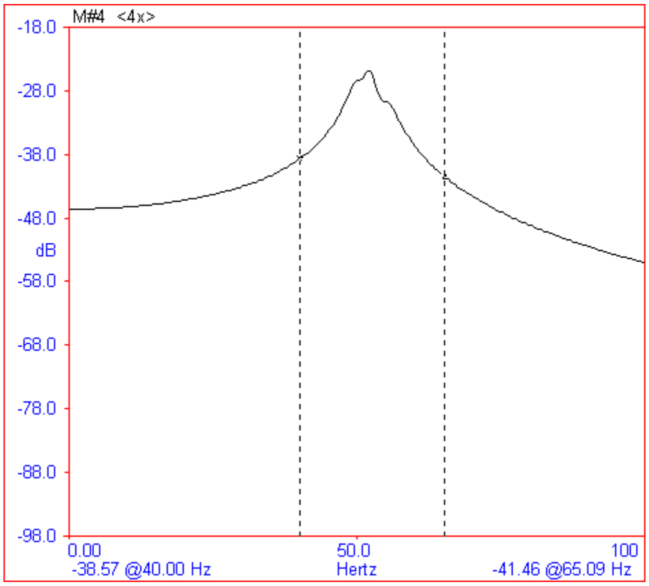

FIGURE 8.d

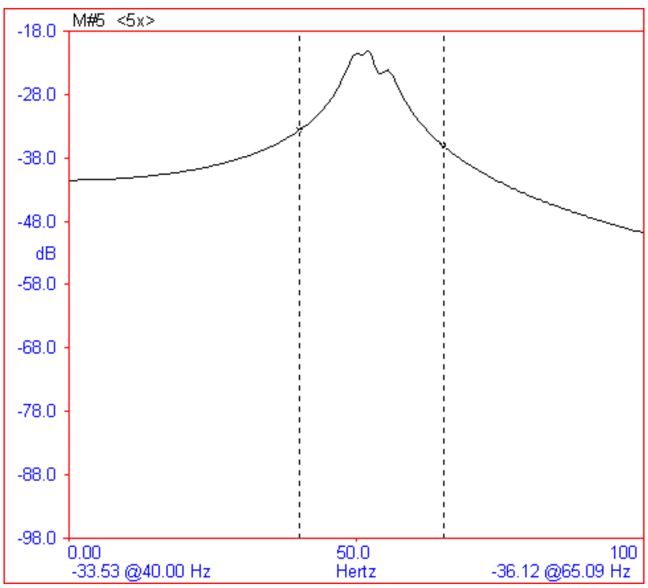

FIGURE 8.e

Figure 9 contains the results of curve fitting each of the measurements using the Local RFP method. The curve fitter was set up to fit 3 modes in all measurements since measurement No.'s. 4 and 5 at least show the evidence of 3 modes (3 modal peaks).

The curve fitting results for measurement No.1 indicate the first problem encountered with fitting these measurements; even though the modal parameters of the two modes in this measurement were estimated quite accurately, the curve fitter assigned these parameters to Mode Nos. 1 and 2 when they should have been assigned to Mode Nos. 1 and 3. This difficulty is always encountered when using a local fitting method on measurements where some of the modes are at, or near, node points. The same problem occurred with the curve fitting results of measurement No.3. Measurement Nos. 2, 4, and 5 all indicate the same curve fitting problem; too many unknown parameters (12 per measurement) are being simultaneously estimated on measurements with heavy modal coupling and some small amount of noise.

Figure 10 contains the results of using the Global RFP method on the five measurements in Figure 8. The global frequency and damping estimates could be obtained by any of the previously described methods. In this case, the known values were merely used.

The residue estimates were obtained by applying equations (11) in Figure 6 to each of the measurements. Comparing these estimates with the correct answers in Figure 7, one can conclude that all of the residues were accurately estimated. All of the estimates are in error by less than 1%.

***** Modal Fit Data *****

MEAS: 1

MODE	FREQ(Hz)	DAMP(%)	AMPL	PHS
1	49.99	3.03	1.998E+00	.33
2	55.01	3.00	1.918E+00	180.20
3	55.26	.95	7.495E-03	43.96

MEAS: 2

MODE	FREQ(Hz)	DAMP(%)	AMPL	PHS
1	50.09	2.38	4.197E-01	339.91
2	54.72	4.26	7.120E-01	146.28
3	55.60	.73	4.585E-02	185.47

MEAS: 3

MODE	FREQ(Hz)	DAMP(%)	AMPL	PHS
1	51.99	2.49	4.914E-01	100.73
2	55.00	2.92	5.873E-01	.73
3	57.77	.95	7.943E-04	7.43

MEAS: 4

MODE	FREQ(Hz)	DAMP(%)	AMPL	PHS
1	49.92	1.36	1.174E-01	207.24
2	51.83	4.04	1.468E+00	178.87
3	56.14	1.07	8.045E-02	167.07

MEAS: 5

MODE	FREQ(Hz)	DAMP(%)	AMPL	PHS
1	50.12	2.61	9.644E-01	109.27
2	52.16	2.99	1.448E+00	190.45
3	55.66	2.28	4.929E-01	175.28

FIGURE 9 Local RFP Results

CONCLUSIONS

In this paper we showed that the RPF method for curve fitting FRF measurements can be reformulated so that it can be used in a global curve fitting scheme. That is, global frequency and damping can be estimated as a first step, and these estimates can then be used to estimate residues in a second step. This global curve fitting approach was then shown by example to give more accurate parameter estimates than simultaneously estimating modal frequency, damping, and residue for each mode in each measurement, which is most commonly done today.

The global curve fitting method was compared with the more conventional local fitting method on measurements that contained heavy modal coupling and additive random noise. In this case, the global method was clearly more accurate than the local method. In cases of less modal coupling and/or less noise, the advantage of the global method over the local method may be less pronounced. Nevertheless, the global method gives generally better results than local methods, especially on measurements where some modes are at node points. The global method does however suffer from a few disadvantages. If modal frequency or damping varies by any substantial amount from one measurement to another, then the global method, which assumes that these parameters are unchanging, will be in error. However, such variations in modal frequency or damping are not characteristic of a linear system, and if this occurs, more care should be taken to obtain a consistent set of measurements.

Secondly, with the global method, the accuracy of the residue estimates strongly depends on the accuracy of both the global frequency and damping estimates. Therefore, the overall success of the method depends on whether global frequency and damping can be accurately estimated first.

These questions and further questions regarding the handling of out-of-band modes will be addressed in future papers.

***** Modal Residues *****

MODE NO.	FREQ(Hz)	DAMP(%)
1	49.999	2.998

MEAS NO.	AMPL	PHS
1	9.917E-01	359.65
2	4.982E-01	359.69
3	1.303E-03	186.34
4	5.046E-01	179.15
5	1.001E+00	180.31

MODE NO.	FREQ(Hz)	DAMP(%)
2	52.000	2.502

MEAS NO.	AMPL	PHS
1	1.315E-02	323.66
2	2.491E-01	179.68
3	4.998E-01	180.45
4	7.530E-01	180.62
5	9.974E-01	177.35

MODE NO.	FREQ(Hz)	DAMP(%)
3	55.000	2.999

MEAS NO.	AMPL	PHS
1	9.990E-01	179.65
2	6.022E-01	180.03
3	6.022E-01	.12
4	2.978E-01	180.00
5	7.970E-01	180.27

FIGURE 10 Global RFP Results

REFERENCES

[1] Richardson, M.H. & Formenti, D.L., "Parameter Estimation from Frequency Response Measurements using Rational Fraction Polynomials", Proceedings of the 1st International Modal Analysis Conference, Orlando, Florida, November 8-10, 1982.

[2] Richardson, M.H., “Modal Analysis using Digital Test Systems", Proceedings of Seminar on Understanding Digital Control and Analysis in Vibration Test Systems, The Shock of Vibration Information Center, Naval Research Lab, Washington, D.C., May, 1975.

[3] Crowley, S.M. & Allemang, R.J., “Application of the PolyReference Technique in Experimental Modal Analysis”, Proceedings of the 2nd International Modal Analysis Conference, Orlando, Florida, February 6-9, 1984.

[4] Spitznogle, F.R. & Quazi, A.H., “Representation and Analysis of Time Limited Signals Using a Complex Exponential Algorithm”, J. of Acoustical Society of America, Vol. 47, No. 5 (Part I), May 1970, pp. 1150-1155.