Convolutional Neural Networks on Graphs with Fast Localized Spectral Filtering

A Discussion

Naganand Y

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    - User data on social networks
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    - Log data on telecommunication networks
    - Text documents on word embeddings

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  - Define convolution and pooling operations on graphs

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  - Ideas from spectral graph theory
  - Filters provable to be strictly localized in a ball of radius K

Proposed Technique

1 Design of localized convolutional filters on graphs

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  - Inverse, $x = U\hat{x}$

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- Learning complexity is $O(K)$ where K is support size of filter
  - Same as classical CNNs

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  - $y = g_\theta(L)x = \left[x'_0, \cdots, x'_{K-1}\right]\theta$ where $x'_k = T_k(\widetilde{L})x$

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- Requires $O(K|E|F_{in}F_{out}S)$ operations

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- Divides # nodes by approximately 2 (there may exist a few singletons, non-matched nodes)

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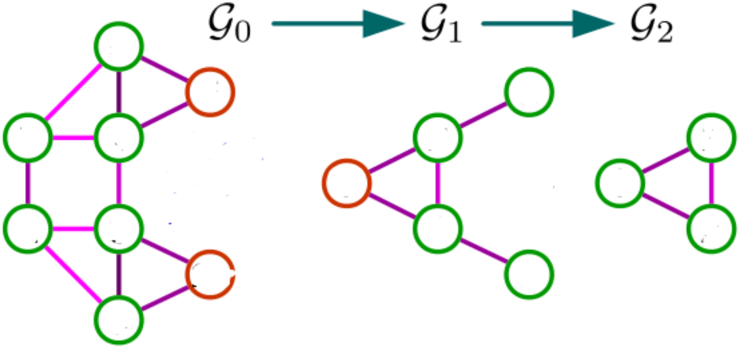
Graph Coarsening: Example



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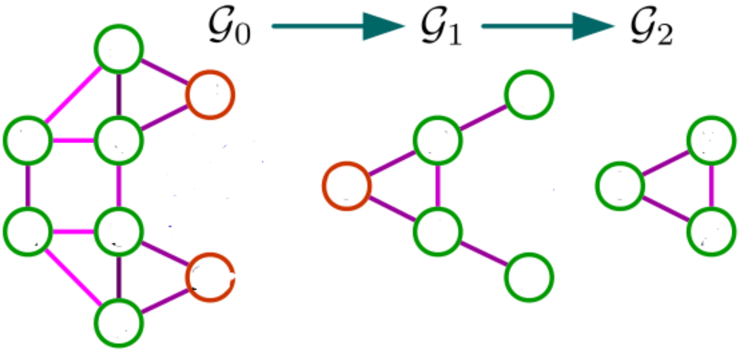
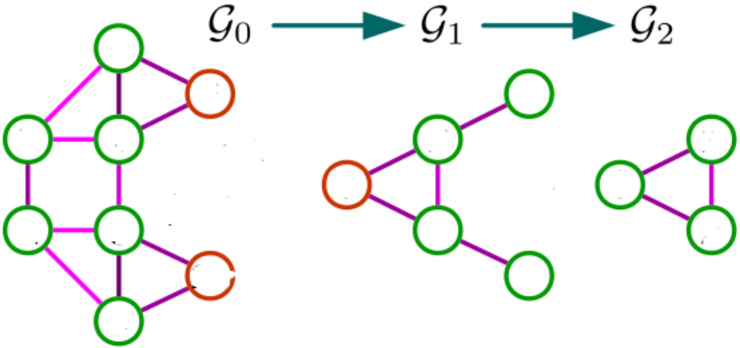


Graclus produces G_1 and G_2 on G_0 as input

Max Pooling: Example



Suppose max pooling of size 4 needs to be carried out (Note: $x \in \mathbb{R}^8$)

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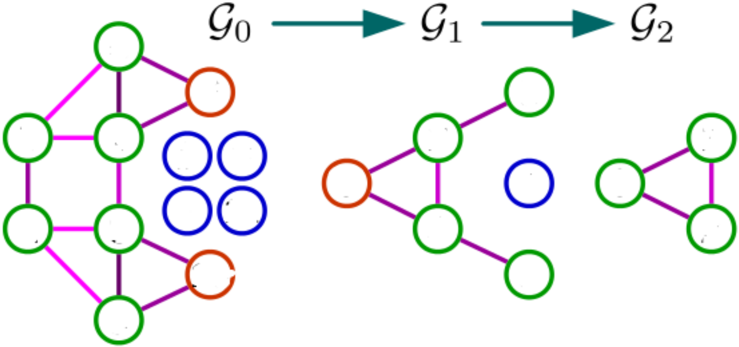

Add fake nodes to pair with singleton nodes

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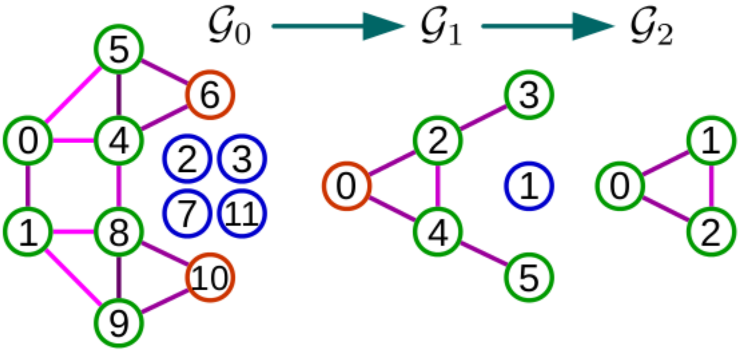

Nodes in V_2: ordered arbitrarily, nodes in V_1, V_0: ordered consequently

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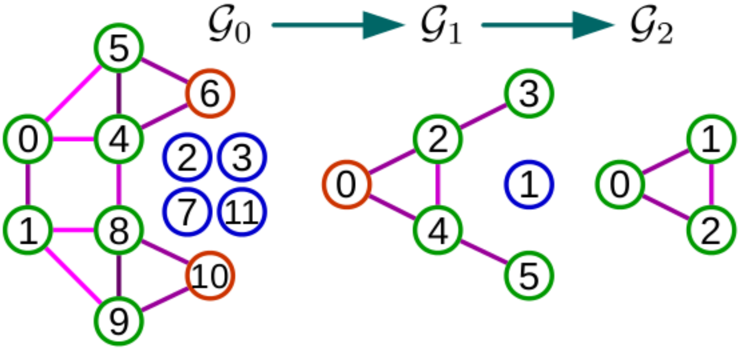

Observation: Node k has $2k$ and $2k+1$ as children

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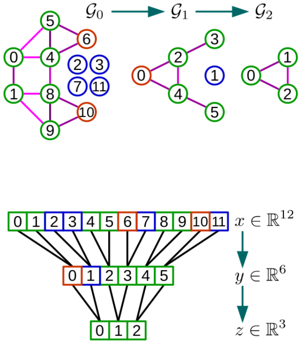

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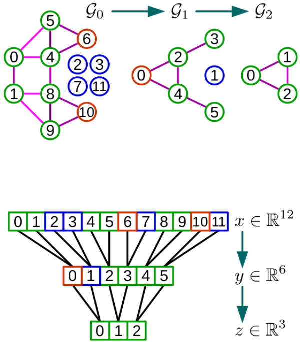

- $z = \Big(max(x_0, x_1), max(x_4, x_5, x_6), max(x_8, x_9, x_{10})\Big) \in \mathbb{R}^3$

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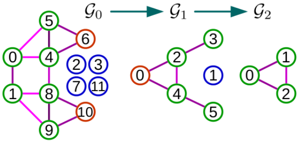

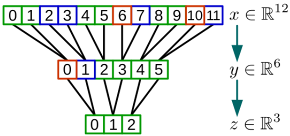

- $z = \Big(max(x_0, x_1), max(x_4, x_5, x_6), max(x_8, x_9, x_{10})\Big) \in \mathbb{R}^3$
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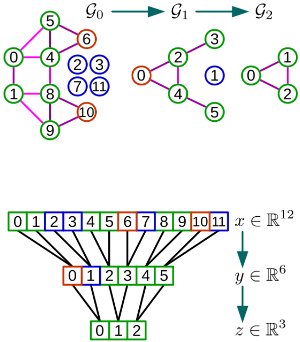

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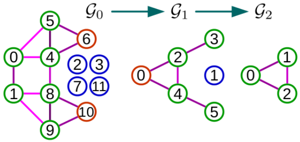

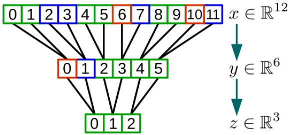

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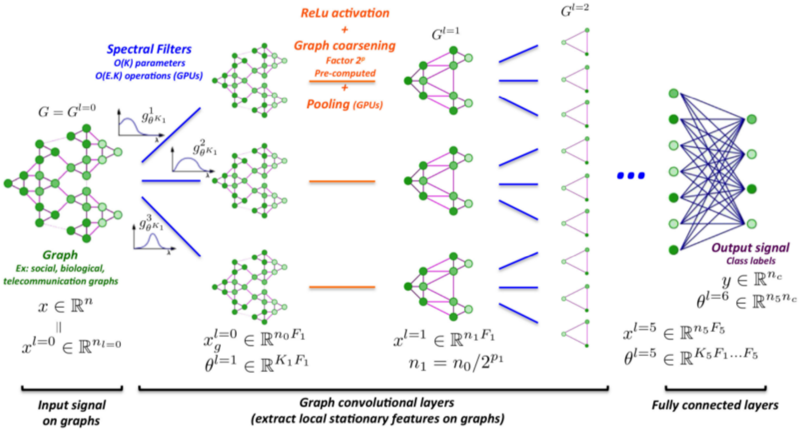
    - Very efficient, satisfies parallel architectures such as GPUs as memory accesses are local

Architecture



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Model	Architecture	Accuracy
Classical CNN	C32-P4-C64-P4-FC512	99.33
Proposed graph CNN	GC32-P4-GC64-P4-FC512	99.14

- Ck: Convolutional layer k feature maps
- GCk: Graph convolutional layer k feature maps
- FCk: Fully connected layer k hidden units
- Pk: Pooling layer size, stride k

Text Categorization on 20NEWS

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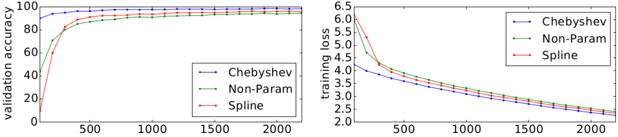

Figure 4: Plots of validation accuracy and training loss for the first 2000 iterations on MNIST.

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Comparison of Computational Efficiency

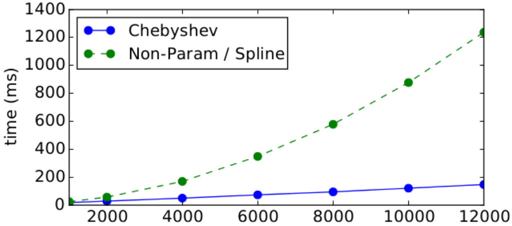

Figure 3: Time to process a mini-batch of $S = 100$ 20NEWS documents w.r.t. the number of words n.

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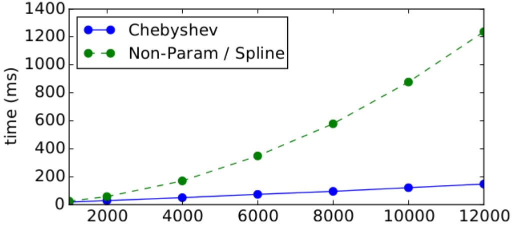

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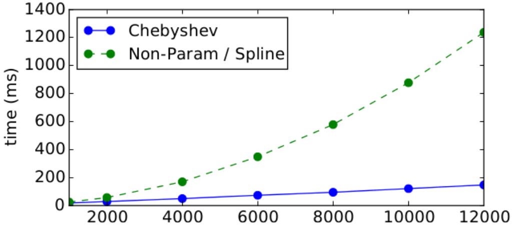

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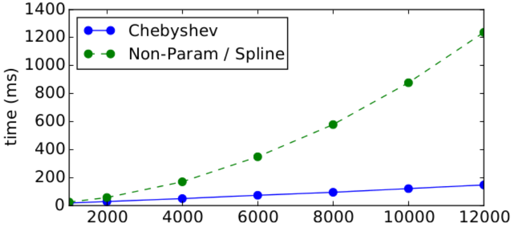

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Model	Architecture	Time (ms) CPU	Time (ms) GPU	Speedup
Classical CNN	C32-P4-C64-P4-FC512	210	31	6.77x
Proposed graph CNN	GC32-P4-GC64-P4-FC512	1600	200	8.00x

Table 4: Time to process a mini-batch of $S = 100$ MNIST images.

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