Lab 4 Report: EXERCISES

Yilin Zhang [1]. Lab Group: 31.

I. INDIVIDUAL

A. Deliverable - Single-segment trajectory optimization

1. Sol.

Since $P(t) = p_1 t$, and $P(1) = 1$, we got

$$P(1) = 1 = p_1 \cdot 1$$
$$\Rightarrow \quad p_1 = 1$$
$$\Rightarrow \quad P(t) = t.$$

Since we have only one solution, this is the optimal solution. Here, $P^{(1)}(t) = t' = 1$, put it to aim function, we have

$$\int_0^1 (P^{(1)}(t))^2 dt = \int_0^1 1^2 dt = 1.$$

That is, the optimal solution is $P(t) = t$, and the value of the cost function is 1.

2. Sol.

(a) We have

$$\int_0^1 (p^{(1)}(t))^2 dt = \int_0^1 ((p_2 t^2 + p_1 t)')^2 dt$$
$$= \int_0^1 (2p_2 t + p_1)^2 dt$$
$$= \int_0^1 (4 p_2{}^2 t^2 + 4 p_1 p_2 t + p_1{}^2) dt$$
$$= [\frac{4}{3} p_2{}^2 t^3 + 2 p_1 p_2 t^2 + p_1{}^2 t]_0^1$$

so, $\int_0^1 (p^{(1)}(t))^2 dt = \frac{4}{3} p_2{}^2 + 2 p_1 p_2 + p_1{}^2.$

Since $\mathbf{p} = [p_1, p_2]^\top$, we got the symmetric matrix $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix},$$
$$s.t. \quad \int_0^1 (p^{(1)}(t))^2 dt = \mathbf{p}^\top \mathbf{Q} \mathbf{p}.$$

(b) Since $P(t) = p_2 t^2 + p_1 t = 1$, we have $p_2 + p_1 = 1$. Therefore, it can be written as $\mathbf{A}\mathbf{p} = \mathbf{b}$, where:

$$\mathbf{A} = [1, 1], \quad \mathbf{b} = 1.$$

(c) For $P(t) = p_2 t^2 + p_1 t = 1$, we have the problem:

$$\min_{\mathbf{p}} \mathbf{p}^\top \mathbf{Q} \mathbf{p} \quad s.t. \quad \mathbf{A}\mathbf{p} = \mathbf{b},$$

[1] Yilin zhang, OUC id: 23020036094, is with Faculty of Robotics and Computer Science, Ocean University of China and Heriot-Watt University, China Mainland zyl8820@stu.ouc.edu.cn

where $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix}, \mathbf{A} = [1,1], \mathbf{b} = 1.$

We import Lagrangian Function:

$$\mathcal{L}(\mathbf{p}, \lambda) = \mathbf{p}^\top \mathbf{Q} \mathbf{p} - \lambda(\mathbf{A}\mathbf{p} - \mathbf{b}). \qquad (1)$$

It has a KKT case: the partial derivatives with respect to $\mathbf{p}$ is zero:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{p}} = 2\mathbf{Q}\mathbf{p} - \mathbf{A}^\top \lambda = 0 \quad \Rightarrow \quad 2\mathbf{Q}\mathbf{p} = \mathbf{A}^\top \lambda.$$

Since $\mathbf{A}\mathbf{p} = \mathbf{b}$,

$$\mathbf{p} = \mathbf{Q}^{-1}\mathbf{A}^\top \left(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^\top \right)^{-1} \mathbf{b}.$$

We got $p_1 = 1$, $p_2 = 0$, here the function is:

$$\mathbf{p}^\top \mathbf{Q} \mathbf{p} = [1, 0] \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1.$$

The result is: the optimal solution is $P(t) = t$, with the value of cost is 1.

3. Sol.

(a) Similar to *2.(a)(b)*, we have

$$\int_0^1 \left(P^{(1)}(t) \right)^2 dt = \int_0^1 \left(p_1 + 2p_2 t + 3p_3 t^2 \right)^2 dt$$
$$= p_1^2 + 2p_1 p_2 + 2p_1 p_3 + \frac{4}{3} p_2^2 + 3 p_2 p_3 + \frac{9}{5} p_3^2.$$

Therefore its symmetric matrix $\mathbf{Q} \in \mathbb{S}^3$ is:

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}.$$

For $P(1) = 1$, there is $p_1 + p_2 + p_3 = 1$, so

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad b = 1.$$

(b) Similar to *2.(c)*, we got

$$\begin{cases} p_1 + p_2 + p_3 & = 1, \\ p_1 + \frac{4}{3}p_2 + \frac{3}{2}p_3 & = 1, \\ p_1 + \frac{3}{2}p_2 + \frac{9}{5}p_3 & = 1. \end{cases}$$

That is $P(t) = t$, with $p_1 = 1$, $p_2 = 0$, $p_3 = 0$.

4. Sol.

(a) The constraints are:

$$P(1) = p_1 + p_2 = 1, \quad P^{(1)}(1) = p_1 + 2p_2 = -2.$$

Solving this linear system yields:

$$p_1 = 4, \quad p_2 = -3.$$

Thus, the optimal solution is:

$$P(t) = 4t - 3t^2.$$

Compute the optimal cost:

$$\begin{aligned}
\int_0^1 (P^{(1)}(t))^2 dt &= \int_0^1 (4 - 6t)^2 dt \\
&= \int_0^1 (16 - 48t + 36t^2) dt \\
&= \left[16t - 24t^2 + 12t^3 \right]_0^1 \\
&= 4.
\end{aligned}$$

That is, the optimal solution is $P(t) = 4t - 3t^2$, and the optimal cost is 4.

(b) Similarly, the constraints are:

$$P(1) = p_1 + p_2 + p_3 = 1, \quad P^{(1)}(1) = p_1 + 2p_2 + 3p_3 = -2.$$

Solving these gives:

$$p_1 = 4 + p_3, \quad p_2 = -3 - 2p_3.$$

The objective function is $\mathbf{p}^\top \mathbf{Q} \mathbf{p}$, where $\mathbf{Q}$ is:

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}.$$

Substituting the parameterization yields a quadratic function in p_3:

$$f(p_3) = \frac{2}{15} p_3^2 + p_3 + 4.$$

Setting the derivative to zero:

$$\frac{df}{dp_3} = \frac{4}{15} p_3 + 1 = 0 \implies p_3 = -\frac{15}{4}.$$

Substituting back:

$$p_1 = 4 - \frac{15}{4} = \frac{1}{4}, \quad p_2 = -3 - 2\left(-\frac{15}{4}\right) = \frac{9}{2}.$$

Thus, the optimal solution is:

$$P(t) = \frac{1}{4}t + \frac{9}{2}t^2 - \frac{15}{4}t^3.$$

Compute the optimal cost:

$$\int_0^1 (P^{(1)}(t))^2 dt = \frac{17}{8}.$$

That is, the optimal solution is $P(t) = \frac{1}{4}t + \frac{9}{2}t^2 - \frac{15}{4}t^3$, and the optimal cost is $\frac{17}{8}$.

B. Deliverable - Multi-segment trajectory optimization

II. TEAM

III. SOME WORDS

APPENDIX A
INDIVIDUAL

A. Deliverable - Single-segment trajectory optimization

Consider the following minimum velocity ($r = 1$) single-segment trajectory optimization problem:

$$\min_{P(t)} \int_0^1 (P^{(1)}(t))^2 dt, \tag{2}$$
$$\text{s.t.} \quad P(0) = 0, \tag{3}$$
$$P(1) = 1, \tag{4}$$

with $P(t) \in \mathbb{R}[t]$, i.e., $P(t)$ is a polynomial function in t with real coefficients:

$$P(t) = p_N t^N + p_{N-1} t^{N-1} + \cdots + p_1 t + p_0. \tag{5}$$

Note that because of constraint (3), we have $P(0) = p_0 = 0$, and we can parametrize $P(t)$ without a scalar part p_0.

1. Suppose we restrict $P(t) = p_1 t$ to be a polynomial of degree 1, what is the optimal solution of problem (2)? What is the value of the cost function at the optimal solution?

2. Suppose now we allow $P(t)$ to have degree 2, i.e., $P(t) = p_2 t^2 + p_1 t$.

(a) Write $\int_0^1 (P^{(1)}(t))^2 dt$, the cost function of problem (2), as $\mathbf{p}^\top \mathbf{Q} \mathbf{p}$, where $\mathbf{p} = [p_1, p_2]^\top$ and $\mathbf{Q} \in \mathbb{S}^2$ is a symmetric 2×2 matrix.

(b) Write $P(1) = 1$, constraint (4), as $\mathbf{A} \mathbf{p} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{1 \times 2}$ and $\mathbf{b} \in \mathbb{R}$.

(c) Solve the Quadratic Program (QP):

$$\min_{\mathbf{p}} \mathbf{p}^\top \mathbf{Q} \mathbf{p} \quad \text{s.t.} \quad \mathbf{A} \mathbf{p} = \mathbf{b}. \tag{6}$$

You can solve it by hand, or you can solve it using numerical QP solvers (e.g., you can easily use the `quadprog` function in Matlab). What is the optimal solution you get for $P(t)$, and what is the value of the cost function at the optimal solution? Are you able to get a lower cost by allowing $P(t)$ to have degree 2?

3. Now suppose we allow $P(t) = p_3 t^3 + p_2 t^2 + p_1 t$:

(a) Let $\mathbf{p} = [p_1, p_2, p_3]^\top$, write down $\mathbf{Q} \in \mathbb{S}^3$, $\mathbf{A} \in \mathbb{R}^{1 \times 3}$, and $b \in \mathbb{R}$ for the QP (6).

(b) Solve the QP. What optimal solution do you get? Does this example agree with the result we learned from the Euler-Lagrange equation in class?

4. Now suppose we are interested in adding one more constraint to problem (2):

$$\min_{P(t)} \int_0^1 (P^{(1)}(t))^2 dt, \tag{7}$$
$$\text{s.t.} \quad P(0) = 0, \quad P(1) = 1, \quad P^{(1)}(1) = -2.$$

Using the QP method above, find the optimal solution and optimal cost of problem (7) in the case of:

(a) $P(t) = p_2 t^2 + p_1 t$, and
(b) $P(t) = p_3 t^3 + p_2 t^2 + p_1 t$.

B. Deliverable - Multi-segment trajectory optimization

1. Assume our goal is to compute the minimum snap trajectory ($r = 4$) over k segments. How many and which type of constraints (at the intermediate points and at the start and end of the trajectory) do we need in order to solve this problem? Specify the number of waypoint constraints, free derivative constraints and fixed derivative constraints.
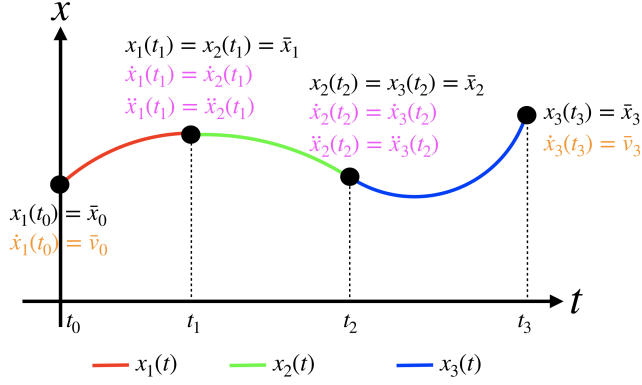
2. Can you extend the previous question to the case in which the cost functional minimizes the r-th derivative and we have k segments?

APPENDIX B
TEAM