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*Professors Gerber and Nesbitt were involved as consultants with the revisions incorporated in the second edition.

AUTHORS' INTRODUCTIONS AND GUIDE TO STUDY

Introduction to First Edition*

This text represents a first step in communicating the revolution in the actuarial profession that is taking place in this age of high-speed computers. During the short period of time since the invention of the microchip, actuaries have been freed from numerous constraints of primitive computing devices in designing and managing insurance systems. They are now able to focus more of their attention on creative solutions to society's demands for financial security.

To provide an educational basis for this focus, the major objectives of this work are to integrate life contingencies into a full risk theory framework and to demonstrate the wide variety of constructs that are then possible to build from basic models at the foundation of actuarial science. Actuarial science is ever evolving, and the procedures for model building in risk theory are at its forefront. Therefore, we examine the nature of models before proceeding with a more detailed discussion of the text.

Intellectual and physical models are constructed either to organize observations into a comprehensive and coherent theory or to enable us to simulate, in a laboratory or a computer system, the operation of the corresponding full-scale entity. Models are absolutely essential in science, engineering, and the management of large organizations. One must, however, always keep in mind the sharp distinction between a model and the reality it represents. A satisfactory model captures enough of reality to give insights into the successful operation of the system it represents.

The insurance models developed in this text have proved useful and have deepened our insights about insurance systems. Nevertheless, we need to always keep

*Chapter references and nomenclature have been changed to be in accord with the second edition. These changes are indicated by italics.

before us the idea that real insurance systems operate in an environment that is more complex and dynamic than the models studied here. Because models are only approximations of reality, the work of model building is never done; approximations can be improved and reality may shift. It is a continuing endeavor of any scientific discipline to revise and update its basic models. Actuarial science is no exception.

Actuarial science developed at a time when mathematical tools (probability and calculus, in particular), the necessary data (especially mortality data in the form of life tables), and the socially perceived need (to protect families and businesses from the financial consequences of untimely death) coexisted. The models constructed at the genesis of actuarial science are still useful. However, the general environment in which actuarial science exists continues to change, and it is necessary to periodically restate the fundamentals of actuarial science in response to these changes.

We illustrate this with three examples:

1. The insurance needs of modern societies are evolving, and, in response, new systems of employee benefits and social insurance have developed. New models for these systems have been needed and constructed.
2. Mathematics has also evolved, and some concepts that were not available for use in building the original foundations of actuarial science are now part of a general mathematics education. If actuarial science is to remain in the mainstream of the applied sciences, it is necessary to recast basic models in the language of contemporary mathematics.
3. Finally, as previously stated, the development of high-speed computing equipment has greatly increased the ability to manipulate complex models. This has far-reaching consequences for the degree of completeness that can be incorporated into actuarial models.

This work features models that are fundamental to the current practice of actuarial science. They are explored with tools acquired in the study of mathematics, in particular, undergraduate level calculus and probability. The proposition guiding Chapters *1–14* is that there is a set of basic models at the heart of actuarial science that should be studied by all students aspiring to practice within any of the various actuarial specialities. These models are constructed using only a limited number of ideas. We will find many relationships among those models that lead to a unity in the foundations of actuarial science. These basic models are followed, in Chapters *15–21*, by some more elaborate models particularly appropriate to life insurance and pensions.

While this book is intended to be comprehensive, it is not meant to be exhaustive. In order to avoid any misunderstanding, we will indicate the limitations of the text:

- Mathematical ideas that could unify and, in some cases, simplify the ideas presented, but which are not included in typical undergraduate courses, are not used. For example, moment generating functions, but not characteristic functions, are used in developments regarding probability distributions. Stieltjes integrals, which could be used in some cases to unify the presentation of discrete and continuous cases, are not used because of this basic decision on mathematical prerequisites.

- The chapters devoted to life insurance stress the randomness of the time at which a claim payment must be made. In the same chapters, the interest rates used to convert future payments to a present value are considered deterministic and are usually taken as constants. In view of the high volatility possible in interest rates, it is natural to ask why probability models for interest rates were not incorporated. Our answer is that the mathematics of life contingencies on a probabilistic foundation (except for interest) does not involve ideas beyond those covered in an undergraduate program. On the other hand, the modeling of interest rates requires ideas from economics and statistics that are not included in the prerequisites of this volume. In addition, there are some technical problems in building models to combine random interest and random time of claim that are in the process of being solved.
- Methods for estimating the parameters of basic actuarial models from observations are not covered. For example, the construction of life tables is not discussed.
- This is not a text on computing. The issues involved in optimizing the organization of input data and computation in actuarial models are not discussed. This is a rapidly changing area, seemingly best left for readers to resolve as they choose in light of their own resources.
- Many important actuarial problems created by long-term practice and insurance regulation are not discussed. This is true in sections treating topics such as premiums actually charged for life insurance policies, costs reported for pensions, restrictions on benefit provisions, and financial reporting as required by regulators.
- Ideas that lead to interesting puzzles, but which do not appear in basic actuarial models, are avoided. Average age at death problems for a stationary population do not appear for this reason.

This text has a number of features that distinguish it from previous fine textbooks on life contingencies. A number of these features represent decisions by the authors on material to be included and will be discussed under headings suggestive of the topics involved.

Probability Approach

As indicated earlier, the sharpest break between the approach taken here and that taken in earlier English language textbooks on actuarial mathematics is the much fuller use of a probabilistic approach in the treatment of the mathematics of life contingencies. Actuaries have usually written and spoken of applying probabilities in their models, but their results could be, and often were, obtained by a deterministic rate approach. In this work, the treatment of life contingencies is based on the assumption that time-until-death is a continuous-type random variable. This admits a rich field of random variable concepts such as distribution function, probability density function, expected value, variance, and moment generating function. This approach is timely, based on the availability of high-speed computers, and is called for, based on the observation that the economic role of life insurance and pensions can be best seen when the random value of time-until-death is stressed. Also, these probability ideas are now part of general education in mathematics, and a fuller realization thereof relates life contingencies to other fields of applied probability, for example, reliability theory in engineering.

Additionally, the deterministic rate approach is described for completeness and is a tool in some developments. However, the results obtained from using a deterministic model usually can be obtained as expected values in a probabilistic model.

Integration with Risk Theory

Risk theory is defined as the study of deviations of financial results from those expected and methods of avoiding inconvenient consequences from such deviations. The probabilistic approach to life contingencies makes it easy to incorporate long-term contracts into risk theory models and, in fact, makes life contingencies only a part, but a very important one, of risk theory. Ruin theory, another important part of risk theory, is included as it provides insight into one source, the insurance claims, of adverse long-term financial deviations. This source is the most unique aspect of models for insurance enterprises.

Utility Theory

This text contains topics on the economics of insurance. The goal is to provide a motivation, based on a normative theory of individual behavior in the face of uncertainty, for the study of insurance models. Although the models used are highly simplified, they lead to insights into the economic role of insurance, and to an appreciation of some of the issues that arise in making insurance decisions.

Consistent Assumptions

The assumption of a uniform distribution of deaths in each year of age is consistently used to evaluate actuarial functions at nonintegral ages. This eliminates some of the anomalies that have been observed when inconsistent assumptions are applied in situations involving high interest rates.

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Introduction to Second Edition

Actuarial science is not static. In the time since the publication of the first edition of *Actuarial Mathematics,* actuarial science has absorbed additional ideas from economics and the mathematical sciences. At the same time, computing and communications have become cheaper and faster, and this has helped to make feasible more complex actuarial models. During this period the financial risks that modern societies seek to manage have also altered as a result of the globalization of business, technological advances, and political shifts that have changed public policies.

It would be impossible to capture the full effect of all these changes in the revision of a basic textbook. Our objective is more modest, but we hope that it is realistic. This edition is a step in an ongoing process of adaptation designed to keep the fundamentals of actuarial science current with changing realities.

In the second edition, changes in notation and nomenclature appear in almost every section. There are also basic changes from the first edition that should be listed.

1. Commutation functions, a classic tool in actuarial calculations, are not used. This is in response to the declining advantages of these functions in an age when interest rates are often viewed as random variables, or as varying deterministically, and the probability distribution of time until decrement may depend on variables other than attained age. Starting in Chapter 3, exercises that illustrate actuarial calculations using recursion formulas that can be implemented with current software are introduced. It is logically necessary that the challenge of implementing tomorrow's software is left to the reader.
2. Utility theory is no longer confined to the first chapter. Examples are given that illustrate how utility theory can be employed to construct consistent models for premiums and reserves that differ from the conventional model that implicitly depends on linear utility of wealth.
3. In the first edition readers were seldom asked to consider more than the first and second moments of loss random variables. In this edition, following the intellectual path used earlier in physics and statistics, the distribution functions and probability density functions of loss variables are illustrated.
4. The basic material on reserves is now presented in two chapters. This facilitates a more complete development of the theory of reserves for general life insurances with varying premiums and benefits.
5. In recent years considerable actuarial research has been done on joint distributions for several future lifetime random variables where mutual independence is not assumed. This work influences the chapters on multiple life actuarial functions and multiple decrement theory.
6. There are potentially serious estimation and interpretation problems in multiple decrement theory when the random times until decrement for competing causes of decrement are not independent. Those problems are illustrated in the second edition.

7. The applications of multiple decrement theory have been consolidated. No attempt is made to illustrate in this basic textbook the variations in benefit formulas driven by rapid changes in pension practice and regulation.
8. The confluence of new research and computing capabilities has increased the use of recursive formulas in calculating the distribution of total losses derived from risk theory models. This development has influenced Chapter 12.
9. The material on pricing life insurance with death and withdrawal benefits and accounting for life insurance operations has been reorganized. Business and regulatory considerations have been concentrated in one chapter, and the foundations of accounting and provisions for expenses in an earlier chapter. The discussion of regulation has been limited to general issues and options for addressing these issues. No attempt has been made to present a definitive interpretation of regulation for any nation, province, or state.
10. The models for some insurance products that are no longer important in the market have been deleted. Models for new products, such as accelerated benefits for terminal illness or long-term care, are introduced.
11. The final chapter contains a brief introduction to simple models in which interest rates are random variables. In addition, ideas for managing interest rate risk are discussed. It is hoped that this chapter will provide a bridge to recent developments within the intersection of actuarial mathematics and financial economics.

As the project of writing this second edition ends, it is clear that a significant new development is under way. This new endeavor is centered on the creation of general models for managing the risks to individuals and organizations created by uncertain future cash flows when the uncertainty derives from any source. This blending of the actuarial/statistical approach to building models for financial security systems with the approach taken in financial economics is a worthy assignment for the next cohort of actuarial students.

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Guide to Study

The reader can consider this text as covering the two branches of risk theory. Individual risk theory views each policy as a unit and allows construction of a model for a group of policies by adding the financial results for the separate policies in the group. Collective risk theory uses a probabilistic model for total claims that avoids the step of adding the results for individual policies. This distinction is sometimes difficult to maintain in practice. The chapters, however, can be classified as illustrated below.

Individual Risk Theory	Collective Risk Theory
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 21	12, 13, 14, 19, 20

It is also possible to divide insurance models into those appropriate for short-term insurance, where investment income is not a significant factor, and long-term insurance, where investment income is important. The following classification scheme provides this division of chapters along with an additional division of long-term models between those for life insurance and those for pensions.

	Long-Term Insurances	
Short-Term Insurances	Life Insurance	Pensions
1, 2, 12, 13, 14	3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 21	9, 10, 11, 19, 20, 21

The selection of topics and their organization do not follow a traditional pattern. As stated previously, the new organization arose from the goal to first cover material considered basic for all actuarial students (Chapters 1–14) and then to include a more in-depth treatment of selected topics for students specializing in life insurance and pensions (Chapters 15–21).

The discussion in Chapter 1 is devoted to two ideas: that random events can disrupt the plans of decision makers and that insurance systems are designed to reduce the adverse financial effects of these events. To illustrate the latter, single insurance policies are discussed and convenient, if not necessarily realistic, distributions of the loss random variable are used. In subsequent chapters, more detailed models are constructed for use with insurance systems.

In Chapter 2, the individual risk model is developed, first in regard to single policies, then in regard to a portfolio of policies. In this model, a random variable, S, the total claims in a single period, is the sum of a fixed number of independent random variables, each of which is associated with a single policy. Each component of the sum S can take either the value 0 or a random claim amount in the course of a single period.

From the viewpoint of risk theory, the ideas developed in Chapters 3 through 11 can be seen as extending the ideas of Chapter 2. Instead of considering the potential claims in a short period from an individual policy, we consider loss variables that take into account the financial results of several periods. Since such random variables are no longer restricted to a short time period, they reflect the time value of money. For groups of individuals, we can then proceed, as in

Chapter 2, to use an approximation, such as the normal approximation, to make probability statements about the sum of the random variables that are associated with the individual members.

In Chapter 3, time-of-death is treated as a continuous random variable, and, after defining the probability density function, several features of the probability distribution are introduced and explored. In Chapters 4 and 5, life insurances and annuities are introduced, and the present values of the benefits are expressed as functions of the time-of-death. Several characteristics of the distributions of the present value of future benefits are examined. In Chapter 6, the equivalence principle is introduced and used to define and evaluate periodic benefit premiums. In Chapters 7 and 8, the prospective future loss on a contract already in force is investigated. The distribution of future loss is examined, and the benefit reserve is defined as the expected value of this loss. In Chapter 9, annuity and insurance contracts involving two lives are studied. (Discussion of more advanced multiple life theory is deferred until Chapter 18.) The discussion in Chapters 10 and 11 investigates a more realistic model in which several causes of decrement are possible. In Chapter 10, basic theory is examined, whereas in Chapter 11 the theory is applied to calculating actuarial present values for a variety of insurance and pension benefits.

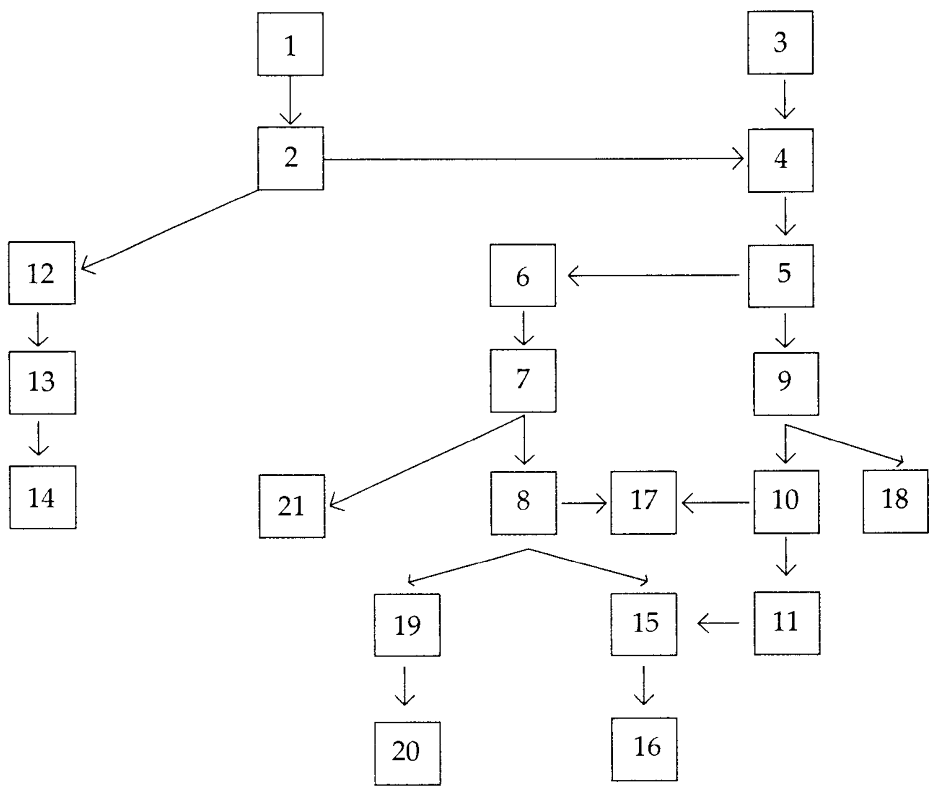
In Chapter 12, the collective risk model is developed with respect to single-period considerations of a portfolio of policies. The distribution of total claims for the period is developed by postulating the characteristics of the portfolio in the aggregate rather than as a sum of individual policies. In Chapter 13, these ideas are extended to a continuous-time model that can be used to study solvency requirements over a long time period. Applications of risk theory to insurance models are given an overview in Chapter 14.

Elaboration of the individual model to incorporate operational constraints such as acquisition and administrative expenses, accounting requirements, and the effects of contract terminations is treated in Chapters 15 and 16. In Chapter 17, individual risk theory models are used to obtain actuarial present values, benefit and contract premiums, and benefit reserves for selected special plans including life annuities with certain periods that depend on the contract premium, variable and flexible products, and accelerated benefits. In Chapter 18, the elementary models for plans involving two lives are extended to incorporate contingencies based on a larger number of lives and more complicated benefits.

In Chapter 19, concepts of population theory are introduced. These concepts are then applied to tracing the progress of life insurance benefits provided on a group, or population, basis. The tools from population theory are applied to tracing the progress of retirement income benefits provided on a group basis in Chapter 20.

Chapter 21 is a step into the future. Interest rates are assumed to be random variables. Several stochastic models are introduced and then integrated into models for basic insurance and annuity contracts.

The following diagram illustrates the prerequisite structure of the chapters. The arrows indicate the direction of the flow. For any chapter, the chapters that are upstream are prerequisite. For example, Chapter 6 has as prerequisites Chapters 1, 2, 3, 4, and 5.



We have a couple of hints for the reader, particularly for one for whom the material is new. The exercises are an important part of the text and include material not covered in the main discussion. In some cases, hints will be offered to aid in the solution. Answers to all exercises are provided except where the answer is given in the formulation of the problem. Writing computer programs and using electronic spreadsheets or mathematical software for the evaluation of basic formulas are excellent ways of enhancing the level of understanding of the material. The student is encouraged to use these tools to work through the computing exercises.

We conclude these introductory comments with some miscellaneous information on the format of the text. First, each chapter concludes with a reference section that provides guidance to those who wish to pursue further study of the topics covered in the chapter. These sections also contain comments that relate the ideas used in insurance models to those used in other areas.

Second, Chapters 1, 12, 13, 14, and 18 contain some theorems with their proofs included as chapter appendices. These proofs are included for completeness, but are not essential to an understanding of the material. They may be excluded from study at the reader's discretion. Exercises associated with these appendices should also be considered optional.

Third, general appendices appear at the end of the text. Included here are numerical tables for computations for examples and exercises, an index to notation, a discussion of general rules for writing actuarial symbols, reference citations, answers to exercises, a subject index, and supplemental mathematical formulas that are not assumed to be a part of the mathematical prerequisites.

Fourth, we observe two notational conventions. A referenced random variable, X, for example, is designated with a capital letter. This notational convention is not used in older texts on probability theory. It will be our practice, in order to indicate the correspondence, to use the appropriate random variable symbol as a subscript on functions and operators that depend on the random variable. We will use the general abbreviation ***log*** to refer to natural (base e) logarithms, because a distinction between natural and common logarithms is unnecessary in the examples and exercises. We assume the natural logarithm in our computations.

Fifth, currencies such as dollar, pound, lira, or yen are not specified in the examples and exercises due to the international character of the required computations.

Finally, since we have discussed prerequisites to this work, some major theorems from undergraduate calculus and probability theory will be used without review or restatement in the discussions and exercises.

1

THE ECONOMICS OF INSURANCE

1.1 Introduction

Each of us makes plans and has expectations about the path his or her life will follow. However, experience teaches that plans will not unfold with certainty and sometimes expectations will not be realized. Occasionally plans are frustrated because they are built on unrealistic assumptions. In other situations, fortuitous circumstances interfere. Insurance is designed to protect against serious financial reversals that result from random events intruding on the plans of individuals.

We should understand certain basic limitations on insurance protection. First, it is restricted to reducing those consequences of random events that can be measured in monetary terms. Other types of losses may be important, but not amenable to reduction through insurance.

For example, pain and suffering may be caused by a random event. However, insurance coverages designed to compensate for pain and suffering often have been troubled by the difficulty of measuring the loss in monetary units. On the other hand, economic losses can be caused by events such as property set on fire by its owner. Whereas the monetary terms of such losses may be easy to define, the events are not insurable because of the nonrandom nature of creating the losses.

A second basic limitation is that insurance does not directly reduce the probability of loss. The existence of windstorm insurance will not alter the probability of a destructive storm. However, a well-designed insurance system often provides financial incentives for loss prevention activities. An insurance product that encouraged the destruction of property or the withdrawal of a productive person from the labor force would affect the probability of these economically adverse events. Such insurance would not be in the public interest.

Several examples of situations where random events may cause financial losses are the following:

- The destruction of property by fire or storm is usually considered a random event in which the loss can be measured in monetary terms.

- A damage award imposed by a court as a result of a negligent act is often considered a random event with resulting monetary loss.
- Prolonged illness may strike at an unexpected time and result in financial losses. These losses will be due to extra health care expenses and reduced earned income.
- The death of a young adult may occur while long-term commitments to family or business remain unfulfilled. Or, if the individual survives to an advanced age, resources for meeting the costs of living may be depleted.

These examples are designed to illustrate the definition:

> An ***insurance system*** is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations.

It is helpful to make certain distinctions between insurance and related systems. Banking institutions were developed for the purpose of receiving, investing, and dispensing the savings of individuals and corporations. The cash flows in and out of a savings institution do not follow deterministic paths. However, unlike insurance systems, savings institutions do not make payments based on the size of a financial loss occurring from an event outside the control of the person suffering the loss.

Another system that does make payments based on the occurrence of random events is gambling. Gambling or wagering, however, stands in contrast to an insurance system in that an insurance system is designed to protect against the economic impact of risks that exist independently of, and are largely beyond the control of, the insured. The typical gambling arrangement is established by defining payoff rules about the occurrence of a contrived event, and the risk is voluntarily sought by the participants. Like insurance, a gambling arrangement typically redistributes wealth, but it is there that the similarity ends.

Our definition of an insurance system is purposefully broad. It encompasses systems that cover losses in both property and human-life values. It is intended to cover insurance systems based on individual decisions to participate as well as systems where participation is a condition of employment or residence. These ideas are discussed in Section 1.4.

The economic justification for an insurance system is that it contributes to general welfare by improving the prospect that plans will not be frustrated by random events. Such systems may also increase total production by encouraging individuals and corporations to embark on ventures where the possibility of large losses would inhibit such projects in the absence of insurance. The development of marine insurance, for reducing the financial impact of the perils of the sea, is an example of this point. Foreign trade permitted specialization and more efficient production, yet mutually advantageous trading activity might be too hazardous for some potential trading partners without an insurance system to cover possible losses at sea.

1.2 Utility Theory

If people could foretell the consequences of their decisions, their lives would be simpler but less interesting. We would all make decisions on the basis of preferences for certain consequences. However, we do not possess perfect foresight. At best, we can select an action that will lead to one set of uncertainties rather than another. An elaborate theory has been developed that provides insights into decision making in the face of uncertainty. This body of knowledge is called ***utility theory***. Because of its relevance to insurance systems, its main points will be outlined here.

One solution to the problem of decision making in the face of uncertainty is to define the value of an economic project with a random outcome to be its expected value. By this ***expected value principle*** the distribution of possible outcomes may be replaced for decision purposes by a single number, the expected value of the random monetary outcomes. By this principle, a decision maker would be indifferent between assuming the random loss X and paying amount $E[X]$ in order to be relieved of the possible loss. Similarly, a decision maker would be willing to pay up to $E[Y]$ to participate in a gamble with random payoff Y. In economics the expected value of random prospects with monetary payments is frequently called the ***fair*** or ***actuarial value*** of the prospect.

Many decision makers do not adopt the expected value principle. For them, their wealth level and other aspects of the distribution of outcomes influence their decisions.

Below is an illustration designed to show the inadequacy of the expected value principle for a decision maker considering the value of accident insurance. In all cases, it is assumed that the probability of an accident is 0.01 and the probability of no accident is 0.99. Three cases are considered according to the amount of loss arising from an accident; the expected loss is tabulated for each.

Case	Possible Losses		Expected Loss
1	0	1	0.01
2	0	1 000	10.00
3	0	1 000 000	10 000.00

A loss of 1 might be of little concern to the decision maker who then might be unwilling to pay more than the expected loss to obtain insurance. However, the loss of 1,000,000, which may exceed his net worth, could be catastrophic. In this case, the decision maker might well be willing to pay more than the expected loss of 10,000 in order to obtain insurance. The fact that the amount a decision maker would pay for protection against a random loss may differ from the expected value suggests that the expected value principle is inadequate to model behavior.

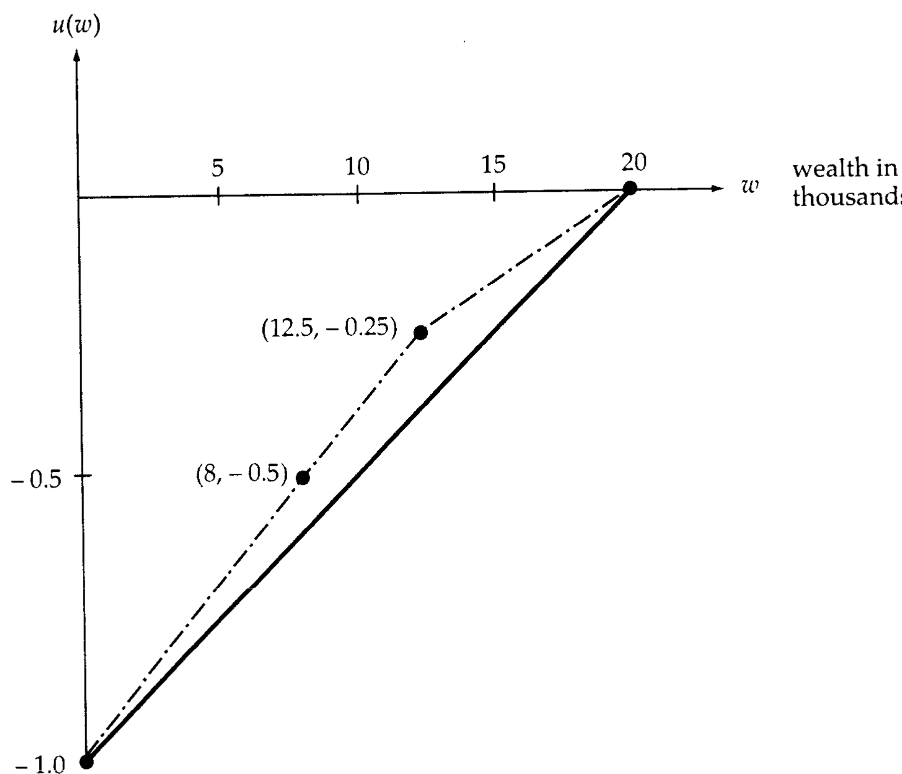
We now study another approach to explain why a decision maker may be willing to pay more than the expected value. At first we simply assume that the value or utility that a particular decision maker attaches to wealth of amount w, measured in monetary units, can be specified in the form of a function $u(w)$, called a ***utility function.*** We demonstrate a procedure by which a few values of such a function can be determined. For this we assume that our decision maker has wealth equal to 20,000. A linear transformation,

$$u^*(w) = a\, u(w) + b \qquad a > 0,$$

???

yields a function $u^*(w)$, which is essentially equivalent to $u(w)$. It then follows by choice of a and b that we can determine arbitrarily the 0 point and one additional point of an individual's utility function. Therefore, we fix $u(0) = -1$ and $u(20{,}000) = 0$. These values are plotted on the solid line in Figure 1.2.1.

Determination of a Utility Function



We now ask a question of our decision maker: Suppose you face a loss of 20,000 with probability 0.5, and will remain at your current level of wealth with probability 0.5. What is the maximum amount* G you would be willing to pay for

*Premium quantities, by convention in insurance literature, are capitalized although they are not random variables.

complete insurance protection against this random loss? We can express this question in the following way: For what value of G does

$$u(20{,}000 - G) = 0.5\,u(20{,}000) + 0.5\,u(0)$$
$$= (0.5)(0) + (0.5)(-1) = -0.5?$$

If he pays amount G, his wealth will certainly remain at $20{,}000 - G$. The equal sign indicates that the decision maker is indifferent between paying G with certainty and accepting the expected utility of wealth expressed on the right-hand side.

Suppose the decision maker's answer is $G = 12{,}000$. Therefore,

$$u(20{,}000 - 12{,}000) = u(8{,}000) = -0.5.$$

This result is plotted on the dashed line in Figure 1.2.1. Perhaps the most important aspect of the decision maker's response is that he is willing to pay an amount for insurance that is greater than

$$(0.5)(0) + (0.5)(20{,}000) = 10{,}000,$$

the expected value of the loss.

This procedure can be used to add as many points $[w, u(w)]$, for $0 \le w \le 20{,}000$, as needed to obtain a satisfactory approximation to the decision maker's utility of wealth function. Once a utility value has been assigned to wealth levels w_1 and w_2, where $0 \le w_1 < w_2 \le 20{,}000$, we can determine an additional point by asking the decision maker the following question: What is the maximum amount you would pay for complete insurance against a situation that could leave you with wealth w_2 with specified probability p, or at reduced wealth level w_1 with probability $1 - p$? We are asking the decision maker to fix a value G such that

$$u(w_2 - G) = (1 - p)u(w_1) + p\,u(w_2). \tag{1.2.1}$$

Once the value $w_2 - G = w_3$ is available, the point $[w_3, (1 - p)u(w_1) + p\,u(w_2)]$ is determined as another point of the utility function. Such a process has been used to assign a fourth point $(12{,}500, -0.25)$ in Figure 1.2.1. Such solicitation of preferences leads to a set of points on the decision maker's utility function. A smooth function with a second derivative may be fitted to these points to provide for a utility function everywhere.

After a decision maker has determined his utility of wealth function by the method outlined, the function can be used to compare two random economic prospects. The prospects are denoted by the random variables X and Y. We seek a decision rule that is consistent with the preferences already elicited in the determination of the utility of wealth function. Thus, if the decision maker has wealth w, and must compare the random prospects X and Y, the decision maker selects X if

$$\mathrm{E}[u(w + X)] > \mathrm{E}[u(w + Y)],$$

and the decision maker is indifferent between X and Y if

$$E[u(w + X)] = E[u(w + Y)].$$

Although the method of eliciting and using a utility function may seem plausible, it is clear that our informal development must be augmented by a more rigorous chain of reasoning if utility theory is to provide a coherent and comprehensive framework for decision making in the face of uncertainty. If we are to understand the economic role of insurance, such a framework is needed. An outline of this more rigorous theory follows.

The theory starts with the assumption that a rational decision maker, when faced with two distributions of outcomes affecting wealth, is able to express a preference for one of the distributions or indifference between them. Furthermore, the preferences must satisfy certain consistency requirements. The theory culminates in a theorem stating that if preferences satisfy the consistency requirements, there is a utility function $u(w)$ such that if the distribution of X is preferred to the distribution of Y, $E[u(X)] > E[u(Y)]$, and if the decision maker is indifferent between the two distributions, $E[u(X)] = E[u(Y)]$. That is, the qualitative preference or indifference relation may be replaced by a consistent numerical comparison. In Section 1.6, references are given for the detailed development of this theory.

Before turning to applications of utility theory for insights into insurance, we record some observations about utility.

Observations:

1. Utility theory is built on the assumed existence and consistency of preferences for probability distributions of outcomes. A utility function should reveal no surprises. It is a numerical description of existing preferences.
2. A utility function need not, in fact, cannot, be determined uniquely. For example, if

$$u^*(w) = a\, u(w) + b \qquad a > 0,$$

then

$$E[u(X)] > E[u(Y)]$$

is equivalent to

$$E[u^*(X)] > E[u^*(Y)].$$

That is, preferences are preserved when the utility function is an increasing linear transformation of the original form. This fact was used in the Figure 1.2.1 illustration where two points were chosen arbitrarily.

3. Suppose the utility function is linear with a positive slope; that is,

$$u(w) = a\,w + b \qquad a > 0.$$

Then, if $E[X] = \mu_X$ and $E[Y] = \mu_Y$, we have

$$E[u(X)] = a\,\mu_X + b > E[u(Y)] = a\,\mu_Y + b$$

if and only if $\mu_X > \mu_Y$. That is, for increasing linear utility functions, preferences for distributions of outcomes are in the same order as the expected values of the distributions being compared. Therefore, the expected value principle for rational economic behavior in the face of uncertainty is consistent with the expected utility rule when the utility function is an increasing linear one.

1.3 Insurance and Utility

In Section 1.2 we outlined utility theory for the purpose of gaining insights into the economic role of insurance. To examine this role we start with an illustration. Suppose a decision maker owns a property that may be damaged or destroyed in the next accounting period. The amount of the loss, which may be 0, is a random variable denoted by X. We assume that the distribution of X is known. Then $E[X]$, the expected loss in the next period, may be interpreted as the long-term average loss if the experiment of exposing the property to damage may be observed under identical conditions a great many times. It is clear that this long-term set of trials could not be performed by an individual decision maker.

Suppose that an insurance organization (***insurer***) was established to help reduce the financial consequences of the damage or destruction of property. The insurer would issue contracts (***policies***) that would promise to pay the owner of a property a defined amount equal to or less than the financial loss if the property were damaged or destroyed during the period of the policy. The contingent payment linked to the amount of the loss is called a ***claim*** payment. In return for the promise contained in the policy, the owner of the property (***insured***) pays a consideration (***premium***).

The amount of the premium payment is determined after an economic decision principle has been adopted by each of the insurer and insured. An opportunity exists for a mutually advantageous insurance policy when the premium for the policy set by the insurer is less than the maximum amount that the property owner is willing to pay for insurance.

Within the range of financial outcomes for an individual insurance policy, the insurer's utility function might be approximated by a straight line. In this case, the insurer would adopt the expected value principle in setting its premium, as indicated in Section 1.2, Observation 3; that is, the insurer would set its basic price for full insurance coverage as the expected loss, $E[X] = \mu$. In this context μ is called the ***pure*** or ***net premium*** for the 1-period insurance policy. To provide for expenses, taxes, and profit and for some security against adverse loss experience, the insurance system would decide to set the premium for the policy by ***loading,*** adding to, the pure premium. For instance, the loaded premium, denoted by H, might be given by

$$H = (1 + a)\mu + c \qquad a > 0, \qquad c > 0.$$

In this expression the quantity $a\mu$ can be viewed as being associated with expenses that vary with expected losses and with the risk that claims experience will deviate from expected. The constant c provides for expected expenses that do not vary with losses. Later, we will illustrate other economic principles for determining premiums that might be adopted by the insurer.

We now apply utility theory to the decision problems faced by the owner of the property subject to loss. The property owner has a utility of wealth function $u(w)$ where wealth w is measured in monetary terms. The owner faces a possible loss due to random events that may damage the property. The distribution of the random loss X is assumed to be known. Much as in (1.2.1), the owner will be indifferent between paying an amount G to the insurer, who will assume the random financial loss, and assuming the risk himself. This situation can be stated as

$$u(w - G) = \mathrm{E}[u(w - X)]. \tag{1.3.1}$$

The right-hand side of (1.3.1) represents the expected utility of not buying insurance when the owner's current wealth is w. The left-hand side of (1.3.1) represents the expected utility of paying G for complete financial protection.

If the owner has an increasing linear utility function, that is, $u(w) = bw + d$ with $b > 0$, the owner will be adopting the expected value principle. In this case the owner prefers, or is indifferent to, the insurance when

$$u(w - G) = b(w - G) + d \geq \mathrm{E}[u(w - X)] = \mathrm{E}[b(w - X) + d],$$

$$b(w - G) + d \geq b(w - \mu) + d,$$

$$G \leq \mu.$$

That is, if the owner has an increasing linear utility function, the premium payments that will make the owner prefer, or be indifferent to, complete insurance are less than or equal to the expected loss. In the absence of a subsidy, an insurer, over the long term, must charge more than its expected losses. Therefore, in this case, there seems to be little opportunity for a mutually advantageous insurance contract. If an insurance contract is to result, the insurer must charge a premium in excess of expected losses and expenses to avoid a bias toward insufficient income. The property owner then cannot use a linear utility function.

In Section 1.2 we mention that the preferences of a decision maker must satisfy certain consistency requirements to ensure the existence of a utility function. Although these requirements were not listed, they do not include any specifications that would force a utility function to be linear, quadratic, exponential, logarithmic, or any other particular form. In fact, each of these named functions might serve as a utility function for some decision maker or they might be spliced together to reflect some other decision maker's preferences.

Nevertheless, it seems natural to assume that $u(w)$ is an increasing function, "more is better." In addition, it has been observed that for many decision makers, each additional equal increment of wealth results in a smaller increment of associated utility. This is the idea of decreasing marginal utility in economics.

The approximate utility function of Figure 1.2.1 consists of straight line segments with positive slopes. It is such that $\Delta^2 u(w) \leq 0$. If these ideas are extended to smoother functions, the two properties suggested by observation are $u'(w) > 0$ and $u''(w) < 0$. The second inequality indicates that $u(w)$ is a strictly concave downward function.

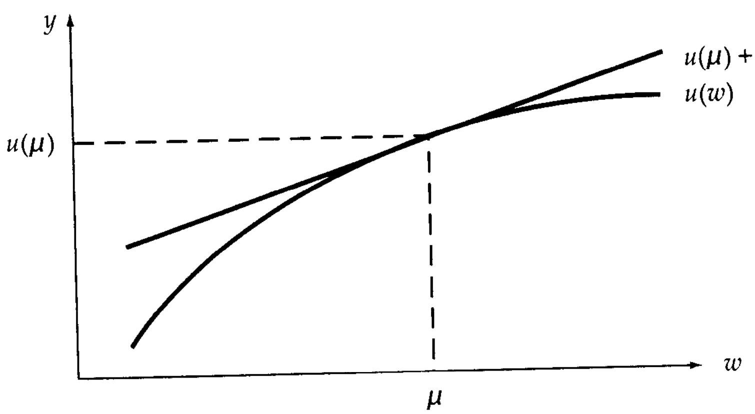
In discussing insurance decisions using strictly concave downward utility functions, we will make use of one form of ***Jensen's inequalities.*** These inequalities state that for a random variable X and function $u(w)$,

$$\text{if } u''(w) < 0, \text{ then } E[u(X)] \le u(E[X]), \tag{1.3.2}$$

$$\text{if } u''(w) > 0, \text{ then } E[u(X)] \ge u(E[X]). \tag{1.3.3}$$

Jensen's inequalities require the existence of the two expected values. Proofs of the inequalities are required by Exercise 1.3. A second proof of (1.3.2) is almost immediate from consideration of Figure 1.3.1 as follows.

Proof of Jensen's Inequalities for the Case $u'(w) > 0$ and $u''(w) < 0$



If $E[X] = \mu$ exists, one considers the line tangent to $u(w)$,

$$y = u(\mu) + u'(\mu)(w - \mu),$$

at the point $(\mu, u(\mu))$. Because of the strictly concave characteristic of $u(w)$, the graph of $u(w)$ will be below the tangent line; that is,

$$u(w) \le u(\mu) + u'(\mu)(w - \mu) \tag{1.3.4}$$

for all values of w. If w is replaced by the random variable X, and the expectation is taken on each side of the inequality (1.3.4), we have $E[u(X)] \le u(\mu)$.

This basic inequality has several applications in actuarial mathematics. Let us apply Jensen's inequality (1.3.2) to the decision maker's insurance problem as formulated in (1.3.1). We will assume that the decision maker's preferences are such that $u'(w) > 0$ and $u''(w) < 0$. Applying Jensen's inequality to (1.3.1), we have

$$u(w - G) = E[u(w - X)] \le u(w - \mu). \tag{1.3.5}$$

Because $u'(w) > 0$, $u(w)$ is an increasing function. Therefore, (1.3.5) implies that $w - G \leq w - \mu$, or $G \geq \mu$ with $G > \mu$ unless X is a constant. In economic terms, we have found that if $u'(w) > 0$ and $u''(w) < 0$, the decision maker will pay an amount greater than the expected loss for insurance. If G is at least equal to the premium set by the insurer, there is an opportunity for a mutually advantageous insurance policy.

Formally we say a decision maker with utility function $u(w)$ is ***risk averse*** if, and only if, $u''(w) < 0$.

We now employ a general utility function for the insurer. We let $u_I(w)$ denote the utility of wealth function of the insurer and w_I denote the current wealth of the insurer measured in monetary terms. Then the minimum acceptable premium H for assuming random loss X, from the viewpoint of the insurer, may be determined from (1.3.6):

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)]. \tag{1.3.6}$$

The left-hand side of (1.3.6) is the utility attached to the insurer's current position. The right-hand side is the expected utility associated with collecting premium H and paying random loss X. In other words, the insurer is indifferent between the current position and providing insurance for X at premium H. If the insurer's utility function is such that $u_I'(w) > 0$, $u_I''(w) < 0$, we can use Jensen's inequality (1.3.2) along with (1.3.6) to obtain

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)] \leq u_I(w_I + H - \mu).$$

Following the same line of reasoning displayed in connection with (1.3.5), we can conclude that $H \geq \mu$. If G, as determined by the decision maker by solving (1.3.5), is such that $G \geq H \geq \mu$, an insurance contract is possible. That is, the expected utility of neither party to the contract is decreased.

A utility function is based on the decision maker's preferences for various distributions of outcomes. An insurer need not be an individual. It may be a partnership, corporation, or government agency. In this situation the determination of $u_I(w)$, the insurer's utility function, may be a rather complicated matter. For example, if the insurer is a corporation, one of management's responsibilities is the formulation of a coherent set of preferences for various risky insurance ventures. These preferences may involve compromises between conflicting attitudes toward risk among the groups of stockholders.

Several elementary functions are used to illustrate properties of utility functions. Here we examine exponential, fractional power, and quadratic functions. Exercises 1.6, 1.8, 1.9, 1.10(b), and 1.13 cover the logarithmic utility function.

An ***exponential utility function*** is of the form

$$u(w) = -e^{-\alpha w} \qquad \text{for all } w \text{ and for a fixed } \alpha > 0$$

and has several attractive features. First,

$$u'(w) = \alpha e^{-\alpha w} > 0.$$

Second,

$$u''(w) = -\alpha^2 e^{-\alpha w} < 0.$$

Therefore, $u(w)$ may serve as the utility function of a risk-averse individual. Third, finding

$$\mathrm{E}[-e^{-\alpha X}] = -\mathrm{E}[e^{-\alpha X}] = -\mathrm{M}_X(-\alpha)$$

is essentially the same as finding the moment generating function (m.g.f.) of X. In this expression,

$$\mathrm{M}_X(t) = \mathrm{E}[e^{tX}]$$

denotes the m.g.f. of X. Fourth, insurance premiums do not depend on the wealth of the decision maker. This statement is verified for the insured by substituting the exponential utility function into (1.3.1). That is,

$$-e^{-\alpha(w-G)} = \mathrm{E}[-e^{-\alpha(w-X)}],$$

$$e^{\alpha G} = \mathrm{M}_X(\alpha),$$

$$G = \frac{\log \mathrm{M}_X(\alpha)}{\alpha}$$

and G does not depend on w.

The verification for the insurer is done by substituting the exponential utility function with parameter α_I into (1.3.6):

$$-e^{-\alpha_I w_I} = \mathrm{E}[-e^{-\alpha_I(w_I+H-X)}],$$

$$-e^{-\alpha_I w_I} = -e^{-\alpha_I(w_I+H)}\,\mathrm{M}_X(\alpha_I),$$

$$H = \frac{\log \mathrm{M}_X(\alpha_I)}{\alpha_I}.$$

Example 1.3.1

A decision maker's utility function is given by $u(w) = -e^{-5w}$. The decision maker has two random economic prospects (gains) available. The outcome of the first, denoted by X, has a normal distribution with mean 5 and variance 2. Henceforth, a statement about a normal distribution with mean μ and variance σ^2 will be abbreviated as $N(\mu, \sigma^2)$. The second prospect, denoted by Y, is distributed as $N(6, 2.5)$. Which prospect will be preferred?

Solution:

We have

$$\begin{aligned}\mathrm{E}[u(X)] &= \mathrm{E}[-e^{-5X}] \\ &= -\mathrm{M}_X(-5) = -e^{[-5(5)+(5^2)(2)/2]} \\ &= -1,\end{aligned}$$

and

$$\begin{aligned} E[u(Y)] &= E[-e^{-5Y}] \\ &= -M_Y(-5) = -e^{[-5(6)+(5^2)(2.5)/2]} \\ &= -e^{1.25}. \end{aligned}$$

Therefore,

$$E[u(X)] = -1 > E[u(Y)] = -e^{1.25},$$

and the distribution of X is preferred to the distribution of Y. ▼

In Example 1.3.1 prospect X is preferred to Y despite the fact that $\mu_X = 5 < \mu_Y = 6$. Since the decision maker is risk averse, the fact that the distribution of Y is more diffuse than the distribution of X is weighted heavily against the distribution of Y in assessing its desirability. If Y had a $N(6, 2.4)$ distribution, $E[u(Y)] = -1$ and the decision maker would be indifferent between the distributions of X and Y.

The family of ***fractional power utility*** functions is given by

$$u(w) = w^{\gamma} \qquad w > 0,\ 0 < \gamma < 1.$$

A member of this family might represent the preferences of a risk-averse decision maker since

$$u'(w) = \gamma w^{\gamma-1} > 0$$

and

$$u''(w) = \gamma(\gamma - 1)w^{\gamma-2} < 0.$$

In this family, premiums depend on the wealth of the decision maker in a manner that may be sufficiently realistic in many situations.

Example 1.3.2

A decision maker's utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of $w = 10$ and faces a random loss X with a uniform distribution on $(0, 10)$. What is the maximum amount this decision maker will pay for complete insurance against the random loss?

Solution:

Substituting into (1.3.1) we have

$$\begin{aligned} \sqrt{10 - G} &= E[\sqrt{10 - X}] \\ &= \int_0^{10} \sqrt{10 - x}\ 10^{-1}\, dx \\ &= \left. \frac{-2(10 - x)^{3/2}}{3(10)} \right|_0^{10} \\ &= \frac{2}{3}\sqrt{10}, \end{aligned}$$

$$G = 5.5556.$$

The decision maker is risk averse and has $u'(w) > 0$. Following the discussion of (1.3.5), we would expect $G > E[X]$, and in this example $G = 5.5556 > E[X] = 5$.▼

The family of ***quadratic utility*** functions is given by

$$u(w) = w - \alpha w^2 \qquad w < (2\alpha)^{-1}, \qquad \alpha > 0.$$

A member of this family might represent the preferences of a risk-averse decision maker since $u''(w) = -2\alpha$. While a quadratic utility function is convenient because decisions depend only on the first two moments of the distributions of outcomes under consideration, there are certain consequences of its use that strike some people as being unreasonable. Example 1.3.3 illustrates one of these consequences.

Example 1.3.3

A decision maker's utility of wealth function is given by

$$u(w) = w - 0.01w^2 \qquad w < 50.$$

The decision maker will retain wealth of amount w with probability p and suffer a financial loss of amount c with probability $1 - p$. For the values of w, c, and p exhibited in the table below, find the maximum insurance premium that the decision maker will pay for complete insurance. Assume $c \leq w < 50$.

Solution:

For the facts stated, (1.3.1) becomes

$$\begin{aligned} u(w - G) &= pu(w) + (1 - p)u(w - c), \\ (w - G) - 0.01(w - G)^2 &= p(w - 0.01w^2) \\ &\quad + (1 - p)[(w - c) - 0.01(w - c)^2]. \end{aligned}$$

For given values of w, p, and c this expression becomes a quadratic equation. Two solutions are shown.

Wealth w	Loss c	Probability p	Insurance Premium G
10	10	0.5	5.28
20	10	0.5	5.37

▼

In Example 1.3.3, as anticipated, G is greater than the expected loss of 5. However, the maximum insurance premium for exactly the same loss distribution increases with the wealth of the decision maker. This result seems unreasonable to some who anticipate that more typical behavior would be a decrease in the amount a decision maker would pay for insurance when an increase in wealth would permit the decision maker to absorb more of a random loss. Unfortunately, a maximum insurance premium that increases with wealth is a property of quadratic utility functions. Consequently, these utility functions should not be selected by a decision maker who perceives that his ability to absorb random losses goes up with increases in wealth.

If we rework Example 1.3.3 using an exponential utility function, we know that the premium G will not depend on w, the amount of wealth. In fact, if $u(w) = -e^{-0.01w}$, it can be shown that $G = 5.12$ for both $w = 10$ and $w = 20$.

Example 1.3.4

The probability that a property will not be damaged in the next period is 0.75. The probability density function (p.d.f.) of a positive loss is given by

$$f(x) = 0.25(0.01e^{-0.01x}) \qquad x > 0.$$

The owner of the property has a utility function given by

$$u(w) = -e^{-0.005w}.$$

Calculate the expected loss and the maximum insurance premium the property owner will pay for complete insurance.

Solution:

The expected loss is given by

$$\begin{aligned} \mathrm{E}[X] &= 0.75(0) + 0.25 \int_0^\infty x(0.01e^{-0.01x})dx \\ &= 25. \end{aligned}$$

We apply (1.3.1) to determine the maximum premium that the owner will pay for complete insurance. This premium will be consistent with the property owner's preferences as summarized in the utility function:

$$\begin{aligned} u(w - G) &= 0.75u(w) + \int_0^\infty u(w - x)f(x)dx, \\ -e^{-0.005(w-G)} &= -0.75e^{-0.005w} - 0.25 \int_0^\infty e^{-0.005(w-x)}(0.01e^{-0.01x})dx, \\ e^{0.005G} &= 0.75 + (0.25)(2) \\ &= 1.25, \\ G &= 200 \log 1.25 \\ &= 44.63. \end{aligned}$$

Therefore, in accord with the property owner's preferences, he will pay up to $44.63 - 25 = 19.63$ in excess of the expected loss to purchase insurance covering all losses in the next period. ▼

In Example 1.3.5 the notion of insurance that covers something less than the complete loss is introduced. A modification is made in (1.3.1) to accommodate the fact that losses are shared by the decision maker and the insurance system.

Example 1.3.5

The property owner in Example 1.3.4 is offered an insurance policy that will pay 1/2 of any loss during the next period. The expected value of the partial loss payment is $E[X/2] = 12.50$. Calculate the maximum premium that the property owner will pay for this insurance.

Solution:

Consistent with his attitude toward risk, as summarized in his utility function, the premium is determined from

$$0.75u(w - G) + \int_0^\infty u\left(w - G - \frac{x}{2}\right) f(x)dx$$
$$= 0.75u(w) + \int_0^\infty u(w - x)f(x)dx.$$

The left-hand side of this equation represents the expected utility with the partial insurance coverage. The right-hand side represents the expected utility with no insurance. For the exponential utility function and p.d.f. of losses specified in Example 1.3.4, it can be shown that $G = 28.62$. The property owner is willing to pay up to $G - \mu = 28.62 - 12.50 = 16.12$ more than the expected partial loss for the partial insurance coverage. ▼

1.4 Elements of Insurance

Individuals and organizations face the threat of financial loss due to random events. In Section 1.3 we saw how insurance can increase the expected utility of a decision maker facing such random losses. Insurance systems are unique in that the alleviation of financial losses in which the number, size, or time of occurrence is random is the primary reason for their existence. In this section we review some of the factors influencing the organization and management of an insurance system.

An insurance system can be organized only after the identification of a class of situations where random losses may occur. The word random is taken to mean, along with other attributes, that the frequency, size, or time of loss is not under the control of the prospective insured. If such control exists, or if a claim payment exceeds the actual financial loss, an incentive to incur a loss will exist. In such a situation, the assumptions under which the insurance system was organized will become invalid. The actual conditions under which premiums are collected and claims paid will be different from those assumed in organizing the system. The system will not achieve its intended objective of not decreasing the expected utilities of both the insured and the insurer.

Once a class of insurable situations is identified, information on the expected utilities and the loss-generating process can be obtained. Market research in insurance can be viewed as an effort to learn about the utility functions, that is, the risk preferences of consumers.

The processes generating size and time of loss may be sufficiently stable over time so that past information can be used to plan the system. When a new insurance system is organized, directly relevant statistics are not often available. However, enough ancillary information from similar risk situations may be obtained to identify the risks and to provide preliminary estimates of the probability distributions needed to determine premiums. Because most insurance systems operate under dynamic conditions, it is important that a plan exist for collecting and analyzing insurance operating data so that the insurance system can adapt. Adaptation in this case may mean changing premiums, paying an experience-based dividend or premium refund, or modifying future policies.

In a competitive economy, market forces encourage insurers to price short-term policies so that deviations of experience from expected value behave as independent random variables. Deviations should exhibit no pattern that might be exploited by the insured or insurer to produce consistent gains. Such consistent deviations would indicate inefficiencies in the insurance market.

As a result, the classification of risks into homogeneous groups is an important function within a market-based insurance system. Experience deviations that are random indicate efficiency or equity in classification. In a competitive insurance market, the continual interaction of numerous buyers and sellers forces experimentation with classification systems as the market participants attempt to take advantage of perceived patterns of deviations. Because insurance losses may be relatively rare events, it is often difficult to identify nonrandom patterns. The cost of classification information for a refined classification system also places a bound on experimentation in this area.

For insurance systems organized to serve groups rather than individuals, the issue is no longer whether deviations in insurance experience are random for each individual. Instead, the question is whether deviations in group experience are random. Consistent deviations in experience from that expected would indicate the need for a revision in the system.

Group insurance decisions do not rest on individual expected utility comparisons. Instead, group insurance plans are based on a collective decision on whether the system increases the total welfare of the group. Group health insurance providing benefits for the employees of a firm is an example.

1.5 Optimal Insurance

The ideas outlined in Sections 1.2, 1.3, and 1.4 have been used as the foundation of an elaborate theory for guiding insurance decision makers to actions consistent with their preferences. In this section we present one of the main results from this theory and review many of the ideas introduced so far.

A decision maker has wealth of amount w and faces a loss in the next period. This loss is a random variable X. The decision maker can buy an insurance contract that will pay $I(x)$ of the loss x. To avoid an incentive to incur the loss, we assume that all ***feasible*** insurance contracts are such that $0 \le I(x) \le x$. We make the simplifying assumption that all feasible insurance contracts with $\mathrm{E}[I(X)] = \beta$ can be purchased for the same amount P.

The decision maker has formulated a utility function $u(w)$ that is consistent with his preferences for distributions of outcomes. We assume that the decision maker is risk averse, $u''(w) < 0$. We further assume that the decision maker has decided on the amount, denoted by P, to be paid for insurance. The question is: which of the insurance contracts from the class of feasible contracts with expected claims, β, and premium, P, should be purchased to maximize the expected utility of the decision maker?

One subclass of the class of feasible insurance contracts is defined as follows:

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d. \end{cases} \tag{1.5.1}$$

This class of contracts is characterized by the fact that claim payments do not start until the loss exceeds the ***deductible*** amount d. For losses above the deductible amount, the excess is paid under the terms of the contract. This type of contract is sometimes called ***stop-loss*** or ***excess-of-loss insurance,*** the choice depending on the application.

In the problem discussed in this section the expected claims are denoted by β. In (1.5.2) the symbol $f(x)$ denotes the p.d.f. and the symbol $F(x)$ denotes the distribution function (d.f.) associated with the random loss X:

$$\beta = \int_d^{\infty} (x - d) f(x) dx \tag{1.5.2A}$$

or

$$\beta = \int_d^{\infty} [1 - F(x)] dx. \tag{1.5.2B}$$

Equation (1.5.2B) is obtained from (1.5.2A) by integration by parts. When β is given, then (1.5.2) provides explicit equations for the corresponding deductible, denoted by d^*. In Exercise 1.17, it is shown that d^* exists and is unique.

The main result of this section can be stated as a theorem.

Theorem 1.5.1

If a decision maker

- has wealth of amount w
- is risk averse; in other words, has utility of wealth function $u(w)$ such that $u''(w) < 0$
- faces a random loss X
- will spend amount P on insurance

and the insurance [illegible] offers for a payment of P all feasible [illegible] contracts of the form [illegible] then the [illegible] expected utility will be maximized by purchasing an insurance [illegible]

$$[illegible] \begin{cases} 0 & x < d^* \\ [illegible] & x \geq d^* \end{cases}$$

where [illegible]

$$P - \int_{d^*}^{\infty} (x - d^*) f(x) dx = 0.$$

The theorem is proved in the Appendix to this chapter.

Theorem 1.5.1 is an important result and illustrates many of the ideas developed in this chapter. However, it is instructive to consider certain limitations on its applicability. First, the ratio of premium to expected claims is the same for all available contracts. In fact, the distributions of the random variables $I(X)$ can be very different, and the provision for risk in the premium usually depends on the characteristics of the distribution of $I(X)$. Second, in Theorem 1.5.1, it is assumed that the premium P is fixed by a budget constraint. Alternatives to amount P are not considered. In Exercise 1.22, relaxation of the budget constraint is considered. Third, while the theorem indicates the form of insurance, it does not help to determine the amount P to spend. In the theorem, P is fixed.

1.6 Notes and References

Definitions and principles of actuarial science can be found in "Principles of Actuarial Science" (SOA Committee on Actuarial Principles 1992).

The role of risk in business was developed in a pioneering thesis by Willett (1951). Borch (1974) has published a series of papers applying utility theory to insurance questions. DeGroot (1970) gives a complete development of utility theory starting from basic axioms for consistency among preferences for various distributions of outcomes. DeGroot and Borch both discuss the historically important St. Petersburg paradox, outlined in Exercise 1.2. A paper by Friedman and Savage (1948) provides many insights into utility theory and human behavior.

Pratt (1964) has studied (1.3.1) and derived several theorems about premiums and utility functions. Exercise 1.10, which uses two rough approximations, is related to one of Pratt's results.

Theorem 1.5.1 on optimal insurance was proved by Arrow (1963) in the context of health insurance. The theorem in Exercise 1.21, in which the goal of insurance is to minimize the variance of retained losses, was the subject of papers by Borch (1960) and Kahn (1961). The use of the variance of losses as a measure of stability is discussed by Beard, Pentikäinen, and Pesonen (1984). Exercise 1.23 is based on their discussion.

Appendix

Lemma:
If $u''(w) < 0$ for all w in $[a, b]$, then for w and z in $[a, b]$,

$$u(w) - u(z) \leq (w - z)u'(z). \tag{1.A.1}$$

Proof:
The lemma may be established with the aid of Figure 1.3.1. Using the point slope form, a line tangent to $u(w)$ at the point $(z, u(z))$ has the equation $y - u(z) = u'(z)(w - z)$ and is above the graph of the function $u(w)$ except at the point $(z, u(z))$. Therefore,

$$u(w) - u(z) \leq u'(z)(w - z). \qquad \blacksquare$$

Figure 1.3.1 shows the case $u'(w) > 0$. The same argument holds for $u'(w) < 0$.

In Exercise 1.20 an alternative proof is required.

Proof of Theorem 1.5.1:
Let $I(x)$ be associated with an insurance policy satisfying the hypothesis of the theorem. Then from the lemma,

$$\begin{aligned} &u(w - x + I(x) - P) - u(w - x + I_{d^*}(x) - P) \\ &\leq [I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P). \end{aligned} \tag{1.A.2}$$

In addition, we claim

$$\begin{aligned} &[I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P) \\ &\leq [I(x) - I_{d^*}(x)]u'(w - d^* - P). \end{aligned} \tag{1.A.3}$$

To establish inequality (1.A.3), we must consider three cases:

Case I. $I_{d^*}(x) = I(x)$
In this case equality holds, (1.A.3) is 0 on both sides.

Case II. $I_{d^*}(x) > I(x)$
In this case $I_{d^*}(x) > 0$ and from (1.5.1), $x - I_{d^*}(x) = d^*$. Therefore, equality holds with each side of (1.A.3) equal to $[I(x) - I_{d^*}(x)]u'(w - d^* - P)$.

Case III. $I_{d^*}(x) < I(x)$
In this case $I(x) - I_{d^*}(x) > 0$. From (1.5.1) we obtain $I_{d^*}(x) - x \geq -d^*$ and $I_{d^*}(x) - x - P \geq -d^* - P$. Therefore,

$$u'(w - x + I_{d^*}(x) - P) \leq u'(w - d^* - P)$$

since the second derivative of $u(x)$ is negative and $u'(x)$ is a decreasing function.

Therefore, in each case

$$[I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P) \leq [I(x) - I_{d^*}(x)]u'(w - P - d^*),$$

establishing inequality (1.A.3).

Now, combining inequalities (1.A.2) and (1.A.3) and taking expectations, we have

$$\begin{aligned} &\mathrm{E}[u(w - X + I(X) - P)] - \mathrm{E}[u(w - X + I_{d^*}(X) - P)] \\ &\quad \leq \mathrm{E}[I(X) - I_{d^*}(X)]u'(w - d^* - P) = (\beta - \beta)u'(w - d^* - P) = 0. \end{aligned}$$

Therefore,

$$\mathrm{E}[u(w - X + I(X) - P)] \leq \mathrm{E}[u(w - X + I_{d^*}(X) - P)]$$

and the expected utility will be maximized by selecting $I_{d^*}(x)$, the stop-loss policy. ■

Exercises

Section 1.2

1.1. Assume that a decision maker's current wealth is 10,000. Assign $u(0) = -1$ and $u(10{,}000) = 0$.

a. When facing a loss of X with probability 0.5 and remaining at current wealth with probability 0.5, the decision maker would be willing to pay up to G for complete insurance. The values for X and G in three situations are given below.

X	G
10 000	6 000
6 000	3 300
3 300	1 700

Determine three values on the decision maker's utility of wealth function u.

b. Calculate the slopes of the four line segments joining the five points determined on the graph $u(w)$. Determine the rates of change of the slopes from segment to segment.

c. Put yourself in the role of a decision maker with wealth 10,000. In addition to the given values of $u(0)$ and $u(10{,}000)$, elicit three additional values on your utility of wealth function u.

d. On the basis of the five values of your utility function, calculate the slopes and the rates of change of the slopes as done in part (b).

1.2. **St. Petersburg paradox:** Consider a game of chance that consists of tossing a coin until a head appears. The probability of a head is 0.5 and the repeated trials are independent. Let the random variable N be the number of the trial on which the first head occurs.

a. Show that the probability function (p.f.) of N is given by

$$f(n) = \left(\frac{1}{2}\right)^n \qquad n = 1, 2, 3, \ldots.$$

b. Find $E[N]$ and $Var(N)$.

c. If a reward of $X = 2^N$ is paid, prove that the expectation of the reward does not exist.

d. If this reward has utility $u(w) = \log w$, find $E[u(X)]$.

Section 1.3

1.3. **Jensen's inequalities:**

a. Assume $u''(w) < 0$, $E[X] = \mu$, and $E[u(X)]$ exist; prove that $E[u(X)] \le u(\mu)$. [Hint: Express $u(w)$ as a series around the point $w = \mu$ and terminate the expansion with an error term involving the second derivative. Note that Jensen's inequalities do not require that $u'(w) > 0$.]

b. If $u''(w) > 0$, prove that $E[u(X)] \ge u(\mu)$.

c. Discuss Jensen's inequalities for the special case $u(w) = w^2$. What is

$$E[u(X)] - u(E[X])?$$

1.4. If a utility function is such that $u'(w) > 0$ and $u''(w) > 0$, use (1.3.1) to show $G \le \mu$. A decision maker with preferences consistent with $u''(w) > 0$ is a ***risk lover.***

1.5. Construct a geometric argument, based on a graph like that displayed in Figure 1.3.1, that if $u'(w) < 0$ and $u''(w) < 0$, then (1.3.4) follows.

1.6. Confirm that the utility function $u(w) = \log w$, $w > 0$, is the utility function of a decision maker who is risk averse for $w > 0$.

1.7. A utility function is given by

$$u(w) = \begin{cases} e^{-(w-100)^2/200} & w < 100 \\ 2 - e^{-(w-100)^2/200} & w \ge 100. \end{cases}$$

a. Is $u'(w) \ge 0$?

b. For what range of w is $u''(w) < 0$?

1.8. If one assumes, as did D. Bernoulli in his comments on the St. Petersburg paradox, that utility of wealth satisfies the differential equation

$$\frac{du(w)}{dw} = \frac{k}{w} \qquad w > 0, k > 0,$$

confirm that $u(w) = k \log w + c$.

1.9. A decision maker has utility function $u(w) = k \log w$. The decision maker has wealth w, $w > 1$, and faces a random loss X, which has a uniform distribution on the interval (0, 1). Use (1.3.1) to show that the maximum insurance premium that the decision maker will pay for complete insurance is

$$G = w - \frac{w^w}{e(w-1)^{w-1}}.$$

1.10. a. In (1.3.1) use the approximations

$$u(w-G) \cong u(w-\mu) + (\mu - G)u'(w-\mu),$$

$$u(w-x) \cong u(w-\mu) + (\mu - x)u'(w-\mu) + \frac{1}{2}(\mu - x)^2 u''(w-\mu)$$

and derive the following approximation for G:

$$G \cong \mu - \frac{1}{2}\frac{u''(w-\mu)}{u'(w-\mu)}\sigma^2.$$

b. If $u(w) = k \log w$, use the approximation developed in part (a) to obtain

$$G \cong \mu + \frac{1}{2}\frac{\sigma^2}{(w-\mu)}.$$

1.11. The decision maker has a utility function $u(w) = -e^{-\alpha w}$ and is faced with a random loss that has a chi-square distribution with n degrees of freedom. If $0 < \alpha < 1/2$, use (1.3.1) to obtain an expression for G, the maximum insurance premium the decision maker will pay, and prove that $G > n = \mu$.

1.12. Rework Example 1.3.4 for
a. $u(w) = -e^{-w/400}$
b. $u(w) = -e^{-w/150}$.

1.13. a. An insurer with net worth 100 has accepted (and collected the premium for) a risk X with the following probability distribution:

$$\Pr(X = 0) = \Pr(X = 51) = \frac{1}{2}.$$

What is the maximum amount G it should pay another insurer to accept 100% of this loss? Assume the first insurer's utility function of wealth is $u(w) = \log w$.

b. An insurer, with wealth 650 and the same utility function, $u(w) = \log w$, is considering accepting the above risk. What is the minimum amount H this insurer would accept as a premium to cover 100% of the loss?

1.14. If the complete insurance of Example 1.3.4 can be purchased for 40 and the 50% coinsurance of Example 1.3.5 can be purchased for 25, the purchase of which insurance maximizes the property owner's expected utility?

Section 1.4

1.15. A hospital expense policy is issued to a group consisting of n individuals. The policy pays B dollars each time a member of the group enters a hospital.

The group is not homogeneous with respect to the expected number of hospital admissions each year. The group may be divided into r subgroups. There are n_i individuals in subgroup i and $\Sigma_1^r\, n_i = n$. For subgroup i the number of annual hospital admissions for each member has a Poisson distribution with parameter λ_i, $i = 1, 2, \ldots, r$. The number of annual hospital admissions for members of the group are mutually independent.

a. Show that the expected claims payment in one year is

$$B \sum_1^r n_i\lambda_i = Bn\bar{\lambda}$$

where

$$\bar{\lambda} = \frac{\sum_1^r n_i\lambda_i}{n}.$$

b. Show that the number of hospital admissions in 1 year for the group has a Poisson distribution with parameter $n\,\bar{\lambda}$.

Section 1.5

1.16. Perform the integration by parts indicated in (1.5.2). Use the fact that if E[X] exists, if and only if, $\lim_{x\to\infty} x[1 - F(x)] = 0$.

1.17. a. Differentiate the right-hand side of (1.5.2B) with respect to d.
b. Let β be a number such that $0 < \beta < \text{E}[X]$. Show that (1.5.2) has a unique solution d^*.

1.18. Let the loss random variable X have a p.d.f. given by

$$f(x) = 0.1e^{-0.1x} \qquad x > 0.$$

a. Calculate E[X] and Var(X).
b. If $P = 5$ is to be spent for insurance to be purchased by the payment of the pure premium, show that

$$I(x) = \frac{x}{2}$$

and

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d, \text{ where } d = 10 \log 2, \end{cases}$$

both represent feasible insurance policies with pure premium $P = 5$. $I(x)$ is called ***proportional insurance.***

1.19. The loss random variable X has a p.d.f. given by

$$f(x) = \frac{1}{100} \qquad 0 < x < 100.$$

a. Calculate E[X] and Var(X).
b. Consider a proportional policy where

$$I(x) = kx \qquad 0 < k < 1,$$

and a stop-loss policy where

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d. \end{cases}$$

Determine k and d such that the pure premium in each case is $P = 12.5$.
c. Show that $\text{Var}[X - I(X)] > \text{Var}[X - I_d(x)]$.

Appendix

1.20. Establish the lemma by using an analytic rather than a geometric argument. [Hint: Expand $u(w)$ in a series as far as a second derivative remainder around the point z and subtract $u(z)$.]

1.21. Adopt the hypotheses of Theorem 1.5.1 with respect to β and insurance contracts $I(x)$ and assume $E[X] = \mu$. Prove that

$$\text{Var}[X - I(X)] = E[(X - I(X) - \mu + \beta)^2]$$

is a minimum when $I(x) = I_{d^*}(x)$. You will be proving that for a fixed pure premium, a stop-loss insurance contract will minimize the variance of retained claims. [Hint: we may follow the proof of Theorem 1.5.1 by first proving that $x^2 - z^2 \geq (x - z)(2z)$ and then establishing that

$$[x - I(x)]^2 - [x - I_{d^*}(x)]^2 \geq [I_{d^*}(x) - I(x)][2x - 2I_{d^*}(x)]$$
$$\geq 2[I_{d^*}(x) - I(x)]d^*.$$

The final inequality may be established by breaking the proof into three cases. Alternatively, by proper choice of wealth level and utility function, the result of this exercise is a special case of Theorem 1.5.1.]

1.22. Adopt the hypotheses of Theorem 1.5.1, except remove the budget constraint; that is, assume that the decision maker will pay premium P, $0 < P \leq E[X] = \mu$, that will maximize expected utility. In addition, assume that any feasible insurance can be purchased for its expected value. Prove that the optimal insurance is $I_0(x)$. This result can be summarized by stating that full coverage is optimal in the absence of a budget constraint if insurance can be purchased for its pure premium. [Hint: Use the lemma with the role of w played by $w - x + I(x) - P$ and that of z played by $w - x + I_0(x) - E[X] = w - \mu$. Take expectations and establish that $E[u(w - X) + I(X) - P] \leq u(w - \mu)$.]

1.23. Optimality properties of stop-loss insurance were established in Theorem 1.5.1 and Exercise 1.21. These results depended on the decision criteria, the constraints, and the insurance alternatives available. In each of these developments, there was a budget constraint. Consider the situation where there is a risk constraint and the price of insurance depends on the insurance risk as measured by the variance.

(i) The insurance premium is $E[I(X)] + f(\mathrm{Var}[I(X)])$, where $f(w)$ is an increasing function. The amount of $f(\mathrm{Var}[I(X)])$ can be interpreted as a ***security loading.***

(ii) The decision maker elects to retain loss $X - I(X)$ such that $\mathrm{Var}[X - I(X)] = V \geq 0$. This requirement imposes a risk rather than a budget constraint. The constant is determined by the degree of risk aversion of the decision maker. Fixing the accepted variance, and then optimizing expected results, is a decision criterion in investment portfolio theory.

(iii) The decision maker selects $I(x)$ to minimize $f(\mathrm{Var}[I(X)])$. The objective is to minimize the security loading, the premium paid less the expected insurance payments. Confirm the following steps:

a. $\mathrm{Var}[I(X)] = V + \mathrm{Var}(X) - 2\,\mathrm{Cov}[X, X - I(X)]$.

b. The $I(x)$ that minimizes $\mathrm{Var}[I(X)]$ and thereby $f(\mathrm{Var}[I(X)])$ is such that the correlation coefficient between X and $X - I(x)$ is 1.

c. It is known that if two random variables W and Z have correlation coefficient 1, then $\Pr\{W = aZ + b, \text{ where } a > 0\} = 1$. In words, the probability of their joint distribution is concentrated on a line of positive slope. In part (b), the correlation coefficient of X and $X - I(X)$ was found to be 1. Thus, $X - I(X) = aX + b$, which implies that $I(X) = (1 - a)X - b$. To be a feasible insurance, $0 \leq I(x) \leq x$ or $0 \leq (1 - a)x - b \leq x$. These inequalities imply that $b = 0$ and $0 \leq 1 - a \leq 1$ and $0 \leq a \leq 1$.

d. To determine a, set the correlation coefficient of X and $X - I(X)$ equal to 1, or equivalently, their covariance equal to the product of their standard deviations. Thus, show that $a = \sqrt{V / \mathrm{Var}(X)}$ and thus that the insurance that minimizes $f(\mathrm{Var}[X])$ is $I(X) = [1 - \sqrt{V / \mathrm{Var}(X)}]X$.

2

INDIVIDUAL RISK MODELS FOR A SHORT TERM

2.1 Introduction

In Chapter 1 we examined how a decision maker can use insurance to reduce the adverse financial impact of some types of random events. That examination was quite general. The decision maker could have been an individual seeking protection against the loss of property, savings, or income. The decision maker could have been an organization seeking protection against those same types of losses. In fact, the organization could have been an insurance company seeking protection against the loss of funds due to excess claims either by an individual or by its portfolio of insureds. Such protection is called ***reinsurance*** and is introduced in this chapter.

The theory in Chapter 1 requires a probabilistic model for the potential losses. Here we examine one of two models commonly used in insurance pricing, reserving, and reinsurance applications.

For an insuring organization, let the random loss of a segment of its risks be denoted by S. Then S is the random variable for which we seek a probability distribution. Historically, there have been two sets of postulates for distributions of S. The ***individual risk model*** defines

$$S = X_1 + X_2 + \cdots + X_n \tag{2.1.1}$$

where X_i is the loss on insured unit i and n is the number of risk units insured. Usually the X_i's are postulated to be independent random variables, because the mathematics is easier and no historical data on the dependence relationship are needed. The other model is the collective risk model described in Chapter 12.

The individual risk model in this chapter does not recognize the time value of money. This is for simplicity and is why the title refers to short terms. Chapters 4–11 cover models for long terms.

In this chapter we discuss only ***closed models;*** that is, the number of insured units n in (2.1.1) is known and fixed at the beginning of the period. If we postulate about migration in and out of the insurance system, we have an ***open model.***

2.2 Models for Individual Claim Random Variables

First, we review basic concepts with a life insurance product. In a ***one-year term life insurance*** the insurer agrees to pay an amount b if the insured dies within a year of policy issue and to pay nothing if the insured survives the year. The probability of a claim during the year is denoted by q. The claim random variable, X, has a distribution that can be described by either its probability function, p.f., or its distribution function, d.f. The p.f. is

$$f_X(x) = \Pr(X = x) = \begin{cases} 1 - q & x = 0 \\ q & x = b \\ 0 & \text{elsewhere,} \end{cases} \tag{2.2.1}$$

and the d.f. is

$$F_X(x) = \Pr(X \le x) = \begin{cases} 0 & x < 0 \\ 1 - q & 0 \le x < b \\ 1 & x \ge b. \end{cases} \tag{2.2.2}$$

From the p.f. and the definition of moments,

$$\begin{aligned} \mathrm{E}[X] &= bq, \\ \mathrm{E}[X^2] &= b^2 q, \end{aligned} \tag{2.2.3}$$

and

$$\mathrm{Var}(X) = b^2 q(1 - q). \tag{2.2.4}$$

These formulas can also be obtained by writing

$$X = Ib \tag{2.2.5}$$

where b is the constant amount payable in the event of death and I is the random variable that is 1 for the event of death and 0 otherwise. Thus, $\Pr(I = 0) = 1 - q$ and $\Pr(I = 1) = q$, the mean and variance of I are q and $q(1 - q)$, respectively, and the mean and variance of X are bq and $b^2q(1 - q)$ as above.

The random variable I with its $\{0, 1\}$ range is widely applicable in actuarial models. In probability textbooks it is called an ***indicator, Bernoulli random variable,*** or ***binomial random variable*** for a single trial. We refer to it as an indicator for the sake of brevity and because it indicates the occurrence, $I = 1$, or nonoccurrence, $I = 0$, of a given event.

We now seek more general models in which the amount of claim is also a random variable and several claims can occur in a period. Health, automobile, and other property and liability coverages provide immediate examples. Extending (2.2.5), we postulate that

$$X = IB \tag{2.2.6}$$

where X is the claim random variable for the period, B gives the total claim amount incurred during the period, and I is the indicator for the event that at least one claim has occurred. As the indicator for this event, I reports the occurrence ($I = 1$) or nonoccurrence ($I = 0$) of claims in this period and not the number of claims in the period. $\Pr(I = 1)$ is still denoted by q.

Let us look at several situations and determine the distributions of I and B for a model. First, consider a 1-year term life insurance paying an extra benefit in case of accidental death. To be specific, if death is accidental, the benefit amount is 50,000. For other causes of death, the benefit amount is 25,000. Assume that for the age, health, and occupation of a specific individual, the probability of an accidental death within the year is 0.0005, while the probability of a nonaccidental death is 0.0020. More succinctly,

$$\Pr(I = 1 \text{ and } B = 50{,}000) = 0.0005$$

and

$$\Pr(I = 1 \text{ and } B = 25{,}000) = 0.0020.$$

Summing over the possible values of B, we have

$$\Pr(I = 1) = 0.0025,$$

and then

$$\Pr(I = 0) = 1 - \Pr(I = 1) = 0.9975.$$

The conditional distribution of B, given $I = 1$, is

$$\Pr(B = 25{,}000|I = 1) = \frac{\Pr(B = 25{,}000 \text{ and } I = 1)}{\Pr(I = 1)} = \frac{0.0020}{0.0025} = 0.8,$$

$$\Pr(B = 50{,}000|I = 1) = \frac{\Pr(B = 50{,}000 \text{ and } I = 1)}{\Pr(I = 1)} = \frac{0.0005}{0.0025} = 0.2.$$

Let us now consider an automobile insurance providing collision coverage (this indemnifies the owner for collision damage to his car) above a 250 deductible up to a maximum claim of 2,000. For illustrative purposes, assume that for a particular individual the probability of one claim in a period is 0.15 and the chance of more than one claim is 0:

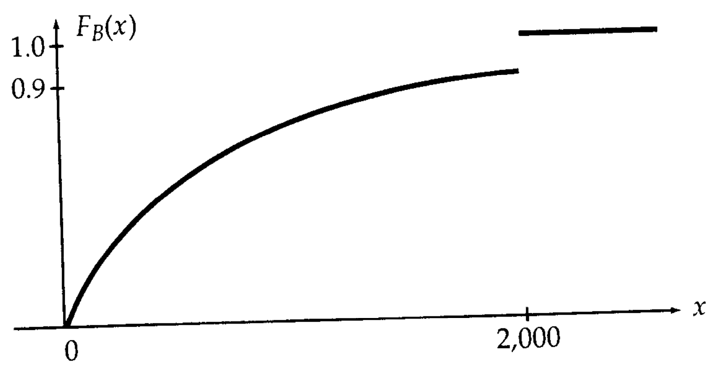
$$\Pr(I = 0) = 0.85,$$

$$\Pr(I = 1) = 0.15.$$

This unrealistic assumption of no more than one claim per period is made to simplify the distribution of B. We remove that assumption in a later section after we discuss the distribution of the sum of a number of claims. Since B is the claim incurred by the insurer, rather than the amount of damage to the car, we can infer two characteristics of I and B. First, the event $I = 0$ includes those collisions in which the damage is less than the 250 deductible. The other inference is that B's distribution has a probability mass at the maximum claim size of 2,000. Assume this probability mass is 0.1. Furthermore, assume that claim amounts between 0 and 2,000 can be modeled by a continuous distribution with a p.d.f. proportional to $1 - x/2{,}000$ for $0 < x < 2{,}000$. (In practice the continuous curve chosen to represent the distribution of claims is the result of a study of claims by size over a recent period.) Summarizing these assumptions about the conditional distribution of B, given $I = 1$, we have a mixed distribution with positive density from 0 to 2,000 and a mass at 2,000. This is illustrated in Figure 2.2.1. The d.f. of this conditional distribution is

$$\Pr(B \le x | I = 1) = \begin{cases} 0 & x \le 0 \\ 0.9 \left[1 - \left(1 - \dfrac{x}{2{,}000}\right)^2\right] & 0 < x < 2{,}000 \\ 1 & x \ge 2{,}000. \end{cases}$$

Distribution Function for B, given $I = 1$



We see in Section 2.4 that the moments of the claim random variable, X, in particular the mean and variance, are extensively used. For this automobile insurance, we shall calculate the mean and the variance by two methods. First, we derive the distribution of X and use it to calculate E[X] and Var(X). Letting $F_X(x)$ be the d.f. of X, we have

$$\begin{aligned} F_X(x) &= \Pr(X \le x) = \Pr(IB \le x) \\ &= \Pr(IB \le x | I = 0) \Pr(I = 0) \\ &\quad + \Pr(IB \le x | I = 1) \Pr(I = 1). \end{aligned} \tag{2.2.7}$$

For $x < 0$,

$$F_X(x) = 0(0.85) + 0(0.15) = 0.$$

For $0 \le x < 2{,}000$,

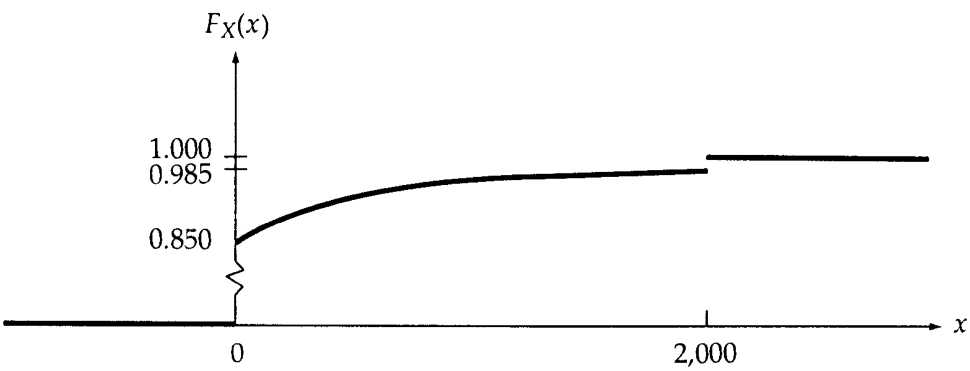
$$F_X(x) = 1(0.85) + 0.9 \left[1 - \left(1 - \frac{x}{2{,}000}\right)^2\right] (0.15).$$

For $x \geq 2{,}000$,

$$F_X(x) = 1(0.85) + 1(0.15) = 1.$$

This is a mixed distribution. It has both probability masses and a continuous part as can be seen in its graph in Figure 2.2.2.

Distribution Function of $X = IB$



Corresponding to this d.f. is a combination p.f. and p.d.f. given by

$$\Pr(X = 0) = 0.85,$$

$$\Pr(X = 2{,}000) = 0.015 \tag{2.2.8}$$

with p.d.f.

$$f_X(x) = \begin{cases} F'_X(x) = 0.000135\left(1 - \dfrac{x}{2{,}000}\right) & 0 < x < 2{,}000 \\ 0 & \text{elsewhere.} \end{cases}$$

Moments of X can then be calculated by

$$\mathrm{E}[X^k] = 0 \times \Pr(X = 0) + (2{,}000)^k \times \Pr(X = 2{,}000) + \int_0^{2{,}000} x^k f_X(x)\,dx, \tag{2.2.9}$$

specifically,

$$\mathrm{E}[X] = 120$$

and

$$\mathrm{E}[X^2] = 150{,}000.$$

Thus,

$$\mathrm{Var}(X) = 135{,}600.$$

There are some formulas relating the moments of random variables to certain conditional expectations. General versions of these formulas for the mean and variance are

$$\mathrm{E}[W] = \mathrm{E}[\mathrm{E}[W|V]] \tag{2.2.10}$$

and

$$\mathrm{Var}(W) = \mathrm{Var}(\mathrm{E}[W|V]) + \mathrm{E}[\mathrm{Var}(W|V)]. \tag{2.2.11}$$

In these equations we think of calculating the terms of the left-hand sides by direct use of W's distribution. In the terms on the right-hand sides, $\mathrm{E}[W|V]$ and $\mathrm{Var}(W|V)$ are calculated by use of W's conditional distribution for a given value of V. These components are then functions of the random variable V, and we can calculate their moments by use of V's distribution.

In many actuarial models conditional distributions are used. This makes the formulas above directly applicable. In our model, $X = IB$, we can substitute X for W and I for V to obtain

$$\mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|I]] \tag{2.2.12}$$

and

$$\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}[X|I]) + \mathrm{E}[\mathrm{Var}(X|I)]. \tag{2.2.13}$$

Now let us write

$$\mu = \mathrm{E}[B|I = 1], \tag{2.2.14}$$

$$\sigma^2 = \mathrm{Var}(B|I = 1), \tag{2.2.15}$$

and look at the conditional means

$$\mathrm{E}[X|I = 0] = 0 \tag{2.2.16}$$

and

$$\mathrm{E}[X|I = 1] = \mathrm{E}[B|I = 1] = \mu. \tag{2.2.17}$$

Formulas (2.2.16) and (2.2.17) define $\mathrm{E}[X|I]$ as a function of I, which can be written by the formula

$$\mathrm{E}[X|I] = \mu I. \tag{2.2.18}$$

Hence,

$$\mathrm{E}[\mathrm{E}[X|I]] = \mu \mathrm{E}[I] = \mu q \tag{2.2.19}$$

and

$$\mathrm{Var}(\mathrm{E}[X|I]) = \mu^2\, \mathrm{Var}(I) = \mu^2 q(1 - q). \tag{2.2.20}$$

Since $X = 0$ for $I = 0$, we have

$$\mathrm{Var}(X|I = 0) = 0. \tag{2.2.21}$$

For $I = 1$, we have $X = B$ and

$$\mathrm{Var}(X|I = 1) = \mathrm{Var}(B|I = 1) = \sigma^2. \tag{2.2.22}$$

Formulas (2.2.21) and (2.2.22) can be combined as

$$\mathrm{Var}(X|I) = \sigma^2 I. \tag{2.2.23}$$

Then

$$\mathrm{E}[\mathrm{Var}(X|I)] = \sigma^2 \mathrm{E}[I] = \sigma^2 q. \tag{2.2.24}$$

Substituting (2.2.19), (2.2.20), and (2.2.24) into (2.2.12) and (2.2.13), we have

$$\mathrm{E}[X] = \mu q \tag{2.2.25}$$

and

$$\mathrm{Var}(X) = \mu^2 q(1 - q) + \sigma^2 q. \tag{2.2.26}$$

Let us now apply these formulas to calculate $\mathrm{E}[X]$ and $\mathrm{Var}(X)$ for the automobile insurance in Figure 2.2.2. Since the p.d.f. for B, given $I = 1$, is

$$f_{B|I}(x|1) = \begin{cases} 0.0009\left(1 - \dfrac{x}{2{,}000}\right) & 0 < x < 2{,}000 \\ 0 & \text{elsewhere,} \end{cases}$$

with $\Pr(B = 2{,}000|I = 1) = 0.1$, we have

$$\mu = \int_0^{2{,}000} 0.0009\, x \left(1 - \frac{x}{2{,}000}\right) dx + (0.1)(2{,}000) = 800,$$

$$\mathrm{E}[B^2|I = 1] = \int_0^{2{,}000} 0.0009\, x^2 \left(1 - \frac{x}{2{,}000}\right) dx + (0.1)(2{,}000)^2 = 1{,}000{,}000,$$

and

$$\sigma^2 = 1{,}000{,}000 - (800)^2 = 360{,}000.$$

Finally, with $q = 0.15$ we obtain the following from (2.2.25) and (2.2.26):

$$\mathrm{E}[X] = 800(0.15) = 120$$

and

$$\begin{aligned} \mathrm{Var}(X) &= (800)^2(0.15)(0.85) + (360{,}000)(0.15) \\ &= 135{,}600. \end{aligned}$$

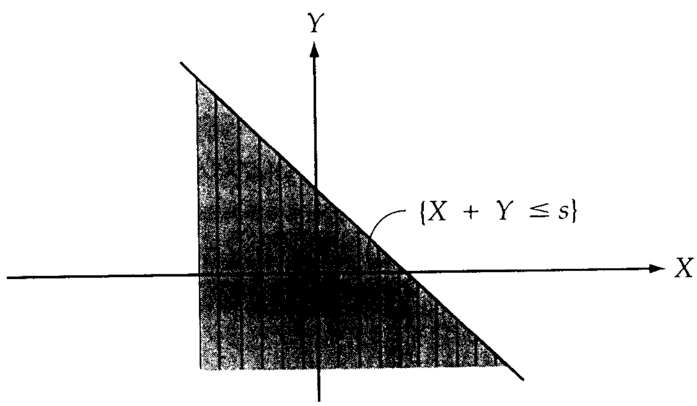
There are other possible models for B in different insurance situations. As an example, let us consider a model for the number of deaths due to crashes during an airline's year of operation. We can start with a random variable for the number of deaths, X, on a single flight and then add up a set of such random variables over the set of flights for the year. For a single flight, the event $I = 1$ will be the event of an accident during the flight. The number of deaths in the accident, B, will be modeled as the product of two random variables, L and Q, where L is the load factor, the number of persons on board at the time of the crash, and Q is the fraction of deaths among persons on board. The number of deaths B is modeled in this way since separate statistical data for the distributions of L and Q may be more readily available than are total data for B. We have $X = ILQ$. While the fraction of passengers killed in a crash and the fraction of seats occupied are probably related, L and Q might be assumed to be independent as a first approximation.

2.3 Sums of Independent Random Variables

In the individual risk model, claims of an insuring organization are modeled as the sum of the claims of many insured individuals.

The claims for the individuals are assumed to be independent in most applications. In this section we review two methods for determining the distribution of the sum of independent random variables. First, let us consider the sum of two random variables, $S = X + Y$, with the sample space shown in Figure 2.3.1.

Event $\{X + Y \leq s\}$



The line $X + Y = s$ and the region below the line represent the event

$$[S \leq X + Y \leq s].$$

Hence the d.f. of S is

$$F_S(s) = \Pr(S \leq s) = \Pr(X + Y \leq s). \tag{2.3.1}$$

For two discrete, non-negative random variables, we can use the law of total probability to write (2.3.1) as

$$\begin{aligned} F_S(s) &= \sum_{\text{all } y \leq s} \Pr(X + Y \leq s | Y = y) \Pr(Y = y) \\ &= \sum_{\text{all } y \leq s} \Pr(X \leq s - y | Y = y) \Pr(Y = y). \end{aligned} \tag{2.3.2}$$

When X and Y are independent, this last sum can be written

$$F_S(s) = \sum_{\text{all } y \leq s} F_X(s - y) f_Y(y). \tag{2.3.3}$$

The p.f. corresponding to this d.f. can be calculated by

$$f_S(s) = \sum_{\text{all } y \leq s} f_X(s - y) f_Y(y). \tag{2.3.4}$$

For continuous, non-negative random variables the formulas corresponding to (2.3.2), (2.3.3), and (2.3.4) are

$$F_S(s) = \int_0^s \Pr(X \le s - y | Y = y) f_Y(y)\, dy, \tag{2.3.5}$$

$$F_S(s) = \int_0^s F_X(s - y) f_Y(y)\, dy, \tag{2.3.6}$$

$$f_S(s) = \int_0^s f_X(s - y) f_Y(y)\, dy. \tag{2.3.7}$$

When either one, or both, of X and Y have a mixed-type distribution (typical in individual risk model applications), the formulas are analogous but more complex. For random variables that may also take on negative values, the sums and integrals in the formulas above are over all y values from $-\infty$ to $+\infty$.

In probability, the operation in (2.3.3) and (2.3.6) is called the ***convolution*** of the pair of distribution functions $F_X(x)$ and $F_Y(y)$ and is denoted by $F_X * F_Y$. Convolutions can also be defined for a pair of probability functions or probability density functions as in (2.3.4) and (2.3.7).

To determine the distribution of the sum of more than two random variables, we can use the convolution process iteratively. For $S = X_1 + X_2 + \cdots + X_n$ where the X_i's are independent random variables, F_i is the d.f. of X_i, and $F^{(k)}$ is the d.f. of $X_1 + X_2 + \cdots + X_k$, we proceed thus:

$$F^{(2)} = F_2 * F^{(1)} = F_2 * F_1$$

$$F^{(3)} = F_3 * F^{(2)}$$

$$F^{(4)} = F_4 * F^{(3)}$$

$$\vdots$$

$$F^{(n)} = F_n * F^{(n-1)}.$$

Example 2.3.1 illustrates the procedure using probability functions for three discrete random variables.

Example 2.3.1

The random variables X_1, X_2, and X_3 are independent with distributions defined by columns (1), (2), and (3) of the table below. Derive the p.f. and d.f. of

$$S = X_1 + X_2 + X_3.$$

Solution:

The notation of the previous paragraph is used in the table:

- Columns (1)–(3) are given information.
- Column (4) is derived from columns (1) and (2) by use of (2.3.4).
- Column (5) is derived from columns (3) and (4) by use of (2.3.4).

The determination of column (5) completes the determination of the distribution of S. Its d.f. in column (8) is the set of partial sums of column (5) from the top.

x	(1) $f_1(x)$	(2) $f_2(x)$	(3) $f_3(x)$	(4) $f^{(2)}(x)$	(5) $f^{(3)}(x)$	(6) $F_1(x)$	(7) $F^{(2)}(x)$	(8) $F^{(3)}(x)$
0	0.4	0.5	0.6	0.20	0.120	0.4	0.20	0.120
1	0.3	0.2	0.0	0.23	0.138	0.7	0.43	0.258
2	0.2	0.1	0.1	0.20	0.140	0.9	0.63	0.398
3	0.1	0.1	0.1	0.16	0.139	1.0	0.79	0.537
4		0.1	0.1	0.11	0.129	1.0	0.90	0.666
5			0.1	0.06	0.115	1.0	0.96	0.781
6				0.03	0.088	1.0	0.99	0.869
7				0.01	0.059	1.0	1.00	0.928
8					0.036	1.0	1.00	0.964
9					0.021	1.0	1.00	0.985
10					0.010	1.0	1.00	0.995
11					0.004	1.0	1.00	0.999
12					0.001	1.0	1.00	1.000

For illustrative purposes we include column (6), the d.f. for column (1), column (7) which can be derived directly from columns (2) and (6) by use of (2.3.3), and column (8), derived similarly from columns (3) and (7). Column (5) can then be obtained by differencing column (8). ▼

We follow with two examples involving continuous random variables.

Example 2.3.2

Let X have a uniform distribution on (0, 2) and let Y be independent of X with a uniform distribution over (0, 3). Determine the d.f. of $S = X + Y$.

Solution:

Since X and Y are continuous, we use (2.3.6):

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

and

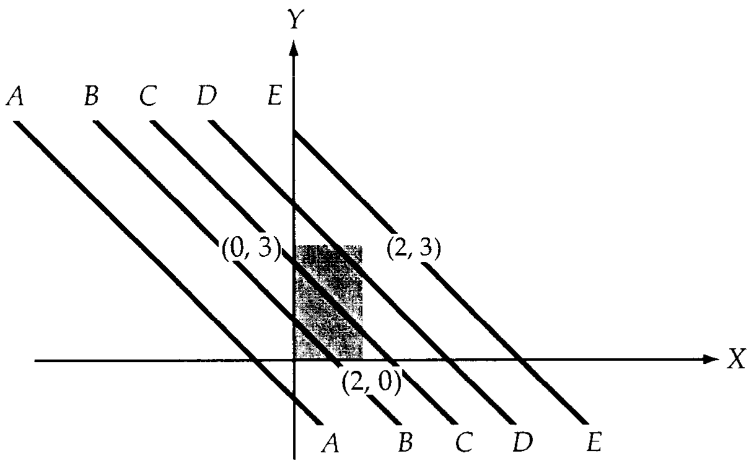
$$f_Y(y) = \begin{cases} \dfrac{1}{3} & 0 < y < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$F_S(s) = \int_0^s F_X(s - y) f_Y(y)\, dy.$$

The X, Y sample space is illustrated in Figure 2.3.2. The rectangular region contains all of the probability for X and Y. The event of interest, $X + Y \le s$, has been

Convolution of Two Uniform Distributions



illustrated in the figure for five values of s. For each value, the line intersects the y-axis at s and the line $x = 2$ at $s - 2$. The values of F_S for these five cases are

$$F_S(s) = \begin{cases} 0 & s < 0 \quad \text{line } A \\ \displaystyle\int_0^s \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s^2}{12} & 0 \le s < 2 \quad \text{line } B \\ \displaystyle\int_0^{s-2} 1\,\frac{1}{3}\,dy + \int_{s-2}^{s} \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s-1}{3} & 2 \le s < 3 \quad \text{line } C \\ \displaystyle\int_0^{s-2} 1\,\frac{1}{3}\,dy + \int_{s-2}^{3} \frac{s-y}{2}\frac{1}{3}\,dy = 1 - \frac{(5-s)^2}{12} & 3 \le s < 5 \quad \text{line } D \\ 1 & s \ge 5 \quad \text{line } E. \end{cases}$$

▼

Example 2.3.3

Consider three independent random variables X_1, X_2, X_3. For $i = 1, 2, 3$, X_i has an exponential distribution and $E[X_i] = 1/i$. Derive the p.d.f. of $S = X_1 + X_2 + X_3$ by the convolution process.

Solution:

$$f_1(x) = e^{-x} \qquad x > 0,$$

$$f_2(x) = 2e^{-2x} \qquad x > 0,$$

$$f_3(x) = 3e^{-3x} \qquad x > 0.$$

Using (2.3.7) twice, we have

$$f^{(2)}(x) = \int_0^x f_1(x-y)f_2(y)\,dy = \int_0^x e^{-(x-y)}\,2e^{-2y}\,dy$$
$$= 2e^{-x}\int_0^x e^{-y}\,dy$$
$$= 2e^{-x} - 2e^{-2x} \qquad x > 0,$$
$$f_S(x) = f^{(3)}(x) = \int_0^x f^{(2)}(x-y)f_3(y)\,dy$$
$$= \int_0^x (2e^{-(x-y)} - 2e^{-2(x-y)})3e^{-3y}\,dy$$
$$= 6e^{-x}\int_0^x e^{-2y}\,dy - 6e^{-2x}\int_0^x e^{-y}\,dy$$
$$= (3e^{-x} - 3e^{-3x}) - (6e^{-2x} - 6e^{-3x})$$
$$= 3e^{-x} - 6e^{-2x} + 3e^{-3x} \qquad x > 0.$$

▼

Another method to determine the distribution of the sum of random variables is based on the uniqueness of the ***moment generating function*** (m.g.f.), which, for the random variable X, is defined by $M_X(t) = \mathrm{E}[e^{tX}]$. If this expectation is finite for all t in an open interval about the origin, then $M_X(t)$ is the only m.g.f. of the distribution of X, and it is not the m.g.f. of any other distribution. This uniqueness can be used as follows. For the sum $S = X_1 + X_2 + \cdots + X_n$,

$$M_S(t) = \mathrm{E}[e^{tS}] = \mathrm{E}[e^{t}(X_1 + X_2 + \cdots + X_n)]$$
$$= \mathrm{E}[e^{tX_1}e^{tX_2}\cdots e^{tX_n}]. \qquad (2.3.8)$$

If $X_1, X_2, \ldots, X_n$ are independent, then the expectation of the product in (2.3.8) is equal to

$$\mathrm{E}[e^{tX_1}]\,\mathrm{E}[e^{tX_2}]\cdots\mathrm{E}[e^{tX_n}]$$

so that

$$M_S(t) = M_{X_1}(t)\,M_{X_2}(t)\cdots M_{X_n}(t). \qquad (2.3.9)$$

Recognition of the unique distribution corresponding to (2.3.9) would complete the determination of S's distribution. If inversion by recognition is not possible, then inversion by numerical methods may be used. (See Section 2.6.)

Example 2.3.4

Consider the random variables of Example 2.3.3. Derive the p.d.f. of $S = X_1 + X_2 + X_3$ by recognition of the m.g.f. of S.

Solution:

By (2.3.9), $M_S(t) = \left(\frac{1}{1-t}\right)\left(\frac{2}{2-t}\right)\left(\frac{3}{3-t}\right)$, which we write, by the method of partial fractions, as

$$M_S(t) = \frac{A}{1-t} + \frac{2B}{2-t} + \frac{3C}{3-t}.$$

The solution for this is $A = 3$, $B = -3$, $C = 1$. But $\beta/(\beta - t)$ is the moment generating function of an exponential distribution with parameter β, so the p.d.f. for S is

$$f_S(x) = 3(e^{-x}) - 3(2e^{-2x}) + (3e^{-3x}).$$ ▼

Example 2.3.5

The ***inverse Gaussian distribution*** was developed in the study of stochastic processes. Here it is used as the distribution of B, the claim amount. It will have a similar role in risk theory in Chapters 12–14. The p.d.f. and m.g.f. associated with the inverse Gaussian distribution are given by

$$f_X(x) = \frac{\alpha}{\sqrt{2\pi\beta}} x^{-3/2} \exp\left[-\frac{(\beta x - \alpha)^2}{2\beta x}\right] \qquad x > 0,$$

$$M_X(t) = \exp\left[\alpha\left(1 - \sqrt{1 - \frac{2t}{\beta}}\right)\right].$$

Find the distribution of $S = X_1 + X_2 + X_3 + \cdots + X_n$ where the random variables $X_1, X_2, \ldots, X_n$ are independent and have identical inverse Gaussian distributions.

Solution:

Using (2.3.9), the m.g.f. of S is given by

$$M_S(t) = [M_X(t)]^n = \exp\left[n\alpha\left(1 - \sqrt{1 - \frac{2t}{\beta}}\right)\right].$$

The m.g.f. $M_S(t)$ can be recognized and shows that S has an inverse Gaussian distribution with parameters $n\alpha$ and β. ▼

2.4 Approximations for the Distribution of the Sum

The *central limit theorem* suggests a method to obtain numerical values for the distribution of the sum of independent random variables. The usual statement of the theorem is for a sequence of independent and identically distributed random variables, $X_1, X_2, \ldots$, with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$. For each n, the distribution of $\sqrt{n}\,(\bar{X}_n - \mu)/\sigma$, where $\bar{X}_n = (X_1 + X_2 + \cdots + X_n)/n$, has mean 0 and variance 1. The sequence of distributions ($n = 1, 2, \ldots$) is known to approach the standard normal distribution. When n is large the theorem is applied to approximate the distribution of $\bar{X}_n$ by a normal distribution with mean μ and variance σ^2/n. Equivalently, the distribution of the sum of the n random variables is approximated by a normal distribution with mean $n\mu$ and variance $n\sigma^2$. The effectiveness of these approximations depends not only on the number of variables but also on the departure of the distribution of the summands from normality. Many elementary statistics textbooks recommend that n be at least 30 for the approximations to be reasonable. One routine used to generate normally distributed random variables for simulation is based on the average of only 12 independent random variables uniformly distributed over (0, 1).

In many individual risk models the random variables in the sum are not identically distributed. This is illustrated by examples in the next section. The central limit theorem extends to sequences of nonidentically distributed random variables.

To illustrate some applications of the individual risk model, we use a normal approximation to the distribution of the sum of independent random variables to obtain numerical answers. If

$$S = X_1 + X_2 + \cdots + X_n,$$

then

$$\mathrm{E}[S] = \sum_{k=1}^{n} \mathrm{E}[X_k],$$

and, further, under the assumption of independence,

$$\mathrm{Var}(S) = \sum_{k=1}^{n} \mathrm{Var}(X_k).$$

For an application we need only

- Evaluate the means and variances of the individual loss random variables
- Sum them to obtain the mean and variance for the loss of the insuring organization as a whole
- Apply the normal approximation.

Illustrations of this process follow.

2.5 Applications to Insurance

In this section four examples illustrate the results of Section 2.2 and use of the normal approximation.

Example 2.5.1

A life insurance company issues 1-year term life contracts for benefit amounts of 1 and 2 units to individuals with probabilities of death of 0.02 or 0.10. The following table gives the number of individuals n_k in each of the four classes created by a benefit amount b_k and a probability of claim q_k.

k	q_k	b_k	n_k
1	0.02	1	500
2	0.02	2	500
3	0.10	1	300
4	0.10	2	500

The company wants to collect, from this population of 1,800 individuals, an amount equal to the 95th percentile of the distribution of total claims. Moreover, it wants each individual's share of this amount to be proportional to that individual's expected claim. The share for individual j with mean $E[X_j]$ would be $(1 + \theta)E[X_j]$. The 95th percentile requirement suggests that $\theta > 0$. This extra amount, $\theta E[X_j]$, is the ***security loading*** and θ is the ***relative security loading.*** Calculate θ.

Solution:

The criterion for θ is $\Pr(S \le (1 + \theta)E[S]) = 0.95$ where $S = X_1 + X_2 + \cdots + X_{1,800}$. This probability statement is equivalent to

$$\Pr\left[\frac{S - E[S]}{\sqrt{\mathrm{Var}(S)}} \le \frac{\theta E[S]}{\sqrt{\mathrm{Var}(S)}}\right] = 0.95.$$

Following the discussion of the central limit theorem in Section 2.4, we approximate the distribution of $(S - E[S])/\sqrt{\mathrm{Var}(S)}$ by the standard normal distribution and use its 95th percentile to obtain

$$\frac{\theta\, E[S]}{\sqrt{\mathrm{Var}(S)}} = 1.645.$$

It remains to calculate the mean and variance of S and to calculate θ by this equation.

For the four classes of insured individuals, we have the results given below.

k	q_k	b_k	Mean $b_k q_k$	Variance $b_k^2 q_k(1 - q_k)$	n_k
1	0.02	1	0.02	0.0196	500
2	0.02	2	0.04	0.0784	500
3	0.10	1	0.10	0.0900	300
4	0.10	2	0.20	0.3600	500

Then

$$E[S] = \sum_{j=1}^{1,800} E[X_j] = \sum_{k=1}^{4} n_k b_k q_k = 160$$

and

$$\mathrm{Var}(S) = \sum_{j=1}^{1,800} \mathrm{Var}(X_j) = \sum_{k=1}^{4} n_k b_k^2 q_k(1 - q_k) = 256.$$

Thus, the relative security loading is

$$\theta = 1.645\,\frac{\sqrt{\mathrm{Var}(S)}}{E[S]} = 1.645\,\frac{16}{160} = 0.1645.$$

▼

Example 2.5.2

The policyholders of an automobile insurance company fall into two classes.

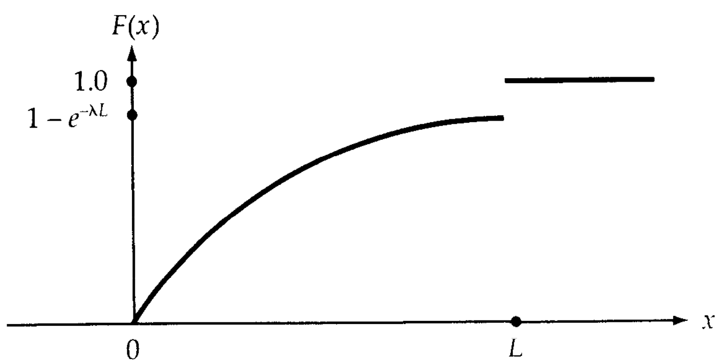
Class k	Number in Class n_k	Claim Probability q_k	Distribution of Claim Amount, B_k, Parameters of Truncated Exponential λ	L
1	500	0.10	1	2.5
2	2 000	0.05	2	5.0

A truncated exponential distribution is defined by the d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & 0 \le x < L \\ 1 & x \ge L. \end{cases}$$

This is a mixed distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $0 < x < L$, and a probability mass $e^{-\lambda L}$ at L. A graph of the d.f. appears in Figure 2.5.1.

Truncated Exponential Distribution



Again, the probability that total claims exceed the amount collected from policyholders is 0.05. We assume that the relative security loading, θ, is the same for the two classes. Calculate θ.

Solution:

This example is much like the previous one. It differs in that the claim amounts are random variables. First, we obtain formulas for the moments of the truncated exponential distribution in preparation for applying (2.2.25) and (2.2.26):

$$\mu = \mathrm{E}[B|I = 1] = \int_0^L x\lambda e^{-\lambda x}\, dx + Le^{-\lambda L} = \frac{1 - e^{-\lambda L}}{\lambda},$$

$$\mathrm{E}[B^2|I = 1] = \int_0^L x^2\lambda e^{-\lambda x}\, dx + L^2e^{-\lambda L} = \frac{2}{\lambda^2}(1 - e^{-\lambda L}) - \frac{2L}{\lambda} e^{-\lambda L},$$

$$\sigma^2 = \mathrm{E}[B^2|I = 1] - (\mathrm{E}[B|I = 1])^2 = \frac{1 - 2\lambda Le^{-\lambda L} - e^{-2\lambda L}}{\lambda^2}.$$

Using the parameter values given and applying (2.2.25) and (2.2.26), we obtain the following results.

k	q_k	μ_k	σ_k^2	Mean $q_k\mu_k$	Variance $\mu_k^2 q_k(1 - q_k) + \sigma_k^2 q_k$	n_k
1	0.10	0.9179	0.5828	0.09179	0.13411	500
2	0.05	0.5000	0.2498	0.02500	0.02436	2 000

Then S, the sum of the claims, has moments

$$\mathrm{E}[S] = 500\ (0.09179) + 2{,}000\ (0.02500) = 95.89,$$

$$\mathrm{Var}(S) = 500\ (0.13411) + 2{,}000\ (0.02436) = 115.78.$$

The criterion for θ is the same as in Example 2.5.1,

$$\Pr(S \le (1 + \theta)\mathrm{E}[S]) = 0.95.$$

Again by the normal approximation,

$$\frac{\theta\,\mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} = 1.645$$

and

$$\theta = \frac{1.645\sqrt{115.78}}{95.89} = 0.1846.$$

▼

Example 2.5.3

A life insurance company covers 16,000 lives for 1-year term life insurance in amounts shown below.

Benefit Amount	Number Covered
10 000	8 000
20 000	3 500
30 000	2 500
50 000	1 500
100 000	500

The probability of a claim q for each of the 16,000 lives, assumed to be mutually independent, is 0.02. The company wants to set a ***retention limit.*** For each life, the retention limit is the amount below which this (the ***ceding***) company will retain the insurance and above which it will purchase ***reinsurance*** coverage from another (the ***reinsuring***) company. For example, assume the retention limit is 20,000. The company will retain up to 20,000 on each life and purchase reinsurance for the difference between the benefit amount and 20,000 for each of the 4,500 individuals with benefit amounts in excess of 20,000. As a decision criterion, the company wants to minimize the probability that retained claims plus the amount that it pays for reinsurance will exceed 8,250,000. Reinsurance is available at a cost of 0.025 per unit of coverage (i.e., at 125% of the expected claim amount per unit, 0.02). We will consider the block of business as closed. New policies sold during the year are not to enter this decision process. Calculate the retention limit that minimizes the probability that the company's retained claims plus cost of reinsurance exceeds 8,250,000.

Partial Solution:

First, do all calculations in benefit units of 10,000. As an illustrative step, let S be the amount of retained claims paid when the retention limit is 2 (20,000). Our portfolio of retained business is given by

k	Retained Amount b_k	Number Covered n_k
1	1	8 000
2	2	8 000

and

$$E[S] = \sum_{k=1}^{2} n_k b_k q_k = 8{,}000\ (1)(0.02) + 8{,}000\ (2)(0.02) = 480$$

and

$$\begin{aligned} \text{Var}(S) &= \sum_{k=1}^{2} n_k b_k^2 q_k(1 - q_k) \\ &= 8{,}000\ (1)(0.02)(0.98) + 8{,}000\ (4)(0.02)(0.98) = 784. \end{aligned}$$

In addition to the retained claims, S, there is the cost of reinsurance premiums. The total coverage in the plan is

$$8{,}000\ (1) + 3{,}500\ (2) + 2{,}500\ (3) + 1{,}500\ (5) + 500\ (10) = 35{,}000.$$

The retained coverage for the plan is

$$8{,}000\ (1) + 8{,}000\ (2) = 24{,}000.$$

Therefore, the total amount reinsured is $35{,}000 - 24{,}000 = 11{,}000$ and the reinsurance cost is $11{,}000(0.025) = 275$. Thus, at retention limit 2, the retained claims plus reinsurance cost is $S + 275$. The decision criterion is based on the probability that this total cost will exceed 825,

$$\begin{aligned}\Pr(S + 275 > 825) &= \Pr(S > 550)\\ &= \Pr\left[\frac{S - E[S]}{\sqrt{\operatorname{Var}(S)}} > \frac{550 - E[S]}{\sqrt{\operatorname{Var}(S)}}\right]\\ &= \Pr\left[\frac{S - E[S]}{\sqrt{\operatorname{Var}(S)}} > 2.5\right].\end{aligned}$$

Using the normal distribution, this is approximately 0.0062. The solution is completed in Exercises 2.13 and 2.14. ▼

In Section 1.5 stop-loss insurance, which is available as a reinsurance coverage, was discussed. The expected value of the claims paid under the stop-loss reinsurance coverage can be approximated by using the normal distribution as the distribution of total claims.

Let total claims, X, have a normal distribution with mean μ and variance σ^2 and let d be the deductible of the stop-loss insurance. Then, by (1.5.2A), the expected claims equal

$$E[I_d(X)] = \frac{1}{\sigma\sqrt{2\pi}} \int_d^\infty (x - d) \exp\left[\frac{-(x - \mu)^2}{2\sigma^2}\right] dx. \tag{2.5.1}$$

Changing the variable of integration to $z = (x - \mu)/\sigma$ and defining β by $d = \mu + \beta\sigma$, we obtain the following general expression for the expected value of stop-loss claims under a normal distribution assumption:

$$E[I_d(X)] = \sigma \left\{\frac{\exp(-\beta^2/2)}{(2\pi)^{0.5}} - \beta[1 - \Phi(\beta)]\right\} \tag{2.5.2}$$

where $\Phi(x)$ is the distribution function for the standard normal distribution.

Example 2.5.4

Consider the portfolio of insurance contracts in Example 2.5.3. Calculate the expected value of the claims provided by a stop-loss reinsurance coverage where

a. There is no individual reinsurance and the deductible amount is 7,500,000
b. There is a retention amount of 20,000 on individual policies and the deductible amount on the business retained is 5,300,000.

Solution:

a. With no individual reinsurance and the use of 10,000 as the unit,

$$E[S] = 0.02[8{,}000(1) + 3{,}500(2) + 2{,}500(3) + 1{,}500(5) + 500(10)] = 700$$

and

$$\begin{aligned}\operatorname{Var}(S) &= (0.02)(0.98)[8{,}000(1) + 3{,}500(4) + 2{,}500(9) + 1{,}500(25) + 500(100)]\\ &= 2{,}587.2\end{aligned}$$

so

$$\sigma(S) = 50.86.$$

Then, with

$$\beta = \frac{(d - \mu)}{\sigma} = \frac{(750 - 700)}{50.86} = 0.983$$

the application of (2.5.2) gives us

$$P = 50.86[0.24608 - (0.983)(0.16280)] = 4.377.$$

This is equivalent to 43,770 in the example as posed.

b. In Example 2.5.3 we determined the mean and the variance of the aggregate claims, after imposing a 20,000 retention limit per individual, to be 480 and 784, respectively, in units of 10,000. Thus $\sigma(S) = 28$.

Then, with

$$\beta = \frac{d - \mu}{\sigma} = \frac{530 - 480}{28} = 1.786$$

the application of (2.5.2) gives us

$$P = 28[0.08100 - (1.786)(0.03707)] = 0.414.$$

This is equivalent to 4,140 in the example as posed. ▼

2.6 Notes and References

The basis of the material in Sections 2.2, 2.3, and 2.4 can be found in a number of post-calculus probability and statistics texts. Mood et al. (1974) prove the theorems given in (2.2.10) and (2.2.11). They also provide an extensive discussion of properties of the moment generating function. For a discussion of the advanced mathematical methods for deriving the distribution function that corresponds to a given moment generating function, see Bellman et al. (1966). Methods are also available to obtain the p.f. of a discrete distribution from its ***probability generating function;*** see Kornya (1983).

DeGroot (1986) provides a discussion of several conditions under which the central limit theorem holds. Kendall and Stuart (1977) give material on ***normal power expansions*** that may be viewed as modifications of the normal approximation to improve numerical results. Bowers (1967) also describes the use of normal power expansions and gives an application to approximate the distribution of present values for an annuity portfolio.

Exercises

Section 2.2

2.1. Use (2.2.3) and (2.2.4) to obtain the mean and variance of the claim random variable X where $q = 0.05$ and the claim amount is fixed at 10.

2.2. Obtain the mean and variance of the claim random variable X where $q = 0.05$ and the claim amount random variable B is uniformly distributed between 0 and 20.

2.3. Let X be the number of heads observed in five tosses of a true coin. Then, X true dice are thrown. Let Y be the sum of the numbers showing on the dice. Determine the mean and variance of Y. [Hint: Apply (2.2.10) and (2.2.11).]

2.4. Let X be the number showing when one true die is thrown. Let Y be the number of heads obtained when X true coins are then tossed. Calculate $E[Y]$ and $Var(Y)$.

2.5. Let X be the number obtained when one true die is tossed. Let Y be the sum of the numbers obtained when X true dice are then thrown. Calculate $E[Y]$ and $Var(Y)$.

2.6. The probability of a fire in a certain structure in a given time period is 0.02. If a fire occurs, the damage to the structure is uniformly distributed over the interval $(0, a)$ where a is its total value. Calculate the mean and variance of fire damage to the structure within the time period.

Section 2.3

2.7. Independent random variables X_k for four lives have the discrete probability functions given below.

x	$\Pr(X_1 = x)$	$\Pr(X_2 = x)$	$\Pr(X_3 = x)$	$\Pr(X_4 = x)$
0	0.6	0.7	0.6	0.9
1	0.0	0.2	0.0	0.0
2	0.3	0.1	0.0	0.0
3	0.0	0.0	0.4	0.0
4	0.1	0.0	0.0	0.1

Use a convolution process on the non-negative integer values of x to obtain $F_S(x)$ for $x = 0, 1, 2, \ldots, 13$ where $S = X_1 + X_2 + X_3 + X_4$.

2.8. Let X_i for $i = 1, 2, 3$ be independent and identically distributed with the d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

Let $S = X_1 + X_2 + X_3$.

a. Show that $F_S(x)$ is given by

$$F_S(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^3}{6} & 0 \le x < 1 \\ \dfrac{x^3 - 3(x-1)^3}{6} & 1 \le x < 2 \\ \dfrac{x^3 - 3(x-1)^3 + 3(x-2)^3}{6} & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

$1 - \frac{(3-s)^3}{6} =$

b. Show that $E[S] = 1.5$ and $\mathrm{Var}(S) = 0.25$.

c. Evaluate the following probabilities using the d.f. of part (a):

(i) $\Pr(S \le 0.5)$
(ii) $\Pr(S \le 1.0)$
(iii) $\Pr(S \le 1.5)$.

2.9. Find the mean and variance of the inverse Gaussian distribution by using its m.g.f. as given in Example 2.3.5.

Section 2.4

2.10. Calculate the mean and variance of X and Y in Example 2.3.2. Use a normal distribution to approximate $\Pr(X + Y > 4)$. Compare this with the exact answer.

2.11. a. Use the central limit theorem to calculate b, c, and d, for given a, in the statement

$$\Pr\left(\sum_1^n X_i \ge n\mu + a\sqrt{n}\sigma\right) \cong c + b\Phi(d)$$

where the X_i's are independent and identically distributed with mean μ and variance σ^2 and $\Phi(z)$ is the d.f. of the standard normal distribution.

b. Evaluate the probabilities in Exercise 2.8(c) by use of the normal approximation developed in part (a).

2.12. A random variable U has m.g.f.

$$M_U(t) = (1 - 2t)^{-9} \qquad t < \frac{1}{2}.$$

a. Use the m.g.f. to calculate the mean and variance of U.

b. Use a normal approximation to calculate points $y_{0.05}$ and $y_{0.01}$ such that $\Pr(U > y_\epsilon) = \epsilon$.

Note the random variable U has a gamma distribution with parameters $\alpha = 9$ and $\beta = 1/2$. Gamma distributions with $\alpha = n/2$ and $\beta = 1/2$ are chi-square distributions with n degrees of freedom. Thus U has a chi-square distribution with 18 degrees of freedom. From tables of d.f.'s of chi-square distributions, we obtain $y_{0.05} = 28.869$ and $y_{0.01} = 34.805$.

Section 2.5

2.13. Calculate the probability that the total cost in Example 2.5.3 will exceed 8,250,000 if the retention limit is
a. 30,000 b. 50,000.

2.14. Calculate the retention limit that minimizes the probability of the total cost in Example 2.5.3 exceeding 8,250,000. Assume that the limit is between 30,000 and 50,000.

2.15. A fire insurance company covers 160 structures against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Contract Amount	Number of Contracts
10 000	80
20 000	35
30 000	25
50 000	15
100 000	5

Assume that for each of the structures, the probability of one claim within a year is 0.04, and the probability of more than one claim is 0. Assume that fires in the structures are mutually independent events. Furthermore, assume that the conditional distribution of the claim size, given that a claim has occurred, is uniformly distributed over the interval from 0 to the contract amount. Let N be the number of claims and let S be the amount of claims in a 1-year period.
a. Calculate the mean and variance of N.
b. Calculate the mean and variance of S.
c. What relative security loading, θ, should be used so the company can collect an amount equal to the 99th percentile of the distribution of total claims? (Use a normal approximation.)

2.16. Consider a portfolio of 32 policies. For each policy, the probability q of a claim is $1/6$ and B, the benefit amount given that there is a claim, has p.d.f.

$$f(y) = \begin{cases} 2(1-y) & 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let S be the total claims for the portfolio. Using a normal approximation, estimate $\Pr(S > 4)$.

3

SURVIVAL DISTRIBUTIONS AND LIFE TABLES

3.1 Introduction

Chapter 1 was dedicated to showing how insurance can increase the expected utility of individuals facing random losses. In Chapter 2 simple models for single-period insurance policies were developed. The foundations of these models were Bernoulli random variables associated with the occurrence or nonoccurrence of a loss. The occurrence of a loss, in some examples, resulted in a second random process generating the amount of the loss. Chapters 4 through 8 deal primarily with models for insurance systems designed to manage random losses where the randomness is related to how long an individual will survive. In these chapters the ***time-until-death*** random variable, $T(x)$, is the basic building block. This chapter develops a set of ideas for describing and using the distribution of time-until-death and the distribution of the corresponding ***age-at-death,*** X.

We show how a distribution of the age-at-death random variable can be summarized by a ***life table.*** Such tables are useful in many fields of science. Consequently a profusion of notation and nomenclature has developed among the various professions using life tables. For example, engineers use life tables to study the reliability of complex mechanical and electronic systems. Biostatisticians use life tables to compare the effectiveness of alternative treatments of serious diseases. Demographers use life tables as tools in population projections. In this text, life tables are used to build models for insurance systems designed to assist individuals facing uncertainty about the times of their deaths. This application determines the viewpoint adopted. However, when it provides a bridge to other disciplines, notes relating the discussion to alternative applications of life tables are added.

A life table is an indispensable component of many models in actuarial science. In fact, some scholars fix the date of the beginning of actuarial science as 1693. In that year, Edmund Halley published "An Estimate of the Degrees of the Mortality of Mankind, Drawn from Various Tables of Births and Funerals at the City of

$\Omega \to R$

$(\omega) \in R.$

Breslau." The life table, called the Breslau Table, contained in Halley's paper remains of interest because of its surprisingly modern notation and ideas.

3.2 Probability for the Age-at-Death

In this section we formulate the uncertainty of age-at-death in probability concepts.

3.2.1 The Survival Function

Let us consider a newborn child. This newborn's age-at-death, X, is a continuous-type random variable. Let $F_X(x)$ denote the distribution function (d.f.) of X,

$$F_X(x) = \Pr(X \le x) \qquad x \ge 0, \tag{3.2.1}$$

and set

$$s(x) = 1 - F_X(x) = \Pr(X > x) \qquad x \ge 0. \tag{3.2.2}$$

We always assume that $F_X(0) = 0$, which implies $s(0) = 1$. The function $s(x)$ is called the ***survival function*** (s.f.). For any positive x, $s(x)$ is the probability a newborn will attain age x. The distribution of X can be defined by specifying either the function $F_X(x)$ or the function $s(x)$. Within actuarial science and demography, the survival function has traditionally been used as a starting point for further developments. Within probability and statistics, the d.f. usually plays this role. However, from the properties of the d.f., we can deduce corresponding properties of the survival function.

Using the laws of probability, we can make probability statements about the age-at-death in terms of either the survival function or the distribution function. For example, the probability that a newborn dies between ages x and z ($x < z$) is

$$\begin{aligned}\Pr(x < X \le z) &= F_X(z) - F_X(x) = 1 - s(z) - 1 + s(x)\\ &= s(x) - s(z).\end{aligned}$$

3.2.2 Time-until-Death for a Person Age x

The conditional probability that a newborn will die between the ages x and z, given survival to age x, is

$$\begin{aligned}\Pr(x < X \le z \mid X > x) &= \frac{F_X(z) - F_X(x)}{1 - F_X(x)}\\ &= \frac{s(x) - s(z)}{s(x)}.\end{aligned} \tag{3.2.3}$$

The symbol (x) is used to denote a ***life-age-x***. The future lifetime of (x), $X - x$, is denoted by $T(x)$. $T(x) = X - x$

Within actuarial science, it is frequently necessary to make probability statements about $T(x)$. For this purpose, and to promote research and communication, a set of symbols, part of the International Actuarial Notation, was originally adopted by the 1898 International Actuarial Congress. Symbols for common actuarial functions and principles to guide the adoption of new symbols were established. This system has been subject to constant review and is revised or extended as necessary by the International Actuarial Association's Permanent Committee on Notation. These notational conventions are followed in this book whenever possible.

These symbols differ from those used for probability notation, and the reader may be unfamiliar with them. For example, a single-variate function that would be written $q(x)$ in probability notation is written q_x in this system. Likewise, a multivariate function is written in actuarial notation using combinations of subscripts, superscripts, and other symbols. The general rules for defining a function in actuarial notation are given in Appendix 4. The reader may want to study these forms before continuing the discussion of the future-lifetime random variable.

To make probability statements about $T(x)$, we use the notations

$$ {}_tq_x = \Pr[T(x) \le t] \qquad t \ge 0, \qquad {}_tq_x + {}_tp_x = 1 \tag{3.2.4}$$

$$ {}_tp_x = 1 - {}_tq_x = \Pr[T(x) > t] \qquad t \ge 0. \tag{3.2.5}$$

$tq_x + tp_x = 1$

The symbol ${}_tq_x$ can be interpreted as the probability that (x) will die within t years; that is, ${}_tq_x$ is the d.f. of $T(x)$. On the other hand, ${}_tp_x$ can be interpreted as the probability that (x) will attain age $x + t$; that is, ${}_tp_x$ is the s.f. for (x). In the special case of a life-age-0, we have $T(0) = X$ and

$t p_x = 1 - {}_t p_x = P(T \ldots > t)$

survival function for (x) for x year.

$$ {}_xp_0 = s(x) \qquad x \ge 0. \quad \text{age is 0 and survive} \tag{3.2.6}$$

$= P(X > x)$

If $t = 1$, convention permits us to omit the prefix in the symbols defined in (3.2.4) and (3.2.5), and we have

$\cdot\ {}_1q_x = q_x$

$\cdot\ {}_1p_x = p_x$

$$ q_x = \Pr[(x) \text{ will die within 1 year}],$$

$$ p_x = \Pr[(x) \text{ will attain age } x + 1]. \qquad x \quad t \quad t+u \quad u$$

There is a special symbol for the more general event that (x) will survive t years and die within the following u years; that is, (x) will die between ages $x + t$ and $x + t + u$. This special symbol is given by

$$\begin{aligned} {}_{t|u}q_x &= \Pr[t < T(x) \le t + u] \\ &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x - {}_{t+u}p_x . \end{aligned} \tag{3.2.7}$$

As before, if $u = 1$, the prefix is deleted in ${}_{t|u}q_x$, and we have ${}_{t|}q_x$.

At this point it appears there are two expressions for the probability that (x) will die between ages x and $x + u$. Formula (3.2.7) with $t = 0$ is one such expression; (3.2.3) with $z = x + u$ is a second expression. Are these two probabilities different? Formula (3.2.3) can be interpreted as the conditional probability that a newborn will die between ages x and $z = x + u$, given survival to age x. The only information on the newborn, now at age x, is its survival to that age. Hence, the probability statement is based on a conditional distribution of survival for newborns.

On the other hand, (3.2.7) with $t = 0$ defines a probability that a life observed at age x will die between ages x and $x + u$. The observation on the life at age x might include information other than simply survival. Such information might be that the life has just passed a physical examination for insurance, or it might be that the life has commenced treatment for a serious illness. Life tables for situations where the observation of a life at age x implies more than simply survival of a newborn to age x are discussed in Section 3.8, where additional notation for those life tables is introduced. We will continue development of the theory without further reference to the distinction between (3.2.3) and (3.2.7), and we assume that until that section, observation of survival at age x will yield the same conditional distribution of survival as the hypothesis that a newborn has survived to age x; that is,

$$ {}_t p_x = \frac{{}_{x+t}p_0}{{}_x p_0} = \frac{s(x+t)}{s(x)}, \tag{3.2.8} $$

$$ {}_t q_x = 1 - \frac{s(x+t)}{s(x)}. \tag{3.2.9} $$

Under this approach, (3.2.7), and its many special cases, can be expressed as

$$ \begin{aligned} {}_{t|u}q_x &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \left[\frac{s(x+t)}{s(x)}\right]\left[\frac{s(x+t) - s(x+t+u)}{s(x+t)}\right] \\ &= {}_t p_x \; {}_u q_{x+t} \, . \end{aligned} \tag{3.2.10} $$

3.2.3 Curtate-Future-Lifetime

A discrete random variable associated with the future lifetime is the number of future years completed by (x) prior to death. It is called the ***curtate-future-lifetime*** of (x) and is denoted by $K(x)$. Because $K(x)$ is the greatest integer in $T(x)$, its p.f. is

$$ \begin{aligned} \Pr[K(x) = k] &= \Pr[k \le T(x) < k+1] \\ &= \Pr[k < T(x) \le k+1] \\ &= {}_k p_x - {}_{k+1}p_x \\ &= {}_k p_x \, q_{x+k} = {}_{k|}q_x \qquad k = 0, 1, 2, \ldots . \end{aligned} \tag{3.2.11} $$

The switching of inequalities is possible since, under our assumption that $T(x)$ is a continuous-type random variable, $\Pr[T(x) = k] = \Pr[T(x) = k+1] = 0$. Expression (3.2.11) is a special case of (3.2.7) where $u = 1$ and k is a non-negative integer. From (3.2.11) we can see that the d.f. of $K(x)$ is the step function

$$ F_{K(x)}(y) = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x, \qquad y \ge 0 \text{ and } k \text{ is the greatest integer in } y. $$

It often follows from the context that $T(x)$ is the future lifetime of (x), in which case we may write T instead of $T(x)$. Likewise, we may write K instead of $K(x)$.

3.2.4 Force of Mortality

Formula (3.2.3) expresses, in terms of the d.f. and in terms of the survival function, the conditional probability that (0) will die between ages x and z, given survival to x. With $z - x$ held constant, say, at c, then considered as a function of x, this conditional probability describes the distribution of the probability of death in the near future (between time 0 and c) for a life of attained age x. An analogue of this function for instantaneous death can be obtained by using the density of probability of death at attained age x, that is, using (3.2.3) with $z = x + \Delta x$,

$$\Pr(x < X \le x + \Delta x | X > x) = \frac{F_X(x + \Delta x) - F_X(x)}{1 - F_X(x)}$$

$$\cong \frac{f_X(x)\Delta x}{1 - F_X(x)}. \tag{3.2.12}$$

In this expression $F'_X(x) = f_X(x)$ is the p.d.f. of the continuous age-at-death random variable. The function

$$\frac{f_X(x)}{1 - F_X(x)}$$

in (3.2.12) has a conditional probability density interpretation. For each age x, it gives the value of the conditional p.d.f. of X at exact age x, given survival to that age, and is denoted by $\mu(x)$.

We have

$$\mu(x) = \frac{f_X(x)}{1 - F_X(x)}$$

$$= \frac{-s'(x)}{s(x)}. \tag{3.2.13}$$

The properties of $f_X(x)$ and $1 - F_X(x)$ imply that $\mu(x) \ge 0$.

In actuarial science and demography $\mu(x)$ is called the ***force of mortality.*** In reliability theory, the study of the survival probabilities of manufactured parts and systems, $\mu(x)$ is called the ***failure rate*** or ***hazard rate*** or, more fully, the ***hazard rate function.***

As is true for the s.f., the force of mortality can be used to specify the distribution of X. To obtain this result, we start with (3.2.13), change x to y, and rearrange to obtain

$$-\mu(y)\, dy = d \log s(y).$$

Integrating this expression from x to $x + n$, we have

$$-\int_x^{x+n} \mu(y)\, dy = \log\left[\frac{s(x+n)}{s(x)}\right]$$
$$= \log {}_np_x,$$

and on taking exponentials obtain

$$ {}_np_x = \exp[-\textstyle\int_x^{x+n} \mu(y)\, dy]. \tag{3.2.14}$$

Sometimes it is convenient to rewrite (3.2.14), with $s = y - x$, as

$$ {}_np_x = \exp[-\textstyle\int_0^n \mu(x+s)\, ds]. \tag{3.2.15}$$

In particular, we will change the notation to conform with that used in (3.2.6) by setting the age already lived to 0 and denoting the time of survival by x. We then have

$$ {}_xp_0 = s(x) = \exp[-\textstyle\int_0^x \mu(s)\, ds]. \tag{3.2.16}$$

In addition,

$$F_X(x) = 1 - s(x) = 1 - \exp[-\textstyle\int_0^x \mu(s)\, ds] \tag{3.2.17}$$

and

$$F'_X(x) = f_X(x) = \exp[-\textstyle\int_0^x \mu(s)\, ds]\, \mu(x)$$
$$= {}_xp_0\, \mu(x). \tag{3.2.18}$$

Let $F_{T(x)}(t)$ and $f_{T(x)}(t)$ denote, respectively, the d.f. and p.d.f. of $T(x)$, the future lifetime of (x). From (3.2.4) we note that $F_{T(x)}(t) = {}_tq_x$; therefore,

$$f_{T(x)}(t) = \frac{d}{dt}\, {}_tq_x$$
$$= \frac{d}{dt}\left[1 - \frac{s(x+t)}{s(x)}\right]$$
$$= \frac{s(x+t)}{s(x)}\left[-\frac{s'(x+t)}{s(x+t)}\right]$$
$$= {}_tp_x\, \mu(x+t) \qquad t \ge 0. \tag{3.2.19}$$

Thus ${}_tp_x\, \mu(x+t)\, dt$ is the probability that (x) dies between t and $t + dt$, and

$$\int_0^\infty {}_tp_x\, \mu(x+t)\, dt = 1$$

where the upper limit on the integral is written as positive infinity (an abbreviation for integrating over all positive probability density).

It follows from (3.2.19) that

$$\frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt}\, {}_tp_x = {}_tp_x\, \mu(x+t). \tag{3.2.20}$$

This equivalent form is useful in several developments in actuarial mathematics.

Since

$$\lim_{n \to \infty} {}_n p_x = 0,$$

we have

$$\lim_{n \to \infty} (-\log {}_n p_x) = \infty;$$

that is,

$$\lim_{n \to \infty} \int_x^{x+n} \mu(y)\, dy = \infty.$$

The developments of this section are summarized in Table 3.2.1.

The lower half of Table 3.2.1 summarizes some of the relationships among functions of general probability theory and those specific to age-at-death applications. There are many other examples where age-at-death questions can be formed in the more general probability setting. The following will illustrate this point.

Probability Theory Functions for Age-at-Death, X

	d.f. $F_X(x)$	**Survival Function** $s(x)$		**p.d.f.** $f_X(x)$	**Force of Mortality** $\mu(x)$
For	Requirements		For	Requirements	
$x < 0$	$F_X(x) = 0$	$s(x) = 1$	$x < 0$	$f_X(x) = 0$	$\mu(x) = 0$
$x = 0$	$F_X(0) = 0$	$s(0) = 1$	$x = 0$	undefined	undefined
$x \geq 0$	nondecreasing	nonincreasing	$x > 0$	$f_X(x) \geq 0$	$\mu(x) \geq 0$
$\lim_{x \to \infty}$	$F_X(\infty) = 1$	$s(\infty) = 0$	$\lim_{x \to \infty}$	$\int_0^\infty f_X(x)\, dt = 1$	$\int_0^\infty \mu(x) dx = \infty$
Functions in Terms of	Relationships				
$F_X(x)$	$F_X(x)$	$1 - F_X(x)$		$F_X'(x)$	$F_X'(x) / [1 - F_X(x)]$
$s(x)$	$1 - s(x)$	$s(x)$		$-s'(x)$	$-s'(x) / s(x)$
$f_X(x)$	$\int_0^x f_X(u) du$	$\int_x^\infty f_X(u) du$		$f_X(x)$	$f_X(x) / \int_x^\infty f_X(u) du$
$\mu(x)$	$1 - \exp[-\int_0^x \mu(t)\, dt]$	$\exp[-\int_0^x \mu(t)\, dt]$		$\mu(x)\exp[-\int_0^x \mu(t)\, dt]$	$\mu(x)$

Example 3.2.1

If $\bar{A}$ refers to the complement of the event A within the sample space and $\Pr(\bar{A}) \neq 0$, the following expresses an identity in probability theory:

$$\Pr(A \cup B) = \Pr(A) + \Pr(\bar{A}) \Pr(B|\bar{A}).$$

Rewrite this identity in actuarial notation for the events $A = [T(x) \leq t]$ and $B = [t < T(x) \leq 1]$, $0 < t < 1$.

Solution:

$\Pr(A \cup B)$ becomes $\Pr[T(x) \le 1] = q_x$, $\Pr(A)$ is ${}_tq_x$, and $\Pr(B|\bar{A})$ is ${}_{1-t}q_{x+t}$; hence

$$q_x = {}_tq_x + {}_tp_x \; {}_{1-t}q_{x+t}.$$

▼

3.3 Life Tables

A published life table usually contains tabulations, by individual ages, of the basic functions q_x, l_x, d_x, and, possibly, additional derived functions. Before presenting such a table, we consider an interpretation of these functions that is directly related to the probability functions discussed in the preceding section.

3.3.1 Relation of Life Table Functions to the Survival Function

In (3.2.9) we expressed the conditional probability that (x) will die within t years by

$${}_tq_x = 1 - \frac{s(x+t)}{s(x)},$$

and, in particular, we have

$$q_x = 1 - \frac{s(x+1)}{s(x)}.$$

We now consider a group of l_0 newborns, $l_0 = 100{,}000$, for instance. Each newborn's age-at-death has a distribution specified by s.f. $s(x)$. In addition, we let $\mathcal{L}(x)$ denote the group's number of survivors to age x. We index these lives by $j = 1, 2, \ldots, l_0$ and observe that

$$\mathcal{L}(x) = \sum_{j=1}^{l_0} I_j$$

where I_j is an indicator for the survival of life j; that is,

$$I_j = \begin{cases} 1 & \text{if life } j \text{ survives to age } x \\ 0 & \text{otherwise.} \end{cases}$$

Since $E[I_j] = s(x)$,

$$E[\mathcal{L}(x)] = \sum_{j=1}^{l_0} E[I_j] = l_0 \, s(x).$$

We denote $E[\mathcal{L}(x)]$ by l_x; that is, l_x represents the expected number of survivors to age x from the l_0 newborns, and we have

$$l_x = l_0 \, s(x). \tag{3.3.1}$$

Moreover, under the assumption that the indicators I_j are mutually independent, $\mathcal{L}(x)$ has a binomial distribution with parameters $n = l_0$ and $p = s(x)$. Note, however, that (3.3.1) does not require the independence assumption.

In a similar fashion, $_n\mathcal{D}_x$ denotes the number of deaths between ages x and $x + n$ from among the initial l_0 lives. We denote $\mathrm{E}[_n\mathcal{D}_x]$ by $_nd_x$. Since a newborn has probability $s(x) - s(x + n)$ of death between ages x and $x + n$ we can, by an argument similar to that for l_x, express

$$
\begin{aligned}
{}_nd_x = \mathrm{E}[_n\mathcal{D}_x] &= l_0[s(x) - s(x+n)] \\
&= l_x - l_{x+n}. \qquad (3.3.2)
\end{aligned}
$$

When $n = 1$, we omit the prefixes on $_n\mathcal{D}_x$ and $_nd_x$.

From (3.3.1), we see that

$$-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{1}{s(x)}\frac{ds(x)}{dx} = \mu(x) \qquad (3.3.3)$$

and

$$-dl_x = l_x\,\mu(x)\,dx. \qquad (3.3.4)$$

Since

$$l_x\,\mu(x) = l_0\,{}_xp_0\,\mu(x) = l_0\,f_X(x),$$

the factor $l_x\,\mu(x)$ in (3.3.4) can be interpreted as the expected density of deaths in the age interval $(x, x + dx)$. We note further that

$$l_x = l_0 \exp[-\textstyle\int_0^x \mu(y)\,dy], \qquad (3.3.5)$$

$$l_{x+n} = l_x \exp[-\textstyle\int_x^{x+n} \mu(y)\,dy], \qquad (3.3.6)$$

$$l_x - l_{x+n} = \textstyle\int_x^{x+n} l_y\,\mu(y)\,dy. \qquad (3.3.7)$$

For convenience of reference, we call this concept of l_0 newborns, each with survival function $s(x)$, a ***random survivorship group.***

3.3.2 Life Table Example

In "Life Table for the Total Population: United States, 1979–81" (Table 3.3.1), the functions $_tq_x$, l_x, and $_td_x$ are presented with $l_0 = 100{,}000$. Except for the first year of life, the value of t in the tabulated functions $_tq_x$ and $_td_x$ is 1. The other functions appearing in the table are discussed in Section 3.5.

The 1979–81 U.S. Life Table was not constructed by observing 100,000 newborns until the last survivor died. Instead, it was based on estimates of probabilities of death, given survival to various ages, derived from the experience of the entire U.S. population in the years around the 1980 census. In using the random survivorship group concept with this table, we must make the assumption that the probabilities derived from the table will be appropriate for the lifetimes of those who belong to the survivorship group.

Life Table for the Total Population: United States, 1979–81

(1) Age Interval: Period of Life between Two Ages, x to $x + t$	(2) Proportion Dying: Proportion of Persons Alive at Beginning of Age Interval Dying during Interval, ${}_tq_x$	(3) Of 100,000 Born Alive: Number Living at Beginning of Age Interval, l_x	(4) Of 100,000 Born Alive: Number Dying during Age Interval, ${}_td_x$	(5) Stationary Population*: Years Lived in the Age Interval, ${}_tL_x$	(6) Stationary Population*: Years Lived in This and All Subsequent Age Intervals, T_x	(7) Average Remaining Lifetime: Average Number of Years of Life Remaining at Beginning of Age Interval, $\mathring{e}_x$
Days						
0–1	0.00463	100 000	463	273	7 387 758	73.88
1–7	0.00246	99 537	245	1 635	7 387 485	74.22
7–28	0.00139	99 292	138	5 708	7 385 850	74.38
28–365	0.00418	99 154	414	91 357	7 380 142	74.43
Years						
0–1	0.01260	100 000	1 260	98 973	7 387 758	73.88
1–2	0.00093	98 740	92	98 694	7 288 785	73.82
2–3	0.00065	98 648	64	98 617	7 190 091	72.89
3–4	0.00050	98 584	49	98 560	7 091 474	71.93
4–5	0.00040	98 535	40	98 515	6 992 914	70.97
5–6	0.00037	98 495	36	98 477	6 894 399	70.00
6–7	0.00033	98 459	33	98 442	6 795 922	69.02
7–8	0.00030	98 426	30	98 412	6 697 480	68.05
8–9	0.00027	98 396	26	98 383	6 599 068	67.07
9–10	0.00023	98 370	23	98 358	6 500 685	66.08
10–11	0.00020	98 347	19	98 338	6 402 327	65.10
11–12	0.00019	98 328	19	98 319	6 303 989	64.11
12–13	0.00025	98 309	24	98 297	6 205 670	63.12
13–14	0.00037	98 285	37	98 266	6 107 373	62.14
14–15	0.00053	98 248	52	98 222	6 009 107	61.16
15–16	0.00069	98 196	67	98 163	5 910 885	60.19
16–17	0.00083	98 129	82	98 087	5 812 722	59.24
17–18	0.00095	98 047	94	98 000	5 714 635	58.28
18–19	0.00105	97 953	102	97 902	5 616 635	57.34
19–20	0.00112	97 851	110	97 796	5 518 733	56.40
20–21	0.00120	97 741	118	97 682	5 420 937	55.46
21–22	0.00127	97 623	124	97 561	5 323 255	54.53
22–23	0.00132	97 499	129	97 435	5 225 694	53.60
23–24	0.00134	97 370	130	97 306	5 128 259	52.67
24–25	0.00133	97 240	130	97 175	5 030 953	51.74
25–26	0.00132	97 110	128	97 046	4 933 778	50.81
26–27	0.00131	96 982	126	96 919	4 836 732	49.87
27–28	0.00130	96 856	126	96 793	4 739 813	48.94
28–29	0.00130	96 730	126	96 667	4 643 020	48.00
29–30	0.00131	96 604	127	96 541	4 546 353	47.06

*Stationary population is a demographic concept treated in Chapter 19.

Life Table for the Total Population: United States, 1979–81

(1) Age Interval — Period of Life between Two Ages x to $x+t$	(2) Proportion Dying — Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	(3) Of 100,000 Born Alive — Number Living at Beginning of Age Interval l_x	(4) Of 100,000 Born Alive — Number Dying during Age Interval ${}_td_x$	(5) Stationary Population* — Years Lived in the Age Interval ${}_tL_x$	(6) Stationary Population* — Years Lived in This and All Subsequent Age Intervals T_x	(7) Average Remaining Lifetime — Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
30–31	0.00133	96 477	127	96 414	4 449 812	46.12
31–32	0.00134	96 350	130	96 284	4 353 398	45.18
32–33	0.00137	96 220	132	96 155	4 257 114	44.24
33–34	0.00142	96 088	137	96 019	4 160 959	43.30
34–35	0.00150	95 951	143	95 880	4 064 940	42.36
35–36	0.00159	95 808	153	95 731	3 969 060	41.43
36–37	0.00170	95 655	163	95 574	3 873 329	40.49
37–38	0.00183	95 492	175	95 404	3 777 755	39.56
38–39	0.00197	95 317	188	95 224	3 682 351	38.63
39–40	0.00213	95 129	203	95 027	3 587 127	37.71
40–41	0.00232	94 926	220	94 817	3 492 100	36.79
41–42	0.00254	94 706	241	94 585	3 397 283	35.87
42–43	0.00279	94 465	264	94 334	3 302 698	34.96
43–44	0.00306	94 201	288	94 057	3 208 364	34.06
44–45	0.00335	93 913	314	93 756	3 114 307	33.16
45–46	0.00366	93 599	343	93 427	3 020 551	32.27
46–47	0.00401	93 256	374	93 069	2 927 124	31.39
47–48	0.00442	92 882	410	92 677	2 834 055	30.51
48–49	0.00488	92 472	451	92 246	2 741 378	29.65
49–50	0.00538	92 021	495	91 773	2 649 132	28.79
50–51	0.00589	91 526	540	91 256	2 557 359	27.94
51–52	0.00642	90 986	584	90 695	2 466 103	27.10
52–53	0.00699	90 402	631	90 086	2 375 408	26.28
53–54	0.00761	89 771	684	89 430	2 285 322	25.46
54–55	0.00830	89 087	739	88 717	2 195 892	24.65
55–56	0.00902	88 348	797	87 950	2 107 175	23.85
56–57	0.00978	87 551	856	87 122	2 019 225	23.06
57–58	0.01059	86 695	919	86 236	1 932 103	22.29
58–59	0.01151	85 776	987	85 283	1 845 867	21.52
59–60	0.01254	84 789	1 063	84 258	1 760 584	20.76
60–61	0.01368	83 726	1 145	83 153	1 676 326	20.02
61–62	0.01493	82 581	1 233	81 965	1 593 173	19.29
62–63	0.01628	81 348	1 324	80 686	1 511 208	18.58
63–64	0.01767	80 024	1 415	79 316	1 430 522	17.88
64–65	0.01911	78 609	1 502	77 859	1 351 206	17.19

*Stationary population is a demographic concept treated in Chapter 19.

Life Table for the Total Population: United States, 1979–81

(1) Age Interval — Period of Life between Two Ages, x to $x+t$	(2) Proportion Dying — Proportion of Persons Alive at Beginning of Age Interval Dying during Interval, ${}_tq_x$	(3) Of 100,000 Born Alive — Number Living at Beginning of Age Interval, l_x	(4) Of 100,000 Born Alive — Number Dying during Age Interval, ${}_td_x$	(5) Stationary Population* — Years Lived in the Age Interval, ${}_tL_x$	(6) Stationary Population* — Years Lived in This and All Subsequent Age Intervals, T_x	(7) Average Remaining Lifetime — Average Number of Years of Life Remaining at Beginning of Age Interval, $\mathring{e}_x$
Years						
65–66	0.02059	77 107	1 587	76 314	1 273 347	16.51
66–67	0.02216	75 520	1 674	74 683	1 197 033	15.85
67–68	0.02389	73 846	1 764	72 964	1 122 350	15.20
68–69	0.02585	72 082	1 864	71 150	1 049 386	14.56
69–70	0.02806	70 218	1 970	69 233	978 236	13.93
70–71	0.03052	68 248	2 083	67 206	909 003	13.32
71–72	0.03315	66 165	2 193	65 069	841 797	12.72
72–73	0.03593	63 972	2 299	62 823	776 728	12.14
73–74	0.03882	61 673	2 394	60 476	713 905	11.58
74–75	0.04184	59 279	2 480	58 039	653 429	11.02
75–76	0.04507	56 799	2 560	55 520	595 390	10.48
76–77	0.04867	54 239	2 640	52 919	539 870	9.95
77–78	0.05274	51 599	2 721	50 238	486 951	9.44
78–79	0.05742	48 878	2 807	47 475	436 713	8.93
79–80	0.06277	46 071	2 891	44 626	389 238	8.45
80–81	0.06882	43 180	2 972	41 694	344 612	7.98
81–82	0.07552	40 208	3 036	38 689	302 918	7.53
82–83	0.08278	37 172	3 077	35 634	264 229	7.11
83–84	0.09041	34 095	3 083	32 553	228 595	6.70
84–85	0.09842	31 012	3 052	29 486	196 042	6.32
85–86	0.10725	27 960	2 999	26 461	166 556	5.96
86–87	0.11712	24 961	2 923	23 500	140 095	5.61
87–88	0.12717	22 038	2 803	20 636	116 595	5.29
88–89	0.13708	19 235	2 637	17 917	95 959	4.99
89–90	0.14728	16 598	2 444	15 376	78 042	4.70
90–91	0.15868	14 154	2 246	13 031	62 666	4.43
91–92	0.17169	11 908	2 045	10 886	49 635	4.17
92–93	0.18570	9 863	1 831	8 948	38 749	3.93
93–94	0.20023	8 032	1 608	7 228	29 801	3.71
94–95	0.21495	6 424	1 381	5 733	22 573	3.51
95–96	0.22976	5 043	1 159	4 463	16 840	3.34
96–97	0.24338	3 884	945	3 412	12 377	3.19
97–98	0.25637	2 939	754	2 562	8 965	3.05
98–99	0.26868	2 185	587	1 892	6 403	2.93
99–100	0.28030	1 598	448	1 374	4 511	2.82

*Stationary population is a demographic concept treated in Chapter 19.

Life Table for the Total Population: United States, 1979–81

(1) Age Interval: Period of Life between Two Ages, x to $x+t$	(2) Proportion Dying: Proportion of Persons Alive at Beginning of Age Interval Dying during Interval, ${}_tq_x$	(3) Of 100,000 Born Alive: Number Living at Beginning of Age Interval, l_x	(4) Of 100,000 Born Alive: Number Dying during Age Interval, ${}_td_x$	(5) Stationary Population*: Years Lived in the Age Interval, ${}_tL_x$	(6) Stationary Population*: Years Lived in This and All Subsequent Age Intervals, T_x	(7) Average Remaining Lifetime: Average Number of Years of Life Remaining at Beginning of Age Interval, $\mathring{e}_x$
Years						
100–101	0.29120	1 150	335	983	3 137	2.73
101–102	0.30139	815	245	692	2 154	2.64
102–103	0.31089	570	177	481	1 462	2.57
103–104	0.31970	393	126	330	981	2.50
104–105	0.32786	267	88	223	651	2.44
105–106	0.33539	179	60	150	428	2.38
106–107	0.34233	119	41	99	278	2.33
107–108	0.34870	78	27	64	179	2.29
108–109	0.35453	51	18	42	115	2.24
109–110	0.35988	33	12	27	73	2.20

*Stationary population is a demographic concept treated in Chapter 19.

Several observations about the 1979–81 U.S. Life Table are instructive.

Observations:

1. Approximately 1% of a survivorship group of newborns would be expected to die in the first year of life.
2. It would be expected that about 77% of a group of newborns would survive to age 65.
3. The maximum number of deaths within a group would be expected to occur between ages 83 and 84.
4. For human lives, there have been few observations of age-at-death beyond 110. Consequently, it is often assumed that there is an age ω such that $s(x) > 0$ for $x < \omega$, and $s(x) = 0$ for $x \geq \omega$. The age ω, if assumed, is called the ***limiting age.*** The limiting age for this table is not defined. It is clear that there is a positive probability of survival to age 110, but the table does not indicate the age ω.
5. Local minimums in the expected number of deaths occur around ages 11 and 27 and a local maximum around age 24.
6. Although the values of l_x have been rounded to integers, there is no compelling reason, according to (3.3.1), to do so.

A display such as Table 3.3.1 is the conventional method for describing the distribution of age-at-death. Alternatively, an s.f. can be described in analytic form such as $s(x) = e^{-cx}$, $c > 0$, $x \geq 0$. However, most studies of human mortality for insurance purposes use the representation $s(x) = l_x / l_0$, as illustrated in Table 3.3.1. Since 100,000 $s(x)$ is displayed for only integer values of x, there is a need to interpolate in evaluating $s(x)$ for noninteger values. This is the subject of Section 3.6.

Example 3.3.1

On the basis of Table 3.3.1, evaluate the probability that (20) will

a. Live to 100
b. Die before 70
c. Die in the tenth decade of life.

Solution:

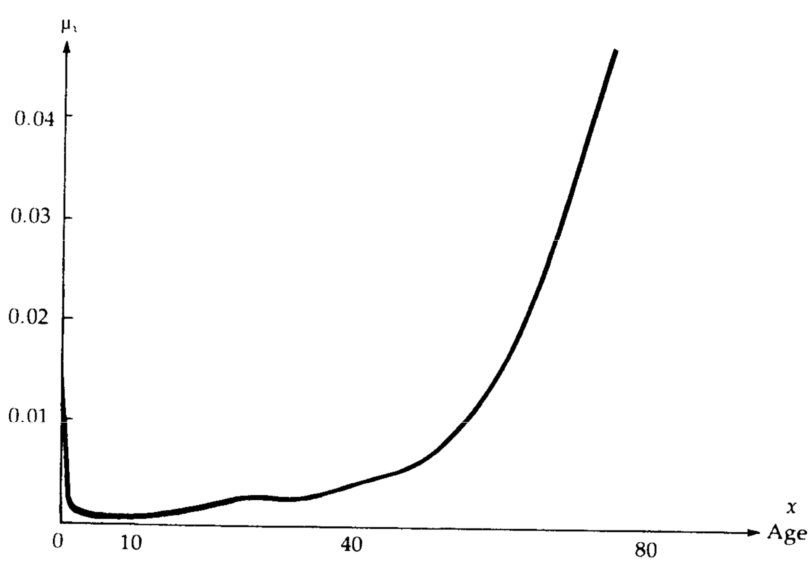
a. $\dfrac{s(100)}{s(20)} = \dfrac{l_{100}}{l_{20}} = \dfrac{1{,}150}{97{,}741} = 0.0118$

b. $\dfrac{[s(20) - s(70)]}{s(20)} = 1 - \dfrac{l_{70}}{l_{20}} = 1 - \dfrac{68{,}248}{97{,}741} = 0.3017$

c. $\dfrac{[s(90) - s(100)]}{s(20)} = \dfrac{(l_{90} - l_{100})}{l_{20}} = \dfrac{(14{,}154 - 1{,}150)}{97{,}741} = 0.1330.$ ▼

Insight into life table functions can be obtained by studying Figures 3.3.1, 3.3.2, and 3.3.3. These are drawn to be representative of current human mortality and are not taken directly from Table 3.3.1.

Graph of $\mu(x)$



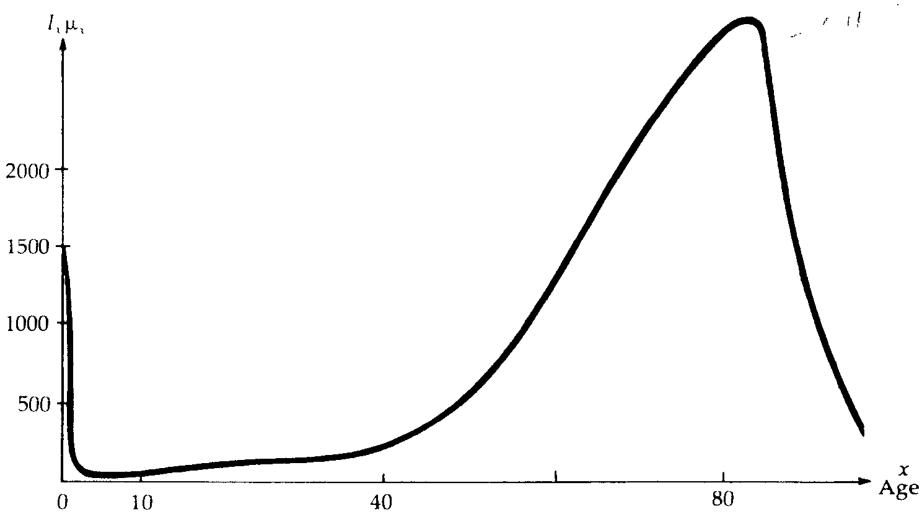
In Figure 3.3.1 note two features:

- The force of mortality is positive and the requirement

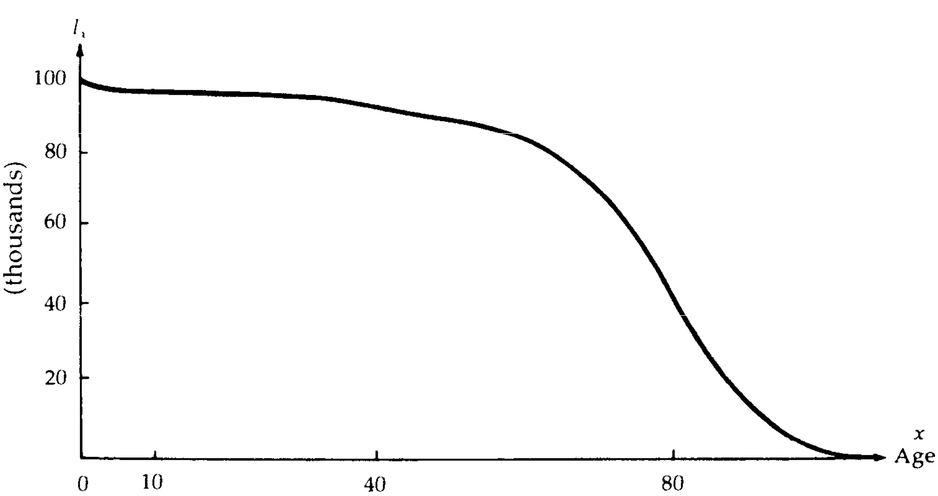
$$\int_0^\infty \mu(x)\, dx = \infty$$

  appears satisfied. (See Table 3.2.1.)
- The force of mortality starts out rather large and then drops to a minimum around age 10.

Graph of $l_x\, \mu(x)$



Graph of l_x



In Figures 3.3.2 and 3.3.3 note the following:

- The function $l_x\ \mu(x)$ is proportional to the p.d.f. of the age-at-death of a newborn. Since $l_x\ \mu(x)$ is the expected density of deaths at age x, under the random survivorship group idea, the graph of $l_x\ \mu(x)$ is called ***the curve of deaths.***
- There is a local minimum of $l_x\ \mu(x)$ at about age 10. The mode of the distribution of deaths—the age at which the maximum of the curve of deaths occurs—is around age 80.
- The function l_x is proportional to the survival function $s(x)$. It can also be interpreted as the expected number living at age x out of an initial group of size l_0.
- Local extreme points of $l_x\ \mu(x)$ correspond to points of inflection of l_x since

$$\frac{d}{dx} l_x\ \mu(x) = \frac{d}{dx}\left(-\frac{d}{dx} l_x\right) = -\frac{d^2}{dx^2} l_x .$$

3.4 The Deterministic Survivorship Group

We proceed now to a second, and nonprobabilistic, interpretation of the life table. This is rooted mathematically in the concept of decrement (negative growth) rates. As such, it is related to growth-rate applications in biology and economics. It is deterministic in nature and leads to the concept of a ***deterministic survivorship group*** or ***cohort.***

A deterministic survivorship group, as represented by a life table, has the following characteristics:

- The group initially consists of l_0 lives age 0.
- The members of the group are subject, at each age of their lives, to effective annual rates of mortality (decrement) specified by the values of q_x in the life table.
- The group is closed. No further entrants are allowed beyond the initial l_0. The only decreases come as a result of the effective annual rates of mortality (decrement).

From these characteristics it follows that the progress of the group is determined by

$$\begin{aligned}
l_1 &= l_0(1 - q_0) = l_0 - d_0, \\
l_2 &= l_1(1 - q_1) = l_1 - d_1 = l_0 - (d_0 + d_1), \\
&\ \ \vdots \qquad\qquad \vdots \\
l_x &= l_{x-1}(1 - q_{x-1}) = l_{x-1} - d_{x-1} = l_0 - \sum_{y=0}^{x-1} d_y \\
&= l_0\left(1 - \frac{\sum_{y=0}^{x-1} d_y}{l_0}\right) = l_0(1 - {}_xq_0) \qquad (3.4.1)
\end{aligned}$$

where l_x is the number of lives attaining age x in the survivorship group. This chain of equalities, generated by a value l_0 called the ***radix*** and a set of q_x values, can be rewritten as

$$l_1 = l_0\, p_0,$$

$$l_2 = l_1\, p_1 = (l_0\, p_0)\, p_1,$$

$$\vdots \quad \vdots \qquad \vdots$$

$$l_x = l_{x-1}\, p_{x-1} = l_0 \left(\prod_{y=0}^{x-1} p_y\right) = l_0\, {}_xp_0. \tag{3.4.2}$$

There is an analogy between the deterministic survivorship group and the model for compound interest. Table 3.4.1 is designed to summarize some of this parallelism.

Related Concepts of the Mathematics of Compound Interest and of Deterministic Survivorship Groups

Compound Interest	Survivorship Group
$A(t)$ = Size of fund at time t, time measured in years	l_x = Size of group at age x, age measured in years
Effective annual rate of interest (increment)	Effective annual rate of mortality (decrement)
$i_t = \dfrac{A(t+1) - A(t)}{A(t)}$	$q_x = \dfrac{l_x - l_{x+1}}{l_x}$
Effective n-year rate of interest, starting at time t	Effective n-year rate of mortality, starting at age x
${}_ni_t^* = \dfrac{A(t+n) - A(t)}{A(t)}$	${}_nq_x = \dfrac{l_x - l_{x+n}}{l_x}$
Force of interest at time t	Force of mortality at age x
$\delta_t = \lim_{\Delta t \to 0} \left[\dfrac{A(t+\Delta t) - A(t)}{A(t)\,\Delta t}\right] = \dfrac{1}{A(t)} \dfrac{dA(t)}{dt}$	$\mu(x) = \lim_{\Delta x \to 0} \left(\dfrac{l_x - l_{x+\Delta x}}{l_x\,\Delta x}\right) = -\dfrac{1}{l_x} \dfrac{dl_x}{dx}$

*There is no universally accepted symbol for an effective n-year rate of interest.

The headings of the ${}_tq_x$, l_x, and ${}_td_x$ columns in Table 3.3.1 refer to the deterministic survivorship group interpretation. Although the mathematical foundations of the random survivorship group and the deterministic survivorship group are different, the resulting functions q_x, l_x, and d_x have the same mathematical properties and subsequent analysis. The random survivorship group concept has the advantage of allowing for the full use of probability theory. The deterministic survivorship group is conceptually simple and easy to apply but does not take account of random variation in the number of survivors.

3.5 Other Life Table Characteristics

In this section we derive expressions for some commonly used characteristics of the distributions of $T(x)$ and $K(x)$ and introduce a general method for computing several of these characteristics.

3.5.1 Characteristics

The expected value of $T(x)$, denoted by $\mathring{e}_x$, is called the ***complete-expectation-of-life***. By definition and an integration by parts, we have

$$\begin{aligned}\mathring{e}_x = \mathrm{E}[T(x)] &= \int_0^\infty t\ {}_tp_x\ \mu(x+t)\ dt \\ &= \int_0^\infty t\ d_t(-{}_tp_x) \\ &= t(-{}_tp_x)\Big|_0^\infty + \int_0^\infty {}_tp_x\ dt.\end{aligned} \tag{3.5.1}$$

The existence of $\mathrm{E}[T(x)]$ is equivalent to the $\lim_{t\to\infty} t(-{}_tp_x) = 0$. Thus

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt. \tag{3.5.2}$$

The complete-expectation-of-life at various ages is often used to compare levels of public health among different populations.

A similar integration by parts yields equivalent expressions for $\mathrm{E}[T(x)^2]$:

$$\begin{aligned}\mathrm{E}[T(x)^2] &= \int_0^\infty t^2\ {}_tp_x\ \mu(x+t)\ dt \\ &= 2\int_0^\infty t\ {}_tp_x\ dt.\end{aligned} \tag{3.5.3}$$

This result is useful in the calculation of Var $[T(x)]$ by

$$\begin{aligned}\mathrm{Var}[T(x)] &= \mathrm{E}[T(x)^2] - \mathrm{E}[T(x)]^2 \\ &= 2\int_0^\infty t\ {}_tp_x\ dt - \mathring{e}_x^2.\end{aligned} \tag{3.5.4}$$

In these developments, we assume that $\mathrm{E}[T(x)]$ and $\mathrm{E}[T(x)^2]$ exist. One can construct s.f.'s such as $s(x) = (1+x)^{-1}$ where this would not be true.

Other characteristics of the distribution of $T(x)$ can be determined. The ***median future lifetime*** of (x), to be denoted by $m(x)$, can be found by solving

$$\Pr[T(x) > m(x)] = \frac{1}{2}$$

or

$$\frac{s[x + m(x)]}{s(x)} = \frac{1}{2} \tag{3.5.5}$$

for $m(x)$. In particular, $m(0)$ is given by solving $s[m(0)] = 1/2$. We can also find the ***mode*** of the distribution of $T(x)$ by locating the value of t that yields a maximum value of ${}_tp_x\ \mu(x + t)$.

The expected value of $K(x)$ is denoted by e_x and is called the ***curtate-expectation-of-life.*** By definition and use of summation by parts as described in Appendix 5, we have

$$\begin{aligned} e_x = \mathrm{E}[K] &= \sum_{k=0}^{\infty} k\ {}_kp_x\ q_{x+k} \\ &= \sum_{k=0}^{\infty} k\ \Delta(-{}_kp_x) \\ &= k(-{}_kp_x)\Big|_0^{\infty} + \sum_{k=0}^{\infty} {}_{k+1}p_x . \end{aligned} \tag{3.5.6}$$

Again, the existence of $\mathrm{E}[K(x)]$ is equivalent to the $\lim\limits_{k\to\infty} k(-{}_kp_x) = 0$. Thus, with a change of the summation variable,

$$e_x = \sum_{k=1}^{\infty} {}_kp_x . \tag{3.5.7}$$

Following the outline used for the continuous model and using summation by parts, we have

$$\begin{aligned} \mathrm{E}[K(x)^2] &= \sum_{k=0}^{\infty} k^2\ {}_kp_x\ q_{x+k} \\ &= \sum_{k=0}^{\infty} k^2\ \Delta(-{}_kp_x) \\ &= k^2(-{}_kp_x)\Big|_0^{\infty} + \sum_{k=0}^{\infty} (\Delta k^2)({}_{k+1}p_x). \end{aligned} \tag{3.5.8}$$

The existence of $\mathrm{E}[K(x)^2]$ is equivalent to $\lim\limits_{k\to\infty} k^2(-{}_kp_x) = 0$. With a change of the summation variable,

$$\mathrm{E}[K(x)^2] = \sum_{k=0}^{\infty} (2k + 1)\ {}_{k+1}p_x = \sum_{k=1}^{\infty} (2k - 1)\ {}_kp_x \tag{3.5.9}$$

Now,

$$\begin{aligned} \mathrm{Var}(K) &= \mathrm{E}[K^2] - \mathrm{E}[K]^2 \\ &= \sum_{k=1}^{\infty} (2k - 1)\ {}_kp_x - e_x^2 . \end{aligned} \tag{3.5.10}$$

To complete the discussion of some of the entries in Table 3.3.1, we must define additional functions. The symbol L_x denotes the total expected number of years lived between ages x and $x + 1$ by survivors of the initial group of l_0 lives. We have

$$L_x = \int_0^1 t\, l_{x+t}\, \mu(x + t)\, dt + l_{x+1} \tag{3.5.11}$$

where the integral counts the years lived of those who die between ages x and $x + 1$, and the term l_{x+1} counts the years lived between ages x and $x + 1$ by those who survive to age $x + 1$. Integration by parts yields

$$\begin{aligned} L_x &= -\int_0^1 t\, dl_{x+t} + l_{x+1} \\ &= -t\, l_{x+t}\Big|_0^1 + \int_0^1 l_{x+t}\, dt + l_{x+1} \\ &= \int_0^1 l_{x+t}\, dt. \end{aligned} \tag{3.5.12}$$

The function L_x is also used in defining the ***central-death-rate*** over the interval from x to $x + 1$, denoted by m_x where

$$m_x = \frac{\int_0^1 l_{x+t}\, \mu(x + t)\, dt}{\int_0^1 l_{x+t}\, dt} = \frac{l_x - l_{x+1}}{L_x}. \tag{3.5.13}$$

An application of this function is found in Chapter 10.

The definitions for m_x and L_x can be extended to age intervals of length other than one:

$$\begin{aligned} {}_nL_x &= \int_0^n t\, l_{x+t}\, \mu(x + t)\, dt + n l_{x+n} \\ &= \int_0^n l_{x+t}\, dt, \end{aligned} \tag{3.5.14}$$

$${}_nm_x = \frac{\int_0^n l_{x+t}\, \mu(x + t)\, dt}{\int_0^n l_{x+t}\, dt} = \frac{l_x - l_{x+n}}{{}_nL_x}. \tag{3.5.15}$$

For the random survivorship group, ${}_nL_x$ is the total expected number of years lived between ages x and $x + n$ by the survivors of the initial group of l_0 lives and ${}_nm_x$ is the average death rate experienced by this group over the interval $(x, x + n)$.

The symbol T_x denotes the total number of years lived beyond age x by the survivorship group with l_0 initial members. We have

$$T_x = \int_0^\infty t\, l_{x+t}\, \mu(x+t)\, dt$$

$$= -\int_0^\infty t\, d_t\, l_{x+t}$$

$$= \int_0^\infty l_{x+t}\, dt. \qquad (3.5.16)$$

The final expression can be interpreted as the integral of the total time lived between ages $x + t$ and $x + t + dt$ by the l_{x+t} lives who survive to that age interval. We also recognize T_x as the limit of ${}_nL_x$ as n goes to infinity.

The average number of years of future lifetime of the l_x survivors of the group at age x is given by

$$\frac{T_x}{l_x} = \frac{\int_0^\infty l_{x+t}\, dt}{l_x}$$

$$= \int_0^\infty {}_tp_x\, dt$$

$$= \mathring{e}_x,$$

as determined in (3.5.1) and (3.5.2).

We can express the average number of years lived between x and $x + n$ by the l_x survivors at age x as

$$\frac{{}_nL_x}{l_x} = \frac{\int_0^n l_{x+t}\, dt}{l_x}$$

$$= \frac{T_x - T_{x+n}}{l_x}$$

$$= \int_0^n {}_tp_x\, dt. \qquad (3.5.17)$$

This function is the ***n-year temporary complete life expectancy*** of (x) and is denoted by $\mathring{e}_{x:\overline{n|}}$. (See Exercise 3.16.)

A final function, related to the interpretation of the life table developed in this section, is the average number of years lived between ages x and $x + 1$ by those of the survivorship group who die between those ages. This function is denoted by $a(x)$ and is defined by

$$a(x) = \frac{\int_0^1 t\, l_{x+t}\, \mu(x+t)\, dt}{\int_0^1 l_{x+t}\, \mu(x+t)\, dt}. \tag{3.5.18}$$

For the probabilistic view of the life table, we would have

$$a(x) = \frac{\int_0^1 t\ {}_tp_x\ \mu(x+t)\, dt}{\int_0^1 {}_tp_x\ \mu(x+t)\, dt} = \mathrm{E}[T|T < 1].$$

If we assume that

$$l_{x+t}\ \mu(x+t)\, dt = d_x\, dt \qquad 0 \le t \le 1,$$

that is, if deaths are uniformly distributed in the year of age, we have

$$a(x) = \int_0^1 t\, dt = \frac{1}{2}.$$

This is the usual approximation for $a(x)$, except for young and old years of age where Figure 3.3.2 shows that the assumption may be inappropriate.

Example 3.5.1

Show that

$$L_x = a(x)\, l_x + [1 - a(x)]\, l_{x+1}$$

and

$$L_x \cong \frac{l_x + l_{x+1}}{2}.$$

Solution:

From (3.5.11), (3.5.12), and (3.5.18), we have

$$a(x) = \frac{L_x - l_{x+1}}{l_x - l_{x+1}}$$

or

$$L_x = a(x)\, l_x + [1 - a(x)]\, l_{x+1}.$$

The formula

$$L_x \cong \frac{l_x + l_{x+1}}{2}$$

can be justified by using the trapezoidal rule for approximate integration on (3.5.12). ▼

Key life table terminology, defined in Sections 3.3–3.5, is summarized as part of Table 3.9.1 in Section 3.9.

3.5.2 Recursion Formulas

Example 3.5.1 illustrates the use of a numerical analysis technique to evaluate a life table characteristic. The trapezoidal rule for approximate integration is used. The calculation of complete and curtate expectations-of-life can be used to illustrate another computational tool called ***recursion formulas.*** The application of recursion formulas in this book typically involves one of two forms:

Backward Recursion Formula

$$u(x) = c(x) + d(x)\,u(x+1) \tag{3.5.19}$$

or

Forward Recursion Formula

$$u(x+1) = -\frac{c(x)}{d(x)} + \frac{1}{d(x)}\,u(x). \tag{3.5.20}$$

The variable x is usually a non-negative integer.

To evaluate a function $u(x)$, for a domain of non-negative integer values of x, we need to have available values of $c(x)$ and $d(x)$ and a starting value of $u(x)$. This procedure is used in subsequent chapters and is illustrated in Table 3.5.1 where backward recursion formulas are developed to compute e_x and $\mathring{e}_x$.

Backward Recursion Formulas for e_x and $\mathring{e}_x$

Step	e_x	$\mathring{e}_x$
1. Basic equation	$e_x = \sum_{k=1}^{\infty} {}_kp_x$	$\mathring{e}_x = \int_0^{\infty} {}_sp_x\,ds$
2. Separate the operation	$e_x = p_x + \sum_{k=2}^{\infty} {}_kp_x$	$\mathring{e}_x = \int_0^1 {}_sp_x\,ds + \int_1^{\infty} {}_sp_x\,ds$
3. Factor p_x and change variable in the operation	$e_x = p_x + p_x \sum_{k=1}^{\infty} {}_kp_{x+1}$ $= p_x + p_x\,e_{x+1}$	$\mathring{e}_x = \int_0^1 {}_sp_x\,ds + p_x \int_0^{\infty} {}_tp_{x+1}\,dt$ $= \int_0^1 {}_sp_x\,ds + p_x\,\mathring{e}_{x+1}$
4. Recursion formula[a]	$u(x) = e_x$, $c(x) = p_x$ $d(x) = p_x$	$u(x) = \mathring{e}_x$, $c(x) = \int_0^1 {}_sp_x\,ds$ $d(x) = p_x$
5. Starting value[b]	$e_\omega = u(\omega) = 0$	$\mathring{e}_\omega = u(x) = 0$

[a]The integral $c(x) = \int_0^1 {}_sp_x\,ds$ can be evaluated using the trapezoidal rule as $c(x) = (1 + p_x)/2$.
[b]From Section 3.3.1 we have $s(x) = 0$, $x \ge \omega$, and $s(x) > 0$, $x < \omega$. In this development we will assume that ω is an integer.

3.6 Assumptions for Fractional Ages

In this chapter we have discussed the continuous random variable remaining lifetime, T, and the discrete random variable curtate-future-lifetime, K. The life table developed in Section 3.3 specifies the probability distribution of K completely. To specify the distribution of T, we must postulate an analytic form or adopt a life table and an assumption about the distribution between integers.

We will examine three assumptions that are widely used in actuarial science. These will be stated in terms of the s.f. and in a form to show the nature of interpolation over the interval $(x, x + 1)$ implied by each assumption. In each statement, x is an integer and $0 \le t \le 1$. The assumptions are the following:

- ***Linear interpolation:*** $s(x + t) = (1 - t)\, s(x) + t\, s(x + 1)$. This is known as the ***uniform distribution*** or, perhaps more properly, a uniform distribution of deaths assumption within each year of age. Under this assumption ${}_tp_x$ is a linear function.
- ***Exponential interpolation,*** or linear interpolation on $\log s(x + t)$: $\log s(x + t) = (1 - t) \log s(x) + t \log s(x + 1)$. This is consistent with the assumption of a ***constant force*** of mortality within each year of age. Under this assumption ${}_tp_x$ is exponential.
- ***Harmonic interpolation:*** $1 / s(x + t) = (1 - t) / s(x) + t / s(x + 1)$. This is what is known as the ***hyperbolic*** (historically ***Balducci****) assumption, for under it ${}_tp_x$ is a hyperbolic curve.

With these basic definitions, formulas can be derived for other standard probability functions in terms of life table probabilities. These results are presented in Table 3.6.1. Note that we just as well could have elected to propose equivalent definitions in terms of the p.d.f., the d.f., or the force of mortality.

The derivations of the entries in Table 3.6.1 are exercises in substituting the stated assumption about $s(x + t)$ into the appropriate formulas of Sections 3.2 and 3.3. We will illustrate the process for the uniform distribution of deaths, an assumption that is used extensively throughout this text.

To derive the first entry in the uniform distribution column, we start with

$$
{}_tq_x = \frac{s(x) - s(x + t)}{s(x)} \qquad 0 \le t \le 1,
$$

then substitute for $s(x + t)$,

$$
{}_tq_x = \frac{s(x) - [(1 - t)\, s(x) + t\, s(x + 1)]}{s(x)} = \frac{t\, [s(x) - s(x + 1)]}{s(x)} = t q_x .
$$

For the second entry, we use (3.2.13) and

$$
\mu(x + t) = - \frac{s'(x + t)}{s(x + t)} ;
$$

*This assumption is named after G. Balducci, an Italian actuary, who pointed out its role in the traditional actuarial method of constructing life tables.

Probability Theory Functions for Fractional Ages

Function	Uniform Distribution	Constant Force	Hyperbolic
	Assumption		
${}_tq_x$	tq_x	$1 - p_x^t$	$\dfrac{tq_x}{1-(1-t)q_x}$
$\mu(x+t)$	$\dfrac{q_x}{1-tq_x}$	$-\log p_x$	$\dfrac{q_x}{1-(1-t)q_x}$
${}_{1-t}q_{x+t}$	$\dfrac{(1-t)\,q_x}{1-tq_x}$	$1-p_x^{1-t}$	$(1-t)q_x$
${}_yq_{x+t}$	$\dfrac{yq_x}{1-tq_x}$	$1-p_x^y$	$\dfrac{yq_x}{1-(1-y-t)q_x}$
${}_tp_x$	$1-tq_x$	p_x^t	$\dfrac{p_x}{1-(1-t)q_x}$
${}_tp_x\mu(x+t)$	q_x	$-p_x^t \log p_x$	$\dfrac{q_x\,p_x}{[1-(1-t)q_x]^2}$

Note that, in this table, x is an integer, $0 < t < 1$, $0 \le y \le 1$, $y + t \le 1$. For rows one, three, four, and five, the relationships also hold for $t = 0$ and $t = 1$.

then, substituting for $s(x+t)$, we have

$$\mu(x+t) = \frac{[s(x) - s(x+1)]}{[(1-t)\,s(x) + t\,s(x+1)]}.$$

Dividing both numerator and denominator of the right-hand side by $s(x)$ yields

$$\mu(x+t) = \frac{q_x}{(1-tq_x)}.$$

The third entry is the special case of the fourth entry with $y = 1 - t$.

For the fourth entry we start with

$${}_yq_{x+t} = \frac{s(x+t) - s(x+t+y)}{s(x+t)},$$

then substitute for $s(x+t)$ and $s(x+t+y)$ to obtain

$$\begin{aligned}
{}_yq_{x+t} &= \frac{[(1-t)\,s(x) + t\,s(x+1)] - [(1-t-y)\,s(x) + (t+y)\,s(x+1)]}{(1-t)\,s(x) + t\,s(x+1)} \\
&= \frac{y[s(x) - s(x+1)]/s(x)}{\{s(x) - t[s(x) - s(x+1)]\}/s(x)} \\
&= \frac{yq_x}{1-tq_x}.
\end{aligned}$$

The fifth entry is the complement of the first, and the final entry for the uniform distribution column is the product of the second and fifth entries.

If, as before, x is an integer, insight can be obtained by defining a random variable $S = S(x)$ by

$$T = K + S \tag{3.6.1}$$

where T is time-until-death, K is the curtate-future-lifetime, and S is the random variable representing the fractional part of a year lived in the year of death. Since K is a non-negative integer random variable and S is a continuous-type random variable with all of its probability mass on the interval (0, 1), we can examine their joint distribution by writing

$$\begin{aligned} \Pr[(K = k) \cap (S \le s)] &= \Pr(k < T \le k + s) \\ &= {}_kp_x\ {}_sq_{x+k}. \end{aligned}$$

Now, using the expression for ${}_sq_{x+k}$ under the uniform distribution assumption as shown in Table 3.6.1, we have

$$\begin{aligned} \Pr[(K = k) \cap (S \le s)] &= {}_kp_x\ s\ q_{x+k} \\ &= {}_{k|}q_x\ s \\ &= \Pr(K = k) \Pr(S \le s). \end{aligned} \tag{3.6.2}$$

Therefore, the joint probability involving K and S can be factored into separate probabilities of K and S. It follows that, under the uniform distribution of deaths assumption, the random variables K and S are independent. Since $\Pr(S \le s) = s$ is the d.f. of a uniform distribution on (0, 1), S has such a uniform distribution.

Example 3.6.1

Under the constant force of mortality assumption, are the random variables K and S independent?

Solution:

Using entries from Table 3.6.1 for the constant force assumption, we obtain

$$\begin{aligned} \Pr[(K = k) \cap (S \le s)] &= {}_kp_x\ {}_sq_{x+k} \\ &= {}_kp_x\ [1 - (p_{x+k})^s]. \end{aligned}$$

To discuss this result, we distinguish two cases:

- If p_{x+k} is not independent of k, we cannot factor the joint probability of K and S into separate probabilities. We conclude that K and S are not independent.
- In the special case where $p_{x+k} = p_x$, a constant,

$$\begin{aligned} \Pr[(K = k) \cap (S \le s)] &= p_x^k(1 - p_x^s) = \frac{(1 - p_x)p_x^k(1 - p_x^s)}{(1 - p_x)} \\ &= \Pr(K = k) \Pr(S \le s). \end{aligned}$$

For this special case we conclude that K and S are independent under the constant force assumption. ▼

Example 3.6.2

Under the assumption of uniform distribution of deaths, show that

a. $\mathring{e}_x = e_x + \frac{1}{2}$

b. $\text{Var}(T) = \text{Var}(K) + \frac{1}{12}$.

Solution:

a. $\mathring{e}_x = \text{E}[T] = \text{E}[K + S]$
$= \text{E}[K] + \text{E}[S]$
$= e_x + \frac{1}{2}$.

b. $\text{Var}(T) = \text{Var}(K + S)$.

From the independence of K and S, under the uniform distribution assumption, it follows that

$$\text{Var}(T) = \text{Var}(K) + \text{Var}(S).$$

Further, since S is uniformly distributed over (0, 1),

$$\text{Var}(T) = \text{Var}(K) + \frac{1}{12}.$$

▼

3.7 Some Analytical Laws of Mortality

There are three principal justifications for postulating an analytic form for mortality or survival functions. The first is philosophical. Many phenomena studied in physics can be explained efficiently by simple formulas. Therefore, using biological arguments, some authors have suggested that human survival is governed by an equally simple law. The second justification is practical. It is easier to communicate a function with a few parameters than it is to communicate a life table with perhaps 100 parameters or mortality probabilities. In addition, some of the analytic forms have elegant properties that are convenient in evaluating probability statements that involve more than one life. The third justification for a simple analytic survival function is the ease of estimating a few parameters of the function from mortality data.

The support for simple analytic survival functions has declined in recent years. Many feel that the belief in universal laws of mortality is naive. With the increasing speed and storage capacity of computers, the advantages of some analytic forms in computations involving more than one life are no longer of great importance. Nevertheless, some interesting research has recently reiterated the biological arguments for analytic laws of mortality.

In Table 3.7.1, several families of simple analytic mortality and survival functions, corresponding to various postulated laws, are displayed. The names of the originators of the laws and the dates of publication are included for identification purposes.

Mortality and Survival Functions under Various Laws

Originator	$\mu(x)$	$s(x)$	Restrictions
De Moivre (1729)	$(\omega - x)^{-1}$	$1 - \dfrac{x}{\omega}$	$0 \le x < \omega$
Gompertz (1825)	Bc^x	$\exp[-m(c^x - 1)]$	$B > 0,\ c > 1,\ x \ge 0$
Makeham (1860)	$A + Bc^x$	$\exp[-Ax - m(c^x - 1)]$	$B > 0,\ A \ge -B,\ c > 1,\ x \ge 0$
Weibull (1939)	kx^n	$\exp(-ux^{n+1})$	$k > 0,\ n > 0,\ x \ge 0$

Note:

- The special symbols are defined as

$$m = \frac{B}{\log c}, \qquad u = \frac{k}{(n+1)}.$$

- Gompertz's law is a special case of Makeham's law with $A = 0$.
- If $c = 1$ in Gompertz's and Makeham's laws, the exponential (constant force) distribution results.
- In connection with Makeham's law, the constant A has been interpreted as capturing the accident hazard, and the term Bc^x as capturing the hazard of aging.

The entries in the $s(x)$ column of Table 3.7.1 were obtained by substituting into (3.2.16). For example, for Makeham's law, we have

$$\begin{aligned} s(x) &= \exp[-\textstyle\int_0^x (A + Bc^s)ds] \\ &= \exp\left[-Ax - B\frac{(c^x - 1)}{\log c}\right] \\ &= \exp[-Ax - m\,(c^x - 1)] \end{aligned}$$

where $m = B/\log c$.

Two objectives governed the development of a mortality table for computational purposes in the examples and exercises of this book. One objective was to have mortality rates in the middle of the range of variation for groups, such variation caused by factors such as residence, gender, insured status, annuity status, marital status, and occupation. The second objective was to have a Makeham law at most ages to illustrate how calculations for multiple lives can be performed.

The Illustrative Life Table in Appendix 2A is based on the Makeham law for ages 13 and greater,

$$1{,}000\ \mu(x) = 0.7 + 0.05\ (10^{0.04})^x. \tag{3.7.1}$$

The calculations of the basic functions q_x, l_x, and d_x from (3.7.1) were all done directly from (3.7.1) instead of calculating l_x and d_x from the truncated values of q_x. It was found that the latter choice would make little difference in the applications. It should be kept in mind that the Illustrative Life Table, as its name implies, is for illustrative purposes only.

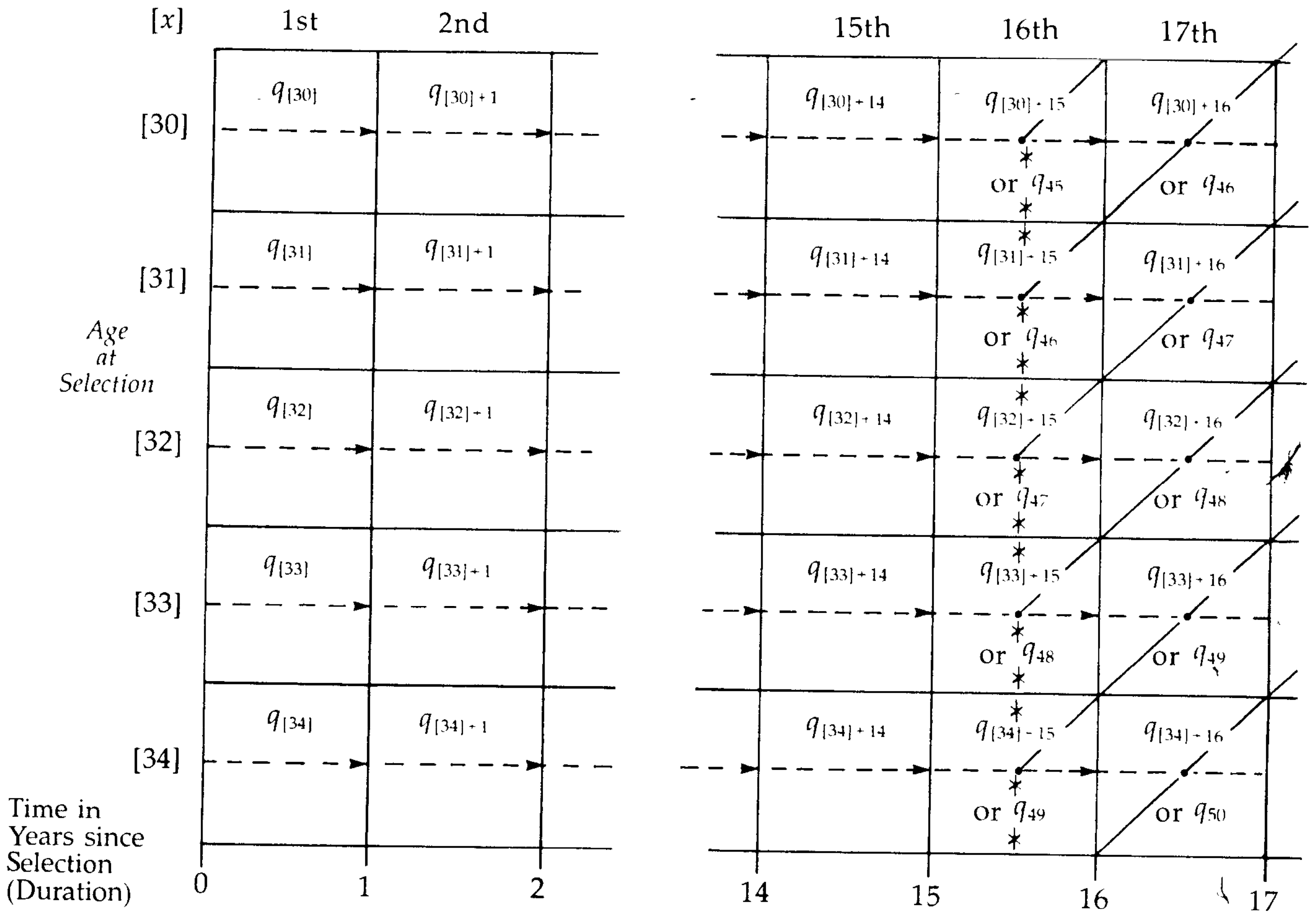
3.8 Select and Ultimate Tables

In Section 3.2 we discussed how ${}_tp_x$ [the probability that (x) will survive to age $x + t$] might be interpreted in two ways. The first interpretation was that the probability can be evaluated by a survival function appropriate for newborns, under the single hypothesis that the newborn has survived to age x. This interpretation has been the basis of the notation and development of the formulas. The second interpretation was that additional knowledge available about the life at age x might make the original survival function inappropriate for evaluating probability statements about the future lifetime of (x). For example, the life might have been underwritten and accepted for life insurance at age x. This information would lead us to believe that (x)'s future-lifetime distribution is different from what we might otherwise assume for lives age x. As a second example, the life might have become disabled at age x. This information would lead us to believe that the future-lifetime distribution for (x) is different from that of those not disabled at age x. In these two illustrations, a special force of mortality that incorporates the particular information available at age x would be preferred. Without this particular information for (x), the form of mortality at duration t would be a function of only the attained age $x + t$, denoted in the previous sections by $\mu(x + t)$. Given the additional information at x, the force of mortality at $x + t$ is a function of this information at x and duration t. Its notation will be $\mu_x(t)$, showing separately the age, x, at which the additional information was available, and the duration, t. The additional information is usually not explicit in the notation but is conveyed by the context. In other words, the complete model for such lives is a set of survival functions including one for each age at which information is available on issue of insurance, disability, and so on. This set of survival functions can be thought of as a function of two variables. One variable is the age at ***selection*** (e.g., at policy issue or the onset of disability), $[x]$, and the second variable is the duration since policy issue or duration since selection, t. Then each of the usual life table functions associated with this bivariate survival function is a two-dimensional array on $[x]$ and t. Note the bracket notation to indicate which variable identifies the age of selection.

The schematic diagram in Figure 3.8.1 illustrates these ideas. For instance, suppose some special information is available about a group of lives age 30. Perhaps they have been accepted for life insurance or perhaps they have become disabled. A special life table can be built for these lives. The conditional probability of death in each year of duration would be denoted by $q_{[30]+i}$, $i = 0, 1, 2, \ldots$, and would be entered on the first row of Figure 3.8.1. The subscript reflects the bivariate nature of this function with the bracketed thirty, [30], denoting that the survival function in the first row is conditional on special information available at age 30. The second row of Figure 3.8.1 would contain the probabilities of death for lives on which the

Select, Ultimate, and Aggregate Mortality, 15-Year Select Period

Year Following Selection



- - - - - → Path followed by a survivorship group selected at [x].

•——• Links (age at selection and duration) cells after a 15-year select period into attained age groups.

×-×-×-× Alternative path followed by survivorship groups after a 15-year select period. These probabilities constitute an ultimate mortality table.

Notes:

1. In biostatistics the select table index [x] may not be age. For example, in cancer research, [x] could be a classification index that depends on the size and location of the tumor, and time following selection would be measured from the time of diagnosis.
2. Ultimate mortality, following a 15-year select period, for age [x] + 15, would be estimated by using observations from all cells identified by [x− j] + 15 + j, for j = 0, 1, 2, ... Therefore, $q_{[x]+15} = q_{x+15}$ is estimated by a weighted average of mortality estimates from several different selection groups. If the effect of selection is not small, the resulting estimate will be influenced by the amount of data from the various cells.

special information became available at age 31. In actuarial science such a two-dimensional life table is called a ***select life*** table.

The impact of selection on the distribution of time-until-death, T, may diminish following selection. Beyond this time period the q's at equal attained ages would be essentially equal regardless of the ages at selection. More precisely, if there is a smallest integer r such that $|q_{[x]+r} - q_{[x-j]+r+j}|$ is less than some small positive constant for all ages of selection $[x]$ and for all $j > 0$, it would be economical to construct a set of ***select-and-ultimate*** tables by truncation of the two-dimensional array after the $(r + 1)$ column. For durations beyond r we would use

$$q_{[x-j]+r+j} \cong q_{[x]+r} \qquad j > 0.$$

The first r years of duration comprise the ***select period.***

The resulting array remains a set of life tables, one for each age at selection. For a single age at selection, the life table entries are horizontal during the select period and then vertical during the ultimate period. This is shown in Figure 3.8.1 by the arrows.

The Society of Actuaries mortality studies of lives who were issued individual life insurance on a standard basis use a 15-year select period as illustrated in Figure 3.8.1; that is, it is accepted that

$$q_{[x-j]+15+j} \cong q_{[x]+15} \qquad j > 0.$$

Beyond the select period, the probabilities of death are subscripted by attained age only; that is, $q_{[x-j]+r+j}$ is written as q_{x+r}. For instance, with $r = 15$, $q_{[30]+15}$ and $q_{[25]+20}$ would both be written as q_{45}.

A life table in which the functions are given only for attained ages is called an ***aggregate table,*** Table 3.3.1, for instance. The last column in a select-and-ultimate table is a special aggregate table that is usually referred to as an ***ultimate table,*** to reflect the select table setting.

Table 3.8.1 contains mortality probabilities and corresponding values of the $l_{[x]+k}$ function, as given in the Permanent Assurances, Females, 1979–82, Table, published by the Institute of Actuaries and the Faculty of Actuaries; it is denoted as the

Excerpt from the AF80 Select-and-Ultimate Table

$[x]$	(1) $1{,}000\,q_{[x]}$	(2) $1{,}000\,q_{[x]+1}$	(3) $1{,}000\,q_{x+2}$	(4) $l_{[x]}$	(5) $l_{[x]+1}$	(6) l_{x+2}	(7) $x + 2$
30	0.222	0.330	0.422	9 906.7380	9 904.5387	9 901.2702	32
31	0.234	0.352	0.459	9 902.8941	9 900.5769	9 897.0919	33
32	0.250	0.377	0.500	9 898.7547	9 896.2800	9 892.5491	34
33	0.269	0.407	0.545	9 894.2903	9 891.6287	9 887.6028	35
34	0.291	0.441	0.596	9 889.4519	9 886.5741	9 882.2141	36

AF80 Table. This table has a 2-year select period and is easier to use for illustrative purposes than tables with a 15-year select period such as the Basic Tables, published by the Society of Actuaries.

In Table 3.8.1 we observe three mortality probabilities for age 32, namely,

$$q_{[32]} = 0.000250 < q_{[31]+1} = 0.000352 < q_{32} = 0.000422.$$

The order among these probabilities is plausible since mortality should be lower for lives immediately after acceptance for life insurance. Column (3) can be viewed as providing ultimate mortality probabilities.

Given the 1-year mortality rates of a select-and-ultimate table, the construction of the corresponding select-and-ultimate life table (survival functions) is started with the ultimate portion. Formulas such as (3.4.1) can be used, which would yield a set of values of $l_{x+r} = l_{[x]+r}$ where r is the length of the select period. We would then complete the select segments by using the relation

$$l_{[x]+r-k-1} = \frac{l_{[x]+r-k}}{p_{[x]+r-k-1}} \qquad k = 0, 1, 2, \ldots, r-1,$$

working from duration $r - 1$ down to 0.

Example 3.8.1

Use Table 3.8.1 to evaluate

a. ${}_2p_{[30]}$ b. ${}_5p_{[30]}$

c. ${}_{1|}q_{[31]}$ d. ${}_3q_{[31]+1}$.

Solution:

Formulas developed earlier in this chapter can be adapted to select-and-ultimate tables yielding

a. $${}_2p_{[30]} = \frac{l_{[30]+2}}{l_{[30]}} = \frac{l_{32}}{l_{[30]}} = \frac{9{,}901.2702}{9{,}906.7380} = 0.99945$$

b. $${}_5p_{[30]} = \frac{l_{35}}{l_{[30]}} = \frac{9{,}887.6028}{9{,}906.7380} = 0.99807$$

c. $${}_{1|}q_{[31]} = \frac{l_{[31]+1} - l_{33}}{l_{[31]}} = \frac{9{,}900.5769 - 9{,}897.0919}{9{,}902.8941} = 0.00035$$

d. $${}_3q_{[31]+1} = \frac{l_{[31]+1} - l_{35}}{l_{[31]+1}} = \frac{9{,}900.5769 - 9{,}887.6028}{9{,}900.5769} = 0.00131.$$ ▼

Chapter 3 Concepts

Symbol	Name or Description of the Concept
(x)	Notation for a life age x
$[x]$	Age, or other status, at selection
X	Age at death, a random variable
$T(x)$	Future lifetime of (x), equals $X - x$
$K(x)$	Curtate-future-lifetime of (x), equals the integer part of $T(x)$
$S(x)$	Future lifetime of (x) within the year of death, equals $T(x) - K(x)$
$s(x)$	Survival function, equal to the probability that a newborn will live to at least x
$\mu(x)$	Force of mortality at age x in an aggregate life table
$\mu_x(t)$	Force of mortality at attained age $x + t$ given selection at age x
${}_tq_x$	Probability that (x) dies within t years
${}_tp_x$	Probability that (x) survives at least t years
${}_{t\vert u}q_x$	Probability that (x) dies between t and $t + u$ years
$\mathring{e}_x$	Complete expectation of life for (x), equals $E[T(x)]$
e_x	Curtate expectation of life for (x), equals $E[K(x)]$
$\mathcal{L}(x)$	Cohort's number of survivors to age x, a random variable
${}_n\mathcal{D}_x$	Cohort's number of deaths between ages x and $x + n$
l_x	Expected number of survivors at age x, equals $E[\mathcal{L}(x)]$
${}_nd_x$	Expected number of deaths between ages x and $x + n$, equals $E[{}_n\mathcal{D}_x]$
${}_nL_x$	Expected number of years lived between ages x and $x + n$ by survivors to age x of the initial group of l_0 lives
T_x	Expected number of years lived beyond age x by the survivors to age x of the initial group of l_0 lives
m_x	Central death rate over the interval $(x, x + 1)$
ω	Omega, the limiting age of a life table

3.9 Notes and References

Table 3.9.1 summarizes this chapter's new concepts with their names, symbols, and descriptions. Life tables are a cornerstone of actuarial science. Consequently they are extensively discussed in several English-language textbooks on life contingencies:

- King (1902)
- Spurgeon (1932)
- Jordan (1967)
- Hooker and Longley-Cook (1953)
- Neill (1977).

These have been used in actuarial education. In addition, life tables are used by biostatisticians. An exposition of this latter approach is given by Chiang (1968) and Elandt-Johnson and Johnson (1980). The deterministic rate function interpretation is discussed by Allen (1907). London (1988) summarizes several methods for estimating life tables from data.

The historically important analytic forms for survival functions are referred to in Table 3.6. Brillinger (1961) provides an argument for certain analytic forms from the viewpoint of statistical life testing. Tenenbein and Vanderhoof (1980) restate the case for analytic laws of mortality and develop formulas for select mortality. Balducci's (1921) contribution was preceded by a remarkable set of papers by Wittstein (1873). Wittstein's papers were published first in German and translated into English by T. B. Sprague. Some of the methods for evaluating probabilities for fractional ages are reviewed by Mereu (1961) and in Batten's textbook on mortality estimation (1978) (see also Seal's 1977 historical review). Discussions of the length of the select period for various types of insurance selection procedures have a long history, for example, Williamson (1942), Thompson (1934), and Jenkins (1943). The Society of Actuaries 1975–80 Basic Tables use a 15-year select period and are published in *TSA Reports 1982*. International Actuarial Notation is outlined in *TASA 48* (1947).

We planned to use the 1989–91 U.S. Life Table for illustrative purposes in Table 3.2.1, but this plan was not realized because the life tables based on the 1990 U.S. Census were not completed when this chapter was revised.

Exercises

Section 3.2

3.1. Using the ideas summarized in Table 3.2.1, complete the entries below.

$s(x)$	$F_X(x)$	$f_X(x)$	$\mu(x)$
			$\tan x,\ 0 \le x \le \frac{\pi}{2}$
$e^{-x},\ x \ge 0$			
	$1 - \frac{1}{1+x},\ x \ge 0$		

3.2. Confirm that each of the following functions can serve as a force of mortality. Show the corresponding survival function. In each case $x \ge 0$.

a. $B\,c^x$ $\quad B > 0 \quad c > 1$ (Gompertz)

b. $k\,x^n$ $\quad n > 0 \quad k > 0$ (Weibull)

c. $a\,(b + x)^{-1}$ $\quad a > 0 \quad b > 0$ (Pareto)

3.3. Confirm that the following can serve as a survival function. Show the corresponding $\mu(x)$, $f_X(x)$, and $F_X(x)$.

$$s(x) = e^{-x^3/12} \qquad x \ge 0.$$

3.4. State why each of the following functions cannot serve in the role indicated by the symbol:

a. $\mu(x) = (1 + x)^{-3} \qquad x \ge 0$

b. $s(x) = 1 - \frac{22x}{12} + \frac{11x^2}{8} - \frac{7x^3}{24} \qquad 0 \le x \le 3$

c. $f_X(x) = x^{n-1} e^{-x/2} \qquad x \ge 0,\ n \ge 1.$

3.5. If $s(x) = 1 - x/100$, $0 \le x \le 100$, calculate
a. $\mu(x)$ b. $F_X(x)$
c. $f_X(x)$ d. $\Pr(10 < X < 40)$.

3.6. Given the survival function of Exercise 3.5, determine the survival function, force of mortality, and p.d.f. of the future lifetime of (40).

3.7. If $s(x) = [1 - (x/100)]^{1/2}$, $0 \le x \le 100$, evaluate
a. ${}_{17}p_{19}$ b. ${}_{15}q_{36}$ c. ${}_{15|13}q_{36}$
d. $\mu(36)$ e. $E[T(36)]$.

3.8. Confirm that ${}_{k|}q_0 = -\Delta s(k)$, and that $\sum_{k=0}^{\infty} {}_{k|}q_0 = 1$.

3.9. If $\mu(x) = 0.001$ for $20 \le x \le 25$, evaluate ${}_{2|2}q_{20}$.

Sections 3.3, 3.4

3.10. If the survival times of 10 lives in a survivorship group are independent with survival defined in Table 3.3.1, exhibit the p.f. of $\mathscr{L}(65)$ and the mean and variance of $\mathscr{L}(65)$.

3.11. If $s(x) = 1 - x/12$, $0 \le x \le 12$, $l_0 = 9$, and the survival times are independent, then $({}_3\mathscr{D}_0, {}_3\mathscr{D}_3, {}_3\mathscr{D}_6, {}_3\mathscr{D}_9)$ is known to have a multinomial distribution. Calculate
a. The expected value of each random variable
b. The variance of each random variable
c. The coefficient of correlation between each pair of random variables.

3.12. On the basis of Table 3.3.1,
a. Compare the values of ${}_5q_0$ and ${}_5q_5$
b. Evaluate the probability that (25) will die between ages 80 and 85.

3.13. Given that l_{x+t} is strictly decreasing in the interval $0 \le t \le 1$, show that
a. If l_{x+t} is concave down, then $q_x > \mu(x)$
b. If l_{x+t} is concave up, then $q_x < \mu(x)$.

3.14. Show that

a. $\frac{d}{dx} l_x \mu(x) < 0 \qquad$ when $\frac{d}{dx} \mu(x) < \mu^2(x)$

b. $\frac{d}{dx} l_x \mu(x) = 0 \qquad$ when $\frac{d}{dx} \mu(x) = \mu^2(x)$

c. $\frac{d}{dx} l_x \mu(x) > 0 \qquad$ when $\frac{d}{dx} \mu(x) > \mu^2(x)$.

3.15. Consider a random survivorship group consisting of two subgroups: (1) the survivors of 1,600 persons joining at birth; (2) the survivors of 540 persons joining at age 10. An excerpt from the appropriate mortality table for both subgroups follows:

x	l_x
0	40
10	39
70	26

If Y_1 and Y_2 are the numbers of survivors to age 70 out of subgroups (1) and (2), respectively, estimate a number c such that $\Pr(Y_1 + Y_2 > c) = 0.05$. Assume the lives are independent and ignore half-unit corrections.

Section 3.5

3.16. Let the random variable

$$T^*(x) = T(x) \qquad 0 < T(x) \le n$$
$$= n \qquad n < T(x)$$

and denote $E[T^*(x)]$ by $\mathring{e}_{x:\overline{n}|}$. This expectation is called a ***temporary complete life expectancy.*** It is used in public health planning; the same expectation, under the name ***limited expected value function,*** is used in the analysis of loss amount distributions. Show that

a. $$\mathring{e}_{x:\overline{n}|} = \int_0^n t \, {}_tp_x \, \mu(x+t) \, dt + n \, {}_np_x$$
$$= \int_0^n {}_tp_x \, dt = \frac{T_x - T_{x+n}}{l_x}$$

b. $$\mathrm{Var}[T^*(x)] = \int_0^n t^2 \, {}_tp_x \, \mu(x+t) \, dt + n^2 \, {}_np_x - (\mathring{e}_{x:\overline{n}|})^2$$
$$= 2\int_0^n t \, {}_tp_x \, dt - \mathring{e}_{x:\overline{n}|}^{\,2}.$$

3.17. Let the random variable

$$K^*(x) = K(x) \qquad K(x) = 0, 1, 2, \ldots, n-1$$
$$= n \qquad K(x) = n, n+1, \ldots$$

and denote $E[K^*(x)]$ by $e_{x:\overline{n}|}$. This expectation is called a ***temporary curtate life expectancy.*** Show that

a. $$e_{x:\overline{n}|} = \sum_0^{n-1} k \, {}_{k|}q_x + n \, {}_np_x$$
$$= \sum_1^n {}_kp_x$$

b. $\operatorname{Var}[K^*(x)] = \sum_{0}^{n-1} k^2 \, {}_{k|}q_x + n^2 \, {}_{n}p_x - (e_{x:\overline{n|}})^2$

$$= \sum_{1}^{n} (2k+1) \, {}_{k}p_x - (e_{x:\overline{n|}})^2.$$

3.18. If the random variable T has p.d.f. given by $f_T(t) = ce^{-ct}$ for $t \ge 0$, $c > 0$, calculate

a. $\mathring{e}_x = \mathrm{E}[T]$ b. $\operatorname{Var}(T)$ c. median (T)

d. The mode of the distribution of T.

3.19. If $\mu(x+t) = t$, $t \ge 0$, calculate

a. ${}_tp_x \, \mu(x+t)$ b. $\mathring{e}_x$.

[Hint: Recall, from the study of probability, that $(1/\sqrt{2\pi}) \, e^{-t^2/2}$ is the p.d.f. for the standard normal distribution.]

3.20. If the random variable $T(x)$ has d.f. given by

$$F_{T(x)}(t) = \begin{cases} \dfrac{t}{(100-x)} & 0 \le t < 100 - x \\ 1 & t \ge 100 - x, \end{cases}$$

calculate

a. $\mathring{e}_x$ b. $\operatorname{Var}[T(x)]$ c. median $[T(x)]$.

3.21. Show that

a. $\dfrac{\partial}{\partial x} \, {}_tp_x = {}_tp_x \, [\mu(x) - \mu(x+t)]$

b. $\dfrac{d}{dx} \mathring{e}_x = \mathring{e}_x \, \mu(x) - 1$

c. $\Delta e_x = q_x \, e_{x+1} - p_x$.

3.22. Confirm the following statements:

a. $a(x) \, d_x = L_x - l_{x+1}$

b. The approximation developed in Example 3.5.1 was not used to calculate L_0 in Table 3.3.1, but was used to calculate L_1

c. $T_x = \sum_{k=0}^{\infty} L_{x+k}$.

3.23. The survival function is given by

$$s(x) = 1 - \frac{x}{10} \qquad 0 \le x \le 10$$
$$= 0 \qquad \text{elsewhere.}$$

Calculate values of $\mathring{e}_x$ and e_x, $x = 0, 1, 2, \ldots, 9$

a. Using formulas (3.5.2) and (3.5.7)

b. Using the formulas developed in Table 3.5.1.

3.24. Find $u(0)$, $-c(x)/d(x)$, and $d(x)$ if $u(x) = \Pr[X = x]$ where (3.5.20) is to be used to produce a table of the p.f. of the random variable X when it has a
a. A Poisson distribution with parameter λ
b. A binomial distribution with parameters n and p.

3.25. Formula (3.5.20) is to be used to produce tables of compound interest functions. Find $u(1)$, $-c(x)/d(x)$, and $1/d(x)$ when
a. $u(x) = \ddot{a}_{\overline{x}|}$
b. $u(x) = \ddot{s}_{\overline{x}|}$.

Section 3.6

3.26. Verify the entries for the constant force of mortality and the hyperbolic assumption in Table 3.6.1. Note that the entry for ${}_sp_x$ in the hyperbolic column provides justification for the hyperbolic name.

3.27. Graph $\mu(x + t)$, $0 < t < 1$, for each of the three assumptions in Table 3.6.1. Also graph the survival function for each assumption.

3.28. Using the l_x column of Table 3.3.1, compute ${}_{1/2}p_{65}$ for each of the three assumptions in Table 3.6.1.

3.29. Use Table 3.3.1 and an assumption of uniform distribution of deaths in each year of age to find the median of the future lifetime of a person
a. Age 0 b. Age 50.

3.30. If $q_{70} = 0.04$ and $q_{71} = 0.05$, calculate the probability that (70) will die between ages $70\frac{1}{2}$ and $71\frac{1}{2}$ under
a. The assumption that deaths are uniformly distributed within each year of age
b. The hyperbolic assumption for each year of age.

3.31. Using the l_x column in Table 3.3.1 and each of the assumptions in Table 3.6.1, compute
a. $\lim_{h \to 0} \mu(60 + h)$
b. $\lim_{h \to 0^-} \mu(60 + h)$
c. $\mu(60 + \frac{1}{2})$.

3.32. If the constant force assumption is adopted, show that
a. $a(x) = \dfrac{[(1 - e^{-\mu})/\mu] - e^{-\mu}}{1 - e^{-\mu}}$ b. $a(x) \cong \dfrac{1}{2} - \dfrac{q_x}{12}$.

3.33. If the hyperbolic assumption is adopted, show
a. $a(x) = -\dfrac{p_x}{q_x^2}(q_x + \log p_x)$ b. $a(x) \cong \dfrac{1}{2} - \dfrac{q_x}{6}$.

Section 3.7

3.34. Verify the entries in Table 3.7.1 for De Moivre's law and Weibull's law.

3.35. Consider a modification of De Moivre's law given by

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \qquad 0 \le x < \omega, \qquad \alpha > 0.$$

Calculate

a. $\mu(x)$ b. $\overset{\circ}{e}_x$.

Section 3.8

3.36. Using Table 3.8.1, calculate

a. ${}_2q_{[32]+1}$ b. ${}_2p_{[31]+1}$.

3.37. The quantity

$$1 - \frac{q_{[x]+k}}{q_{x+k}} = I(x, k)$$

has been called the ***index of selection.*** When it is close to 0, the indication is that selection has worn off. From Table 3.8.1, calculate the index for $x = 32$, $k = 0, 1$.

3.38. The force of mortality for a life selected at age (x) is given by $\mu_{\mathbf{x}}(t) = \Psi(\mathbf{x})\mu(t)$, $t > 0$. In this formula $\mu(t)$ is the standard force of mortality. The symbol $\mathbf{x}$ denotes a vector of numerical information about the life at the time of selection. This information would include the age and other classification information. It is required that $\Psi(\mathbf{x}) > 0$ and $\Psi(\mathbf{x}_0) = 1$, where $\mathbf{x}_0$ denotes standard information. Show that the select survival function is

$${}_tp_{[\mathbf{x}]} = ({}_tp_{[\mathbf{x}_0]})^{\Psi(\mathbf{x})}$$

and the p.d.f. of $T(\mathbf{x})$, the random variable time-until-death given the information x, is $-\Psi(\mathbf{x})\ {}_tp'_{[\mathbf{x}_0]}({}_tp_{[\mathbf{x}_0]})^{\Psi(\mathbf{x})-1}$, where ${}_tp'_{[\mathbf{x}_0]}$ is the derivative with respect to t of ${}_tp_{[\mathbf{x}_0]}$. This is called a ***proportional hazard model.***

Miscellaneous

3.39. A life at age 50 is subject to an extra hazard during the year of age 50 to 51. If the standard probability of death from age 50 to 51 is 0.006, and if the extra risk may be expressed by an addition to the standard force of mortality that decreases uniformly from 0.03 at the beginning of year to 0 at the end of the year, calculate the probability that the life will survive to age 51.

3.40. If the force of mortality $\mu_x(t)$, $0 \le t \le 1$, changes to $\mu_x(t) - c$ where c is a positive constant, find the value of c for which the probability that (x) will die within a year will be halved. Express the answer in terms of $q_{[x]}$.

3.41. From a standard mortality table, a second table is prepared by doubling the force of mortality of the standard table. Is the rate of mortality, q'_x, at any given age under the new table, more than double, exactly double, or less than double the mortality rate, q_x, of the standard table?

3.42. If $\mu(x) = B\, c^x$, $c > 1$, show that the function $l_x\, \mu(x)$ has its maximum at age x_0 where $\mu(x_0) = \log c$. [Hint: This exercise makes use of Exercise 3.14.]

3.43. Assume $\mu(x) = \dfrac{A\, c^x}{1 + B\, c^x}$ for $x > 0$.

a. Calculate the survival function, $s(x)$.
b. Verify that the mode of the distribution of X, the age-at-death, is given by

$$x_0 = \frac{\log(\log c) - \log A}{\log c}.$$

3.44. If $\mu(x) = \dfrac{3}{100 - x} - \dfrac{10}{250 - x}$ for $40 < x < 100$, calculate

a. ${}_{40}p_{50}$
b. The mode of the distribution of X, the age-at-death.

3.45. a. Show that, under the uniform distribution of deaths assumption,

$$m_x = \frac{q_x}{1 - (1/2)q_x} \quad \text{and} \quad q_x = \frac{m_x}{1 + (1/2)m_x}.$$

b. Calculate m_x in terms of q_x under the constant force assumption.
c. Calculate m_x in terms of q_x under the hyperbolic assumption.
d. If $l_x = 100 - x$ for $0 \le x \le 100$, calculate ${}_{10}m_{50}$ where

$${}_nm_x = \frac{\int_0^n l_{x+t}\, \mu(x+t)\, dt}{\int_0^n l_{x+t}\, dt}.$$

3.46. Show that K and S are independent if and only if the expression

$$\frac{{}_sq_{[x]+k}}{q_{[x]+k}}$$

does not depend on k for $0 \le s \le 1$.

Computing Exercises:

These are the first in a series of exercises that involve sufficient computation to make it worthwhile to use a computer. The series will continue in the following chapters, and in each exercise it is assumed that the results of previous exercises are available. For example, in Exercise 3.47, you are asked to set up a life table that will then be used in risk analysis in Chapters 4 and 5.

3.47. Using spreadsheet or other mathematical software, set up an object that will accept input values for the Makeham law parameters and then calculate and display the values of p_x and q_x for ages 0 to 140. As a check on your output, input the parameter values given in (3.7.1) and compare your q_x values with those for $x = 13, 14, \ldots$ in the Illustrative Life Table in Appendix 2A. We will refer to this computing object as your Illustrative Life Table. When the Makeham parameter values are not stated, those of (3.7.1) are implied. [Remark: With a Makeham Table, $s(x) > 0$ for all $x > 0$, so ω does not exist as defined in Section 3.3.1. For the parameter values of the Illustrative Life Table, q_{140} is zero to eight decimal places; thus we choose $\omega = 140$ for our Illustrative Life Table, that is, Table 2A.]

3.48. In your Illustrative Life Table use the forward recursion formula $l_{x+1} = (p_x)(l_x)$ and initial value $l_{13} = 96{,}807.88$ to calculate the l_x values of Table 2A. [Remark: The Makeham law was not realistic for ages less than 13, so the Illustrative Life Table is a blend of some ad hoc values from 0 through 12 and the Makeham law table from age 13 up.]

3.49. Illustrate the result of Exercise 3.41 by doubling the A and B parameter values in your Illustrative Life Table.

3.50. Use the backward recursion formula of Table 3.5.1 to calculate values of e_x in your Illustrative Life Table for ages 13 to 140.

3.51. Compare the values of e_x at $x = 20, 40, 60, 80$, and 100 in your Illustrative Life Table with those found when the force of mortality is doubled.

3.52. Use the backward recursion formula of Table 3.5.1 and the trapezoidal rule to calculate values of $\mathring{e}_x$ in your Illustrative Life Table for ages 13 to 110.

3.53. Verify the following backward recursion formula for the temporary curtate life expectancy to age y:

$$e_{x:\overline{y-x}|} = p_x + p_x\, e_{x+1:\overline{y-(x+1)}|} \quad \text{for } x = 0, 1, \ldots, y-1.$$

Determine an appropriate starting value for use with this formula. For your Illustrative Life Table calculate the curtate temporary life expectancy up to age 45 for ages 13 to 44.

3.54. Verify the following backward recursion formula for the n-year temporary curtate life expectancy:

$$e_{x:\overline{n}|} = p_x\,(1 - {}_{n}p_{x+1}) + p_x\, e_{x+1:\overline{n}|} \quad \text{for } x = 0, 1, \ldots, \omega - 1.$$

Determine an appropriate starting value for use with this formula. For your Illustrative Life Table calculate the 10-year temporary curtate life expectancy for ages 13 to 139.

3.55. "Look up" $e_{15:\overline{25}|}$ in your Illustrative Life Table. [Hint: Since the $c(x)$ term in the relation in Exercise 3.53 does not depend on n, it may be more efficient to view $e_{15:\overline{25}|}$ as a curtate temporary life expectancy to age 40 for (15).]

4

LIFE INSURANCE

4.1 Introduction

We have stated that insurance systems are established to reduce the adverse financial impact of some types of random events. Within these systems individuals and organizations adopt utility models to represent preferences, stochastic models to represent uncertain financial impact, and economic principles to guide pricing. Agreements are reached after analyses of these models.

In Chapter 2 we developed an elementary model for the financial impact of random events in which the occurrence and the size of impact are both uncertain. In that model, the policy term is assumed to be sufficiently short so the uncertainty of investment income from a random payment time could be ignored.

In this chapter we develop models for life insurances designed to reduce the financial impact of the random event of untimely death. Due to the long-term nature of these insurances, the amount of investment earnings, up to the time of payment, provides a significant element of uncertainty. This uncertainty has two causes: the unknown rate of earnings over, and the unknown length of, the investment period. A probability distribution is used to model the uncertainty in regards to the investment period throughout this book. In this chapter a deterministic model is used for the unknown investment earnings, and in Chapter 21 stochastic models for this uncertainty are discussed. In other words, our model will be built in terms of functions of T, the insured's future-lifetime random variable.

While everything in this chapter will be stated as insurances on human lives, the ideas would be the same for other objects such as equipment, machines, loans, and business ventures. In fact, the general model is useful in any situation where the size and time of a financial impact can be expressed solely in terms of the time of the random event.

4.2 Insurances Payable at the Moment of Death

In this chapter, the amount and the time of payment of a life insurance benefit depend only on the length of the interval from the issue of the insurance to the death of the insured. Our model will be developed with a ***benefit function,*** b_t, and a ***discount function,*** v_t. In our model, v_t is the interest discount factor from the time of payment back to the time of policy issue, and t is the length of the interval from issue to death. In the case of endowments, covered in this section, t can be greater than or equal to the length of the interval from issue to payment.

For the discount function we assume that the underlying force of interest is deterministic; that is, the model does not include a probability distribution for the force of interest. Moreover, we usually show the simple formulas resulting from the assumption of a constant, as well as a deterministic, force of interest.

We define the ***present-value function,*** z_t, by

$$z_t = b_t v_t. \tag{4.2.1}$$

Thus, z_t is the present value, at policy issue, of the benefit payment. The elapsed time from policy issue to the death of the insured is the insured's future-lifetime random variable, $T = T(x)$, defined in Section 3.2.2. Thus, the present value, at policy issue, of the benefit payment is the random variable z_T. Unless the context requires a more elaborate symbol, we denote this random variable by Z and base the model for the insurance on the equation

$$Z = b_T v_T. \tag{4.2.2}$$

The random variable Z is an example of a claim random variable and, as such, of an X_i term in the sum of the individual risk model, as defined by (2.1.1). This model is used in later sections when we consider applications involving portfolios. We now turn to the development of the probability model for Z.

The first step in our analysis of a life insurance will be to define b_t and v_t. The next step is to determine some characteristics of the probability distribution of Z that are consequences of an assumed distribution for T, and we work through these steps for several conventional insurances. A summary is provided in Table 4.2.1 on page 109.

4.2.1 Level Benefit Insurance

An ***n-year term life insurance*** provides for a payment only if the insured dies within the n-year term of an insurance commencing at issue. If a unit is payable at the moment of death of (x), then

$$b_t = \begin{cases} 1 & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t \ge 0,$$

$$Z = \begin{cases} v^T & T \le n \\ 0 & T > n. \end{cases}$$

These definitions use three conventions. First, since the future lifetime is a non-negative variable, we define b_t, v_t, and Z only on non-negative values. Second, for a t value where b_t is 0, the value of v_t is irrelevant. At these values of t, we adopt definitions of v_t by convenience. Third, unless stated otherwise, the force of interest is assumed to be constant.

The expectation of the present-value random variable, Z, is called the ***actuarial present value*** of the insurance. The reader will find that the expectation of the present value of a set of payments contingent on the occurrence of a set of events is referred to by different terms in different actuarial contexts. In Chapter 1, the expected loss was called the pure premium. This vocabulary is commonly used in property-liability insurance. A more exact term, but more cumbersome, would be ***expectation of the present value of the payments***. We denote actuarial present values by their symbols according to the International Actuarial Notation (see Appendix 4).

The principal symbol for the actuarial present value of an insurance paying a unit benefit is A. The subscript includes the age of the insured life at the time of the calculation. How this age is displayed depends upon the form of the mortality assumption. For the actuarial present value of an insurance on (40), the age might be displayed as [40], 40, or [20] + 20, for example. As in Section 3.8, the bracket indicates selection at that age and hence the use of a select table commencing at that age. The unbracketed age indicates the use of an aggregate or ultimate table. Thus [20] + 20 indicates the calculation for a 40-year-old on the basis of a select table commencing at age 20.

The actuarial present value for the n-year term insurance with a unit payable at the moment of death of (x), E[Z], is denoted by $\bar{A}^{1}_{x:\overline{n|}}$. This can be calculated by recognizing Z as a function of T so that $E[Z] = E[z_T]$. Then we use the p.d.f. of T to obtain

$$\bar{A}^{1}_{x:\overline{n|}} = E[Z] = E[z_T] = \int_0^\infty z_t \, f_T(t) \, dt = \int_0^n v^t \, {}_tp_x \, \mu_x(t) \, dt. \tag{4.2.3}$$

The j-th moment of the distribution of Z can be found by

$$E[Z^j] = \int_0^n (v^t)^j \, {}_tp_x \, \mu_x(t) \, dt$$
$$= \int_0^n e^{-(\delta j)t} \, {}_tp_x \, \mu_x(t) \, dt.$$

- The second integral shows that the j-th moment of Z is equal to the actuarial present value for an n-year term insurance for a unit amount payable at the moment of death of (x), calculated at a force of interest equal to j times the given force of interest, or $j\,\delta$.

This property, which we call the ***rule of moments,*** holds generally for insurances paying only a unit amount when the force of interest is deterministic, constant or not. More precisely,

$$\mathrm{E}[Z^j] \text{ @ } \delta_t = \mathrm{E}[Z] \text{ @ } j\,\delta_t. \tag{4.2.4}$$

In addition to the existence of the moments, the sufficient condition for the rule of moments is $b_t^j = b_t$ for all $t \geq 0$, that is, for each t the benefit amount is 0 or 1. Demonstration that this is sufficient is left to Exercise 4.30.

It follows from the rule of moments that

$$\mathrm{Var}(Z) = {}^2\bar{A}^{1}_{x:\overline{n}|} - (\bar{A}^{1}_{x:\overline{n}|})^2 \tag{4.2.5}$$

where ${}^2\bar{A}^{1}_{x:\overline{n}|}$ is the actuarial present value for an n-year term insurance for a unit amount calculated at force of interest $2\,\delta$.

Whole life insurance provides for a payment following the death of the insured at any time in the future. If the payment is to be a unit amount at the moment of death of (x), then

$$\begin{aligned} b_t &= 1 && t \geq 0, \\ v_t &= v^t && t \geq 0, \\ Z &= v^T && T \geq 0. \end{aligned}$$

The actuarial present value is

$$\bar{A}_x = \mathrm{E}[Z] = \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt. \tag{4.2.6}$$

For a life selected at x and now age $x + h$, the expression would be

$$\bar{A}_{[x]+h} = \int_0^\infty v^t \, {}_tp_{[x]+h} \, \mu_x(h + t) \, dt.$$

Whole life insurance is the limiting case of n-year term insurance as $n \rightarrow \infty$.

Example 4.2.1

The p.d.f. of the future lifetime, T, for (x) is assumed to be

$$f_T(t) = \begin{cases} 1/80 & 0 \leq t \leq 80 \\ 0 & \text{elsewhere.} \end{cases}$$

At a force of interest, δ, calculate for Z, the present-value random variable for a whole life insurance of unit amount issued to (x):

a. The actuarial present value

b. The variance

c. The 90th percentile, $\xi_Z^{0.9}$.

Solution:

a. $\bar{A}_x = \mathrm{E}[Z] = \int_0^{\infty} v^t f_T(t)\, dt = \int_0^{80} e^{-\delta t} \frac{1}{80}\, dt = \frac{1 - e^{-80\delta}}{80\,\delta} \qquad \delta \neq 0.$

b. By the rule of moments,

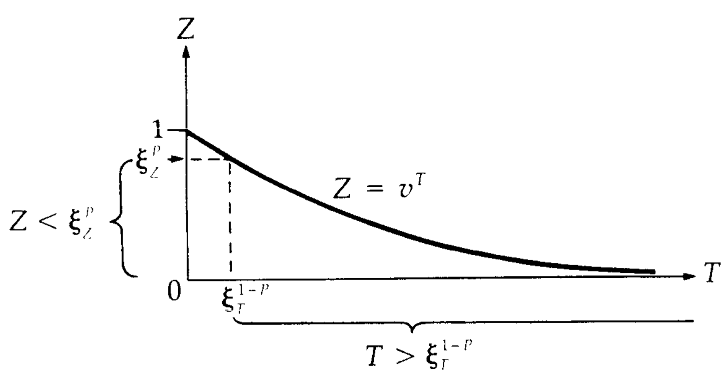
$$\mathrm{Var}(Z) = \frac{1 - e^{-160\delta}}{160\,\delta} - \left(\frac{1 - e^{-80\delta}}{80\,\delta}\right)^2 \qquad \delta \neq 0.$$

c. For the continuous random variable, Z, we have $\Pr(Z \le \xi_Z^{0.9}) = 0.9$.

Since we have the p.d.f. for T and not for Z, we proceed by finding the event for T which corresponds to $Z \le \xi_Z^{0.9}$. From Figure 4.2.1, which shows the general relationship between the sample space of T (on the horizontal axis) and the sample space of Z (on the vertical axis), we see that $\xi_Z^{0.9} = v^{\xi_T^{0.1}}$. Because Z is a strictly decreasing function of T for whole life insurance, the percentile from T's distribution that is related to 90th percentile of Z's distribution is at the complementary probability level, 0.1. In this example T is uniformly distributed over the interval (0, 80), so $\xi_T^{0.1} = 8.0$ and thus $\xi_Z^{0.9} = v^{8.0}$. ▼

The graph in Figure 4.2.1 can be used to establish relationships between the d.f. and p.d.f. of Z and those of T:

Relationship of *Z* to *T* for Whole Life Insurance



For $z \le 0$, $\{Z \le z\}$ is the null event

For $0 < z < 1$, $\{Z \le z\} = \{T \ge \log z/\log v\}$, and

For $z \ge 1$, $\{Z \le z\}$ is the certain event.

Therefore,

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ 1 - F_T(\log z/\log v) & 0 < z < 1 \\ 1 & 1 \le z. \end{cases} \tag{4.2.7}$$

By differentiation of (4.2.7),

$$f_Z(z) = \begin{cases} f_T[(\log z)/(\log v)][1/(\delta z)] & 0 < z < 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{4.2.8}$$

Example 4.2.2

For the assumptions in Example 4.2.1, determine

a. Z's d.f.

b. Z's p.d.f.

Solution:

a. From $F_T(t) = \begin{cases} t/80 & 0 \le t \le 80 \\ 1 & t \ge 80, \end{cases}$

we see that $\Pr\{T > 80\} = 0.0$, so $\Pr\{0 < Z < v^{80}\} = 0.0$. Therefore, from (4.2.7)

$$F_Z(z) = \begin{cases} 0 & z < v^{80} \\ 1 - [(\log z)/(\log v)]/80 & v^{80} < z < 1 \\ 1 & z \ge 1. \end{cases}$$

b. By differentiation of the d.f. in part (a),

$$f_Z(z) = \begin{cases} (1/80)(1/\delta z) & v^{80} < z < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

▼

We now turn our attention to a common application involving portfolios of risk: determining an initial investment fund for a segment of insurances in the total portfolio. The individual risk model and the normal approximation (as discussed in Section 2.4) are used.

Example 4.2.3

Assume that each of 100 independent lives

- Is age x
- Is subject to a constant force of mortality, $\mu = 0.04$, and
- Is insured for a death benefit amount of 10 units, payable at the moment of death.

The benefit payments are to be withdrawn from an investment fund earning $\delta = 0.06$. Calculate the minimum amount at $t = 0$ so that the probability is approximately 0.95 that sufficient funds will be on hand to withdraw the benefit payment at the death of each individual.

Solution:

For each life,

$$\begin{aligned} b_t &= 10 & t &\geq 0, \\ v_t &= v^t & t &\geq 0, \\ Z &= 10v^T & T &\geq 0. \end{aligned}$$

If we think of the lives as numbered, perhaps by the order of issuing policies, then at $t = 0$ the present value of all payments to be made is

$$S = \sum_1^{100} Z_j$$

where Z_j is the present value at $t = 0$ for the payment to be made at the death of the life numbered j.

We can use the fact that Z is 10 times the present-value random variable for the unit amount whole life insurance to calculate the mean and variance. For constant forces of interest, δ, and mortality, μ, the actuarial present value for the unit amount whole life insurance is

$$\bar{A}_x = \int_0^\infty e^{-\delta t} e^{-\mu t} \mu \, dt = \frac{\mu}{\mu + \delta}.$$

Then, for this example,

$$\mathrm{E}[Z] = 10\bar{A}_x = 10 \frac{0.04}{0.1} = 4,$$

$$\mathrm{E}[Z^2] = 10^2 \, {}^2\bar{A}_x = 100 \frac{0.04}{0.04 + 2(0.06)} = 25$$

and $\mathrm{Var}(Z) = 9$.

Using these values for the mean and the variance of each term in the sum for S, we have

$$\mathrm{E}[S] = 100(4) = 400,$$

$$\mathrm{Var}(S) = 100(9) = 900.$$

Analytically, the required minimum amount is a number, h, such that

$$\Pr(S \leq h) = 0.95,$$

or equivalently

$$\Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} \leq \frac{h - 400}{30}\right] \doteq 0.95.$$

By use of a normal approximation, we obtain

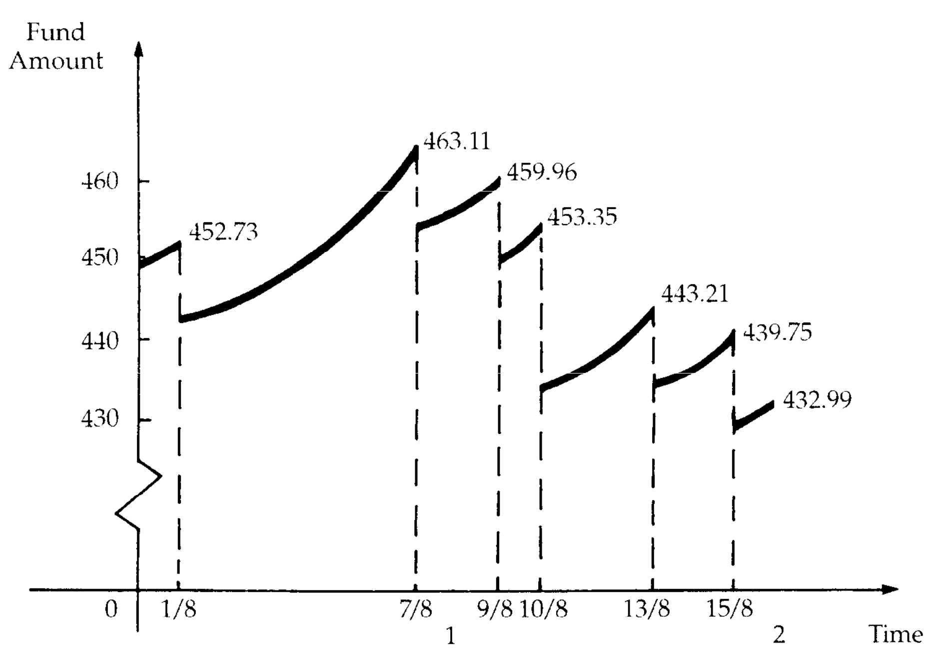
$$\frac{h - 400}{30} = 1.645,$$

$$h = 449.35. \qquad ▼$$

Observations:

1. The 49.35 difference between this initial fund of 449.35 and the expectation of the present value of all payments, 400, is the risk loading of Chapter 1. The loading is 0.4935 per life, or 4.935% per unit payment, or 12.34% of the actuarial present value.
2. This example, like Examples 2.5.2 and 2.5.3, used the individual risk model and a normal approximation to the probability distribution of S. In the short-period examples, the collected income, equal to expected claims plus a risk loading, was determined to have a high probability of being in excess of claims. In this long-period life insurance example, the collected income plus interest income on it at the assumed interest rate is determined to be sufficient to cover the benefit payments. The initial fund of 449.35 will cover less than 45% of the eventual certain payout of 1,000.
3. A graph of the amount in the fund during the first 2 years for a payout pattern when one death occurs at each of times 1/8, 7/8, 9/8, 13/8, and 15/8, and two deaths occur at time 10/8 is shown in Figure 4.2.2. Between the benefit payments, represented by the discontinuities, are exponential arcs representing the growth of the fund at $\delta = 0.06$.

Graph of an Outcome for the Fund



4. There are infinitely many payout patterns, each with its own graph. Both the number of claims and the times of those claims affect the fund. For example, had the seven claims all occurred within the first instant, instead of the payout pattern of Figure 4.2.2, the fund would have dropped immediately to 379.35 and then grown to 427.72 by the end of the second year.

These examples illustrate the different roles of the three random elements in risk model building, that is, whether or not a claim will occur, the size, and the time of payment if one occurs. In Example 2.5.2 there was uncertainty about only the occurrence of the claim. In Example 4.2.2 there was uncertainty about only the time of claim payment. Other uncertainties have been ignored in these models. In Examples 4.2.1, 4.2.2, and 4.2.3 we have ignored the possibility of the fund earning interest at rates different from the deterministic rates assumed.

4.2.2 Endowment Insurance

An ***n*-year pure endowment** provides for a payment at the end of the n years if and only if the insured survives at least n years from the time of policy issue. If the amount payable is a unit, then

$$b_t = \begin{cases} 0 & t \le n \\ 1 & t > n, \end{cases}$$

$$v_t = v^n \qquad t \ge 0,$$

$$Z = \begin{cases} 0 & T \le n \\ v^n & T > n. \end{cases}$$

The only element of uncertainty in the pure endowment is whether or not a claim will occur. The size and time of payment, if a claim occurs, are predetermined. In the expression $Z = v^n Y$, Y is the indicator of the event of survival to age $x + n$. This Y has the value 1 if the insured survives to age $x + n$ and has the value 0 otherwise. The n-year pure endowment's actuarial present value has two symbols. In an insurance context it is $A_{x:\overset{1}{\overline{n|}}}$. We see in the next chapter that it is denoted by ${}_nE_x$ in an annuity context. This distinction is not strict; the reader will have to be ready for either:

$$A_{x:\overset{1}{\overline{n|}}} = \mathrm{E}[Z] = v^n\ \mathrm{E}[Y] = v^n\ {}_np_x,$$

and

$$\begin{aligned} \mathrm{Var}(Z) &= v^{2n}\ \mathrm{Var}(Y) = v^{2n}\ {}_np_x\ {}_nq_x \\ &= {}^2A_{x:\overset{1}{\overline{n|}}} - (A_{x:\overset{1}{\overline{n|}}})^2. \end{aligned} \tag{4.2.9}$$

An ***n*-year endowment insurance** provides for an amount to be payable either following the death of the insured or upon the survival of the insured to the end of the n-year term, whichever occurs first. If the insurance is for a unit amount and the death benefit is payable at the moment of death, then

$$b_t = 1 \qquad t \ge 0,$$

$$v_t = \begin{cases} v^t & t \le n \\ v^n & t > n, \end{cases}$$

$$Z = \begin{cases} v^T & T \le n \\ v^n & T > n. \end{cases}$$

The actuarial present value is denoted by $\bar{A}_{x:\overline{n}|}$. Since $b_t = 1$ for the endowment insurance, we have by the rule of moments

$$\mathrm{E}[Z^j] @ \delta = \mathrm{E}[Z] @ j\delta.$$

Moreover,

$$\mathrm{Var}(Z) = {}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2. \tag{4.2.10}$$

This insurance can be viewed as the combination of an n-year term insurance and an n-year pure endowment—each for a unit amount. Let Z_1, Z_2, and Z_3 denote the present-value random variables of the term, the pure endowment, and the endowment insurances, respectively, with death benefits payable at the moment of death of (x). From the preceding definitions we have

$$Z_1 = \begin{cases} v^T & T \le n \\ 0 & T > n, \end{cases}$$

$$Z_2 = \begin{cases} 0 & T \le n \\ v^n & T > n, \end{cases}$$

$$Z_3 = \begin{cases} v^T & T \le n \\ v^n & T > n. \end{cases}$$

It follows that

$$Z_3 = Z_1 + Z_2, \tag{4.2.11}$$

and by taking expectations of both sides

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;1}. \tag{4.2.12}$$

We can also find the $\mathrm{Var}(Z_3)$ by using (4.2.11),

$$\mathrm{Var}(Z_3) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) + 2\,\mathrm{Cov}(Z_1, Z_2). \tag{4.2.13}$$

By use of the formula

$$\mathrm{Cov}(Z_1, Z_2) = \mathrm{E}[Z_1 Z_2] - \mathrm{E}[Z_1]\,\mathrm{E}[Z_2] \tag{4.2.14}$$

and the observation that

$$Z_1 Z_2 = 0$$

for all T, we have

$$\mathrm{Cov}(Z_1, Z_2) = -\mathrm{E}[Z_1]\,\mathrm{E}[Z_2] = -\bar{A}^{1}_{x:\overline{n}|} A_{x:\overline{n}|}^{\;\;\;1}. \tag{4.2.15}$$

Substituting (4.2.5), (4.2.9), and (4.2.15) into (4.2.13) produces a formula for $\mathrm{Var}(Z_3)$ in terms of actuarial present values for an n-year term insurance and a pure endowment.

Since the actuarial present values are positive, the $\text{Cov}(Z_1, Z_2)$ is negative. This is to be anticipated since, of the pair Z_1 and Z_2, one is always zero and the other positive. On the other hand, the correlation coefficient of Z_1 and Z_2 is not -1 since they are not linear functions of each other; recall Exercise 1.23(c).

4.2.3 Deferred Insurance

An ***m*-year deferred insurance** provides for a benefit following the death of the insured only if the insured dies at least m years following policy issue. The benefit payable and the term of the insurance may be any of those discussed above. For example, an m-year deferred whole life insurance with a unit amount payable at the moment of death has

$$b_t = \begin{cases} 1 & t > m \\ 0 & t \le m, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T & T > m \\ 0 & T \le m. \end{cases}$$

The actuarial present value is denoted by ${}_{m|}\bar{A}_x$ and is equal to

$$\int_m^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt. \tag{4.2.16}$$

Example 4.2.4

Consider a 5-year deferred whole life insurance payable at the moment of the death of (x). The individual is subject to a constant force of mortality $\mu = 0.04$. For the distribution of the present value of the benefit payment, at $\delta = 0.10$:

a. Calculate the expectation
b. Calculate the variance
c. Display the distribution function
d. Calculate the median $\xi_Z^{0.5}$.

Solution:

a. For arbitrary forces μ and δ,

$${}_{5|}\bar{A}_x = \int_5^\infty e^{-\delta t} \, e^{-\mu t} \, \mu \, dt = \frac{\mu}{\mu + \delta} \, e^{-5(\mu+\delta)};$$

thus for $\mu = 0.04$ and $\delta = 0.10$,

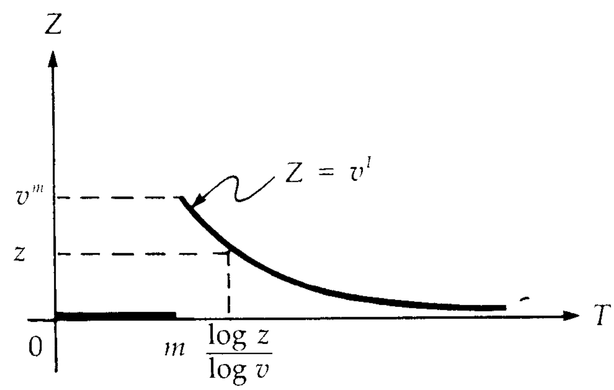
$${}_{5|}\bar{A}_x = \frac{2}{7} \, e^{-0.7} = 0.1419.$$

b. By the rule of moments,

$$\text{Var}(Z) = \frac{0.04}{0.04 + 0.20} \, e^{-5(0.04+0.20)} - \frac{4}{49} \, e^{-1.4} = 0.0301.$$

c. As for the case of whole life insurance, a graph of the relation between Z and T provides an outline for the solution. For the general m-year deferred whole life insurance, the graph is given in Figure 4.2.3.

Relationship of Z to T for Deferred Whole Life Insurance



Although T is a continuous random variable, Z is mixed with a probability mass at 0 because $Z = 0$ corresponds to $T \le m$.

For general mortality assumptions and a constant force of interest, we have for $Z = 0$,

$$F_Z(0) = \Pr(T \le m) = F_T(m); \tag{4.2.17}$$

for $0 < z < v^m$,

$$\begin{aligned} F_Z(z) &= \Pr(Z \le z) = \Pr(Z = 0) + \Pr(0 < Z \le z) \\ &= \Pr(T \le m) + \Pr(0 < v^T \le z) \\ &= \Pr(T \le m) + \Pr\left(T > \frac{\log z}{\log v}\right) \\ &= F_T(m) + 1 - F_T\left(\frac{\log z}{\log v}\right); \end{aligned} \tag{4.2.18}$$

for $z > v^m$,

$$F_Z(z) = 1. \tag{4.2.19}$$

In this example of 5-year deferred whole life insurance where $\mu = 0.04$ and $\delta = 0.10$, we have

from (4.2.17),

$$F_Z(0) = F_T(5) = 1 - e^{-0.2} = 0.1813;$$

from (4.2.18) for $0 < z < v^5$,

$$F_Z(z) = F_T(5) + 1 - F_T\left(\frac{\log z}{-0.1}\right)$$

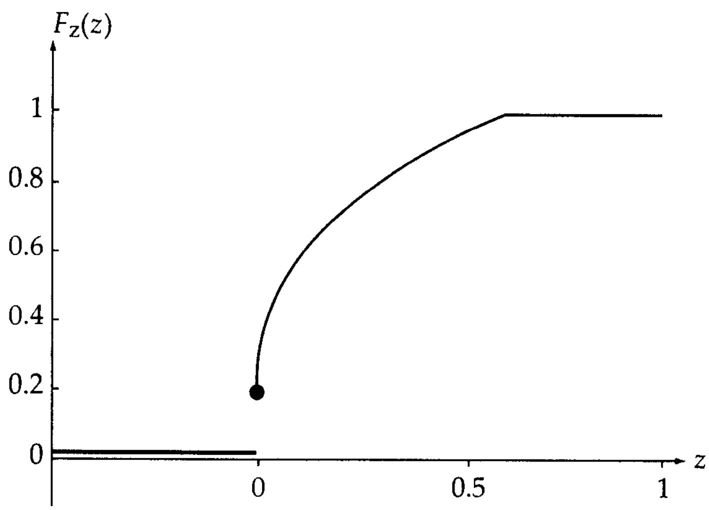
$$= 1 - e^{-0.2} + z^{0.04/0.10} = 0.1813 + z^{0.4}; \tag{4.2.20}$$

from (4.2.19) for $z > v^5$,

$$F_Z(z) = 1.$$

The graph of this d.f. is shown in Figure 4.2.4.

Distribution Function of *Z*



d. From Figure 4.2.4 or (4.2.20), we see that the median is the solution of

$$0.5 = 0.1813 + z^{0.4}.$$

Thus, $\xi_Z^{0.5} = 0.0573$. ▼

Observations:

1. The largest value of Z with nonzero probability density in this example is $e^{-0.1(5)} = 0.6065$, corresponding to $T = 5$.
2. The distribution of Z in this example is highly skewed to the right. While its total mass is in the interval [0, 0.6065] and its mean is 0.1419, its median is only 0.0573. This skewness in the direction of large positive values is characteristic of many claim distributions in all fields of insurance.

4.2.4 Varying Benefit Insurance

The general model given by (4.2.1) can be used for analysis in most applications. We have used it with level benefit life insurances. It can also be applied to insurances where the level of the death benefit either increases or decreases in arithmetic progression over all or a part of the term of the insurance. Such insurances are often sold as an additional benefit when a basic insurance provides for the return of periodic premiums at death or when an annuity contract contains a guarantee of sufficient payments to match its initial premium.

An ***annually increasing whole life insurance*** providing 1 at the moment of death during the first year, 2 at the moment of death in the second year, and so on, is characterized by the following functions:

$$
\begin{aligned}
b_t &= \lfloor t + 1 \rfloor & t &\geq 0, \\
v_t &= v^t & t &\geq 0, \\
Z &= \lfloor T + 1 \rfloor v^T & T &\geq 0,
\end{aligned}
$$

where the $\lfloor \ \rfloor$ denote the greatest integer function.

The actuarial present value for such an insurance is

$$(I\bar{A})_x = \mathrm{E}[Z] = \int_0^\infty \lfloor t + 1 \rfloor \, v^t \, {}_tp_x \, \mu_x(t) \, dt.$$

The higher order moments are not equal to the actuarial present value at an adjusted force of interest as was the case for insurances with benefit payments equal to 0 or 1. These moments can be calculated directly from their definitions.

The increases in the benefit of the insurance can occur more, or less, frequently than once per year. For an m-thly increasing whole life insurance the benefit would be $1/m$ at the moment of death during the first m-th of a year of the term of the insurance, $2/m$ at the moment of death during the second m-th of a year during the term of the insurance, and so on, increasing by $1/m$ at m-thly intervals throughout the term of the insurance. For such a whole life insurance the functions are

$$
\begin{aligned}
b_t &= \frac{\lfloor tm + 1 \rfloor}{m} & t &\geq 0, \\
v_t &= v^t & t &\geq 0, \\
Z &= \frac{v^T \lfloor Tm + 1 \rfloor}{m} & T &\geq 0.
\end{aligned}
$$

The actuarial present value is

$$(I^{(m)} \bar{A})_x = \mathrm{E}[Z].$$

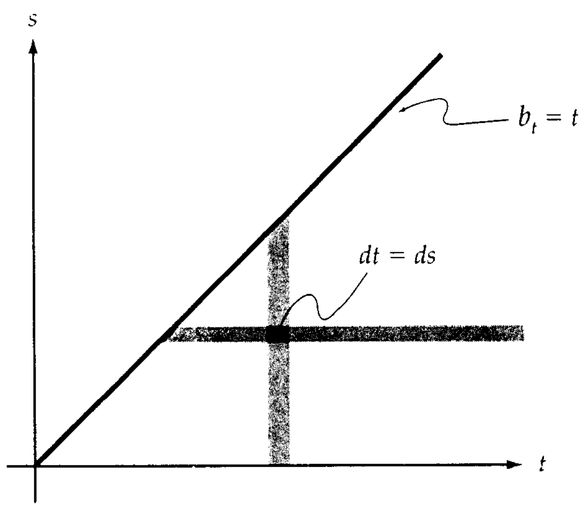
The limiting case, as $m \rightarrow \infty$ in the m-thly increasing whole life insurance, is an insurance paying t at the time of death, t. Its functions are

$$
\begin{aligned}
b_t &= t & t &\geq 0, \\
v_t &= v^t & t &\geq 0, \\
Z &= T v^T & T &\geq 0.
\end{aligned}
$$

Its actuarial present-value symbol is $(\bar{I}\bar{A})_x$.

This continuously increasing whole life insurance is equivalent to a set of deferred level whole life insurances. This equivalence is shown graphically in Figure 4.2.5 where the region between the line $b_t = t$ and the t-axis represents the insurance over the future lifetime. If the infinitesimal regions are joined in the vertical direction for a fixed t, the total benefit payable at t is obtained. If they are joined in the horizontal direction for a fixed s, an s-year deferred whole life insurance for the level amount ds is obtained.

Continuously Increasing Insurance



This equivalence implies that the actuarial present values for the coverages are equal. The equality can be established as follows.

By definition,

$$(\bar{I}\bar{A})_x = \int_0^\infty t\, v^t\, {}_tp_x\, \mu_x(t)\, dt,$$

and interpreting t in the integrand as the integral from zero to t in Figure 4.2.5 we have

$$(\bar{I}\bar{A})_x = \int_0^\infty \left(\int_0^t ds \right) v^t\, {}_tp_x\, \mu_x(t)\, dt.$$

If we interchange the order of integration and, for each s value, integrate on t from s to x, we have

$$\begin{aligned}(\bar{I}\bar{A})_x &= \int_0^\infty \int_s^\infty v^t\, {}_tp_x\, \mu_x(t)\, dt\, ds \\ &= \int_0^\infty {}_{s|}\bar{A}_x\, ds\end{aligned}$$

by (4.2.16).

If, for any of these m-thly increasing life insurances, the benefit is payable only if death occurs within a term of n years, the insurance is an **m-*thly increasing* n-*year term life insurance.***

Complementary to the anually increasing n-year term life insurance is the ***annually decreasing n-year term life insurance*** providing n at the moment of death during the first year, $n - 1$ at the moment of death during the second year, and so on, with coverage terminating at the end of the n-th year. Such an insurance has the following functions:

$$b_t = \begin{cases} n - \lfloor t \rfloor & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T(n - \lfloor T \rfloor) & T \le n \\ 0 & T > n. \end{cases}$$

The actuarial present value for this insurance is

$$(D\bar{A})^1_{x:\overline{n}|} = \int_0^n v^t \,(n - \lfloor t \rfloor)\; {}_tp_x\; \mu_x(t)\, dt.$$

This insurance is complementary to the anually increasing n-year term insurance in the sense that the sum of their benefit functions is the constant $n + 1$ for the n-year term.

Table 4.2.1 is a summary of the models in this section. The insurance plan name appears in the first column followed by the benefit and discount functions that define it in terms of the future lifetime of the insured at policy issue. The present-value function, which is always derived as the product of the previous two functions, is shown next. In the fifth column the International Actuarial Notation for the actuarial present value is shown. In the last column, a reference is given to a footnote stating whether or not the rule of moments can be used in the calculation of higher order moments.

4.3 Insurances Payable at the End of the Year of Death

In the previous section we developed models for life insurances with death benefits payable at the moment of death. In practice, most benefits are considered payable at the moment of death and then earn interest until the payment is actually made. The models were built in terms of T, the future lifetime of the insured at policy issue. In most life insurance applications, the best information available on the probability distribution of T is in the form of a discrete life table. This is the probability distribution of K, the curtate-future-lifetime of the insured at policy issue, a function of T. In this and the following section we bridge this gap by building models for life insurances in which the size and time of payment of the death benefits depend only on the number of complete years lived by the insured from policy issue up to the time of death. We refer to these insurances simply as ***payable at the end of the year of death.***

Summary of Insurances Payable Immediately on Death

(1) Insurance Name	(2) Benefit Function b_t	(3) Discount Function v_t	(4) Present-Value Function z_t	(5) Actuarial Present Value	(6) Higher Moments
Whole life	1	v^t	v^t	$\bar{A}_x$	*
n-Year term	$1 \quad t \le n$ $0 \quad t > n$	v^t	$v^t \quad t \le n$ $0 \quad t > n$	$\bar{A}^{1}_{x:\overline{n}\vert}$	*
n-Year pure endowment	$0 \quad t \le n$ $1 \quad t > n$	v^n	$0 \quad t \le n$ $v^n \quad t > n$	$A_{x:\overline{n}\vert}^{\ \ 1}$, ${}_nE_x$	*
n-Year endowment	1	$v^t \quad t \le n$ $v^n \quad t > n$	$v^t \quad t \le n$ $v^n \quad t > n$	$\bar{A}_{x:\overline{n}\vert}$	*
m-Year deferred n-Year term	$1 \quad m < t \le n + m$ $0 \quad t \le m,\ t > n + m$	v^t	$v^t \quad m < t \le n + m$ $0 \quad t \le m,\ t > n + m$	${}_{m\vert n}\bar{A}_x$	*
n-Year term increasing annually	$\lfloor t + 1 \rfloor \quad t \le n$ $0 \quad t > n$	v^t	$\lfloor t + 1 \rfloor v^t \quad t \le n$ $0 \quad t > n$	$(I\bar{A})^{1}_{x:\overline{n}\vert}$	†
n-Year term decreasing annually	$n - \lfloor t \rfloor \quad t \le n$ $0 \quad t > n$	v^t	$(n - \lfloor t \rfloor) v^t \quad t \le n$ $0 \quad t > n$	$(D\bar{A})^{1}_{x:\overline{n}\vert}$	†
Whole life increasing m-thly	$\lfloor tm + 1 \rfloor / m$	v^t	$v^t \lfloor tm + 1 \rfloor / m$	$(I^{(m)}\bar{A})_x$	†

Note: b_t, v_t, and z_t are defined only for $t \ge 0$.
*The j-th moment is equal to the actuarial present value at j times the given force of interest, denoted by ${}^jA$ for $j > 1$. Then the variance is ${}^2A - A^2$, symbolically.
†Calculated directly from the definition, $E[Z^j]$.

Our model is in terms of functions of the curtate-future-lifetime of the insured. The benefit function, b_{k+1}, and the discount function, v_{k+1}, are, respectively, the benefit amount payable and the discount factor required for the period from the time of payment back to the time of policy issue when the insured's curtate-future-lifetime is k, that is, when the insured dies in year $k+1$ of insurance. The present value, at policy issue, of this benefit payment, denoted by z_{k+1}, is

$$z_{k+1} = b_{k+1} v_{k+1}. \tag{4.3.1}$$

Measured from the time of policy issue, the insurance year of death is 1 plus the curtate-future-lifetime random variable, K, defined in Section 3.2.3. As in the previous section, we denote the present-value random variable z_{K+1}, by Z.

For an n-year term insurance providing a unit amount at the end of the year of death, we have

$$b_{k+1} = \begin{cases} 1 & k = 0, 1, \ldots, n-1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$v_{k+1} = v^{k+1},$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & \text{elsewhere.} \end{cases}$$

The actuarial present value for this insurance is given by

$$A^{1}_{x:\overline{n|}} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k}. \tag{4.3.2}$$

Note that the International Actuarial Notation symbol for the actuarial present value of an insurance payable at the end of the year of death is the symbol for the corresponding insurance payable at the moment of death with the bar removed.

The rule of moments, with the appropriate changes in notation, also holds for insurances payable at the end of the year of death. For example, for the n-year term insurance above,

$$\mathrm{Var}(Z) = {}^2A^{1}_{x:\overline{n|}} - (A^{1}_{x:\overline{n|}})^2$$

where

$${}^2A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} e^{-2\delta(k+1)} \, {}_kp_x \, q_{x+k}.$$

In Section 3.5 recursion relations for life expectancies are derived and used to determine their values. Recursion relations for the term insurance actuarial present values can be derived algebraically from (4.3.2):

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k} = vq_x + \sum_{k=1}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k}$$

$$= vq_x + vp_x \sum_{k=1}^{n-1} v^k \, {}_{k-1}p_{x+1} \, q_{x+k}$$

$$= vq_x + vp_x \sum_{j=0}^{n-2} v^{j+1} \, {}_jp_{x+1} \, q_{x+1+j} = vq_x + vp_x \, A^{1}_{x+1:\overline{n-1}|}. \tag{4.3.3}$$

For (4.3.3) to be true at $n = 1$, we define $A^{1}_{x:\overline{0}|} = 0.0$ for all x.

Note: On a select table basis, all x's in the subscripts in (4.3.3) would be enclosed in brackets.

Example 4.3.1

On the basis of the Illustrative Life Table and $i = 0.04$, determine the mean and variance of the present-value random variable for a 10-year term insurance with a unit benefit payable at the end of the year of death issued on (30).

Solution:

Starting with the initial value $A^{1}_{40:\overline{0}|} = 0.0$ and using (4.3.3) adapted to this insurance,

$$A^{1}_{30-k:\overline{10-k}|} = vq_{30+k} + vp_{30+k} \, A^{1}_{30+k+1:\overline{10-(k+1)}|} \quad k = 0, 1, \ldots, 8, 9,$$

we have by working from age 40 to age 30,

$$A^{1}_{30:\overline{10}|} = 0.01577285$$

and

$$\text{Var}(Z) = 0.01271978 - (0.01577285)^2 = 0.1247099.$$

These values were determined by the spreadsheet constructed in the Computing Exercises. ▼

For a whole life insurance issued to (x), the model may be obtained by letting $n \to \infty$ in the n-year term insurance model. For the actuarial present value we have

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \, {}_kp_x \, q_{x+k}. \tag{4.3.4}$$

Multiplication of both sides of (4.3.4) by l_x yields

$$l_x A_x = \sum_{k=0}^{\infty} v^{k+1} \, d_{x+k}. \tag{4.3.5}$$

Formula (4.3.5) shows the balance, at the time of policy issue, between the aggregate fund of actuarial present values for l_x lives insured at age x and the outflow of funds in accordance with the expected deaths of the l_x lives. It is a compound interest equation of value that is stated on an expected value basis.

The expression

$$\sum_{k=r}^{\infty} v^{k+1}\, d_{x+k} \tag{4.3.6}$$

is that part of the fund at issue that, together with interest at the assumed rate, will provide the payments for the expected deaths after the r-th insurance year.

Accumulation of (4.3.6) at the assumed interest rate for r years yields

$$\sum_{k=r}^{\infty} v^{k-r+1}\, d_{x+k}, \tag{4.3.7}$$

the expected amount in the fund after r insurance years. A comparison of expression (4.3.7) with (4.3.5) shows it to be $l_{x+r}\, A_{x+r}$. The difference between this amount and an actual fund is due to deviations of the actual deaths from the expected deaths (according to the life table adopted), and deviations of the actual interest income from the interest income at the assumed rate.

Example 4.3.2

A group of 100 lives age 30 set up a fund to pay 1,000 at the end of the year of death of each member to a designated survivor. Their mutual agreement is to pay into the fund an amount equal to the whole life insurance actuarial present value calculated on the basis of the Illustrative Life Table at 6% interest. The members, not selected by an insurance company, decided to use this population table as the basis of their plan. The actual experience of the fund is one death in each of the second and fifth years; interest income is at 6% in the first year, 6-1/2% in the second and third years, 7% in the fourth and fifth years. What is the difference, at the end of the first 5 years, between the expected size of the fund as determined at the inception of the plan and the actual fund?

Solution:

On the agreed bases, 1,000 A_{30} = 102.4835, so, for the 100 lives, the fund starts at 10,248.35. Also, A_{35} = 0.1287194 and l_{35}/l_{30} = 0.9915040.

For 100 lives age 30, the expected size of the fund after 5 years will be

$$(1{,}000)(100)\,\frac{l_{35}}{l_{30}}\,A_{35} = 12{,}762.58.$$

The development of the actual fund would be as follows, where F_k denotes its size at the end of insurance year k:

$$F_0 = 10{,}248.35$$

$$F_1 = (10{,}248.35)(1.06) = 10{,}863.25$$

$$F_2 = (10{,}863.25)(1.065) - 1{,}000 = 10{,}569.36$$

$$F_3 = (10{,}569.36)(1.065) = 11{,}256.37$$

$$F_4 = (11{,}256.37)(1.07) = 12{,}044.32$$

$$F_5 = (12{,}044.32)(1.07) - 1{,}000 = 11{,}887.42.$$

Thus the required difference is 12,762.58 − 11,887.42 = 875.16. This result combines the investment experience and the mortality experience for the 5-year period. There were gains from the investment earnings in excess of the assumed rate of 6%. On the other hand, there were losses on the mortality experience of two deaths as compared to the expected number of 0.8496. The interpretation of such results in terms of the various sources such as investment earnings and mortality is an actuarial responsibility. ▼

We derived the recursion relations for n-year term insurance actuarial present values (4.3.3) algebraically. Whereas the relationship will hold for whole life insurance actuarial present values as the limiting case of n-year term insurance, as n goes to ∞, we will establish the whole life insurance relationship independently to illustrate a probabilistic derivation.

Consider A_x from its definition $\mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1}]$. For emphasis we now write this as

$$A_x = \mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1} | K(x) \geq 0],$$

which is redundant since all of $K(x)$'s probability is on the non-negative integers.

E[Z] can be calculated by considering the event that (x) dies in the first year, that is, $K(x) = 0$, and its complement, that (x) survives the first year, that is, $K(x) \geq 1$. We can write

$$\begin{aligned} \mathrm{E}[Z] = {} & \mathrm{E}[v^{K(x)+1} | K(x) = 0] \Pr[K(x) = 0] \\ & + \mathrm{E}[v^{K(x)+1} | K(x) \geq 1] \Pr[K(x) \geq 1]. \end{aligned} \tag{4.3.8}$$

In this expression we can readily substitute

$$\mathrm{E}[v^{K(x)+1} | K(x) = 0] = v,$$

$$\Pr[K(x) = 0] = q_x,$$

and

$$\Pr[K(x) \geq 1] = p_x.$$

To find an expression for the remaining factor, we rewrite it as

$$\mathrm{E}[v^{K(x)+1} | K(x) \geq 1] = v\,\mathrm{E}[v^{(K(x)-1)+1} | K(x) - 1 \geq 0].$$

Since $K(x)$ is the curtate-future-lifetime of (x), given $K(x) \geq 1$, $K(x) - 1$ must be the curtate-future-lifetime of $(x + 1)$.

If we are willing to use the same probabilities for the conditional distribution of $K(x) - 1$ given $K(x) \geq 1$, as we would for a newly considered life age $x + 1$, then we may write

$$\mathrm{E}[v^{(K(x)-1)+1} | K(x) - 1 \geq 0] = A_{x+1} \tag{4.3.9}$$

and substitute it into (4.3.8) to obtain

$$A_x = vq_x + vA_{x+1}p_x. \tag{4.3.10}$$

The assumed equality,

(the distribution of the future lifetime of a newly insured life aged $x + 1$)

= (the distribution of the future lifetime of a life now age $x + 1$ who was insured 1 year ago),

was discussed in Section 3.8. In terms of select tables, the right-hand side of (4.3.9) would be $A_{[x]+1}$. In (4.3.10) every x would be $[x]$.

Note that (4.3.10) is the same backward recursion formula as (4.3.3). That is,

$$u(x) = vq_x + vp_x\, u(x+1).$$

It is the starting value that makes the solution the actuarial present value of whole life insurance or of n-year term insurance. We see this same recursion formula for the actuarial present values of n-year endowment insurance where the starting values are the endowment maturity value.

Analysis of relationship (4.3.10) can give more insight into the nature of A_x. After replacement of p_x by $1 - q_x$ and multiplication of both sides by $(1 + i)l_x$, (4.3.10) can be rearranged as

$$l_x(1 + i)A_x = l_x A_{x+1} + d_x(1 - A_{x+1}). \tag{4.3.11}$$

For the random survivorship group, this equation has the following interpretation. Together with 1 year's interest, A_x will provide A_{x+1} for all l_x lives and an additional $1 - A_{x+1}$ for those expected to die within the year. This latter amount for each expected death, that is, $q_x(1 - A_{x+1})$, is considered the ***annual cost of insurance.*** The A_{x+1} is set aside for survivors and deaths, the $1 - A_{x-1}$ is required only for a death.

Dividing by l_x and then subtracting $A_x + q_x(1 - A_{x+1})$ from both sides of (4.3.11), we have

$$A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1}). \tag{4.3.12}$$

In words, the difference between the actuarial present values at age x and one later at age $x + 1$ is equal to the interest on the actuarial present value at x less the annual cost of insurance for the year.

Another expression for A_x can be obtained from (4.3.10) by replacing p_x by $1 - q_x$, multiplying both sides by v^x, and rearranging the terms to get

$$v^{x+1}A_{x+1} - v^x A_x = -v^{x+1}q_x(1 - A_{x+1}),$$

or

$$\Delta v^x A_x = -v^{x+1}q_x(1 - A_{x+1}).$$

Summing from $x = y$ to ∞ (see Appendix 5), we obtain

$$-v^y A_y = -\sum_{x=y}^{\infty} v^{x+1}q_x(1 - A_{x+1})$$

and thus

$$A_y = \sum_{x=y}^{\infty} v^{x+1-y}q_x(1 - A_{x+1}).$$

This expression shows that the actuarial present value at y is the present value at y of the annual costs of insurance over the remaining lifetime of the insured.

The n-year endowment insurance with a unit amount payable at the end of the year of death is a combination of the n-year term insurance of this section and the n-year pure endowment for a unit amount that was discussed in the previous section. Thus the functions for it are

$$b_{k+1} = 1 \qquad k = 0, 1, \ldots,$$

$$v_{k+1} = \begin{cases} v^{k+1} & k = 0, 1, \ldots, n-1 \\ v^n & k = n, n+1, \ldots, \end{cases}$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ v^n & K = n, n+1, \ldots. \end{cases}$$

The actuarial present value is

$$A_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\ {}_kp_x\ q_{x+k} + v^n\ {}_np_x. \tag{4.3.13}$$

The annually increasing whole life insurance, paying $k + 1$ units at the end of insurance year $k + 1$ provided the insured dies in that insurance year, has the benefit and discount functions and present-value random variable as follows:

$$b_{k+1} = k + 1 \qquad k = 0, 1, 2, \ldots,$$

$$v_{k+1} = v^{k+1} \qquad k = 0, 1, 2, \ldots,$$

$$Z = (K + 1)\, v^{K+1} \qquad K = 0, 1, 2, \ldots.$$

The actuarial present value is denoted by $(IA)_x$.

The annually decreasing n-year term insurance, during the n-year period, provides a benefit at the end of the year of death in an amount equal to $n - k$ where

k is the number of complete years lived by the insured since issue. Its functions are

$$b_{k+1} = \begin{cases} n - k & k = 0, 1, \ldots, n-1 \\ 0 & k = n, n+1, \ldots, \end{cases}$$

$$v_{k+1} = v^{k+1} \qquad k = 0, 1, \ldots,$$

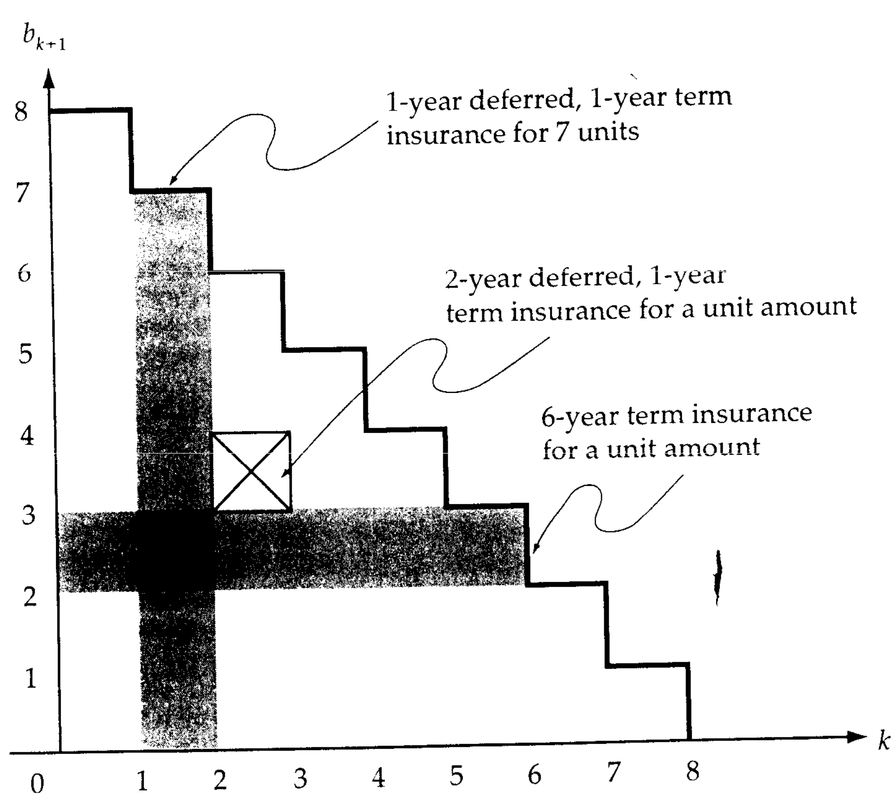
$$Z = \begin{cases} (n - K)v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots. \end{cases}$$

The actuarial present-value symbol for this insurance is $(DA)^{1}_{x:\overline{n}|}$.

As illustrated by Figure 4.2.5 for insurances payable at the moment of death, annually increasing insurances payable at the end of the year of death are equivalent to a combination of deferred level insurances each for a unit amount. Similarly, annually decreasing term insurances are equivalent to a combination of level term insurances of various term lengths. Figure 4.3.1 illustrates this for an annually decreasing 8-year term insurance.

Figure 4.3.1 shows the graph of the benefit function b_{k+1}. Each unit square region between the horizontal steps and the k-axis represents a deferred 1-year term insurance. When these are summed vertically, the deferred 1-year term insurances

Annually Decreasing 8-Year Term Insurance



for the decreasing amounts are obtained. When the squares are summed horizontally, the level amount term insurances of varying duration are obtained. These vertical and horizontal sums are also indicated in Figure 4.3.1.

The equality of the actuarial present values for the combination of level term insurances and the combination of deferred term insurances can be demonstrated analytically. Thus, by definition

$$
\begin{aligned}
(DA)^1_{x:\overline{n}|} &= \sum_{k=0}^{n-1} (n - k)\, v^{k+1}\, {}_kp_x\, q_{x+k} \\
&= \sum_{k=0}^{n-1} (n - k)\, (v^k\, {}_kp_x)\, (vq_{x+k}) \\
&= \sum_{k=0}^{n-1} (n - k)\, {}_{k|1}A_x, \qquad (4.3.14)
\end{aligned}
$$

the total of the column sums.

In (4.3.14) we can substitute

$$n - k = \sum_{j=0}^{n-k-1} (1)$$

to obtain

$$(DA)^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (1)\, v^{k+1}\, {}_kp_x\, q_{x+k}.$$

By interchanging the order of summation we obtain

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} (1)\, v^{k+1}\, {}_kp_x\, q_{x+k},$$

and then by comparing the inner summation to (4.3.2), we can write

$$(DA)^1_{x:\overline{n}|} = \sum_{j=0}^{n-1} A^1_{x:\overline{n-j}|}.$$

Table 4.3.1 provides a summary of functions and symbols for the elementary insurances payable at the end of the year of death.

We close this section with a summary of the recursion relations for the actuarial present values of the insurances payable at the end of the year of death. Consider the list on page 119 arranged in the order of the entries of Table 4.3.1. Each entry is arranged across its line as the recursion relation, the domain for the relation, and the initial condition. Values for the actuarial present value would be generated from the lowest age of the mortality table to age y or ω.

Summary of Insurances Payable at End of Year of Death

(1) Insurance Name	(2) Benefit Function b_{k+1}	(3) Discount Function v_{k+1}	(4) Present-Value Function z_{k+1}	(5) Actuarial Present Value	(6) Higher Moments
(a) Whole life	1	v^{k+1}	v^{k+1}	A_x	*
(b) n-Year term	$1 \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	v^{k+1}	$v^{k+1} \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	$A^{1}_{x:\overline{n}\rceil}$	*
(c) n-Year endowment	1	$v^{k+1} \quad k = 0, 1, \ldots, n-1$ $v^{n} \quad k = n, n+1, \ldots$	$v^{k+1} \quad k = 0, 1, \ldots, n-1$ $v^{n} \quad k = n, n+1, \ldots$	$A_{x:\overline{n}\rceil}$	*
(d) m-Year deferred n-year term	$1 \quad k = m, m+1, \ldots, m+n-1$ $0 \quad k = 0, \ldots, m-1$ $k = m+n, \ldots$	v^{k+1}	$v^{k+1} \quad k = m, m+1, \ldots, m+n-1$ $0 \quad k = 0, \ldots, m-1$ $k = m+n, \ldots$	${}_{m\mid n}A_x$	*
(e) n-Year term increasing annually	$k+1 \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	v^{k+1}	$(k+1)v^{k+1} \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	$(IA)^{1}_{x:\overline{n}\rceil}$	†
(f) n-Year term decreasing annually	$n-k \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	v^{k+1}	$(n-k)v^{k+1} \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	$(DA)^{1}_{x:\overline{n}\rceil}$	†
(g) Whole life increasing annually	$k+1 \quad k = 0, 1, \ldots$	v^{k+1}	$(k+1)v^{k+1} \quad k = 0, 1, \ldots$	$(IA)_x$	†

b_{k+1}, v_{k+1}, and z_{k+1} are defined only for non-negative integral values of k.
*Rule of moments holds, thus $\mathrm{Var}(Z) = {}^2A - A^2$ symbolically.
†Rule of moments does not hold.

(a) $$A_x = vq_x + vp_x\, A_{x+1} \qquad x = 0, 1, \ldots, \omega - 1,$$

and $A_\omega = 0$.

(b) $$A^{1}_{x:\overline{y-x}|} = vq_x + vp_x\, A^{1}_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1,$$

and $A^{1}_{y:\overline{0}|} = 0$.

(c) $$A_{x:\overline{y-x}|} = vq_x + vp_x\, A_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1,$$

and $A_{y:\overline{0}|} = 1$.

(d) $${}_{y-x|n}A_x = 0 + vp_x\, {}_{y-(x+1)|n}A_{x+1} \qquad x = 0, 1, \ldots, y - 1,$$

and ${}_{0|n}A_y = A^{1}_{y:\overline{n}|}$.

(e) $$(IA)^{1}_{x:\overline{y-x}|} = [vq_x + vp_x\, A^{1}_{x+1:\overline{y-(x+1)}|}] + vp_x\, (IA)^{1}_{x+1:\overline{y-(x+1)}|}$$

$x = 0, 1, \ldots, y - 1$, and $(IA)^{1}_{y:\overline{0}|} = 0$.

(f) $$(DA)^{1}_{x:\overline{y-x}|} = (y - x)vq_x + vp_x\, (DA)^{1}_{x+1:\overline{y-(x+1)}|}$$

$x = 0, 1, \ldots, y - 1$, and $(DA)^{1}_{y:\overline{0}|} = 0$.

(g) $$(IA)_x = [vq_x + vp_x\, A_{x+1}] + vp_x\, (IA)_{x+1} \qquad x = 0, 1, \ldots, \omega - 1,$$

and $(IA)_\omega = 0$.

Observations:

1. Only (a) and (b) have been justified in this section. Arguments for (c) through (g) are similar to those for (a) and (b).
2. All seven equations are of the form

$$u(x) = c(x) + vp_x\, u(x + 1),$$

where $c(x)$ is a given function defined for the domain of the relation. In the language of difference equations, all seven equations have the same corresponding homogeneous equation, $u(x) = vp_x\, u(x + 1)$. It is linear but does not have constant coefficients.

3. Since $c(x) = vq_x$ for (a), (b), and (c), those actuarial present values are all solutions of the same recursion formula and are distinguished only by their starting values.

4.4 Relationships between Insurances Payable at the Moment of Death and the End of the Year of Death

We begin the study of these relationships with an analysis of the actuarial present value for whole life insurance paying a unit benefit at the moment of death. From (4.2.6) we have

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + \int_1^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt.$$

The change of variables $s = t - 1$ in the second integral gives

$$\bar{A}_x = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + v \int_0^\infty v^s \, {}_{s+1}p_x \, \mu_x(s+1) \, ds. \tag{4.4.1}$$

On an aggregate mortality basis

$${}_{s+1}p_x \, \mu_x(s+1) = p_x \, {}_sp_{x+1} \, \mu(x+s+1)$$

so the second term of (4.4.1) would be $vp_x \bar{A}_{x+1}$. On a select mortality basis the second term would be $vp_{[x]}\bar{A}_{[x]+1}$. Returning to (4.4.1) and using aggregate notation, we have

$$\bar{A}_x = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + vp_x \, \bar{A}_{x+1} = \bar{A}^{1}_{x:\overline{1}|} + vp_x \, \bar{A}_{x+1}. \tag{4.4.2}$$

The integral in (4.4.2) can be expressed in discrete life table functions by adopting one of the assumptions about the form of the mortality function between integers as discussed in Section 3.6.

Under the assumption of a uniform distribution of deaths over each year of age,

$${}_tp_y \, \mu_y(t) = q_y \qquad 0 \le t \le 1, \text{ and } y = 0, 1, \ldots$$

which can be placed in (4.4.2) to obtain

$$\begin{aligned}\bar{A}_x &= q_x \int_0^1 v^t \, dt + vp_x \bar{A}_{x+1}, \\ &= \frac{i}{\delta} vq_x + vp_x \, \bar{A}_{x+1}.\end{aligned} \tag{4.4.3}$$

The domain for this relationship is $x = 0, 1, \ldots, \omega - 1$, and the starting value is $\bar{A}_\omega = 0$.

If we multiply both sides of recursion formula (a) by i/δ, we have

$$\frac{i}{\delta} A_x = \frac{i}{\delta} vq_x + vp_x \left(\frac{i}{\delta} A_{x+1} \right).$$

Since (a) and (4.4.3) embody the same recursion formula and have the same domain and the same initial value of 0 at ω, $(i/\delta)A_x$ is the solution for (4.4.3), and

$$\bar{A}_x = \frac{i}{\delta} A_x. \tag{4.4.4}$$

Formula (4.4.4) might have been anticipated under the assumption of a uniform distribution of deaths between integral ages. The effect of the assumption is to make the unit payable at the moment of death equivalent to a unit payable continuously throughout the year of death. With respect to interest, a unit payable continuously over the year is equivalent to i/δ at the end of the year.

The identity in (4.4.4) can be reached using the properties of the future-lifetime random variable under the assumption of a uniform distribution of deaths in each year of age as developed in Section 3.6. From (3.6.1) we write $T = K + S$. We observed there that, under the assumption of a uniform distribution of deaths in each year of age, K and S are independent and S has a uniform distribution over the unit interval. As corollaries to these observations, $K + 1$ and $1 - S$ are also independent, and $1 - S$ has a uniform distribution over the unit interval. In the identity

$$\bar{A}_x = \mathrm{E}[v^T] = \mathrm{E}[v^{K+1}(1 + i)^{1-S}],$$

we can use the independence of $K + 1$ and $1 - S$ to calculate the expectation of the product as the product of the expectations,

$$\mathrm{E}[v^{K+1}(1 + i)^{1-S}] = \mathrm{E}[v^{K+1}]\,\mathrm{E}[(1 + i)^{1-S}]. \tag{4.4.5}$$

The first factor on the right-hand side is A_x. Since $1 - S$ has the uniform distribution over the unit interval, the second factor is

$$\mathrm{E}[(1 + i)^{1-S}] = \int_0^1 (1 + i)^t\, 1\, dt = \frac{i}{\delta}.$$

Hence, again we have $\bar{A}_x = (i/\delta)A_x$ under the assumption of uniform distribution of deaths in each year of age.

A similar argument, again based on the assumption of a uniform distribution of deaths in each year of age, can be used to show that the actuarial present value of a whole life insurance which pays a unit at the end of the m-th of a year of death is equal to

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x. \tag{4.4.6}$$

This argument is outlined in Exercise 4.19.

In Section 3.6 we also discussed the assumption that the force of mortality is constant between integral ages. The relationship between the actuarial present values for whole life insurances payable at the moment of death and at the end of the year of death under this assumption is developed in Exercise 4.19. Since the hyperbolic assumption implies that the force of mortality decreases over the year of age (see Exercise 3.27), it is seldom realistic for human lives. Moreover, it leads to more complicated relationships that we will not develop here.

Next we turn to an analysis of the annually increasing n-year term insurance payable at the moment of death. For this insurance, the present-value random variable is

$$Z = \begin{cases} \lfloor T + 1 \rfloor v^T & T < n \\ 0 & T \geq n. \end{cases}$$

Since $\lfloor T + 1 \rfloor = K + 1$, we can use the relation $T = K + S$ to obtain

$$Z = \begin{cases} (K+1)v^{K+1}v^{S-1} & T < n \\ 0 & T \ge n. \end{cases}$$

If we let W be the present-value random variable for the annually increasing n-year term insurance payable at the end of the year of death,

$$W = \begin{cases} (K+1)v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots . \end{cases}$$

Then

$$Z = W(1+i)^{1-S}$$

and

$$\mathrm{E}[Z] = \mathrm{E}[W(1+i)^{1-S}].$$

Since W is a function of $K+1$ alone and $K+1$ and $1-S$ are independent,

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{E}[W]\,\mathrm{E}[(1+i)^{1-S}] \\ &= (IA)^{1}_{x:\overline{n}|}\,\frac{i}{\delta}. \end{aligned}$$

These results for the whole life and the increasing term insurances payable at the moment of death, under the assumption of a uniform distribution of deaths over each year of age, are very similar,

$$\bar{A}_x = \frac{i}{\delta} A_x$$

and

$$(I\bar{A})^{1}_{x:\overline{n}|} = \frac{i}{\delta}(IA)^{1}_{x:\overline{n}|}.$$

Let us look at the general model to find the basis of the similarities. From (4.2.2),

$$Z = b_T v_T. \tag{4.4.7}$$

For the two continuous insurances above, the conditions used were

- $v_T = v^T$, and
- b_T was a function of only the integral part of T, the curtate-future-lifetime, K.

Writing this latter property as $b_T = b^{*}_{K+1}$ we can write (4.4.7) for these insurances as

$$\begin{aligned} Z &= b^{*}_{K+1} v^{T} \\ &= b^{*}_{K+1} v^{K+1}(1+i)^{1-S} \end{aligned}$$

and

$$\mathrm{E}[Z] = \mathrm{E}[b^{*}_{K+1} v^{K+1}(1+i)^{1-S}]. \tag{4.4.8}$$

Under the assumption of a uniform distribution of deaths over each year of age, we can infer the independence of K and S and that $1-S$ also has a uniform distribution. Then we can write (4.4.8) as

$$
\begin{aligned}
E[Z] &= E[b^*_{K+1}v^{K+1}]\, E[(1+i)^{1-S}] \\
&= E[b^*_{K+1}v^{K+1}]\, \frac{i}{\delta}. \qquad (4.4.9)
\end{aligned}
$$

Example 4.4.1

Calculate the actuarial present value and the variance for a 10,000 benefit, 30-year endowment insurance providing the death benefit at the moment of death of a male age 35 at issue of the policy. Use the Illustrative Life Table, the uniform distribution of deaths over each year of age assumption, and $i = 0.06$.

Solution:

For endowment insurance, $v_T \neq v^T$. Therefore, we cannot apply (4.4.9) directly. Recalling (4.2.11), which showed the endowment insurance as the sum of a term insurance and pure endowment, we can apply (4.4.9) to the term insurance component and then calculate the pure endowment insurance part. Thus, using (4.2.12) and (4.2.10), we can calculate the actuarial present value as follows:

$$
\begin{aligned}
\bar{A}_{35:\overline{30|}} &= \frac{i}{\delta} A^{1}_{35:\overline{30|}} + A_{35:\overset{\ \ 1}{\overline{30|}}} \\
&= (1.0297087)[0.06748179] + 0.1392408 \\
&= 0.208727,
\end{aligned}
$$

and the variance as

$$
\begin{aligned}
\mathrm{Var}(Z) &= {}^2\bar{A}_{35:\overline{30|}} - (\bar{A}_{35:\overline{30|}})^2 \\
&= 0.0309294 + 0.0242432 - (0.208727)^2 \\
&= 0.011606.
\end{aligned}
$$

For the 10,000 sum insured, $10{,}000\ \bar{A}_{35:\overline{30|}} = 2{,}087.27$ and $(10{,}000)^2\ \mathrm{Var}(Z) = 1{,}160{,}600$. ▼

Example 4.4.2

Calculate, for a life age 50, the actuarial present value for an annually decreasing 5-year term insurance paying 5,000 at the moment of death in the first year, 4,000 in the second year, and so on. Use the Illustrative Life Table, uniform distribution of deaths over each year of age assumption, and $i = 0.06$.

Solution:

Referring to Table 4.2.1, we see that

$$
b_t = \begin{cases} 5 - \lfloor t \rfloor & t \le 5 \\ 0 & t > 5 \end{cases}
$$

is a function of only k, the integral part of t, and hence we may write it as

$$b_t = \begin{cases} 5 - k & k = 0, 1, 2, 3, 4 \\ 0 & k > 4. \end{cases}$$

The discount function is v^t, so we have

$$\begin{aligned}
(D\bar{A})^1_{50:\overline{5}|} &= \frac{i}{\delta} (DA)^1_{50:\overline{5}|} \\
&= (1.0297087) \sum_{k=0}^{4} (5 - k)\, v^{k+1}\, {}_kp_{50}\, q_{50+k} \\
&= (1.0297087) \frac{\sum_{k=0}^{4} (5 - k)\, v^{k+1}\, d_{50+k}}{l_{50}} \\
&= 0.088307.
\end{aligned}$$

Then, $1{,}000\, (D\bar{A})^1_{50:\overline{5}|} = 88.307$. ▼

For an insurance providing a death benefit at the moment of death that is not a function of K, further analysis is required to express its values in terms of those for an insurance payable at the end of the year of death. Consider the continuously increasing whole life insurance payable at the moment of death. This insurance is discussed extensively in Section 4.2, and its benefit function analyzed in Figure 4.2.5. Its functions are

$$\begin{aligned}
b_t &= t && t > 0, \\
v_t &= v^t && t > 0, \\
z_t &= tv^t && t > 0.
\end{aligned}$$

To find $(\bar{I}\bar{A})_x$ we rewrite

$$\begin{aligned}
Z &= (K + S)v^{K+S} \\
&= (K + 1)v^{K+S} - (1 - S)v^{K+1}(1 + i)^{1-S} \\
&= (K + 1)v^{K+1}(1 + i)^{1-S} - v^{K+1}(1 - S)(1 + i)^{1-S}.
\end{aligned}$$

Now taking expectations, under the assumption of a uniform distribution of deaths over each year of age, we have

$$\begin{aligned}
\mathrm{E}[Z] &= \mathrm{E}[(K + 1)v^{K+1}]\, \mathrm{E}[(1 + i)^{1-S}] - \mathrm{E}[v^{K+1}]\, \mathrm{E}[(1 - S)(1 + i)^{1-S}] \\
&= (IA)_x \frac{i}{\delta} - A_x\, \mathrm{E}[(1 - S)(1 + i)^{1-S}].
\end{aligned}$$

We can simplify the last factor directly since $1 - S$ has a uniform distribution,

$$\mathrm{E}[(1 - S)(1 + i)^{1-S}] = \int_0^1 u(1 + i)^u\, du = (\bar{D}\bar{s})_{\overline{1}|} = \frac{1 + i}{\delta} - \frac{i}{\delta^2}.$$

Thus, we can write

$$(\bar{I}\bar{A})_x = \frac{i}{\delta} \left[(IA)_x - \left(\frac{1}{d} - \frac{1}{\delta} \right) A_x \right].$$

4.5 Differential Equations for Insurances Payable at the Moment of Death

Recursive-type expressions can be established for insurances payable at the moment of death. These are developed using calculus and lead to differential equations.

For a whole life insurance on (x),

$$\frac{d}{dx}\bar{A}_x = -\mu(x) + \bar{A}_x[\delta + \mu(x)] = \delta\bar{A}_x - \mu(x)(1 - \bar{A}_x), \tag{4.5.1}$$

which are the continuous analogues of (4.3.12). The notation used here is for an aggregate mortality basis. Verification of these expressions has been left to Exercise 4.21.

On the other hand, (4.5.1) can be developed from the definition of $\bar{A}_x$ by using conditional expectation as we did for A_x:

$$\begin{aligned} \bar{A}_x &= \mathrm{E}[v^T] \\ &= \mathrm{E}[v^T|T \le h]\ \Pr(T \le h) + \mathrm{E}[v^T|T > h]\ \Pr(T > h). \end{aligned} \tag{4.5.2}$$

Now,

$$\Pr(T \le h) = {}_hq_x \quad \text{and} \quad \Pr(T > h) = {}_hp_x, \tag{4.5.3}$$

and the conditional p.d.f. of T given $T \le h$ is

$$f_T(t|T \le h) = \begin{cases} \dfrac{f_T(t)}{F_T(h)} = \dfrac{{}_tp_x\ \mu(x+t)}{{}_hq_x} & 0 \le t \le h \\ 0 & \text{elsewhere.} \end{cases}$$

Thus,

$$\mathrm{E}[v^T|T \le h] = \int_0^h v^t\ \frac{{}_tp_x\ \mu(x+t)}{{}_hq_x}\ dt. \tag{4.5.4}$$

As we did in the expression for A_x, we will write

$$\begin{aligned} \mathrm{E}[v^T|T > h] &= v^h\mathrm{E}[v^{T-h}|(T-h) > 0] \\ &= v^h\bar{A}_{x+h}. \end{aligned} \tag{4.5.5}$$

Substitution of (4.5.3), (4.5.4), and (4.5.5) into (4.5.2) yields

$$\bar{A}_x = \int_0^h v^t\ \frac{{}_tp_x\ \mu(x+t)}{{}_hq_x}\ dt\ {}_hq_x + v^h\ \bar{A}_{x+h}\ {}_hp_x. \tag{4.5.6}$$

Then, on both sides of (4.5.6), we multiply by -1, add $\bar{A}_{x+h}$, and divide by h to obtain

$$\frac{\bar{A}_{x+h} - \bar{A}_x}{h} = \frac{-1}{h}\int_0^h v^t\ {}_tp_x\ \mu(x+t)\,dt + \bar{A}_{x+h}\frac{1 - v^h\ {}_hp_x}{h}. \tag{4.5.7}$$

Next,

$$\lim_{h\to 0}\frac{1}{h}\int_0^h v^t\ {}_tp_x\ \mu(x+t)\,dt = \frac{d}{ds}\int_0^s v^t\ {}_tp_x\ \mu(x+t)\,dt\Big|_{s=0} = \mu(x)$$

and

$$\lim_{h\to 0}\frac{1 - v^h\ {}_hp_x}{h} = -\frac{d}{dt}(v^t\ {}_tp_x)\Big|_{t=0} = \mu(x) + \delta.$$

Using these two limits as $h \to 0$ in (4.5.7) we obtain (4.5.1),

$$\frac{d}{dx}\bar{A}_x = -\mu(x) + \bar{A}_x[\mu(x) + \delta].$$

Solutions of this differential equation are outlined in Exercise 4.22.

4.6 Notes and References

The life contingencies textbooks listed in Appendix 6 give other developments of formulas for life insurance actuarial present values. Commutation functions, which were fundamental to actuarial calculations until this quarter century, are employed extensively in Jordan (1967).

There is little material in these textbooks on the concept of the time-until-death of an insured as a random variable. Until recently, research and exposition on this concept has been called ***individual risk theory.*** Cramér (1930) gives a detailed exposition of the ideas up to that time. Kahn (1962) and Seal (1969) give concise bibliographical information on both research and expository papers over a 100-year span.

Since 1970 there has been interest in actuarial models that consider both the time-until-death and the investment-rate-of-return as random variables. This combination is discussed in Chapter 21.

Exercises

Assume, unless otherwise stated, that insurances are payable at the moment of death, and that the force of interest is a constant δ with i and d as the equivalent rates of interest and discount.

Section 4.2

4.1. If $\mu(x) = \mu$, a positive constant, for all $x > 0$, show that $\bar{A}_x = \mu/(\mu + \delta)$.

4.2. Let $\mu(x) = 1/(1 + x)$, for all $x > 0$.

a. Integrate by parts to show that

$$\bar{A}_x = 1 - \delta \int_0^\infty e^{-\delta t} \frac{1 + x}{1 + x + t} \, dt.$$

b. Use the expression in (a) to show that $d\bar{A}_x / dx < 0$ for all $x > 0$.

4.3. Show that $d\bar{A}_x / di = -v(\bar{I}\bar{A})_x$.

4.4. Show that the expressions for the variance of the present value of an n-year endowment insurance paying a unit benefit, as given by (4.2.10) and (4.2.13), are identical.

4.5. Let Z_1 and Z_2 be as defined for equation (4.2.11).
a. Show that $\lim_{n \to 0} \text{Cov}(Z_1, Z_2) = \lim_{n \to \infty} \text{Cov}(Z_1, Z_2) = 0$.
b. Develop an implicit equation for the term of the endowment for which $\text{Cov}(Z_1, Z_2)$ is minimized.
c. Develop a formula for the minimum in (b).
d. Simplify the formulas in (b) and (c) for the case when the force of mortality is a constant μ.

4.6. Assume mortality is described by $l_x = 100 - x$ for $0 \le x \le 100$ and that the force of interest is $\delta = 0.05$.
a. Calculate $\bar{A}^{1}_{40:\overline{25}|}$.
b. Determine the actuarial present value for a 25-year term insurance with benefit amount for death at time t, $b_t = e^{0.05t}$, for a person age 40 at policy issue.

4.7. Assuming De Moivre's survival function with $\omega = 100$ and $i = 0.10$, calculate
a. $\bar{A}^{1}_{30:\overline{10}|}$
b. The variance of the present value, at policy issue, of the benefit of the insurance in (a).

4.8. If $\delta_t = 0.2/(1 + 0.05t)$ and $l_x = 100 - x$ for $0 \le x \le 100$, calculate
a. For a whole life insurance issued at age x, the actuarial present value and the variance of the present value of the benefits
b. $(\bar{I}\bar{A})_x$.

4.9. a. Show that $\bar{A}_x$ is the moment generating function of T, the future lifetime of (x), evaluated at $-\delta$.
b. Show that if T has a gamma distribution with parameters α and β, then $\bar{A}_x = (1 + \delta/\beta)^{-\alpha}$.

4.10. Given $b_t = t$, $\mu_x(t) = \mu$, and $\delta_t = \delta$ for all $t > 0$, derive expressions for
a. $(\bar{I}\bar{A})_x = \text{E}[b_T v^T]$ b. $\text{Var}(b_T v^T)$.

4.11. The random variable Z is the present-value random variable for a whole life insurance of unit amount payable at the moment of death and issued to (x). If $\delta = 0.05$ and $\mu_x(t) = 0.01$:
a. Display the formula for the p.d.f. of Z.
b. Graph the p.d.f. of Z.
c. Calculate $\bar{A}_x = E[Z]$ and $Var(Z)$.

4.12. The random variable Z is the present-value random variable for an n-year endowment insurance as defined in Section 4.2.2. Exhibit the d.f. of Z in terms of the d.f. of T.

4.13. The random variable Z is defined as in Exercise 4.12. If $\delta = 0.05$, $\mu_x(t) = 0.01$, and $n = 20$:
a. Display the d.f. of Z.
b. Graph the d.f. of Z.
c. Calculate $\bar{A}_{x:\overline{n}|} = E[Z]$ using the distribution of Z. [Hint: Consider using the complement of the d.f.]

Section 4.3

4.14. If $l_x = 100 - x$ for $0 \le x \le 100$ and $i = 0.05$, evaluate
a. $A_{40:\overline{25}|}$ b. $(IA)_{40}$.

4.15. Show that $A_{x:\overline{n}|} = A^{\,1}_{x:\overline{m}|} + v^m \, {}_mp_x \, A_{x+m:\overline{n-m}|}$ for $m < n$ and interpret the result in words.

4.16. If $A_x = 0.25$, $A_{x+20} = 0.40$, and $A_{x:\overline{20}|} = 0.55$, calculate
a. $A_{x:\overset{1}{\overline{20}|}}$ b. $A^{\,1}_{x:\overline{20}|}$.

4.17. a. Describe the benefits of the insurance with actuarial present value given by the symbol $(IA)_{x:\overline{m}|}$.
b. Express the actuarial present value of (a) in terms of the symbols given in Tables 4.2.1 and 4.3.1.

4.18. In Example 4.3.2, let the expected size of the fund k years after the agreement, and immediately after the payment of death claims, be denoted by E_k, where $E_0 = (100)(1{,}000\, A_{30}) = 10{,}248.35$.
a. Start with (4.3.10) and develop the forward recursion formula

$$E_k = 1.06E_{k-1} - 100{,}000_{k-1|}q_{30}.$$

b. Use the recursion formula developed in (a) to confirm that $E_5 = 12{,}762.58$.

Section 4.4

4.19. Consider the timescale measured in intervals of length $1/m$ where the unit is a year. Let a whole life insurance for a unit amount be payable at the end of the m-thly interval in which death occurs. Let k be the number of complete insurance years lived prior to death and let j be the number of complete m-ths of a year lived in the year of death.

a. What is the present-value function for this insurance?

b. Set up a formula analogous to (4.4.2) for the actuarial present value, $A_x^{(m)}$, for this insurance.

c. Show algebraically that, under the assumption of a uniform distribution of deaths over the insurance year of age,

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

4.20. Show, under the assumption of a constant force of mortality between integral ages, that

$$\bar{A}_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x \, \mu_x(k) \frac{i + q_{x+k}}{\delta + \mu_x(k)}$$

where $\mu_x(k) = -\log p_{x+k}$.

Section 4.5

4.21. a. Show that (4.2.6), for an aggregate mortality basis, can be rewritten as

$$\bar{A}_x = \frac{1}{{}_xp_0 \, v^x} \int_x^{\infty} v^y \, {}_yp_0 \, \mu(y) \, dy \qquad x \geq 0.$$

b. Differentiate the formula of (a) to establish (4.5.1),

$$\frac{d\bar{A}_x}{dx} = [\mu(x) + \delta]\bar{A}_x - \mu(x) \qquad x \geq 0.$$

c. Use the same technique to show

$$\frac{d\bar{A}^{1}_{x:\overline{n}|}}{dx} = [\mu(x) + \delta]\bar{A}^{1}_{x:\overline{n}|} + \mu(x + n)A_{x:\overline{n}|}^{\ \ \ 1} - \mu(x) \qquad x \geq 0.$$

4.22. Solve the differential equation (4.5.1) as follows:

a. Use the integrating factor

$$\exp\left[-\int_y^x (\delta + \mu(z) \, dz\right]$$

to obtain

$$\bar{A}_y = \int_y^{\infty} \mu(x) \exp\left\{-\int_y^x [\delta + \mu(z)] \, dz\right\} dx.$$

b. Use the integrating factor $e^{-\delta x}$ to obtain

$$\bar{A}_y = \int_y^{\infty} \mu(x) \, v^{x-y}(1 - \bar{A}_x) \, dx.$$

Miscellaneous

4.23. Display the actuarial present value of a Double Protection to Age 65 policy that provides a benefit of 2 in the event of death prior to age 65 and a benefit of 1 after age 65 in symbols of Table 4.3.1. Assume benefits are paid at the end of the year of death.

4.24. A policy is issued at age 20 with the following graded scale of death benefits payable at the moment of death.

Age	Death Benefits
20	1 000
21	2 000
22	4 000
23	6 000
24	8 000
25–40	10 000
41 and over	50 000

$_{m|n}A_x = v^m \, _mP_x \, A^{1}_{x+m:\overline{n}|}$

$A_{x+m:\overline{n}|} = A_{x+m} - v^n \, _nP_{x+m} \cdot A_{x+m+n}.$

Calculate the actuarial present value on the basis of the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.05$. [Hint: A backward recursion formula for the actuarial present value will include a function $c(x)$ that incorporates the death benefit scale.]

4.25. a. Determine whether or not a constant increase in the force of mortality has the same effect on A_x as the same increase in the force interest.

b. Show that if the single probability of death q_{x+n} is increased to $q_{x+n} + c$, then A_x will be increased by

$$c\, v^{n+1} \, _np_x \, (1 - A_{x+n+1}).$$

4.26. The actuarial present value for a modified pure endowment of 1,000 issued at age x for n years is 700 with a death benefit equal to the actuarial present value in event of death during the n-year period and is 650 with no death benefit.

a. Calculate the actuarial present value for a modified pure endowment of 1,000 issued at age x for n years if $100k\%$ of the actuarial present value is to be paid at death during the period.

b. For the modified pure endowment in (a), express the variance of the present value at policy issue in terms of actuarial present values for pure endowments and term insurances.

4.27. An appliance manufacturer sells his product with a 5-year warranty promising the return of cash equal to the pro rata share of the initial purchase price for failure within 5 years. For example, if failure is reported 3-3/4 years following purchase, 25% of the purchase price will be returned. From statistical studies, the probability of failure of a new product during the first year is estimated to be 0.2, in each of the second, third, and fourth years 0.1, and in the fifth year 0.2.

a. Assuming that failures are reported uniformly within each year since purchase, determine the fraction of the purchase price that equals the actuarial present value for this warranty. Assume $i = 0.10$.

b. If the warranted return is the reduction on the purchase price of a new product with a 5-year warranty, would the answer to (a) change?

4.28. Assume that $T(x)$ has a p.d.f. given by

$$f_{T(x)}(t) = \frac{2}{10\sqrt{2\pi}} e^{-t^2/200} \qquad t > 0$$

and $\delta = 0.05$.
Show:
a. $\bar{A}_x = 2e^{0.125}[1 - \Phi(0.5)] = 0.6992$
b. ${}^2\bar{A}_x = 2e^{-0.5}[1 - \Phi(1)] = 0.5232$
c. $\mathrm{Var}(Z) = 0.0343$, where $Z = v^T$
d. $\xi_Z^{0.5} = 0.7076$ [Hint: Use Figure 4.2.1]
e. $v^{\mathring{e}_x} = 0.6710 < 0.6992 = \bar{A}_x$.

4.29. Generalize Exercise 4.28(e) by showing that if $\delta > 0$, then

$$v^{\mathring{e}_x} \le \bar{A}_x.$$

[Hint: Use Jensen's Inequality in Section 1.3 when $u''(x) > 0$.]

4.30. For a whole life insurance the benefit amount, b_t, is 0 or 1 for each $t \ge 0$. For calculations at force of interest δ_t:
a. Express the discount function in terms of δ_t.
b. Express the present-value random variable, Z, in terms of T.
c. Show that Z^j@force of interest δ_t equals Z@force of interest $j\delta_t$.

Computing Exercises

4.31. Augment your Illustrative Life Table to include input of a constant interest rate, i, and a payment frequency, m. It will be helpful for the equivalent rates δ, $d^{(m)}$, $i^{(m)}$, d, and v to be shown in your Illustrative Life Table output.

4.32. Develop a set of values of $\ddot{a}_{\overline{n}|}$, $n = 1, 2, \ldots, 100$ at $i = 0.06$ by using the forward recursion formula (3.5.20). [Hint: Review Exercise 3.25 or use $\ddot{a}_{\overline{n+1}|} = 1 + v\ddot{a}_{\overline{n}|}$.]

4.33. a. Using the backward recursion formula of (4.3.10) in your Illustrative Life Table and the appropriate starting value, calculate $1{,}000A_x$ for $x = 13$ to 140 for an interest rate of 0.06.
b. Compare your values to those in Appendix 2A.

4.34. a. Use (4.3.3) and your Illustrative Life to determine $A^{1}_{20:\overline{20}|}$ and ${}^2A^{1}_{20:\overline{20}|}$ at $i = 0.05$.
b. What is the variance for the present-value random variable for a 20-year term insurance with benefit amount 100,000 issued to (20)?

4.35. a. By an algebraic or probabilistic argument, verify the following backward recursion formula of an n-year term insurance with a unit benefit:

$$A^{1}_{x:\overline{n}|} = vq_x - v^{n+1}{}_{n}p_x\, q_{x+n} + vp_x\, A^{1}_{x+1:\overline{n}|}.$$

b. Determine an appropriate starting value for use with this formula.

c. Use your Illustrative Life Table with $i = 0.06$ to calculate the actuarial present value of a 10-year term insurance issued at ages $x = 13, \ldots, 130$.

4.36. a. Use recursion relation (g) at the end of Section 4.3 and your Illustrative Life Table to calculate $(IA)_{28}$ at $i = 0.06$.

b. Modify the recursion relation of part (a) to obtain one for $(I\bar{A})_x$ and determine a starting value for it.

c. Modify the recursion relation of part (b) to obtain one for $(\bar{I}\bar{A})_x$ and determine a starting value for it.

d. Make the recursion relations of parts (b) and (c) specific to the assumption of a uniform distribution of deaths over each year of age.

4.37. Use your Illustrative Life Table to verify the numerical solutions to parts (a) and (b) of Example 4.2.4. [Hint: Set $B = 0.00$ and $A = 0.04$ in your Makeham law parameters, $i = e^{0.10} - 1$, and use recursion formula (d) at the end of Section 4.3. Remember that the insurance in Example 4.2.4 is payable at the moment of death.]

4.38. a. By an algebraic or probabilistic argument, verify the following backward recursion formula for the actuarial present value of a unit benefit endowment insurance to age y with the death benefit payable at the moment of death:

$$\bar{A}_{x:\overline{y-x}|} = \bar{A}^{1}_{x:\overline{1}|} + vp_x\, \bar{A}_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y-1.$$

b. Determine an appropriate starting value for use with this formula.

c. Use your Illustrative Life Table with the assumption of uniform distribution of deaths over each year of age and $i = 0.06$ to calculate the actuarial present value of a unit benefit endowment insurance to age 65 with the death benefit payable at the moment of death for issue ages $x = 13, \ldots,$ 64.

d. By an algebraic or probabilistic argument, verify the following backward recursion formula for the actuarial present value of a unit benefit n-year endowment insurance with the death benefit payable at the moment of death:

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{1}|} + v^{n}\, {}_{n}p_x\, (1 - vp_{x+n} - \bar{A}^{1}_{x+1:\overline{n}|}) + vp_x\, \bar{A}_{x+1:\overline{n}|}$$
$$x = 0, 1, \ldots, \omega - 1.$$

4.39. Let Z be the present-value random variable for a 100,000 unit 20-year endowment insurance with the death benefit payable at the moment of death. Use your Illustrative Life Table to calculate the mean and the variance of Z on the basis of a Makeham law with $A = 0.001$, $B = 0.00001$, $c = 1.10$, and $\delta = 0.05$.

5

LIFE ANNUITIES

5.1 Introduction

In the preceding chapter we studied payments contingent on death, as provided by various forms of life insurances. In this chapter we study payments contingent on survival, as provided by various forms of life annuities. A ***life annuity*** is a series of payments made continuously or at equal intervals (such as months, quarters, years) while a given life survives. It may be temporary, that is, limited to a given term of years, or it may be payable for the whole of life. The payment intervals may commence immediately or, alternatively, the annuity may be deferred. Payments may be due at the beginnings of the payment intervals (***annuities-due***) or at the ends of such intervals (***annuities-immediate***).

Through the study of ***annuities-certain*** in the theory of interest, the reader already has a knowledge of annuity terminology, notation, and theory. Life annuity theory is analogous but brings in survival as a condition for payment. This condition has been encountered in Chapter 4 in connection with pure endowments and the maturity payments under endowment insurances.

Life annuities play a major role in life insurance operations. As we see in the next chapter, life insurances are usually purchased by a life annuity of premiums rather than by a single premium. The amount payable at the time of claim may be converted through a settlement option into some form of life annuity for the beneficiary. Some types of life insurance carry this concept even further and, instead of featuring a lump sum payable on death, provide stated forms of income benefits. Thus, for example, there may be a monthly income payable to a surviving spouse or to a retired insured.

Annuities are even more central in pension systems. In fact, a retirement plan can be regarded as a system for purchasing deferred life annuities (payable during retirement) by some form of temporary annuity of contributions during active service. The temporary annuity may consist of varying contributions, and valuation of it may take into account not only interest and mortality, but other factors such as salary increases and the termination of participation for reasons other than death.

Life annuities also have a role in disability and workers' compensation insurances. In the case of disability insurance, termination of the annuity benefit by reason of recovery of the disabled insured may need to be considered. For surviving spouse benefits under workers' compensation, remarriage may terminate the annuity.

We proceed in this chapter as we did in Chapter 4 and express the present value of benefits to be received by the annuitant as a function of T, the annuitant's future-lifetime random variable. It then will be possible to study properties of the distribution of this financial value random variable. The expectation, still called the actuarial present value, can be evaluated in an alternative way using either integration by parts or summation by parts depending, respectively, on whether a continuous or discrete set of payments is being evaluated. The results of this process have a useful interpretation and lead to an alternative method of obtaining actuarial present values called the ***current payment technique.***

As in the preceding chapter on life insurance, unless otherwise stated we assume a constant effective annual rate of interest i (or the equivalent constant force of interest δ). We also assume aggregate mortality for most of the development in this chapter and indicate those situations where a select mortality assumption makes a major difference.

In most applications of the theory developed in this chapter, annuity payments continue while a human life remains in a particular status. However, the possible applications of the theory are much wider. It may be applied to any set of periodic payments where the payments are not made with certainty. Examples of some of these applications are seen in later chapters dealing with multiple lives or multiple causes of decrement.

5.2 Continuous Life Annuities

We start with annuities payable continuously at the rate of 1 per year. This is of course an abstraction but makes use of familiar mathematical tools and as a practical matter closely approximates annuities payable on a monthly basis. A ***whole life annuity*** provides for payments until death. Hence, the present value of payments to be made is $Y = \bar{a}_{\overline{T|}}$ for all $T \ge 0$ where T is the future lifetime of (x). The distribution function of Y can be obtained from that for T as follows:

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) = \Pr(\bar{a}_{\overline{T|}} \le y) = \Pr(1 - v^T \le \delta y) \\ &= \Pr(v^T \ge 1 - \delta y) = \Pr\left[T \le \frac{-\log(1 - \delta y)}{\delta}\right] \\ &= F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right) \quad \text{for } 0 < y < \frac{1}{\delta}. \end{aligned} \tag{5.2.1}$$

From this we obtain the probability density function for Y as

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_T\left(\frac{-\log(1-\delta y)}{\delta}\right)$$

$$= \frac{f_T([-\log(1-\delta y)]/\delta)}{1-\delta y} \quad \text{for } 0 < y < \frac{1}{\delta}. \tag{5.2.2}$$

The distribution function for Y depends on the distribution of T but would resemble that shown at the end of this section in Figure 5.2.1(a).

The actuarial present value for a continuous whole life annuity is denoted by $\bar{a}_x$ where the post fixed subscript, x, indicates that the annuity ceases at the death of (x) and that the distribution of $T(x)$ may depend on information available at age x. Under aggregate mortality the p.d.f. of T is ${}_tp_x\, \mu(x+t)$, and the actuarial present value can be calculated by

$$\bar{a}_x = \mathrm{E}[Y] = \int_0^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt. \tag{5.2.3}$$

Using integration-by-parts with $f(t) = \bar{a}_{\overline{t}|}$, $dg(t) = {}_tp_x\, \mu(x+t)\, dt$, $g(t) = -{}_tp_x$, and $df(t) = v^t\, dt$, we obtain

$$\bar{a}_x = \int_0^\infty v^t\, {}_tp_x\, dt = \int_0^\infty {}_tE_x\, dt. \tag{5.2.4}$$

This integral can be considered as involving a momentary payment of 1 dt made at time t, discounted at interest back to time zero by multiplying by v^t and further multiplied by ${}_tp_x$ to reflect the probability that a payment is made at time t. This is the current payment form of the actuarial present value for the whole life annuity. In general, the current payment technique for determining an actuarial present value for an annuity gives

$$\mathrm{APV} = \int_0^\infty v^t\, \Pr[\text{payments are being made at time } t] \times [\text{Payment rate at time } t]\, dt. \tag{5.2.5}$$

Let us rewrite (5.2.4) by splitting off that portion of the integral involving t values from 0 to 1. Thus

$$\begin{aligned}\bar{a}_x &= \int_0^1 v^t\, {}_tp_x\, dt + \int_1^\infty v^t\, {}_tp_x\, dt \\ &= \bar{a}_{x:\overline{1}|} + v\, p_x \int_0^\infty v^s\, {}_sp_{x+1}\, ds \\ &= \bar{a}_{x:\overline{1}|} + v\, p_x\, \bar{a}_{x+1}.\end{aligned} \tag{5.2.6}$$

The actuarial present-value symbol, $\bar{a}_{x:\overline{1}|}$, used above is introduced below in (5.2.11). Expression (5.2.6) is an example of the backward recursion formula first observed in Section 3.5 and explored more fully in Section 4.3. Here $u(x) = \bar{a}_x$, $c(x) = \bar{a}_{x:\overline{1}|}$, and $d(x) = v\, p_x$. The initial value to use for the whole life annuity is $\bar{a}_\omega = 0$. There are several ways to evaluate the $c(x)$ term. A simple approach is to use a trapezoid approximation for the integral

$$\bar{a}_{x:\overline{1}|} = \int_0^1 v^t\, {}_tp_x\, dt = \frac{1 + v\, p_x}{2}.$$

Another approach, based on the assumption of a uniform distribution of deaths within each year of age, is examined in Section 5.4.

A relationship, familiar from compound interest theory, is that

$$1 = \delta\, \bar{a}_{\overline{t}|} + v^t.$$

This can be interpreted as indicating that a unit invested now will produce annual interest of δ payable continuously for t years at which point interest ceases and the investment is repaid. This relationship holds for all values of t and thus is true for the random variable T:

$$1 = \delta\, \bar{a}_{\overline{T}|} + v^T. \tag{5.2.7}$$

Then, taking expectations, we obtain

$$1 = \delta\, \bar{a}_x + \bar{A}_x. \tag{5.2.8}$$

This is subject to the same kind of interpretation as above. A unit invested now will produce annual interest of δ payable continuously for as long as (x) survives, and, at the time of death, interest ceases and the investment of 1 is repaid.

To measure, on the basis of the assumptions in our model, the mortality risk in a continuous life annuity, we are interested in $\mathrm{Var}(\bar{a}_{\overline{T}|})$. We determine

$$\begin{aligned}\mathrm{Var}(\bar{a}_{\overline{T}|}) &= \mathrm{Var}\left(\frac{1 - v^T}{\delta}\right) \\ &= \frac{\mathrm{Var}(v^T)}{\delta^2} \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}.\end{aligned} \tag{5.2.9}$$

Further, we can observe that since $1 = \delta\, \bar{a}_{\overline{T}|} + v^T$, $\mathrm{Var}(\delta\, \bar{a}_{\overline{T}|} + v^T) = 0$. Thus there is no mortality risk for the combination of a continuous life annuity of δ per year and a life insurance of 1 payable at the moment of death.

Example 5.2.1

Under the assumptions of a constant force of mortality, μ, and of a constant force of interest, δ, evaluate

a. $\bar{a}_x = \mathrm{E}[\bar{a}_{\overline{T}|}]$
b. $\mathrm{Var}(\bar{a}_{\overline{T}|})$
c. The probability that $\bar{a}_{\overline{T}|}$ will exceed $\bar{a}_x$.

Solution:

a. $\bar{a}_x = \int_0^\infty v^t\, {}_tp_x\, dt = \int_0^\infty e^{-\delta t} e^{-\mu t}\, dt = \dfrac{1}{\delta + \mu}.$

b. $\bar{A}_x = 1 - \delta\,\bar{a}_x = \dfrac{\mu}{\delta + \mu}$

$${}^2\bar{A}_x = \frac{\mu}{2\delta + \mu} \quad \text{by the rule of moments}$$

$$\operatorname{Var}(\bar{a}_{\overline{T}|}) = \frac{1}{\delta^2}\left[\frac{\mu}{2\delta + \mu} - \left(\frac{\mu}{\delta + \mu}\right)^2\right] = \frac{\mu}{(2\delta + \mu)(\delta + \mu)^2}.$$

c. $$\Pr(\bar{a}_{\overline{T}|} > \bar{a}_x) = \Pr\left(\frac{1 - v^T}{\delta} > \bar{a}_x\right) = \Pr\left[T > -\frac{1}{\delta}\log\left(\frac{\mu}{\delta + \mu}\right)\right]$$

$$= {}_{t_0}p_x \quad \text{where } t_0 = -\frac{1}{\delta}\log\left(\frac{\mu}{\delta + \mu}\right)$$

$$= \left(\frac{\mu}{\delta + \mu}\right)^{\mu/\delta}.$$

▼

We now turn to temporary and deferred life annuities. The present value of a benefits random variable for an ***n-year temporary life annuity*** of 1 per year, payable continuously while (x) survives during the next n years, is

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & 0 \le T < n \\ \bar{a}_{\overline{n}|} & T \ge n. \end{cases} \tag{5.2.10}$$

The distribution of Y in this case is a mixed distribution. In particular, the maximum value of Y is limited to $\bar{a}_{\overline{n}|}$, and there is a positive probability associated with $\bar{a}_{\overline{n}|}$ of $\Pr(T \ge n) = {}_np_x$. A typical distribution function for this random variable is illustrated in Figure 5.2.1(b).

The actuarial present value of an n-year temporary life annuity is denoted by $\bar{a}_{x:\overline{n}|}$ and equals

$$\bar{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \int_0^n \bar{a}_{\overline{t}|}\,{}_tp_x\,\mu(x + t)\,dt + \bar{a}_{\overline{n}|}\,{}_np_x. \tag{5.2.11}$$

Integration by parts gives

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t\,{}_tp_x\,dt. \tag{5.2.12}$$

This is the current payment integral for the actuarial present value for the n-year temporary annuity. It can be considered as involving a momentary payment 1 dt made at time t, discounted at interest back to time 0 by multiplying by v^t and further multiplied by ${}_tp_x$ to reflect the probability that a payment is made at time t for times up to time n. No payments are to be made after time n so the probability of such payments is 0.

The same recursion formula as indicated for (5.2.6) applies here with $u(x) = \bar{a}_{x:\overline{y-x}|}$ and the same $c(x)$ function which we now recognize as $\bar{a}_{x:\overline{1}|}$. We use here $n = y - x$. The only thing that needs to be changed is the initial value, for which we use $u(y) = \bar{a}_{y:\overline{0}|} = 0$. Another form of a recursion formula for a temporary annuity with the n-year period fixed is examined in Exercise 5.7.

Returning to (5.2.10) we note that

$$Y = \begin{cases} \bar{a}_{\overline{T}|} = \dfrac{1 - Z}{\delta} & 0 \le T < n \\ \bar{a}_{\overline{n}|} = \dfrac{1 - Z}{\delta} & T \ge n \end{cases} \tag{5.2.13}$$

where

$$Z = \begin{cases} v^T & 0 \le T < n \\ v^n & T \ge n. \end{cases} \tag{5.2.14}$$

In (5.2.14), Z is the present-value random variable for an n-year endowment insurance. Hence

$$\mathrm{E}[Y] = \bar{a}_{x:\overline{n}|} = \mathrm{E}\left[\frac{1 - Z}{\delta}\right] = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta} \tag{5.2.15}$$

and

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)}{\delta^2} = \frac{{}^2\bar{A}_{x:\overline{n}|} - \bar{A}_{x:\overline{n}|}^2}{\delta^2}. \tag{5.2.16}$$

In terms of annuity values, (5.2.16) becomes

$$\mathrm{Var}(Y) = \frac{1 - 2\delta\, {}^2\bar{a}_{x:\overline{n}|} - (1 - \delta\, \bar{a}_{x:\overline{n}|})^2}{\delta^2}$$

$$= \frac{2}{\delta}\,(\bar{a}_{x:\overline{n}|} - {}^2\bar{a}_{x:\overline{n}|}) - (\bar{a}_{x:\overline{n}|})^2.$$

The analysis for an ***n-year deferred whole life annuity*** is similar. The present-value random variable Y is defined as

$$Y = \begin{cases} 0 & = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{T}|} & 0 \le T < n \\ v^n\, \bar{a}_{\overline{T-n}|} & = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & T \ge n. \end{cases} \tag{5.2.17}$$

Here the random variable Y can take on a value no larger than $(1/\delta) - \bar{a}_{\overline{n}|} = v^n/\delta$, and the probability that it takes on a zero value is $\Pr(T \le n) = {}_nq_x$. A typical distribution function is illustrated in Figure 5.2.1(c).

Then,

$${}_{n|}\bar{a}_x = \mathrm{E}[Y] = \int_n^\infty v^n\, \bar{a}_{\overline{t-n}|}\, {}_tp_x\, \mu(x + t)\, dt$$

$$= \int_0^\infty v^n\, \bar{a}_{\overline{s}|}\, {}_{n+s}p_x\, \mu(x + n + s)\, ds$$

$$= v^n\, {}_np_x \int_0^\infty \bar{a}_{\overline{s}|}\, {}_sp_{x+n}\, \mu(x + n + s)\, ds$$

which shows that

$${}_{n|}\bar{a}_x = {}_nE_x\, \bar{a}_{x+n}. \tag{5.2.18}$$

An alternative development would be to note that, from the definitions of Y,

$$\begin{aligned}(Y \text{ for an } n\text{-year deferred} &= (Y \text{ for a whole life annuity}) \\ \text{whole life annuity}) &\\ &\qquad - (Y \text{ for an } n\text{-year temporary} \\ &\qquad\quad \text{life annuity}).\end{aligned}$$

Taking expectations gives

$$ {}_{n|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n|}}. \tag{5.2.19}$$

Integration by parts can be employed to verify the result given by the current payment technique. Since the annuity will be paying after time n if x survives, the actuarial present value can be written as

$$ {}_{n|}\bar{a}_x = \int_n^\infty v^t \, {}_tp_x \, dt = \int_n^\infty {}_tE_x \, dt. \tag{5.2.20}$$

To develop the backward recursion formula for deferred annuities with $n = y - x > 1$, we note that we have no term corresponding to the integral for t values between 0 and 1. Thus, for $u(x) = {}_{y-x|}\bar{a}_x$ at ages less than y, $c(x) = 0$, and $d(x) = v\,p_x$. For a starting value we would use $u(y) = \bar{a}_y$.

One way to calculate the variance of Y for the deferred annuity is the following:

$$\begin{aligned}\mathrm{Var}(Y) &= \int_n^\infty v^{2n} (\bar{a}_{\overline{t-n|}})^2 \, {}_tp_x \, \mu(x+t) \, dt - ({}_{n|}\bar{a}_x)^2 \\ &= v^{2n} \, {}_np_x \int_0^\infty (\bar{a}_{\overline{s|}})^2 \, {}_sp_{x+n} \, \mu(x+n+s) \, ds - ({}_{n|}\bar{a}_x)^2,\end{aligned}$$

and using integration by parts,

$$\begin{aligned} &= v^{2n} \, {}_np_x \int_0^\infty 2\bar{a}_{\overline{s|}} \, v^s \, {}_sp_{x+n} \, ds - ({}_{n|}\bar{a}_x)^2 \\ &= \frac{2}{\delta} v^{2n} \, {}_np_x \int_0^\infty (v^s - v^{2s}) \, {}_sp_{x+n} \, ds - ({}_{n|}\bar{a}_x)^2 \\ &= \frac{2}{\delta} v^{2n} \, {}_np_x \, (\bar{a}_{x-n} - {}^2\bar{a}_{x+n}) - ({}_{n|}\bar{a}_x)^2. \end{aligned} \tag{5.2.21}$$

For an alternative development of this formula, see Exercise 5.37.

We now turn to analysis of an ***n*-year certain and life annuity.** This is a whole life annuity with a guarantee of payments for the first n years. The present value of annuity payments is

$$ Y = \begin{cases} \bar{a}_{\overline{n|}} & T \le n \\ \bar{a}_{\overline{T|}} & T > n. \end{cases} \tag{5.2.22}$$

A typical distribution function is shown in Figure 5.2.1(d), which reflects the mixed nature of the distribution and the minimum value and upper bound of Y, which are $\bar{a}_{\overline{n|}}$ and $1/\delta$, respectively.

The actuarial present value is denoted by $\bar{a}_{\overline{\overline{x:\overline{n}|}}}$. This symbol is adopted to indicate that payments continue until max$[T(x), n]$:

$$\bar{a}_{\overline{\overline{x:\overline{n}|}}} = \mathrm{E}[Y] = \int_0^n \bar{a}_{\overline{n}|}\, {}_tp_x\, \mu(x+t)\, dt$$

$$+ \int_n^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt$$

$$= {}_nq_x\, \bar{a}_{\overline{n}|} + \int_n^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt. \tag{5.2.23}$$

Integration by parts can be used to obtain

$$\bar{a}_{\overline{\overline{x:\overline{n}|}}} = \bar{a}_{\overline{n}|} + \int_n^\infty v^t\, {}_tp_x\, dt. \tag{5.2.24}$$

This is the current payment form for the actuarial present value, since at times 0 to n payment is certain, whereas for times greater than n payment is made if (x) survives.

Further insight can be obtained by rewriting Y as

$$Y = \begin{cases} \bar{a}_{\overline{n}|} + 0 & T \le n \\ \bar{a}_{\overline{n}|} + (\bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|}) & T > n. \end{cases}$$

Here Y is the sum of a constant $\bar{a}_{\overline{n}|}$ and the random variable for the n-year deferred annuity. Thus,

$$\begin{aligned} \bar{a}_{\overline{\overline{x:\overline{n}|}}} &= \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x \\ &= \bar{a}_{\overline{n}|} + {}_nE_x\, \bar{a}_{x+n} && \text{by (5.2.18)} \\ &= \bar{a}_{\overline{n}|} + (\bar{a}_x - \bar{a}_{x:\overline{n}|}) && \text{by (5.2.19).} \end{aligned} \tag{5.2.25}$$

Furthermore, since $\mathrm{Var}(Y - \bar{a}_{\overline{n}|}) = \mathrm{Var}(Y)$, the variance for the n-year certain and life annuity is the same as that of the n-year deferred annuity given by (5.2.21).

A backward recursion for $\bar{a}_{\overline{\overline{x:\overline{n}|}}}$ with a fixed n-year certain period is examined in Exercise 5.9.

Analogous to the function

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^{n-t}\, dt$$

in the theory of interest, we have for life annuities

$$\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_nE_x} = \int_0^n \frac{1}{{}_{n-t}E_{x+t}}\, dt, \tag{5.2.26}$$

representing the actuarial accumulated value at the end of the term of an n-year temporary life annuity of 1 per year payable continuously while (x) survives.

Such accumulated value, which is often said to have been accumulated under (or with the benefit of) interest and survivorship, is available at age $x + n$ if (x) survives.

We obtain an expression for $d\bar{a}_x/dx$ by differentiating the integral in (5.2.4), assuming that the probabilities are derived from an aggregate table:

$$\frac{d}{dx}\bar{a}_x = \int_0^\infty v^t \left(\frac{\partial}{\partial x}\,{}_tp_x\right) dt = \int_0^\infty v^t\,{}_tp_x\,[\mu(x) - \mu(x+t)]\,dt$$

$$= \mu(x)\,\bar{a}_x - \bar{A}_x = \mu(x)\,\bar{a}_x - (1 - \delta\,\bar{a}_x).$$

Therefore,

$$\frac{d}{dx}\bar{a}_x = [\mu(x) + \delta]\,\bar{a}_x - 1. \tag{5.2.27}$$

The interpretation of (5.2.27) is that the actuarial present value changes at a rate that is the sum of the rate of interest income $\delta\,\bar{a}_x$ and the rate of survivorship benefit $\mu(x)\,\bar{a}_x$, less the rate of payment outgo.

Example 5.2.2

Assuming that probabilities come from an aggregate table, obtain formulas for

a. $\dfrac{\partial}{\partial x}\bar{a}_{x:\overline{n}|}$ b. $\dfrac{\partial}{\partial n}\,{}_{n|}\bar{a}_x$.

Solution:

a. Proceeding as in the development of (5.2.27), we obtain

$$\frac{\partial}{\partial x}\bar{a}_{x:\overline{n}|} = \mu(x)\,\bar{a}_{x:\overline{n}|} - \bar{A}^{\,1}_{x:\overline{n}|}$$

$$= \mu(x)\,\bar{a}_{x:\overline{n}|} - (1 - \delta\,\bar{a}_{x:\overline{n}|} - {}_nE_x)$$

$$= [\mu(x) + \delta]\,\bar{a}_{x:\overline{n}|} - (1 - {}_nE_x).$$

b. $\dfrac{\partial}{\partial n}\,{}_{n|}\bar{a}_x = \dfrac{\partial}{\partial n}\displaystyle\int_n^\infty v^t\,{}_tp_x\,dt = -v^n\,{}_np_x.$ ▼

Table 5.2.1 summarizes concepts for continuous life annuities.

Figure 5.2.1 shows typical distribution functions for the several types of continuous life annuities studied in this section. Limiting values and points of discontinuities are indicated on one or both axes.

When $F_Y(0) = 0$, $E[Y] = \int_0^\infty [1 - F_Y(y)]\,dy$, the actuarial present value of Y can be visualized as the area above the graph of $z = F_Y(y)$, below $z = 1$, and to the right of the line $y = 0$. This interpretation can provide a bridge between the actuarial

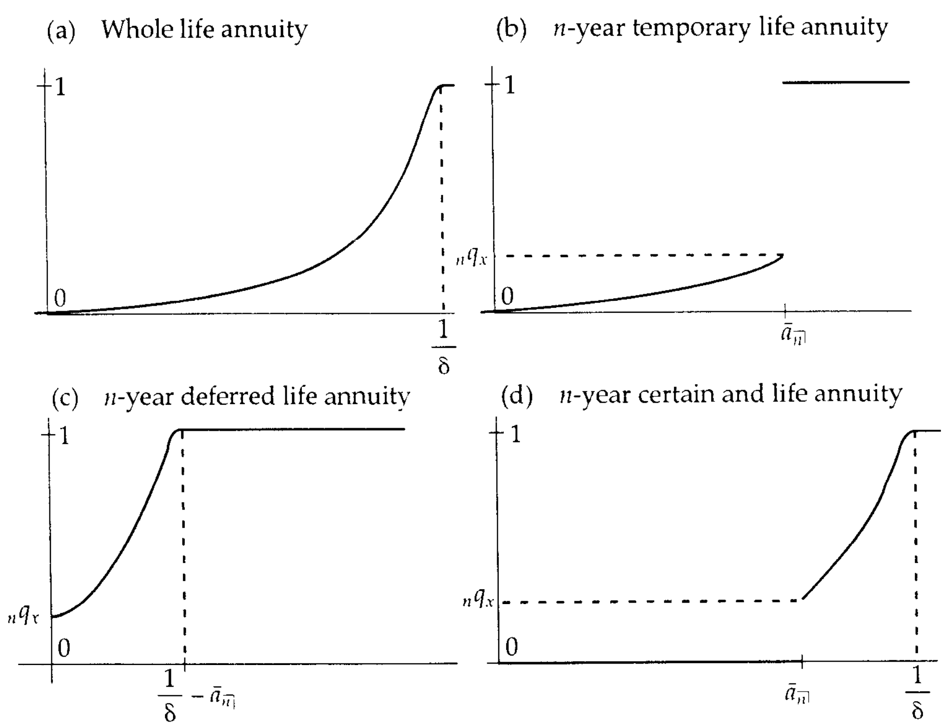
Summary of Continuous Life Annuities (Annuity of 1 per Annum Payable Continuously)

Annuity Name	Present-Value Random Variable Y	Actuarial Present Value E[Y] Equal to
Whole Life Annuity	$\bar{a}_{\overline{T}\rvert} \quad T \ge 0$	$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$
n-Year Temporary Life Annuity	$\begin{cases} \bar{a}_{\overline{T}\rvert} & 0 \le T < n \\ \bar{a}_{\overline{n}\rvert} & T \ge n \end{cases}$	$\bar{a}_{x:\overline{n}\rvert} = \int_0^n v^t \, {}_tp_x \, dt$
n-Year Deferred Whole Life Annuity	$\begin{cases} 0 & 0 \le T < n \\ \bar{a}_{\overline{T}\rvert} - \bar{a}_{\overline{n}\rvert} & T \ge n \end{cases}$	${}_{n\rvert}\bar{a}_x = \int_n^\infty v^t \, {}_tp_x \, dt$
n-Year Certain and Life Annuity	$\begin{cases} \bar{a}_{\overline{n}\rvert} & 0 \le T < n \\ \bar{a}_{\overline{T}\rvert} & T \ge n \end{cases}$	$\bar{a}_{\overline{x:\overline{n}\rvert}} = \bar{a}_{\overline{n}\rvert} + \int_n^\infty v^t \, {}_tp_x \, dt$

Additional relations are

- $1 = \delta \, \bar{a}_x + \bar{A}_x$
- $1 = \delta \, \bar{a}_{x:\overline{n}\rvert} + \bar{A}_{x:\overline{n}\rvert}$
- ${}_{n\rvert}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}\rvert}$
- $\bar{s}_{x:\overline{n}\rvert} = \dfrac{\bar{a}_{x:\overline{n}\rvert}}{{}_nE_x} = \int_0^n (1+i)^{n-t} \dfrac{l_{x+t}}{l_{x+n}} \, dt.$

Typical Distribution Functions for the Present-Value Random Variables, Y



present value as evaluated from the definition of the random variable and the current payment form for the actuarial present value.

5.3 Discrete Life Annuities

The theory of discrete life annuities is analogous, step-by-step, to the theory of continuous life annuities, with integrals replaced by sums, integrands by summands, and differentials by differences. For continuous annuities there was no distinction between payments at the beginning of payment intervals or at the ends, that is, between annuities-due and annuities-immediate. For discrete annuities, the distinction is meaningful, and we start with annuities-due as they have a more prominent role in actuarial applications. For example, most individual life insurances are purchased by an annuity-due of periodic premiums.

We consider an annuity that pays a unit amount at the beginning of each year that the annuitant (x) survives. In the nomenclature this is called a ***whole life annuity-due.*** The present-value random variable, Y, for such an annuity, is given by $Y = \ddot{a}_{\overline{K+1}|}$ where the random variable K is the curtate-future-lifetime of (x). The possible values of this random variable are a discrete set of values ranging from $\ddot{a}_{\overline{1}|} = 1$ to $\ddot{a}_{\overline{\omega-x}|}$, a value which is less than $1/d$. The probability associated with the value $\ddot{a}_{\overline{k+1}|}$ is $\Pr(K = k) = {}_kp_x\, q_{x+k}$.

Let us now consider $\ddot{a}_x$, the actuarial present value of the annuity:

$$\ddot{a}_x = \mathrm{E}[Y] = \mathrm{E}[\ddot{a}_{\overline{K+1}|}]$$

$$= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\ {}_kp_x\, q_{x+k}, \tag{5.3.1}$$

since $\Pr(K = k) = {}_kp_x\, q_{x+k}$. By summation-by-parts (see Appendix 5) with $\Delta f(k) = {}_kp_x\, q_{x+k} = {}_kp_x - {}_{k+1}p_x$ and $g(k) = \ddot{a}_{\overline{k+1}|}$ and use of the relations

$$\Delta\, g(k) = \Delta\, \ddot{a}_{\overline{k+1}|} = v^{k+1} \text{ and } f(k) = -{}_kp_x,$$

(5.3.1) converts to

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} v^{k+1}\ {}_{k+1}p_x \tag{5.3.2}$$

$$= \sum_{k=0}^{\infty} v^k\ {}_kp_x\, . \tag{5.3.3}$$

The expression (5.3.3) is the current payment form of the actuarial present value for a whole life annuity-due where the ${}_kp_x$ term is the probability of a payment of size 1 being made at time k.

Starting with the sum in (5.3.2) above we have

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} v^{k+1}\ {}_{k+1}p_x = 1 + v\ p_x \sum_{k=0}^{\infty} v^k\ {}_kp_{x+1}$$

$$= 1 + v\ p_x\ \ddot{a}_{x+1}. \tag{5.3.4}$$

This expression is an example of the backward recursion formula first observed in Section 3.5 and explored more fully in Sections 4.3 and 5.2. Here $u(x) = \ddot{a}_x$, $c(x) = 1$, and $d(x) = v\ p_x$. The initial value to use for the whole life annuity is $\ddot{a}_\omega = 0$.

From (5.3.1) we obtain in succession

$$\ddot{a}_x = \mathrm{E}\left[\frac{1 - v^{K+1}}{d}\right]$$

$$= \frac{1 - A_x}{d}, \tag{5.3.5}$$

and

$$\ddot{a}_x = \ddot{a}_{\overline{\infty}|} - \ddot{a}_{\overline{\infty}|}\ A_x, \tag{5.3.6}$$

$$1 = d\ddot{a}_x + A_x. \tag{5.3.7}$$

This should be compared with its continuous counterpart (5.2.8). Formula (5.3.7) indicates that a unit invested now will produce interest-in-advance of d per year while (x) survives plus repayment of the unit at the end of the year of death of (x).

The variance formula is

$$\mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \mathrm{Var}\left(\frac{1 - v^{K+1}}{d}\right) = \frac{\mathrm{Var}(v^{K+1})}{d^2}$$

$$= \frac{{}^2A_x - (A_x)^2}{d^2}; \tag{5.3.8}$$

see (5.2.9).

The present-value random variable of an **n-*year temporary life annuity-due*** of 1 per year is

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & 0 \le K < n \\ \ddot{a}_{\overline{n}|} & K \ge n, \end{cases}$$

and its actuarial present value is

$$\ddot{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_kp_x\ q_{x+k} + \ddot{a}_{\overline{n}|}\ {}_np_x. \tag{5.3.9}$$

Summation by parts can be used to transform (5.3.9) into

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k\ {}_kp_x, \tag{5.3.10}$$

which is the actuarial present value in the current payment form.

Again separating out the first term and factoring out vp_x, we obtain the backward recursion formula for a temporary annuity-due payable to age $y = x + n$:

$$\ddot{a}_{x:\overline{y-x}|} = 1 + vp_x\, \ddot{a}_{x+1:\overline{y-(x+1)}|}. \tag{5.3.11}$$

This recursive formula for the actuarial present values is the same as (5.3.4) but differs in that here we use an initial value of $\ddot{a}_{y:\overline{0}|} = 0$.

Since $Y = (1 - Z)/d$, where

$$Z = \begin{cases} v^{K+1} & 0 \le K < n \\ v^n & K \ge n \end{cases}$$

is the present-value random variable for a unit of endowment insurance, payable at the end of the year of death or at maturity, we have

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - \mathrm{E}[Z]}{d} = \frac{1 - A_{x:\overline{n}|}}{d}; \tag{5.3.12}$$

see (5.2.15).

Rearrangement of (5.3.12) yields

$$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}. \tag{5.3.13}$$

To calculate the variance, we can use

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)}{d^2} = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}. \tag{5.3.14}$$

For an ***n-year deferred whole life annuity-due*** of 1 payable at the beginning of each year while (x) survives from age $x + n$ onward, the present-value random variable is

$$Y = \begin{cases} 0 & 0 \le K < n \\ {}_{n|}\ddot{a}_{\overline{K+1-n}|} & K \ge n, \end{cases}$$

and its actuarial present value is

$$\mathrm{E}[Y] = {}_{n|}\ddot{a}_x = {}_nE_x\, \ddot{a}_{x+n} \tag{5.3.15}$$

$$= \ddot{a}_x - \ddot{a}_{x:\overline{n}|} \tag{5.3.16}$$

$$= \sum_{k=n}^{\infty} v^k\, {}_kp_x; \tag{5.3.17}$$

see (5.2.18)–(5.2.20).

The backward recursion formula for a deferred annuity-due with $n = y - x > 1$ is identical to that for the continuous version in that it uses $c(x) = 0$ and $d(x) = vp_x$. The change is that we use the actuarial present value for an annuity-due, $u(y) = \ddot{a}_y$, for the starting value.

The variance of Y can be developed in a manner completely analogous to that used in formula (5.2.21) and leads to the expression

$$\text{Var}(Y) = \frac{2}{d} v^{2n} {}_n p_x (\ddot{a}_{x+n} - {}^2\ddot{a}_{x+n}) + {}_{n|}{}^2\ddot{a}_x - ({}_{n|}\ddot{a}_x)^2. \tag{5.3.18}$$

We turn now to analysis of an **n-*year certain and life annuity-due.*** This is a life annuity with a guarantee of payments for at least n years. The present value of the annuity payments is

$$Y = \begin{cases} \ddot{a}_{\overline{n}|} & 0 \le K < n \\ \ddot{a}_{\overline{K+1}|} & K \ge n. \end{cases} \tag{5.3.19}$$

Then

$$\ddot{a}_{\overline{x:\overline{n}|}} = \text{E}[Y] = \ddot{a}_{\overline{n}|}\, {}_n q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_k p_x\, q_{x+k}, \tag{5.3.20}$$

and this can be transformed by summation by parts into the current payment version of the actuarial present value

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k\, {}_k p_x. \tag{5.3.21}$$

This can be written as

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}.$$

The actuarial accumulated value at the end of the term of an n-year temporary life annuity-due of 1 per year, payable while (x) survives, is denoted by $\ddot{s}_{x:\overline{n}|}$. Formulas for this function are

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_n E_x} = \sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}}, \tag{5.3.22}$$

which are analogous to formulas for $\ddot{s}_{\overline{n}|}$ in the theory of interest.

The procedures used above for annuities-due can be adapted for annuities-immediate where payments are made at the ends of the payment periods. For instance, for a ***whole life annuity-immediate,*** the present-value random variable is $Y = a_{\overline{K}|}$. Then,

$$a_x = \text{E}[Y] = \sum_{k=0}^{\infty} {}_k p_x\, q_{x+k}\, a_{\overline{k}|}, \tag{5.3.23}$$

and a summation by parts will give the current payment form of the actuarial present value as

$$a_x = \sum_{k=1}^{\infty} v^k\, {}_k p_x. \tag{5.3.24}$$

Since Y equals $(1 - v^K)/i = [1 - (1+i)\, v^{K+1}]/i$, we have, taking expectations,

$$a_x = \text{E}[Y] = \frac{1 - (1+i)\, A_x}{i}. \tag{5.3.25}$$

This formula can be rewritten as $1 = i a_x + (1+i) A_x$. A comparison of this formula with (5.3.7) shows that an interest payment of i is made at the end of each year while (x) remains alive and that at the end of the year of death an interest payment of i along with the principal amount of 1 must be paid. This formula has significance for estate taxation. For each unit of an estate, define $i a_x$ as the ***life estate*** and $(1 + i) A_x = 1 - i a_x$ as the ***remainder,*** which, if designated for a qualified charitable organization, is exempt from estate taxation.

The analysis for the other forms of the annuity-immediate is similar. The present-value random variable can be formed in a manner analogous to that for the annuity-due. Formulas for the actuarial present value from the definition and by summation by parts can be obtained. Formulas for the variances of the annuities-immediate in this section can also be obtained.

Example 5.3.1

Find formulas for the expectation and variance of the present-value random variable for the temporary life annuity-immediate.

Solution:

We start with the present-value random variable for an n-year temporary annuity-immediate:

$$Y = \begin{cases} a_{\overline{K}|} = \dfrac{1 - v^K}{i} & 0 \le K < n \\ a_{\overline{n}|} = \dfrac{1 - v^n}{i} & K \ge n. \end{cases}$$

We now introduce two new random variables

$$Z_1 = \begin{cases} (1 + i) v^{K+1} & 0 \le K < n \\ 0 & K \ge n \end{cases}$$

and

$$Z_2 = \begin{cases} 0 & 0 \le K < n \\ v^n & K \ge n \end{cases}$$

so that $Y = (1 - Z_1 - Z_2)/i$ for all K. Now taking expectations, we have

$$\mathrm{E}[Y] = a_{x:\overline{n}|} = \frac{1 - (1 + i)\, A^{1}_{x:\overline{n}|} - A_{x:\overline{n}|}^{\;\;1}}{i}.$$

This can be rewritten, following (5.3.13),

$$1 = i\, a_{x:\overline{n}|} + i\, A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}.$$

The variance calculation is as follows:

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z_1 + Z_2)}{i^2} = \frac{\mathrm{Var}(Z_1) + 2\,\mathrm{Cov}(Z_1, Z_2) + \mathrm{Var}(Z_2)}{i^2}.$$

Recall $\mathrm{Var}(Z_1) = (1+i)^2\,[{}^2A^{\;1}_{x:\overline{n|}} - (A^{\;1}_{x:\overline{n|}})^2]$ and $\mathrm{Var}(Z_2) = v^{2n}\,{}_np_x\,(1 - {}_np_x)$. Since $Z_1 Z_2 = 0$ for all K, we have $\mathrm{Cov}(Z_1, Z_2) = -(1+i)\,A^{\;1}_{x:\overline{n|}}\,v^n\,{}_np_x$. Combining, we obtain

$$\mathrm{Var}(Y) = \frac{(1+i)^2\,[{}^2A^{\;1}_{x:\overline{n|}} - (A^{\;1}_{x:\overline{n|}})^2] - 2(1+i)\,A^{\;1}_{x:\overline{n|}}\,v^n\,{}_np_x + v^{2n}\,{}_np_x\,(1 - {}_np_x)}{i^2}.$$

▼

Summary of Discrete Life Annuities [Annuity of 1 Per Annum Payable at the Beginning of Each Year (Annuity-Due) or at the End of Each Year (Annuity-Immediate)]

Annuity Name	Present-Value Random Variable Y	Actuarial Present Value E[Y] Equal to
Whole Life Annuity		
• Due	$\ddot{a}_{\overline{K+1\vert}} \quad K \ge 0$	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x$
• Immediate	$a_{\overline{K\vert}} \quad K \ge 0$	$a_x = \sum_{k=1}^{\infty} v^k\,{}_kp_x$
n-Year Temporary Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{K+1\vert}} & 0 \le K < n \\ \ddot{a}_{\overline{n\vert}} & K \ge n \end{cases}$	$\ddot{a}_{x:\overline{n\vert}} = \sum_{k=0}^{n-1} v^k\,{}_kp_x$
• Immediate	$\begin{cases} a_{\overline{K\vert}} & 0 \le K < n \\ a_{\overline{n\vert}} & K \ge n \end{cases}$	$a_{x:\overline{n\vert}} = \sum_{k=1}^{n} v^k\,{}_kp_x$
n-Year Deferred Whole Life Annuity		
• Due	$\begin{cases} 0 & 0 \le K < n \\ \ddot{a}_{\overline{K+1\vert}} - \ddot{a}_{\overline{n\vert}} & K \ge n \end{cases}$	${}_{n\vert}\ddot{a}_x = \sum_{k=n}^{\infty} v^k\,{}_kp_x$
• Immediate	$\begin{cases} 0 & 0 \le K < n \\ a_{\overline{K\vert}} - a_{\overline{n\vert}} & K \ge n \end{cases}$	${}_{n\vert}a_x = \sum_{k=n+1}^{\infty} v^k\,{}_kp_x$
n-Year Certain and Whole Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{n\vert}} & 0 \le K < n \\ \ddot{a}_{\overline{K+1\vert}} & K \ge n \end{cases}$	$\ddot{a}_{\overline{x:\overline{n\vert}}} = \ddot{a}_{\overline{n\vert}} + \sum_{k=n}^{\infty} v^k\,{}_kp_x$
• Immediate	$\begin{cases} a_{\overline{n\vert}} & 0 \le K < n \\ a_{\overline{K\vert}} & K \ge n \end{cases}$	$a_{\overline{x:\overline{n\vert}}} = a_{\overline{n\vert}} + \sum_{k=n+1}^{\infty} v^k\,{}_kp_x$

Additional relations are

- $1 = d\,\ddot{a}_x + A_x$
- $A_x = v\,\ddot{a}_x - a_x$
- $1 = d\,\ddot{a}_{x:\overline{n|}} + A_{x:\overline{n|}}$
- $\ddot{a}_{x:\overline{n|}} = 1 + a_{x:\overline{n-1|}}$
- $A^{\;1}_{x:\overline{n|}} = v\,\ddot{a}_{x:\overline{n|}} - a_{x:\overline{n|}}$
- $A_{x:\overline{n|}} = v\,\ddot{a}_{x:\overline{n|}} - a_{x:\overline{n-1|}}$
- ${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n|}}$
- $\ddot{a}_{\overline{x:\overline{n|}}} = \ddot{a}_{\overline{n|}} + \ddot{a}_x - \ddot{a}_{x:\overline{n|}}$
- $\ddot{s}_{x:\overline{n|}} = \dfrac{\ddot{a}_{x:\overline{n|}}}{{}_nE_x} = \sum_{k=0}^{n-1} (1+i)^{n-k}\,\dfrac{l_{x+k}}{l_{x+n}}.$

5.4 Life Annuities with m-thly Payments

In practice, life annuities are often payable on a monthly, quarterly, or semi-annual basis. In International Actuarial Notation, the actuarial present value of a life annuity of 1 per year, payable in installments of 1 / m at the beginning of each m-th of the year while (x) survives, is denoted by $\ddot{a}_x^{(m)}$.

We start the analysis of the distribution of Y, the present value of the life annuity-due, with payments made on an m-thly basis, by expressing Y in terms of the interest rate and the random variables, K and $J = \lfloor (T - K)m \rfloor$. The "$\lfloor\ \rfloor$" in the expression for J denote the greatest integer function so that J is the number of complete m-ths of a year lived in the year of death. For an annuity-due there would be m payments for each of the K complete years and then $J + 1$ payments of 1 / m in the year of death; thus,

$$Y = \sum_{j=0}^{mK+J} \frac{1}{m} v^{j/m} = \ddot{a}^{(m)}_{\overline{K+(J+1)/m}|} = \frac{1 - v^{K+(J+1)/m}}{d^{(m)}}. \tag{5.4.1}$$

The actuarial present value, E[Y], which can be determined using Exercise 4.19, is

$$\mathrm{E}[Y] = \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}. \tag{5.4.2}$$

The current payment form, which is the sum of the actuarial present value of the set of payments, is

$$\ddot{a}_x^{(m)} = \frac{1}{m} \sum_{h=0}^{\infty} v^{h/m}\ {}_{h/m}p_x . \tag{5.4.3}$$

Again using (5.4.1), we obtain

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(v^{K+(J+1)/m})}{(d^{(m)})^2} = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}. \tag{5.4.4}$$

It is convenient to use (5.3.7) and (5.4.2) to obtain various relationships between the actuarial present values for m-thly annuities and those with annual payments:

$$1 = d\,\ddot{a}_x + A_x \doteq d^{(m)}\,\ddot{a}_x^{(m)} + A_x^{(m)}. \tag{5.4.5}$$

These show that an investment of 1 will produce interest-in-advance at the beginning of each interest period and repayment of the unit at the end of the period in which death occurs.

From the right two members of (5.4.5) we obtain

$$\begin{aligned} \ddot{a}_x^{(m)} &= \frac{d}{d^{(m)}}\,\ddot{a}_x - \frac{1}{d^{(m)}}\,(A_x^{(m)} - A_x) \\ &= \ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \ddot{a}_{\overline{\infty}|}^{(m)}\,(A_x^{(m)} - A_x). \end{aligned} \tag{5.4.6}$$

This can be interpreted as follows: The m-thly payment life annuity is equivalent to a series of 1-year annuities-certain in each year that (x) begins, with cancellation in the year of death of installments payable beyond the m-th (month, quarter, half-year) of death. The cancellation is accomplished by an m-thly payment perpetuity beginning at the end of the m-th of death less a similar perpetuity beginning at the end of the year of death. Alternatively, from (5.4.2), we can write

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \ddot{a}_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)} A_x^{(m)}, \tag{5.4.7}$$

which is left to the reader to interpret.

Remark:
In the study of interest theory the calculation of the present value of an annuity with payment periods and effective interest periods of different lengths was reduced to the calculation of the present value of an annuity with payment periods and interest periods of equal length in one of two ways. The first was to replace the payments corresponding to an interest period by a single equivalent payment (at the given interest rate) at one end or the other of the interest period. The expression for the m-thly whole life annuity in (5.4.6) is an extension of this method to a set of contingent payments. When calculators with exponentiation keys replaced interest tables, the second method, using the equivalent effective interest rate per payment period, became the preferred way to match payment period and interest period lengths. The extension of this second method to m-thly whole life annuities would be to use an m-thly mortality table along with the equivalent effective interest rate per m-th of a year. Using this, the recursion relations of Section 5.3 could be used to obtain the actuarial present value of the m-thly whole life annuity. An advantage of this second approach is that it frees us to use other assumptions about the distribution of deaths within each year of age, like constant force or Makeham's, in place of the uniform distribution that is central in the discussion below.

Now let us assume that deaths have a uniform distribution in each year of age. This means that S has a uniform distribution on (0, 1) so that J is uniformly distributed on the integers $\{0, 1, \ldots, m-1\}$. Exercise 4.19 shows that this implies

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x = s_{\overline{1}|}^{(m)} A_x,$$

and from (5.4.6) we have

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} \ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} A_x. \tag{5.4.8}$$

Formula (5.4.8) shows that the value of the m-thly life annuity-due is the difference between

a. The value of an annual life annuity-due with each annual payment sufficient to pay a 1-year annuity certain of $1/m$ at the beginning of each m-th; and

b. The value of an insurance payable at the end of the year of death with the benefit equal to the coefficient of A_x. It can be shown that, under the assumption of uniform distribution of deaths in each year of age, this coefficient is the actuarial accumulated value of those payments of $1/m$ for the m-ths after death. See the bracketed expression in Exercise 5.15.

By substituting $1 - d\,\ddot{a}_x$ for A_x, in (5.4.8) and noting that $d^{(m)}\,\ddot{a}_{\overline{1}|}^{(m)} = d$, we obtain a formula involving only annuity functions, namely,

$$\begin{aligned}\ddot{a}_x^{(m)} &= \frac{1 - s_{\overline{1}|}^{(m)}(1 - d\,\ddot{a}_x)}{d^{(m)}} \\ &= s_{\overline{1}|}^{(m)}\,\ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \frac{(s_{\overline{1}|}^{(m)} - 1)}{d^{(m)}}.\end{aligned} \tag{5.4.9}$$

An alternative, widely used, formula is

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}. \tag{5.4.10}$$

This result can be obtained by assuming that the function $v^{k+(j/m)}\,{}_{k+(j/m)}p_x$ is linear in j for $j = 0, 1, 2, \ldots, m-1$. In that case,

$$\begin{aligned}\sum_{j=0}^{m-1} \frac{1}{m}\, v^{k+(j/m)}\,{}_{k+(j/m)}p_x &= \sum_{j=0}^{m-1} \frac{1}{m}\left[\left(1 - \frac{j}{m}\right) v^k\,{}_kp_x + \frac{j}{m}\, v^{k+1}\,{}_{k+1}p_x\right] \\ &= v^k\,{}_kp_x - (v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x)\sum_{j=0}^{m-1}\frac{j}{m^2} \\ &= v^k\,{}_kp_x - \frac{m-1}{2m}(v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x).\end{aligned}$$

Thus

$$\begin{aligned}\ddot{a}_x^{(m)} &= \sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\frac{1}{m}\, v^{k+(j/m)}\,{}_{k+(j/m)}p_x \\ &= \sum_{k=0}^{\infty} v^k\,{}_kp_x - \frac{m-1}{2m}\sum_{k=0}^{\infty}(v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x) \\ &= \ddot{a}_x - \frac{m-1}{2m}.\end{aligned}$$

Note that this is not the same assumption as that of linearity of ${}_tp_x$, which would follow if a uniform distribution of deaths within each year of age is assumed. Consistent use of the assumption of a uniform distribution of deaths in each year of age assures that relations such as (5.4.5) are satisfied exactly. It has also been observed that formulas derived from (5.4.10) can, for high rates of interest and low rates of mortality, produce distorted annuity values, such as $a_{x:\overline{1}|}^{(m)} > a_{\overline{1}|}^{(m)}$; see Exercise 5.50. For these reasons, (5.4.8) and equivalently (5.4.9) are presented as replacements for the widely used formula (5.4.10).

It is convenient for writing purposes to express (5.4.9) in the form

$$\ddot{a}_x^{(m)} = \alpha(m)\,\ddot{a}_x - \beta(m), \tag{5.4.11}$$

where

$$\alpha(m) = s_{\overline{1}|}^{(m)} \ddot{a}_{\overline{1}|}^{(m)} = \frac{id}{i^{(m)}\, d^{(m)}}, \tag{5.4.12}$$

and

$$\beta(m) = \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)}\, d^{(m)}}. \tag{5.4.13}$$

We note that $\alpha(m)$ and $\beta(m)$ depend only on m and the rate of interest and are independent of the year of age. Further, for $m = 1$, (5.4.11) is an identity where $\alpha(1) = 1$ and $\beta(1) = 0$. Also, $\beta(m)$ is the coefficient of the cancellation term in (5.4.8); that is, (5.4.8) can be written as

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_x - \beta(m) A_x. \tag{5.4.14}$$

For series expansions of $\alpha(m)$ and $\beta(m)$, see Exercise 5.41.

Example 5.4.1

On the basis of the Illustrative Life Table, with interest at the effective annual rate of 6%, calculate the actuarial present value of a whole life annuity-due of 1,000 per month for a retiree age 65 by (5.4.9) and (5.4.10) and its standard deviation by (5.4.4).

Solution:

Here

$$\alpha(12) = s_{\overline{1}|}^{(12)}\, \ddot{a}_{\overline{1}|}^{(12)} = (1.02721070)(0.97378368) = 1.0002810,$$

$$\beta(12) = \frac{s_{\overline{1}|}^{(12)} - 1}{d^{(12)}} = 0.46811951,$$

$$\frac{11}{24} = 0.45833333.$$

Observe that $\alpha(12) \cong 1$, and $\beta(12)$ is fairly close to the $11/24$ that appears in the traditional approximation.

By the Illustrative Life Table, as defined by (3.7.1), with interest at 6%,

$$\ddot{a}_{65} = 9.89693,$$

$$A_{65} = 1 - d\, \ddot{a}_{65} = 0.4397965.$$

Then, $12{,}000\ddot{a}_{65}^{(12)}$ can be calculated as follows:

by (5.4.11)
$$12{,}000[\alpha(12)\ddot{a}_{65} - \beta(12)]$$
$$= 12{,}000[(1.0002810)(9.89693) - 0.46811951]$$
$$= 113{,}179 \text{ and}$$

by (5.4.10)
$$12{,}000\left(\ddot{a}_{65} - \frac{11}{24}\right) = 113{,}263.$$

The variance of $12{,}000\,Y = 12{,}000(1 - v^{K+(J+1)/12})/d^{(12)}$ is equal to

$$\left(\frac{12{,}000}{d^{(12)}}\right)^2 \operatorname{Var}[v^{K+1}(1+i)^{1-(J+1)/12}]$$
$$= \left(\frac{12{,}000}{d^{(12)}}\right)^2 \{\mathrm{E}[v^{2(K+1)}(1+i)^{2(1-(J+1)/12)}] - (\mathrm{E}[v^{(K+1)}(1+i)^{(1-(J+1)/12)}])^2\}$$
$$= \left(\frac{12{,}000}{d^{(12)}}\right)^2 \left\{ {}^2A_{65}\,\mathrm{E}[(1+i)^{2(1-(J+1)/12)}] - \left(\frac{A_{65}\,i}{i^{(12)}}\right)^2\right\}$$
$$= (206{,}442.14)^2\,[(0.2360299 \times 1.055458268) - (0.4397965 \times 1.02721069)^2]$$
$$= 1{,}919{,}074{,}762.$$

This means that the standard deviation for the present value for the payments to an individual is 43,807 as compared to the actuarial present value of 113,179. ▼

The development starting with random variables can be followed for m-thly temporary and deferred annuities-due. However, if all we seek is formulas for their actuarial present values, we can proceed by starting with (5.4.14). Thus

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_{x} - {}_nE_x\,\ddot{a}^{(m)}_{x+n}\\ &= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \beta(m)A_x - {}_nE_x[\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x+n} - \beta(m)A_{x+n}]\\ &= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x:\overline{n}|} - \beta(m)A^{\,1}_{x:\overline{n}|};\end{aligned} \tag{5.4.15}$$

and similarly,

$${}_{n|}\ddot{a}^{(m)}_x = \ddot{a}^{(m)}_{\overline{1}|}\,{}_{n|}\ddot{a}_x - \beta(m)\,{}_{n|}A_x. \tag{5.4.16}$$

Alternately from (5.4.11),

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x), \tag{5.4.17}$$
$${}_{n|}\ddot{a}^{(m)}_x = \alpha(m)\,{}_{n|}\ddot{a}_x - \beta(m)\,{}_nE_x. \tag{5.4.18}$$

Backward recursion formulas can be developed directly for m-thly life annuities, and the reader is asked to do this in Exercise 5.16 for an m-thly annuity-due. A more direct approach, however, would be to use the recursions of Section 5.3 and then adjust from annual to m-thly life annuities by means of (5.4.11), (5.4.17), and (5.4.18) or the equivalent formulas (5.4.14), (5.4.15), and (5.4.16).

The distribution of the present value of the payments of a life annuity-immediate with m-thly payments can be explored in steps analogous with those for the annuity-due. For example, the present-value random variable, Y, would be

$$Y = a^{(m)}_{\overline{K+(J/m)}|} = \frac{1 - v^{K+(J/m)}}{i^{(m)}} \tag{5.4.19}$$

for the whole life annuity-immediate with m-thly payments. This exploration leads to the following formula analogous to (5.4.5):

$$1 = i\,a_x + (1+i)\,A_x = i^{(m)}\,a^{(m)}_x + \left(1 + \frac{i^{(m)}}{m}\right)A^{(m)}_x. \tag{5.4.20}$$

The meaning here is that an investment of 1 will produce interest at the end of each interest period plus the repayment of the unit together with interest then due at the end of the interest period in which death occurs.

The actuarial present values for the annuities-immediate can also be obtained by adjusting the actuarial present values of the corresponding life annuities-due. For instance,

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m},$$

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}(1 - {}_nE_x). \tag{5.4.21}$$

5.5 Apportionable Annuities-Due and Complete Annuities-Immediate

With discrete annuities each payment is made either for the following period (an annuity-due) or for the preceding period (an annuity-immediate). A question may arise about having an adjustment for the payment period of death. For instance, suppose that a life insurance contract is purchased by annual payments payable at the beginning of each contract year. If the insured dies 1 month after making an annual payment, a refund of premium for the 11 months the insured did not complete in the policy year of death might seem appropriate. As another example, if a retirement income life annuity-immediate provides annual payments and the annuitant dies 1 month before the due date of the next payment, there might be a final payment for the 11-month period that the annuitant has survived since the last full payment. Consider first the appropriate size for the adjustment payment in such cases.

Let us examine the first case above. The insured dies at time T after paying a full yearly premium of 1 at time K. Assume that the premium is earned or accrued at a uniform rate over the year following the payment. In this case the rate of accrual, c, is given by $c\,\bar{a}_{\overline{1}|} = 1$. If accrual ceases at death, then $c\,\bar{s}_{\overline{T-K}|}$ has been earned to that date, while

$$(1+i)^{T-K} - c\,\bar{s}_{\overline{T-K}|} = 1 \times (1+i)^{T-K} - \frac{\bar{s}_{\overline{T-K}|}}{\bar{a}_{\overline{1}|}} = \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}$$

is unearned and is the amount to be refunded. The present-value random variable, at time 0, of all the payments less the refund is

$$\begin{aligned} Y &= \ddot{a}_{\overline{K+1}|} - v^T \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}} \\ &= \ddot{a}_{\overline{K+1}|} - \frac{v^T - v^{K+1}}{d} \\ &= \frac{1-v^T}{d} = \ddot{a}_{\overline{T}|}. \end{aligned} \tag{5.5.1}$$

When the annual rate of payments is 1, the actuarial present value at time 0 of the payments is denoted by $\ddot{a}_x^{\{1\}}$:

$$\ddot{a}_x^{\{1\}} = \mathrm{E}[\ddot{a}_{\overline{T}|}] = \mathrm{E}\left[\frac{\delta}{d}\,\bar{a}_{\overline{T}|}\right] = \frac{\delta}{d}\,\bar{a}_x. \tag{5.5.2}$$

This type of life annuity-due, one with a refund for the period between the time of death and the end of the period represented by the last full regular payment, is called an ***apportionable annuity-due.***

We can extend this idea to annuities that are paid more frequently than once a year. As in Section 5.4, we define $J = \lfloor (T - K)m \rfloor$ to be the number of m-ths of a year completed in the year of death, so $K + (J + 1)/m - T$ is the length of the period to be compensated for by the refund. The accrual rate is given by $c\,\bar{a}_{\overline{1/m}|} = 1/m$. Then, proceeding as above,

$$\begin{aligned} Y &= \ddot{a}_{\overline{K+(J+1)/m}|} - v^T\left(\frac{\bar{a}_{\overline{K+(J+1)/m-T}|}}{m\bar{a}_{\overline{1/m}|}}\right) \\ &= \ddot{a}_{\overline{K+(J+1)/m}|} - \frac{v^T - v^{K+(J+1)/m}}{d^{(m)}} \\ &= \frac{1 - v^T}{d^{(m)}} = \ddot{a}_{\overline{T}|}^{(m)}. \end{aligned} \tag{5.5.3}$$

When the annual rate of payments is 1, the actuarial present value of payments, less the refund, is $\ddot{a}_x^{\{m\}}$:

$$\ddot{a}_x^{\{m\}} = \mathrm{E}\left[\frac{1 - v^T}{d^{(m)}}\right] = \frac{\delta}{d^{(m)}}\,\bar{a}_x. \tag{5.5.4}$$

Alternatively, using the second line of (5.5.3),

$$\begin{aligned} \ddot{a}_x^{\{m\}} &= \mathrm{E}\left[\frac{1 - v^{K+(J+1)/m}}{d^{(m)}}\right] - \mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right] \\ &= \ddot{a}_x^{(m)} - \mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right]. \end{aligned} \tag{5.5.5}$$

The second term on the right-hand side of (5.5.5) is the actuarial present value of the refund. Using the ideas developed in Exercise 4.19 we have

$$\mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right] = \frac{\bar{A}_x - A_x^{(m)}}{d^{(m)}}. \tag{5.5.6}$$

Under the uniform distribution of death assumption for each year of age, this becomes

$$\frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right) A_x$$

and

$$\ddot{a}_x^{\{m\}} = \ddot{a}_x^{(m)} - \frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right) A_x. \tag{5.5.7}$$

Let us now develop a parallel theory for annuities-immediate. Assume the annuitant dies at time T after receiving a last regular payment of size $1/m$ at time $K + J/m$, where $J = \lfloor (T - K)m \rfloor$. Now $T - K - (J/m)$ is the length of the period to be compensated for by an additional payment. Assume that each payment is accrued at a uniform rate over the m-th of the year preceding its payment. In this case the rate of accrual, c, is given by $c\,\bar{s}_{\overline{1/m}|} = 1/m$. If accrual ceases at death, an appropriate payment at death is that portion of the next payment that has been accrued to date and is given by $c\,\bar{s}_{\overline{T-K-(J/m)}|} = \bar{s}_{\overline{T-K-(J/m)}|}/(m\bar{s}_{\overline{1/m}|})$. The present value, at time 0, of all of the payments is

$$
\begin{aligned}
Y &= a^{(m)}_{\overline{K+J/m}|} + v^T \left(\frac{\bar{s}_{\overline{T-K-(J/m)}|}}{m\bar{s}_{\overline{1/m}|}} \right) \\
&= a^{(m)}_{\overline{K+J/m}|} + \frac{v^{K+J/m} - v^T}{i^{(m)}} \\
&= \frac{1 - v^T}{i^{(m)}} = a^{(m)}_{\overline{T}|}.
\end{aligned}
\tag{5.5.8}
$$

When the annual rate of payments is 1, the actuarial present value at 0 of the payments is denoted by $\mathring{a}^{(m)}_x$. When $m = 1$, the (1) in the notation is omitted for this annuity:

$$\mathring{a}^{(m)}_x = \mathrm{E}\left[\frac{1 - v^T}{i^{(m)}}\right] = \frac{\delta}{i^{(m)}}\,\bar{a}_x. \tag{5.5.9}$$

Alternatively, using the second line of (5.5.8),

$$
\begin{aligned}
\mathring{a}^{(m)}_x &= \mathrm{E}[a^{(m)}_{\overline{K+J/m}|}] + \mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}}\right] \\
&= a^{(m)}_x + \mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}}\right].
\end{aligned}
\tag{5.5.10}
$$

The second term on the right-hand side of (5.5.10) is the actuarial present value of the final partial payment. Using the ideas developed in Exercise 4.19, we have

$$\mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}}\right] = \frac{(1+i)^{1/m} A^{(m)}_x - \bar{A}_x}{i^{(m)}}. \tag{5.5.11}$$

Under the uniform distribution of death assumption for each year of age, this becomes

$$\frac{i}{i^{(m)}}\left(\frac{1}{d^{(m)}} - \frac{1}{\delta}\right) A_x$$

and

$$\mathring{a}^{(m)}_x = a^{(m)}_x + \frac{i}{i^{(m)}}\left(\frac{1}{d^{(m)}} - \frac{1}{\delta}\right) A_x. \tag{5.5.12}$$

This type of life annuity-immediate, one with a partial payment for the period between the last full payment and the time of death, is called a ***complete annuity-immediate.***

For use in subsequent material, (5.5.3) and (5.5.8) seem to be most useful. We illustrate this in the following example.

Example 5.5.1

Compare the variances of the present-value random variables for the complete annuity-immediate and apportionable annuity-due.

Solution:

For the apportionable annuity-due, we have

$$\begin{aligned} \mathrm{Var}(\ddot{a}_{\overline{T}|}^{\{m\}}) &= \mathrm{Var}\left(\frac{1-v^T}{d^{(m)}}\right) && \text{from (5.5.3)} \\ &= \frac{\mathrm{Var}(v^T)}{(d^{(m)})^2} \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(d^{(m)})^2}. \end{aligned}$$

For the complete annuity-immediate, we have

$$\begin{aligned} \mathrm{Var}(\mathring{a}_{\overline{T}|}^{(m)}) &= \mathrm{Var}\left(\frac{1-v^T}{i^{(m)}}\right) && \text{from (5.5.8)} \\ &= \frac{\mathrm{Var}(v^T)}{(i^{(m)})^2} \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(i^{(m)})^2}. \end{aligned}$$

Since $i^{(m)}$ is larger than $d^{(m)}$, and in fact $i^{(m)} = d^{(m)}\,(1+i)^{1/m}$, the variance of the complete annuity-immediate is the smaller. ▼

5.6 Notes and References

Taylor (1952) presents new formulas analogous to (5.4.6). Various inquiries into the probability distribution of annuity costs are made by Boermeester (1956), Fretwell and Hickman (1964), and Bowers (1967). This work is summarized by McCrory (1984). Mereu (1962) gives a means of calculating annuity values directly from Makeham constants. The use of the floor function, $\lfloor t \rfloor$, in actuarial science, in particular with respect to actuarial present values of life annuities, is found in Shiu (1982) and in the discussions to that paper. Complete and apportionable annuities are involved, explicitly or implicitly, in papers by Rasor and Greville (1952), Lauer (1967), and Scher (1974) and in the discussions thereto.

Exercises

Section 5.2

5.1. Using the assumption of a uniform distribution of deaths in each year of age and the Illustrative Life Table with interest at the effective annual rate of 6%, calculate

a. $\bar{a}_{20}$, $\bar{a}_{50}$, $\bar{a}_{80}$

b. $\text{Var}(\bar{a}_{\overline{T}|})$ for $x = 20, 50, 80$.
[Hint: Use (5.2.8), (5.2.9), and (4.4.4).]

5.2. Using the values obtained in Exercise 5.1, calculate the standard deviation and the coefficient of variation, σ / μ, of the following present-value random variables.

a. Individual annuities issued at ages 20, 50, 80 with life incomes of 1,000 per year payable continuously.

b. A group of 100 annuities, each issued at age 50, with life income of 1,000 per year payable continuously.

5.3. Show that $\text{Var}(\bar{a}_{\overline{T}|})$ can be expressed as

$$\frac{2}{\delta}\,(\bar{a}_x - {}^2\bar{a}_x) - \bar{a}_x^2,$$

where ${}^2\bar{a}_x$ is based on the force of interest 2δ.

5.4. Calculate $\text{Cov}(\delta\, \bar{a}_{\overline{T}|}, v^T)$.

5.5. If a deterministic (rate function) approach is adopted, (5.2.27) could be taken as the starting point for the development of a theory of continuous life annuities. For this, we would begin with

$$\frac{d\bar{a}_y}{dy} = [\mu(y) + \delta]\bar{a}_y - 1 \qquad x \le y < \omega,$$

$$\bar{a}_y = 0 \qquad \omega \le y.$$

a. Use the integrating factor $\exp\{-\int_0^y[\mu(z) + \delta]dz\}$ to solve the differential equation to obtain (5.2.3).

b. Use the integrating factor $e^{-\delta y}$ to obtain

$$\bar{a}_x = \bar{a}_{\overline{\omega - x}|} - \int_x^{\omega} e^{-\delta(y-x)}\, \bar{a}_y\, \mu(y)\, dy$$

and give an interpretation of it in words.

5.6. Assume that $\mu(x + t) = \mu$ and the force of interest is δ for all $t \ge 0$.

a. If $Y = \bar{a}_{\overline{T}|}$, $0 \le T$, display the formula for the distribution function of Y.

b. If

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & 0 \le T < n \\ \bar{a}_{\overline{n}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

c. If

$$Y = \begin{cases} 0 & 0 \le T < n \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

d. If

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & 0 \le T < n \\ \bar{a}_{\overline{T}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

5.7. By considering the integral $\int_0^{n+1} v^t\, {}_tp_x\, dt$ and breaking it in two different ways into subintervals (first, from 0 to 1 and 1 to $n + 1$ and then 0 to n and n to $n + 1$), establish a backward recursion formula for the n-year temporary life annuity based on a fixed n-year temporary period. What starting value is appropriate for this recursion formula?

5.8. By considering the integral $\int_n^{\infty} v^t\, {}_tp_x\, dt$ and breaking it into subintervals from n to $n + 1$ and $n + 1$ to ∞, establish a backward recursion formula for the n-year deferred whole life annuity based on a fixed n-year deferral period. What starting value is appropriate for this recursion formula?

5.9. Combine the result from Exercise 5.7 with the first line of (5.2.25) to establish a backward recursion formula for $u(x) = \bar{a}_{x:\overline{n}|}$. What starting value is appropriate for this recursion formula?

5.10. If the probabilities come from an aggregate table, establish (5.2.6) by a probabilistic derivation starting with a rewrite of (5.2.3) as

$$\bar{a}_x = \mathrm{E}[\bar{a}_{\overline{T}|}] = \mathrm{E}[\bar{a}_{\overline{T}|}\,|0 \le T < 1] \Pr(0 \le T < 1) + \mathrm{E}[\bar{a}_{\overline{T}|}\,|1 \le T] \Pr(1 \le T).$$

Section 5.3

5.11. Show that

$$\mathrm{Var}(a_{\overline{K}|}) = \mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \frac{\mathrm{Var}(v^{K+1})}{d^2}.$$

5.12. Prove and interpret the given relations:

a. $a_{x:\overline{n}|} = {}_1E_x\, \ddot{a}_{x+1:\overline{n}|}$

b. ${}_{n|}a_x = \dfrac{A_{x:\overline{n}|} - A_x}{d} - {}_nE_x.$

5.13. Using (5.3.13), prove and interpret the following relation in words:

$$A_{x:\overline{n}|} = v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}\,.$$

5.14. Obtain an alternative expression for the variance given in Example 5.3.1 by starting with

$$Y^2 = \begin{cases} \dfrac{1 - 2v^K + v^{2K}}{i^2} = \dfrac{2(1 - v^K) - (1 - v^{2K})}{i^2} & K = 0, 1, n - 1 \\ (a_{\overline{n}|})^2 & K = n, n + 1 \ldots . \end{cases}$$

Section 5.4

5.15. Assume a uniform distribution of deaths over each year of age. Simplify

$$\sum_{k=0}^{\infty} {}_kp_x\, v^{k+1} \left[\sum_{j=0}^{m-1} ({}_{j/m}p_{x+k}\ {}_{1/m}q_{x+k+j/m})\ \ddot{s}^{(m)}_{\overline{1-(j+1)/m}|} \right]$$

for use in interpretation of (5.4.8).

5.16. Consider an m-thly temporary life annuity-due that pays 1 per annum to an annuitant age x for $y - x$ years.

a. Express the current payment form of the actuarial present value of the above annuity as a sum of that for payments in the first year and for the remaining $y - x - 1$ years.

b. Express the actuarial present value of payments in the first year in terms of $\alpha(m)$ and $\beta(m)$ under an assumption of the uniform distribution of deaths within each year of age.

c. Find the form of the $c(x)$ and $d(x)$ expressions for a recursion relation for such an annuity and indicate a starting value.

5.17. Using (5.4.10), derive alternative formulas to (5.4.17) and (5.4.18).

5.18. Show that the annuity-immediate analogue for (5.4.6) is

$$a_x^{(m)} = s_{\overline{1}|}^{(m)}\, a_x + \frac{1}{i^{(m)}} \left[(1 + i)A_x - \left(1 + \frac{i^{(m)}}{m}\right) A_x^{(m)} \right],$$

and that under the assumption of a uniform distribution of deaths in each year of age, this becomes

$$a_x^{(m)} = s_{\overline{1}|}^{(m)}\, a_x + (1 + i)\, \frac{1 - \ddot{a}^{(m)}_{\overline{1}|}}{i^{(m)}}\, A_x\,.$$

5.19. Show that the annuity-immediate analogues for (5.4.7) are

$$a_x^{(m)} = \frac{1 - (1 + i^{(m)}/m)A_x^{(m)}}{i^{(m)}} = a_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)}\, A_x^{(m)}$$

and that under the assumption of a uniform distribution of deaths in each year of age these become

$$a_x^{(m)} = \alpha(m) a_x + \frac{1 - \ddot{a}_{\overline{1}|}^{(m)}}{i^{(m)}}.$$

5.20. a. Use (5.4.3) as a starting point to verify that

$$\lim_{m\to\infty} \ddot{a}_x^{(m)} = \bar{a}_x .$$

b. Use (5.4.10) and the result in (a) to show

$$\bar{a}_x \cong a_x + \frac{1}{2}.$$

5.21. Using the traditional approximation given in (5.4.10), establish the following:

a. $a_x^{(m)} \cong a_x + \dfrac{m-1}{2m}$

b. $a_{x:\overline{n}|}^{(m)} \cong a_{x:\overline{n}|} + \dfrac{m-1}{2m} (1 - {}_nE_x)$

c. ${}_{n|}a_x^{(m)} \cong {}_{n|}a_x + \dfrac{m-1}{2m} \; {}_nE_x .$

5.22. a. Develop a formula for $\ddot{s}_{25:\overline{40}|}^{(m)}$ in terms of $\ddot{s}_{25:\overline{40}|}$.

b. On the basis of the Illustrative Life Table with interest at the effective annual rate of 6%, calculate the values of
(i) $\ddot{a}_{25:\overline{40}|}^{(12)}$ (ii) $\ddot{s}_{25:\overline{40}|}^{(12)}$.

5.23. The actuarial present value of a standard increasing temporary life annuity with respect to (x) with

- Yearly income of 1 in the first year, 2 in the second year, and so on, ending with n in the n-th year,
- Payments made m-thly on a due basis

is denoted by $(I\ddot{a})_{x:\overline{n}|}^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_{x:\overline{n-k}|}^{(m)} .$$

5.24. The actuarial present value of a standard decreasing temporary life annuity with respect to (x) with

- Yearly income of n in the first year, $n - 1$ in the second year, and so on, ending with 1 in the n-th year,
- Payments, made m-thly on a due basis

is denoted by $(D\ddot{a})_{x:\overline{n}|}^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=1}^{n} \ddot{a}_{x:\overline{k}|}^{(m)}.$$

5.25. If in Exercise 5.23 the yearly income does not cease at age $x + n$ but continues at the level n while (x) survives therafter, the actuarial present value is denoted by $(I_{\overline{n}|}\, \ddot{a})_x^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_x^{(m)}.$$

5.26. Verify the formula

$$\delta(\bar{I}\bar{a})_{\overline{T}|} + T\, v^T = \bar{a}_{\overline{T}|},$$

where T represents the future lifetime of (x). Use it to prove that

$$\delta(\bar{I}\bar{a})_x + (\bar{I}\bar{A})_x = \bar{a}_x,$$

where $(\bar{I}\bar{a})_x$ is the actuarial present value of a life annuity to (x) under which payments are being made continuously at the rate of t per annum at time t.

5.27. From $a_{\overline{x:\overline{n}|}}^{(m)} = a_x^{(m)} + a_{\overline{n}|}^{(m)} - a_{x:\overline{n}|}^{(m)}$, show that the assumption of uniform distribution of deaths in each year of age leads to

$$a_{\overline{x:\overline{n}|}}^{(m)} = \frac{i}{i^{(m)}}\left[a_{\overline{n}|} + v^n\, {}_n p_x\, a_{x+n} + \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) v^n\, {}_n p_x\, A_{x+n}\right].$$

Section 5.5

5.28. Establish and interpret the following formulas:

a. $1 = i^{(m)}\, \mathring{a}_x^{(m)} + \bar{A}_x$

b. $1 = d^{(m)}\, \ddot{a}_x^{\{m\}} + \bar{A}_x$

c. $\mathring{a}_{x:\overline{n}|}^{(m)} = (\delta / i^{(m)})\, \bar{a}_{x:\overline{n}|}$

d. $\ddot{a}_{x:\overline{n}|}^{\{m\}} = (\delta / d^{(m)})\, \bar{a}_{x:\overline{n}|}$

e. $\ddot{a}_{x:\overline{n}|}^{\{m\}} = (1 + i)^{1/m}\, \mathring{a}_{x:\overline{n}|}^{(m)}.$

5.29. Let $H(m) = \ddot{a}_x^{\{m\}} - \mathring{a}_x^{\{m\}}$. Prove that $H(m) \geq 0$ and $\lim_{m\to\infty} H(m) = 0$.

Miscellaneous

5.30. For $0 \leq t \leq 1$ and the assumption of a uniform distribution of deaths in each year of age, show that

a. $\ddot{a}_{x+t} = \dfrac{(1 + it)\ddot{a}_x - t(1 + i)}{1 - t\, q_x}$

b. ${}_{t|}\ddot{a}_x = v^t[(1 + it)\ddot{a}_x - t(1 + i)]$

c. ${}_{1\,|t}\ddot{a}_{x+t} = \dfrac{(1+i)^t}{1 - tq_x}(\ddot{a}_x - 1)$

d. $A_{x+t} = \dfrac{1+it}{1-tq_x} A_x - \dfrac{t\, q_x}{1 - t\, q_x}.$

5.31. Obtain formulas for the evaluation of a life annuity-due to (x) with an initial payment of 1 and with annual payments increasing thereafter by

a. 3% of the initial annual payment

b. 3% of the previous year's annual payment.

5.32. Express $(\bar{D}\bar{a})_{x:\overline{n}|}$ as an integral and prove the formula

$$\frac{\partial}{\partial n}(\bar{D}\bar{a})_{x:\overline{n}|} = \bar{a}_{x:\overline{n}|}.$$

5.33. Give an expression for the actuarial accumulated value at age 70 of an annuity with the following monthly payments:

- 100 at the end of each month from age 30 to 40
- 200 at the end of each month from age 40 to 50
- 500 at the end of each month from age 50 to 60
- 1,000 at the end of each month from age 60 to 70.

5.34. Derive a simplified expression for the actuarial present value for a 25-year term insurance payable immediately on the death of (35), under which the death benefit in case of death at age $35 + t$ is $\bar{s}_{\overline{t}|}$, $0 \le t \le 25$. Interpret your result.

5.35. Derive a simplified expression for the actuarial present value for an n-year term insurance payable at the end of the year of death of (x), under which the death benefit in case of death in year $k+1$ is $\ddot{s}_{\overline{k+1}|}$, $0 \le k < n$. Interpret your result.

5.36. Obtain a simplified expression for

$$(I\ddot{a})^{(12)}_{x:\overline{25}|} - (Ia)^{(12)}_{x:\overline{25}|}.$$

5.37. Consider an n-year deferred continuous life annuity of 1 per year as an insurance with probability of claim, ${}_np_x$, and random amount of claim, $v^n\,\bar{a}_{\overline{T}|}$. Here T has p.d.f., ${}_tp_{x+n}\,\mu_x(n+t)$. Apply (2.2.13) to show that the variance of the insurance equals

$$v^{2n}\,{}_np_x\,(1 - {}_np_x)\,\bar{a}^2_{x+n} + v^{2n}\,{}_np_x\,\frac{{}^2\bar{A}_{x+n} - (\bar{A}_{x+n})^2}{\delta^2}$$

and verify that this reduces to (5.2.21).

5.38. Write the discrete analogue of the variance formula in Exercise 5.37.

5.39. Consider the indicator random variable, I_k, defined by

$$I_k = \begin{cases} 1 & T(x) \geq k \\ 0 & T(x) < k. \end{cases}$$

Show the following:

a. The present value of a life annuity to (x), with annual payment b_k on survival to age $x + k$, $k = 0, 1, 2, \ldots$, can be written as

$$\sum_{k=0}^{\infty} v^k\, b_k\, I_k.$$

b. $\mathrm{E}[I_j I_k] = {}_k p_x \qquad j \leq k$

$\mathrm{Cov}(I_j, I_k) = {}_k p_x\, {}_j q_x \qquad j \leq k.$

c. $\mathrm{Var}\left(\sum_{k=0}^{\infty} v^k\, b_k\, I_k\right) = \sum_{k=0}^{\infty} v^{2k}\, b_k^2\, {}_k p_x\, {}_k q_x + 2 \sum_{k=0}^{\infty} \sum_{j<k} v^{j+k}\, b_j\, b_k\, {}_k p_x\, {}_j q_x.$

5.40. If a left superscript 2 indicates that interest is at force 2δ, show that

a. ${}^2A_x = 1 - (2\,d - d^2)\, {}^2\ddot{a}_x$

b. $\mathrm{Var}\,(v^{K+1}) = 2\,d(\ddot{a}_x - {}^2\ddot{a}_x) - d^2\,(\ddot{a}_x^2 - {}^2\ddot{a}_x)$

c. $\mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \dfrac{2}{d}\,(\ddot{a}_x - {}^2\ddot{a}_x) - (\ddot{a}_x^2 - {}^2\ddot{a}_x).$

5.41. a. Expand, in terms of powers of δ, the annuity coefficients $\alpha(m)$ and $\beta(m)$.

b. What do the expansions in (a) become for $m = \infty$?

5.42. Use Jensen's inequality to show, for $\delta > 0$, that

a. $\bar{a}_x < \bar{a}_{\overline{\mathring{e}_x}|}$ b. $a_x < a_{\overline{e_x}|}$ $\quad x < \omega - 1.$

5.43. If $g(x)$ is a non-negative function and X is a random variable with p.d.f. $f(x)$, justify the inequality

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx \geq k \Pr[g(X) \geq k] \qquad k > 0$$

and use it to show that

$$\bar{a}_x \geq \bar{a}_{\overline{\mathring{e}_x}|} \Pr(\bar{a}_{\overline{T}|} \geq \bar{a}_{\overline{\mathring{e}_x}|}) = \bar{a}_{\overline{\mathring{e}_x}|} \Pr(T \geq \mathring{e}_x).$$

5.44. A unit is to be used to purchase a combination benefit consisting of a life income of I per year payable continuously while (x) survives and an insurance of J payable immediately on the death of (x). Write the present-value random variable for this combination and give its mean and variance.

5.45. Using the assumption of a uniform distribution of deaths in each year of age and the Illustrative Life Table with interest at the effective annual rate of 6%, calculate

a. $\ddot{a}_{40}^{(12)}$ b. $\ddot{a}_{40:\overline{30}|}^{(12)}$ c. ${}_{30|}\ddot{a}_{40}^{(12)}$.

5.46. If $A''_{x:\overline{m}|}$ and $\ddot{a}''_{x:\overline{m}|}$ are actuarial present values calculated using

- An interest rate of i for the first n years, $n < m$, and
- An interest rate i' for the remaining $m - n$ years,

show algebraically and interpret

a. $A''_{x:\overline{m}|} = 1 - d\,\ddot{a}_{x:\overline{n}|} - v^n\,{}_np_x\,d'\,\ddot{a}'_{x+n:\overline{m-n}|}$

b. $A''_{x:\overline{m}|} = 1 - d'\,\ddot{a}''_{x:\overline{m}|} + (d' - d)\,\ddot{a}_{x:\overline{n}|}$.

5.47. Show that

$$\frac{d\ddot{a}_x}{di} = -v(Ia)_x,$$

where

$$(Ia)_x = \sum_{t=1}^{\infty} t\,v^t\,{}_tp_x,$$

and interpret the relation.

5.48. Show that a constant increase in the force of mortality has the same effect on $\ddot{a}_x$ as a constant increase in the force of interest, but that this is not the case for $\ddot{a}_x^{(m)}$ evaluated by $\alpha(m)\,\ddot{a}_x - \beta(m)$.

5.49. Show that

$$\alpha(m) - \beta(m)d = \ddot{a}_{\overline{1}|}^{(m)}.$$

5.50. Show that, if $q_x < (i^{(2)}/2)^2$, the approximation

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x),$$

in the special case with $n = 1$, $m = 2$, leads to

$$\ddot{a}_{x:\overline{1}|}^{(2)} > \ddot{a}_{\overline{1}|}^{(2)}.$$

5.51. Consider the following portfolio of annuities-due currently being paid from the assets of a pension fund.

Age	Number of Annuitants
65	30
75	20
85	10

Each annuity has an annual payment of 1 as long as the annuitant survives. Assume an earned interest rate of 6% and a mortality as given in the Illustrative Life Table. For the present value of these obligations of the pension fund, calculate

a. The expectation

b. The variance

c. The 95th percentile of its distribution.

For parts (b) and (c), assume the lives are mutually independent.

Computing Exercises

5.52. a. For your Illustrative Life Table with $i = 0.06$, calculate the actuarial present value of a life annuity-due of 1 per annum for ages 13 to 140.
b. Compare your values to those given in Table 2A.

5.53. For your Illustrative Life Table with $i = 0.06$, calculate the actuarial present value of a temporary life annuity-due of 1 per annum payable to age 65 for ages 13 to 64.

5.54. Using your Illustrative Life Table with the assumption of a uniform distribution of deaths within each year of age and $i = 0.06$, calculate the actuarial present value of a 10-year temporary life annuity-due of 1 per annum issued at ages 13 to 99 using the results of Exercise 5.7.

5.55. a. Add $\alpha(m)$ and $\beta(m)$ to the interest functions calculated and stored in your Illustrative Life Table. Refer to Exercise 4.31.
b. Determine $\alpha(12)$ and $\beta(12)$ at $i = 0.06$ and compare your results to those given in Example 5.4.1.
[Remark: We suggest using the series derived in Exercise 5.41 for accurate results with small interest rates.]

5.56. Using your Illustrative Life Table with $i = 0.06$ and the assumption of uniform distribution of deaths over each year of age, calculate the actuarial present value of a temporary life annuity of 1 per annum payable continuously to age 65 for ages 13 to 64.

5.57. Let Y be the present-value random variable for a continuous 10-year temporary life annuity of 1 per annum commencing at age 60. On the basis of your Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.08$, calculate the mean and variance of Y.

5.58. Let Y be the present-value random variable for a life annuity-due of 1 per annum, payable monthly to (65). On the basis of your Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.05$, calculate the mean and variance of Y.

5.59. Use the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.07$ to determine $a_{30:\overline{20|}}$.

6

BENEFIT PREMIUMS

6.1 Introduction

In Chapters 4 and 5 we discussed actuarial present values of the payments of various life insurances and annuities. These ideas are combined in this chapter to determine the level of life annuity payments necessary to buy, or fund, the benefits of an insurance or annuity contract. In practice individual life insurance is usually purchased by a life annuity of ***contract premiums***—the insurance contract specifies the premium to be paid. Contract premiums provide for benefits, expenses of initiating and maintaining the insurance, and margins for profit and for offsetting possible unfavorable experience. The premiums studied in this chapter are determined only by the pattern of benefits and premiums and do not consider expenses, profit, or contingency margins.

In Chapter 1 we discussed the idea that determination of the insurance premium requires the adoption of a ***premium principle.*** Example 6.1.1 illustrates the application of three such premium principles. All three principles are based on the impact of the insurance on the wealth of the insuring organization. The random variable that gives the present value at issue of the insurer's loss, if the insurance is contracted at a certain premium level, is the key in the model for the principles. Principle I requires that the loss random variable be positive with no more than a specified probability. Principles II and III are based on the expected utility of the insurer's wealth as discussed in Section 1.3. We will see that Principle II, which uses a linear utility function, could also be characterized as requiring that the loss random variable have zero expected value.

Example 6.1.1

An insurer is planning to issue a policy to a life age 0 whose curtate-future-lifetime, K, is governed by the p.f.

$$ {}_{k|}q_0 = 0.2 \qquad k = 0, 1, 2, 3, 4. $$

The policy will pay 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Find the annual premium, P, as determined by:

a. Principle I: P will be the least annual premium such that the insurer has probability of a positive financial loss of at most 0.25.
b. Principle II: P will be the annual premium such that the insurer, using a utility of wealth function $u(x) = x$, will be indifferent between accepting and not accepting the risk.
c. Principle III: P will be the annual premium such that the insurer, using a utility of wealth function $u(x) = -e^{-0.1x}$, will be indifferent between accepting and not accepting the risk.

For all three parts assume the insurer will use an annual effective interest rate of $i = 0.06$.

Solution:

For $K = k$ and an arbitrary premium, P, the present value of the financial loss at policy issue is $l(k) = v^{k+1} - P\ddot{a}_{\overline{k+1}|} = (1 + P/d)\, v^{k+1} - P/d$, $k = 0, 1, 2, 3, 4$. The corresponding loss random variable is $L = v^{K+1} - P\ddot{a}_{\overline{K+1}|}$.

a. Since $l(k)$ decreases as k increases, the requirement of Principle I will hold if P is such that $v^2 - P\,\ddot{a}_{\overline{2}|} = 0$. Then the financial loss is positive for only $K = 0$, which has probability $0.2 < 0.25$. Thus, for this principle, $P = 1/\ddot{s}_{\overline{2}|} = 0.45796$.

b. By an extension of (1.3.6), we seek the premium P such that $u(w) = \mathrm{E}[u(w - L)]$. By Principle II $u(x) = x$ so we have

$$w = \mathrm{E}[w - L] = w - \mathrm{E}[L].$$

Thus, Principle II is equivalent to requiring that P be chosen so that $\mathrm{E}[L] = 0$. For this example, we require

$$\sum_{k=0}^{4} (v^{k+1} - P\,\ddot{a}_{\overline{k+1}|}) \Pr(K = k) = 0 \tag{6.1.1}$$

which gives $P = 0.30272$.

c. Again by (1.3.6) and now using the utility function in Principle III, we have

$$-e^{-0.1w} = \mathrm{E}[-e^{-0.1(w-L)}] = -e^{-0.1w}\,\mathrm{E}[e^{0.1L}].$$

Thus, Principle III is equivalent to requiring that P be chosen so that $\mathrm{E}[e^{0.1L}] = 1$. Here, we require

$$\sum_{k=0}^{4} \exp[0.1\,(v^{k+1} - P\,\ddot{a}_{\overline{k+1}|})] \Pr(K = k) = 1, \tag{6.1.2}$$

which gives $P = 0.30628$.

These three results are summarized below.

Outcome k	Probability ${}_{k\mid}q_0$	General Formula	Present Value of Financial Loss When Premium Is by Principle I	II	III	$\exp(0.1L)$ III
0	0.2	$v^1 - P\,\ddot{a}_{\overline{1}\rceil}$	0.48544	0.64067	0.63712	1.06579
1	0.2	$v^2 - P\,\ddot{a}_{\overline{2}\rceil}$	0	0.30169	0.29477	1.02992
2	0.2	$v^3 - P\,\ddot{a}_{\overline{3}\rceil}$	−0.45796	−0.01811	−0.02819	0.99718
3	0.2	$v^4 - P\,\ddot{a}_{\overline{4}\rceil}$	−0.89000	−0.31981	−0.33287	0.96726
4	0.2	$v^5 - P\,\ddot{a}_{\overline{5}\rceil}$	−1.29758	−0.60443	−0.62031	0.93985
	Premium		0.45796	0.30272	0.30628	
	Expected Value		−0.43202	0.00000	−0.12020	1.00000

The table shows that for this example the decision makers adopting principles I and III reduce their risk in the sense that they are demanding an expected present value of loss to be negative. ▼

Premiums defined by Principle I are known as ***percentile premiums.*** Although the principle is attractive on the surface, it is easy to show that it can lead to quite unsatisfactory premiums. Such cases are examined in Example 6.2.3.

Principle II has many applications in practice. To formalize its concepts, we define the insurer's loss, L, as the random variable of the present value of benefits to be paid by the insurer less the annuity of premiums to be paid by the insured. Principle II is called the ***equivalence principle*** and has the requirement that

$$E[L] = 0. \tag{6.1.3}$$

We will speak of ***benefit premiums*** as those satisfying (6.1.3). Equivalently, benefit premiums will be such that

$$\text{E[present value of benefits]} = \text{E[present value of benefit premiums]}.$$

Methods developed in Chapters 4 and 5 for calculating these actuarial present values can be used to reduce this equality to a form that can be solved for the premiums. For instance, in Example 6.1.1, which has constant benefit premiums and constant benefits of 1, equation (6.1.1) can be rewritten as $A_0 = P\,\ddot{a}_0$, and $\ddot{a}_0$ can be calculated as

$$\sum_{k=0}^{4} v^k\,{}_kp_0.$$

When the equivalence principle is used to determine a single premium at policy issue for a life insurance or a life annuity, the premium is equal to the actuarial present value of benefit payments and is called the ***single benefit premium.***

Premiums based on Principle III, using an exponential utility function, are known as ***exponential premiums.*** Exponential premiums are nonproportional in the sense that the premium for the policy with a benefit level of 10 is more than 10 times the premium for a policy with a benefit level of 1 (see Exercise 6.2). This is consistent for a risk averse utility function.

6.2 Fully Continuous Premiums

The basic concepts involved in the determination of annual benefit premiums using the equivalence principle will be illustrated first for the case of the fully continuous level annual benefit premium for a unit whole life insurance payable immediately on the death of (x). For any continuously paid premium, $\bar{P}$, consider

$$l(t) = v^t - \bar{P}\,\bar{a}_{\overline{t}|}, \tag{6.2.1}$$

the present value of the loss to the insurer if death occurs at time t.

We note that $l(t)$ is a decreasing function of t with $l(0) = 1$ and $l(t)$ approaching $-\bar{P}/\delta$ as $t \to \infty$. If t_0 is the time when $l(t_0) = 0$, death before t_0 results in a positive loss, whereas death after t_0 produces a negative loss, that is, a gain. Figure 6.2.1 later in this section illustrates these ideas.

We now consider the loss random variable,

$$L = l(T) = v^T - \bar{P}\,\bar{a}_{\overline{T}|}, \tag{6.2.2}$$

corresponding to the loss function $l(t)$. If the insurer determines his premium by the equivalence principle, the premium is denoted by $\bar{P}(\bar{A}_x)$ and is such that

$$\mathrm{E}[L] = 0. \tag{6.2.3}$$

It follows from (4.2.6) and (5.2.3) that

$$\bar{A}_x - \bar{P}(\bar{A}_x)\bar{a}_x = 0,$$

or

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}. \tag{6.2.4}$$

Remark:

In this chapter we continue to suppress the select notation except in situations in which it is necessary or helpful to eliminate ambiguity.

The variance of L can be used as a measure of the variability of losses on an individual whole life insurance due to the random nature of time-until-death. When $\mathrm{E}[L] = 0$,

$$\mathrm{Var}(L) = \mathrm{E}[L^2]. \tag{6.2.5}$$

For the loss in (6.2.2), we have

$$\begin{aligned}\mathrm{Var}(v^T - \bar{P}\,\bar{a}_{\overline{T}|}) &= \mathrm{Var}\left[v^T - \frac{\bar{P}(1 - v^T)}{\delta}\right] \\ &= \mathrm{Var}\left[v^T\left(1 + \frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right]\end{aligned}$$

$$= \text{Var}\left[v^T\left(1 + \frac{\bar{P}}{\delta}\right)\right]$$

$$= \text{Var}\,(v^T)\left(1 + \frac{\bar{P}}{\delta}\right)^2$$

$$= [{}^2\bar{A}_x - (\bar{A}_x)^2]\left(1 + \frac{\bar{P}}{\delta}\right)^2. \tag{6.2.6}$$

For the premium determined by the equivalence principle, we can use (6.2.4) and (5.2.8), $\delta\bar{a}_x + \bar{A}_x = 1$, to rewrite (6.2.6) as

$$\text{Var}(L) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta\bar{a}_x)^2}. \tag{6.2.7}$$

Example 6.2.1

Calculate $\bar{P}(\bar{A}_x)$ and $\text{Var}(L)$ with the assumptions that the force of mortality is a constant $\mu = 0.04$ and the force of interest $\delta = 0.06$.

Solution:

These assumptions yield $\bar{a}_x = 10$, $\bar{A}_x = 0.4$, and ${}^2\bar{A}_x = 0.25$. Using (6.2.4), we obtain

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = 0.04,$$

and from (6.2.7)

$$\text{Var}(L) = \frac{0.25 - 0.16}{(0.6)^2} = 0.25.$$

▼

By reference to (6.2.6) we can see that the numerator of this last expression can be interpreted as the variance of the loss, $v^T - \bar{A}_x$, associated with a single-premium whole life insurance. This latter variance is 0.09, and hence the standard deviation of the loss associated with this annual premium insurance is $\sqrt{0.25/0.09} = 5/3$ times the standard deviation of the loss in the single-premium case. Additional uncertainty about the present value of the premium income increases the variability of losses due to the random nature of time-until-death.

In Example 6.2.1, $\bar{P}(\bar{A}_x) = 0.04$, the constant force of mortality. We can confirm this as a general result by using parts of Examples 4.2.3 and 5.2.1. Under the constant force of mortality assumption,

$$\bar{A}_x = \frac{\mu}{\mu + \delta}$$

and

$$\bar{a}_x = \frac{1}{\mu + \delta},$$

thus

$$\bar{P}(\bar{A}_x) = \frac{\mu(\mu + \delta)^{-1}}{(\mu + \delta)^{-1}} = \mu,$$

which does not depend on the force of interest or the age at issue.

Using the equivalence principle, as in (6.1.3), we can determine formulas for annual premiums of a variety of fully continuous life insurances. Our general loss is

$$b_T v_T - \bar{P}\,Y = Z - \bar{P}\,Y \tag{6.2.8}$$

where

- b_t and v_t are, respectively, the benefit amount and discount factor defined in connection with (4.2.1)
- $\bar{P}$ is a general symbol for a fully continuous net annual premium
- Y is a continuous annuity random variable as defined, for example, in (5.2.13), and
- Z is defined by (4.2.2).

Application of the equivalence principle yields

$$\mathrm{E}[b_T v_T - \bar{P}\,Y] = 0$$

or

$$\bar{P} = \frac{\mathrm{E}[b_T v_T]}{\mathrm{E}[Y]}.$$

These ideas are used to display annual premium formulas in Table 6.2.1.

It is of interest to note how these steps can be used for an n-year deferred whole life annuity of 1 per year payable continuously. In this case $b_T v_T = 0$, $T \le n$ and $b_T v_T = \bar{a}_{\overline{T-n}|} v^n$, $T > n$. Then,

$$\begin{aligned}\mathrm{E}[b_T v_T] &= {}_n p_x\, \mathrm{E}[\bar{a}_{\overline{T-n}|} v^n | T > n] \\ &= v^n\, {}_n p_x\, \bar{a}_{x+n} = A_{x:\overline{n}|}^{\;\;1}\, \bar{a}_{x+n}.\end{aligned}$$

In practice, however, deferred life annuities usually provide some type of death benefit during the period of deferment. One contract of this type is examined in Example 6.6.2.

Example 6.2.2

Express the variance of the loss, L, associated with an n-year endowment insurance, in terms of actuarial present values (see the third row of Table 6.2.1).

Fully Continuous Benefit Premiums

Plan	Loss Components: $b_T v_T$	Loss Components: $\bar{P}\,Y$ Where Y Is	Premium Formula $\bar{P} = \dfrac{\mathrm{E}[b_T v_T]}{\mathrm{E}[Y]}$
Whole life insurance	$1\ v^T$	$\bar{a}_{\overline{T}\rceil}$	$\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_x}$
n-Year term insurance	$1\ v^T$ 0	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}(\bar{A}^{1}_{x:\overline{n}\rceil}) = \dfrac{\bar{A}^{1}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
n-Year endowment insurance	$1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \dfrac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
h-Payment* whole life insurance	$1\ v^T$ $1\ v^T$	$\bar{a}_{\overline{T}\rceil},\ T \le h$ $\bar{a}_{\overline{h}\rceil},\ T > h$	${}_h\bar{P}(\bar{A}_x) = \dfrac{\bar{A}_x}{\bar{a}_{x:\overline{h}\rceil}}$
h-Payment,* n-year endowment insurance	$1\ v^T$ $1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le h$ $\bar{a}_{\overline{h}\rceil},\ h < T \le n$ $\bar{a}_{\overline{h}\rceil},\ T > n$	${}_h\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \dfrac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{h}\rceil}}$
n-Year pure endowment	0 $1\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}(A_{x:\overline{n}\rceil}^{\ \ 1}) = \dfrac{A_{x:\overline{n}\rceil}^{\ \ 1}}{\bar{a}_{x:\overline{n}\rceil}}$
n-Year† deferred whole life annuity	0 $\bar{a}_{\overline{T-n}\rceil}\ v^n$	$\bar{a}_{\overline{T}\rceil},\ T \le n$ $\bar{a}_{\overline{n}\rceil},\ T > n$	$\bar{P}({}_{n\vert}\bar{a}_x) = \dfrac{A_{x:\overline{n}\rceil}^{\ \ 1}\,\bar{a}_{x+n}}{\bar{a}_{x:\overline{n}\rceil}}$

*The insurances described in the fourth and fifth rows provide for a premium paying period that is shorter than the period over which death benefits are paid.

†The annuity product described above provides no death benefits and has a level premium with premiums payable for n years. A different, perhaps more realistic, design for an n-year level premium-deferred annuity is given in Example 6.6.2.

Solution:

Using the notation of (4.2.11), we have

$$\mathrm{Var}(L) = \mathrm{Var}\left\{Z_3\left[1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta}\right] - \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta}\right\}.$$

We now use (4.2.10) to obtain

$$\mathrm{Var}(L) = \left[1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta}\right]^2 [{}^2\bar{A}_{x:\overline{n}\rceil} - (\bar{A}_{x:\overline{n}\rceil})^2].$$

Formula (5.2.14) can be rewritten as

$$(\delta\,\bar{a}_{x:\overline{n}\rceil})^{-1} = 1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta},$$

which implies that

$$\mathrm{Var}(L) = \frac{{}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2}{(\delta\, \bar{a}_{x:\overline{n|}})^2}.$$

▼

The two identities, (5.2.8) and (5.2.15), can be used to derive relationships among continuous benefit premiums. For example, starting with (5.2.8),

$$\delta\, \bar{a}_x + \bar{A}_x = 1,$$

$$\delta + \bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x},$$

$$\bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x} - \delta$$

$$= \frac{1 - \delta\, \bar{a}_x}{\bar{a}_x}$$

$$= \frac{\delta\, \bar{A}_x}{1 - \bar{A}_x}. \tag{6.2.9}$$

Starting with (5.2.14) we obtain

$$\delta\, \bar{a}_{x:\overline{n|}} + \bar{A}_{x:\overline{n|}} = 1,$$

$$\delta + \bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{1}{\bar{a}_{x:\overline{n|}}},$$

$$\bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{1}{\bar{a}_{x:\overline{n|}}} - \delta$$

$$= \frac{1 - \delta\, \bar{a}_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}}$$

$$= \frac{\delta\, \bar{A}_{x:\overline{n|}}}{1 - \bar{A}_{x:\overline{n|}}}. \tag{6.2.10}$$

Verbal interpretations of the discrete analogues of (6.2.9) and (6.2.10) are given in Example 6.3.4.

The premiums discussed so far in this section are benefit premiums, those derived from the equivalence principle. We now turn to an example that describes two ways that percentile premiums give unsatisfactory results.

Example 6.2.3

Find the 25th percentile premium for an insured age 55 for the following plans of insurance:

a. 20-year endowment
b. 20-year term
c. 10-year term.

Assume a fully continuous basis with a force of interest, $\delta = 0.06$ and mortality following the Illustrative Life Table.

Solution:

a. The loss function for 20-year endowment insurance is

$$\begin{aligned} L &= v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 20 \\ &= v^{20} - \bar{P}\bar{a}_{\overline{20}|} \qquad T \geq 20 \end{aligned}$$

and is a nonincreasing function of T. Thus the values of T for which the loss L is to be positive, which are to have probability of 0.25, are those values below $\xi_T^{0.25}$. Since $l_{55} = 86{,}408.60$ and $l_{70.617} = 64{,}806.45$ (by linear interpolation), $\Pr(T < 15.617) = 0.25$. Thus, the premium required by the 25th percentile principle is that which sets the loss at $T = 15.617$ equal to zero and is $v^{15.617}/\bar{a}_{\overline{15.617}|} = 0.03865$.

b. The loss function for 20-year term insurance is

$$\begin{aligned} L &= v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 20 \\ &= \quad - \bar{P}\bar{a}_{\overline{20}|} \qquad T \geq 20. \end{aligned}$$

This is still a nonincreasing function of T, and since $\Pr(T < 15.617) = 0.25$, the premium required by the 25th percentile principle is again $v^{15.617}/\bar{a}_{\overline{15.617}|} = 0.03865$. It is, of course, unsatisfactory that the same premium is generated for two different plans of insurance, particularly since the benefit premium at this age for 20-year endowment is almost two times that for 20-year term.

c. The loss function for a 10-year term insurance is

$$\begin{aligned} L &= v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 10 \\ &= - \bar{P}\bar{a}_{\overline{10}|} \qquad T \geq 10. \end{aligned}$$

If the premium is set at zero, then $\Pr(L > 0) = \Pr(T < 10)$, and this probability equals, by the Illustrative Life Table,

$$\frac{(l_{55} - l_{65})}{l_{55}} = 0.1281.$$

Thus, zero is the least non-negative annual premium such that the insurer's probability of financial loss is at most 0.25. In this case $\Pr(L > 0) = 0.1281$, and $\bar{P} = 0$ the 25th percentile premium. The benefit premium in this case is 70% of that for 20-year term insurance. ▼

The conclusion from this example is that the percentile premium principle yields conflicting results for insurances on a single individual. Its use will be minimal in what follows.

For a whole life insurance, as defined in the first row of Table 6.2.1,

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T \geq 0.$$

The d.f. of L can be developed as follows:

$$\begin{aligned}
F_L(u) &= \Pr(L \le u) \\
&= \Pr\left[v^T - \bar{P}\left(\frac{1 - v^T}{\delta}\right) \le u\right] \\
&= \Pr\left(v^T \le \frac{\delta u + \bar{P}}{\delta + \bar{P}}\right) \\
&= \Pr\left[T \ge -\frac{1}{\delta}\log\left(\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right)\right] \\
&= 1 - F_T\left(-\frac{1}{\delta}\log\left[\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right]\right) \qquad -\frac{\bar{P}}{\delta} < u.
\end{aligned} \tag{6.2.11}$$

The p.d.f. of L is

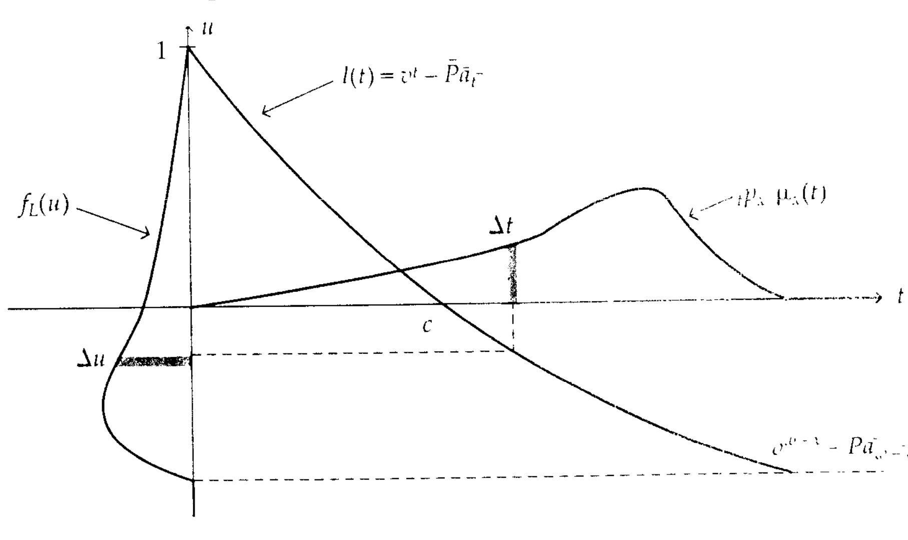
$$\frac{d}{du}F_L(u) = f_L(u) = f_T\left(-\frac{1}{\delta}\log\left[\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right]\right)\left(\frac{1}{\delta u + \bar{P}}\right) \qquad -\frac{\bar{P}}{\delta} < u. \tag{6.2.12}$$

Using the language of decision analysis, we can say that the determination of the premium $\bar{P}$ is equivalent to selecting the distribution of L, given by (6.2.11), that is optimal from the viewpoint of the premium principle adopted by the decision maker. This principle reflects the preferences of the decision maker.

Schematic diagrams of $l(t)$, the p.d.f. of $T(x)$, and the induced p.d.f. of L are combined in Figure 6.2.1.

The set of d.f.'s of L is indexed by the parameter $\bar{P}$. The value of $\bar{P}$ is selected by the premium principle adopted. For illustration, use Figure 6.2.1 where $\Pr(T \le c) = \Pr(L > 0)$ and this probability is taken as 0.25. We assume that the

Schematic Diagrams of $l(t)$ and the p.d.f.'s of $T(x)$ and L



value of $\bar{P}$ will be obtained by solving $F_L(0) = 1 - 0.25 = 0.75$. This illustration uses a percentile premium principle with the probability for a positive value of L set at 0.25.

It is evident from Figure 6.2.1 that the events $(T \le c)$ and $(L > 0)$ are equivalent in the sense that the occurrence of one of the two events implies the occurrence of the other. To continue our illustration, if the decision maker has adopted the percentile premium principle with $\Pr(L > 0) = p$, then $\Pr(T \le c) = p$, where $c = \xi_T^p$, the $100p$-th percentile of the distribution of T. Furthermore, because of the equivalence of these two events, the premium can be determined from an equation involving the loss function, that is, from

$$v^{\xi_T^p} - \bar{P}\bar{a}_{\overline{\xi_T^p}|} = 0,$$

or

$$\boxed{\bar{P} = \frac{v^{\xi_T^p}}{\bar{a}_{\overline{\xi_T^p}|}} = \frac{1}{\bar{s}_{\overline{\xi_T^p}|}}.} \tag{6.2.13}$$

Because $\bar{P}$ is the rate of payment into a fund that will provide a unit payment at time ξ_T^p, there is intuitive support for the result. The accumulation $\bar{s}_{\overline{T}|}/\bar{s}_{\overline{\xi_T^p}|}$ will be less than 1 with probability p and greater than 1 with probability $1 - p$.

Example 6.2.4

This example builds on Example 6.2.3, except that $T(55)$ has a De Moivre distribution, with p.d.f.

$${}_tp_{55}\ \mu_{55}(t) = 1/45 \qquad 0 < t < 45.$$

For the three loss variables, display the d.f. of L and determine the parameter $\bar{P}$ as the smallest non-negative number such that $\Pr(L > 0) \le 0.25$.

Solution:

a. Adapting (6.2.11), with recognition of the jump in the d.f. at $u = v^{20} - \bar{P}\bar{a}_{\overline{20}|}$ induced by the constraint on L if $T \ge 20$, we have the following set of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 && u \le v^{20} - \bar{P}\bar{a}_{\overline{20}|} \\ &= 1 + \frac{1}{0.06}\,\frac{\log[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} && v^{20} - \bar{P}\bar{a}_{\overline{20}|} < u \le 1 \\ &= 1 && 1 < u. \end{aligned}$$

Figure 6.2.2a is a diagram of the d.f. associated with a 20-year endowment insurance. This figure provides a graphical way of thinking of premium determination using the percentile principle. The d.f. within the set of d.f.'s indexed by $\bar{P}$ that crosses the vertical axis at 0.75 is sought. Analytically this means that the premium is determined by solving, for $\bar{P}$,

$$
\begin{aligned}
F_L(0) &= 0.75 \\
&= 1 + \frac{1}{0.06}\,\frac{\log[\bar{P}/(0.06+\bar{P})]}{45} = 0.75,
\end{aligned}
$$

or

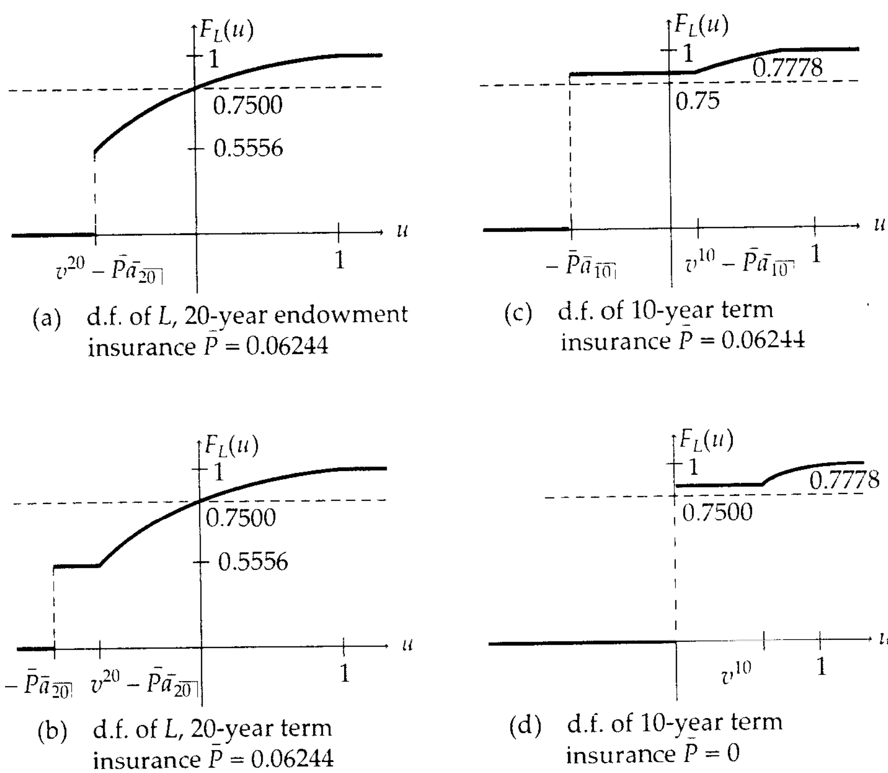
$$\log\left(\frac{\bar{P}}{0.06+\bar{P}}\right) = -0.675,$$

and

$$
\begin{aligned}
\bar{P} &= \frac{0.06e^{-0.675}}{(1-e^{-0.675})} \\
&= \frac{1}{\bar{s}_{\overline{11.25}|}} \\
&= 0.06224.
\end{aligned}
$$

In view of the discussion about (6.2.11), this is not surprising. The 25th percentile of the De Moivre distribution of T in this example is $\xi_T^{0.25} = 11.25$.

Distribution Functions of L Developed in Example 6.2.4



(a) d.f. of L, 20-year endowment insurance $\bar{P} = 0.06244$

(c) d.f. of 10-year term insurance $\bar{P} = 0.06244$

(b) d.f. of L, 20-year term insurance $\bar{P} = 0.06244$

(d) d.f. of 10-year term insurance $\bar{P} = 0$

For comparison, the benefit, or equivalence principle, premium is

$$\bar{P}(\bar{A}_{55:\overline{20|}}) = \frac{\int_0^{20} (v^t / 45)\, dt + (25/45)\, v^{20}}{\int_0^{20} v^t\, [1 - (t/45)]\, dt} = 0.04456.$$

b. Adapting (6.2.11) with recognition of the jump in the d.f. of L at $u = -\bar{P}\bar{a}_{\overline{20|}}$, induced by the constraint on the 20-year term insurance loss variable, we have the following set of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 & u &\le -\bar{P}\bar{a}_{\overline{20|}} \\ &= \frac{25}{45} & -\bar{P}\bar{a}_{\overline{20|}} &< u \le v^{20} - \bar{P}\bar{a}_{\overline{20|}} \\ &= 1 + \frac{1}{0.06}\frac{\log[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} & v^{20} - \bar{P}\bar{a}_{\overline{20|}} &< u \le 1 \\ &= 1 & 1 &< u. \end{aligned}$$

A diagram of the d.f. associated with a 20-year term insurance is shown in Figure 6.2.2b. The premium is determined by solving $F_L(0) = 0.75$ for $\bar{P}$. Using part (a) we find, once more, that $\bar{P} = 0.06224$.

c. Adapting (6.2.11), with recognition of the jump in the d.f. at $u = -\bar{P}\bar{a}_{\overline{10|}}$ induced by the constraint on L for 10-year term insurance, we have the family of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 & u &\le -\bar{P}\bar{a}_{\overline{10|}} \\ &= \frac{35}{45} & -\bar{P}\bar{a}_{\overline{10|}} &< u \le v^{10} - \bar{P}\bar{a}_{\overline{10|}} \\ &= 1 + \frac{1}{0.06}\frac{\log[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} & v^{10} - \bar{P}\bar{a}_{\overline{10|}} &< u \le 1 \\ &= 1 & 1 &< u. \end{aligned}$$

It is tempting to conjecture that $\bar{P} = 0.06224$, as it was in parts (a) and (b), when we observe that the only nonconstant values of the d.f. have the same formula as in the earlier parts of this example. When we observe that for any u in the interval $(-\bar{P}\bar{a}_{\overline{10|}}, v^{10} - \bar{P}\bar{a}_{\overline{10|}})$, $F_L(u) = 35/45 > 0.75$, it appears that the conjecture is wrong. Figure 6.2.2c displays the d.f. of L when $\bar{P} = 0.06224$ and confirms this judgment. As in Example 6.2.3, try $\bar{P} = 0$. The corresponding d.f. of L is

$$\begin{aligned} F_L(u) &= 0 & u &\le 0 \\ &= \frac{35}{45} & 0 &< u \le v^{10} \\ &= 1 + \frac{1}{0.06}\frac{\log u}{45} & v^{10} &< u \le 1 \\ &= 1 & 1 &< u \end{aligned}$$

and the probability of a positive value of L is

$$\Pr(L > 0) = \frac{10}{45} < 0.25.$$

This is illustrated in Figure 6.2.2d.

As in Example 6.2.3c, the specifications for applying the percentile premium principle leads to $\bar{P} = 0$, an anomalous result from a business perspective. ▼

6.3 Fully Discrete Premiums

In Section 6.2 we have discussed the theory of fully continuous benefit premiums. In this section we consider annual premium insurances like the one that appeared in Example 6.1.1; that is, the sum insured is payable at the end of the policy year in which death occurs, and the first premium is payable when the insurance is issued. Subsequent premiums are payable on anniversaries of the policy issue date while the insured survives during the contractual premium payment period. The set of annual premiums form a life annuity-due. This model does not conform to practice but is of historic importance in the development of actuarial theory.

Under these circumstances, the level annual benefit premium for a unit whole life insurance is denoted by P_x, where the absence of $(\bar{A}_x)$ means that the insurance is payable at the end of the policy year of death. The loss for this insurance is

$$L = v^{K+1} - P_x\, \ddot{a}_{\overline{K+1}|} \qquad K = 0, 1, 2, \ldots. \tag{6.3.1}$$

The equivalence principle requires that $\mathrm{E}[L] = 0$, or

$$\mathrm{E}[v^{K+1}] - P_x\, \mathrm{E}[\ddot{a}_{\overline{K+1}|}] = 0,$$

which yields

$$P_x = \frac{A_x}{\ddot{a}_x}. \tag{6.3.2}$$

This is the discrete analogue of (6.2.4).

Using (5.3.7) in place of (5.2.8) in steps parallel to those taken in obtaining (6.2.7), we obtain

$$\mathrm{Var}(L) = \frac{{}^2A_x - (A_x)^2}{(d\,\ddot{a}_x)^2}. \tag{6.3.3}$$

Example 6.3.1

If

$$_{k|}q_x = c(0.96)^{k+1} \qquad k = 0, 1, 2, \ldots$$

where $c = 0.04/0.96$ and $i = 0.06$, calculate P_x and $\text{Var}(L)$.

Solution:

First we exhibit the components of (6.3.2),

$$A_x = c \sum_{k=0}^{\infty} (1.06)^{-k-1} (0.96)^{k+1} = 0.40,$$

$$\ddot{a}_x = \frac{1 - A_x}{d} = 10.60.$$

Then using (6.3.2) we obtain

$$P_x = \frac{A_x}{\ddot{a}_x} = 0.0377.$$

For $\text{Var}(L)$, we calculate

$$^2A_x = c \sum_{k=0}^{\infty} [(1.06)^2]^{-k-1} (0.96)^{k+1} = 0.2445.$$

Therefore,

$$\begin{aligned} \text{Var}(L) &= \frac{0.2445 - 0.1600}{[(0.06)(10.60)/(1.06)]^2} \\ &= 0.2347. \end{aligned}$$

▼

There is a connection between Examples 6.2.1 and 6.3.1. Since

$$_{k|}q_x = \int_k^{k+1} {}_tp_x \, \mu_x(t) \, dt \qquad k = 0, 1, 2, \ldots \tag{6.3.4}$$

for the situation described in Example 6.3.1, we have

$$\frac{0.04}{0.96} (0.96)^{k+1} = \int_k^{k-1} {}_tp_x \, \mu_x(t) \, dt.$$

If the force of mortality is a constant, μ, it follows that

$$\frac{0.04}{0.96} (0.96)^{k+1} = e^{-(k+1)\mu} (e^{\mu} - 1),$$

and then $e^{-\mu} = 0.96$ and $\mu = 0.0408$. The geometric distribution, with p.f.

$$_{k|}q_x = \frac{0.04}{0.96} (0.96)^{k+1},$$

is a discrete version of the exponential distribution with $\mu = 0.0408$. Formula (6.3.4) provides the bridge between the discrete and continuous versions. The fully continuous annual benefit premium corresponding to $P_x = 0.0377$ in Example 6.3.1 would be $\bar{P}(\bar{A}_x) = \mu = 0.0408$.

Continuing to use the equivalence principle, we can determine formulas for annual benefit premiums for a variety of fully discrete life insurances. Our general loss will be

$$b_{K+1}v_{K+1} - PY$$

where

- b_{k+1} and v_{k+1} are, respectively, the benefit and discount functions defined in (4.3.1)
- P is a general symbol for an annual premium paid at the beginning of each policy year during the premium paying period while the insured survives and
- Y is a discrete annuity random variable as defined, for example, in connection with (5.3.9).

Application of the equivalence principle yields

$$\mathrm{E}[b_{K+1}v_{K+1} - PY] = 0,$$

or

$$P = \frac{\mathrm{E}[b_{K+1}v_{K+1}]}{\mathrm{E}[Y]}.$$

These ideas are used in Table 6.3.1 to display premium formulas for fully discrete insurances.

Example 6.3.2

Express the variance of the loss, L, associated with an n-year endowment insurance, in terms of actuarial present values (see the third row of Table 6.3.1).

Solution:

We start with the notation of Table 6.3.1. Let

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ v^n & K = n, n+1, \ldots. \end{cases}$$

Then we can write, by reference to the third row of Table 6.3.1,

$$L = Z - P_{x:\overline{n}|}\frac{1-Z}{d};$$

therefore we have

$$\mathrm{Var}(L) = \mathrm{Var}\left[Z\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d}\right].$$

We can use the rule of moments to find Var(Z), as indicated in Table 4.3.1, and then obtain

Fully Discrete Annual Benefit Premiums

	Loss Components		Premium Formula
Plan	$b_{K+1}v_{K+1}$	$P\,Y$ Where Y Is	$P = \dfrac{E[b_{K+1}v_{K+1}]}{E[Y]}$
Whole life insurance	$1\ v^{K+1}$	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, 2, \ldots$	$P_x = \dfrac{A_x}{\ddot{a}_x}$
n-Year term insurance	$1\ v^{K+1}$ 0	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert}$, $K = n, n+1, \ldots$	$P^{1}_{x:\overline{n}\rvert} = \dfrac{A^{1}_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}$
n-Year endowment insurance	$1\ v^{K+1}$ $1\ v^{n}$	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert}$, $K = n, n+1, \ldots$	$P_{x:\overline{n}\rvert} = \dfrac{A_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}$
h-Payment whole life insurance	$1\ v^{K+1}$ $1\ v^{K+1}$	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\rvert}$, $K = h, h+1, \ldots$	${}_hP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{h}\rvert}}$
h-Payment, n-year endowment insurance	$1\ v^{K+1}$ $1\ v^{K+1}$ $1\ v^{n}$	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\rvert}$, $K = h, \ldots, n-1$ $\ddot{a}_{\overline{h}\rvert}$, $K = n, n+1, \ldots$	${}_hP_{x:\overline{n}\rvert} = \dfrac{A_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{h}\rvert}}$
n-Year pure endowment	0 $1\ v^{n}$	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert}$, $K = n, n+1, \ldots$	$P_{x:\overline{n}\rvert}^{\ \ 1} = \dfrac{A_{x:\overline{n}\rvert}^{\ \ 1}}{\ddot{a}_{x:\overline{n}\rvert}}$
n-Year deferred whole life annuity	0 $\ddot{a}_{\overline{K+1-n}\rvert}v^{n}$	$\ddot{a}_{\overline{K+1}\rvert}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rvert}$, $K = n, n+1, \ldots$	$P({}_{n\rvert}\ddot{a}_x) = \dfrac{A_{x:\overline{n}\rvert}^{\ \ 1}\,\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\rvert}}$

$$\text{Var}(L) = \left(1 + \frac{P_{x:\overline{n}\rvert}}{d}\right)^2 [{}^2A_{x:\overline{n}\rvert} - (A_{x:\overline{n}\rvert})^2].$$

Formula (5.3.13) and the entry from the third row of Table 6.3.1 can be combined as follows:

$$d\,\ddot{a}_{x:\overline{n}\rvert} + A_{x:\overline{n}\rvert} = 1,$$

$$1 + \frac{P_{x:\overline{n}\rvert}}{d} = \frac{1}{d\,\ddot{a}_{x:\overline{n}\rvert}}.$$

Therefore, the variance we seek is

$$\frac{{}^2A_{x:\overline{n}\rvert} - (A_{x:\overline{n}\rvert})^2}{(d\,\ddot{a}_{x:\overline{n}\rvert})^2}. \tag{6.3.5}$$

▼

Example 6.3.3

Consider a 10,000 fully discrete whole life insurance. Let π denote an annual premium for this policy and $L(\pi)$ denote the loss-at-issue random variable for one such policy on the basis of the Illustrative Life Table, 6% interest and issue age 35.

a. Determine the premium, π_a, such that the distribution of $L(\pi_a)$ has mean 0. Calculate the variance of $L(\pi_a)$.
b. Approximate the smallest non-negative premium, π_b, such that the probability is less than 0.5 that the loss $L(\pi_b)$ is positive. Find the variance of $L(\pi_b)$.
c. Determine the premium, π_c, such that the probability of a positive total loss on 100 such independent policies is 0.05 by the normal approximation.

Solution:

a. By the equivalence principle, (6.1.3),

$$\pi_a = 10{,}000\, P_{35} = 10{,}000\, \frac{A_{35}}{\ddot{a}_{35}}$$

$$= \frac{1287.194}{15.39262}$$

$$= 83.62.$$

From (6.3.3)

$$\text{Var}[L(\pi_a)] = (10{,}000)^2\, \frac{{}^2A_{35} - (A_{35})^2}{(d\,\ddot{a}_{35})^2}$$

$$= 10^8\, \frac{0.0348843 - (0.1287194)^2}{[(0.06/1.06)(15.39262)]^2}$$

$$= \frac{1{,}831{,}562}{0.7591295}$$

$$= 2{,}412{,}713.$$

b. We want π_b such that

$$\Pr[L(\pi_b) > 0] < 0.5,$$

or in terms of curtate-future-lifetime, K,

$$\Pr(10{,}000 v^{K+1} - \pi_b\, \ddot{a}_{\overline{K+1}|} > 0) < 0.5.$$

From the Illustrative Life Table, ${}_{42}p_{35} = 0.5125101$ and ${}_{43}p_{35} = 0.4808964$. Therefore, if π_b is chosen so that

$$10{,}000 v^{43} - \pi_b\, \ddot{a}_{\overline{43}|} = 0,$$

then $\Pr[L(\pi_b) > 0)] = \Pr(K < 42) < 0.5$. Thus,

$$\pi_b = \frac{10{,}000}{\ddot{s}_{\overline{43}|}} = 50.31.$$

Using the fully discrete analogue of (6.2.6) we can write

$$\operatorname{Var}[L(\pi_b)] = (10{,}000)^2[{}^2A_{35} - (A_{35})^2]\left(1 + \frac{\pi_b}{10{,}000}\frac{1}{d}\right)^2$$

$$= (1{,}831{,}562)(1.18567)$$

$$= 2{,}171{,}630.$$

c. With a premium π_c, the loss on one policy is

$$L(\pi_c) = 10{,}000v^{K+1} - \pi_c\ddot{a}_{\overline{K+1|}} = \left(10{,}000 + \frac{\pi_c}{d}\right)v^{K+1} - \frac{\pi_c}{d},$$

and its expectation and variance are as follows:

$$\mathrm{E}[L(\pi_c)] = \left(10{,}000 + \frac{\pi_c}{d}\right)A_{35} - \frac{\pi_c}{d}$$

$$= (0.1287194)\left(10{,}000 + \frac{\pi_c}{d}\right) - \frac{\pi_c}{d}$$

and

$$\operatorname{Var}[L(\pi_c)] = \left(10{,}000 + \frac{\pi_c}{d}\right)^2[{}^2A_{35} - (A_{35})^2]$$

$$= \left(10{,}000 + \frac{\pi_c}{d}\right)^2(0.01831562).$$

For each of 100 such policies each loss $L_i(\pi_c)$ is distributed like $L(\pi_c)$, $i = 1, 2, \ldots, 100$ and

$$S = \sum_{i=1}^{100} L_i(\pi_c)$$

for the total loss on the portfolio. Then

$$\mathrm{E}[S] = 100\ \mathrm{E}[L(\pi_c)],$$

and, using the assumption of independent policies,

$$\operatorname{Var}(S) = 100\ \operatorname{Var}[L(\pi_c)].$$

To determine π_c so that $\Pr(S > 0) = 0.05$ by the normal approximation, we want

$$\frac{0 - \mathrm{E}[S]}{\sqrt{\operatorname{Var}(S)}} = 1.645,$$

$$10\left\{\frac{-\mathrm{E}[L(\pi_c)]}{\sqrt{\operatorname{Var}[L(\pi_c)]}}\right\} = 1.645,$$

$$10\left[\frac{-A_{35}[10{,}000 + (\pi_c/d)] + (\pi_c/d)}{[10{,}000 + (\pi_c/d)]\sqrt{{}^2A_{35} - (A_{35})^2}}\right] = 1.645.$$

Thus

$$\pi_c = 10{,}000\, d \left[\frac{(0.1645)\sqrt{{}^2A_{35} - (A_{35})^2} + A_{35}}{1 - (A_{35} + 0.1645\sqrt{{}^2A_{35} - (A_{35})^2})} \right]$$

$$= 100.66.$$

▼

The two identities, (5.3.7) and (5.3.13), can be used to derive relationships among discrete premiums. For example, starting with (5.3.7), we have for whole life insurances

$$\begin{aligned} d\,\ddot{a}_x + A_x &= 1, \\ d + P_x &= \frac{1}{\ddot{a}_x}, \\ P_x &= \frac{1}{\ddot{a}_x} - d \\ &= \frac{1 - d\,\ddot{a}_x}{\ddot{a}_x} \\ &= \frac{d\,A_x}{1 - A_x}. \end{aligned} \tag{6.3.6}$$

Starting with (5.3.13) we obtain a similar chain of equalities for n-year endowment insurances:

$$\begin{aligned} d\,\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|} &= 1, \\ d + P_{x:\overline{n}|} &= \frac{1}{\ddot{a}_{x:\overline{n}|}}, \\ P_{x:\overline{n}|} &= \frac{1}{\ddot{a}_{x:\overline{n}|}} - d \\ &= \frac{1 - d\,\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \\ &= \frac{d\,A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}. \end{aligned} \tag{6.3.7}$$

Example 6.3.4

Give interpretations in words of the following equations from the (6.3.6) set:

$$\frac{1}{\ddot{a}_x} = P_x + d \tag{6.3.8}$$

and

$$P_x = \frac{d\,A_x}{1 - A_x}. \tag{6.3.9}$$

Solution:

We will use the word equivalent to mean equal in terms of actuarial present value. For (6.3.8), first note that a unit now is equivalent to a life annuity of $\ddot{a}_x^{-1}$ payable at the beginning of each year while (x) survives. A unit now is also equivalent to interest-in-advance of d at the beginning of each year while (x) survives with the repayment of the unit at the end of the year of (x)'s death; that is, $1 = \ddot{a}_x / \ddot{a}_x = d\ddot{a}_x + A_x$. The repayment of the unit at the end of the year of death is, in turn, equivalent to a life annuity-due of P_x while (x) survives. Therefore, the unit now is equivalent to $P_x + d$ at the beginning of each year during the lifetime of (x). Then $\ddot{a}_x^{-1} = P_x + d$, for each side of the equality, represents the annual payment of a life annuity produced by a unit available now.

For (6.3.9), we consider an insured (x) who borrows the single benefit premium A_x for the purchase of a single-premium unit whole life insurance. The insured agrees to pay interest-in-advance in the amount of $d\,A_x$ on the loan at the beginning of each year during survival and to repay the A_x from the unit death benefit at the end of the year of death. In essence, the insured is paying an annual benefit premium of $d\,A_x$ for an insurance of amount $1 - A_x$. Then for a full unit of insurance, the annual benefit premium must be $d\,A_x/(1 - A_x)$. ▼

Similar interpretations exist for corresponding relationships involving endowment insurances as given in the second and fifth equalities in the (6.3.7) set. There is an analogy between (6.3.8), the corresponding formula involving endowment insurances,

$$\ddot{a}_{x:\overline{n}|}^{-1} = P_{x:\overline{n}|} + d,$$

and the interest-only formula

$$\ddot{a}_{\overline{n}|}^{-1} = \ddot{s}_{\overline{n}|}^{-1} + d.$$

Example 6.3.5

Prove and interpret the formula

$$P_{x:\overline{n}|} = {}_nP_x + P_{x:\overline{n}|}^{\ \ \ 1}(1 - A_{x+n}). \tag{6.3.10}$$

Solution:

The proof is completed using entries from Table 6.3.1:

$$P_{x:\overline{n}|}\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1},$$

$${}_nP_x\ddot{a}_{x:\overline{n}|} = A_x = A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1}\,A_{x+n}.$$

By subtraction,

$$(P_{x:\overline{n}|} - {}_nP_x)\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|}^{\ \ \ 1}(1 - A_{x+n}),$$

from which (6.3.10) follows.

The interpretation is that both $P_{x:\overline{n}|}$ and ${}_nP_x$ are payable during the survival of (x) to a maximum of n years. During these years, both insurances provide a death benefit of 1 payable at the end of the year of the death of (x). If (x) survives the n years, $P_{x:\overline{n}|}$ provides a maturity benefit of 1, while ${}_nP_x$ provides whole life insurance without further premiums, that is, an insurance with an actuarial present value of A_{x+n}. Hence, the difference $P_{x:\overline{n}|} - {}_nP_x$ is the level annual premium for a pure endowment of $1 - A_{x+n}$. ▼

In practice, life insurances are payable soon after death rather than at the end of the policy year of death, so there is a need for annual payment, semicontinuous benefit premiums. Such premiums, following the same order used in Tables 6.2.1 and 6.3.1, are denoted by $P(\bar{A}_x)$, $P(\bar{A}^{\,1}_{x:\overline{n}|})$, $P(\bar{A}_{x:\overline{n}|})$, ${}_hP(\bar{A}_x)$, and ${}_hP(\bar{A}_{x:\overline{n}|})$. There is no need for a semicontinuous annual premium n-year pure endowment since no death benefit is involved. The equivalence principle can be applied to produce formulas like those in Table 6.3.1, but with the general symbol A replaced by $\bar{A}$. For example,

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x}. \tag{6.3.11}$$

We observe that the notation for this premium is not $\bar{P}_x$, the annual premium payable continuously for a unit whole life insurance benefit payable at the end of the year of death and equal to $A_x/\bar{a}_x$. If a uniform distribution of deaths is assumed over each year of age, we can use the notations of Section 4.4 to write

$$P(\bar{A}_x) = \frac{i}{\delta}\frac{A_x}{\ddot{a}_x} = \frac{i}{\delta}P_x,$$

$$P(\bar{A}^{\,1}_{x:\overline{n}|}) = \frac{i}{\delta}P^{\,1}_{x:\overline{n}|},$$

and

$$P(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}P^{\,1}_{x:\overline{n}|} + P_{x:\overline{n}|}^{\ \ \,1}. \tag{6.3.12}$$

6.4 True *m*-thly Payment Premiums

If premiums are payable m times a policy year, rather than annually, with no adjustment in the death benefit, the resulting premiums are called ***true fractional premiums.*** Thus $P^{(m)}_x$ denotes the ***true level annual benefit premium,*** payable in m-thly installments at the beginning of each m-thly period, for a unit whole life insurance payable at the end of the year of death. The symbol $P^{(m)}(\bar{A}_x)$ would have the same interpretation except that the insurance is payable at the moment of death. Typically, m is 2, 4, or 12.

The development in this section stresses the payment of insurance benefits at the end of the policy year of death. Table 6.4.1 specifies the symbols and formulas for true fractional premiums for common life insurances. The premium formulas can be obtained by applying the equivalence principle.

True Fractional Benefit Premiums*

Plan	Payment of Proceeds: At End of Policy Year	Payment of Proceeds: At Moment of Death
Whole life insurance	$P_x^{(m)} = \dfrac{A_x}{\ddot{a}_x^{(m)}}$	$P^{(m)}(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}_x^{(m)}}$
n-Year term insurance	$P_{x:\overline{n}\vert}^{1\,(m)} = \dfrac{A_{x:\overline{n}\vert}^{1}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$	$P^{(m)}(\bar{A}_{x:\overline{n}\vert}^{1}) = \dfrac{\bar{A}_{x:\overline{n}\vert}^{1}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$
n-Year endowment insurance	$P_{x:\overline{n}\vert}^{(m)} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$	$P^{(m)}(\bar{A}_{x:\overline{n}\vert}) = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{n}\vert}^{(m)}}$
h-Payment years, whole life insurance	${}_hP_x^{(m)} = \dfrac{A_x}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$	${}_hP^{(m)}(\bar{A}_x) = \dfrac{\bar{A}_x}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$
h-Payment years, n-year endowment insurance	${}_hP_{x:\overline{n}\vert}^{(m)} = \dfrac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$	${}_hP^{(m)}(\bar{A}_{x:\overline{n}\vert}) = \dfrac{\bar{A}_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}$

*The actual amount of each fractional premium, payable m times each policy year, during the premium paying period and the survival of (x), is $P^{(m)}/m$. Note that here h refers to the number of payment years, not to the number of payments.

In some applications it is useful to write the m-thly payment premium as a multiple of the annual premium. This will be illustrated for ${}_hP_{x:\overline{n}\vert}^{(m)}$, the premium for a rather general insurance. The resulting formula can be modified to produce premium formulas for other common insurances. From the last row of Table 6.4.1 we have

$$ {}_hP_{x:\overline{n}\vert}^{(m)} = \frac{A_{x:\overline{n}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}. \tag{6.4.1}$$

Since

$$ A_{x:\overline{n}\vert} = {}_hP_{x:\overline{n}\vert}\, \ddot{a}_{x:\overline{h}\vert}, $$

(6.4.1) can be rearranged as

$$ {}_hP_{x:\overline{n}\vert}^{(m)} = \frac{{}_hP_{x:\overline{n}\vert}\, \ddot{a}_{x:\overline{h}\vert}}{\ddot{a}_{x:\overline{h}\vert}^{(m)}}. \tag{6.4.2}$$

Formula (6.4.2) is used in the next chapter; it expresses the m-thly payment premium as equal to the corresponding annual payment premium times a ratio of annuity values. This ratio can be arranged in various ways each corresponding to a different formula used to express the relationship between $\ddot{a}_{x:\overline{h}\vert}^{(m)}$ and $\ddot{a}_{x:\overline{h}\vert}$ (see Exercise 6.14).

Example 6.4.1

a. Calculate the level annual benefit premium payable in semiannual installments for a 10,000, 20-year endowment insurance with proceeds paid at the end of the policy year of death (discrete) issued to (50), on the basis of the Illustrative Life Table with interest at the effective annual rate of 6%.

b. Determine the corresponding premium with proceeds paid at the moment of death (semicontinuous).

For both parts, assume a uniform distribution of deaths in each year of age.

Solution:

a. We require 10,000 $P^{(2)}_{50:\overline{20}|}$. As preliminary steps we calculate

$$d = 0.056603774,$$

$$i^{(2)} = 0.059126028,$$

$$d^{(2)} = 0.057428275,$$

$$\ddot{a}^{(2)}_{\overline{1}|} = 0.98564294,$$

$$s^{(2)}_{\overline{1}|} = 1.01478151,$$

$$\alpha(2) = s^{(2)}_{\overline{1}|}\ddot{a}^{(2)}_{\overline{1}|} = 1.0002122,$$

$$\beta(2) = \frac{s^{(2)}_{\overline{1}|} - 1}{d^{(2)}} = 0.25739081,$$

and the following actuarial present values:

$$\ddot{a}_{50:\overline{20}|} = 11.291832,$$

$$A^{\ 1}_{50:\overline{20}|} = 0.13036536,$$

$$P^{\ 1}_{50:\overline{20}|} = 0.01154510,$$

$${}_{20}E_{50} = 0.23047353,$$

$$A_{50:\overline{20}|} = 0.36083889,$$

$$P_{50:\overline{20}|} = 0.03195574.$$

Then, under the assumption of a uniform distribution of deaths for each year of age, the required premium can be calculated by use of (6.4.1), with $x = 50$, $n = 20$, $h = 20$, and $m = 2$. For this purpose, we calculate

$$\ddot{a}^{(2)}_{50:\overline{20}|} = \alpha(2)\ddot{a}_{50:\overline{20}|} - \beta(2)(1 - {}_{20}E_{50}) = 11.096159,$$

and then

$$10{,}000\, P^{(2)}_{50:\overline{20}|} = 325.19.$$

b. The corresponding semicontinuous premium can be obtained by multiplying the values in (a) by the ratio

$$\frac{P(\bar{A}_{50:\overline{20}|})}{P_{50:\overline{20}|}} = \frac{\bar{A}_{50:\overline{20}|}}{A_{50:\overline{20}|}}.$$

Under the uniform distribution of deaths assumption this ratio is

$$\frac{(i/\delta)\,P^{\;1}_{50:\overline{20}|} + P_{50:\overline{20}|}^{\quad\;\;1}}{P_{50:\overline{20}|}}, \tag{6.4.3}$$

and the result is

$$10{,}000\,P^{(2)}(\bar{A}_{50:\overline{20}|}) = 328.68.$$

▼

6.5 Apportionable Premiums

A second type of fractional premium is the ***apportionable premium.*** Here, at death, a refund is made of a portion of the premium related to the length of time between the time of death and the time of the next scheduled premium payment. In practice this may be on a pro rata basis without interest. In this section we consider interest and view the sequence of m-thly premiums as an apportionable life annuity-due in the sense of Section 5.5. The symbols used to denote these level apportionable annual benefit premiums payable m-thly are like the symbols for true fractional premiums on the semicontinuous basis. They differ in that the superscript m is enclosed in braces rather than parentheses, for example, $P^{\{m\}}(\bar{A}_x)$. In view of the premium refund feature, it is natural to assume that the death benefit is payable at the moment of death.

Again, we use an h-payment years, n-year endowment insurance to illustrate the development of formulas for apportionable premiums paid m-thly. The equivalence principle leads to the formulas

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|})\,\ddot{a}^{\{m\}}_{x:\overline{h}|} = \bar{A}_{x:\overline{n}|}$$

and

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}^{\{m\}}_{x:\overline{h}|}}. \tag{6.5.1}$$

Utilizing the temporary annuity version of (5.5.4), we obtain

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{(\delta/d^{(m)})\bar{a}_{x:\overline{h}|}} = \frac{d^{(m)}}{\delta}\,{}_h\bar{P}(\bar{A}_{x:\overline{n}|}). \tag{6.5.2}$$

This implies that the m-thly installment is

$$\frac{1}{m}\,{}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\,\frac{1-v^{1/m}}{\delta} = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{1/m}|}, \tag{6.5.3}$$

and in particular, for $m = 1$,

$${}_hP^{\{1\}}(\bar{A}_{x:\overline{n}|}) = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{1}|}. \tag{6.5.4}$$

Formulas (6.5.3) and (6.5.4) demonstrate that these apportionable premiums are equivalent to fully continuous premiums, discounted for interest to the start of each payment period. Similar formulas exist for other types of insurance. For example, by letting h and $n \to \infty$, (6.5.4) becomes

$$P^{\{1\}}(\bar{A}_x) = \bar{P}(\bar{A}_x)\,\bar{a}_{\overline{1}|}. \tag{6.5.5}$$

The apportionable benefit premium $P^{\{1\}}(\bar{A}_x)$ and the semicontinuous benefit premium $P(\bar{A}_x)$ are both payable annually at the beginning of each year while (x) survives. Each insurance provides a unit at the death of (x). The two insurances differ only in respect to the refund provided by $P^{\{1\}}(\bar{A}_x)$. Thus, the difference

$$P^{\{1\}}(\bar{A}_x) - P(\bar{A}_x) \tag{6.5.6}$$

is a level annual benefit premium paid at the beginning of each year for the refund-of-premium feature. We verify this assertion about the expression in (6.5.6) in the following analysis.

From (5.5.1), we note that the random variable for the present value of the refund-of-premium feature is

$$\frac{P^{\{1\}}(\bar{A}_x)\,v^T\,\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}$$

where K and T are defined as in Chapter 3. The actuarial present value for this feature is

$$\bar{A}_x^{PR} = P^{\{1\}}(\bar{A}_x)\,\mathrm{E}\left[v^T\,\frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}\right].$$

Using (6.5.5) we obtain

$$\begin{aligned}\bar{A}_x^{PR} &= \bar{P}(\bar{A}_x)\,\mathrm{E}\left[\frac{v^T - v^{K+1}}{\delta}\right]\\ &= \bar{P}(\bar{A}_x)\left(\frac{\bar{A}_x - A_x}{\delta}\right).\end{aligned} \tag{6.5.7}$$

The level annual benefit premium is then, by the equivalence principle,

$$P(\bar{A}_x^{PR}) = \frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta\ddot{a}_x}. \tag{6.5.8}$$

Formula (6.5.7) has the following interpretation: The actuarial present value of the refund feature is the difference between the value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ per year beginning at the death of (x), and the value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ payable from the end of the year of death of (x).

We return now to (6.5.6) where, by (6.5.5), we have

$$\begin{aligned}
P^{\{1\}}(\bar{A}_x) - P(\bar{A}_x) &= \bar{P}(\bar{A}_x)\frac{d}{\delta} - \frac{\bar{A}_x}{\ddot{a}_x} \\
&= \bar{P}(\bar{A}_x)\left(\frac{d}{\delta} - \frac{\bar{a}_x}{\ddot{a}_x}\right) \\
&= \bar{P}(\bar{A}_x)\frac{d\,\ddot{a}_x - \delta\,\bar{a}_x}{\delta\,\ddot{a}_x} \\
&= \bar{P}(\bar{A}_x)\frac{\bar{A}_x - A_x}{\delta\ddot{a}_x} \\
&= P(\bar{A}_x^{PR}),
\end{aligned} \tag{6.5.9}$$

as obtained in (6.5.8). This confirms our assertion about (6.5.6).

This analysis can be extended to m-thly payment premiums and to other life insurance in addition to whole life. In general,

$$P^{\{m\}}(\bar{A}) - P^{(m)}(\bar{A})$$

is an m-thly payment premium for the refund feature.

Example 6.5.1

If the policy of Example 6.4.1(b) is to have apportionable premiums, what increase occurs in the annual benefit premium?

Solution:

The apportionable annual benefit premium per unit of insurance is given by (6.5.2),

$$P^{\{2\}}(\bar{A}_{50:\overline{20}|}) = \bar{P}(\bar{A}_{50:\overline{20}|})\frac{d^{(2)}}{\delta} = \frac{\bar{A}_{50:\overline{20}|}}{\bar{a}_{50:\overline{20}|}}\frac{d^{(2)}}{\delta}.$$

Under the assumption of a uniform distribution of deaths in each age interval, this becomes

$$\begin{aligned}
&= \frac{(i/\delta)A^{1}_{50:\overline{20}|} + A_{50:\overset{\;1}{\overline{20}|}}}{\alpha(\infty)\ddot{a}_{50:\overline{20}|} - \beta(\infty)(1 - {}_{20}E_{50})}\frac{d^{(2)}}{\delta} \\
&= \frac{(i/\delta)\,P^{1}_{50:\overline{20}|} + P_{50:\overset{\;1}{\overline{20}|}}}{\alpha(\infty) - \beta(\infty)(P^{1}_{50:\overline{20}|} + d)}\frac{d^{(2)}}{\delta}.
\end{aligned}$$

Here $\alpha(\infty) = \bar{s}_{\overline{1}|}\bar{a}_{\overline{1}|} = id/\delta^2 = 1.00028$, $\beta(\infty) = (\bar{s}_{\overline{1}|} - 1)/\delta = 0.50985$. Using other values available in Example 6.4.1 we find

$$10{,}000\;P^{\{2\}}(\bar{A}_{50:\overline{20}|}) = 329.69.$$

Then the increase in annual premium is

$$10{,}000[P^{(2)}(\bar{A}_{50:\overline{20}|}) - P^{(2)}(\bar{A}_{50:\overline{20}|})] = 1.01,$$

which is the annual benefit premium payable semiannually for the refund feature. ▼

6.6 Accumulation-Type Benefits

The analysis in this section is in terms of annual premiums for insurances payable at the end of the year of death. An analogous development is possible for fully continuous premiums and, with some adjustment, for semicontinuous premiums. We first seek the actuarial present value for an n-year term insurance on (x) for which the sum insured, in case death occurs in year $k+1$, is $\ddot{s}_{\overline{k+1}|j}$. The present-value random variable of this benefit, at policy issue, is

$$W = \begin{cases} v^{K+1}\ddot{s}_{\overline{K+1}|j} = \dfrac{1}{d_{(j)}}[v^{K+1}(1+j)^{K+1} - v^{K+1}] & 0 \le K < n \\ 0 & K \ge n \end{cases}$$

where the insurer's present values are computed at interest rate i and $d_{(j)}$ is the discount rate equivalent to interest rate j. The actuarial present value is

$$\mathrm{E}[W] = \frac{A'^{\,1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|}}{d_{(j)}} \tag{6.6.1}$$

where $A'^{\,1}_{x:\overline{n}|}$ is calculated at the rate of interest $i' = (i-j)/(1+j)$.

If $i = j$, then $i' = 0$, and the actuarial present value is

$$\frac{{}_nq_x - A^{1}_{x:\overline{n}|}}{d} = \frac{1 - {}_np_x - A_{x:\overline{n}|} + v^n\,{}_np_x}{d}$$

$$= \ddot{a}_{x:\overline{n}|} - {}_np_x\,\ddot{a}_{\overline{n}|}$$

$$= \ddot{a}_{x:\overline{n}|} - {}_nE_x\,\ddot{s}_{\overline{n}|}. \tag{6.6.2}$$

Formula (6.6.2) indicates that, when $j = i$, this special term insurance is equivalent to an n-year life annuity-due except for the event that (x) survives the n years. Then the term insurance would provide a benefit of zero, whereas the life annuity payments, given survival for n years, would have value $\ddot{s}_{\overline{n}|}$ at time n.

Now let us consider the situation where (x) has the choice of purchasing an n-year unit endowment insurance with an annual premium of $P_{x:\overline{n}|}$ or of establishing a savings fund with deposits of $1/\ddot{s}_{\overline{n}|}$ at the beginning of each of n years and purchasing a special decreasing term insurance. The special insurance will provide, in the event of death in year $k+1$, the difference,

$$1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} \qquad k = 0, 1, 2, \ldots, n-1,$$

between the unit benefit under the endowment insurance and the accumulation in the savings fund. We suppose further that the same interest rate i is applicable in valuing all these transactions. The same benefits are provided by the endowment insurance and by the combination of the special term insurance and the savings fund. Therefore one would anticipate that

(the annual benefit premium $P_{x:\overline{n}|}$ for the endowment insurance) = (the annual benefit premium for the special term insurance) + (the annual savings fund deposit $1/\ddot{s}_{\overline{n}|}$).

To verify this conjecture, we consider the present-value random variable for the special decreasing term insurance,

$$\tilde{W} = \begin{cases} v^{K+1}\left(1 - \dfrac{\ddot{s}_{\overline{K+1}|}}{\ddot{s}_{\overline{n}|}}\right) = v^{K+1} - \dfrac{\ddot{a}_{\overline{K+1}|}}{\ddot{s}_{\overline{n}|}} & 0 \le K < n \\ 0 & K \ge n. \end{cases} \tag{6.6.3}$$

The actuarial present value of $\tilde{W}$ is denoted by $\tilde{A}^{1}_{x:\overline{n}|}$ and given by

$$\begin{aligned} \tilde{A}^{1}_{x:\overline{n}|} &= \mathrm{E}[\tilde{W}] \\ &= A^{1}_{x:\overline{n}|} - \frac{\ddot{a}_{x:\overline{n}|} - {}_{n}p_{x}\,\ddot{a}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}} \\ &= A^{1}_{x:\overline{n}|} - \frac{\ddot{a}_{x:\overline{n}|} - {}_{n}E_{x}\,\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}} \end{aligned}$$

[see (6.6.2)].

The annual benefit premium for the special term insurance is therefore

$$\begin{aligned} \tilde{P}^{1}_{x:\overline{n}|} = \frac{\tilde{A}^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} &= P^{1}_{x:\overline{n}|} - \frac{1}{\ddot{s}_{\overline{n}|}} + P_{x:\overline{n}|}^{\;\;1} \\ &= P_{x:\overline{n}|} - \frac{1}{\ddot{s}_{\overline{n}|}}, \end{aligned}$$

and then

$$P_{x:\overline{n}|} = \tilde{P}^{1}_{x:\overline{n}|} + \frac{1}{\ddot{s}_{\overline{n}|}}. \tag{6.6.4}$$

We have already seen that

$$P_{x:\overline{n}|} = P^{1}_{x:\overline{n}|} + P_{x:\overline{n}|}^{\;\;1},$$

and now (6.6.4) provides an alternative decomposition of $P_{x:\overline{n}|}$. The components are the annual premium for the special term insurance and the annual savings fund deposits, $1/\ddot{s}_{\overline{n}|}$, which accumulate to one at the end of n years.

Example 6.6.1

Derive formulas for the annual benefit premium for a 5,000, 20-year term insurance on (x) providing, in case of death within the 20 years, the return of the annual benefit premiums paid:

a. Without interest
b. Accumulated at the interest rate used in the determination of premiums.
   In each case, the return of premiums is in addition to the 5,000 sum insured and benefit payments are made at the end of the year of death.

Solution:

a. Let π_a be the benefit premium. Then

$$\pi_a\, \ddot{a}_{x:\overline{20}|} = 5{,}000\, A^{\,1}_{x:\overline{20}|} + \pi_a (IA)^{\,1}_{x:\overline{20}|}$$

and

$$\pi_a = 5{,}000\, \frac{A^{\,1}_{x:\overline{20}|}}{\ddot{a}_{x:\overline{20}|} - (IA)^{\,1}_{x:\overline{20}|}}.$$

b. Let π_b be the benefit premium. We use (6.6.2) to obtain

$$\pi_b\, \ddot{a}_{x:\overline{20}|} = 5{,}000\, A^{\,1}_{x:\overline{20}|} + \pi_b(\ddot{a}_{x:\overline{20}|} - {}_{20}E_x\, \ddot{s}_{\overline{20}|}),$$

$$\pi_b\, {}_{20}E_x \ddot{s}_{\overline{20}|} = 5{,}000\, A^{\,1}_{x:\overline{20}|},$$

$$\pi_b = 5{,}000\, \frac{A^{\,1}_{x:\overline{20}|}}{{}_{20}E_x\, \ddot{s}_{\overline{20}|}}$$

$$= 5{,}000\, \frac{A^{\,1}_{x:\overline{20}|}}{{}_{20}p_x\, \ddot{a}_{\overline{20}|}}.$$

In practice, annual contract premiums would be refunded, and the formulas would take this into account. ▼

Example 6.6.2

A deferred annuity issued to (x) for an annual income of 1 commencing at age $x + n$ is to be paid for by level annual benefit premiums during the deferral period. The benefit for death during the premium paying period is the return of annual benefit premiums accumulated with interest at the rate used for the premium. Assuming the death benefit is paid at the end of the year of death, determine the annual benefit premium.

Solution:

Equating the actuarial present value of the annual benefit premiums, π, to the actuarial present value of the benefits, we have

$$\pi \ddot{a}_{x:\overline{n}|} = {}_nE_x\, \ddot{a}_{x+n} + \pi(\ddot{a}_{x:\overline{n}|} - {}_nE_x\, \ddot{s}_{\overline{n}|})$$

where the second term on the right-hand side comes from (6.6.2). Solving for π yields

$$\pi = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$ ▼

6.7 Notes and References

Lukacs (1948) provides a survey of the development of the equivalence principle. Premiums derived by an application of the equivalence principle are often called actuarial premiums in the literature of the economics of uncertainty. Gerber (1976, 1979) discussed exponential premiums and reserves; these were illustrated in Example 6.1.1 under Principle III. Fractional premiums of various types are important in practice. Scher (1974) has discussed developments in this area, namely, the relations among fully continuous, apportionable, and semicontinuous premiums. The decomposition of an endowment insurance premium appeared in a paper by Linton (1919).

Exercises

Section 6.1

6.1. Calculate the expectation and the variance of the present value of the financial loss for the insurance in Example 6.1.1, when the premium is determined by Principle I.

6.2. Verify that the exponential premium (with $\alpha = 0.1$) for the insurance in Example 6.1.1, modified so that the benefit amount is 10, is 3.45917. (Note that this is roughly 11.3 times as large as the exponential premium for a benefit amount of 1 found in Example 6.1.1.)

6.3. Using the assumptions of Example 6.1.1, determine the annual premium that maximizes the expected utility of an insurer with initial wealth $w = 10$ and utility function $u(x) = x - 0.01x^2$, $x < 50$. [Hint: Use (1.3.6), $w - 0.01w^2 = \mathrm{E}[(w - L) - 0.01(w - L)^2]$.]

Section 6.2

6.4. A fully continuous whole life insurance with unit benefit has a level premium. The time-until-death random variable, $T(x)$, has an exponential distribution with $\mathrm{E}[T(x)] = 50$ and the force of interest is $\delta = 0.06$.
a. If the principle of equivalence is used, find the benefit premium rate.
b. Find the premium rate if it is to be such that $\Pr(L > 0) = 0.50$.
c. Repeat part (b) if the force of interest, δ, equals 0.

6.5. If the force of mortality strictly increases with age, show that $\bar{P}(\bar{A}_x) > \mu_x(0)$. [Hint: Show that $\bar{P}(\bar{A}_x)$ is a weighted average of $\mu_x(t)$, $t > 0$.]

6.6. Following Example 6.2.1, derive a general expression for

$$\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta \bar{a}_x)^2}$$

where $\mu_x(t) = \mu$ and δ is the force of interest for $t > 0$.

6.7. If $\delta = 0$, show that

$$\bar{P}(\bar{A}_x) = \frac{1}{\mathring{e}_x}.$$

6.8. Prove that the variance of the loss associated with a single premium whole life insurance is less than the variance of the loss associated with an annual premium whole life insurance. Assume immediate payment of claims on death and continuous payment of benefit premiums.

6.9. Show that

$$\left(1 + \frac{d\bar{a}_x}{dx}\right) \bar{P}(\bar{A}_x) - \frac{d\bar{A}_x}{dx} = \mu(x).$$

Section 6.3

6.10. On the basis of the Illustrative Life Table and an interest rate of 6%, calculate values for the annual premiums in the following table. Note any patterns of inequalities that appear in the matrix of results.

Fully Continuous	**Semicontinuous**	**Fully Discrete**
$\bar{P}(\bar{A}_{35:\overline{10}\vert})$	$P(\bar{A}_{35:\overline{10}\vert})$	$P_{35:\overline{10}\vert}$
$\bar{P}(\bar{A}_{35:\overline{30}\vert})$	$P(\bar{A}_{35:\overline{30}\vert})$	$P_{35:\overline{30}\vert}$
$\bar{P}(\bar{A}_{35:\overline{60}\vert})$	$P(\bar{A}_{35:\overline{60}\vert})$	$P_{35:\overline{60}\vert}$
$\bar{P}(\bar{A}_{35})$	$P(\bar{A}_{35})$	P_{35}
$\bar{P}(\bar{A}^{1}_{35:\overline{30}\vert})$	$P(\bar{A}^{1}_{35:\overline{30}\vert})$	$P^{1}_{35:\overline{30}\vert}$
$\bar{P}(\bar{A}^{1}_{35:\overline{10}\vert})$	$P(\bar{A}^{1}_{35:\overline{10}\vert})$	$P^{1}_{35:\overline{10}\vert}$

6.11. Show that

$${}_{20}P^{1}_{x:\overline{30}\vert} - P^{1}_{x:\overline{20}\vert} = {}_{20}P({}_{20|10}A_x).$$

6.12. Generalize Example 6.3.1 where

$${}_{k|}q_x = (1 - r)r^k \qquad k = 0, 1, 2, \ldots;$$

that is, derive expressions in terms of r and i for A_x, $\ddot{a}_x$, P_x, and $[{}^2A_x - (A_x)^2]/(d\,\ddot{a}_x)^2$.

Section 6.4

6.13. Using the information given in Example 6.4.1, calculate the value $P^{(2)}_{50}$.

6.14. Using various formulas for $\ddot{a}^{(m)}_{x:\overline{n}\vert}$ under the assumption of a uniform distribution of deaths in each year of age, show that the ratio

$$\frac{\ddot{a}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$$

in (6.4.2) can be expressed as the reciprocal of

a. $\ddot{a}_{\overline{1}|}^{(m)} - \beta(m)P_{x:\overline{h}|}^{1}$

b. $\alpha(m) - \beta(m)(P_{x:\overline{h}|}^{1} + d)$

c. $1 - \dfrac{m-1}{2m}(P_{x:\overline{h}|}^{1} + d)$.

6.15. Refer to Example 6.4.1(b) and directly calculate

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}^{(2)}}$$

using the Illustrative Life Table for the actuarial present values in the numerator and the denominator.

6.16. If

$$\frac{P_{x:\overline{20}|}^{1\,(12)}}{P_{x:\overline{20}|}^{1}} = 1.032$$

and $P_{x:\overline{20}|} = 0.040$, what is the value of $P_{x:\overline{20}|}^{(12)}$?

Section 6.5

6.17. Arrange in order of magnitude and indicate your reasoning:
$P^{(2)}(\bar{A}_{40:\overline{25}|})$, $\bar{P}(\bar{A}_{40:\overline{25}|})$, $P^{\{4\}}(\bar{A}_{40:\overline{25}|})$, $P(\bar{A}_{40:\overline{25}|})$, $P^{\{12\}}(\bar{A}_{40:\overline{25}|})$.

6.18. Given that

$$\frac{d}{d^{(12)}} = \frac{99}{100},$$

evaluate

$$\frac{P^{\{12\}}(\bar{A}_x)}{P^{\{1\}}(\bar{A}_x)}.$$

6.19. If $\bar{P}(\bar{A}_x) = 0.03$, and if interest is at the effective annual rate of 5%, calculate the semiannual benefit premium for a 50,000 whole life insurance on (x) where premiums are apportionable.

6.20. Show that

$$P^{\{m\}}(\bar{A}_{x:\overline{n}|}) - P^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{P}(\bar{A}_{x:\overline{n}|})\left(\frac{\bar{A}_{x:\overline{n}|} - A_{x:\overline{n}|}^{(m)}}{\delta\,\ddot{a}_{x:\overline{n}|}^{(m)}}\right)$$

$$= \bar{P}(\bar{A}_{x:\overline{n}|})\left(\frac{\bar{A}_{x:\overline{n}|}^{1} - A_{x:\overline{n}|}^{1\,(m)}}{\delta\,\ddot{a}_{x:\overline{n}|}^{(m)}}\right).$$

Section 6.6

6.21. Express

$$1 - \frac{\ddot{s}_{\overline{20}|}}{\ddot{s}_{45:\overline{20}|}}$$

as an annual premium. Interpret your result.

6.22. On the basis of the Illustrative Life Table and an interest rate of 6%, calculate the components of the two decompositions

a. $1{,}000\ P_{50:\overline{20}|} = 1{,}000(P^{1}_{50:\overline{20}|} + P_{50:\overset{\ 1}{\overline{20}|}})$

b. $1{,}000\ P_{50:\overline{20}|} = 1{,}000\left(\tilde{P}^{1}_{50:\overline{20}|} + \dfrac{1}{\ddot{s}_{\overline{20}|}}\right).$

6.23. Consider the continuous random variable analogue of (6.6.3),

$$\tilde{\bar{W}} = \begin{cases} v^{T}\left(1 - \dfrac{\bar{s}_{\overline{T}|}}{\bar{s}_{\overline{n}|}}\right) & 0 \le T < n \\ 0 & T \ge n. \end{cases}$$

The loss,

$$L = \tilde{\bar{W}} - \tilde{\bar{A}}^{1}_{x:\overline{n}|},$$

can be used with the equivalence principle to determine $\tilde{\bar{A}}^{1}_{x:\overline{n}|}$, the single benefit premium for this special policy. Show that

a. $\tilde{\bar{A}}^{1}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} - \dfrac{\bar{a}_{x:\overline{n}|} - {}_{n}p_{x}\, \bar{a}_{\overline{n}|}}{\bar{s}_{\overline{n}|}}$

b. $\mathrm{E}[\tilde{\bar{W}}^{2}] = \dfrac{(1 + i)^{2n}\ {}^{2}\bar{A}^{1}_{x:\overline{n}|} - 2(1 + i)^{n}\ \bar{A}^{1}_{x:\overline{n}|} + (1 - {}_{n}p_{x})}{[(1 + i)^{n} - 1]^{2}}.$

Miscellaneous

6.24. Express

$$A_{40}\ P_{40:\overline{25}|} + (1 - A_{40})P_{40}$$

as an annual benefit premium. Interpret your result.

6.25. a. Show that

$$\frac{1}{\ddot{a}_{65:\overline{10}|}} - \frac{1}{\ddot{s}_{65:\overline{10}|}} = P^{1}_{65:\overline{10}|} + d.$$

b. What is the corresponding formula for

$$\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}?$$

c. Show that the amount of annual income provided by a single benefit premium of 100,000 where

- The income is payable at the beginning of each month while (65) survives during the next 10 years, and

- The single premium is returned at the end of 10 years if (65) reaches age 75,

is given by

$$100{,}000\left(\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}\right) = 100{,}000(\beta)$$

where (β) denotes the answer to part (b) of this exercise.

6.26. An insurance issued to (35) with level premiums to age 65 provides
- 100,000 in case the insured survives to age 65, and
- The return of the annual contract premiums with interest at the valuation rate to the end of the year of death if the insured dies before age 65.

If the annual contract premium G is 1.1π where π is the annual benefit premium, write an expression for π.

6.27. If ${}_{15}P_{45} = 0.038$, $P_{45:\overline{15}|} = 0.056$, and $A_{60} = 0.625$, calculate $P^{\;1}_{45:\overline{15}|}$.

6.28. A 20-payment life policy is designed to return, in the event of death, 10,000 plus all contract premiums without interest. The return-of-premium feature applies both during the premium paying period and after. Premiums are annual and death claims are paid at the end of the year of death. For a policy issued to (x), the annual contract premium is to be 110% of the benefit premium plus 25. Express in terms of actuarial present-value symbols the annual contract premium.

6.29. Express in terms of actuarial present-value symbols the initial annual benefit premium for a whole life insurance issued to (25), subject to the following provisions:
- The face amount is to be one for the first 10 years and two thereafter.
- Each premium during the first 10 years is 1/2 of each premium payable thereafter.
- Premiums are payable annually to age 65.
- Claims are paid at the end of the year of death.

6.30. Let L_1 be the insurer's loss on a unit of whole life insurance, issued to (x) on a fully continuous basis. Let L_2 be the loss to (x) on a continuous life annuity purchased for a single premium of one. Show that $L_1 \equiv L_2$ and give an explanation in words.

6.31. An ordinary life contract for a unit amount on a fully discrete basis is issued to a person age x with an annual premium of 0.048. Assume $d = 0.06$, $A_x = 0.4$, and ${}^2A_x = 0.2$. Let L be the insurer's loss function at issue of this policy.

a. Calculate $\mathrm{E}[L]$.

b. Calculate $\mathrm{Var}(L)$.

c. Consider a portfolio of 100 policies of this type with face amounts given below.

Face Amount	Number of Policies
1	80
4	20

Assume the losses are independent and use a normal approximation to calculate the probability that the present value of gains for the portfolio will exceed 20.

6.32. Express, in terms of actuarial present-value symbols, the initial annual benefit premium for a unit of whole life insurance for (x) if after 5 years the annual benefit premium is double that payable during the first 5 years. Assume that death claims are made at the moment of death.

6.33. Repeat Exercise 6.20 for h-payment whole life insurance.

6.34. The function $l(t)$ is given by (6.2.1).

a. Establish that $l''(t) \geq 0$.

b. Adapt Jensen's inequality from Section 1.3 to establish that if $\bar{P} = \bar{P}(\bar{A}_x)$, then $\bar{P}(\bar{A}_x) \geq v^{\mathring{e}_x} / \bar{a}_{\overline{\mathring{e}_x}|}$.

6.35. If $T(x)$ has an exponential distribution with parameter μ,

a. Exhibit the p.d.f. of L as shown in (6.2.12)

b. Show that $\mathrm{E}[L] = (\mu - \bar{P}) / (\mu + \delta)$.

c. Use part (b) to confirm that $\mathrm{E}[L] = 0$ when $\bar{P} = \bar{P}(\bar{A}_x)$.

6.36. Use the assumptions of Exercise 6.35, with $\mu = 0.03$ and $\delta = 0.06$.

a. Evaluate $\Pr(L \leq 0)$ when $\bar{P} = \bar{P}(\bar{A}_x)$.

b. Determine $\bar{P}$ so that $\Pr(L > 0) = 0.5$.

7

BENEFIT RESERVES

7.1 Introduction

In Chapter 6 we introduced several principles that can be used for the determination of benefit premiums. The equivalence principle was used extensively in our discussion in Chapter 6. By it, an equivalence relation is established on the date a long-term contract is entered into by two parties who agree to exchange a set of payments. For example, under an amortized loan, a borrower may pay a series of equal monthly payments equivalent to the single payment by a lender at the date of the loan. An insured may pay a series of benefit premiums to an insurer equivalent, at the date of policy issue, to the sum to be paid on the death of the insured, or on survival of the insured to the maturity date. An individual may purchase a deferred life annuity by means of level premiums payable to an annuity organization equivalent, at the date of contract agreement, to monthly payments by the annuity organization to the individual when that person survives beyond a specified age. Equivalence in the loan example is in terms of present value, whereas in the insurance and annuity examples it is an equivalence between two actuarial present values.

After a period of time, however, there will no longer be an equivalence between the future financial obligations of the two parties. A borrower may have payments remaining to be made, whereas the lending organization has already performed its responsibilities. In other settings both parties may still have obligations. The insured may still be required to pay further benefit premiums, whereas the insurer has the duty to pay the face amount on maturity or the death of the insured. In our deferred annuity example, the individual may have completed the payments, whereas the annuity organization still has monthly remunerations to make.

In this chapter we study payments in time periods beyond the date of initiation. For this, a balancing item is required, and this item is a liability for one of the parties and an asset for the other. In the loan case, the balancing item is the outstanding principal, an asset for the lender and a liability for the borrower. In the other two cases, if the individual continues to survive, the balancing item is called a reserve. This is typically a liability that should be recognized in any financial statement of an insurer or annuity organization, as the case may be. It is also typically an asset for the insured or individual purchasing the annuity.

We illustrate the determination of the balancing item spoken of above by continuation of Example 6.1.1 in the two cases where a utility function was used to define the premium principle.

Example 7.1.1

An insurer has issued a policy paying 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Assume that the insured is still alive 1 year after entering into the contract. Further, assume that the insurer continues to use $i = 0.06$ and the following mortality assumption for K:

$$_{k|}q_0 = 0.2 \qquad k = 0, 1, 2, 3, 4.$$

Find the reserve, $_1V$, as determined by the following:

a. Principle II: The insurer, using a utility of wealth function $u(x) = x$, will be indifferent between continuing with the risk while receiving premiums of 0.30272 (from Example 6.1.1) and paying the amount $_1V$ to a reinsurer to assume the risk.
b. Principle III: The insurer, using a utility of wealth function $u(x) = -e^{-0.1x}$, will be indifferent between continuing with the risk while receiving premiums of 0.30628 (from Example 6.1.1) and paying the amount $_1V$ to a reinsurer to assume the risk.

Solution:

The conditional probability function for K, the curtate-future-lifetime, given that $K \geq 1$, is

$$\Pr(K = k|K \geq 1) = \frac{\Pr(K = k)}{\Pr(K \geq 1)} = \frac{0.2}{0.8} = 0.25 \qquad k = 1, 2, 3, 4.$$

The present value at duration 1 of the insurer's future financial loss is $_1L = v^{(K-1)+1} - P\, \ddot{a}_{\overline{(K-1)+1}|}$, where P is the premium determined in Example 6.1.1.

a. According to (1.3.1), we seek the amount $_1V$ such that $u(w - {_1V}) = \mathrm{E}[u(w - {_1L})|K \geq 1]$. By Principle II $u(x) = x$, so we have

$$w - {_1V} = \mathrm{E}[w - {_1L}|K \geq 1] = w - \mathrm{E}[{_1L}|K \geq 1].$$

Thus, Principle II is equivalent to requiring that $_1V$ be chosen so that $_1V = \mathrm{E}[{_1L}|K \geq 1]$. For this example, this requirement is

$$_1V = \sum_{k=1}^{4} (v^{(k-1)+1} - 0.30272\, \ddot{a}_{\overline{(k-1)+1}|}) \times \Pr(K = k|K \geq 1)$$

$$= \sum_{k=1}^{4} v^{(k-1)+1} \Pr(K = k|K \geq 1) - 0.30272 \sum_{k=1}^{4} \ddot{a}_{\overline{(k-1)+1}|} \times \Pr(K = k|K \geq 1), \tag{7.1.1}$$

which gives ${}_1V = 0.15111$ as shown in the following calculation.

Outcome k	Conditional Probability	Present Value (1 Year after Issue) of Future Obligations of		Insurer's Prospective Loss
		Insurer	Insured	
1	0.25	$v = 0.94340$	$P\,\ddot{a}_{\overline{1}\rceil} = 0.30272$	0.64067
2	0.25	$v^2 = 0.89000$	$P\,\ddot{a}_{\overline{2}\rceil} = 0.58831$	0.30169
3	0.25	$v^3 = 0.83962$	$P\,\ddot{a}_{\overline{3}\rceil} = 0.85773$	−0.01811
4	0.25	$v^4 = 0.79209$	$P\,\ddot{a}_{\overline{4}\rceil} = 1.11191$	−0.31981
Expected Value		0.86628	0.71517	0.15111

The actuarial present value of the insurer's prospective loss is

$$0.86628 - 0.71517 = \boxed{0.15111}$$

b. Again by (1.3.1) and now using the utility function in Principle III, we have

$$-e^{-0.1(w-{}_1V)} = \mathrm{E}[-e^{-0.1(w-{}_1L)}|K \geq 1] = -e^{-0.1w}\,\mathrm{E}[e^{0.1{}_1L}|K \geq 1].$$

Thus, Principle III is equivalent to requiring that ${}_1V$ be chosen so that $e^{0.1\,{}_1V} = \mathrm{E}[e^{0.1\,{}_1L}|K \geq 1]$ or that ${}_1V = 10 \log \mathrm{E}[e^{0.1\,{}_1L}|K \geq 1]$.

The following table summarizes the calculation of ${}_1V$ using the premium (0.30628) from part (c) of Example 6.1.1.

Outcome k	Conditional Probability	Insurer's Prospective Loss, ${}_1L$	$e^{(0.1{}_1L)}$
1	0.25	0.63712	1.06579
2	0.25	0.29477	1.02992
3	0.25	−0.02819	0.99718
4	0.25	−0.33287	0.96726

Thus, $\mathrm{E}[e^{0.1\,{}_1L}|K \geq 1] = (1.06579 + 1.02992 + 0.99718 + 0.96726)(0.25) = 1.01504$ and ${}_1V = (\log 1.01504)/0.1 = 0.14925.$ ▼

Henceforth, benefit reserves will be based on benefit premiums as determined by the equivalence principle in part (a) of Example 7.1.1. Thus, the ***benefit reserve at time t*** is the conditional expectation of the difference between the present value of future benefits and the present value of future benefit premiums, the conditioning event being survivorship of the insured to time t. The type of reserve found in part (b) of Example 7.1.1 is called an ***exponential reserve.***

The sections in Chapter 7 parallel sections of Chapter 6 on benefit premiums. We assume, as we do in Example 7.1.1, that the mortality and interest rates adopted at policy issue for the determination of benefit premiums continue to be appropriate and are used in the determination of benefit reserves.

7.2 Fully Continuous Benefit Reserves

We now develop the benefit reserves related to the fully continuous benefit premiums developed in Section 6.2 by application of the equivalence principle.

Let us consider reserves for a whole life insurance of 1 issued to (x) on a fully continuous basis with an annual benefit premium rate of $\bar{P}(\bar{A}_{[x]})$. The corresponding reserve for an insured surviving t years later is defined under the equivalence principle as the conditional expected value of the prospective loss at time t, given that (x) has survived to t. More formally, for $T(x) > t$ the prospective loss is

$$ {}_tL = v^{T(x)-t} - \bar{P}(\bar{A}_{[x]})\, \bar{a}_{\overline{T(x)-t|}}. \tag{7.2.1}$$

The reserve, as a conditional expectation, is calculated using the conditional distribution of the future lifetime at t for a life selected at x, given it has survived to t. In International Actuarial Notation symbols this is

$$\begin{aligned} {}_t\bar{V}(\bar{A}_{[x]}) &= \mathrm{E}[{}_tL|T(x) > t] \\ &= \mathrm{E}[v^{T(x)-t}|T(x) > t] - \bar{P}(\bar{A}_{[x]})\, \mathrm{E}[\bar{a}_{\overline{T(x)-t|}}|T(x) > t] \\ &= \bar{A}_{[x]+t} - \bar{P}(\bar{A}_{[x]})\, \bar{a}_{[x]+t}. \end{aligned} \tag{7.2.2}$$

If the attained age was the only given information at issue of the insurance at age x, or for some other reason an aggregate mortality table is used for the distribution of the future lifetime, then the conditional distribution of $T(x) - t$ is the same as the distribution of $T(x + t)$, and (7.2.2) in symbols is

$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}. \tag{7.2.3}$$

Formulas (7.2.2) and (7.2.3) state that

(the benefit reserve) = (the actuarial present value for the whole life insurance from age $x + t$)

− (the actuarial present value of future benefit premiums payable at an annual rate of $\bar{P}(\bar{A}_x)$).

The formulations of $\bar{P}(\bar{A}_x)$ and ${}_t\bar{V}(\bar{A}_x)$ are related. When $t = 0$, (7.2.3) yields ${}_0\bar{V}(\bar{A}_x) = 0$. This is a consequence of applying the equivalence principle as of the time the contract was established.

Remark on Notation (Restated):

In this book, we will simplify the appearance of the formulas by suppressing the select notation unless its use is necessary or helpful in the particular situation. The symbol $\mu_x(t)$ will be used for the force of mortality in the development of benefit reserves to reinforce the idea that the conditional distributions used in reserve calculations are derived from the distribution of $T(x)$.

Benefit reserves are defined in Section 7.1 as the conditional expectation of loss variables. In evaluating these conditional expected values in this section, the distribution of $T(x) - t$, given $T(x) > t$, was used. The interest rate and the distribution of $T(x)$ used with the equivalence principle at time $t = 0$ to determine the benefit premium were used again. The survival of the life (x) to time t was the only new information incorporated into the expected value calculation. A comprehensive reserve principle would require that all new information relevant to the loss variables and their distributions be incorporated into the reserve calculation. The objective of this requirement would be to estimate liabilities appropriate for the time that the valuation is made. In Chapters 7 and 8 the process of learning from experience to modify the assumptions under which benefit reserves are estimated are not studied.

By steps analogous to those used to obtain (6.2.6), we can determine the variance of ${}_tL$ as follows:

$$ {}_tL = v^{T(x)-t}\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}, \tag{7.2.4}$$

thus

$$\begin{aligned} \operatorname{Var}[{}_tL|T(x) > t] &= \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 \operatorname{Var}[v^{T(x)-t}|T(x) > t] \\ &= \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2]. \end{aligned} \tag{7.2.5}$$

Note the relation to (6.2.6) and that the development holds for all premium levels. It is not dependent on the equivalence principle.

Example 7.2.1

Follow up Example 6.2.1 by calculating ${}_t\bar{V}(\bar{A}_x)$ and Var $[{}_tL|T(x) > t]$.

Solution:

Since $\bar{A}_x$, $\bar{a}_x$, and $\bar{P}(\bar{A}_x)$ are independent of age x, (7.2.3) becomes

$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_x - \bar{P}(\bar{A}_x)\,\bar{a}_x = 0 \qquad t \ge 0.$$

In this case, future premiums are always equivalent to future benefits, and no reserve is needed.

Also, in this case, (7.2.5) reduces to

$$\operatorname{Var}[{}_tL|T(x) > t] = \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_x - (\bar{A}_x)^2] = \operatorname{Var}(L) = 0.25,$$

as in Example 6.2.1. Here the variance of ${}_tL$ depends on neither the age x nor the duration t. ▼

Example 7.2.2

On the basis of De Moivre's law with $l_x = 100 - x$ and the interest rate of 6%, calculate

a. $\bar{P}(\bar{A}_{35})$

b. ${}_t\bar{V}(\bar{A}_{35})$ and $\mathrm{Var}[{}_tL|T(x) > t]$ for $t = 0, 10, 20, \ldots, 60$.

Solution:

a. From $l_x = 100 - x$, we obtain ${}_tp_{35} = 1 - t/65$ and ${}_tp_{35}\,\mu(35 + t) = 1/65$ for $0 \le t < 65$. It follows that

$$\bar{A}_{35} = \int_0^{65} v^t \frac{1}{65}\,dt = \frac{\bar{a}_{\overline{65}|0.06}}{65} = 0.258047.$$

Then

$$\bar{P}(\bar{A}_{35}) = \frac{\delta\,\bar{A}_{35}}{1 - \bar{A}_{35}} = 0.020266.$$

b. At age $35 + t$, we have $\bar{A}_{35+t} = \bar{a}_{\overline{65-t}|}/(65 - t)$ and

$${}_t\bar{V}(\bar{A}_{35}) = \bar{A}_{35+t} - 0.020266\,\frac{1 - \bar{A}_{35+t}}{\log(1.06)}.$$

Further,

$${}^2\bar{A}_{35+t} = \int_0^{65-t} v^{2u}\,\frac{1}{65 - t}\,du = \frac{{}^2\bar{a}_{\overline{65-t}|}}{65 - t}$$

and, from (7.2.5),

$$\mathrm{Var}[{}_tL|T(x) > t] = \left[1 + \frac{0.020266}{\log(1.06)}\right]^2 [{}^2\bar{A}_{35+t} - (\bar{A}_{35+t})^2].$$

Applying these formulas, we obtain the following results.

t	${}_t\bar{V}(\bar{A}_{35})$	$\mathrm{Var}[{}_tL\|T(35) > t]$
0	0.0000	0.1187
10	0.0557	0.1201
20	0.1289	0.1173
30	0.2271	0.1073
40	0.3619	0.0861
50	0.5508	0.0508
60	0.8214	0.0097

Note the convergence of $\mathrm{Var}[{}_tL|T(35) > t]$ toward zero. There is a rendezvous with certainty. ▼

The table in Example 7.2.2 provides the mean and the variance of the conditional distributions of ${}_tL$ for selected values of t. To gain more insight into the nature of reserves, let us explore these distributions of ${}_tL$ in more depth. We previously studied the case for $t = 0$ at (6.2.11) following Example 6.2.3. From (7.2.1),

$$\begin{aligned} {}_tL &= v^{T(x)-t} - \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{T(x)-t}|} \\ &= v^{T(x)-t} \left[\frac{\delta + \bar{P}(\bar{A}_x)}{\delta} \right] - \frac{\bar{P}(\bar{A}_x)}{\delta}. \end{aligned} \tag{7.2.6}$$

If $\delta > 0$, ${}_tL$ is a decreasing function of $T(x) - t$ and lies in the interval

$$-\frac{\bar{P}(\bar{A}_x)}{\delta} < {}_tL \le 1. \tag{7.2.7}$$

Using Figure 6.2.1 as a guide we can repeat the steps of (6.2.11) to establish the following relationship between $F_{T(x)}(u)$ and the d.f. of the conditional distribution of ${}_tL$, given $T(x) > t$, which we denote by $F_{{}_tL}(y)$. For a y in the interval given by (7.2.7),

$$\begin{aligned} F_{{}_tL}(y) &= \Pr[{}_tL \le y | T(x) > t] \\ &= \Pr\left[v^{T(x)-t} \left[\frac{\delta + \bar{P}(\bar{A}_x)}{\delta} \right] - \frac{\bar{P}(\bar{A}_x)}{\delta} \le y \,\middle|\, T(x) > t \right] \\ &= \Pr\left[v^{T(x)-t} \le \frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \,\middle|\, T(x) > t \right] \\ &= \Pr\left[T(x) \ge t - \frac{1}{\delta} \log \left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \right] \,\middle|\, T(x) > t \right] \\ &= \frac{\Pr[T(x) \ge t - (1/\delta)\{\log[\delta y + \bar{P}(\bar{A}_x)]/[\delta + \bar{P}(\bar{A}_x)]\}]}{\Pr[T(x) > t]} \quad (7.2.8) \\ &= \frac{1 - F_{T(x)}(t - (1/\delta)\log\{[\delta y + \bar{P}(\bar{A}_x)]/[\delta + \bar{P}(\bar{A}_x)]\})}{1 - F_{T(x)}(t)}. \quad (7.2.9) \end{aligned}$$

Differentiation of (7.2.9) with respect to y derives the p.d.f. for the conditional distribution of ${}_tL$, given $T(x) > t$:

$$f_{{}_tL}(y) = \left\{ \frac{1}{[\delta y + \bar{P}(\bar{A}_x)][1 - F_{T(x)}(t)]} \right\} f_{T(x)} \left(t - \frac{1}{\delta} \log \left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \right] \right). \tag{7.2.10}$$

For an aggregate mortality law the conditional distribution of $T(x) - t$, given $T(x) > t$, is the same as the distribution of $T(x + t)$, so (7.2.8), (7.2.9), and (7.2.10) reduce to

$$F_{{}_tL}(y) = \Pr\left[T(x+t) \ge -\frac{1}{\delta} \log \left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \right] \right] \tag{7.2.11}$$

$$= 1 - F_{T(x+t)} \left(-\frac{1}{\delta} \log \left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \right] \right), \tag{7.2.12}$$

$$f_{{}_tL}(y) = \frac{1}{[\delta y + \bar{P}(\bar{A}_x)]} f_{T(x+t)} \left(-\frac{1}{\delta} \log \left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \right] \right). \tag{7.2.13}$$

To illustrate these concepts we will extend Example 7.2.2.

Example 7.2.3

For the insurance contract and assumptions in Example 7.2.2:

a. Exhibit the formulas for the d.f. and the p.d.f. of the conditional distribution for ${}_tL$, given $T(x) > t$.

b. Display graphs of these conditional p.d.f.'s for $t = 0$, 20, 40, and 60.

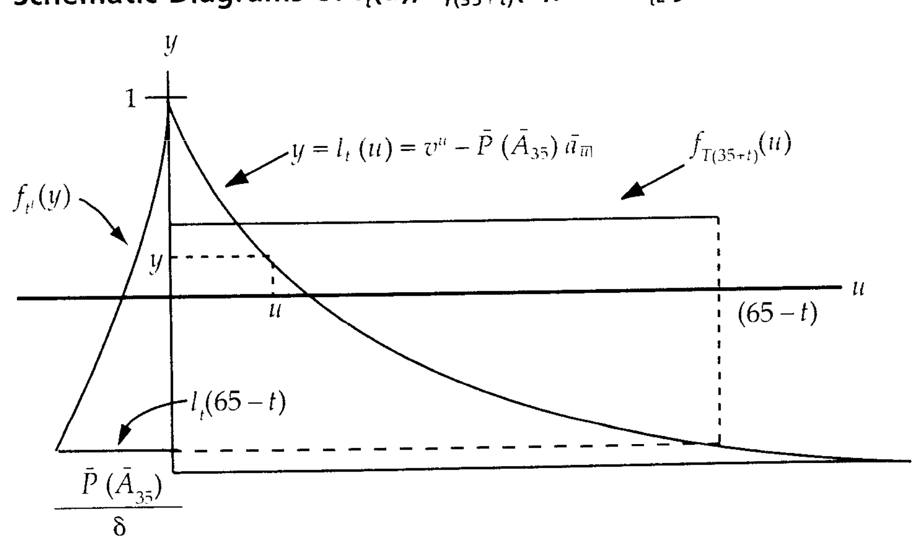
Solution:

a. Since Example 7.2.2 specifies an aggregate mortality law, we use formulas (7.2.12) and (7.2.13). In Example 7.2.2,

$$
\begin{aligned}
F_{T(35+t)}(u) &= \frac{u}{65 - t} && \text{for } 0 \le u \le 65 - t \\
&= 1 && \text{for } 65 - t < u, \\
f_{T(35+t)}(u) &= \frac{1}{65 - t} && \text{for } 0 \le u \le 65 - t \\
&= 0 && \text{elsewhere.}
\end{aligned}
$$

Figure 7.2.1 shows Figure 6.2.1 as it applies to this example. In this figure the outcome space of $T(35 + t)$ is on the u-axis, and the outcome space of the loss random variable, ${}_tL$, is on the y-axis. The relationship between the outcomes of $T(35 + t)$ and the outcomes of ${}_tL$ is given by the loss function $y = l_t(u)$ and is indicated by the dashed line connecting u and y in the figure. The p.d.f. $f_{T(35+t)}(u)$ has its domain on the u-axis and its range on the y-axis. The domain of the p.d.f. $f_{{}_tL}(y)$ is on the y-axis, and its range is to be imagined on an axis perpendicular to the u-y plane, but for the sketch it has been laid perpendicular to the y-axis in the u-y plane.

Schematic Diagrams of $l_t(u)$, $f_{T(35+t)}(u)$, and $f_{{}_tL}(y)$



To determine the d.f. by ${}_tL$ we start with a value of y corresponding to a value of u in the interval $(0, 65 - t)$. For such a y we have, by (7.2.12),

$$F_{{}_tL}(y) = 1 - \frac{(-1/\delta) \log \{[\delta y + \bar{P}(\bar{A}_{35})] / [\delta + \bar{P}(\bar{A}_{35})]\}}{65 - t} \qquad 0 \le y \le 1.$$

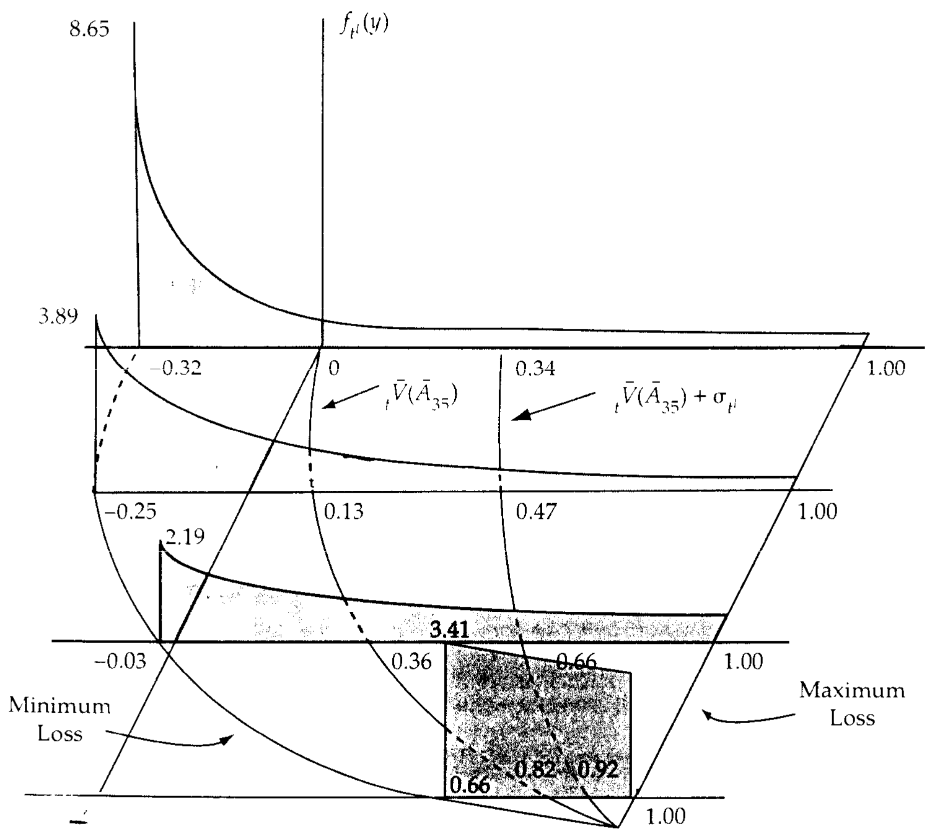
For a $y > 1$, $F_{{}_tL}(y) = 1$.

Again, for a value of y corresponding to a value of u in the interval $(0, 65 - t)$, and using (7.2.13), we have

$$f_{{}_tL}(y) = \begin{cases} \left(\dfrac{1}{65 - t}\right)\left[\dfrac{1}{\delta y + \bar{P}(\bar{A}_{35})}\right] & -\dfrac{\bar{P}(\bar{A}_{35})}{\delta} \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

b. Figure 7.2.2 is the composite of the required graphs of the p.d.f.'s $f_{{}_tL}(y)$, for $t = 0, 20, 40$, and 60. Figure 7.2.1 is a graph for one value of t. Compare Figures 7.2.1 and 7.2.2 as follows: The vertical y-axis of Figure 7.2.1 is the horizontal axis in Figure 7.2.2. The axis that was imagined to be perpendicular to the u-y plane in Figure 7.2.1 is the vertical axis in Figure 7.2.2. Note the curves in the y-t plane that indicate the minimum and maximum possible losses, the expected loss (benefit reserve), and the expected loss plus one standard deviation of the loss. ▼

$f_{{}_tL}(y)$ for t = 0, 20, 40, and 60



Corresponding to Table 6.2.1, Table 7.2.1 for benefit reserves is presented. We have not tabulated details of the prospective loss, ${}_tL$, and explicit formulas for $\mathrm{Var}[{}_tL|T(x) > t]$, corresponding to the several benefit reserves, are not displayed.

Fully Continuous Benefit Reserves; Age at Issue x, Duration t, Unit Benefit

Plan	International Actuarial Notation	Prospective Formula	
Whole life insurance	${}_t\bar{V}(\bar{A}_x)$	$\bar{A}_{x+t} - \bar{P}(\bar{A}_x)\,\bar{a}_{x+t}$	
n-Year term insurance	${}_t\bar{V}(\bar{A}^{1}_{x:\overline{n}\rvert})$	$\begin{cases} \bar{A}^{1}_{x+t:\overline{n-t}\rvert} - \bar{P}(\bar{A}^{1}_{x:\overline{n}\rvert})\,\bar{a}_{x+t:\overline{n-t}\rvert} \\ 0 \end{cases}$	$t < n$ $t = n$
n-Year endowment insurance	${}_t\bar{V}(\bar{A}_{x:\overline{n}\rvert})$	$\begin{cases} \bar{A}_{x+t:\overline{n-t}\rvert} - \bar{P}(\bar{A}_{x:\overline{n}\rvert})\,\bar{a}_{x+t:\overline{n-t}\rvert} \\ 1 \end{cases}$	$t < n$ $t = n$
h-Payment years, whole life insurance	${}^{h}_{t}\bar{V}(\bar{A}_x)$	$\begin{cases} \bar{A}_{x+t} - {}_h\bar{P}(\bar{A}_x)\,\bar{a}_{x+t:\overline{h-t}\rvert} \\ \bar{A}_{x+t} \end{cases}$	$t \le h$ $t > h$
h-Payment years, n-year endowment insurance	${}^{h}_{t}\bar{V}(\bar{A}_{x:\overline{n}\rvert})$	$\begin{cases} \bar{A}_{x+t:\overline{n-t}\rvert} - {}_h\bar{P}(\bar{A}_{x:\overline{n}\rvert})\,\bar{a}_{x+t:\overline{h-t}\rvert} \\ \bar{A}_{x+t:\overline{n-t}\rvert} \\ 1 \end{cases}$	$t \le h < n$ $h < t < n$ $t = n$
n-Year pure endowment	${}_t\bar{V}(A_{x:\overline{n}\rvert}^{\;\;1})$	$\begin{cases} A_{x+t:\overline{n-t}\rvert}^{\;\;1} - \bar{P}(A_{x:\overline{n}\rvert}^{\;\;1})\,\bar{a}_{x+t:\overline{n-t}\rvert} \\ 1 \end{cases}$	$t < n$ $t = n$
Whole life annuity	${}_t\bar{V}({}_{n\rvert}\bar{a}_x)$	$\begin{cases} {}_{n-t\rvert}\bar{a}_{x+t} - \bar{P}({}_{n\rvert}\bar{a}_x)\,\bar{a}_{x+t:\overline{n-t}\rvert} \\ \bar{a}_{x+t} \end{cases}$	$t \le n$ $t > n$

7.3 Other Formulas for Fully Continuous Benefit Reserves

So far we have defined the benefit reserve as the conditional expectation of the prospective loss random variable and developed only one method to write formulas for fully continuous benefit reserves, namely, the ***prospective method,*** stating that the benefit reserve is the difference between the actuarial present values of future benefits and of future benefit premiums. From the prospective method, we can easily develop three other general formulas for policies with level benefits and level benefit premium rates. We illustrate these for the case of n-year endowment insurances.

The ***premium-difference formula*** for ${}_t\bar{V}(\bar{A}_{x:\overline{n}\rvert})$ is obtained by factoring $\bar{a}_{x+t:\overline{n-t}\rvert}$ out of the prospective formula for ${}_t\bar{V}(\bar{A}_{x:\overline{n}\rvert})$:

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}\rvert}) &= \left[\frac{\bar{A}_{x+t:\overline{n-t}\rvert}}{\bar{a}_{x+t:\overline{n-t}\rvert}} - \bar{P}(\bar{A}_{x:\overline{n}\rvert})\right]\bar{a}_{x+t:\overline{n-t}\rvert} \\
&= [\bar{P}(\bar{A}_{x+t:\overline{n-t}\rvert}) - \bar{P}(\bar{A}_{x:\overline{n}\rvert})]\,\bar{a}_{x+t:\overline{n-t}\rvert}. \qquad (7.3.1)
\end{aligned}
$$

Formula (7.3.1) exhibits the benefit reserve as the actuarial present value of a premium difference payable over the remaining premium-payment term. The premium difference is obtained by subtracting the original annual benefit premium from the benefit premium for an insurance issued at the attained age $x + t$ for the remaining benefits.

A second formula is obtained by factoring the actuarial present value of future benefits out of the prospective formula. Thus, for ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$ we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) &= \left[1 - \bar{P}(\bar{A}_{x:\overline{n}|}) \frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{A}_{x+t:\overline{n-t}|}}\right] \bar{A}_{x+t:\overline{n-t}|} \\
&= \left[1 - \frac{\bar{P}(\bar{A}_{x:\overline{n}|})}{\bar{P}(\bar{A}_{x+t:\overline{n-t}|})}\right] \bar{A}_{x+t:\overline{n-t}|}. \qquad (7.3.2)
\end{aligned}
$$

This exhibits the benefit reserve as the actuarial present value of a portion of the remaining future benefits, that portion which is not funded by the future benefit premiums still to be collected. Note that $\bar{P}(\bar{A}_{x+t:\overline{n-t}|})$ is the benefit premium required if the future benefits were to be funded from only the future benefit premiums, but $\bar{P}(\bar{A}_{x:\overline{n}|})$ is the benefit premium actually payable. Thus, $\bar{P}(\bar{A}_{x:\overline{n}|}) / \bar{P}(\bar{A}_{x+t:\overline{n-t}|})$ is the portion of future benefits funded by future benefit premiums. This is called a ***paid-up insurance formula,*** named from the paid-up insurance nonforfeiture benefit to be discussed in Chapter 16. Formulas analogous to (7.3.1) and (7.3.2) exist for a wide variety of benefit reserves.

A third expression is the ***retrospective formula.*** We develop this from a more general relationship. We have, from Exercise 4.12 and from formulas (5.2.18) and (5.2.19), for $t < n - s$,

$$\bar{A}_{x+s:\overline{n-s}|} = \bar{A}^{\,1}_{x+s:\overline{t}|} + {}_tE_{x+s}\, \bar{A}_{x+s+t:\overline{n-s-t}|}$$

and

$$\bar{a}_{x+s:\overline{n-s}|} = \bar{a}_{x+s:\overline{t}|} + {}_tE_{x+s}\, \bar{a}_{x+s+t:\overline{n-s-t}|}.$$

Substituting these expressions into the prospective formula for ${}_s\bar{V}(\bar{A}_{x:\overline{n}|})$, we obtain

$$
\begin{aligned}
{}_s\bar{V}(\bar{A}_{x:\overline{n}|}) &= \bar{A}^{\,1}_{x+s:\overline{t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+s:\overline{t}|} \\
&\quad + {}_tE_{x+s}\, [\bar{A}_{x+s+t:\overline{n-t-s}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+s+t:\overline{n-s-t}|}] \\
&= \bar{A}^{\,1}_{x+s:\overline{t}|} + {}_tE_{x+s}\; {}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|}) - \bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+s:\overline{t}|}. \qquad (7.3.3)
\end{aligned}
$$

Thus the benefit reserve at the beginning and end of a t-year interval are connected by the following argument:

(the benefit reserve at the beginning of the interval) = (the actuarial present value at the beginning of the interval of benefits payable during the interval)

+ (the actuarial present value at the beginning of the interval of a pure endowment for the amount of the benefit reserve at the end of the interval)

− (the actuarial present value of benefit premiums payable during the interval).

$\bar{A}_{x+s:\overline{t}|}$?

The rearranged symbolic form,

$$ {}_s\bar{V}(\bar{A}_{x:\overline{n}|}) + \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s:\overline{n-t}|} \qquad \bar{P}_{x+s:\overline{t}|}$$

$$= \bar{A}_{x+s:\overline{t}|} + {}_tE_{x+s}\ {}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|}), \tag{7.3.4}$$

shows that the actuarial present values of the insurer's resources and obligations are equal.

The retrospective formula is obtained from (7.3.4) by setting $s = 0$, noting that ${}_0\bar{V}(\bar{A}_{x:\overline{n}|}) = 0$ by the equivalence principle, and solving for ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$. Thus,

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \frac{1}{{}_tE_x}\,[\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x:\overline{t}|} - \bar{A}^{\,1}_{x:\overline{t}|}].$$

Further, $\bar{s}_{x:\overline{t}|} = \bar{a}_{x:\overline{t}|}/{}_tE_x$ so the formula reduces to

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x. \tag{7.3.5}$$

Here

$$ {}_t\bar{k}_x = \frac{\bar{A}^{\,1}_{x:\overline{t}|}}{{}_tE_x} \tag{7.3.6}$$

is called the ***accumulated cost of insurance***. One notes that

$$ {}_t\bar{k}_x = \int_0^t \frac{v^s\,{}_sp_x\,\mu_x(s)\,ds}{v^t\,{}_tp_x}$$

$$= \int_0^t \frac{(1+i)^{t-s}\,l_{x+s}\,\mu_x(s)\,ds}{l_{x+t}}. \tag{7.3.7}$$

This can be interpreted as the assessment against each of the l_{x+t} survivors to provide for the accumulated value of the death claims in the survivorship group between ages x and $x + t$. Thus, the reserve can be viewed as the difference between the benefit premiums, accumulated with interest and shared among only the survivors at age $x + t$, and the accumulated cost of insurance.

We conclude this section with some special formulas that express the whole life insurance benefit reserves in terms of a single actuarial function. Analogous formulas hold for n-year endowment insurance benefit reserves when benefit premiums are payable continuously for the n years. Because we used (5.2.8) to express $\bar{P}(\bar{A}_x)$ in terms of δ and either $\bar{a}_x$ or $\bar{A}_x$, we can use those ideas here to express ${}_t\bar{V}(\bar{A}_x)$ in terms of one of the actuarial functions $\bar{a}_x$, $\bar{A}_x$, or $\bar{P}(\bar{A}_x)$ and δ.

For an annuity function formula, substitute (6.2.9) and (5.2.8) into the prospective formula (7.2.3) to obtain

$$ {}_t\bar{V}(\bar{A}_x) = 1 - \delta\bar{a}_{x+t} - \left(\frac{1}{\bar{a}_x} - \delta\right)\bar{a}_{x+t}$$

$$= 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}. \tag{7.3.8}$$

Further, substituting (6.2.9) into the premium-difference formula, we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_x) &= [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\,\bar{a}_{x+t} \\
&= \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}.
\end{aligned} \tag{7.3.9}
$$

Finally, we can rewrite (7.3.8) using $\bar{A}_{x+t} = 1 - \delta\bar{a}_{x+t}$ to obtain

$$
{}_t\bar{V}(\bar{A}_x) = 1 - \frac{1 - \bar{A}_{x+t}}{1 - \bar{A}_x} = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x}. \tag{7.3.10}
$$

These last three formulas depend on relationship (5.2.8) between the annuity for the premium paying period and the insurance for the benefit period. Thus they are available only for whole life and endowment insurances where the premium-paying period and the benefit period are the same. Moreover, the frequency of premium payment must be the same as the "frequency" of benefit payment. We will see that apportionable premiums satisfy these requirements in their own way.

Remark:

Although benefit reserves are non-negative in most applications, there is no mathematical theorem that guarantees this property. Indeed, the reader can combine Exercise 4.2 and formula (7.3.10) for a quick verification that negative benefit reserves are a real possibility.

7.4 Fully Discrete Benefit Reserves

The benefit reserves of this section are for the insurances of Section 6.3 which have annual premium payments and payment of the benefit at the end of the year of death. As in Section 7.2 the underlying mortality assumption can be on a select or aggregate basis. We will display the formulas for the aggregate case, which has simpler notation. Let us consider a whole life insurance with benefit issued to (x) with benefit premium P_x. Following the development in Section 7.2, for an insured surviving k years later, we define the benefit reserve, denoted by ${}_kV_x$, as the conditional expectation of the prospective loss, ${}_kL$, at duration k. More precisely,

$$
{}_kL = v^{K(x)-k+1} - P_x\,\ddot{a}_{\overline{K(x)-k+1}|} \tag{7.4.1}
$$

$$
{}_kV_x = \mathrm{E}[{}_kL \mid K(x) = k, k+1, \ldots]. \tag{7.4.2}
$$

The prospective formula for the benefit reserve is

$$
{}_kV_x = A_{x+k} - P_x\,\ddot{a}_{x+k}. \tag{7.4.3}
$$

As in Section 7.2 this formula is the actuarial present value of future benefits less the actuarial present value of future benefit premiums.

Analogous to (7.2.4), we have

$$\begin{aligned}\operatorname{Var}[{}_kL|K(x) &= k, k+1, \ldots] \\ &= \operatorname{Var}\left[v^{[K(x)-k]+1}\left(1+\frac{P_x}{d}\right)\Bigg| K(x) = k, k+1, \ldots\right] \\ &= \left(1+\frac{P_x}{d}\right)^2 \operatorname{Var}[v^{[K(x)-k]+1}|K(x) = k, k+1, \ldots] \\ &= \left(1+\frac{P_x}{d}\right)^2 [{}^2A_{x+k} - (A_{x+k})^2]. \qquad (7.4.4)\end{aligned}$$

Example 7.4.1

Follow up Example 6.4.1 by calculating ${}_kV_x$ and $\operatorname{Var}[{}_kL|K(x) = k, k+1, \ldots]$.

Solution:

Here A_x, $\ddot{a}_x$, and P_x are independent of age x so that $A_{x+k} = A_x$ and

$${}_kV_x = A_x - P_x\,\ddot{a}_x = 0 \qquad k = 0, 1, 2, \ldots.$$

Also, from (7.4.4), $\operatorname{Var}[{}_kL|K(x) = k, k+1, \ldots] = \operatorname{Var}(L) = 0.2347$. ▼

The benefit reserve formulas tabulated in Table 7.4.1 correspond to the benefit premiums in Table 6.3.1 and are analogous to the benefit reserves in Table 7.2.1.

Fully Discrete Benefit Reserves; Age at Issue x, Duration k, Unit Benefit

Plan	International Actuarial Notation	Prospective Formula	
Whole life insurance	${}_kV_x$	$A_{x+k} - P_x\,\ddot{a}_{x+k}$	
n-Year term insurance	${}_kV^{1}_{x:\overline{n}\vert}$	$A^{1}_{x+k:\overline{n-k}\vert} - P^{1}_{x:\overline{n}\vert}\,\ddot{a}_{x+k:\overline{n-k}\vert}$ 0	$k < n$ $k = n$
n-Year endowment insurance	${}_kV_{x:\overline{n}\vert}$	$A_{x+k:\overline{n-k}\vert} - P_{x:\overline{n}\vert}\,\ddot{a}_{x+k:\overline{n-k}\vert}$ 1	$k < n$ $k = n$
h-Payment years, whole life insurance	${}^{h}_{k}V_x$	$A_{x+k} - {}_hP_x\,\ddot{a}_{x+k:\overline{h-k}\vert}$ A_{x+k}	$k < h$ $k \ge h$
h-Payment years, n-year endowment insurance	${}^{h}_{k}V_{x:\overline{n}\vert}$	$A_{x+k:\overline{n-k}\vert} - {}_hP_{x:\overline{n}\vert}\,\ddot{a}_{x+k:\overline{h-k}\vert}$ $A_{x+k:\overline{n-k}\vert}$ 1	$k < h < n$ $h \le k < n$ $k = n$
n-Year pure endowment	${}_kV_{x:\overline{n}\vert}^{\ \ 1}$	$A_{x+k:\overline{n-k}\vert}^{\ \ \ \ 1} - P_{x:\overline{n}\vert}^{\ \ 1}\,\ddot{a}_{x+k:\overline{n-k}\vert}$ 1	$k < n$ $k = n$
Whole life annuity	${}_kV({}_{n\vert}\ddot{a}_x)$	${}_{n-k\vert}\ddot{a}_{x+k} - P({}_{n\vert}\ddot{a}_x)\,\ddot{a}_{x+k:\overline{n-k}\vert}$ $\ddot{a}_{x+k}$	$k < n$ $k \ge n$

Example 7.4.2

Determine an expression in actuarial present values and benefit premiums for the $\mathrm{Var}[{}_kL|K(x) = k, k+1, \ldots]$ for a fully discrete n-year endowment insurance with a unit benefit.

Solution:

$$
\begin{aligned}
{}_kL &= v^{K(x)-k+1}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = k, k+1, \ldots, n-1 \\
&= v^{n-k}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = n, n+1, \ldots,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Var}[{}_kL|K(x) = k, k+1, \ldots] \\
&\quad = \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 [{}^2A_{x+k:\overline{n-k}|} - (A_{x+k:\overline{n-k}|})^2].
\end{aligned}
$$

▼

In cases other than whole life or endowment insurances with premiums payable throughout the insurance term, the expressions for the variance of the loss may be difficult to summarize in convenient notation.

Formulas similar to those of Section 7.3 can be developed for fully discrete benefit reserves. We illustrate these by writing the formulas for ${}_kV_{x:\overline{n}|}$ with a minimum of discussion. Verbal interpretations and algebraic developments closely parallel those for fully continuous benefit reserves.

The premium difference formula is

$$
{}_kV_{x:\overline{n}|} = (P_{x+k:\overline{n-k}|} - P_{x:\overline{n}|})\, \ddot{a}_{x+k:\overline{n-k}|}. \tag{7.4.5}
$$

The paid-up insurance formula is

$$
{}_kV_{x:\overline{n}|} = \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}\right) A_{x+k:\overline{n-k}|}. \tag{7.4.6}
$$

For the retrospective formula, we first establish a result analogous to (7.3.3), namely, for $h < n - j$,

$$
{}_jV_{x:\overline{n}|} = A^{1}_{x+j:\overline{h}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+j:\overline{h}|} + {}_hE_{x+j}\; {}_{j+h}V_{x:\overline{n}|}. \tag{7.4.7}
$$

Then, if $j = 0$, we have, since ${}_0V_{x:\overline{n}|} = 0$,

$$
\begin{aligned}
{}_hV_{x:\overline{n}|} &= \frac{1}{{}_hE_x}\,(P_{x:\overline{n}|}\, \ddot{a}_{x:\overline{h}|} - A^{1}_{x:\overline{h}|}) \\
&= P_{x:\overline{n}|}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x.
\end{aligned} \tag{7.4.8}
$$

Here the accumulated cost of insurance is ${}_hk_x = A^{1}_{x:\overline{h}|} / {}_hE_x$, and a survivorship group interpretation is possible.

An interesting observation follows from the retrospective formula for the benefit reserve. Let us consider two different policies issued to (x), each for a unit of insurance during the first h years. Here, h is less than or equal to the shorter of the two premium-payment periods. The retrospective formula for the benefit reserve on policy one is

$$ {}_hV_{(1)} = P_{(1)}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x $$

and that for the benefit reserve on policy two is

$$ {}_hV_{(2)} = P_{(2)}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x. $$

It follows that

$$ {}_hV_{(1)} - {}_hV_{(2)} = (P_{(1)} - P_{(2)})\, \ddot{s}_{x:\overline{h}|}, \tag{7.4.9} $$

which shows that the difference in the two benefit reserves equals the actuarial accumulated value of the difference in the benefit premiums $P_{(1)} - P_{(2)}$. Since $1/\ddot{s}_{x:\overline{h}|} = P_{x:\overline{h}|}^{\;\;1}$, formula (7.4.9) can be rearranged as

$$ P_{(1)} - P_{(2)} = P_{x:\overline{h}|}^{\;\;1}\,({}_hV_{(1)} - {}_hV_{(2)}). \tag{7.4.10} $$

The difference in the benefit premiums is expressed as the benefit premium for an h-year pure endowment of the difference in the benefit reserves at the end of h years. Formula (6.3.10) is a special case of (7.4.10) with ${}_nV_{x:\overline{n}|} = 1$ and ${}_n^nV_x = A_{x+n}$. Another illustration of (7.4.10) is

$$ P_x = P^1_{x:\overline{n}|} + P_{x:\overline{n}|}^{\;\;1}\,{}_nV_x \tag{7.4.11} $$

since ${}_nV^1_{x:\overline{n}|} = 0$.

As in the fully continuous case, there are special formulas for whole life and endowment insurance benefit reserves in the fully discrete case. Parallel to (7.3.8)–(7.3.10), we have, by use of the relations $A_y = 1 - d\,\ddot{a}_y$ and $1/\ddot{a}_y = P_y + d$,

$$ {}_kV_x = 1 - d\,\ddot{a}_{x+k} - \left(\frac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+k} $$

$$ = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}, \tag{7.4.12} $$

$$ {}_kV_x = 1 - \frac{P_x + d}{P_{x+k} + d} = \frac{P_{x+k} - P_x}{P_{x+k} + d}, \tag{7.4.13} $$

and

$$ {}_kV_x = 1 - \frac{1 - A_{x+k}}{1 - A_x} = \frac{A_{x+k} - A_x}{1 - A_x}. \tag{7.4.14} $$

Similar special formulas also hold for fully discrete n-year endowment insurance benefit reserves, but not for insurance benefit reserves in general.

Fully discrete insurances provide instructive examples for the deterministic, or expected cash flow, model of the operations of benefit reserves. This is displayed in Examples 7.4.3 and 7.4.4.

Example 7.4.3

Assume that a 5-year term life insurance of 1,000 is issued on a fully discrete basis to each member of a group of l_{50} persons at age 50. Trace the cash flow expected for this group on the basis of the Illustrative Life Table with interest at 6%, and, as a by-product, obtain the benefit reserves.

Solution:

We first calculate the annual benefit premium $\pi = 1{,}000\ P^{1}_{50:\overline{5}|} = 6.55692$. Then the expected accumulation of funds for the group through the collection of benefit premiums, the crediting of interest, and the payment of claims is as stated in the following:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Year h	Expected Benefit Premiums at Beginning of Year $l_{50+h-1}\pi$	Expected Fund at Beginning of Year $(2)_h + (6)_{h-1}$	Expected Interest $(0.06)\ (3)_h$	Expected Death Claims $1{,}000\ d_{50+h-1}$	Expected Fund at End of Year $(3)_h + (4)_h - (5)_h$	Expected Number of Survivors at End of Year l_{50+h}	$1{,}000\ {}_hV^{1}_{50:\overline{5}|}$ $(6)_h/(7)_h$
1	586 903	586 903	35 214	529 884	92 233	88 979.11	1.04
2	583 429	675 662	40 540	571 432	144 770	88 407.68	1.64
3	579 682	724 452	43 467	616 416	151 503	87 791.26	1.73
4	575 640	727 143	43 629	665 065	105 707	87 126.20	1.21
5	571 280	676 987	40 619	717 606	0	86 408.60	0.00

Note that the benefit reserves derived in the table match those calculated by formula. For example, at duration 2 we have

$$1{,}000\ A^{1}_{52:\overline{3}|} = 20.09 \text{ and } \ddot{a}_{52:\overline{3}|} = 2.81391.$$

Then

1.64

$$1{,}000\ {}_2V^{1}_{50:\overline{5}|} = 20.09 - (6.55692)(2.81391) = \cancel{1.54} \qquad \blacktriangledown$$

Example 7.4.4

Assume that a 5-year endowment insurance of 1,000 is issued on a fully discrete basis to each member of a group of l_{50} persons at age 50. Trace the cash flow expected for this group on the basis of the Illustrative Life Table with interest at 6%, and as a by-product obtain the benefit reserves.

Solution:

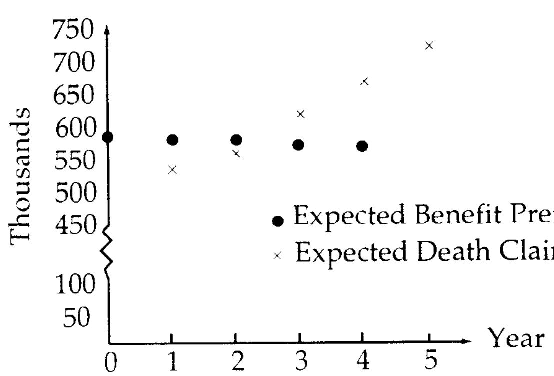
Here the annual benefit premium is $\pi = 1{,}000\ P_{50:\overline{5}|} = 170.083$. The expected cash flow is displayed in the following table:

(1) Year h	(2) Expected Benefit Premiums at Beginning of Year $l_{50+h-1}\,\pi$	(3) Expected Fund at Beginning of Year $(2)_h + (6)_{h-1}$	(4) Expected Interest $(0.06)\,(3)_h$	(5) Expected Death Claims $1{,}000\,d_{50+h-1}$	(6) Expected Fund at End of Year $(3)_h + (4)_h - (5)_h$	(7) Expected Number of Survivors at End of Year l_{50+h}	(8) $1{,}000\;{}_hV^{1}_{50:\overline{5}|}$ $(6)_h \div (7)_h$
1	15 223 954	15 223 954	913 437	529 884	15 607 507	88 979.11	175.14
2	15 133 829	30 741 336	1 844 480	571 432	32 014 384	88 407.68	362.12
3	15 036 638	47 051 022	2 823 061	616 416	49 257 667	87 791.26	561.08
4	14 931 796	64 189 463	3 851 368	665 065	67 375 766	87 126.20	773.31
5	14 818 680	82 194 446	4 931 667	717 606	86 408 507	86 408.60	1 000.00

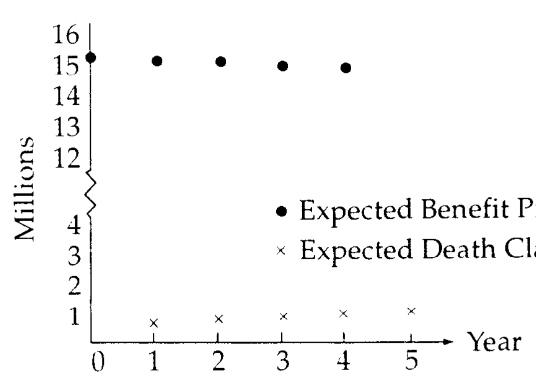
▼

Figures 7.4.1 and 7.4.2 display the expected benefit premiums and expected death claims for the preceding two examples. In Example 7.4.3, expected benefit premiums exceed expected death claims for 2 years, but thereafter are less than expected claims. The excess benefit premiums accumulate a fund in the early years to be drawn on in the later years when expected claims are higher. At the end of 5 years, the fund is expected to be exhausted.

Expected Benefit Premiums and Expected Death Claims for Example 7.4.3



Expected Benefit Premiums and Expected Death Claims for Example 7.4.4



For the 5-year endowment case of Example 7.4.4, the picture is much different. As shown in Figure 7.4.2, the expected benefit premiums remain far in excess of expected death claims throughout. The expected fund at the end of 5 years is sufficient to provide 1,000 in maturity payments to each of the expected survivors.

The 5-year term insurance exemplifies a low-premium, low-accumulation life insurance, whereas the 5-year endowment insurance exemplifies a high-premium, high-accumulation form. Most life insurances would fall between these two extremes.

7.5 Benefit Reserves on a Semicontinuous Basis

We noted at the end of Section 6.3 that, in practice, there is a need for semicontinuous annual benefit premiums $P(\bar{A}_x)$, $P(\bar{A}_{x:\overline{n}|})$, $P(\bar{A}^{\,1}_{x:\overline{n}|})$, ${}_hP(\bar{A}_x)$, and ${}_hP(\bar{A}_{x:\overline{n}|})$ to take account of immediate payment of death claims. In such cases, the benefit reserve formulas in Table 7.4.1 need to be revised by replacement of A by $\bar{A}$ and of P by $P(\bar{A})$. Moreover, the principal symbol for the benefit reserve is now $V(\bar{A})$ with a subscript on the $\bar{A}$ to indicate the type of insurance as in the benefit premium symbol. For example, for an h-payment years, n-year endowment insurance

$$ {}^h_kV(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}_{x:\overline{n}|})\,\ddot{a}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}_{x+k:\overline{n-k}|} & h \le k < n \\ 1 & k = n. \end{cases} \tag{7.5.1} $$

If a uniform distribution of deaths over each year of age is assumed, we have, from (4.4.2) and (6.3.12),

$$ {}^h_kV(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}\,{}^h_kV^{\,1}_{x:\overline{n}|} + {}^h_kV_{x:\overline{n}|}^{\ \ \,1}. \tag{7.5.2} $$

Under this circumstance benefit reserves on a semicontinuous basis are easily calculated from the corresponding fully discrete benefit reserves.

7.6 Benefit Reserves Based on True *m*-thly Benefit Premiums

In this section we examine the benefit reserve formulas corresponding to the formulas for true m-thly benefit premiums discussed in Section 6.4. By the prospective method, one can write a direct formula for ${}^h_kV^{(m)}_{x:\overline{n}|}$, namely,

$$ {}^h_kV^{(m)}_{x:\overline{n}|} = A_{x+k:\overline{n-k}|} - {}_hP^{(m)}_{x:\overline{n}|}\,\ddot{a}^{(m)}_{x+k:\overline{h-k}|} \qquad k < h. \tag{7.6.1} $$

This can be evaluated after obtaining ${}_hP^{(m)}_{x:\overline{n}|}$ by means of (6.4.1) or (6.4.2), and $\ddot{a}^{(m)}_{x+k:\overline{h-k}|}$ by means of (5.4.15) or (5.4.17).

We now consider the difference between ${}^h_kV^{(m)}_{x:\overline{n}|}$ and ${}^h_kV_{x:\overline{n}|}$ in the general case of a limited payment endowment insurance. We have, for $k < h$,

$$ {}^h_kV^{(m)}_{x:\overline{n}|} - {}^h_kV_{x:\overline{n}|} = {}_hP_{x:\overline{n}|}\, \ddot{a}_{x+k:\overline{h-k}|} - {}_hP^{(m)}_{x:\overline{n}|}\, \ddot{a}^{(m)}_{x+k:\overline{h-k}|} $$
$$ = {}_hP^{(m)}_{x:\overline{n}|} \frac{\ddot{a}^{(m)}_{x:\overline{h}|}}{\ddot{a}_{x:\overline{h}|}}\, \ddot{a}_{x+k:\overline{h-k}|} - {}_hP^{(m)}_{x:\overline{n}|}\, \ddot{a}^{(m)}_{x+k:\overline{h-k}|}. \tag{7.6.2} $$

Under the assumption of a uniform distribution of deaths in each year of age, (7.6.2) becomes

$$ {}^h_kV^{(m)}_{x:\overline{n}|} - {}^h_kV_{x:\overline{n}|} = {}_hP^{(m)}_{x:\overline{n}|} \left\{ \frac{\ddot{a}^{(m)}_{\overline{1}|}\, \ddot{a}_{x:\overline{h}|} - \beta(m)\, A^{1}_{x:\overline{h}|}}{\ddot{a}_{x:\overline{h}|}}\, \ddot{a}_{x+k:\overline{h-k}|} - \left[\ddot{a}^{(m)}_{\overline{1}|}\, \ddot{a}_{x+k:\overline{n-k}|} - \beta(m)\, A^{\;1}_{x+k:\overline{h-k}|} \right] \right\}. $$

The terms involving $\ddot{a}^{(m)}_{\overline{1}|}$ cancel to yield

$$ {}^h_kV^{(m)}_{x:\overline{n}|} - {}^h_kV_{x:\overline{n}|} = \beta(m)\, {}_hP^{(m)}_{x:\overline{n}|}\, {}_kV^{1}_{x:\overline{h}|}. \tag{7.6.3} $$

Thus,

(the benefit reserve for an insurance with true m-thly benefit premiums) = (the corresponding fully discrete benefit reserve)

+ (a fully discrete benefit reserve for term insurance over the premium paying period for a fraction, $\beta(m)$, of the true m-thly benefit premium for the plan of insurance).

A similar result holds for benefit reserves on a semicontinuous basis with true m-thly benefit premiums under the assumption of uniform distribution of deaths in each year of age. By the prospective method, we have for $k < h$,

$$ {}^h_kV^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_hP^{(m)}(\bar{A}_{x:\overline{n}|})\, \ddot{a}^{(m)}_{x+k:\overline{h-k}|}. \tag{7.6.4} $$

By steps analogous to those connecting (7.6.1) and (7.6.3), we obtain

$$ {}^h_kV^{(m)}(\bar{A}_{x:\overline{n}|}) = {}^h_kV(\bar{A}_{x:\overline{n}|}) + \beta(m)\, {}_hP^{(m)}(\bar{A}_{x:\overline{n}|})\, {}_kV^{1}_{x:\overline{h}|}. \tag{7.6.5} $$

Further, by letting $m \to \infty$ above, we obtain for a fully continuous basis

$$ {}^h_k\bar{V}(\bar{A}_{x:\overline{n}|}) = {}^h_kV(\bar{A}_{x:\overline{n}|}) + \beta(\infty)\, {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\, {}_kV^{1}_{x:\overline{h}|}. \tag{7.6.6} $$

Note again that the term insurance benefit reserve is on a fully discrete basis.

Example 7.6.1

On the basis of the Illustrative Life Table with the assumption of uniform distribution of deaths over each year of age and $i = 0.06$ calculate the following for a 20-year endowment insurance issued to (50) with a unit benefit and true semiannual benefit premiums:

a. The benefit reserve at the end of the tenth year if the benefit is payable at the end of the year of death.
b. The benefit reserve at the end of the tenth year if the benefit is payable at the moment of death.

Also verify (7.6.5) in relation to the benefit reserve in part (b).

Solution:

a. In addition to the values calculated in Example 6.4.1, we require

$$A^{\ 1}_{60:\overline{10}|} = 0.13678852$$

$$A_{60:\overline{10}|} = 0.58798425$$

$$\ddot{a}_{60:\overline{10}|} = 7.2789425$$

$${}_{10}V^{\ 1}_{50:\overline{20}|} = A^{\ 1}_{60:\overline{10}|} - P^{\ 1}_{50:\overline{20}|}\,\ddot{a}_{60:\overline{10}|} = 0.052752$$

$${}_{10}V_{50:\overline{20}|} = A_{60:\overline{10}|} - P_{50:\overline{20}|}\,\ddot{a}_{60:\overline{10}|} = 0.355380.$$

Then, under the assumption of a uniform distribution of deaths over each year of age, we have

$$\ddot{a}^{(2)}_{60:\overline{10}|} = \alpha(2)\,\ddot{a}_{60:\overline{10}|} - \beta(2)\,(1 - {}_{10}E_{60}) = 7.1392299.$$

The benefit reserve, ${}_{10}V^{(2)}_{50:\overline{20}|}$, can be calculated using either

(7.6.1): $$A_{60:\overline{10}|} - P^{(2)}_{50:\overline{20}|}\,\ddot{a}^{(2)}_{60:\overline{10}|} = 0.355822$$

or

(7.6.3): $${}_{10}V_{50:\overline{20}|} + \beta(2)\,P^{(2)}_{50:\overline{20}|}\,{}_{10}V^{\ 1}_{50:\overline{20}|} = 0.355822.$$

b. We need additional calculated values:

$$\frac{i}{\delta}\,A^{\ 1}_{50:\overline{20}|} = 0.13423835$$

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}^{(2)}_{50:\overline{20}|}} = 0.03286830$$

$$\frac{i}{\delta}\,A^{\ 1}_{60:\overline{10}|} = 0.14085233$$

$$\bar{A}_{50:\overline{20}|} = 0.36471188$$

$$P(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} = 0.03229873$$

$$\bar{A}_{60:\overline{10}|} = 0.59204806$$

$${}_{10}V(\bar{A}_{50:\overline{20}|}) = \bar{A}_{60:\overline{10}|} - P(\bar{A}_{50:\overline{20}|})\,\ddot{a}_{60:\overline{10}|} = 0.3569475$$

$${}_{10}V^{(2)}(\bar{A}_{50:\overline{20}|}) = \bar{A}_{60:\overline{10}|} - P^{(2)}(\bar{A}_{50:\overline{20}|})\,\ddot{a}^{(2)}_{60:\overline{10}|} = 0.3573937$$

$$\beta(2)\,P^{(2)}(\bar{A}_{50:\overline{20}|})\,{}_{10}V^{\ 1}_{50:\overline{20}|} = 0.000446.$$

This last value is the difference between the two directly above it, as shown in (7.6.5). ▼

7.7 Benefit Reserves on an Apportionable or Discounted Continuous Basis

In Section 6.5 we discussed apportionable, or discounted continuous, benefit premiums, and we now consider the corresponding benefit reserves. For integer k, we have by the prospective method

$$ {}_k^hV^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|})\, \ddot{a}^{\{m\}}_{x+k:\overline{h-k}|} \qquad k < h. \tag{7.7.1}$$

But by (6.5.2),

$$ {}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{d^{(m)}}{\delta}\, {}_h\bar{P}(\bar{A}_{x:\overline{n}|}),$$

and by (5.5.4),

$$ \ddot{a}^{\{m\}}_{x+k:\overline{h-k}|} = \frac{\delta}{d^{(m)}}\, \bar{a}_{x+k:\overline{h-k}|}.$$

Substitution into (7.7.1) yields, for an integer k,

$$ {}_k^hV^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+k:\overline{h-k}|} = {}_k^h\bar{V}(\bar{A}_{x:\overline{n}|}). \tag{7.7.2}$$

This means that, on anniversaries of the issue date, fully continuous benefit reserves can be used for all apportionable cases, independent of the premium-paying mode. The condition that k be an integer can be relaxed to being at the end of an m-th for m-thly premiums.

In Section 6.5, it was noted that the apportionable benefit premium could be decomposed as

$$ P^{\{1\}}(\bar{A}_x) = P(\bar{A}_x) + P(\bar{A}_x^{\mathrm{PR}}) \tag{7.7.3}$$

where the superscript PR is used to denote an insurance for the benefit premium refund feature. A similar decomposition for the benefit reserves can be verified by use of the prospective method and (6.5.7). The steps are:

$$\begin{aligned} {}_kV(\bar{A}_x^{\mathrm{PR}}) &= \bar{P}(\bar{A}_x)\, \frac{\bar{A}_{x+k} - A_{x+k}}{\delta} - P(\bar{A}_x^{\mathrm{PR}})\, \ddot{a}_{x+k} \\ &= \bar{P}(\bar{A}_x)\, \frac{d\ddot{a}_{x+k} - \delta\bar{a}_{x+k}}{\delta} - [P^{\{1\}}(\bar{A}_x) - P(\bar{A}_x)]\, \ddot{a}_{x+k}. \end{aligned}$$

Since

$$ \frac{d}{\delta}\, \bar{P}(\bar{A}_x) = P^{\{1\}}(\bar{A}_x),$$

the expression can be reduced to

$$\begin{aligned} {}_kV(\bar{A}_x^{\mathrm{PR}}) &= -\bar{P}(\bar{A}_x)\, \bar{a}_{x+k} + P(\bar{A}_x)\, \ddot{a}_{x+k} \\ &= \bar{A}_{x+k} - \bar{P}(\bar{A}_x)\, \bar{a}_{x+k} - [\bar{A}_{x+k} - P(\bar{A}_x)\, \ddot{a}_{x+k}] \end{aligned}$$

$$= {}_k\bar{V}(\bar{A}_x) - {}_kV(\bar{A}_x)$$
$$= {}_kV^{(1)}(\bar{A}_x) - {}_kV(\bar{A}_x).$$

Thus we have

$${}_kV^{(1)}(\bar{A}_x) = {}_kV(\bar{A}_k) + {}_kV(\bar{A}_x^{PR}). \tag{7.7.4}$$

7.8 Notes and References

This chapter has developed the idea of a reserve in parallel to the development of premiums in Chapter 6. Discussion of recursion formulas for reserves is deferred to Chapter 8. Reserve principles based on the utility functions used in Chapter 6 were first applied. Gerber (1976, 1979) develops these reserves in a more abstract setting. Benefit reserves, which followed from a linear utility function, were studied extensively. Scher (1974) explored the apportionable benefit premium reserves as discounted fully continuous benefit reserves.

Exercises

Section 7.1

7.1. Determine the benefit reserve for $t = 2, 3, 4$, and 5 for the insurance in Example 6.1.1.

7.2. Determine the exponential reserve for $t = 2, 3, 4$, and 5 for the insurance in Example 6.1.1.

7.3. Determine the exponential reserve for $t = 1, 2, 3, 4$, and 5 for the insurance in Exercise 6.2.

7.4. Consider the insurance in Example 7.1.1 and the insurer of Exercise 6.3 with utility function $u(x) = x - 0.01\,x^2$, $0 < x < 50$. Determine the reserve, ${}_kV$, for $k = 1, 2, 3$, and 4 such that the insurer, with wealth 10 at each duration, will be indifferent between continuing the risk while receiving premiums of 0.30360 (from Exercise 6.3) and paying the amount ${}_kV$ to a reinsurer to assume the risk.

7.5. Consider a unit insurance issued to (0) on a fully continuous basis using the following assumptions:
   i. De Moivre's law with $\omega = 5$
   ii. $i = 0.06$
   iii. Principle III of Example 6.1.1 with $\alpha = 0.1$.
   a. Display equations which can be solved for the exponential premium and the exponential reserve at $t = 1$.

b. Solve the equations of (a) for the numerical values for the exponential premium and exponential reserve. Numerical methods must be used to obtain these required solutions.

Section 7.2

7.6. For an n-year unit endowment insurance issued on a fully continuous basis to (x), define ${}_tL$, the prospective loss after duration t. Confirm that

$$\mathrm{Var}({}_tL|T > t) = \frac{{}^2\bar{A}_{x+t:\overline{n-t|}} - (\bar{A}_{x+t:\overline{n-t|}})^2}{(\delta \bar{a}_{x:\overline{n|}})^2}.$$

7.7. The prospective loss, after duration t, for a single benefit premium n-year continuous temporary life annuity of 1 per annum issued to (x) is given by

$${}_tL = \begin{cases} \bar{a}_{\overline{T-t|}} & t \le T < n \\ \bar{a}_{\overline{n-t|}} & T \ge n. \end{cases}$$

Express $\mathrm{E}[{}_tL|T > t]$ and $\mathrm{Var}({}_tL|T > t)$ in symbols of actuarial present values.

7.8. Write prospective formulas for
a. ${}^{20}_{10}\bar{V}(\bar{A}_{35:\overline{30|}})$
b. the benefit reserve at the end of 5 years for a unit benefit 10-year term insurance issued to (45) on a single premium basis.

7.9. a. For the fully continuous whole life insurance with the benefit premium determined by the equivalence principle, determine the outcome $u_0 = T(x) - t$ such that the loss is zero. [Caution: For large values of t, a solution may not exist.]
b. Determine the value of u_0 for $t = 20$ in Example 7.2.3 and compare it to Figure 7.2.1 for reasonableness.

7.10. The assumptions of Example 7.2.3 are repeated. Find the value of t such that the minimum loss is zero. Check your result by examining Figure 7.2.2.

7.11. a. Repeat the development leading to (7.2.9) to obtain the d.f. for the loss variable associated with an n-year fully continuous endowment insurance.
b. Draw the sketch that corresponds to Figure 7.2.1 for this endowment insurance.

7.12. Repeat Exercise 7.11 for an n-year fully continuous term insurance.

7.13. Confirm that (7.2.10) satisfies the conditions for a p.d.f.

Section 7.3

7.14. Write four formulas for ${}^{20}_{10}\bar{V}(\bar{A}_{40})$.

7.15. Write seven formulas for ${}_{10}\bar{V}(\bar{A}_{40:\overline{20|}})$.

7.16. Give the retrospective formula for ${}^{30}_{20}\bar{V}({}_{30|}\bar{a}_{35})$.

7.17. For $0 < t \le m$, show

a. $\bar{P}(\bar{A}_{x:\overline{m+n}|}) = \bar{P}(\bar{A}^{\,1}_{x:\overline{m}|}) + \bar{P}_{x:\overline{m}|}^{\;\;1}\ {}_m\bar{V}(\bar{A}_{x:\overline{m+n}|})$

b. ${}_t\bar{V}(\bar{A}_{x:\overline{m+n}|}) = {}_t\bar{V}(\bar{A}^{\,1}_{x:\overline{m}|}) + {}_t\bar{V}_{x:\overline{m}|}^{\;\;1}\ {}_m\bar{V}(\bar{A}_{x:\overline{m+n}|})$

and give an interpretation in words.

7.18. State what formula in Section 7.3 the following equation is related to, and give an interpretation in words:

$${}^{20}_{10}\bar{V}(\bar{A}_{30}) = \bar{A}^{\,1}_{40:\overline{5}|} + {}_5E_{40}\ {}^{20}_{15}\bar{V}(\bar{A}_{30}) - {}_{20}\bar{P}(\bar{A}_{30})\ \bar{a}_{40:\overline{5}|}.$$

Section 7.4

7.19. Write four formulas for ${}^{20}_{10}V_{40}$.

7.20. Write seven formulas for ${}_{10}V_{40:\overline{20}|}$.

7.21. For $0 < k \le m$, show

$${}_kV_{x:\overline{m+n}|} = {}_kV^{\,1}_{x:\overline{m}|} + {}_kV_{x:\overline{m}|}^{\;\;1}\ {}_mV_{x:\overline{m+n}|}.$$

7.22. If $k < n/2$, ${}_kV_{x:\overline{n}|} = 1/6$, and $\ddot{a}_{x:\overline{n}|} + \ddot{a}_{x+2k:\overline{n-2k}|} = 2\ \ddot{a}_{x+k:\overline{n-k}|}$, calculate ${}_kV_{x+k:\overline{n-k}|}$.

Section 7.5

7.23. On the basis of the Illustrative Life Table and interest of 6%, calculate values for the benefit reserves in the following table. (See Exercise 6.10.)

Fully Continuous	**Semicontinuous**	**Fully Discrete**
${}_{10}\bar{V}(\bar{A}_{35:\overline{30}|})$	${}_{10}V(\bar{A}_{35:\overline{30}|})$	${}_{10}V_{35:\overline{30}|}$
${}_{10}\bar{V}(\bar{A}_{35})$	${}_{10}V(\bar{A}_{35})$	${}_{10}V_{35}$
${}_{10}\bar{V}(\bar{A}^{\,1}_{35:\overline{30}|})$	${}_{10}V(\bar{A}^{\,1}_{35:\overline{30}|})$	${}_{10}V^{\,1}_{35:\overline{30}|}$

7.24. Under the assumption of a uniform distribution of deaths in each year of age, which of the following are correct?

a. ${}_kV(\bar{A}_{x:\overline{n}|}) = \dfrac{i}{\delta}\ {}_kV_{x:\overline{n}|}$

b. ${}_kV(\bar{A}_x) = \dfrac{i}{\delta}\ {}_kV_x$

c. ${}_kV(\bar{A}^{\,1}_{x:\overline{n}|}) = \dfrac{i}{\delta}\ {}_kV^{\,1}_{x:\overline{n}|}$

Section 7.6

7.25. Show that, under the assumption of a uniform distribution of deaths in each year of age,

$$\frac{{}_5V^{(4)}_{30:\overline{20}|} - {}_5V_{30:\overline{20}|}}{{}^{20}_{5}V^{(4)}_{30} - {}^{20}_{5}V_{30}} = \frac{A_{30:\overline{20}|}}{A_{30}}.$$

(The assumption is sufficient but not necessary.)

7.26. Which of the following are correct formulas for ${}_{15}V_{40}^{(m)}$?

a. $(P_{55}^{(m)} - P_{40}^{(m)})\, \ddot{a}_{55}^{(m)}$ b. $\left(1 - \dfrac{P_{40}^{(m)}}{P_{55}^{(m)}}\right) A_{55}$

c. $P_{40}^{(m)}\, \ddot{s}_{40:\overline{15}|}^{(m)} - {}_{15}k_{40}$ d. $1 - \dfrac{\ddot{a}_{55}^{(m)}}{\ddot{a}_{40}^{(m)}}$

Section 7.7

7.27. Which of the following are correct formulas for ${}_{15}V^{\{4\}}(\bar{A}_{40})$?

a. ${}_{15}\bar{V}(\bar{A}_{40})$ b. $[P^{\{4\}}(\bar{A}_{55}) - P^{\{4\}}(\bar{A}_{40})]\ddot{a}_{55}^{\{4\}}$

c. $[\bar{P}(\bar{A}_{55}) - \bar{P}(\bar{A}_{40})]\bar{a}_{55}$ d. $\left[1 - \dfrac{\bar{P}(\bar{A}_{40})}{\bar{P}(\bar{A}_{55})}\right] \bar{A}_{55}$

e. $1 - \dfrac{\bar{a}_{55}}{\bar{a}_{40}}$ f. $\bar{P}(\bar{A}_{40})\, \bar{s}_{40:\overline{15}|} - {}_{15}\bar{k}_{40}$

7.28. Show that
a. $P^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_{n}P^{\{m\}}(\bar{A}_x) + (1 - \bar{A}_{x+n})P_{x:\overline{n}|}^{\{m\}\,\,\,1}$
b. ${}_{k}V^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_{k}^{n}V^{\{m\}}(\bar{A}_x) + (1 - \bar{A}_{x+n})_{k}V_{x:\overline{n}|}^{\{m\}\,\,\,1}$.
Give an interpretation in words.

Miscellaneous

7.29. Calculate the value of $P_{x:\overline{n}|}^{1}$ if ${}_{n}V_x = 0.080$, $P_x = 0.024$, and $P_{x:\overline{n}|}^{\,\,\,\,1} = 0.2$.

7.30. If ${}_{10}V_{35} = 0.150$ and ${}_{20}V_{35} = 0.354$, calculate ${}_{10}V_{45}$.

7.31. A whole life insurance issued to (25) pays a unit benefit at the end of the year of death. Premiums are payable annually to age 65. The benefit premium for the first 10 years is P_{25} followed by an increased level annual benefit premium for the next 30 years. Use your Illustrative Life Table and $i = 0.06$ to find the following.
a. The annual benefit premium payable at ages 35 through 64.
b. The tenth-year benefit reserve.
c. At the end of 10 years the policyholder has the option to continue with the benefit premium P_{25} until age 65 in return for reducing the death benefit to B for death after age 35. Calculate B.
d. If the option in (c) is selected, calculate the twentieth-year benefit reserve.

7.32. Assuming $\delta = 0.05$, $q_x = 0.05$, and a uniform distribution of deaths in each year of age, calculate
a. $(\bar{I}\bar{A})_{x:\overline{1}|}^{1}$ b. ${}_{1/2}V(\bar{I}\bar{A})_{x:\overline{1}|}^{1}$.

8

ANALYSIS OF BENEFIT RESERVES

8.1 Introduction

In Chapter 3 probability distributions for future lifetime random variables were developed. Chapters 4 and 5 studied the present-value random variables for insurances and annuities. The funding of insurance and annuities with a system of periodic payments was explored in Chapter 6, and in Chapter 7 the evolution of the liabilities under the periodic payments to fund an insurance or annuity was discussed. In these last two chapters the emphasis was on ***level*** benefits funded by ***level*** periodic payments that are usually determined by an application of the equivalence principle.

Why was the emphasis on level payments? First, traditional insurance products are purchased with level contract premiums. It is natural to think of a constant portion of each premium being for the benefit, hence a level benefit premium. Second, the single equation of the equivalence principle yields a solution for only one parameter. It is natural to think of this parameter as the benefit premium. Third, until the incidence of expenses is discussed in Chapter 15, one of the motivations of nonlevel benefit premiums is not present. Fourth, historically some regulatory standards have been specified in terms of benefit reserves defined by level benefit premiums.

In this chapter we define benefit reserves as we did in Chapter 7, but the definition is applied to general contracts with possibly nonlevel benefits and premiums. Of course, level premiums are a special case of nonlevel premiums, so the ideas here apply to the examples of Chapter 7. However, the reverse is not true. The special technique and relationships of Chapter 7 may not apply to the more general contracts of this chapter.

We start with definitions of general fully continuous and fully discrete insurances, and recursion relations are developed for these general models. The general discrete model is used to obtain formulas for benefit reserves at durations other than a contract anniversary, something that was not obtained in Chapter 7. An allocation of loss and of risk of the contract to the various periods of the contract duration is obtained by use of the general fully discrete model. Again, these ideas apply to the contracts of Chapter 7, and the reader is encouraged to exercise this application.

8.2 Benefit Reserves for General Insurances

Consider a general fully discrete insurance on (x) in which

- The death benefit is payable at the end of the policy year of death
- Premiums are payable annually, at the beginning of the policy year
- The death benefit in the j-th policy year is b_j, $j = 1, 2, \ldots$
- The benefit premium payment in the j-th policy year is π_{j-1}, $j = 1, 2, \ldots$.

Note that the subscripts of b and π are the times of payment.

For a non-negative integer, h, the prospective loss, ${}_hL$, is the present value at h of the future benefits less the present value at h of the future benefit premiums. Expressed as a function of $K(x)$, it is

$${}_hL = \begin{cases} 0 & K(x) = 0, 1, \ldots h-1 \\ b_{K(x)+1}\, v^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j\, v^{j-h} & K(x) = h, h+1, \ldots. \end{cases} \tag{8.2.1}$$

Note: This definition extends the one given in (7.4.1) by including values (zeros) of ${}_hL$ for $K(x)$ less than h. Of course, this extension will not change the value of the benefit reserve because it is the conditional expectation given $K(x) \geq h$. The extension will be used in the development of recursion relations.

The benefit reserve at h, which we will denote by ${}_hV$, is defined as

$$\begin{aligned} {}_hV &= \mathrm{E}[{}_hL \mid K(x) \geq h] \\ &= \mathrm{E}\left[b_{K(x)+1}\, v^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j\, v^{j-h} \mid K(x) \geq h \right] \\ &= \mathrm{E}\left[b_{(K(x)-h)+h+1}\, v^{(K(x)-h)+1} - \sum_{j=0}^{K(x)-h} \pi_{h+j}\, v^j \mid K(x) \geq h \right]. \end{aligned} \tag{8.2.2}$$

Under the assumption that the conditional distribution of $K(x) - h$, given $K(x) = h, h+1, \ldots$, is equal to the distribution of $K(x+h)$, this last expression can be rewritten as

$$\begin{aligned} {}_hV &= \mathrm{E}\left[b_{K(x+h)+h+1}\, v^{K(x+h)+1} - \sum_{j=0}^{K(x+h)} \pi_{h+j}\, v^j \right] \\ &= \sum_{j=0}^{\infty} \left(b_{h+j+1}\, v^{j+1} - \sum_{k=0}^{j} \pi_{h+k}\, v^k \right) {}_jp_{x+h}\, q_{x+h+j}. \end{aligned} \tag{8.2.3}$$

Note that if this assumption fails, we are in the select mortality mode. By applying summation by parts (see Appendix 5) or reversing the order of summation, (8.2.3) can be rewritten as

$$ {}_hV = \sum_{j=0}^{\infty} b_{h+j+1}\, v^{j+1}\, {}_jp_{x+h}\, q_{x+h+j} - \sum_{j=0}^{\infty} \pi_{h+j}\, v^j\, {}_jp_{x+h}. \tag{8.2.4} $$

Thus, ${}_hV$ as defined by (8.2.2) converts readily to the prospective formula: the actuarial present value of future benefits less the actuarial present value of future benefit premiums.

In Chapter 7 we discussed four types of formulas for the benefit reserve: prospective, retrospective, premium-difference, and paid-up insurance. These were applicable to benefit reserves for contracts with level benefit premiums and level benefits. Only the prospective and retrospective forms extend naturally to the general fully discrete insurance. The retrospective formula will be developed in the next section.

Example 8.2.1

A fully discrete whole life insurance with a unit benefit issued to (x) has its first year's benefit premium equal to the actuarial present value of the first year's benefit, and the remaining benefit premiums are level and determined by the equivalence principle. Determine formulas for (a) the first year's benefit premium, (b) the level benefit premium after the first year, and (c) the benefit reserve at the first duration.

Solution:

a. From Chapter 4, $\pi_0 = A^{1}_{x:\overline{1}|}$.
b. By the equivalence principle $A^{1}_{x:\overline{1}|} + \pi a_x = A_x$, so $\pi = (A_x - A^{1}_{x:\overline{1}|}) / a_x = A_{x+1} / \ddot{a}_{x+1} = P_{x+1}$.
c. By the prospective formula, ${}_1V = A_{x+1} - \pi \ddot{a}_{x+1} = 0$. ▼

Example 8.2.1 illustrates one approach to nonlevel premiums. Another approach would be to set a premium pattern in the loss variable ${}_0L$ as defined in (8.2.1) by a set of weights, w_j, for $j = 0, 1, 2, \ldots$. Applying the equivalence principle, we have in a special case of (8.2.1)

$$ \mathrm{E}[{}_0L] = 0, $$

or

$$ \sum_{j=0}^{\infty} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j} = \pi \sum_{j=0}^{\infty} w_j\, v^j\, {}_jp_x \tag{8.2.5} $$

and

$$ \pi = \frac{\sum_{j=0}^{\infty} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j}}{\sum_{j=0}^{\infty} w_j\, v^j\, {}_jp_x}. \tag{8.2.6} $$

By different selections of sequences $\{b_{j+1}; j = 0, 1, 2, \ldots\}$ and $\{w_j; j = 0, 1, 2, \ldots\}$, the various benefit premium formulas can be obtained.

If we consider the sequence $\{b_{j+1}; j = 0, 1, 2, \ldots\}$ to be fixed, there remains great flexibility in selecting the sequence $\{w_j; j = 0, 1, 2, \ldots\}$, which in turn determines the sequence $\{\pi_j; j = 0, 1, 2, \ldots\}$. There may be commercial considerations to require that $w_j \geq 0$ for all j, but the equivalence principle does not impose this condition. In Example 8.2.1, $b_{j+1} = 1$ for all j, but $\pi w_0 = A^{1}_{x:\overline{1}|}$ and $\pi w_j = (A_x - A^{1}_{x:\overline{1}|}) / a_x$, $j = 1, 2, \ldots,$ and (8.2.6) is satisfied. A different application is found in Example 8.2.2.

Example 8.2.2

The annual benefit premiums for a fully discrete whole life insurance with a unit benefit issued to (x) are $\pi_j = \pi w_j$, where $w_j = (1 + r)^j$. The rate r might be selected to estimate the expected growth rate in the insured's income.

Develop formulas for

a. π
b. ${}_hV$ and
c. ${}_hV$ when $r = i$.

Solution:

a. Using (8.2.5), $\pi = A_x / \ddot{a}^*_x$, where $\ddot{a}^*_x$ is valued at the rate of interest $i^* = (i - r) / (1 + r)$. When $r = i$, $\pi = A_x / (e_x + 1)$.
b. Using (8.2.4),

$$
\begin{aligned}
{}_hV &= A_{x+h} - \sum_{j=0}^{\infty} \pi_{j+h}\, v^j\, {}_jp_{x+h} \\
&= A_{x+h} - \frac{A_x}{\ddot{a}^*_x} (1 + r)^h \sum_{j=0}^{\infty} \left(\frac{1 + r}{1 + i}\right)^j {}_jp_{x+h} \\
&= A_{x+h} - \frac{A_x}{\ddot{a}^*_x} (1 + r)^h \ddot{a}^*_{x+h}.
\end{aligned}
$$

c. ${}_hV = A_{x+h} - [A_x / (e_x + 1)](1 + r)^h (e_{x+h} + 1)$. ▼

In Example 8.2.2, negative benefit reserves are possible with higher values of r. See Exercise 8.32 for a variation of this policy.

Now consider a general fully continuous insurance on (x) under which
- The death benefit payable at the moment of death, t, is b_t, and
- Benefit premiums are payable continuously at t at the annual rate, π_t.

The prospective loss for a life insured at x and surviving at t is the present value at t of the future benefits less the present value at t of the future benefit premiums:

$$
{}_tL = \begin{cases} 0 & T(x) \le t \\ b_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} \pi_u\, v^{u-t}\, du & T(x) > t. \end{cases} \tag{8.2.7}
$$

The benefit reserve for this general case, which we will denote by ${}_t\bar{V}$, is then

$$
\begin{aligned}
{}_t\bar{V} &= \mathrm{E}[{}_tL \mid T(x) > t] \\
&= \mathrm{E}\left[b_{T(x)}\, v^{T(x)-t} - \int_t^{T(x)} \pi_u\, v^{u-t}\, du \Big| T(x) > t \right] \\
&= \mathrm{E}\left[b_{(T(x)-t)+t}\, v^{T(x)-t} - \int_0^{T(x)-t} \pi_{t+r}\, v^r\, dr \Big| T(x) > t \right].
\end{aligned} \tag{8.2.8}
$$

As assumed to obtain (8.2.3) for the fully discrete insurance, we assume here that the conditional distribution of $T(x) - t$, given $T(x) > t$, is the same as the distribution of $T(x + t)$ and proceed to

$$
\begin{aligned}
{}_t\bar{V} &= \mathrm{E}\left[b_{T(x+t)+t}\, v^{T(x+t)} - \int_0^{T(x+t)} \pi_{t+r}\, v^r\, dr \right] \\
&= \int_0^\infty \left(b_{t+u}\, v^u - \int_0^u \pi_{t+r}\, v^r\, dr \right) {}_up_{x+t}\, \mu_x(t+u)\, du \\
&= \int_0^\infty b_{t+u}\, v^u\, {}_up_{x+t}\, \mu_x(t+u)\, du - \int_0^\infty \pi_{t+r}\, v^r\, {}_rp_{x+t}\, dr.
\end{aligned} \tag{8.2.9}
$$

The second integral in (8.2.9) is obtained by integration by parts, or, alternatively, by reversing the order of integration. In other words, ${}_t\bar{V}$ can be expressed as the actuarial present value of future benefits less the actuarial present value of future benefit premiums. If the assumption about the conditional distribution of $T(x) - t$, given $T(x) > t$, does not hold, we are in the select mortality mode.

8.3 Recursion Relations for Fully Discrete Benefit Reserves

One objective of this chapter is to explore recursion relations among the loss random variables, their expected values, and variances. We start with a definition for the insurer's net cash loss (negative cash flow) within each insurance year for the fully discrete model as defined for (8.2.1). Figure 8.3.1 is a time diagram that shows the annual cash income and cash outgo.

Insurer's Cash Income and Outgo for General Fully Discrete Insurance

Outgo		0	0	0	0	0	$b_{K(x)+1}$	0	0	
Income	π_0	π_1	π_2	π_3	$\pi_{K(x)-1}$	$\pi_{K(x)}$	0	0	0	etc.
	0	1	2	3	$K(x)-1$	$K(x)$	$K(x)+1$	$K(x)+2$	$K(x)+3$	

Let C_h denote the present value at h of the net cash loss during the year $(h, h + 1)$. If $(h, h + 1)$ is before the year of death $[h < K(x)]$, then $C_h = -\pi_h$. If $(h, h + 1)$ is the year of death $[h = K(x)]$, then $C_h = v\, b_{h+1} - \pi_h$. And if $(h, h + 1)$ is after the year of death, of course, $C_h = 0$. Restating this definition as an explicit function of $K(x)$,

$$C_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h - 1 \\ v\, b_{h+1} - \pi_h & K(x) = h \\ -\pi_h & K(x) = h + 1, h + 2, \ldots . \end{cases} \tag{8.3.1}$$

For the conditional distribution of C_h, given $K(x) > h$, we observe that

$$C_h = vb_{h+1}\, I - \pi_h,$$

where

$$I = \begin{cases} 1 & \text{with probability } q_{x+h} \\ 0 & \text{with probability } p_{x+h}. \end{cases}$$

Therefore,

$$\mathrm{E}[C_h | K(x) \geq h] = v\, b_{h+1}\, q_{x+h} - \pi_h \tag{8.3.2}$$

and

$$\mathrm{Var}[C_h | K(x) \geq h] = (v\, b_{h+1})^2\, q_{x+h}\, p_{x+h}. \tag{8.3.3}$$

Moreover, using (2.2.10) and (2.2.11) along with (8.3.1)–(8.3.3),

$$\mathrm{E}[C_h] = (v\, b_{h+1}\, q_{x+h} - \pi_h)\, {}_hp_x \tag{8.3.4}$$

and

$$\mathrm{Var}(C_h) = (v\, b_{h+1}\, q_{x+h} - \pi_h)^2\, {}_hp_x\, {}_hq_x + (v\, b_{h+1})^2 q_{x+h}\, p_{x+h}\, {}_hp_x. \tag{8.3.5}$$

Finally, for $j > h$, C_j and C_h are correlated, an assertion that is left for the student to verify in Exercises 8.5 and 8.6.

As previously defined in (8.2.1), ${}_hL$ is the present value at h of the insurer's future cash outflow less the present value at h of the insurer's future cash income. By rearranging the terms in this definition, an equivalent one that states ${}_hL$ as the sum of the present values at h of the insurer's future net annual cash losses is obtained. This is

$${}_hL = \sum_{j=h}^{\infty} v^{j-h}\, C_j. \tag{8.3.6}$$

For $h < K(x)$, we have from (8.3.1),

$$\begin{aligned} {}_hL &= \sum_{j=h}^{K(x)} v^{j-h}\, C_j = v^{K(x)-h}\, (v\, b_{K(x)+1} - \pi_{K(x)}) - \sum_{j=h}^{K(x)-1} v^{j-h} \pi_j \\ &= v^{K(x)+1-h}\, b_{K(x)+1} - \sum_{j=h}^{K(x)} v^{j-h}\, \pi_j \end{aligned}$$

as before. For $h = K(x)$, ${}_hL = C_{K(x)} = (v\, b_{K(x)+1} - \pi_{K(x)})$. And for $h > K(x)$, both sides of (8.3.6) are zero.

A recursion relation for the loss variables follows from (8.3.6):

$$_hL = C_h + v \sum_{j=h+1}^{\infty} v^{j-(h+1)} C_j = C_h + v \ _{h+1}L. \tag{8.3.7}$$

A recursion relation for the benefit reserves can be obtained from (8.3.7) by

$$\begin{aligned} _hV &= \mathrm{E}[_hL|K(x) \ge h] \\ &= \mathrm{E}[C_h + v \ _{h+1}L|K(x) \ge h] \\ &= v \ b_{h+1} \ q_{x+h} - \pi_h + v \ \mathrm{E}[_{h+1}L|K(x) \ge h]. \end{aligned} \tag{8.3.8}$$

Since $_{h+1}L$ is zero when $K(x)$ is h, we have

$$\begin{aligned} _hV &= v \ b_{h+1} \ q_{x+h} - \pi_h + v \ \mathrm{E}[_{h+1}L|K(x) \ge h + 1] \ p_{x+h} \\ &= v \ b_{h+1} \ q_{x+h} - \pi_h + v \ _{h+1}V \ p_{x+h}. \end{aligned} \tag{8.3.9}$$

Formula (8.3.9) is a backward recursion formula $[u(h) = c(h) + d(h) \times u(h + 1)]$ for the general fully discrete benefit reserve. Note that $d(h) = vp_{x+h}$ again and $c(h) = vb_{h+1}q_{x+h} - \pi_h$. A forward recursion formula can be obtained by solving (8.3.9) for $_{h+1}V$. (See Exercise 8.2.) This forward formula was used in Examples 7.4.3 and 7.4.4 in an aggregate mode; that is, the mortality functions were in life table form.

Further insight to the progress of benefit reserves can be gained by rearrangements of (8.3.9). First, add π_h to both sides, to see

$$_hV + \pi_h = b_{h+1} \ v \ q_{x+h} + \ _{h+1}V \ v \ p_{x+h}. \tag{8.3.10}$$

In words, the resources required at the beginning of insurance year $h + 1$ equal the actuarial present value of the year-end requirements. The sum $_hV + \pi_h$ is called the ***initial benefit reserve*** for the policy year $h + 1$. In contrast, $_hV$ and $_{h+1}V$ are called the ***terminal benefit reserves*** for insurance years h and $h + 1$ to indicate that they are year-end benefit reserves.

Formula (8.3.10) can be rearranged to separate the benefit premium π_h into components for insurance year $h + 1$, namely,

$$\pi_h = b_{h+1} \ v \ q_{x+h} + (_{h+1}V \ v \ p_{x+h} - \ _hV). \tag{8.3.11}$$

The first component on the right-hand side of (8.3.11) is the 1-year term insurance benefit premium for the sum insured b_{h+1}. The second component, $_{h+1}V \ v \ p_{x+h} - \ _hV$, represents the amount which, if added to $_hV$ at the beginning of the year, would accumulate under interest and survivorship to $_{h+1}V$ at the end of the year.

For the purpose of subsequent comparison with formulas for a fully continuous insurance, we multiply both sides of (8.3.11) by $1 + i$ and rearrange the formula to

$$\pi_h + (_hV + \pi_h)i + \ _{h+1}Vq_{x+h} = b_{h+1} \ q_{x+h} + \Delta(_hV). \tag{8.3.12}$$

The left-hand side of (8.3.12) indicates resources for insurance year $h + 1$, namely, the benefit premium, interest for the year on the initial benefit reserve, and the expected release by death of the terminal benefit reserve. The right-hand side consists of the expected payment of the death benefit at the end of the year and the increment ${}_{h+1}V - {}_hV$ in the benefit reserve.

An analysis different than (8.3.10)–(8.3.12) results if one considers that the benefit reserve ${}_{h+1}V$ is to be available to offset the death benefit b_{h+1}, and that only the ***net amount at risk***, $b_{h+1} - {}_{h+1}V$, needs to be covered by 1-year term insurance. For this analysis we have, on substituting $1 - q_{x+h}$ for p_{x+h} in (8.3.10) and multiplying through by $1 + i$,

$$ {}_{h+1}V = ({}_hV + \pi_h)(1 + i) - (b_{h+1} - {}_{h+1}V)q_{x+h}. \tag{8.3.13}$$

Corresponding to (8.3.11), we now have

$$\pi_h = (b_{h+1} - {}_{h+1}V)v\, q_{x+h} + (v\, {}_{h+1}V - {}_hV). \tag{8.3.14}$$

The first component on the right-hand side of (8.3.14) is the 1-year term insurance benefit premium for the net amount of risk. The second component, $v\, {}_{h+1}V - {}_hV$, is the amount which, if added to ${}_hV$ at the beginning of the year, would accumulate under interest to ${}_{h+1}V$ at the end of the year. In this formulation ${}_{h+1}V$ is used, in case of death, to offset the death benefit. Consequently, the benefit reserve accumulates as a savings fund. This is shown again by the formula corresponding to (8.3.12), namely,

$$\pi_h + ({}_hV + \pi_h)i = (b_{h+1} - {}_{h+1}V)q_{x+h} + \Delta({}_hV), \tag{8.3.15}$$

which is left for the reader to interpret.

The analysis by (8.3.11) does not use the benefit reserve to offset the death benefit, and consequently the benefit reserve accumulates under interest and survivorship. Both components of the right-hand side of (8.3.11) involve mortality risk, whereas in (8.3.14) only the first component does. We see in Section 8.5 that (8.3.14) is related to a flexible means for calculating the variance of loss attributable to the random nature of time until death.

Formulas (8.3.10)–(8.3.15) are all recursion relations for the benefit reserve at integral durations. None of the six is written in the form of an explicit backward or forward formula; rather, each is written to give an insight. In Example 8.3.1, recursion relation (8.3.14) is used to obtain an explicit formula for the benefit premium and benefit reserve.

Example 8.3.1

A deferred whole life annuity-due issued to (x) for an annual income of 1 commencing at age $x + n$ is to be paid for by level annual benefit premiums during the deferral period. The benefit for death prior to age $x + n$ is the benefit reserve. Assuming the death benefit is paid at the end of the year of death, determine the annual benefit premium and the benefit reserve at the end of year k for $k \le n$.

Solution:

Using the fact that $b_{h+1} = {}_{h+1}V$ for $h = 0, 1, 2, \ldots, n-1$, in (8.3.14), we have

$$\pi = v\ {}_{h+1}V - {}_hV.$$

On multiplication by v^h, we have

$$\pi v^h = v^{h+1}\ {}_{h+1}V - v^h\ {}_hV = \Delta(v^h\ {}_hV). \qquad (8.3.16)$$

Summing over $h = 0, 1, 2, \ldots, n-1$, we obtain

$$v^n\ {}_nV - v^0\ {}_0V = \pi \sum_{h=0}^{n-1} v^h = \pi \ddot{a}_{\overline{n}|},$$

and, since ${}_0V = 0$ and ${}_nV = \ddot{a}_{x+n}$, it follows that

$$\pi = v^n \frac{\ddot{a}_{x+n}}{\ddot{a}_{\overline{n}|}} = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$

Thus, this annuity is identical to that described in Example 6.6.2. The benefit reserve at the end of k years can be found by summing (8.3.16) over $h = 0, 1, 2, \ldots, k-1$ to give

$$v^k\ {}_kV = \pi \ddot{a}_{\overline{k}|},$$

from which

$$_kV = \pi \ddot{s}_{\overline{k}|}.$$

▼

Example 8.3.2

A fully discrete n-year endowment insurance on (x) provides, in case of death within n years, a payment of 1 plus the benefit reserve. Obtain formulas for the level benefit premium and the benefit reserve at the end of k years, given that the maturity value is 1.

Solution:

In this case $b_h = 1 + {}_hV$, and the net amount at risk has constant value 1. Denoting the annual benefit premium by π and using (8.3.14), we have

$$v\ {}_{h+1}V - {}_hV = \pi - v\ q_{x+h} \qquad h = 0, 1, \ldots, n-1.$$

On multiplication by v^h, this becomes

$$\Delta(v^h\ {}_hV) = \pi\ v^h - v^{h+1}\ q_{x+h}. \qquad (8.3.17)$$

Summing this over $h = 0, 1, 2, \ldots, n-1$, we obtain

$$v^n\ {}_nV = \pi\ \ddot{a}_{\overline{n}|} - \sum_{h=0}^{n-1} v^{h+1} q_{x+h}$$

so that, with ${}_nV = 1$ (the maturity value is 1),

$$\pi = \frac{v^n + \sum_{h=0}^{n-1} v^{h+1} q_{x+h}}{\ddot{a}_{\overline{n}|}}.$$

By summing (8.3.17) over $h = 0, 1, 2, \ldots, k-1$, and solving for ${}_kV$, we have

$$ {}_kV = \pi \ddot{s}_{\overline{k|}} - \sum_{h=0}^{k-1} (1+i)^{k-h-1} q_{x+h}. $$

▼

Just before Example 8.2.1 we promised to develop a retrospective formula for the benefit reserve of the general fully discrete insurance in this section. We start by rewriting recursion relation (8.3.11) in the form

$$ \pi_h - b_{h+1}\, v\, q_{x+h} = {}_{h+1}V\, v\, p_{x+h} - {}_hV $$

and then multiplying both sides by $v^h\, {}_hp_x$ to obtain

$$ \begin{aligned} \pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h} &= {}_{h+1}V\, v^{h+1}\, {}_{h+1}p_x - {}_hV\, v^h\, {}_hp_x \\ &= \Delta({}_hV\, v^h\, {}_hp_x), \end{aligned} \tag{8.3.18} $$

which holds for $h = 0, 1, 2, \ldots$. When we sum both sides of (8.3.18) over the values from 0 to $k-1$, we have

$$ \sum_{h=0}^{k-1} (\pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h}) = {}_kV\, v^k\, {}_kp_x - {}_0V. $$

With equivalence principle premiums, ${}_0V = 0$, so we can rewrite this last equation for the terminal benefit reserve of the general fully discrete insurance as

$$ {}_kV = \sum_{h=0}^{k-1} \frac{\pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h}}{v^k\, {}_kp_x} $$

and then as

$$ {}_kV = \sum_{h=0}^{k-1} (\pi_h - v b_{h+1}\, q_{x+h}) \frac{(1+i)^{k-h}}{{}_{k-h}p_{x+h}}. \tag{8.3.19} $$

Formula (8.3.19) shows the benefit reserve at k as the sum over the first k years of each year's premium less its expected death benefit accumulated with respect to interest and mortality to k.

8.4 Benefit Reserves at Fractional Durations

We consider again the general fully discrete insurance of (8.2.1) on (x) for a death benefit of b_{j+1} at the end of insurance year $j+1$, purchased by annual benefit premiums of π_j, $j = 0, 1, \ldots$, payable at the beginning of the insurance year. We seek a formula for, and an approximation to, the ***interim benefit reserve***, that is, ${}_{h+s}V$ for $h = 0, 1, 2, \ldots$ and $0 < s < 1$. Extending the earlier definition of the benefit reserve as stated in (8.2.1) and (8.2.2), we have for the interim case

$$ {}_{h+s}L = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ v^{1-s}\, b_{K(x)+1} & K(x) = h \\ v^{K(x)+1-(h+s)}\, b_{K(x)+1} - \displaystyle\sum_{j=h+1}^{K(x)} v^{j-(h+s)}\, \pi_j & K(x) = h+1, h+2, \ldots \end{cases} \tag{8.4.1} $$

and

$$_{h+s}V = \mathrm{E}[_{h+s}L|T(x) > h + s]. \tag{8.4.2}$$

From (8.4.2),

$$_{h+s}V = v^{1-s}\, b_{h+1}\ {}_{1-s}q_{x+h+s} + v^{1-s}\ _{h+1}V\ {}_{1-s}p_{x+h+s}. \tag{8.4.3}$$

Now multiply both sides of (8.4.3) by $v^s\ {}_sp_{x+h}$ to obtain

$$v^s{}_sp_{x+h}\ {}_{h+s}V = v\ b_{h+1}({}_{s|1-s}q_{x+h}) + v({}_{h+1}V)\ p_{x+h}. \tag{8.4.4}$$

Equation (8.3.9) provides an expression for $vb_{h+1}\ q_{x+h}$, which can be substituted into (8.4.4) to obtain

$$v^s{}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h - {}_{h+1}V\ v\ p_{x+h})\ \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + v\ {}_{h+1}V\ p_{x+h},$$

which can be rearranged to

$$v^s{}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h)\ \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + ({}_{h+1}V\ v\ p_{x+h})\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{8.4.5}$$

This exact expression shows that when the interim benefit reserve at $h + s$ is discounted with respect to interest and mortality to h, the result is equal to an interpolated value between the initial benefit reserve at h and the value of the terminal benefit reserve at $h + 1$ discounted to h.

We emphasize that the interpolation is, in general, not linear; however, under the assumption of uniform distribution of deaths over the age interval the interpolation weights are linear and (8.4.5) is

$$v^s{}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h)(1 - s) + ({}_{h+1}V\ v\ p_{x+h})(s). \tag{8.4.6}$$

By replacing i and q_{x+h} with zeros in (8.4.6), as an approximation, the result is linear interpolation between the initial benefit reserve at h and the terminal benefit reserve. The approximate result is

$$_{h+s}V = (1 - s)({}_hV + \pi_h) + s({}_{h+1}V), \tag{8.4.7}$$

which is often written in the form

$$_{h+s}V = (1 - s)({}_hV) + s({}_{h+1}V) + (1 - s)\pi_h. \tag{8.4.8}$$

Here the interim benefit reserve is the sum of the value obtained by linear interpolation between the terminal benefit reserves,

$$(1 - s)({}_hV) + s({}_{h+1}V),$$

and the ***unearned benefit premium*** $(1 - s)\pi_h$. In general,

(the unearned benefit premium at a given time during the year) = (the benefit premium for the year)
× (the difference between the time through which the premium has been paid and the given time).

Thus, on an annual premium basis, the benefit premium has been paid to the end of the year so at time s the unearned benefit premium is $(1 - s)\pi_h$. This notion of an unearned benefit premium will be used in discussing approximations to benefit reserves when the premiums are collected by installments more frequent than an annual basis.

We consider now one such case, that of true semiannual premiums with claims paid at the end of the insurance year of death. For $0 < s \le 1/2$ we could start with the random variable giving the present value of prospective losses as of time $h + s$ and then calculate its conditional expectation given that (x) has survived to $h + s$. This is a bit more complex than it was for (8.4.4), so we start with the equation corresponding to (8.4.3) by noting that it is the prospective formula

$$\begin{aligned} {}_{h+s}V^{(2)} &= v^{1-s}\, b_{h+1}({}_{1-s}q_{x+h+s}) + v^{1-s}({}_{h+1}V^{(2)})\ {}_{1-s}p_{x+h+s} \\ &\quad - \frac{\pi_h}{2}\,(v^{0.5-s})({}_{0.5-s}p_{x+h+s}). \end{aligned} \tag{8.4.9}$$

The first two terms of (8.4.9) can be viewed as the actuarial present value of the death benefit and an endowment benefit of amount equal to the reserve, and the third term is the actuarial present value of the future benefit premium for the $1 - s$ year endowment insurance. Multiplying both sides of (8.4.9) by ${}_sp_{x+h}\, v^s$, we have

$$\begin{aligned} {}_sp_{x+h}\, v^s\ {}_{h+s}V^{(2)} &= v b_{h+1}({}_{s|1-s}q_{x+h}) + v({}_{h+1}V^{(2)})\ p_{x+h} \\ &\quad - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h}). \end{aligned} \tag{8.4.10}$$

For the semiannual premium policy the equation corresponding to (8.3.10) is also a prospective benefit reserve formula:

$$ {}_hV^{(2)} = b_{h+1} v q_{x+h} + {}_{h+1}V^{(2)} v p_{x+h} - \frac{\pi_h}{2}\,(1 + v^{0.5}\,{}_{0.5}p_{x+h}). \tag{8.4.11}$$

Equation (8.4.11) provides an expression for vb_{h+1} to substitute in (8.4.10), which yields

$$\begin{aligned} {}_sp_{x+h}\, v^s\ {}_{h+s}V^{(2)} &= \left({}_hV^{(2)} + \frac{\pi_h}{2}\right) \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} \\ &\quad + \left[v({}_{h+1}V^{(2)})\ p_{x+h} - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h}) \right] \\ &\quad \times \left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \end{aligned} \tag{8.4.12}$$

Formula (8.4.12) corresponds to (8.4.5), which showed that the interim benefit reserve at $h + s$, discounted with respect to interest and mortality to h, is equal to a nonlinear interpolated value between the initial benefit reserve at h and the discounted value to h of the terminal benefit reserve at $h + 1$. For the semiannual premium case, this terminal reserve has been reduced by the amount of the discounted value of the midyear benefit premium. Under the assumption of a uniform distribution of deaths over the age interval, we have linear interpolation on the right-hand side:

$$ {}_{s}p_{x+h}\, v^{s}\, {}_{h+s}V^{(2)} = \left({}_{h}V^{(2)} + \frac{\pi_h}{2}\right)(1 - s) $$
$$ + \left[v({}_{h+1}V^{(2)})\, p_{x+h} - \frac{\pi_h}{2}(v^{0.5})({}_{0.5}p_{x+h})\right](s). \qquad (8.4.13) $$

Again setting i and q_{x+h} equal to zero, as an approximation, we obtain simple linear interpolation between the initial benefit reserve and the terminal benefit reserve reduced by the benefit premium due at midyear:

$$ {}_{h+s}V^{(2)} = \left({}_{h}V^{(2)} + \frac{\pi_h}{2}\right)(1 - s) + \left({}_{h+1}V^{(2)} - \frac{\pi_h}{2}\right)(s). $$

This formula can be rearranged as the interpolated value between the terminal benefit reserves plus the unearned benefit premium $(\pi_h)(1/2 - s)$:

$$ {}_{h+s}V^{(2)} = [(1 - s)_{h}V^{(2)} + s\ {}_{h+1}V^{(2)}] + \left(\frac{1}{2} - s\right)\pi_h. \qquad (8.4.14) $$

For the last half of the year when $1/2 < s \le 1$, we can proceed as above to obtain the following exact formula for the benefit reserve at $h + s$ discounted with respect to interest and mortality to h:

$$ {}_{s}p_{x+h}\, v^{s}\, {}_{h+s}V^{(2)} = \left[{}_{h}V^{(2)} + \frac{\pi_h}{2}(1 + v^{0.5}\,{}_{0.5}p_{x+h})\right]\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} $$
$$ + [v({}_{h+1}V^{(2)})\, p_{x+h}]\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \qquad (8.4.15) $$

Again under uniform distribution of death in the year of age, we have the linear interpolation

$$ {}_{s}p_{x+h}\, v^{s}\, {}_{h+s}V^{(2)} = (1 - s)\left[{}_{h}V^{(2)} + \frac{\pi_h}{2}(1 + v^{0.5}\,{}_{0.5}p_{x+h})\right] $$
$$ + s[v({}_{h+1}V^{(2)})\, p_{x+h}], \qquad (8.4.16) $$

and with i and q_{x+h} set equal to zero, as an approximation, we have the simple linear interpolation plus the unearned benefit premium

$$ {}_{h+s}V^{(2)} = {}_{h}V^{(2)}(1 - s) + {}_{h+1}V^{(2)}(s) + \pi_h(1 - s). \qquad (8.4.17) $$

(For the general result for m-thly reserves see Exercise 8.12.)

8.5 Allocation of the Risk to Insurance Years

In Section 8.3 recursion relations for benefit reserves are developed by an analysis of the insurer's annual cash income and cash outflow. Now we extend this analysis to an accrual or incurred basis and develop allocations of the risk, as measured by the variance of the loss variables, to the insurance years. Figure 8.5.1 shows in a time diagram the insurer's annual cash incomes, cash outflows, and changes in liability for the general fully discrete insurance of (8.2.1). The random variable C_h is related to the cash flows of the policy year $(h, h+1)$. We now define a random variable related to the total change in liability, cash flow, and reserves.

Insurer's Cash Incomes, Outflows, and Changes in Liability for Fully Discrete General Insurance

Outflow		0	0	0	$b_{K(x)+1}$	0	
Income	π_0	π_1	π_2	$\pi_{K(x)}$	0	0	
ΔLiability		${}_1V$	${}_2V-(1+i)\,{}_1V$	${}_{K(x)}V-(1+i)_{K(x)-1}V$	$-(1+i)_{K(x)}V$	0	etc.
	0	1	2	$K(x)$	$K(x)+1$	$K(x)+2$	

Let Λ_h denote the present value at h (a non-negative integer) of the insurer's cash loss plus change in liability during the year $(h, h+1)$. If $(h, h+1)$ is before the year of death $[h < K(x)]$, then

$$\Lambda_h = C_h + v\ \Delta\text{Liability} = -\pi_h + v\ {}_{h+1}V - {}_hV.$$

If $(h, h+1)$ is the year of death $[h = K(x)]$, then

$$\Lambda_h = C_h + v\Delta\text{Liability} = v\ b_{h+1} - \pi_h - {}_hV.$$

And if $(h, h+1)$ is after the year of death, of course $\Lambda_h = 0$. Restating this definition as a function of $K(x)$, and rearranging the terms,

$$\Lambda_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ (v\ b_{h+1} - \pi_h) + \qquad (-\,{}_hV) & K(x) = h \\ (-\pi_h) \qquad + (v\ {}_{h+1}V - {}_hV) & K(x) = h+1, h+2, \ldots. \end{cases} \tag{8.5.1}$$

The definition of Λ_h in (8.5.1) can be rewritten to display Λ_h as the loss variable for a 1-year term insurance with a benefit equal to the amount at risk on the basic policy. See Exercise 8.31.

It follows that

$$E[\Lambda_h | K(x) \ge h] = v\ b_{h+1}\ q_{x+h} + v\ {}_{h+1}V\ p_{x+h} - (\pi_h + {}_hV), \tag{8.5.2}$$

which is zero by (8.3.10).

Since the conditional distribution of Λ_h, given $K(x) = h, h+1, \ldots$, is a two-point distribution, then

$$\text{Var}[\Lambda_h | K(x) \ge h] = [v(b_{h+1} - {}_{h+1}V)]^2\ p_{x+h}\ q_{x+h}. \tag{8.5.3}$$

With $j \le h$ we can use (2.2.10) and (2.2.11) to obtain

$$E[\Lambda_h | K(x) \ge j] = 0 \tag{8.5.4}$$

and

$$\mathrm{Var}[\Lambda_h | K(x) \geq j] = \mathrm{Var}[\Lambda_h | K(x) \geq h]\ {}_{h-j}p_{x+j}. \tag{8.5.5}$$

Unlike the C_h's of Section 8.3, the Λ_h's are uncorrelated, an assertion that is proved in the following lemma. This fact conveys some sense of the role of reserves in stabilizing financial reporting of insurance operations.

Lemma 8.5.1:

For non-negative integers satisfying $g \leq h < j$,

$$\mathrm{Cov}[\Lambda_h, \Lambda_j | K(x) \geq g] = 0. \tag{8.5.6}$$

Proof:

From (8.5.4), $\mathrm{E}[\Lambda_h | K(x) \geq g] = 0$; therefore,

$$\mathrm{Cov}[\Lambda_h, \Lambda_j | K(x) \geq g] = \mathrm{E}[\Lambda_h \Lambda_j | K(x) \geq g].$$

From (8.5.1) we see that Λ_h is equal to the constant $(v\ {}_{h+1}V - {}_hV - \pi_h)$ where Λ_j is nonzero. Thus,

$$\Lambda_h \Lambda_j = (v\ {}_{k+1}V - {}_hV - \pi_h)\Lambda_j \qquad \text{for all } K(x), \tag{8.5.7}$$

$$\mathrm{E}[\Lambda_h \Lambda_j | K(x) \geq g] = (v\ {}_{h+1}V - {}_hV - \pi_h)\mathrm{E}[\Lambda_j | K(x) \geq g] = 0,$$

and

$$\mathrm{Cov}[\Lambda_k \Lambda_j | K(x) \geq g] = 0. \qquad \blacksquare$$

We now express the loss variables ${}_hL$ in terms of the Λ_h's. From the definition of the Λ_h's and formula (8.3.6),

$$\sum_{j=h}^{\infty} v^{j-h}\ \Lambda_j = \sum_{j=h}^{\infty} v^{j-h}[C_j + v\Delta\text{Liability } (j, j+1)]$$

$$= {}_hL + \sum_{j=h}^{\infty} v^{j-h+1}\ \Delta\text{Liability } (j, j+1). \tag{8.5.8}$$

Conceptually the last term will be the present value of the final liability minus the liability at h, that is, $0 - {}_hV$. Thus, we have the relationship

$${}_hL = \sum_{j=h}^{\infty} v^{j-h}\ \Lambda_j + {}_hV. \tag{8.5.9}$$

The following recursion relations for the ${}_hL$'s follow from the definition of the Λ_h's and (8.3.7):

$${}_hL = C_h + v\ {}_{h+1}L = [\Lambda_h - (v\ {}_{h+1}V - {}_hV)] + v\ {}_{h+1}L$$

$$= \Lambda_h + v\ {}_{h+1}L + ({}_hV - v\ {}_{h+1}V). \tag{8.5.10}$$

Relationship (8.5.10) can be extended by iteration or by direct use of (8.5.9) to obtain, for $h < j$,

$${}_hL = \sum_{i=h}^{h+j-1} v^{i-h}\ \Lambda_i + v^j\ {}_{h+j}L + ({}_hV - v^j\ {}_{h+j}V). \tag{8.5.11}$$

The following result, attributed to Hattendorf, provides a means for expressing the variance of ${}_hL$ in terms of the benefit reserve values. Equally significant is the fact that the variance of ${}_hL$ can be allocated to the individual insurance years. This allocation facilitates risk management planning for a limited number of future insurance years rather than for the entire insurance period. This option permits sequential risk management decisions.

Theorem 8.5.1

$$\mathrm{Var}({}_hL|K(x) \geq h)$$

$$\text{a.} = \sum_{i=h}^{\infty} v^{2(i-h)} \mathrm{Var}[\Lambda_i|K(x) \geq h] \tag{8.5.12}$$

$$\text{b.} = \mathrm{Var}[\Lambda_h|K(x) \geq h] + v^2 \mathrm{Var}[{}_{h+1}L|K(x) \geq h] \tag{8.5.13}$$

$$\text{c.} = \sum_{i=h}^{h+j-1} v^{2(i-h)} \mathrm{Var}[\Lambda_i|K(x) \geq h] + v^{2j} \mathrm{Var}[{}_{h+j}L|K(x) \geq h]. \tag{8.5.14}$$

Proof:

(a) follows from (8.5.9) and Lemma 8.5.1.
(b) is a special case ($j = 1$) of (c) which follows from (8.5.11) and Lemma 8.5.1. ■

Theorem 8.5.2

$$\mathrm{Var}[{}_{h+j}L|K(x) \geq h] = {}_jp_{x+h} \mathrm{Var}[{}_{h+j}L|K(x) \geq h + j]. \tag{8.5.15}$$

Proof:
By (8.5.9),

$$\mathrm{Var}[{}_{h+j}L|K(x) \geq h] = \mathrm{Var}\left[\sum_{i=h+j}^{\infty} v^{i-(h+j)} \Lambda_i + {}_{h+j}V|K(x) \geq h\right]$$

$$= \sum_{i=h+j}^{\infty} v^{2[i-(h+j)]} \mathrm{Var}[\Lambda_i|K(x) \geq h],$$

which, from (8.5.5),

$$= \sum_{i=h+j}^{\infty} v^{2[i-(h+j)]} {}_jp_{x+h} \mathrm{Var}[\Lambda_i|K(x) \geq h + j]$$

$$= {}_jp_{x+h} \mathrm{Var}[{}_{h+j}L|K(x) \geq h + j]. \quad ■$$

Combining the results of these two theorems, we have the following corollary.

Corollary:

$$\mathrm{Var}[{}_hL|K(x) \geq h]$$

$$\text{a. } = \sum_{j=h}^{\infty} v^{2(j-h)}\ {}_{j-h}p_{x+h}\ \mathrm{Var}[\Lambda_j | K(x) \ge j] \tag{8.5.16}$$

$$\text{b. } = \mathrm{Var}[\Lambda_h | K(x) \ge h] + v^2\ p_{x+h} \mathrm{Var}[{}_{h+1}L | K(x) \ge h+1] \tag{8.5.17}$$

$$\text{c. } = \sum_{i=h}^{h+j-1} v^{2(i-h)}\ {}_{i-h}p_{x+h}\ \mathrm{Var}[\Lambda_i | K(x) \ge i] + v^{2j}\ {}_{j}p_{x+h}\ \mathrm{Var}[{}_{h+j}L | K(x) \ge h+j]. \tag{8.5.18}$$

Proof:

- For (a), apply (8.5.5) to each term of (8.5.12).
- For (b), apply Theorem 8.5.2 to (8.5.13).
- For (c), apply Theorem 8.5.2 to (8.5.14). ■

We refer to these theorems and their corollary as the Hattendorf theorem, and we illustrate their application in the following two examples. Items (b) and (c) of the corollary can be used as backward recursion formulas that are useful for understanding the duration allocation of risk and, perhaps, for computing.

Example 8.5.1

Consider an insured from Example 7.4.3 who has survived to the end of the second policy year. For this insured, evaluate

a. $\mathrm{Var}[{}_2L | K(50) \ge 2]$ directly
b. $\mathrm{Var}[{}_2L | K(50) \ge 2]$ by means of the Hattendorf theorem
c. $\mathrm{Var}[{}_3L | K(50) \ge 3]$
d. $\mathrm{Var}[{}_4L | K(50) \ge 4]$.

Solution:

a. For the direct calculation, we need a table of values for ${}_2L$.

Outcome of $K(50) - 2 = j$	${}_2L$	Conditional Probability of Outcome
0	$1{,}000v - 6.55692\ \ddot{a}_{\overline{1}\rceil} = 936.84$	${}_{0\vert}q_{52} = 0.0069724$
1	$1{,}000v^2 - 6.55692\ \ddot{a}_{\overline{2}\rceil} = 877.25$	${}_{1\vert}q_{52} = 0.0075227$
2	$1{,}000v^3 - 6.55692\ \ddot{a}_{\overline{3}\rceil} = 821.04$	${}_{2\vert}q_{52} = 0.0081170$
≥ 3	$0 - 6.55692\ \ddot{a}_{\overline{3}\rceil} = -18.58$	${}_{3}p_{52} = 0.9773879$

Then $\mathrm{E}[{}_2L | K(50) \ge 2] = 1.64$, in agreement with the value shown in Example 7.4.3 and

$$\begin{aligned}\mathrm{Var}[{}_2L | K(50) \ge 2] &= \mathrm{E}[{}_2L^2 | K(50) \ge 2] - (\mathrm{E}[{}_2L | K(50) \ge 2])^2 \\ &= 17{,}717.82 - (1.64)^2 \\ &= 17{,}715.1.\end{aligned}$$

b. To apply the Hattendorf theorem, we can use the benefit reserves from Example 7.4.3 to calculate the variances of the losses associated with the 1-year term insurances.

j	q_{52+j}	$v^2\,(1{,}000 - 1{,}000\ {}_{2+j+1}V^{1}_{50:\overline{5}|})^2\ p_{52+j}\ q_{52+j}$
0	0.0069724	6 140.842
1	0.0075755	6 674.910
2	0.0082364	7 269.991

Then by (8.5.16),

$$\mathrm{Var}[{}_2L|K(50) \geq 2] = 6{,}140.842 + (1.06)^{-2}(6{,}674.910)p_{52}$$

$$+ (1.06)^{-4}(7{,}269.991)\,{}_2p_{52} = 17{,}715.1,$$

which agrees with the value found by the direct calculation in part (a).

Note that in the direct method it was necessary to consider the gain in the event of survival to age 55; but for the Hattendorf theorem, we need to consider only the losses associated with the 1-year term insurances for the net amounts at risk in the remaining policy years. Thereafter, the net amount at risk is 0, and the corresponding terms in (8.5.16) vanish.

Also note that the standard deviation, $\sqrt{17{,}751.1} = 133.1$, for a single policy is more than 80 times the benefit reserve, $\mathrm{E}[{}_2L|K(50)=2, 3, \ldots] = 1.64$.

Similarly, we use (8.5.16) to calculate

c. $\mathrm{Var}[{}_3L|K(50) \geq 3] = 6{,}674.910 + (1.06)^{-2}(7{,}269.991)\ p_{53} = 13{,}096.2$

d. $\mathrm{Var}[{}_4L|K(50) \geq 4] = 7{,}269.991$, or after rounding, 7,270.0. ▼

Example 8.5.2

Consider a portfolio of 1,500 policies of the type described in Example 7.4.3 and discussed in Example 8.5.1. Assume all policies have annual premiums due immediately. Further, assume 750 policies are at duration 2, 500 are at duration 3, and 250 are at duration 4, and that the policies in each group are evenly divided between those with 1,000 face amount and those with 3,000 face amount.

a. Calculate the aggregate benefit reserve.

b. Calculate the variance of the prospective losses over the remaining periods of coverage of the policies assuming such losses are independent. Also, calculate the amount which, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the future obligations to this block of business.

c. Calculate the variance of the losses associated with the 1-year term insurances for the net amounts at risk under the policies and the amount of supplement to the aggregate benefit reserve that, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the obligations to this block of business for the 1-year period.

d. Redo (b) and (c) with each set of policies increased 100-fold in number.

Solution:

a. Let Z be the sum of the prospective losses on the 1,500 policies. The symbols E[Z] and Var(Z) used below for the mean and variance of the portfolio of 1,500 policies are abridged, for in both cases the expectations are to be computed with respect to the set of conditions given above for the insureds. Using the results of Example 7.4.3, we have for the aggregate benefit reserve

$$
\begin{aligned}
E[Z] &= [375(1) + 375(3)](1.64) + [250(1) + 250(3)](1.73) \\
&\quad + [125(1) + 125(3)](1.21) \\
&= 4{,}795.
\end{aligned}
$$

b. From Example 8.5.1, we have

$$
\begin{aligned}
\mathrm{Var}(Z) &= [375(1) + 375(9)](17{,}715.1) \\
&\quad + [250(1) + 250(9)](13{,}096.2) \\
&\quad + [125(1) + 125(9)](7{,}270.0) \\
&= (1.0825962) \times 10^8
\end{aligned}
$$

and $\sigma_Z = 10{,}404.8$.

Then, if

$$0.05 = \Pr(Z > c) = \Pr\left(\frac{Z - 4{,}795.0}{10{,}404.8} > \frac{c - 4{,}795.0}{10{,}404.8}\right),$$

the normal approximation would imply

$$\frac{c - 4{,}795.0}{10{,}404.8} = 1.645,$$

or

$$c = 21{,}911,$$

which is 4.6 times the aggregate benefit reserve, E[Z].

c. Here we take account of only the next year's risk. For each policy, we consider a variable equal to the loss associated with a 1-year term insurance for the net amount at risk. Let Z_1 be the sum of these loss variables. The expected loss for each of the 1-year term insurances is 0, hence $E[Z_1] = 0$.

From the table in part (b) of Example 8.5.1 we can obtain the variances of the losses in regard to the 1-year term insurances, and hence

$$
\begin{aligned}
\mathrm{Var}(Z_1) &= [375(1) + 375(9)](6{,}140.8) + [250(1) + 250(9)](6{,}674.9) \\
&\quad + [125(1) + 125(9)](7{,}270.0) \\
&= (4.880275) \times 10^7
\end{aligned}
$$

and $\sigma_{Z_1} = 6985.9$.

If c_1 is the required supplement to the aggregate benefit reserve, then

$$0.05 = \Pr(Z_1 > c_1) = \Pr\left(\frac{Z_1 - 0}{6{,}985.9} > \frac{c_1 - 0}{6{,}985.9}\right),$$

and we determine, again by the normal approximation,

$$c_1 = (1.645)(6{,}985.9) = 11{,}492,$$

which is 2.4 times the aggregate benefit reserve 4,795.

d. In this case, E[Z] = 479,500 and Var(Z) = (1.0825962) $\times$ 10^{10}. By the normal approximation the amount c required to provide a probability of 0.95 that all future obligations will be met is

$$479{,}500 + 1.645\sqrt{1.0825962} \times 10^5 = 650{,}659,$$

which is 1.36 times the aggregate benefit reserve E[Z].

Also, Var(Z_1) is now (4.880275) $\times$ 10^9. The amount c_1 of supplement to the aggregate benefit reserve required to give a 0.95 probability that the insurer can meet policy obligations for the next year is $1.645\sqrt{4.880275} \times 10^{4.5} = 114{,}918$, or 24% of the aggregate benefit reserve. ▼

8.6 Differential Equations for Fully Continuous Benefit Reserves

In Section 8.2 a general fully discrete insurance and a general fully continuous model is developed. Section 8.3 contains the recursion relations for the fully discrete model. The parallel results for the fully continuous model are developed in this section.

The expression for the benefit reserve at t, ${}_t\bar{V}$, is given in (8.2.9) and is restated here:

$${}_t\bar{V} = \int_0^\infty b_{t+u}\, v^u\, {}_u p_{x+t}\, \mu_x(t+u)\, du - \int_0^\infty \pi_{t+u}\, v^u\, {}_u p_{x+t}\, du.$$

To simplify the calculation of the derivative with respect to t of ${}_t\bar{V}$, we combine the two integrals, replace the variable of integration by the substitution $s = t + u$, and then multiply inside and divide outside by the factor $v^t\, {}_t p_x$ to obtain

$${}_t\bar{V} = \frac{\int_t^\infty [b_s\, \mu_x(s) - \pi_s] v^s\, {}_s p_x\, ds}{v^t\, {}_t p_x}. \tag{8.6.1}$$

Now

$$\frac{d\,{}_t\bar{V}}{dt} = (-1)[b_t\, \mu_x(t) - \pi_t] + \frac{\mu_x(t) + \delta}{v^t\, {}_t p_x} \int_t^\infty [b_s\, \mu_x(s) - \pi_s] v^s\, {}_s p_x\, ds,$$

$$\frac{d\,{}_t\bar{V}}{dt} = \pi_t + [\delta + \mu_x(t)]\,{}_t\bar{V} - b_t\, \mu_x(t). \tag{8.6.2}$$

Here the rate of change of the benefit reserve is made up of three components: the benefit premium rate, the rate of increase of the benefit reserve under interest and survivorship, and the rate of benefit outgo. A rearrangement of formula (8.6.2) provides a formula corresponding to (8.3.12):

$$\pi_t + \delta\ {}_t\bar{V} + {}_t\bar{V}\ \mu_x(t) = b_t\ \mu_x(t) + \frac{d\ {}_t\bar{V}}{dt}. \tag{8.6.3}$$

This balances the sum of income rates to the sum of the rate of benefit outgo and the rate of change in the benefit reserve.

If the benefit reserve is treated as a savings fund available to offset the death benefit, we have

$$\pi_t + \delta\ {}_t\bar{V} = (b_t - {}_t\bar{V})\ \mu_x(t) + \frac{d\ {}_t\bar{V}}{dt}. \tag{8.6.4}$$

Here the income rates are in respect to benefit premiums and to interest on the benefit reserve, and these balance with the outgo rate, $(b_t - {}_t\bar{V})\ \mu_x(t)$, based on the net amount at risk and the rate of change in the benefit reserve. Formula (8.6.4) corresponds to (8.3.15). Again, the left side represents the resources available, benefit premiums, and investment income, and the right side represents their allocation to benefits and benefit resources.

Example 8.6.1

Use (8.6.2) to develop a retrospective formula for the benefit reserve for the general fully continuous insurance.

Solution:

We start by moving all of the benefit reserve terms of (8.6.2) to the left-hand side and then multiplying both sides by the integrating factor $\exp\{-\int_0^t[\delta + \mu_x(s)]\ ds\}$. Thus,

$$v^t\ {}_tp_x \left\{\frac{d\ {}_t\bar{V}}{dt} - [\delta + \mu_x(t)]\ {}_t\bar{V}\right\} = [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x,$$

or

$$\frac{d}{dt}(v^t\ {}_tp_x\ {}_t\bar{V}) = [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x.$$

Integration of both sides of this last equation over the interval $(0, r)$ yields

$$v^r\ {}_rp_x\ {}_r\bar{V} - {}_0\bar{V} = \int_0^r [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x\ dt.$$

For equivalence principle benefit premium rates, ${}_0V = 0$, so

$${}_r\bar{V} = \frac{\displaystyle\int_0^r [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x\ dt}{v^r\ {}_rp_x}. \tag{8.6.5}$$

▼

8.7 Notes and References

Recursive formulas and differential equations for the loss variables as functions of duration and the expectations and variances of these loss variables provide basic insight into long-term insurance and annuity processes. In particular, one of these recursive formulas is applied to develop Hattendorf's theorem (1868); for references, see Steffensen (1929), Hickman (1964), and Gerber (1976). This formula allocates the variance of the loss to the separate insurance years. This discussion of reserves can be easily extended to more general insurances using martingales of probability theory; see, for example, Gerber (1979). Another application of the recursion relations is to the formulation of the interim reserves at fractional durations, which is discussed for the fully discrete case in Section 8.4.

Exercises

Section 8.2

8.1. Assume that ${}_jp_x = r^j$, $b_{j+1} = 1$, $j = 0, 1, 2, 3, \ldots$, and $0 < r < 1$.
   a. If $w_0 = w_1 = w_2 = \ldots = 1$, use (8.2.6) to calculate π at the interest rate i.
   b. If $w_j = (-1)^j$, $j = 0, 1, 2, \ldots$, use (8.2.6) to calculate π at the interest rate i.

8.2. Develop a continuous analogue of (8.2.6) by applying the equivalence principle to the loss variable of (8.2.7) with $t = 0$ and $\pi_t = \pi w(t)$, where $w(t)$ is given.

8.3. If ${}_0L = T(x)\, v^{T(x)} - \pi \bar{a}_{\overline{T(x)|}}$ and the forces of mortality and interest are constant, express (a) π and (b) ${}_t\bar{V}$ in terms of μ and δ.

8.4. For the general fully discrete insurance of Section 8.2, show that for $j < h$,

$$\text{Cov}(C_j, C_h) = (\pi_h - v b_{h+1}\, q_{x+h})\, {}_hp_x(\pi_j\, {}_jq_x + v b_{j+1}\, {}_jp_x\, q_{x+j}).$$

Section 8.3

8.5. Consider the life insurance policy described in Example 8.2.1. Display, for $0 < j < h$:
   a. The covariance of C_0 and C_h.
   b. Repeat part (a) for C_j and C_h.
   c. Give a rule for determining h such that the covariance of C_j and C_h is negative.

8.6. Consider the deferred annuity described in Example 8.3.1. Find $\text{Cov}(C_j, C_h)$, $j < h \le n$ and, for a fixed j, determine a condition on h such that $\text{Cov}(C_j, C_h) < 0$. [Note that this condition on h does not depend on j.]

8.7. Show that (8.3.9), with h replaced by $h + 1$, can be rearranged as

$$_{h+1}V = ({}_hV + \pi_h)\,\frac{1+i}{p_{x+h}} - b_{h+1}\,\frac{q_{x+h}}{p_{x+h}}.$$

Give an interpretation in words. (This is called the ***Fackler reserve*** accumulation formula, after the American actuary David Parks Fackler.)

8.8. For a fully discrete whole life insurance of 1 issued to (x), use recursion relations [(8.3.11) in (a) and (8.3.14 in (b)] to prove that

a. ${}_kV_x = \sum_{h=0}^{k-1} \dfrac{P_x - vq_{x+h}}{{}_{k-h}E_{x+h}}$

b. ${}_kV_x = \sum_{h=0}^{k-1} [P_x - vq_{x+h}(1 - {}_{h+1}V_x)](1 + i)^{k-h}.$

Give an interpretation of the formulas in words.

8.9. If $b_{h+1} = {}_{h+1}V$, ${}_0V = 0$, and $\pi_h = \pi$, for $h = 0, 1, \ldots, k-1$, prove that ${}_kV = \pi\ddot{s}_{\overline{k}|}$. [Hint: Use (8.3.14).]

8.10. Show that if π is the (8.3.14) level annual benefit premium for an n-year term insurance with $b_h = \ddot{a}_{\overline{n-h}|}$, $h = 1, 2, \ldots, n$, ${}_0V = {}_nV = 0$, then

a. $\pi = \dfrac{\ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$

b. ${}_kV = \ddot{a}_{\overline{n-k}|} - \ddot{a}_{x+k:\overline{n-k}|} - \pi\ddot{a}_{x+k:\overline{n-k}|}.$

[Hint: This can be shown directly or by use of (8.3.10).]

Section 8.4

8.11. Starting with (8.4.3), establish the equation

$${}_sp_{x+h}\;{}_{h+s}V + v^{1-s}\;{}_sq_{x+h}\;b_{h+1} = (1 + i)^s({}_hV + \pi_h) \qquad 0 < s < 1.$$

Explain the result by general reasoning.

8.12. Interpret the formulas

a. $${}_{k+(h/m)+r}V^{(m)} \cong \left(1 - \frac{h}{m} - r\right){}_kV^{(m)} + \left(\frac{h}{m} + r\right){}_{k+1}V^{(m)} + \left(\frac{1}{m} - r\right)P^{(m)}$$

b. $${}_{k+(h/m)+r}V^{\{m\}} \cong \left(1 - \frac{h}{m} - r\right){}_kV^{\{m\}} + \left(\frac{h}{m} + r\right){}_{k+1}V^{\{m\}} + \left(\frac{1}{m} - r\right)P^{\{m\}},$$

where $0 < r < 1/m$.

8.13. For each of the following benefit reserves, develop formulas similar to one or more of (8.4.8), (8.4.14), and (8.4.18).

a. ${}_{20^1/_2}V(\bar{A}_{x:\overline{40|}})$

b. ${}_{20^1/_2}V(\bar{A}_{x:\overline{40|}}^{\;1})$

c. ${}_{20^1/_2}V^{(2)}(\bar{A}_{x:\overline{40|}})$

d. ${}_{20^2/_3}V^{(2)}(\bar{A}_{x:\overline{40|}})$

e. ${}_{20^1/_2}V^{\{2\}}(\bar{A}_{x:\overline{40|}})$

f. ${}_{20^2/_3}V^{\{2\}}(\bar{A}_{x:\overline{40|}})$

8.14. On the basis of the Illustrative Life Table and interest of 6%, approximate ${}_{10^1/_6}V^{\{4\}}(\bar{A}_{25})$.

Section 8.5

8.15. For a fully discrete whole life insurance of amount 1 issued to (x) with premiums payable for life, show that

a. $\text{Var}[L] = \sum_{h=0}^{\infty} \left(\frac{\ddot{a}_{x+h+1}}{\ddot{a}_x}\right)^2 v^{2(h+1)}\ {}_hp_x\ p_{x+h}\ q_{x+h}$

b. $\text{Var}[{}_kL | K(x) \geq k] = \sum_{h=0}^{\infty} \left(\frac{\ddot{a}_{x+k+h+1}}{\ddot{a}_x}\right)^2 v^{2(h+1)}\ {}_hp_{x+k}\ p_{x+k+h}\ q_{x+k+h}\,.$

8.16. For a life annuity-due of 1 per annum payable while (x) survives, consider the whole life loss

$$L = \ddot{a}_{\overline{K+1|}} - \ddot{a}_x \qquad K = 0, 1, 2, \ldots$$

and the loss Λ_h, valued at time h, that is allocated to annuity year h, namely,

$$\Lambda_h = \begin{cases} 0 & & K \leq h - 1 \\ -(\ddot{a}_{x+h} - 1) & = -vp_{x+h}\ \ddot{a}_{x+h+1} & K = h \\ v\ddot{a}_{x+h+1} - (\ddot{a}_{x+h} - 1) = vq_{x+h}\ \ddot{a}_{x+h+1} & & K \geq h + 1. \end{cases}$$

a. Interpret the formulas for Λ_h.

b. Show that

$$\text{(i) } L = \sum_{h=0}^{\infty} v^h\, \Lambda_h$$

$$\text{(ii) } \text{E}[\Lambda_h] = 0$$

$$\text{(iii) } \text{Var}(\Lambda_h) = v^2\ (\ddot{a}_{x+h+1})^2\ {}_hp_x\ p_{x+h}\ q_{x+h}\,.$$

8.17. a. For the insurance of Example 8.3.2, establish that

$$\text{Var}(L) = \sum_{h=0}^{n-1} v^{2(h+1)}\ {}_hp_x\ p_{x+h}\ q_{x+h}\,.$$

b. If $\delta = 0.05$, $n = 20$, and $\mu_x(t) = 0.01$, $t \geq 0$, calculate $\text{Var}(L)$ for the insurance in (a).

8.18. A 20-payment whole life policy with unit face amount was issued on a fully discrete basis to a person age 25. On the basis of your Illustrative Life Table and interest of 6%, calculate

a. ${}_{20}P_{25}$

b. ${}^{20}_{19}V_{25}$

c. ${}^{20}_{20}V_{25}$

d. $\mathrm{Var}[{}_{20}L|K(25) \geq 20]$ e. $\mathrm{Var}[{}_{18}L|K(25) \geq 18]$, using Theorem 8.2.

Section 8.6

8.19. Interpret the differential equations

a. $\dfrac{d}{dt}\,{}_t\bar{V} = \pi_t + [\delta + \mu_x(t)]\,{}_t\bar{V} - b_t\,\mu_x(t)$

b. $\dfrac{d}{dt}\,{}_t\bar{V} = \pi_t + \delta\,{}_t\bar{V} - (b_t - {}_t\bar{V})\,\mu_x(t)$.

8.20. If $b_t = {}_t\bar{V}$, ${}_0\bar{V} = 0$, and $\pi_t = \pi$, $t \geq 0$, show that ${}_t\bar{V} = \pi\bar{s}_{\overline{t|}}$.

8.21. Evaluate $(d/dt)\,\{[1 - {}_t\bar{V}(\bar{A}_x)]\,{}_tp_x\}$.

8.22. Use (8.6.2) to write expressions for

a. $\dfrac{d}{dt}\,({}_tp_x\,{}_t\bar{V})$ b. $\dfrac{d}{dt}\,(v^t\,{}_t\bar{V})$ c. $\dfrac{d}{dt}\,(v^t\,{}_tp_x\,{}_t\bar{V})$

and interpret the results.

Miscellaneous

8.23. Show that the formula equivalent to (8.4.6) under the hyperbolic assumption for mortality within the year of age is

$${}_{k+s}V = v^{1-s}[(1 - s)({}_kV + \pi_k)(1 + i) + s\;{}_{k+1}V].$$

8.24. Prove that

$$\int_0^{\infty} [v^t - \bar{P}(\bar{A}_x)\bar{a}_{\overline{t|}}]^2\,{}_tp_x\,\mu_x(t)\,dt = \int_0^{\infty} [1 - {}_t\bar{V}(\bar{A}_x)]^2\,v^{2t}\,{}_tp_x\,\mu_x(t)\,dt$$

and interpret the result.

8.25. For a different form of the Hattendorf theorem, consider the following:

$${}_{k,m}L = \begin{cases} b_{K+1}\,v^{(K-k)+1} - {}_kV - \sum\limits_{h=0}^{(K-k)} \pi_{k+h}\,v^h & K(x) = k, k+1, \ldots, k+m-1 \\ {}_{k+m}V\,v^m - {}_kV - \sum\limits_{h=0}^{m-1} \pi_{k+h}\,v^h & K(x) = k+m, k+m+1, \ldots, \end{cases}$$

and, for $h = 0, 1, \ldots, m-1$,

$$\Lambda_{k+h} = \begin{cases} 0 & K(x) = k, k+1, \ldots, k+h-1 \\ vb_{k+h+1} - ({}_{k+h}V + \pi_{k+h}) & K(x) = k+h \\ v\;{}_{k+h+1}V - ({}_{k+h}V + \pi_{k+h}) & K(x) = k+h+1, k+h+2, \ldots. \end{cases}$$

Show that

a. ${}_{k,m}L = \sum\limits_{h=0}^{m-1} v^h\,\Lambda_{k+h}$

b. $\mathrm{Var}[{}_{k,m}L|K(x) \geq k] = \sum\limits_{h=0}^{m-1} v^{2h}\,\mathrm{Var}[\Lambda_{k+h}|K(x) \geq k]$.

8.26. Repeat Example 8.5.1 in terms of an insured from Example 7.4.4 who has survived to the end of the second policy year.

8.27. Repeat Example 8.5.2 in terms of a portfolio of 1,500 policies of the type described in Example 7.4.4 and discussed in Exercise 8.26.

8.28. In Exercise 8.27 there is no uncertainty about the amount or time of payment for the insureds who have survived to the end of the fourth policy year. Redo Exercise 8.27 for just those insureds at durations 2 and 3.

8.29. Write a formula, in terms of benefit premium and terminal benefit reserve symbols, for the benefit reserve at the middle of the eleventh policy year for a 10,000 whole life insurance with apportionable premiums payable annually issued to (30).

8.30. A 3-year endowment policy for a face amount of 3 has the death benefit payable at the end of the year of death and a benefit premium of 0.94 payable annually. Using an interest rate of 20%, the following benefit reserves are generated:

End of Year	Benefit Reserve
1	0.66
2	1.56
3	3.00

Calculate

a. q_x
b. q_{x+1}
c. The variance of the loss at policy issue, ${}_0L$
d. The variance of the loss at the end of the first year, ${}_1L$.

8.31. a. Use (8.3.10) to transform (8.5.1) to

$$\Lambda_h = \begin{cases} 0 & K(x) \le h - 1 \\ (b_{h+1} - {}_{h+1}V)\, v - (b_{h+1} - {}_{h+1}V)\, v\, q_{x+h} & K(x) = h \\ 0 \qquad - (b_{h+1} - {}_{h+1}V)\, v\, q_{x+h} & K(x) \ge h + 1. \end{cases}$$

In this interpretation Λ_h is the loss on the 1-year term insurance for the amount at risk in the year $(h, h + 1)$.

b. Use the display in (a) to verify $E[\Lambda_h] = 0$.
c. Use parts (a) and (b) to obtain $\mathrm{Var}(\Lambda_h)$.

Computing Exercises

8.32. Consider a variation of the insurance of Example 8.2.2 which provides a unit benefit whole life insurance to (20) with geometrically increasing benefit premiums payable to age 65. On the basis of your Illustrative Life Table and $i = 0.06$, determine the maximum value of r such that the benefit reserve is non-negative at all durations.

8.33. Use the backward recursion formula (8.3.9) to calculate the benefit reserves of

a. Example 7.4.3 [Hint: ${}_5V = 0.0$.]

b. Example 7.4.4. [Hint ${}_5V = 1.0$.]

8.34. A decreasing term insurance to age 65 with immediate payment of death claims is issued to (30) with the following benefits:

For Death between Ages	Benefit
30–50	100,000
50–55	90,000
55–60	80,000
60–65	60,000

On the basis of your Illustrative Life Table with uniform distribution of deaths within each year of age and $i = 0.06$, determine

a. The annual apportionable benefit premium, payable semiannually and

b. The reserve at the end of 30 years, if benefit premiums are as in (a).

8.35. A single premium insurance contract issued to (35) provides 100,000 in case the insured survives to age 65, and it returns (at the end of the year of death) the single benefit premium without interest if the insured dies before age 65. If the single benefit premium is denoted by S, write expressions, in terms of actuarial functions, for

a. S

b. The prospective formula for the benefit reserve at the end of k years

c. The retrospective formula for the benefit reserve at the end of k years

d. On the basis of your Illustrative Life Table and $d = 0.05$, calculate S and the benefit reserve ${}_{20}V$.

8.36. In terms of $P = {}_{20}P^{(12)}(\bar{A}_{30:\overline{35}|})$ and actuarial functions, write prospective and retrospective formulas for the following:

a. ${}^{20}_{10}V^{(12)}(\bar{A}_{30:\overline{35}|})$

b. ${}^{20}_{25}V^{(12)}(\bar{A}_{30:\overline{35}|})$

c. On the basis of your Illustrative Life Table with uniform distribution of deaths within each year of age and $\delta = 0.05$, calculate P and the benefit reserves of parts (a) and (b).

9

MULTIPLE LIFE FUNCTIONS

9.1 Introduction

In Chapters 3 through 8 we developed a theory for the analysis of financial benefits contingent on the time of death of a single life. We can extend this theory to benefits involving several lives. An application of this extension commonly found in pension plans is the joint-and-survivor annuity option. Other applications of multiple life actuarial calculations are common. In estate and gift taxation, for example, the investment income from a trust can be paid to a group of heirs as long as at least one of the group survives. Upon the last death, the principal from the trust is to be donated to a qualified charitable institution. The amount of the charitable deduction allowed for estate tax purposes is determined by an actuarial calculation. There are family policies in which benefits differ due to the order of the deaths of the insured and the spouse, and there are insurance policies with benefits payable on the first or last death providing cash in accordance with an estate plan.

In this chapter we discuss models involving two lives. Actuarial present values for basic benefits are derived by applying the concepts and techniques developed in Chapters 3 through 5. Models built on the assumption that the two future lifetime random variables are independent constitute most of the chapter. Section 9.6 introduces special models in which the two future lifetime random variables are dependent. Annual benefit premiums, reserves, and models involving three or more lives are covered in Chapter 18.

A useful abstraction in the theory of life contingencies, particularly as it is applied to several lives, is that of ***status*** for which there are definitions of survival and failure. Two elements are necessary for a status to be defined. The general term ***entities*** is used in the definition because of the broad range of application of the concept:

- There must be a finite set of entities, and for each member it must be possible to define a future lifetime random variable.
- There must be a rule by which the survival of the status can be determined at any future time.

To compute probabilities or actuarial present values associated with the survival of a status, the joint distribution of the future lifetime random variables must be available. Some of these random variables may have a marginal distribution such that all the probability is at one point.

Several illustrations of the status concept may be helpful. A single life age x defines a status that survives while (x) lives. Thus, the random variable $T(x)$, used in Chapter 3 to denote the future lifetime of (x), can be interpreted as the period of survival of the status and also as the time-until-failure of the status. A term certain, $\overline{n|}$, defines a status surviving for exactly n years and then failing. More complex statuses can be defined in terms of several lives in various ways. Survival can mean that all members survive or, alternatively, that at least one member survives. Still more complicated statuses can be in regard to two men and two women with the status considered to survive only as long as at least one man and at least one woman survive.

After a status and its survival have been defined, we can apply the definition to develop models for annuities and insurances. An annuity is payable as long as the status survives, whereas an insurance is payable upon the failure of the status. Insurances also can be restricted so they are payable only if the individuals die in a specific order.

9.2 Joint Distributions of Future Lifetimes

The time-until-failure of a status is a function of the future lifetimes of the lives involved. In theory these future lifetimes will be dependent random variables. We will explore the consequences of that dependence. For convenience, or because of the lack of data on dependent lives, in practice, an assumption of independence among the future lifetimes has traditionally been made. With the independence assumption numerical values from the marginal distributions (life tables) for single lives can be used.

Example 9.2.1

While the distribution in this example is not realistic, it is offered as a vehicle to explore a joint distribution for two dependent future lifetimes. For two lives (x) and (y), the joint p.d.f. of their future lifetimes, $T(x)$ and $T(y)$, is

$$f_{T(x)T(y)}(s, t) = \begin{cases} 0.0006(t - s)^2 & 0 < s < 10,\ 0 < t < 10 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the following:

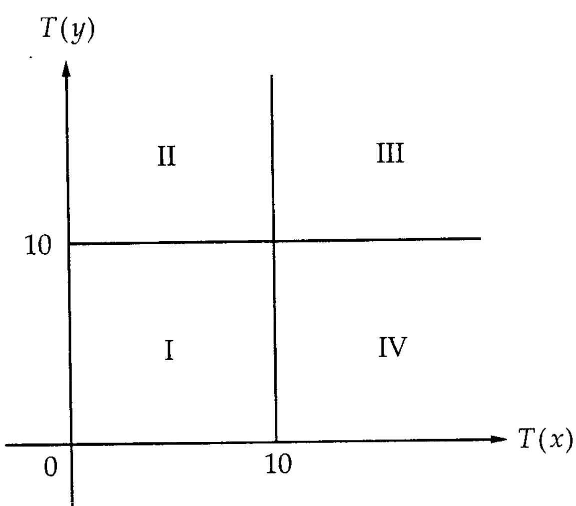
a. The joint d.f. of $T(x)$ and $T(y)$
b. The p.d.f., d.f., ${}_sp_x$, and $\mu(x + s)$ for the marginal distribution of $T(x)$. Note the symmetry of the distribution in s and t, which implies that $T(x)$ and $T(y)$ are identically distributed.
c. The correlation coefficient of $T(x)$ and $T(y)$.

Solution:

a. Before calculating, we look at the sample space of $T(x)$ and $T(y)$ in Figure 9.2.1 and observe the region where the joint p.d.f. is positive. At points outside the first quadrant, the d.f. will be 0. In the first quadrant we start by calculating the d.f. at a point in Region I where both s and t are between 0 and 10:

$$\begin{aligned} F_{T(x)T(y)}(s, t) &= \Pr[T(x) \le s \text{ and } T(y) \le t] \\ &= \int_{-\infty}^{s} \int_{-\infty}^{t} f_{T(x)T(y)}(u, v)\, dv\, du \\ &= \int_{0}^{s} \int_{0}^{t} 0.0006(v - u)^2\, dv\, du \\ &= 0.00005[s^4 + t^4 - (t - s)^4] \\ &\quad 0 < s \le 10,\ 0 < t \le 10. \end{aligned}$$

Sample Space of $T(x)$ and $T(y)$



Sample Space of $T(x)$ and $T(y)$

Since the joint p.d.f. is 0 in regions II, III, and IV, we have

$$\begin{aligned} F_{T(x)T(y)}(s, t) &= \left.\begin{aligned} &F_{T(x)T(y)}(s, 10) = F_{T(x)}(s) \\ &= \frac{1}{2} + 0.00005[s^4 - (10 - s)^4] \end{aligned}\right\} && \text{in Region II} \\ &= \left.\begin{aligned} &F_{T(x)T(y)}(10, t) = F_{T(y)}(t) \\ &= \frac{1}{2} + 0.00005[t^4 - (10 - t)^4] \end{aligned}\right\} && \text{in Region IV} \\ &= 1 && \text{in Region III.} \end{aligned}$$

b. Using the d.f. obtained in part (a), we have

$$
\begin{aligned}
F_{T(x)T(y)}(s, 10) &= F_{T(x)}(s) \\
&= 0 && s \le 0 \\
&= \frac{1}{2} + 0.00005[s^4 - (10 - s)^4] && 0 < s \le 10 \\
&= 1 && s > 10
\end{aligned}
$$

and

$$
f_{T(x)}(s) = F'_{T(x)}(s) = \begin{cases} 0.0002[s^3 + (10 - s)^3] & 0 < s \le 10 \\ 0 & \text{elsewhere.} \end{cases}
$$

The survival probability and force of mortality are given by

$$
\begin{aligned}
{}_sp_x &= 1 - F_{T(x)}(s) \\
&= \frac{1}{2} + 0.00005[(10 - s)^4 - s^4] && 0 < s \le 10 \\
&= 0 && s > 10,
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(x + t) &= \frac{f_{T(x)}(t)}{1 - F_{T(x)}(t)} \\
&= \frac{0.0002[s^3 + (10 - s)^3]}{1/2 + 0.00005[(10 - s)^4 - s^4]} && 0 < s \le 10.
\end{aligned}
$$

c.

$$
\begin{aligned}
\mathrm{E}[T(x)] &= \int_0^{10} s(0.0002)[s^3 + (10 - s)^3]\, ds = 5 = \mathrm{E}[T(y)], \\
\mathrm{E}[T(x)^2] &= \int_0^{10} s^2(0.0002)[s^3 + (10 - s)^3]\, ds = \frac{110}{3} = \mathrm{E}[T(y)^2], \\
\mathrm{Var}[T(x)] &= \frac{35}{3} = \mathrm{Var}[T(y)], \\
\mathrm{E}[T(x)T(y)] &= \int_0^{10}\int_0^{10} st(0.0006)(t - s)^2\, ds\, dt = \frac{50}{3}, \\
\mathrm{Cov}[T(x), T(y)] &= \mathrm{E}[T(x)T(y)] - \mathrm{E}[T(x)]\mathrm{E}[T(y)] = -\frac{25}{3}, \\
\rho_{T(x)T(y)} &= \frac{\mathrm{Cov}[T(x), T(y)]}{\sigma_{T(x)}\sigma_{T(y)}} = \frac{-25/3}{35/3} = -\frac{5}{7}.
\end{aligned}
$$

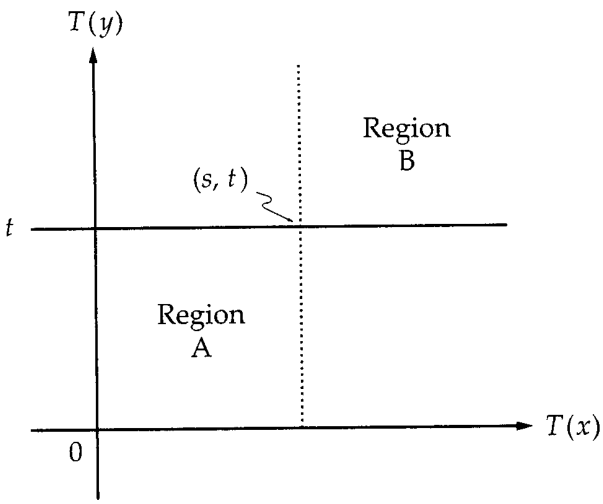
▼

For the joint life distribution, we define the ***joint survival function*** as

$$
s_{T(x)T(y)}(s, t) = \Pr[T(x) > s \text{ and } T(y) > t]. \tag{9.2.1}
$$

Unlike the single life distribution, the d.f. and survival function do not necessarily add up to 1. Their relationship for the joint life distribution can be illustrated by a graph of their joint sample space as shown in Figure 9.2.2. The d.f. $F_{T(x)\,T(y)}(s, t)$ gives the probability of Region A, "southwest" of the point (s, t), and $s_{T(x)\,T(y)}(s, t)$ gives the probability of Region B, "northeast" of (s, t).

Sample Space of Future Lifetime Random Variables $T(x)$ and $T(y)$



Sample Space of Future Lifetime Random Variables $T(x)$ and $T(y)$

Example 9.2.2

For the distribution of $T(x)$ and $T(y)$ in Example 9.2.1 determine the joint survival function.

Solution:

For $0 < s < 10$ and $0 < t < 10$,

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= \Pr[T(x) > s \cap T(y) > t] \\ &= \int_s^{\infty} \int_t^{\infty} f_{T(x)T(y)}(u, v)\, dv\, du \\ &= \int_s^{10} \int_t^{10} 0.0006(v - u)^2\, dv\, du \\ &= 0.00005[(10 - t)^4 + (10 - s)^4 - (t - s)^4]. \end{aligned}$$

For other points in the first quadrant, $s_{T(x)\,T(y)}(s, t)$ will be 0 and for all points in the third quadrant it will be 1. In the second quadrant, where $s < 0$ and $t > 0$,

$$s_{T(x)\,T(y)}(s, t) = s_{T(y)}(t) = {}_tp_y.$$

In the fourth quadrant, where $s > 0$ and $t < 0$,

$$s_{T(x)\,T(y)}(s, t) = s_{T(x)}(s) = {}_sp_x.$$

▼

In Example 9.2.1 we were given the joint distribution of two dependent future lifetimes and then determined their marginal distributions and their correlation coefficient, which indicated their degree of dependence. In applications, the dependence of the time-until-death random variables may be difficult to quantify. Consequently, the future lifetimes are usually assumed to be independent, and then their joint distribution is obtained from their marginal single life distributions that we discussed in Chapter 3. This is illustrated in the next example.

Example 9.2.3

The future lifetimes $T(x)$ and $T(y)$ are independent, and each has the distribution defined by the p.d.f.

$$f(t) = \begin{cases} 0.02\,(10 - t) & 0 < t < 10 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Determine the d.f., survival function, and force of mortality of this distribution.
b. Determine the joint p.d.f., d.f., and survival function for $T(x)$ and $T(y)$.

Solution:

a.

$$F_{T(x)}(t) = \int_{-\infty}^{t} f_{T(x)}(s)\, ds$$

$$= \begin{cases} 0 & t \le 0 \\ 1 - 0.01(10 - t)^2 = 0.2t - 0.01t^2 & 0 < t \le 10 \\ 1 & t > 10, \end{cases}$$

$$s_{T(x)}(t) = 1 - F_{T(x)}(t) = \begin{cases} 1 & t < 0 \\ 0.01(10 - t)^2 & 0 \le t < 10 \\ 0 & t \ge 10, \end{cases}$$

$$\mu(x + t) = \frac{f_{T(x)}(t)}{s_{T(x)}(t)} = \frac{2}{10 - t} \qquad 0 < t < 10.$$

b. $f_{T(x)T(y)}(s, t) = f_{T(x)}(s) f_{T(y)}(t)$

$$= \begin{cases} (0.02)^2(10 - s)(10 - t) & 0 < s < 10,\ 0 < t < 10 \\ 0 & \text{elsewhere,} \end{cases}$$

$$\begin{aligned} F_{T(x)T(y)}(s, t) &= F_{T(x)}(s)F_{T(y)}(t) \\ &= (0.2)^2(t - 0.05t^2)(s - 0.05s^2) && 0 < s \le 10,\ 0 < t \le 10 \\ &= F_{T(x)}(s) = (0.2)(s - 0.05s^2) && 0 < s \le 10,\ t > 10 \\ &= F_{T(y)}(t) = (0.2)(t - 0.05t^2) && s > 10,\ 0 < t \le 10, \end{aligned}$$

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= s_{T(x)}(s)s_{T(y)}(t) \\ &= (0.01)^2(10 - s)^2(10 - t)^2 && 0 \le s < 10,\ 0 \le t < 10 \\ &= s_{T(x)}(s) = (0.01)(10 - s)^2 && 0 \le s < 10,\ t < 0 \\ &= s_{T(y)}(t) = (0.01)(10 - t)^2 && s < 0,\ 0 \le t < 10 \\ &= 0 && s \ge 10,\ t \ge 10. \end{aligned}$$

▼

9.3 The Joint-Life Status

A status that survives as long as all members of a set of lives survive and fails upon the first death is called a ***joint-life status.*** It is denoted by $(x_1, x_2, \ldots, x_m)$, where x_i represents the age of member i of the set and m represents the number of members. Notation introduced in Chapters 3 through 5 is used here with the subscript listing several ages rather than a single age. For example, A_{xy} and ${}_tp_{xy}$ have the same meaning for the joint-life status (xy) as A_x and ${}_tp_x$ have for the single life (x).

A joint-life status is an example of what we call a ***survival status,*** that is, a status for which there is a future lifetime random variable and, therefore, a survival function can be defined. For the future lifetime of a survival status, the concepts and relationships established in Sections 3.2.2 through 3.5 (excluding the life table example in Section 3.3.2) apply to the distribution of the survival status. These concepts will be used here without new proofs.

We now consider the distribution of the time-until-failure of a joint-life status. For m lives, $T(x_1, x_2, \ldots, x_m) = \min[T(x_1), T(x_2), \ldots, T(x_m)]$, where $T(x_i)$ is the time of death of individual i. For the special case of two lives, (x) and (y), we have $T(xy) = \min[T(x), T(y)]$. When clear by context, we denote the future lifetime of the joint-life status by simply T. The student can interpret the time-until-failure of the joint-life status as the ***smallest order statistic*** of the m lifetimes in the set. In previous studies of order statistics, the random variables in the sample have usually been independent and identically distributed. Here the random variables are typically independent by assumption but are rarely identically distributed.

We begin by expressing the distribution function of T, for $t > 0$, in terms of the joint distribution of $T(x)$ and $T(y)$ for the general (dependent) case:

$$\begin{aligned} F_T(t) &= {}_tq_{xy} = \Pr(T \le t) \\ &= \Pr\{\min[T(x), T(y)] \le t\} \\ &= 1 - \Pr\{\min[T(x), T(y)] > t\} \\ &= 1 - \Pr\{T(x) > t \text{ and } T(y) > t\} \\ &= 1 - s_{T(x)\,T(y)}(t, t). \end{aligned} \tag{9.3.1}$$

Another equation can be obtained from the second line by recognizing that the event $\{\min[T(x), T(y)] \le t\}$ is the union of $\{T(x) \le t\}$ and $\{T(y) \le t\}$. Then,

$$F_T(t) = \Pr\{\min[T(x), T(y)] \le t\},$$

and using a basic result in probability, we have

$$\begin{aligned} F_T(t) &= \Pr[T(x) \le t] + \Pr[T(y) \le t] - \Pr[T(x) \le t \cap T(y) \le t] \\ &= {}_tq_x + {}_tq_y - F_{T(x)\,T(y)}(t, t). \end{aligned} \tag{9.3.2}$$

The mixture of IAN and standard probability/statistics notation in (9.3.2) demonstrates that although the IAN system may accommodate survival statuses, it does not provide for the joint distribution of several statuses except in the independent case which can be expressed in single survival status symbols.

When $T(x)$ and $T(y)$ are independent, the two expressions for the d.f. of T can be written in terms of single life functions as:

$$F_T(t) = \Pr\{\min[T(x), T(y)] \leq t\}$$
$$= 1 - s_{T(x)\ T(y)}(t, t) = 1 - {}_tp_x\ {}_tp_y, \tag{9.3.3}$$

and

$$F_T(t) = {}_tq_x + {}_tq_y - F_{T(x)\ T(y)}(t, t) = {}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y. \tag{9.3.4}$$

The survival function for the joint-life status, ${}_tp_{xy}$, is obtained by subtracting the d.f. from 1.

For the general case, ${}_tp_{xy} = s_{T(x)\ T(y)}(t, t)$ using (9.3.1). In the independent case, we have by (9.3.3)

$${}_tp_{xy} = {}_tp_x\ {}_tp_y. \tag{9.3.5}$$

Expression (9.3.5) is the convenient starting point for the independent case since the joint-life status survives to t if, and only if, both (x) and (y) survive to t.

Example 9.3.1

Determine the d.f., survival function, and complete expectation for the joint-life status, $T(xy)$, for the lives of Example 9.2.1.

Solution:

For $t \leq 0$ and $t > 10$, the value of $F_{T(xy)}(t)$ would be 0 and 1, respectively. For $0 < t \leq 10$, we have by the results of Example 9.2.1(a) and (b) and (9.3.2)

$$F_{T(xy)}(t) = 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - 0.0001\ t^4$$
$$= 1 - 0.0001\ (10 - t)^4$$

for $0 < t \leq 10$.

Now,

$${}_tp_{xy} = 1 - F_{T(xy)}(t) = 0.0001(10 - t)^4 \qquad 0 \leq t < 10.$$

From (3.5.2),

$$\mathring{e}_{xy} = \mathrm{E}[T(xy)] = \int_0^\infty {}_tp_{xy}\ dt = \int_0^{10} 0.0001(10 - t)^4\ dt = 2. \qquad \blacktriangledown$$

Example 9.3.2

Determine the d.f. and survival function and the complete expectation for the joint-life status, $T(xy)$, for the distribution of Example 9.2.3.

Solution:

For independent lives we use (9.3.5) and the results of Example 9.2.3 to obtain

$$ {}_tp_{xy} = [0.01(10 - t)^2]^2 = 0.0001(10 - t)^4 \qquad \text{for } 0 \le t < 10. $$

Then

$$ F_{T(xy)}(t) = 1 - (0.0001)(10 - t)^4 \qquad \text{for } 0 < t \le 10 $$

and

$$ \mathring{e}_{xy} = \int_0^{10} 0.0001(10 - t)^4\, dt = 2. $$

▼

An insight can be gained from the two previous examples. Although the joint distributions of the two underlying future lifetime random variables of the two examples are not the same, the distributions of their joint-life statuses are the same. This is a important point in practice when only the first-to-die can be observed. In such case the underlying joint distribution is not uniquely determined. In statistics this is called ***nonidentifiability*** because of the difficulty in distinguishing among two or more models for the same observed data.

The p.d.f. for T can be obtained by differentiating its d.f. as displayed in either (9.3.1) or (9.3.2). For (9.3.1) we will need the derivative, with respect to t, of

$$ s_{T(x)\,T(y)}(t, t) = \int_t^\infty \int_t^\infty f_{T(x)T(y)}(u, v)\, du\, dv. $$

Using the formula from calculus in Appendix 5, we have

$$ \frac{d}{dt}\, s_{T(x)\,T(y)}(t, t) = -\left[\int_t^\infty f_{T(x)T(y)}(t, v)\, dv + \int_t^\infty f_{T(x)T(y)}(u, t)\, du\right]. $$

Hence,

$$ f_{T(xy)}(t) = \int_t^\infty f_{T(x)T(y)}(t, v)\, dv + \int_t^\infty f_{T(x)T(y)}(u, t)\, du. \tag{9.3.6} $$

Using (9.3.2), the reader can show that the p.d.f. of T can also be written as

$$ f_{T(xy)}(t) = f_{T(x)}(t) + f_{T(y)}(t) - \left[\int_0^t f_{T(x)T(y)}(t, v)\, dv + \int_0^t f_{T(x)T(y)}(u, t)\, du\right], $$

or with actuarial notation as

$$ \begin{aligned} f_{T(xy)}(t) &= {}_tp_x\, \mu(x + t) + {}_tp_y\, \mu(y + t) \\ &\quad - \left[\int_0^t f_{T(x)T(y)}(t, v)\, dv + \int_0^t f_{T(x)T(y)}(u, t)\, du\right]. \end{aligned} \tag{9.3.7} $$

When $T(x)$ and $T(y)$ are independent, $f_{T(x)T(y)}(u, v) = {}_up_x\, \mu(x + u)\, {}_vp_y\, \mu(y + v)$, and (9.3.6) reduces directly to

$$ = {}_tp_y\, {}_tp_x\, [\mu(x + t) + \mu(y + t)]. \tag{9.3.8} $$

Example 9.3.3

By use of (9.3.6) determine the p.d.f. of $T(xy)$ for Example 9.2.1. Verify your result by examination of the d.f. in Example 9.3.1.

Solution:

Using (9.3.6) we obtain

$$f_T(t) = \begin{cases} \displaystyle\int_t^{10} 0.0006\,(t - v)^2\,dv + \int_t^{10} 0.0006\,(u - t)^2\,du \\ \qquad = 0.0004(10 - t)^3 & \text{for } 0 < t < 10, \\ 0 & \text{elsewhere.} \end{cases}$$

This is the derivative of the d.f. in Example 9.3.1. ▼

We saw that $T(xy)$ has the same distribution in Examples 9.3.1 and 9.3.2. If we use (9.3.8) to obtain the p.d.f. of $T(xy)$ for Example 9.2.3, we will see this again. This is left as Exercise 9.8.

As explained in Chapter 3, the distribution of $T = T(xy)$ can also be specified by the force of "mortality," or more generally, the force of "failure." First we consider a notation for the force of failure of the status at time t. The traditional notation for this force is $\mu_{x+t:y+t}$ (in analogy with μ_{x+t}), but, in preparation for discussing other statuses where duration must be recognized, and in accordance with the notational convention adopted in Chapter 3, we use the notation $\mu_{xy}(t)$. The notation $\mu_{xy}(t)$ does not necessarily mean that (x) and (y) or the survival status (xy) were subject to a selection process, but the status did come into existence at these ages.

By analogy with the first formula of (3.2.12) and with $f_{T(x)}(x)$ and $F_{T(x)}(x)$ replaced by $f_{T(xy)}(t)$ and $F_{T(xy)}(t)$, we have

$$\mu_{xy}(t) = \frac{f_{T(xy)}(t)}{1 - F_{T(xy)}(t)}. \tag{9.3.9}$$

For dependent $T(x)$, $T(y)$, this expression does not simplify beyond the general form. However, using (9.3.3) and (9.3.8) for the independent case, we have $\mu_{xy}(t) = \mu(x + t) + \mu(y + t)$.

In words, if the future lifetimes are independent, the force of failure for their joint-life status is the sum of the forces of mortality for the individuals. As in Chapter 3 with the single life case, we can characterize the distribution of $T(xy)$ by the p.d.f., the d.f., the survival function, or the force of failure.

Example 9.3.4

Determine the force of failure for the joint-life statuses of (x) and (y) in Examples 9.2.1 and 9.2.3.

Solution:

Since the joint distributions of the two examples produce the same distribution for $T(xy)$, we will use the independent case. From the results of part (a) of Example 9.2.3 and (9.3.9),

$$\mu_{xy}(t) = \frac{4}{10 - t} \qquad 0 < t < 10.$$

▼

We now turn to the curtate future lifetime of the joint-life status.

The probability that the joint-life status fails during the time k to $k + 1$ is determined by

$$\begin{aligned} \Pr(k < T \le k + 1) &= \Pr(T \le k + 1) - \Pr(T \le k) \\ &= {}_kp_{xy} - {}_{k+1}p_{xy} \\ &= {}_kp_{xy}\, q_{x+k:y+k}. \end{aligned} \tag{9.3.10}$$

When the future lifetimes of (x) and (y) are independent, the probability of the joint-life status $(x + k:y + k)$ failing within the next year can be written in terms of the probabilities of independent failure of the individual lives as follows:

$$\begin{aligned} q_{x+k:y+k} &= 1 - p_{x+k:y+k} \\ &= 1 - p_{x+k}\, p_{y+k} \\ &= 1 - (1 - q_{x+k})(1 - q_{y+k}) \\ &= q_{x+k} + q_{y+k} - q_{x+k}\, q_{y+k} \\ &= q_{x+k} + (1 - q_{x+k})\, q_{y+k}. \end{aligned} \tag{9.3.11}$$

From the discussion of curtate-future-lifetime of (x) in Section 3.2.3, we see that (9.3.10) also provides the p.f. of the random variable K, the number of years completed prior to failure of the joint-life status; that is, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \Pr(K = k) &= \Pr(k \le T < k + 1) \\ &= \Pr(k < T \le k + 1) \\ &= {}_kp_{xy}\, q_{x+k:y+k} \\ &= {}_{k|}q_{xy}. \end{aligned} \tag{9.3.12}$$

Example 9.3.5

Determine the p.f. and curtate expectation of $K(xy)$ using the common d.f. of Examples 9.3.1 and 9.3.2.

Solution:

From Example 9.3.2,

$${}_kp_{xy} = 0.0001(10 - k)^4.$$

Hence

$$\Pr[K(xy) = k] = 0.0001[(10 - k)^4 - (9 - k)^4],$$
$$k = 0, 1, 2, 3, \ldots, 9.$$

From (3.5.5),

$$e_{xy} = \mathrm{E}[K(xy)] = \sum_{k=0}^{\infty} {}_{k+1}p_{xy} = \sum_{k=0}^{9} 0.0001(9 - k)^4 = 1.5333.$$

▼

9.4 The Last-Survivor Status

In addition to benefits defined in terms of the time of the first death, there are those defined in terms of the time of the last death. In this section we will examine situations in which the random variable is the time of the last death.

A survival status that exists as long as at least one member of a set of lives is alive and fails upon the last death is called the ***last-survivor status***. It is denoted by $(\overline{x_1 x_2 \cdots x_m})$, where x_i represents the age of member i and m represents the number of members of the set. We consider the distribution of the time-until-failure of the last-survivor status, the random variable $T = \max[T(x_1), T(x_2), \ldots, T(x_m)]$, where $T(x_i)$ is the time-until-death of individual i. The random variable T can be interpreted as the ***largest order statistic*** associated with $[T(x_1), T(x_2), \ldots, T(x_m)]$. Unlike the typical situation in the study of inferential statistics, the m random variables here are not necessarily independent and identically distributed.

For the case of two lives (x) and (y), $T(\overline{xy}) = \max[T(x), T(y)]$. General relationships exist among $T(xy)$, $T(\overline{xy})$, $T(x)$, and $T(y)$. For each outcome, $T(xy)$ equals either $T(x)$ or $T(y)$ and $T(\overline{xy})$ equals the other. Thus, for all joint distributions of $T(x)$ and $T(y)$, the following relationships hold:

$$T(xy) + T(\overline{xy}) = T(x) + T(y), \tag{9.4.1}$$

$$T(xy)\, T(\overline{xy}) = T(x)\, T(y), \tag{9.4.2}$$

$$a^{T(xy)} + a^{T(\overline{xy})} = a^{T(x)} + a^{T(y)} \quad \text{for } a > 0. \tag{9.4.3}$$

There are also some general relationships among the distributions of these four random variables that come from the method of inclusion-exclusion of probability; that is,

$$\Pr(A \cup B) + \Pr(A \cap B) = \Pr(A) + \Pr(B). \tag{9.4.4}$$

Defining A as $\{T(x) \le t\}$ and B as $\{T(y) \le t\}$, we have $A \cap B = \{T(\overline{xy}) \le t\}$ and $A \cup B = \{T(xy) \le t\}$, which lead to

$$F_{T(xy)}(t) + F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t). \tag{9.4.5}$$

From this it follows that

$${}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y \tag{9.4.6}$$

and

$$f_{T(xy)}(t) + f_{T(\overline{xy})}(t) = f_{T(x)}(t) + f_{T(y)}(t). \tag{9.4.7}$$

The relationships developed above allow the distribution of the last-survivor status to be explored by use of the distribution of the joint-life status that is developed in the previous section. An illustration of this fact is provided by substituting (9.3.2) into (9.4.5) to obtain

$$F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(x)}(t) - F_{T(xy)}(t) = F_{T(x)T(y)}(t, t),$$

a relationship that also follows from $F_{T(\overline{xy})}(t) = \Pr[T(x) \le t \cap T(y) \le t]$.

Example 9.4.1

Determine the d.f., survival function of p.d.f. of $T(\overline{xy})$ for the lives in Example 9.2.1.

Solution:

From (9.4.5) and the solutions to part (b) of Example 9.2.1 and Example 9.3.1,

$$\begin{aligned} F_{T(\overline{xy})}(t) &= 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - [1 - 0.0001(10 - t)^4] \\ &= 0.0001t^4 = F_{T(x)T(y)}(t, t) \qquad 0 \le t < 10, \\ {}_tp_{\overline{xy}} &= 1 - F_{T(\overline{xy})}(t) = 1 - 0.0001t^4 \qquad 0 < t \le 10. \end{aligned}$$

By differentiation,

$$f_{T(\overline{xy})}(t) = 0.0004\, t^3 \qquad 0 < t < 10.$$

▼

Example 9.4.2

Determine the d.f., survival function and p.d.f. of $T(\overline{xy})$ for the lives in Example 9.2.3.

Solution:

From (9.4.5) and the solutions to Examples 9.2.3 and 9.3.2 for $0 < t \le 10$,

$$\begin{aligned} F_{T(\overline{xy})}(t) &= 2[1 - 0.01(10 - t)^2] - [1 - 0.0001(10 - t)^4] \\ &= [1 - 0.01(10 - t)^2]^2 \\ &= t^2(0.2 - 0.01t)^2 = F_{T(x)T(y)}(t, t), \\ {}_tp_{\overline{xy}} &= 1 - F_{T(\overline{xy})}(t) = 1 - t^2(0.2 - 0.01t)^2, \\ f_{T(\overline{xy})}(t) &= 0.04t(2 - 0.1t)(1 - 0.1t) \qquad 0 < t < 10. \end{aligned}$$

▼

An observation derived by comparing Examples 9.3.1 and 9.3.2 with Examples 9.4.1 and 9.4.2 is that two different joint distributions may produce the same distribution for the joint-life status but different distributions for the last-survivor status. This possibility could have been anticipated by the general nature of (9.4.5).

For applications it is preferable to rearrange and to restate (9.4.5) and (9.4.7) in actuarial notation:

$$ {}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}, \qquad \text{(9.4.5) restated} $$

$$ {}_tp_{\overline{xy}}\, \mu_{\overline{xy}}(t) = {}_tp_x\, \mu(x+t) + {}_tp_y\, \mu(y+t) - {}_tp_{xy}\, \mu_{xy}(t). \qquad \text{(9.4.7) restated} $$

The force of failure for the last-survivor status is implicitly defined in this restatement of (9.4.7) as

$$ \mu_{\overline{xy}}(t) = \frac{f_{T(\overline{xy})}(t)}{1 - F_{T(\overline{xy})}(t)} $$

$$ = \frac{{}_tp_x\, \mu(x+t) + {}_tp_y\, \mu(y+t) - {}_tp_{xy}\, \mu_{xy}(t)}{{}_tp_{\overline{xy}}}. \qquad (9.4.8) $$

When $T(x)$ and $T(y)$ are independent $\mu_{xy}(t)$ can be replaced by $\mu_x(t) + \mu_y(t)$ in (9.4.7) and (9.4.8), and they can be rewritten as

$$ {}_tp_{\overline{xy}}\, \mu_{\overline{xy}}(t) = {}_tq_y\, {}_tp_x\, \mu(x+t) + {}_tq_x\, {}_tp_y\, \mu(y+t) \qquad (9.4.9) $$

and

$$ \mu_{\overline{xy}}(t) = \frac{{}_tq_y\, {}_tp_x\, \mu(x+t) + {}_tq_x\, {}_tp_y\, \mu(y+t)}{{}_tq_y\, {}_tp_x + {}_tq_x\, {}_tp_y + {}_tp_x\, {}_tp_y}. \qquad (9.4.10) $$

In this form the force of failure of the last-survivor status is a weighted average of forces of mortality. Forces of mortality are conditional p.d.f.'s, and the probability density that (x) and (y) will die at t is 0. As a result, the force associated with ${}_tp_x\, {}_tp_y$ in the weighted average is 0.

Example 9.4.3

Determine the force of failure for the last survivor of the two lives in
a. Example 9.2.1, and
b. Example 9.2.3.

Solution:

a. Using the results of Example 9.4.1,

$$ \mu_{\overline{xy}}(t) = \frac{0.0004t^3}{1 - 0.0001t^4} = \frac{4t^3}{10{,}000 - t^4}. $$

b. Using the results of Example 9.4.2,

$$ \mu_{\overline{xy}}(t) = \frac{0.04t(2 - 0.1t)(1 - 0.1t)}{1 - t^2(0.2 - 0.01t)^2} = \frac{4t(20 - t)(10 - t)}{10{,}000 - t^2(20 - t)^2}. $$ ▼

Discrete analogues of relationships (9.4.1)–(9.4.3) and (9.4.5)–(9.4.7) exist for the curtate-future-lifetimes. These are

$$ K(xy) + K(\overline{xy}) = K(x) + K(y), \qquad (9.4.11) $$

$$ K(xy)\, K(\overline{xy}) = K(x)\, K(y), \qquad (9.4.12) $$

$$a^{K(xy)} + a^{K(\overline{xy})} = a^{K(x)} + a^{K(y)} \qquad \text{for } a > 0, \tag{9.4.13}$$

$$F_{K(xy)}(k) + F_{K(\overline{xy})}(k) = F_{K(x)}(k) + F_{K(y)}(k). \tag{9.4.14}$$

From (9.4.14) it follows that

$$f_{K(xy)}(k) + f_{K(\overline{xy})}(k) = f_{K(x)}(k) + f_{K(y)}(k). \tag{9.4.15}$$

The distribution of $K(\overline{xy})$, the number of years completed prior to failure of the last-survivor status, that is, the number of years completed prior to the last death, can now be determined from these relationships and the results for the curtate-future-lifetime of the joint-life survivor status. From (9.4.15),

$$\Pr[K(\overline{xy}) = k] = f_{K(\overline{xy})}(k) = {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}. \tag{9.4.16}$$

For independent lives, (9.3.5) and (9.3.12) allow us to write (9.4.16) as

$$\begin{aligned}\Pr[K(\overline{xy}) = k] &= {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_x\, {}_kp_y\, (q_{x+k} + q_{x+k} - q_{x+k}q_{y+k}) \\ &= (1 - {}_kp_y)\, {}_kp_x\, q_{x+k} + (1 - {}_kp_x)\, {}_kp_y\, q_{y+k} + {}_kp_x\, {}_kp_y\, q_{x+k}\, q_{x+k}.\end{aligned}$$

In this last form, the first two terms are the probability that only the second death occurs between times k and $k + 1$. The third term is the probability that both deaths occur during that year. This expression for $\Pr[K(\overline{xy}) = k]$ is analogous to (9.4.9) for the p.d.f. of $T(\overline{xy})$ where, since the probability that two deaths occur in the same instant is 0, there are only two terms.

9.5 More Probabilities and Expectations

In Sections 9.3 and 9.4 we express the p.d.f.'s and the d.f.'s of the future lifetimes of the joint-life status and the last-survivor status in terms of the functions of the probability distributions for the single lives. In this section we use these expressions to solve probability problems and to obtain expectations, variances, and, for independent individual future lifetimes, the covariance of the joint-life and last-survivor future lifetimes.

Example 9.5.1

Assuming the future lifetimes of (80) and (85) are independent, obtain an expression, in single life table functions, for the probability that their

a. First death occurs after 5 and before 10 years from now, and
b. Last death occurs after 5 and before 10 years from now.

Solution:

a. With $T = T(80:85)$ we obtain

$$\begin{aligned}\Pr(5 < T \le 10) &= \Pr(T > 5) - \Pr(T > 10) \\ &= {}_5p_{80:85} - {}_{10}p_{80:85} \\ &= {}_5p_{80}\, {}_5p_{85} - {}_{10}p_{80}\, {}_{10}p_{85}.\end{aligned}$$

Note that the independence assumption is used only in the last step.

b. With $T = T(\overline{80:85})$

$$\Pr(5 < T \le 10) = \Pr(T > 5) - \Pr(T > 10)$$

$$= {}_5p_{\overline{80:85}} - {}_{10}p_{\overline{80:85}},$$

and from (9.4.6) we obtain

$$= {}_5p_{80} - {}_{10}p_{80} + {}_5p_{85} - {}_{10}p_{85} - ({}_5p_{80:85} - {}_{10}p_{80:85}).$$

Using the independence assumption, we can substitute ${}_5p_{80}\ {}_5p_{85}$ for ${}_5p_{80:85}$ and ${}_{10}p_{80}\ {}_{10}p_{85}$ for ${}_{10}p_{80:85}$. ▼

The results of Section 3.5 concerning expected values of the distribution of T, the time-until-the death of (x), are also valid if $T = T(u)$ the time-until-failure of a survival status (u). We used some of these ideas in the previous two sections; now we will state them explicitly.

From (3.5.2) we have that $\mathring{e}_u = \mathrm{E}[T(u)]$, which for a survival status (u) can be obtained from the formula

$$\mathring{e}_u = \int_0^\infty {}_tp_u \, dt. \tag{9.5.1}$$

If (u) is the joint-life status (xy), then

$$\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy} \, dt, \tag{9.5.2}$$

and for the last-survivor status $(\overline{xy})$ we have

$$\mathring{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}} \, dt. \tag{9.5.3}$$

Upon taking expectation of both sides of (9.4.1), we see that

$$\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}. \tag{9.5.4}$$

From (3.5.5), the expected value of $K = K(u)$ is

$$e_u = \sum_1^\infty {}_kp_u$$

for a survival status, (u). Special cases include

$$e_{xy} = \sum_1^\infty {}_kp_{xy}$$

and

$$e_{\overline{xy}} = \sum_1^\infty {}_kp_{\overline{xy}}.$$

It follows from (9.4.11) that

$$e_{\overline{xy}} = e_x + e_y - e_{xy}. \tag{9.5.5}$$

The variance formulas derived in Section 3.5 can be used to calculate the variance of the future lifetime, or curtate-future-lifetime, of any survival status, (u). Thus,

$$\operatorname{Var}[T(xy)] = 2\int_0^{\infty} t\ {}_tp_{xy}\,dt - (\mathring{e}_{xy})^2 \tag{9.5.6}$$

and

$$\operatorname{Var}[T(\overline{xy})] = 2\int_0^{\infty} t\ {}_tp_{\overline{xy}}\,dt - (\mathring{e}_{\overline{xy}})^2. \tag{9.5.7}$$

In Example 9.2.1 we calculated the covariance of $T(x)$ and $T(y)$ for dependent future lifetimes. For the moment, we return to dependent future lifetimes to explore the covariance of $T(xy)$ and $T(\overline{xy})$ in the general case:

$$\operatorname{Cov}[T(xy), T(\overline{xy})] = \mathrm{E}[T(xy)\,T(\overline{xy})] - \mathrm{E}[T(xy)]\,\mathrm{E}[T(\overline{xy})]. \tag{9.5.8}$$

On the basis of (9.4.2),

$$\mathrm{E}[T(xy)\,T(\overline{xy})] = \mathrm{E}[T(x)\,T(y)].$$

Using this result and (9.5.4), we can write

$$\operatorname{Cov}[T(xy), T(\overline{xy})] = \mathrm{E}[T(x)\,T(y)] - \mathrm{E}[T(xy)]\,\{\mathrm{E}[T(x)] + \mathrm{E}[T(y)] - \mathrm{E}[T(xy)]\},$$

which can be rewritten as

$$= \operatorname{Cov}[T(x), T(y)] + \{\mathrm{E}[T(x)] - \mathrm{E}[T(xy)]\}\,\{\mathrm{E}[T(y)] - \mathrm{E}[T(xy)]\}. \tag{9.5.9}$$

If $T(x)$ and $T(y)$ are uncorrelated, then

$$\operatorname{Cov}[T(xy), T(\overline{xy})] = (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}). \tag{9.5.10}$$

Since both factors of (9.5.10) must be non-negative, we can see that when $T(x)$ and $T(y)$ are uncorrelated, $T(xy)$ and $T(\overline{xy})$ are positively correlated, except in the trivial case where $\mathring{e}_x$ or $\mathring{e}_y$ equals $\mathring{e}_{xy}$.

Example 9.5.2

For $T(x)$ and $T(y)$ of Examples 9.2.1 and 9.2.3 determine (a) $\operatorname{Cov}[T(xy), T(\overline{xy})]$, and (b) the correlation coefficient of $T(xy)$ and $T(\overline{xy})$.

Solution:

Most of the required calculations have been done in previous examples. The remaining calculations can be done readily with the p.d.f.'s that were determined there. We will complete the calculations by displaying intermediate results in tabular form along with the number of the formula being illustrated.

Item	Distribution: Examples 9.2.1, 9.3.1, 9.4.1	Distribution: Examples 9.2.3, 9.3.2, 9.4.2
a. $\mathring{e}_x = \mathring{e}_y = \mathrm{E}[T(x)] = \mathrm{E}[T(y)]$	5	10/3
$\mathring{e}_{xy} = \mathrm{E}[T(xy)]$	2	2
$\mathrm{Cov}[T(x), T(y)]$	−25/3	0
$\mathrm{Cov}[T(xy), T(\overline{xy})]$	2/3	16/9
b. $\mathrm{E}[T(xy)^2]$	20/3	20/3
$\mathrm{Var}[T(xy)]$	8/3	8/3
$\mathrm{E}[T(\overline{xy})]$	8	14/3
$\mathrm{E}[T(\overline{xy})^2]$	200/3	80/3
$\mathrm{Var}[T(\overline{xy})]$	8/3	44/9
$\rho_{T(xy)\ T(\overline{xy})}$	1/4	$\sqrt{8/33}$

9.6 Dependent Lifetime Models

In Section 9.2 the concept of the joint distribution of $[T(x), T(y)]$ was introduced. Two examples, which did not appear plausible as models for the joint distribution of $[T(x), T(y)]$ for human lives, were used to illustrate the ideas. Example 9.2.1 illustrated the ideas when $T(x)$ and $T(y)$ are dependent, and Example 9.2.3 provided similar illustrations for independent random variables.

In this section, two general approaches to specifying the joint distribution of $[T(x), T(y)]$ will be studied. The convenient independent case is included within each model.

9.6.1 Common Shock

Let $T^*(x)$ and $T^*(y)$ denote two future lifetime random variables that, in the absence of the possibility of a common shock, are independent; that is,

$$\begin{aligned} s_{T^*(x)\ T^*(y)}(s, t) &= \Pr[T^*(x) > s \cap T^*(y) > t] \\ &= s_{T^*(x)}(s)\ s_{T^*(y)}(t). \end{aligned} \tag{9.6.1}$$

In addition, there is a ***common shock*** random variable, to be denoted by Z, that can affect the joint distribution of the time-until-death of lives (x) and (y). This common shock random variable is independent of $[T^*(x), T^*(y)]$ and has an exponential distribution; that is,

$$s_Z(z) = e^{-\lambda z} \quad z > 0, \lambda \geq 0.$$

We can picture the random variable Z as being associated with the time of a catastrophe such as an earthquake or aircraft crash. The random variables of interest in building models for life insurance or annuities to (x) and (y) are $T(x) = \min[T^*(x), Z]$ and $T(y) = \min[T^*(y), Z]$. The joint survival function of $[T(x), T(y)]$ is

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= \Pr\{\min[T^*(x), Z] > s \cap \min[T^*(y), Z] > t\} \\ &= \Pr\{[T^*(x) > s \cap Z > s] \cap [T^*(y) > t \cap Z > t]\} \\ &= \Pr[T^*(x) > s \cap T^*(y) > t \cap Z > \max(s, t)] \\ &= s_{T^*(x)}(s)\, s_{T^*(y)}(t)\, e^{-\lambda[\max(s,t)]}. \end{aligned} \tag{9.6.2}$$

The final line of (9.6.2) follows from the independence of $[T^*(x), T^*(y), Z]$.

Using the joint survival function displayed in (9.6.2), we can determine the joint p.d.f. of $[T(x), T(y)]$. Except when $s = t$, the p.d.f. can be found by partial differentiation. We have

$$f_{T(x)T(y)}(s, t) = \frac{\partial^2}{\partial s \partial t} s_{T^*(x)}(s)\, s_{T^*(y)}(t)\, e^{-\lambda[\max(s,t)]}$$

$$= [s'_{T^*(x)}(s)\, s'_{T^*(y)}(t) - \lambda s_{T^*(x)}(s)\, s'_{T^*(y)}(t)]\, e^{-\lambda s} \quad 0 < t < s \tag{9.6.3a}$$

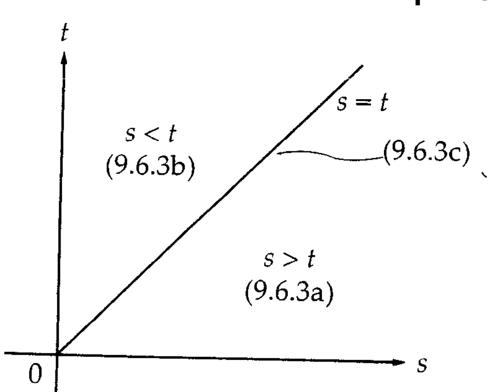
$$= [s'_{T^*(x)}(s)\, s'_{T^*(y)}(t) - \lambda s'_{T^*(x)}(s)\, s_{T^*(y)}(t)]\, e^{-\lambda t} \quad 0 < s < t. \tag{9.6.3b}$$

This display does not complete the definition of the p.d.f. When $s = t$, the common shock contribution to the p.d.f. is

$$f_{T(x)T(y)}(t, t) = \lambda e^{-\lambda t}\, s_{T^*(x)}(t)\, s_{T^*(y)}(t) \qquad t \geq 0. \tag{9.6.3c}$$

The domain of this mixed p.d.f., with a ridge of density along the line $s = t$, is shown in Figure 9.6.1.

Domain of Common Shock p.d.f.



Remark:

The interpretation of the p.d.f. given in (9.6.3.a), (9.6.3.b), and (9.6.3c) requires a careful analysis of the distribution of $[T^*(x), T^*(y)]$ and the derived distribution of $\{T(x) = \min[T^*(x), Z], T(y) = \min[T^*(y), Z]\}$. Table 9.6.1 facilitates this analysis. Because of the intervention of the common shock, realizations of $T^*(x)$ and $T^*(y)$ may not be observed because of the prior realization of Z. The common shock random variable can mask or censor observations of $T^*(x)$ and $T^*(y)$. In addition, events such as $T^*(x) < Z < T^*(y)$ and $T^*(y) < Z < T^*(x)$ can contribute to the p.d.f.

The marginal survival functions are given by

$$s_{T(x)}(s) = \Pr\{[T(x) > s] \cap [T(y) > 0]\}$$
$$= s_{T^*(x)}(s)e^{-\lambda s} \tag{9.6.4a}$$

and

$$s_{T(y)}(t) = \Pr\{[T(x) > 0] \cap [T(y) > t]\}$$
$$= s_{T^*(x)}(t)e^{-\lambda t}. \tag{9.6.4b}$$

Interpretation of p.d.f. of $(T(x), T(y))$, Common Shock Model

Formula	Interpretation	Domain
9.6.3a	$\Pr\{[(s < T^*(x) \le s + \Delta s) \cap (t < T^*(y) \le t + \Delta t) \cap (Z > s)] \cup [(T^*(x) > s) \cap (t < T^*(y) \le t + \Delta t) \cap (s < Z \le s + \Delta s)]\} \approx [f_{T^*(x)}(s) f_{T^*(y)}(t) s_Z(s) + s_{T^*(x)}(s) f_{T^*(y)}(t) f_Z(s)]\Delta s\Delta t$	$0 < t < s$
9.6.3b	$\Pr\{[(s < T^*(x) \le s + \Delta s) \cap (t < T^*(y) \le t + \Delta t) \cap (Z > t)] \cup [(s < T^*(x) \le s + \Delta s) \cap (T^*(y) > t) \cap (t < Z \le t + \Delta t)]\} \approx [f_{T^*(x)}(s) f_{T^*(y)}(t) s_Z(t) + f_{T^*(x)}(s) s_{T^*(y)}(t) f_Z(t)]\Delta s\Delta t$	$0 < s < t$
9.6.3c	$\Pr\{(T^*(x) > t) \cap (T^*(y) > t) \cap (t < Z \le t + \Delta t)\} \approx s_{T^*(x)}(t) s_{T^*(y)}(t) f_Z(t) \Delta t$	$s = t$

If we are interested in the distribution of $T(xy) = \min[T(x), T(y)]$, the joint-life status in the common shock model, the survival function can be obtained using (9.3.1) and (9.6.3):

$$s_{T(xy)}(t) = s_{T^*(x)}(t)\, s_{T^*(y)}(t)\, e^{-\lambda t} \quad 0 < t. \tag{9.6.5}$$

The distribution of $T(\overline{xy}) = \max[T(x), T(y)]$, the last-survivor status in the common shock model, can be derived by using (9.4.5), (9.6.5), (9.6.4a), and (9.6.4b):

$$s_{T(\overline{xy})}(t) = [s_{T^*(x)}(t) + s_{T^*(y)}(t) - s_{T^*(x)T^*(y)}(t, t)]\, e^{-\lambda t} \quad 0 < t. \tag{9.6.6}$$

If the common shock parameter $\lambda = 0$, formulas (9.6.5) and (9.6.6) revert to the form where $T(x)$ and $T(y)$ are independent. When $\lambda > 0$ the joint-life and last-survivor survival functions are each less than the corresponding survival function when $\lambda = 0$.

Example 9.6.1

Exhibit $\mu_{xy}(t)$ as derived from (9.6.5).

Solution:

$$\mu_{xy}(t) = -\frac{d}{dt}\log\left[s_{T^*(x)}(t)\, s_{T^*(y)}(t)e^{-\lambda t}\right]$$
$$= \mu(x+t) + \mu(y+t) + \lambda. \quad \blacktriangledown$$

Example 9.6.2

The random variables $T^*(x)$, $T^*(y)$, and Z are independent and have exponential distributions with, respectively, parameter μ_1, μ_2, and λ. These three random variables are components of a common shock model.

a. Exhibit the marginal p.d.f. of $T(y)$ by evaluating

$$f_{T(y)}(t) = \int_0^t f_{T(x)T(y)}(s, t)\,ds + f_{T(x)T(y)}(t, t) + \int_t^\infty f_{T(x)T(y)}(s, t)\,ds.$$

b. Exhibit the marginal survival function of $T(y)$ by evaluating

$$s_{T(y)}(t) = \int_t^\infty f_{T(y)}(u)\,du.$$

c. Evaluate

$$\Pr[T(x) = T(y)] = \int_0^\infty f_{T(x)T(y)}(t, t)\,dt.$$

Solution:

a. We use the three elements of (9.6.3), adapted for exponential distributions, to obtain

$$f_{T(y)}(t) = \textstyle\int_0^t \mu_1(\mu_2+\lambda)e^{-\mu_1 s-(\mu_2+\lambda)t}\,ds + \lambda e^{-(\mu_1+\mu_2+\lambda)t}$$
$$+ \textstyle\int_t^\infty \mu_2(\mu_1+\lambda)e^{-(\mu_1+\lambda)s-\mu_2 t}\,ds$$
$$= (\mu_2+\lambda)e^{-(\mu_2+\lambda)t}(1-e^{-\mu_1 t}) + \lambda e^{-(\mu_1+\mu_2+\lambda)t} + \mu_2 e^{-(\mu_1+\mu_2+\lambda)t}$$
$$= (\mu_2+\lambda)e^{-(\mu_2+\lambda)t} \quad 0 < t.$$

b. $s_{T(y)} = \int_t^\infty f_{T(y)}(u)\,du = e^{-(\mu_2+\lambda)t} = s_{T^*(y)}(t)e^{-\lambda t}$, which agrees with (9.6.4.b).

c. $\Pr[T(x) = T(y)] = \int_0^\infty \lambda e^{-(\mu_1+\mu_2+\lambda)t}\,dt = \dfrac{\lambda}{\lambda+\mu_1+\mu_2}$. ▼

9.6.2 Copulas

The word ***copula*** means something that connects or joins together. Copula is used in multivariate statistical analysis to define a class of bivariate distributions with specified marginal distributions.

In this section we will illustrate actuarial applications of Frank's copula. The notation will be that used in Section 9.2. It is claimed that

$$F_{T(x)T(y)}(s, t) = \frac{1}{\alpha} \log \left[1 + \frac{(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)}{e^{\alpha} - 1} \right] \tag{9.6.7}$$

when $\alpha \neq 0$ is a joint d.f. with marginal distribution $F_{T(x)}(s)$ and $F_{T(y)}(t)$. This claim can be verified by confirming that

$$F_{T(x)T(y)}(0, 0) = 0, \tag{9.6.8}$$

$$F_{T(x)T(y)}(\infty, \infty) = 1, \tag{9.6.9}$$

$$\frac{\partial}{\partial s \partial t} F_{T(x)T(y)}(s, t) = f_{T(x)T(y)}(s, t)$$

$$= \frac{\alpha f_{T(x)}(s)\, f_{T(x)}(t)[e^{\alpha[F_{T(x)}(s)+F_{T(y)}(t)]}]}{[(e^{\alpha} - 1) + (e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)]^2} (e^{\alpha} - 1) \geq 0, \tag{9.6.10}$$

and

$$F_{T(x)T(y)}(s, \infty) = F_{T(x)}(s),$$

$$F_{T(x)T(y)}(\infty, t) = F_{T(y)}(t).$$

Statements (9.6.8) and (9.6.9) are necessary for $F_{T(x)T(y)}(s, t)$ to be a d.f. of two future lifetime random variables. Statement (9.6.10) exhibits the joint p.d.f. and shows that it is non-negative.

The parameter α in the d.f. displayed in (9.6.7) and the p.d.f. displayed in (9.6.10) controls the dependence of $T(x)$ and $T(y)$. This can be appreciated by finding

$$\lim_{\alpha \to 0} f_{T(x)T(y)}(s, t) = f_{T(x)}(s)\, f_{T(y)}(t) \left\{ \lim_{\alpha \to 0} [A(\alpha)\, B(\alpha)\, C(\alpha)] \right\},$$

where

$$A(\alpha) = e^{\alpha[F_{T(x)}(s)+F_{T(y)}(t)]},$$

$$B(\alpha) = \frac{(e^{\alpha} - 1)\alpha}{(e^{\alpha} - 1)^2},$$

and

$$C(\alpha) = \frac{1}{\{1 + [(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)/(e^{\alpha} - 1)]\}^2}.$$

We have

$$\lim_{\alpha \to 0} A(\alpha) = 1,$$

$$\lim_{\alpha \to 0} B(\alpha) = 1,$$

and $\lim_{\alpha \to 0} C(\alpha)$ depends on the term in the denominator

$$\lim_{\alpha \to 0}\left[\frac{(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)}{e^{\alpha} - 1}\right]$$
$$= \lim_{\alpha \to 0}\left\{\frac{F_{T(x)}(s)\, e^{\alpha F_{T(x)}(s)}(e^{\alpha F_{T(y)}(t)} - 1) + F_{T(y)}(t) e^{\alpha F_{T(y)}(t)}(e^{\alpha F_{T(x)}(s)} - 1)}{e^{\alpha}}\right\}$$
$$= 0.$$

Therefore, $\lim_{\alpha \to 0} C(\alpha) = 1$, and $\lim_{\alpha \to 0} f_{T(x)T(y)}(s, t) = f_{T(x)}(s)\, f_{T(y)}(t)$. This means that $T(x)$ and $T(y)$ are independent in the limit as $\alpha \to 0$. We interpret the joint p.d.f. in this fashion when $\alpha = 0$.

Example 9.6.3

Let $F_{T(x)}(s) = s,\ 0 < s \leq 1$, and $F_{T(y)}(t) = t,\ 0 < t \leq 1$, in (9.6.7) and $T(xy) = \min[T(x), T(y)]$.

a. Find the d.f. of $T(xy)$.
b. Find the p.d.f. of $T(xy)$.

Solution:

a. Using (9.3.2),

$$F_{T(xy)}(t) = F_{T(x)}(t) + F_{T(y)}(t) - F_{T(x)T(y)}(t, t)$$
$$= 2t - \frac{1}{\alpha}\log\left[1 + \frac{(e^{\alpha t} - 1)^2}{e^{\alpha} - 1}\right] \qquad 0 < t < 1.$$

b.
$$f_{T(xy)}(t) = \frac{d}{dt} F_{T(xy)}(t)$$
$$= 2 - \frac{2\alpha(e^{\alpha t} - 1)e^{\alpha t} / (e^{\alpha} - 1)}{\alpha\{1 + [(e^{\alpha t} - 1)^2 / (e^{\alpha} - 1)]\}}$$
$$= 2 - \left[\frac{2(e^{\alpha t} - 1)e^{\alpha t}}{(e^{\alpha} - 1) + (e^{\alpha t} - 1)^2}\right] \qquad 0 < t < 1.$$

▼

9.7 Insurance and Annuity Benefits

Insurances and annuities, previously discussed for an individual life, can be defined for the survival status, (u), and are discussed in this section. We also investigate more complicated examples in which an annuity, payable to a survival status, (u), is deferred until another status, (v), has failed.

9.7.1 Survival Statuses

With the single-life status, (x), replaced by the survival status, (u), the models and formulas of Chapters 4 and 5 are applicable here. Expressions for the actuarial present values, variances, and percentiles in terms of the distribution of (u) are immediately available. The relationships of Sections 9.3 and 9.4 can then be used to be these expressions in terms of functions for the individual lives of the survival status.

For an insurance of unit amount payable at the end of the year in which the "survival" status fails, the model and formulas of Section 4.3 apply. Thus if K denotes the curtate-future-lifetime of (u), then the

- Time of payment is $K + 1$,
- Present value at issue of the payment is $Z = v^{K+1}$,
- Actuarial present value, A_u, is $E[Z] = \sum_0^{\infty} v^{k+1} \Pr(K = k)$, (9.7.1)
- $\operatorname{Var}(Z) = {}^2A_u - (A_u)^2$. (9.7.2)

As an illustration, consider a unit sum insured payable at the end of the year in which the last survivor of (x) and (y) dies. From (9.4.16) and (9.7.1) we have

$$A_{\overline{xy}} = \sum_0^{\infty} v^{k+1} \left({}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}\right),$$

which can be used at forces of interest δ and 2δ to obtain the variance by (9.7.2).

The numerous formulas for discrete annuities in Section 5.3 are valid when the annuity payments are contingent on the survival of a survival status. For example, if we replace x with u to emphasize that K is the curtate-future-lifetime of the survival status, (u), we can restate the following formulas for an n-year temporary life annuity in regard to (u):

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & K = 0, 1, \ldots, n-1 \\ \ddot{a}_{\overline{n}|} & K = n, n+1, \ldots, \end{cases} \qquad \text{(5.3.9) restated}$$

$$\ddot{a}_{u:\overline{n}|} = E[Y] = \sum_0^{n-1} \ddot{a}_{\overline{K+1}|}\, {}_{k|}q_u + \ddot{a}_{\overline{n}|}\, {}_np_u, \qquad (9.7.3)$$

$$\ddot{a}_{u:\overline{n}|} = \sum_0^{n-1} {}_kE_u = \sum_0^{n-1} v^k\, {}_kp_u, \qquad \text{(5.3.9) restated}$$

$$\ddot{a}_{u:\overline{n}|} = \frac{1}{d}\left(1 - A_{u:\overline{n}|}\right), \qquad \text{(5.3.12) restated}$$

$$\operatorname{Var}(Y) = \frac{1}{d^2}\left[{}^2A_{u:\overline{n}|} - (A_{u:\overline{n}|})^2\right]. \qquad \text{(5.3.14) restated}$$

As an illustration, consider an annuity of 1, payable at the beginning of each year to which both (x) and (y) survive during the next n years. This is an annuity to the joint-life status (xy). By substituting ${}_tp_{xy}$, or ${}_tp_x\, {}_tp_y$ if the lifetimes are independent, for ${}_tp_u$ in the above formulas, we can obtain the actuarial present value of the annuity. For the variance as given in (5.3.14), we can use

$$A_{xy:\overline{n}|} = 1 - d\ddot{a}_{xy:\overline{n}|}$$

and

$$^2A_{xy:\overline{n}|} = 1 - (2d - d^2)\, ^2\ddot{a}_{xy:\overline{n}|},$$

or we can calculate the actuarial present values directly.

In addition, we can establish relationships among the present-value random variables for annuities and insurances on the last-survivor status and the joint-life status. From relationship (9.4.13), we have

$$v^{K(\overline{xy})+1} + v^{K(xy)+1} = v^{K(x)+1} + v^{K(y)+1}, \tag{9.7.4}$$

$$\ddot{a}_{\overline{K(\overline{xy})+1}|} + \ddot{a}_{\overline{K(xy)+1}|} = \ddot{a}_{\overline{K(x)+1}|} + \ddot{a}_{\overline{K(y)+1}|}. \tag{9.7.5}$$

By taking the expectations of both sides of (9.7.4) and (9.7.5), we have

$$A_{\overline{xy}} + A_{xy} = A_x + A_y$$

and

$$\ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y.$$

These formulas allow us to express the actuarial present values of last-survivor annuities and insurances in terms of those for the individual lives and the joint-life status. Note that these formulas hold for all joint distributions; independence is not required.

We now consider continuous insurances and annuities. If T, the future-lifetime random variable in Sections 4.2 and 5.2, is reinterpreted as $T(u)$, the time-until-failure of the survival status, (u), the formulas of those sections for present values, actuarial present values, percentiles, and variances hold for insurances and annuities for the status (u).

For an insurance paying a unit amount at the moment of failure of (u), the present value at policy issue, the actuarial present value and the variance are given by

$$Z = v^T,$$

$$\bar{A}_u = \int_0^\infty v^t\, {}_tp_u\, \mu_u(t)\, dt, \qquad \text{(4.2.6) restated}$$

$$\text{Var}(Z) = {}^2\bar{A}_u - (\bar{A}_u)^2.$$

As an illustration, the restated (4.2.6) for the last survivor of (x) and (y) would be

$$\bar{A}_{\overline{xy}} = \int_0^\infty v^t\, {}_tp_{\overline{xy}}\, \mu_{\overline{xy}}(t)\, dt.$$

From (9.4.7), this is

$$\bar{A}_{\overline{xy}} = \int_0^\infty v^t\, [{}_tp_x\, \mu(x + t) + {}_tp_y\, \mu(y + t) - {}_tp_{xy}\, \mu_{xy}(t)]\, dt.$$

For an annuity payable continuously at the rate of 1 per annum until the time-of-failure of (u), we have

$$Y = \bar{a}_{\overline{T}|},$$

$$\bar{a}_u = \int_0^\infty \bar{a}_{\overline{t}|}\ {}_tp_u\ \mu_u(t)\ dt \qquad (5.2.3) \text{ restated}$$

$$= \int_0^\infty v^t\ {}_tp_u\ dt, \qquad (5.2.4) \text{ restated}$$

$$\text{Var}(Y) = \frac{{}^2\bar{A}_u - (\bar{A}_u)^2}{\delta^2}. \qquad (5.2.9) \text{ restated}$$

The interest identity,

$$\delta\ \bar{a}_{\overline{T}|} + v^T = 1, \qquad (5.2.7) \text{ restated}$$

is also available for $T = T(u)$ and provides the connection between the models for insurances and annuities.

As an application, consider an annuity payable continuously at the rate of 1 per year as long as at least one of (x) or (y) survives. This is an annuity in respect to $(\overline{xy})$, so we have from the above formulas with $T = T(\overline{xy})$

$$Y = \bar{a}_{\overline{T}|},$$

$$\bar{a}_{\overline{xy}} = \int_0^\infty \bar{a}_{\overline{t}|}\ [{}_tp_x\ \mu(x+t) + {}_tp_y\ \mu(y+t) - {}_tp_{xy}\ \mu_{xy}(t)]\ dt$$

$$= \int_0^\infty v^t\ {}_tp_{\overline{xy}}\ dt,$$

$$\text{Var}(Y) = \frac{{}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2}{\delta^2}.$$

Formula (9.4.3) implies that

$$v^{T(\overline{xy})} + v^{T(xy)} = v^{T(x)} + v^{T(y)} \qquad (9.7.6)$$

and

$$\bar{a}_{\overline{T(\overline{xy})}|} + \bar{a}_{\overline{T(xy)}|} = \bar{a}_{\overline{T(x)}|} + \bar{a}_{\overline{T(y)}|}, \qquad (9.7.7)$$

and (9.4.1) implies that

$$v^{T(\overline{xy})}\ v^{T(xy)} = v^{T(x)}\ v^{T(y)}. \qquad (9.7.8)$$

These identities can be used to obtain the relations among the actuarial present values, variances, and covariances of insurances and annuities for the various statuses. For example, taking the expectations of both sides of (9.7.6) and (9.7.7), we obtain

$$\bar{A}_{\overline{xy}} + \bar{A}_{xy} = \bar{A}_x + \bar{A}_y, \qquad (9.7.9)$$

$$\bar{a}_{\overline{xy}} + \bar{a}_{xy} = \bar{a}_x + \bar{a}_y. \qquad (9.7.10)$$

In the same way that $\text{Cov}[T(\overline{xy}), T(xy)]$ was expressed as

$$\text{Cov}[T(\overline{xy}), T(xy)] = \text{Cov}[T(x), T(y)] + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}),$$

it can be shown that

$$\text{Cov}(v^{T(\overline{xy})}, v^{T(xy)}) = \text{Cov}(v^{T(x)}, v^{T(y)}) + (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy}). \qquad (9.7.11)$$

Both factors in the second term of (9.7.11) are nonpositive, so it will be non-negative and will be zero only in the trivial case where either $\bar{A}_x$ or $\bar{A}_y$ equals $\bar{A}_{xy}$.

The actuarial present value of a continuous annuity paid with respect to $(\overline{xy})$ where, because (x) and (y) are subject to common shock, the joint survival function is given by (9.6.5) can be written in current payment form as

$$\bar{a}_{\overline{xy}} = \int_0^\infty e^{-(\delta+\lambda)t} s_{T^*(x)}(t)\, dt + \int_0^\infty e^{-(\delta+\lambda)t} s_{T^*(y)}(t)\, dt$$

$$- \int_0^\infty e^{-(\delta+\lambda)t} s_{T^*(x)}(t) s_{T^*(y)}(t)\, dt. \qquad (9.7.12)$$

Formula (9.7.12) illustrates how the common shock parameter λ can be combined with the force of interest in some calculations.

Example 9.7.1

a. Extend (9.7.9) to the actuarial present value of an n-year term insurance paying a death benefit of 1 at the moment of the last death of (x) and (y) if this death occurs before time n. If at least one individual survives to time n, no payment is made.
b. Use the formula to calculate the actuarial present value, on the basis of $\delta = 0.05$, of a 5-year term insurance payable on the death of the last survivor of the two lives in Example 9.2.1.

Solution:

a. By restating (9.7.6) for n-year term insurance random variables and then taking expectation of both sides, we have

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n}|} = \bar{A}^{\;1}_{x:\overline{n}|} + \bar{A}^{\;1}_{y:\overline{n}|} - \bar{A}^{\;1}_{xy:\overline{n}|}.$$

The symbol $\bar{A}^{\;1}_{xy:\overline{n}|}$ represents the actuarial present value of an n-year term insurance payable at the failure of the joint-life status if it occurs prior to n.

b. From Example 9.2.1, $T(x)$ and $T(y)$ each has p.d.f. $f_T(t) = 0.0002\,[t^3 + (10-t)^3]$, $0 \le t < 10$. Therefore,

$$\bar{A}^{\;1}_{x:\overline{5}|} = \bar{A}^{\;1}_{y:\overline{5}|} = \int_0^5 e^{-0.05t}\{0.0002[t^3 + (10-t)^3]\}\, dt$$

$$= 0.4563.$$

From Example 9.3.3, we have that $f_{T(xy)}(t) = 0.0004(10-t)^3$, $0 < t < 10$, so

$$\bar{A}^{\;1}_{xy:\overline{5}|} = \int_0^5 e^{-0.05t}[0.0004(10-t)^3]\, dt = 0.8614.$$

Using the result from part (a),

$$\bar{A}^{\;1}_{\overline{xy}:\overline{5}|} = 2(0.4563) - 0.8614 = 0.0512.$$

▼

9.7.2 Special Two-Life Annuities

In this section we illustrate by an example special annuities in which the payment rate depends on the survival of two lives.

Example 9.7.2

An annuity is payable continuously at the rate of

- 1 per year while both (x) and (y) are alive,
- 2/3 per year while one of (x) or (y) is alive and the other is dead.

Derive expressions for

a. The annuity's present-value random variable
b. The annuity's actuarial present value
c. The variance of the random variable in (a), under the assumption that $T(x)$ and $T(y)$ are independent.

Solution:

a. The annuity is a combination of one that is payable at the rate of 2/3 per year while at least one of (x) and (y) is alive—until time $T(\overline{xy})$]—and one that is payable at the rate of 1/3 per year while both individuals are alive—until time $T(xy)$. The present value of the payments is

$$Z = \frac{2}{3}\,\bar{a}_{\overline{T(\overline{xy})}|} + \frac{1}{3}\,\bar{a}_{\overline{T(xy)}|}.$$

b. The actuarial present value is

$$\text{E}[Z] = \frac{2}{3}\,\bar{a}_{\overline{xy}} + \frac{1}{3}\,\bar{a}_{xy}.$$

Using (9.7.10) to substitute for $\bar{a}_{\overline{xy}}$, we have

$$\text{E}[Z] = \frac{2}{3}\,\bar{a}_x + \frac{2}{3}\,\bar{a}_y - \frac{1}{3}\,\bar{a}_{xy}.$$

Alternatively, from (5.3.2B) restated, we have

$$\text{E}[Z] = \frac{2}{3}\int_0^\infty v^t\,{}_tp_{\overline{xy}}\,dt + \frac{1}{3}\int_0^\infty v^t\,{}_tp_{xy}\,dt.$$

Then, by considering the three mutually exclusive cases as to which of the lives may be surviving when $(\overline{xy})$ is surviving at time t, we can write

$${}_tp_{\overline{xy}} = {}_tp_{xy} + ({}_tp_x - {}_tp_{xy}) + ({}_tp_y - {}_tp_{xy}).$$

Substitution of this expression into the first integral and combining the results with the second give

$$\mathrm{E}[Z] = \int_0^\infty v^t \, {}_tp_{xy}\, dt + \frac{2}{3}\int_0^\infty v^t \, ({}_tp_x - {}_tp_{xy})\, dt$$

$$+ \frac{2}{3}\int_0^\infty v^t \, ({}_tp_y - {}_tp_{xy})\, dt.$$

This expression of E[Z] can be directly obtained by considering the three cases. The first term is the actuarial present value of the payments at the rate of 1 per year while both (x) and (y) survive. The second term is the actuarial present value of the payments at the rate of 2/3 per year at those times t when (x) is alive (with probability ${}_tp_x$) but not both of (x) and (y) are alive (with probability ${}_tp_{xy}$). The third term has a similar interpretation with x and y interchanged.

c.
$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{2}{3}\,\bar{a}_{\overline{T(\overline{xy})}|} + \frac{1}{3}\,\bar{a}_{\overline{T(xy)}|}\right)$$

$$= \frac{4}{9}\,\mathrm{Var}(\bar{a}_{\overline{T(\overline{xy})}|}) + \frac{1}{9}\,\mathrm{Var}(\bar{a}_{\overline{T(xy)}|}) + \frac{4}{9}\,\mathrm{Cov}(\bar{a}_{\overline{T(\overline{xy})}|},\, \bar{a}_{\overline{T(xy)}|}).$$

But, by Exercise 9.23, for independent $T(x)$ and $T(y)$,

$$\mathrm{Cov}(\bar{a}_{\overline{T(\overline{xy})}|},\, \bar{a}_{\overline{T(xy)}|}) = \frac{\mathrm{Cov}(v^{T(\overline{xy})},\, v^{T(xy)})}{\delta^2}$$

$$= \frac{(\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})}{\delta^2}.$$

Hence

$$\mathrm{Var}(Z) = \left\{\frac{4/9\,[{}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2] + 1/9\,[{}^2\bar{A}_{xy} - (\bar{A}_{xy})^2] + 4/9\,(\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})}{\delta^2}\right\}.$$

▼

9.7.3 Reversionary Annuities

A ***reversionary annuity*** is payable during the existence of a status (u), but only after the failure of a second status (v). Conceptually, this is a deferred life annuity with a random deferment period equal to the time until failure of the second status. In fact, it is a generalization of the deferred life annuity, for if (v) is an n-year term certain, then reversionary annuity reduces to an n-year deferred annuity. If (u) is a term certain, the reversionary annuity reduces to a form of insurance, family income insurance, studied in Chapter 17. The basic notation for the actuarial present value of this annuity is $a_{v|u}$ with adornments to indicate frequency and timing of payments. The concept has been useful to obtain expressions for the more complex annuity arrangements in terms of single and joint status annuities (see Example 9.7.3). Here we will study the present-value random variables for reversionary annuities.

We start with an annuity of 1 per year payable continuously to (y) after the death of (x). The present value at 0, denoted by Z, is

$$Z = \begin{cases} {}_{T(x)|}\bar{a}_{\overline{T(y)-T(x)}|} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases} \tag{9.7.13}$$

This can be written as

$$Z = \begin{cases} \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(x)}|} & T(x) \le T(y) \\ 0 & T(x) > T(y), \end{cases} \tag{9.7.14}$$

or as

$$Z = \begin{cases} \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(x)}|} & T(x) \le T(y) \\ \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(y)}|} & T(x) > T(y), \end{cases} \tag{9.7.15}$$

which is the same as

$$Z = \bar{a}_{\overline{T(y)}|} - \bar{a}_{\overline{T(xy)}|}. \tag{9.7.16}$$

Thus,

$$\bar{a}_{x|y} = \mathrm{E}[Z] = \mathrm{E}[\bar{a}_{\overline{T(y)}|}] - \mathrm{E}[\bar{a}_{\overline{T(xy)}|}] = \bar{a}_y - \bar{a}_{xy}. \tag{9.7.17}$$

Formulas (9.7.16) and (9.7.17) hold for survival statuses (u) and (v). For example,

$$\bar{a}_{x:\overline{n}||y} = \bar{a}_y - \bar{a}_{xy:\overline{n}|},$$

$$\bar{a}_{x|y:\overline{n}|} = \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}.$$

We note that (9.7.17) holds for dependent future lifetimes.

Example 9.7.3

Calculate the actuarial present value of an annuity payable continuously at the rates shown in the following display.

Case, Rate, and Condition

1. 1 per year with certainty until time n,
2. 1 per year after time n if both (x) and (y) are alive,
3. 3/4 per year after time n if (x) is alive and (y) is dead, and
4. 1/2 per year after time n if (y) is alive and (x) is dead.

Solution:

We use the reversionary annuity idea to write the actuarial present value of this arrangement in terms of single-life and joint-life annuities.

- Case 1: This is an n-year annuity-certain: $\bar{a}_{\overline{n}|}$.
- Case 2: This is n-year deferred joint life annuity to (xy):

$${}_{n|}\bar{a}_{xy} = \bar{a}_{xy} - \bar{a}_{xy:\overline{n}|}.$$

- Case 3: This is a reversionary annuity of 3/4 per annum to x after $\overline{y:\overline{n}|}$:

$$\frac{3}{4}\,\bar{a}_{\overline{y:\overline{n}|}|x} = \frac{3}{4}\,(\bar{a}_x - \bar{a}_{x:(\overline{y:\overline{n}|})}) = \frac{3}{4}\,(\bar{a}_x - \bar{a}_{xy} - \bar{a}_{x:\overline{n}|} + \bar{a}_{xy:\overline{n}|}).$$

- Case 4: This is a reversionary annuity of 1/2 per annum to y after $\overline{x:\overline{n|}}$:

$$\frac{1}{2}\,\bar{a}_{\overline{x:\overline{n|}}|y} = \frac{1}{2}\,(\bar{a}_y - \bar{a}_{xy} - \bar{a}_{y:\overline{n|}} + \bar{a}_{xy:\overline{n|}}).$$

Adding the results for the four cases together, we obtain the required actuarial present value,

$$\bar{a}_{\overline{n|}} + \frac{3}{4}\,\bar{a}_x + \frac{1}{2}\,\bar{a}_y - \frac{1}{4}\,\bar{a}_{xy} - \frac{3}{4}\,\bar{a}_{x:\overline{n|}} - \frac{1}{2}\,\bar{a}_{y:\overline{n|}} + \frac{1}{4}\,\bar{a}_{xy:\overline{n|}}.$$ ▼

9.8 Evaluation—Special Mortality Assumptions

In Section 9.7 the actuarial present values of a variety of insurance and annuity benefits that involve two lifetime random variables were developed. These developments typically culminated in an integral or summation. In this section we study several assumptions about the distribution of $T(u)$ that will simplify the evaluation of these integrals and summations.

9.8.1 Gompertz and Makeham Laws

Here we examine the assumption that mortality follows Makeham's law, or its important special case, Gompertz's law, and the implications for the computations of actuarial present values with respect to multiple life statuses. Independent future lifetime random variables will be assumed.

We begin with the assumption that mortality for each life follows Gompertz's law, $\mu(x) = Bc^x$. We seek to substitute a single-life survival status (w) that has a force of failure equal to the force of failure of the joint-life status (xy) for all $t \geq 0$. Consider

$$\mu_{xy}(s) = \mu(w + s) \qquad s \geq 0; \tag{9.8.1}$$

that is,

$$Bc^{x+s} + Bc^{y+s} = Bc^{w+s},$$

or

$$c^x + c^y = c^w, \tag{9.8.2}$$

which defines the desired w. It follows that for $t > 0$,

$$\begin{aligned} {}_tp_w &= \exp\left[-\int_0^t \mu(w + s)\,ds\right] \\ &= \exp\left[-\int_0^t \mu_{xy}(s)\,ds\right] \\ &= {}_tp_{xy}. \end{aligned} \tag{9.8.3}$$

Thus for w defined in (9.8.2), all probabilities, expected values, and variances for the joint-life status (xy) equal those for the single life (w). For tabled values, the need for a two-dimensional array has been replaced by the need for a one-dimensional array, but typically w is nonintegral, and therefore the determination of its values requires interpolation in the single array.

The assumption that mortality for each life follows Makeham's law (see Table 3.6) makes the search more complex. The force of mortality for the joint-life status is

$$\mu_{xy}(s) = \mu(x + s) + \mu(y + s) = 2A + Bc^s\,(c^x + c^y). \tag{9.8.4}$$

We cannot substitute a single life for the two lives because of the $2A$. Instead, we replace (xy) with another joint-life status (ww), and then

$$\mu_{ww}(s) = 2\mu(w + s) = 2(A + Bc^s c^w), \tag{9.8.5}$$

and we choose w such that

$$2c^w = c^x + c^y. \tag{9.8.6}$$

Unlike the Gompertz case where the one-dimensional array is based on functions from a single life table, this one-dimensional array is based on functions for a joint-life status (ww) involving equal-age lives.

Example 9.8.1

Use (3.7.1) and the $\ddot{a}_{xx}$ values based on the Illustrative Life Table (Appendix 2A) with interest at 6% to calculate the value of $\ddot{a}_{60:70}$. Compare your result with the values of $\ddot{a}_{60:70}$ in the table of $\ddot{a}_{x:x+10}$.

Solution:

From $c = 10^{0.04}$ and $c^{60} + c^{70} = 2c^w$, we obtain $w = 66.11276$. Then using linear interpolation, $\ddot{a}_{60:70} = 0.88724\ddot{a}_{66:66} + 0.11276\ddot{a}_{67:67} = 7.55637$. The value by the $\ddot{a}_{x:x+10}$ table is 7.55633. ▼

9.8.2 Uniform Distribution

We retain the independence assumption, and in addition we assume a uniform distribution of deaths in each year of age for each individual in the joint-life status. With this additional assumption, we can evaluate the actuarial present values of annuities payable more frequently than once a year and insurance benefits payable at the moment of death.

We recall from Table 3.6 that, under the assumption of a uniform distribution of deaths for each year of age, ${}_tp_x = 1 - tq_x$ and

$${}_tp_x\,\mu(x + t) = \frac{d}{dt}(1 - {}_tp_x) = q_x. \tag{9.8.7}$$

When we apply this assumption to a joint-life status (xy), with independent $T(x)$ and $T(y)$, we obtain, for $0 \le t \le 1$,

$$\begin{aligned}
{}_tp_{xy}\,\mu_{xy}(t) &= {}_tp_x\,{}_tp_y\,[\mu(x+t)+\mu(y+t)] \\
&= {}_tp_y\,{}_tp_x\,\mu(x+t) + {}_tp_x\,{}_tp_y\,\mu(y+t) \\
&= (1-tq_y)q_x + (1-tq_x)q_y \\
&= q_x + q_y - q_xq_y + (1-2t)q_xq_y \\
&= q_{xy} + (1-2t)q_xq_y. \qquad (9.8.8)
\end{aligned}$$

On the basis of (4.4.1), the actuarial present value for an insurance benefit in regard to a survival status, (u), can be written as

$$\bar{A}_u = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_u \int_0^1 (1+i)^{1-s}\,\frac{{}_{k+s}p_u}{{}_kp_u}\,\mu_u(k+s)\,ds.$$

Using (9.8.8), we can rewrite this for the joint-life status, (xy), as

$$\begin{aligned}
\bar{A}_{xy} &= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\left[q_{x+k:y+k}\int_0^1 (1+i)^{1-s}\,ds \right. \\
&\qquad\qquad \left. + \, q_{x+k}\,q_{y+k}\int_0^1 (1+i)^{1-s}\,(1-2s)\,ds\right] \\
&= \frac{i}{\delta}A_{xy} + \frac{i}{\delta}\left(1-\frac{2}{\delta}+\frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \qquad (9.8.9)
\end{aligned}$$

To interpret the right-hand side of (9.8.9), we see from (4.4.2) that the first term is equal to $\bar{A}_{xy}$ if $T(xy)$, the time-until-failure of (xy), is uniformly distributed in each year of future lifetime. Such is not the case for $T(xy) = \{\min[T(x), T(y)]\}$ when $T(x)$ and $T(y)$ are distributed independently and uniformly over such years. Under this latter assumption, the conditional distribution of $T(xy)$, given that $T(x)$ and $T(y)$ are in different yearly intervals, is also uniform over each year of future lifetime. However, given that $T(x)$ and $T(y)$ are within the same interval, the distribution of their minimum is shifted toward the beginning of the interval (see Exercise 9.38). A consequence of this shift is to require the second term in (9.8.9) to cover the additional expected costs of the earlier claims in those years. The second term, which is the product of an interest term that is close to $i/6$ (see Exercise 9.39) and a actuarial present value for an insurance payable if both individuals die in the same future year, is very small. The actuarial present value $\bar{A}_{xy}$ is often approximated by ignoring the small correction term, thereby simplifying (9.8.9) to

$$\bar{A}_{xy} \cong \frac{i}{\delta}A_{xy}, \qquad (9.8.10)$$

which is exact, as noted previously, if $T(xy)$ is uniformly distributed in each year of future lifetime.

To evaluate $\bar{a}_{xy}$, we have from (5.2.8), with survival status (x) replaced by (xy),

$$\bar{a}_{xy} = \frac{1}{\delta}(1-\bar{A}_{xy}),$$

and, on substitution from (9.8.9), obtain

$$\bar{a}_{xy} = \frac{1}{\delta}\left\{1 - \frac{i}{\delta}\left[A_{xy} + \left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k}\, q_{y+k}\right]\right\}.$$

Now, on the basis of (5.3.7) for the status (xy), we substitute $1 - d\ddot{a}_{xy}$ for A_{xy} and use (5.4.12) and (5.4.13) to write

$$\begin{aligned}\bar{a}_{xy} &= [\alpha(\infty)\ddot{a}_{xy} - \beta(\infty)] \\ &\quad - \frac{i}{\delta^2}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k}\, q_{y+k}. \qquad (9.8.11)\end{aligned}$$

Formula (9.8.11) follows from the assumption that $T(x)$ and $T(y)$ are distributed independently and uniformly over future years. If we assume that $T(xy)$ itself is uniformly distributed over each future year, then from the continuous case of (5.4.11), $m = \infty$, we would have immediately

$$\bar{a}_{xy} = \alpha(\infty)\ddot{a}_{xy} - \beta(\infty). \qquad (9.8.12)$$

Formula (9.8.12) differs from (9.8.11) by a small amount, which approximates the product of $i/(6\delta)$ and the actuarial present value for an insurance payable if both individuals die in the same future year.

To use the same approach to evaluate the actuarial present value of an annuity-due payable m-thly, we need an expression for $A_{xy}^{(m)}$ under the assumption of a uniform distribution of deaths for each of the individuals in each year of age. In analogy to the continuous case, we start with

$$A_{xy}^{(m)} = \sum_{k=0}^{\infty} v^k\, {}_kp_{xy} \sum_{j=1}^{m} v^{j/m}\left({}_{(j-1)/m}p_{x+k:y+k} - {}_{j/m}p_{x+k:y+k}\right). \qquad (9.8.13)$$

In Exercise 9.40 this expression, under the assumption that $T(x)$ and $T(y)$ are independently and uniformly distributed over each year of age, is reduced to

$$A_{xy}^{(m)} = \frac{i}{i^{(m)}} A_{xy} + \frac{i}{i^{(m)}}\left(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{xy}\, q_{x+k}\, q_{y+k}. \qquad (9.8.14)$$

As $m \to \infty$, the expression in (9.8.14) approach their counterparts in (9.8.9). To interpret the right-hand side of (9.8.14), we see by analogy to (9.8.9) that the first term is the usual approximation for $A_{xy}^{(m)}$ and is exact if $T(xy)$ is uniformly distributed in each year. Then,

$$\frac{i}{i^{(m)}}\left(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}\right) \cong \frac{m^2 - 1}{6m^2}\, i,$$

which is less than $i/6$.

By substituting (9.8.14) into (5.4.2) ~~(5.4.4)~~ restated for (xy), and replacing A_{xy} by $1 - d\ddot{a}_{xy}$, we obtain the formula for $\ddot{a}_{xy}^{(m)}$ that is analogous to (9.8.11). If the second term of (9.8.14) is ignored, the formula for $\ddot{a}_{xy}^{(m)}$ reduces to

$$\ddot{a}_{xy}^{(m)} = \alpha(m)\ddot{a}_{xy} - \beta(m). \qquad (9.8.15)$$

Again by (5.4.11), this is exact under the assumption that the distribution of $T(xy)$ is uniform over each year of future lifetime.

9.9 Simple Contingent Functions

In this section we study insurances that, in addition to being dependent on the time of failure of the status, are contingent on the order of the deaths of the individuals in the group. In this section we will assume that $T(x)$ and $T(y)$ have a continuous joint d.f. This is done to exclude the common shock model of Section 9.6.1.

We begin with an evaluation of the probability that (x) dies before (y) and before n years from now. In IAN this probability is denoted by ${}_nq^{\,1}_{xy}$, where the 1 over the x indicates the probability is for an event in which (x) dies before (y), and n indicates that the event occurs within n years. Then ${}_nq^{\,1}_{xy}$ equals the double integral of the joint p.d.f. of $T(x)$ and $T(y)$ over the set of outcomes such that $T(x) \le T(y)$ and $T(x) \le n$:

$$\begin{aligned}
{}_nq^{\,1}_{xy} &= \int_0^n \int_s^\infty f_{T(x)T(y)}(s, t)\, dt\, ds \\
&= \int_0^n \int_s^\infty f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[T(y) > s|T(x) = s]\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[T(y) > s|T(x) = s]\, {}_sp_x\, \mu(x + s)\, ds. \qquad (9.9.1)
\end{aligned}$$

For the independent case, $\Pr[T(y) \ge s|T(x) = s] = {}_sp_y$, so

$${}_nq^{\,1}_{xy} = \int_0^n {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds. \qquad (9.9.2)$$

An interpretation of (9.9.2) involves three elements. First, because s is the time of death of (x), the probability ${}_sp_x\, {}_sp_y$ indicates that both (x) and (y) survive to time s. Second, $\mu(x + s)\, ds$ is the probability that (x), now age $x + s$, will die in the interval $(s, s + ds)$. Third, the probabilities are summed for all times s between 0 and n.

Example 9.9.1

Calculate ${}_5q^{\,1}_{xy}$ for the lives in Example 9.2.1.

Solution:

From (9.9.1),

$$\begin{aligned}
{}_5q^{\,1}_{xy} &= \int_0^5 \int_s^{10} 0.0006(t - s)^2\, dt\, ds \\
&= \int_0^5 0.0002(10 - s)^3\, ds = 0.46875.
\end{aligned}$$

▼

We can also evaluate the probability that (y) dies after (x) and before n years from now. This probability is denoted by ${}_nq_{xy}^{\;2}$, the 2 above the y indicating that (y) dies second and n requiring that this occurs within n years. To evaluate ${}_nq_{xy}^{\;2}$, we integrate the joint p.d.f. of $T(x)$ and $T(y)$ over the event $[0 \le T(x) \le T(y) \le n]$:

$$
\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n \int_0^t f_{T(x)T(y)}(s, t)\, ds\, dt \\
&= \int_0^n \int_0^t f_{T(x)|T(y)}(s|t)\, ds\, f_{T(y)}(t)\, dt \\
&= \int_0^n \Pr[T(x) \le t | T(y) = t]\, f_{T(y)}(t)\, dt \\
&= \int_0^n \Pr[T(x) \le t | T(y) = t]\, {}_tp_y\, \mu(y + t)\, dt. \qquad (9.9.3)
\end{aligned}
$$

Again in the independent case, we have

$$
\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n {}_tq_x\, {}_tp_y\, \mu(y + t)\, dt \\
&= {}_nq_y - {}_nq_{xy}^{\;1} \qquad (9.9.4)
\end{aligned}
$$

If the integration of (9.9.3) is set up in the reverse order, we have

$$
\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n \int_s^n f_{T(x)T(y)}(s, t)\, dt\, ds \\
&= \int_0^n \int_s^n f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[s < T(y) \le n | T(x) = s]\, {}_sp_x\, \mu(x + s)\, ds.
\end{aligned}
$$

Making the assumption of independence for $T(x)$ and $T(y)$, we can rewrite this as

$$
\begin{aligned}
{}_nq_{xy}^{\;2} &= \int_0^n ({}_sp_y - {}_np_y)\, {}_sp_x\, \mu(x + s)\, ds \\
&= {}_nq_{xy}^{1} - {}_np_y\, {}_nq_x. \qquad (9.9.5)
\end{aligned}
$$

In (9.9.5) the integrand is interpreted as the probability that (x) dies at time s, with $0 < s < n$, and (y) survives to time s but not to time n. Moreover, we now have that

$$
{}_nq_{xy}^{1} = {}_nq_{xy}^{\;2} + {}_np_y\, {}_nq_x.
$$

This implies

$$
{}_nq_{xy}^{1} \ge {}_nq_{xy}^{\;2}.
$$

Similar integrals can be written for the actuarial present values of contingent insurances, but some do not simplify to the same extent. Consider the actuarial present value of an insurance of 1 payable at the moment of death of (x) provided that (y) is still alive. This actuarial present value, denoted by $\bar{A}_{xy}^{1}$, is E[Z] where

$$Z = \begin{cases} v^{T(x)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

Since Z is a function of $T(x)$ and $T(y)$, the expectation of Z can be obtained by using the joint p.d.f. of $T(x)$ and $T(y)$:

$$\begin{aligned} \bar{A}^{1}_{xy} &= \int_0^{\infty} \int_s^{\infty} v^s f_{T(x)T(y)}(s, t)\, dt\, ds \\ &= \int_0^{\infty} \int_s^{\infty} v^s f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\ &= \int_0^{\infty} \left[\int_s^{\infty} F_{T(y)|T(x)}(t|s)\, dt \right] v^s\, {}_sp_x\, \mu(x + s)\, ds. \end{aligned} \tag{9.9.6}$$

In the case of independent future lifetimes, $T(x)$ and $T(y)$, we can simplify (9.9.6) and express it in IAN as

$$\begin{aligned} \bar{A}^{1}_{xy} &= \int_0^{\infty} \left[\int_s^{\infty} {}_tp_y\, \mu(y + t)\, dt \right] v^s\, {}_sp_x\, \mu(x + s)\, ds \\ &= \int_0^{\infty} v^s\, {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds. \end{aligned} \tag{9.9.7}$$

The final expression can be interpreted as follows: If (x) dies at any future time s and (y) is still surviving, then a payment of 1, with present value v^s, is made. When $\delta = 0$, $\bar{A}^{1}_{xy} = {}_{\infty}q^{1}_{xy}$.

Example 9.9.2

Determine the actuarial present value of a payment of 1,000 at the moment of death of (x) providing that (y) is still alive for (x) and (y) in Example 9.2.3 and on the basis of $\delta = 0.04$.

Solution:

Since $T(x)$ and $T(y)$ are independent in Example 9.2.3, we can use the results of that example in (9.9.7) to have

$$\begin{aligned} 1{,}000\, \bar{A}^{1}_{xy} &= 1{,}000 \int_0^{\infty} e^{-0.04s}\, {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds \\ &= 1{,}000 \int_0^{10} e^{-0.04s}\, 0.01(10 - s)^2\, 0.02(10 - s)\, ds \\ &= 0.2 \int_0^{10} e^{-0.04s}\, (10 - s)^3\, ds = 462.52. \end{aligned}$$

▼

Example 9.9.3

a. Derive the single integral expression for the actuarial present value of an insurance of 1 payable at the time of death of (y) if predeceased by (x).
b. Simplify the integral under the assumption of independent future lifetimes.

c. Obtain a second answer for part (b) by reversing the order of integration in the part (b) double integral.

Solution:

a. The actuarial present value, denoted by $\bar{A}_{xy}^{\;2}$, is E[Z] where

$$Z = \begin{cases} v^{T(y)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

Again, Z is a function of $T(x)$ and $T(y)$, so we write an integral for the expectation of Z by using the joint p.d.f. of $T(x)$ and $T(y)$,

$$\begin{aligned} \bar{A}_{xy}^{\;2} &= \int_0^\infty \int_0^t v^t f_{T(x)T(y)}(s, t)\, ds\, dt \\ &= \int_0^\infty v^t f_{T(x)|T(y)}(s|t)\, ds\, f_{T(y)}(t)\, dt \\ &= \int_0^\infty v^t \Pr[T(x) \le t | T(y) = t]\, f_{T(y)}(t)\, dt. \end{aligned}$$

b. Invoking the independence assumption and writing in IAN we have

$$\begin{aligned} \bar{A}_{xy}^{\;2} &= \int_0^\infty v^t\, {}_tq_x\, {}_tp_y\, \mu(y + t)\, dt \\ &= \int_0^\infty v^t(1 - {}_tp_x)\, {}_tp_y\, \mu(y + t)\, dt \\ &= \bar{A}_y - \bar{A}_{xy}^{1}. \end{aligned}$$

We note for the independent case that we can express the actuarial present value for a simple contingent insurance, payable on a death other than the first death, in terms of the actuarial present values for insurances payable on the first death. This is the initial step in the numerical evaluation of simple contingent insurances for independent lives.

c. We have

$$\bar{A}_{xy}^{\;2} = \int_0^\infty \int_s^\infty v^t\, {}_sp_x\, \mu(x + s)\, {}_tp_y\, \mu(y + t)\, dt\, ds.$$

To simplify we replace t with $r + s$ in the inner integral and rewrite the expression

$$\begin{aligned} \bar{A}_{xy}^{\;2} &= \int_0^\infty \int_0^\infty v^{r+s}\, {}_{r+s}p_y\, \mu(y + r + s)\, {}_sp_x\, \mu(x + s)\, dr\, ds \\ &= \int_0^\infty v^s\, {}_sp_y\, {}_sp_x\, \mu(x + s) \left[\int_0^\infty v^r\, {}_rp_{y+s}\, \mu(y + s + r)\, dr \right] ds \\ &= \int_0^\infty v^s\, \bar{A}_{y+s}\, {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds. \end{aligned}$$

This last integral is an application of the general result given in (2.2.10), $E[W] = E[E[W|V]]$. Here $V = T(x)$, $W = Z$, and we see that the conditional expectation of Z, given $T(x) = s$, is the actuarial present value, $v^s\,{}_sp_y\,\bar{A}_{y+s}$, of the pure endowment for an amount A_{y+s} sufficient to fund a unit insurance on $(y + s)$. ▼

9.10 Evaluation—Simple Contingent Functions

We now turn to the evaluation of simple contingent probabilities and actuarial present values, noting the effects of assuming Gompertz's law, Makeham's law, and a uniform distribution of deaths. The ubiquitous assumption of independence will be made.

Example 9.10.1

Assuming Gompertz's law for the forces of mortality, calculate

a. The actuarial present value for an n-year term contingent insurance paying a unit amount at the moment of death of (x) only if (x) dies before (y)
b. The probability that (x) dies within n years and predeceases (y).

Solution:

a.
$$\bar{A}^{\;1}_{xy:\overline{n}|} = \int_0^n v^t\,{}_tp_{xy}\,\mu(x + t)\,dt.$$

Under Gompertz's law,

$$\begin{aligned}\bar{A}^{\;1}_{xy:\overline{n}|} &= \int_0^n v^t\,{}_tp_{xy}\,Bc^x\,c^t\,dt\\ &= \frac{c^x}{c^x + c^y}\int_0^n v^t\,{}_tp_{xy}\,B(c^x + c^y)c^t\,dt\\ &= \frac{c^x}{c^x + c^y}\int_0^n v^t\,{}_tp_{xy}\,\mu_{xy}(t)\,dt\\ &= \frac{c^x}{c^x + c^y}\,\bar{A}^{\;\;1}_{\overline{xy}:\overline{n}|}.\end{aligned} \tag{9.10.1}$$

Furthermore, if (9.8.2) holds, then

$$\bar{A}^{\;\;1}_{\overline{xy}:\overline{n}|} = \bar{A}^{\;1}_{w:\overline{n}|},$$

and

$$\bar{A}^{\;1}_{xy:\overline{n}|} = \frac{c^x}{c^w}\,\bar{A}^{\;1}_{w:\overline{n}|}. \tag{9.10.2}$$

b. Referring to (9.9.2) we see that ${}_nq^{\;1}_{xy}$ is $\bar{A}^{\;1}_{xy:\overline{n}|}$ with $v = 1$. Thus, it follows from (9.10.2) that, under Gompertz's law,

$${}_nq^{\;1}_{xy} = \frac{c^x}{c^w}\,{}_nq_w, \tag{9.10.3}$$

where $c^w = c^x + c^y$. ▼

Example 9.10.2

Assuming Makeham's law for the forces of mortality, repeat Example 9.10.1.

Solution:

a.
$$\bar{A}^{1}_{xy:\overline{n}|} = \int_0^n v^t\, {}_tp_{xy}\,(A + Bc^x c^t)\,dt$$
$$= A\int_0^n v^t\, {}_tp_{xy}\,dt + \frac{c^x}{c^x + c^y}\int_0^n v^t\, {}_tp_{xy}\,B(c^x + c^y)c^t\,dt$$
$$= A\left(1 - \frac{2c^x}{c^x + c^y}\right)\int_0^n v^t\, {}_tp_{xy}\,dt$$
$$+ \frac{c^x}{c^x + c^y}\int_0^n v^t\, {}_tp_{xy}\,[2A + B(c^x + c^y)c^t]\,dt$$
$$= A\left(1 - \frac{2c^x}{c^x + c^y}\right)\bar{a}_{xy:\overline{n}|} + \frac{c^x}{c^x + c^y}\,\bar{A}\,{}^{\;1}_{\overline{xy}:\overline{n}|}.$$

Then by using (9.8.6), we obtain

$$\bar{A}^{1}_{xy:\overline{n}|} = A\left(1 - \frac{c^x}{c^w}\right)\bar{a}_{ww:\overline{n}|} + \frac{c^x}{2c^w}\,\bar{A}\,{}^{\;1}_{\overline{ww}:\overline{n}|}. \tag{9.10.4}$$

b. Again, we set $v = 1$ in the result of part (a) to have

$${}_nq^{1}_{xy} = A\left(1 - \frac{c^x}{c^w}\right)\mathring{e}_{ww:\overline{n}|} + \frac{c^x}{2c^w}\,{}_nq_{ww}. \tag{9.10.5}$$

▼

The actuarial present value for a contingent insurance payable at the end of the year of death is

$$A^{1}_{xy} = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q^{\;1}_{x+k:y+k}. \tag{9.10.6}$$

Under the assumption of a uniform distribution of deaths for each individual and independence between the pair of future lifetime random variables, we have

$$q^{\;1}_{x+k:y+k} = \int_0^1 {}_sp_{x+k:y+k}\,\mu(x + k + s)\,ds$$
$$= \int_0^1 q_{x+k}(1 - s\,q_{y+k})\,ds$$
$$= q_{x+k}\left(1 - \frac{1}{2}\,q_{y+k}\right). \tag{9.10.7}$$

We can now rewrite ${}_sp_{x+k:y+k}\,\mu(x + k + s)$ in terms of $q^{\;1}_{x+k:y+k}$,

$$
\begin{aligned}
{}_sp_{x+k:y+k}\,\mu(x+k+s) &= q_{x+k}(1 - {}_sq_{y+k}) \\
&= q_{x+k}\left(1 - \frac{1}{2}\,q_{y+k}\right) \\
&\quad + \left(\frac{1}{2} - s\right) q_{x+k}q_{y+k} \\
&= q_{\overset{1}{x+k}:y+k} + \left(\frac{1}{2} - s\right) q_{x+k}q_{y+k}\,. \qquad (9.10.8)
\end{aligned}
$$

When the benefit is payable at the moment of death, the actuarial present value is

$$
\begin{aligned}
\bar{A}^{1}_{xy} &= \sum_{k=0}^{\infty} v^k\,{}_kp_{xy} \int_0^1 v^s\,{}_sp_{x+k:y+k}\,\mu(x+k+s)\,ds \\
&= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy} \left[q_{\overset{1}{x+k}:y+k} \int_0^1 (1+i)^{1-s}\,ds \right. \\
&\quad \left. + q_{x+k}\,q_{y+k} \int_0^1 (1+i)^{1-s} \left(\frac{1}{2} - s\right) ds \right] \\
&= \frac{i}{\delta}\,A^{1}_{xy} + \frac{1}{2}\frac{i}{\delta}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right) \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}\,. \qquad (9.10.9)
\end{aligned}
$$

The second term (9.10.9) is very small relative to the total amount. It is 1/2 of the second term in (9.8.9).

9.11 Notes and References

The concept of the future-lifetime random variable, developed for a single life in previous chapters, has been extended to a survival status, particular cases of which are statuses defined by several lives. Probability distributions, actuarial present values, variances, and covariances based on these new random variables were obtained for statuses defined by two lives by adaptation of the single life theory. These concepts are developed for more than two lives in Chapter 18.

Discussions of the ideas of this chapter without the use of random variables can be found in Chapters 9–13 of Jordan (1967) and Chapters 7–8 of Neill (1977). A general analysis of laws of mortality, which simplify the formulas for actuarial functions based on more than one life, is given by Greville (1956). Exercise 9.36 illustrates Greville's analysis.

Marshall and Olkin (1967, 1988) contributed to the literature on families of bivariate distributions. In particular, they wrote about the common shock model. Frank's family of bivariate distributions is named for M. J. Frank, who developed it. This family is reviewed by Genest (1987).

Frees, Carriere, and Valdez (1996) used Frank's copula to analyze data from last-survivor annuity experience. They assumed that the marginal distributions belong to the Gompertz family. The estimation process yielded an estimate of α of approximately -3.4. Comparing this estimate with the approximate standard error of the estimate leads to the conclusion that $T(x)$ and $T(y)$ were dependent in that experience.

The parameter α is not a conventional measure of association. The value -3.4 is associated with a positive correlation between $T(x)$ and $T(y)$. This might have been expected because in practice lives receiving payments until the last survivor dies live in the same environment.

Frees et al. also found that the assumption of independence between $T(x)$ and $T(y)$ resulted in higher estimated last-survivor annuity actuarial present values than those estimated using a model that permits dependence. The difference was in the range of 3% to 5%.

Exercises

Unless otherwise indicated, all lives in question are subject to the same table of mortality rates, and their times-until-death are independent random variables.

Section 9.2

9.1. The joint p.d.f. of $T(x)$ and $T(y)$ is given by

$$f_{T(x)T(y)}(s, t) = \frac{(n-1)(n-2)}{(1+s+t)^n} \qquad 0 < s,\ 0 < t,\ n > 2.$$

Find:

a. The joint d.f. of $T(x)$ and $T(y)$.
b. The p.d.f., d.f., and $\mu(x+s)$ for the marginal distribution of $T(x)$. Note the symmetry in s and t which implies that $T(x)$ and $T(y)$ are identically distributed.
c. The covariance and correlation coefficients of $T(x)$ and $T(x)$, given that $n > 4$.

9.2. Display the joint survival function of $[T(x), T(y)]$ where the distribution is defined in Exercise 9.1.

9.3. The future lifetime random variables $T(x)$ and $T(y)$ are independent and identically distributed with p.d.f.

$$f(t) = \frac{n-2}{(1+t)^{n-1}} \qquad n > 3,\ t > 0.$$

Determine the joint d.f. and the joint survival function.

Section 9.3

9.4. In terms of the single life probabilities ${}_np_x$ and ${}_np_y$, express
  a. The probability that (xy) will survive n years
  b. The probability that exactly one of the lives (x) and (y) will survive n years
  c. The probability that at least one of the lives (x) and (y) will survive n years
  d. The probability that (xy) will fail within n years
  e. The probability that at least one of the lives will die within n years
  f. The probability that both lives will die within n years.

9.5. Show that the probability that (x) survives n years and (y) survives $n-1$ years may be expressed either as

$$\frac{{}_np_{x:y-1}}{p_{y-1}}$$

or as

$$p_x \; {}_{n-1}p_{x+1:y}.$$

9.6. Evaluate

$$\int_0^n {}_tp_{xx} \, \mu_{xx}(t) \, dt.$$

9.7. Using the distribution of $T(x)$ and $T(y)$ shown in Exercise 9.1 display (a) the d.f., (b) the survival function, and (c) $E[T(xy)]$ for $T(xy)$. Assume $n > 3$.

9.8. Use (9.3.8) to obtain the p.d.f. of $T(xy)$ for the joint distributions of $T(x)$ and $T(y)$ given in Example 9.2.3.

Section 9.4

9.9. Show

$${}_tp_{\overline{xy}} = {}_tp_{xy} + {}_tp_x\,(1 - {}_tp_y) + {}_tp_y\,(1 - {}_tp_x)$$

algebraically and by general reasoning.

9.10. Find the probability that at least one of two lives (x) and (y) will die in the year $(n+1)$. Is this the same as ${}_{n|}q_{\overline{xy}}$? Explain.

9.11. The random variables $T(x)$ and $T(y)$ have the joint p.d.f. displayed in Exercise 9.1. Find (a) the d.f. and the p.d.f. of $T(\overline{xy})$, (b) $E[T(\overline{xy})]$, and (c) $\mu_{\overline{xy}}(t)$.

Section 9.5

9.12. Given that ${}_{25}p_{25:50} = 0.2$ and ${}_{15}p_{25} = 0.9$, calculate the probability that a person age 40 will survive to age 75.

9.13. If $\mu(x) = 1/(100 - x)$ for $0 \le x < 100$, calculate

a. ${}_{10}p_{40:50}$

b. ${}_{10}p_{\overline{40:50}}$

c. $e_{40:50}$

d. $e_{\overline{40:50}}$

e. $\mathrm{Var}[T(40:50)]$

f. $\mathrm{Var}[T(\overline{40:50})]$

g. $\mathrm{Cov}[T(40:50), T(\overline{40:50})]$

h. The correlation between $T(40:50)$ and $T(\overline{40:50})$.

9.14. Evaluate $\dfrac{d\mathring{e}_{xx}}{dx}$.

9.15. Show that the probability of two lives (30) and (40) dying in the same year can be expressed as

$$1 + e_{30:40} - p_{30}(1 + e_{31:40}) - p_{40}(1 + e_{30:41}) + p_{30:40}(1 + e_{31:41}).$$

9.16. Show that the probability of two lives (30) and (40) dying at the same age last birthday can be expressed as

$${}_{10}p_{30}(1 + e_{40:40}) - 2\ {}_{11}p_{30}\ (1 + e_{40:41}) + p_{40}\ {}_{11}p_{30}\ (1 + e_{41:41}).$$

9.17. Assume that the forces of mortality that apply to individuals I and II, respectively, are

$$\mu^{\mathrm{I}}(x) = \log \frac{10}{9} \quad \text{for all } x$$

and

$$\mu^{\mathrm{II}}(x) = (10 - x)^{-1} \quad \text{for } 0 \le x < 10.$$

Evaluate the probability that, if both individuals are of exact age 1, the first death will occur between exact ages 3 and 5.

Section 9.6

9.18. This is a continuation of Example 9.6.3. Exhibit (a) the d.f. and (b) the p.d.f. of $T(\overline{xy})$.

9.19. If ${}_5q_x = 0.05$ and ${}_5q_y = 0.03$, calculate the corresponding value of $F_{T(x)T(y)}(5, 5)$ using (9.6.7). Your answer will depend on the parameter α.

9.20. Use the result of Exercise 9.19 to evaluate ${}_5q_{\overline{xy}}$ for (a) $\alpha = 0$, (b) $\alpha = 3$, and (c) $\alpha = -3$. [Hint: Recall (9.4.5) and (9.3.2).]

Section 9.7

9.21. Show that

$$a_{\overline{\overline{xy}:\overline{n|}}} = a_{\overline{n|}} + {}_{n|}a_{xy}.$$

Describe the underlying benefit.

9.22. For an actuarial present value denoted by $\bar{A}_{\overline{x:\overline{n|}}}$, describe the benefit. Show that

$$\bar{A}_{\overline{x:\overline{n|}}} = \bar{A}_x - \bar{A}_{x:\overline{n|}} + v^n.$$

9.23. For independent lifetimes $T(x)$ and $T(y)$, show that

$$\mathrm{Cov}(v^{T(\overline{xy})}, v^{T(xy)}) = (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy}).$$

9.24. Express, in terms of single- and joint-life annuity values, the actuarial present value of an annuity payable continuously at a rate of 1 per year while at least one of (25) and (30) survives and is below age 50.

9.25. Express, in terms of single- and joint-life annuity values, the actuarial present value of a deferred annuity of 1 payable at the end of any year as long as either (25) or (30) is living after age 50.

9.26. Express, in terms of single- and joint-life annuity values, the actuarial present value of an n-year temporary annuity-due, payable in respect to $(\overline{xy})$, providing annual payments of 1 while both lives survive, reducing to 1/2 on the death of (x) and to 1/3 on the death of (y).

9.27. An annuity-immediate of 1 is payable to (x) as long as he lives jointly with (y) and for n years after the death of (y), except that in no event will payments be made after m years from the present time, $m > n$. Show that the actuarial present value is

$$a_{x:\overline{n|}} + {}_nE_x\, a_{x+n:y:\overline{m-n|}}.$$

9.28. Obtain an expression for the actuarial present value of a continuous annuity of 1 per annum payable while at least one of two lives (40) and (55) is living and is over age 60, but not if (40) is alive and under age 55.

9.29. A joint-and-survivor annuity to (x) and (y) is payable at an initial rate per year while (x) lives, and, if (y) survives (x), is continued at the fraction p, $1/2 \le p \le 1$, of the initial rate per year during the lifetime of (y) following the death of (x).

a. Express the actuarial present value of such an annuity-due with an initial rate of 1 per year, payable in m-thly installments, in terms of the actuarial present values of single-life and joint-life annuities.

b. A joint-and-survivor annuity to (x) and (y) and a life annuity to (x) are said to be actuarially equivalent on the basis of stated assumptions if they have equal actuarial present values on such basis. Derive an expression for the ratio of the initial payment of the joint-and-survivor annuity to the payment rate of the actuarially equivalent life annuity to (x).

9.30. Show that

a. $\bar{A}^{\,2}_{xy} = \bar{A}^{\,1}_{xy} - \delta \bar{a}_{y|x}$

b. $\dfrac{\partial}{\partial x}\, \bar{a}_{y|x} = \mu(x)\, \bar{a}_{y|x} - \bar{A}^{\,2}_{xy}.$

9.31. When, under Makeham's law, the status (xy) is replaced by the status (ww), show that

$$w - y = \frac{\log(c^{\Delta} + 1) - \log 2}{\log c}$$

where $\Delta = x - y \geq 0$. (This indicates that w can be obtained from the younger age y by adding an amount that is a function of $\Delta = x - y$. Such a property is referred to as a ***law of uniform seniority***.)

9.32. On the basis of your Illustrative Life Table with interest of 6% calculate $\ddot{a}_{50:60:\overline{10}|}$. In your solution, use

a. Values interpolated in the $\ddot{a}_{xx}$ table
b. Values from the $\ddot{a}_{x:x+10}$ table.

9.33. Given a mortality table that follows Makeham's law and ages x and y for which (ww) is the equivalent equal-age status, show that

a. ${}_tp_w$ is the geometric mean of ${}_tp_x$ and ${}_tp_y$
b. ${}_tp_x + {}_tp_y > 2 {}_tp_w$ for $x \neq y$
c. $a_{\overline{xy}} > a_{\overline{ww}}$ for $x \neq y$.

9.34. Given a mortality table that follows Makeham's law, show that $\bar{a}_{xy}$ is equal to the actuarial present value of an annuity with a single life (w) where $c^w = c^x + c^y$ and force of interest $\delta' = \delta + A$. Further, show that

$$\bar{A}_{xy} = \bar{A}'_w + A\bar{a}'_w$$

where the primed functions are evaluated at force of interest δ'.

9.35. Consider two mortality tables, one for males, M, and one for females, F, with

$$\mu^M(z) = 3a + \frac{3bz}{2} \quad \text{and} \quad \mu^F(z) = a + bz.$$

We wish to use a table of actuarial present values for two lives, one male and one female, each of age w, to evaluate the actuarial present value of a joint-life annuity for a male age x and a female age y. Express w in terms of x and y.

9.36. From Section 9.5 we know that if $T(x)$ and $T(y)$ are independent,

$$\bar{a}_{xy} = \int_0^{\infty} e^{-\int_0^t [\delta + \mu(x+s) + \mu(y+s)]ds}\, dt.$$

If we could find δ', k, and w such that

$$\delta + \mu(x + t) + \mu(y + t) = \delta' + k\mu(w + t) \qquad (*)$$

we would have

$$\bar{a}_{xy} = \int_0^\infty v^t ({}_tp_w)^k \, dt$$
$$= \bar{a}'_{w(k)},$$

where the prime on the discount factor indicates that it is valued at force of interest δ' and $w(k)$ indicates a joint-life status with k "lives" (k is not necessarily an integer).

If $\mu(x + t) = a + b(x + t) + c(x + t)^2$, confirm that (*) is satisfied if

$$k = 2,$$
$$w = \frac{x + y}{2},$$
$$\delta' = \delta + c(x^2 + y^2 - 2w^2).$$

9.37. Find $\mathring{e}_{xy}$ if $q_x = q_y = 1$ and the deaths are uniformly distributed over the year of age for each of (x) and (y).

9.38. Let $T(x)$ and $T(y)$ be independent and uniformly distributed in the next year of age. Given that both (x) and (y) die within the next year, demonstrate that the time-of-failure of (xy) is not uniformly distributed over the year. [Hint: Show that $\Pr[T(xy) \le t | (T(x) \le 1) \cap (T(y) \le 1)] = 2t - t^2$.]

9.39. Show

$$\frac{1}{\delta} = \frac{1}{i[1 - (i/2 - i^2/3 + i^2/4 - i^4/5 + \cdots)]}$$
$$= \frac{1}{i}\left(1 + \frac{i}{2} - \frac{i^2}{12} + \frac{i^3}{24} - \frac{19i^4}{720} + \cdots\right).$$

Hence, show

$$\frac{i}{\delta}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right) \cong \frac{i}{6} - \frac{i^3}{360} + \cdots.$$

9.40. Show that if deaths are uniformly distributed over each year of age, then

$${}_{(j-1)/m}p_{xy} - {}_{j/m}p_{xy} = \frac{1}{m} q_{xy} + \frac{m + 1 - 2j}{m^2} q_x \, q_y$$

for any x and y and $j = 1, 2, 3, \ldots, m$. Hence, verify (9.8.14).

Section 9.9

9.41. Show by general reasoning that

$${}_nq^1_{xy} = {}_nq^{\;2}_{xy} + {}_nq_x \, {}_np_y.$$

When $n \to \infty$, what does the equation become?

9.42. Show that the actuarial present value for an insurance of 1 payable at the end of the year of death of (x), provided that (y) survives to the time of payment, can be expressed as $vp_y\ddot{a}_{x:y+1} - a_{xy}$.

9.43. Show that $A^{1}_{xy} - A^{\;2}_{xy} = A_{xy} - A_y$.

9.44. Express, in terms of actuarial present values for single life and first death contingent insurances, the net single premium for an insurance of 1 payable at the moment of death of (50), provided that (20), at that time, has died or attained age 40.

9.45. Express, in terms of actuarial present values for pure endowment and first death contingent insurances, the actuarial present value for an insurance of 1 payable at the time of the death of (x) after (y), providing (y) died during the n years preceding the death of (x).

9.46. If $\mu(x) = 1/(100 - x)$ for $0 \le x < 100$, calculate ${}_{25}q_{25:\overset{2}{50}}$.

Section 9.10

9.47. In a mortality table known to follow Makeham's law, you are given that $A = 0.003$ and $c^{10} = 3$.

a. If $\mathring{e}_{40:50} = 17$, calculate ${}_{\infty}q_{\overset{1}{40}:50}$.

b. Express $\bar{A}_{\overset{1}{40}:50}$ in terms of $\bar{A}_{40:50}$ and $\bar{a}_{40:50}$.

9.48. Given that mortality follows Gompertz's law with $\mu(x) = 10^{-4} \times 2^{x/8}$ for $x > 35$ and that by (9.10.12)

$$\bar{A}_{\overset{1}{40}:48:\overline{10}|} = f\bar{A}^{1}_{w:\overline{10}|},$$

calculate f and w.

Miscellaneous

9.49. The survival status $(\overline{n}|)$ is one that exists for exactly n years. It has been used in conjunction with life statuses, for example, in $\bar{A}_{x:\overline{n}|}$, $\bar{A}^{1}_{x:\overline{n}|}$, $\bar{A}_{x:\overset{1}{\overline{n}|}}$, $\ddot{a}_{x:\overline{n}|}$, $A_{xy:\overline{n}|}$. Simplify and interpret the following:

a. $\bar{a}_{\overline{x:\overline{n}|}}$

b. $\bar{A}_{x:\overset{2}{\overline{n}|}}$.

9.50. Use the probability rule $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ to obtain (9.4.6).

9.51. Evaluate $\dfrac{\partial}{\partial x}\mathring{e}_{xy}$.

9.52. The random variables $T^*(x)$, $T^*(y)$, and Z are independent and have exponential distributions with, respectively, parameters μ_1, μ_2, and λ. These three random variable are components of a common shock model.

a. Exhibit the marginal p.d.f. of $T(y)$ by evaluating

$$f_{T(y)}(t) = \int_0^t f_{T(x)T(y)}(s, t)\, ds + f_{T(x)T(y)}(t, t) + \int_t^{\infty} f_{T(x)T(y)}(s, t)\, ds.$$

b. Exhibit the marginal survival function of $T(y)$ by evaluating

$$s_{T(y)}(t) = \int_t^{\infty} f_{T(y)}(u)\, du.$$

Compare the result with (9.6.4b).

c. Evaluate

$$\Pr[T(x) = T(y)] = \int_0^{\infty} f_{T(x)T(y)}(t, t)\, dt.$$

10

MULTIPLE DECREMENT MODELS

10.1 Introduction

In Chapter 9 we extended the theory of Chapters 3 through 8 from an individual life to multiple lives, subject to a single contingency of death. We now return to the case of a single life, but here subject to multiple contingencies. As an application of this extension, we observe that the number of workers for an employer is reduced when an employee withdraws, becomes disabled, dies, or retires. In manpower planning, it might be necessary to estimate only the numbers of those presently at work who will remain active to various years into the future. For this task, the model for survivorship developed in Chapter 3 is adequate, with time-until-termination of employment rather than time-until-death as the interpretation of the basic random variable. Employee benefit plans, however, provide benefits paid on termination of employment that may depend on the cause of termination. For example, the benefits on retirement often differ from those payable on death or disability. Therefore, survivorship models for employee benefit systems include random variables for both time-of-termination and cause of termination. Also, the benefit structure often depends on earnings, which is another and different kind of uncertainty that is discussed in Chapter 11.

As another application, most individual life insurances provide payment of a nonforfeiture benefit if premiums stop before the end of the specified premium payment term. A comprehensive model for such insurances incorporates both time-until-termination and cause of termination as random variables.

Disability income insurance provides periodic payments to insureds who satisfy the definition of disability contained in the policy. In some cases, the amount of the periodic payments may depend on whether the disability was caused by illness or accident. A person may cease to be an active insured by dying, withdrawing, becoming disabled, or reaching the end of the coverage period. A complete model for disability insurance incorporates a random variable for time-until-termination, when the insured ceases to be a member of the active insureds, as well as a random variable for the cause of termination.

Public health planners are interested in the analysis of mortality and survivorship by cause of death. Public health goals may be set by a study of the joint distribution of time-until-death and cause of death. Priorities in cardiovascular and cancer research were established by this type of analysis.

The main purpose of this chapter is to build a theory for studying the distribution of two random variables in regard to a single life: time-until-termination from a given status and cause of the termination. The resulting model is used in each of the applications described in this section. Within actuarial science, the termination from a given status is called *decrement*, and the subject of this chapter is called ***multiple decrement theory.*** Within biostatistics it is referred to as the ***theory of competing risks.***

It is also possible to develop multiple decrement theory in terms of deterministic rates and rate functions. There is some recapitulation of the theory from this point of view in Section 10.4.

10.2 Two Random Variables

Chapter 3 was devoted in part to methods for specifying and using the distribution of the continuous random variable $T(x)$, the time-until-death of (x). The same methods can be used to study time-until-termination from a status, such as employment with a particular employer, with only minor changes in vocabulary. In fact, we use the same notation $T(x)$, or T, to denote the time random variable in this new setting.

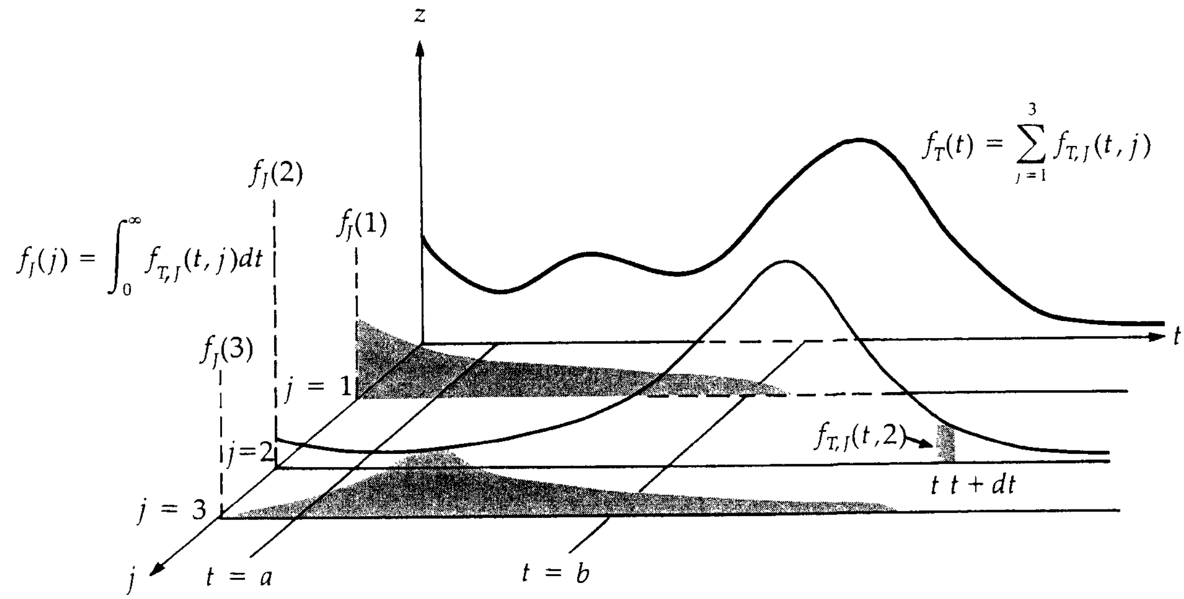
In this section we expand the basic model by introducing a second random variable, cause of decrement, to be denoted by $J(x) = J$. We assume that J is a discrete random variable.

The applications in Section 10.1 provide examples of these random variables. For employee benefit plan applications, the random variable J could be assigned the values 1, 2, 3, or 4 depending on whether termination is due to withdrawal, disability, death, or retirement, respectively. In the life insurance application, J could be assigned the values 1 or 2, depending on whether the insured dies or chooses to terminate payment of premiums. For the disability insurance application, J could be assigned the values 1, 2, 3, or 4 depending on whether the insured dies, withdraws, becomes disabled, or reaches the end of the coverage period. Finally, in the public health application, there are many possibilities for causes of decrement. For example, in a given study, J could be assigned the values 1, 2, 3, or 4 depending on whether death was caused by cardiovascular disease, cancer, accident, or all other causes.

Our purpose is to describe the joint distribution of T and J and the related marginal and conditional distributions. We denote the joint p.d.f. of T and J by $f_{T,J}(t, j)$, the marginal p.f. of J by $f_J(j)$, and the marginal p.d.f. of T by $f_T(t)$. Figure

10.2.1 illustrates these distributions. They may seem strange at first because J is a discrete random variable and T is continuous.

Graph of $f_{T,J}(t, j)$



The joint p.d.f. of T and J, $f_{T,J}(t, j)$, can be pictured as falling on m parallel sheets, as illustrated in Figure 10.2.1 for three causes of decrement ($m = 3$). There is a separate sheet for each of the m causes of decrement recognized in the model. In Figure 10.2.1 the following relations hold:

$$\sum_{j=1}^{3} f_J(j) = 1$$

and

$$\int_0^\infty f_T(t)\, dt = 1.$$

The p.d.f. $f_{T,J}(t, j)$ can be used in the usual ways to calculate the probabilities of events defined by T and J. For example,

$$f_{T,J}(t, j)\, dt = \Pr\{(t < T \le t + dt) \cap (J = j)\} \tag{10.2.1}$$

expresses the probability of decrement due to cause j between t and $t + dt$,

$$\int_0^t f_{T,J}(s, j)\, ds = \Pr\{(0 < T \le t) \cap (J = j)\} \tag{10.2.2}$$

expresses the probability of decrement due to cause j before time t, and

$$\sum_{j=1}^{m} \int_a^b f_{T,J}(t, j)\, dt = \Pr\{a < T \le b\}$$

expresses the probability of decrement due to all causes between a and b.

The probability of decrement before time t due to cause j given in (10.2.2) has the special symbol

$$_tq_x^{(j)} = \int_0^t f_{T,J}(s, j)\, ds \qquad t \ge 0,\ j = 1, 2, \ldots, m, \tag{10.2.3}$$

which is illustrative of the use of the superscript to denote the cause of decrement in multiple decrement theory.

The use of information given at age x to select a distribution is similar to the concepts in Chapter 3. If being in the survival status at age x is the only information available, then an aggregate distribution would be used. On the other hand, if there is additional information, then the distribution would be a select distribution and the age of selection would be enclosed in brackets.

By the definition of the marginal distribution for J, appearing as $f_J(j)$ in the (j, z) plane of Figure 10.2.1, we have the probability of decrement due to cause j at any time in the future to be

$$f_J(j) = \int_0^\infty f_{T,J}(s, j)\, ds = {}_\infty q_x^{(j)} \qquad j = 1, 2, \ldots, m. \tag{10.2.4}$$

This is new and without a counterpart in Chapter 3, unlike the marginal p.d.f. for T, $f_T(t)$ in the (t, z) plane of Figure 10.2.1. For $f_T(t)$, and the d.f., $F_T(t)$, we have for $t \ge 0$

$$f_T(t) = \sum_{j=1}^{m} f_{T,J}(t, j) \tag{10.2.5}$$

and

$$F_T(t) = \int_0^t f_T(s)\, ds.$$

The notations introduced in Chapter 3 for the functions of the distribution of the future-lifetime random variable, T, can be extended to accommodate those of the time-until-decrement random variable of the multiple decrement model. Using the superscript (τ) to indicate that a function refers to all causes, or total force, of decrement, we obtain

$$_tq_x^{(\tau)} = \Pr\{T \le t\} = F_T(t) = \int_0^t f_T(s)\, ds, \tag{10.2.6}$$

$$_tp_x^{(\tau)} = \Pr\{T > t\} = 1 - {}_tq_x^{(\tau)}, \tag{10.2.7}$$

$$\mu_x^{(\tau)}(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tq_x^{(\tau)}$$

$$= -\frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tp_x^{(\tau)}$$

$$= -\frac{d}{dt} \log {}_tp_x^{(\tau)}, \tag{10.2.8}$$

and

$$ {}_tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)\, ds}. \tag{10.2.9}$$

Mathematically, these functions for the random variable T of this chapter are identical to those for the T of Chapter 3; the difference is in their interpretation in the applications. The choice of the symbol $\mu_x^{(\tau)}(t)$ for the force of total decrement is influenced by these applications. For example, in pension applications (x) is an age of entry into the pension plan, and although no special selection information may be available, subsequent causes of decrement may depend on this age.

As with the applications in previous chapters, the statement in (10.2.1) can be analyzed by conditioning on survival in the given status to time t. In this way, we have

$$f_{T,J}(t, j)\, dt = \Pr\{T > t\} \Pr\{[(t < T \le t + dt) \cap (J = j)] | T > t\}. \tag{10.2.10}$$

By analogy with (3.2.12) this suggests the definition of the ***force of decrement due to cause j*** as

$$\mu_x^{(j)}(t) = \frac{f_{T,J}(t, j)}{1 - F_T(t)} = \frac{f_{T,J}(t, j)}{{}_tp_x^{(\tau)}}. \tag{10.2.11}$$

The force of decrement at age $x + t$ due to cause j has a conditional probability interpretation. It is the value of the joint conditional p.d.f. of T and J at $x + t$ and j, given survival to $x + t$. Then (10.2.10) can be rewritten as

$$f_{T,J}(t, j)\, dt = {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt \qquad j = 1, 2, \ldots, m,\ t \ge 0. \qquad \text{(10.2.10) restated}$$

Restated in words,

(the probability of decrement between t and $t + dt$ due to cause j) = (the probability, ${}_tp_x^{(\tau)}$, that (x) remains in the given status until time t)

× (the conditional probability, $\mu_x^{(j)}(t)$ that decrement occurs between t and $t + dt$ due to cause j, given that decrement has not occurred before time t).

It follows, from differentiation of (10.2.3) and use of (10.2.11), that

$$\mu_x^{(j)}(t) = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tq_x^{(j)}. \tag{10.2.12}$$

Now, from (10.2.6), (10.2.5), and (10.2.3),

$$
\begin{aligned}
{}_tq_x^{(\tau)} &= \int_0^t f_T(s)\, ds = \int_0^t \sum_{j=1}^m f_{T,J}(s, j)\, ds \\
&= \sum_{j=1}^m \int_0^t f_{T,J}(s, j)\, ds = \sum_{j=1}^m {}_tq_x^{(j)}. \qquad (10.2.13)
\end{aligned}
$$

That the first and last members of (10.2.13) are equal is immediately interpretable. Combining (10.2.8), (10.2.13), and (10.2.12), we have

$$
\mu_x^{(\tau)}(t) = \sum_{j=1}^m \mu_x^{(j)}(t); \qquad (10.2.14)
$$

that is, the total force of decrement is the sum of the forces of decrement due to the m causes.

We can summarize the definitions here by expressing the joint, marginal, and conditional p.d.f.'s in actuarial notation and repeating the defining equation numbers:

$$
\begin{aligned}
f_{T,J}(t, j) &= {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t), && \text{(10.2.11) restated} \\
f_J(j) &= {}_\infty q_x^{(j)}, && \text{(10.2.4) restated} \\
f_T(t) &= {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t). && \text{(10.2.8) restated}
\end{aligned}
$$

The conditional p.f. of J, given decrement at time t, is

$$
\begin{aligned}
f_{J|T}(j|t) &= \frac{f_{T,J}(t, j)}{f_T(t)} = \frac{{}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)}{{}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)} \\
&= \frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)}. \qquad (10.2.15)
\end{aligned}
$$

Finally, we note that the probability in (10.2.3) can be rewritten as

$$
{}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\, \mu_x^{(j)}(s)\, ds. \qquad (10.2.16)
$$

Example 10.2.1

Consider a multiple decrement model with two causes of decrement; the forces of decrement are given by

$$
\mu_x^{(1)}(t) = \frac{t}{100} \qquad t \ge 0,
$$

$$
\mu_x^{(2)}(t) = \frac{1}{100} \qquad t \ge 0.
$$

For this model, calculate the p.f. (or p.d.f.) for the joint, marginal, and conditional distributions.

Solution:

Since

$$\mu_x^{(\tau)}(s) = \mu_x^{(1)}(s) + \mu_x^{(2)}(s) = \frac{s+1}{100},$$

the survival probability ${}_tp_x^{(\tau)}$ is

$$\begin{aligned} {}_tp_x^{(\tau)} &= \exp\left(-\int_0^t \frac{s+1}{100}\,ds\right) \\ &= \exp\left(\frac{-(t^2+2t)}{200}\right) \qquad t \ge 0, \end{aligned}$$

and the joint p.d.f. of T and J is

$$f_{T,J}(t, j) = \begin{cases} \dfrac{t}{100}\exp\left[\dfrac{-(t^2+2t)}{200}\right] & t \ge 0,\ j = 1 \\ \dfrac{1}{100}\exp\left[\dfrac{-(t^2+2t)}{200}\right] & t \ge 0,\ j = 2. \end{cases}$$

The marginal p.d.f. of T is

$$f_T(t) = \sum_{j=1}^{2} f_{T,J}(t, j) = \frac{t+1}{100}\exp\left[\frac{-(t^2+2t)}{200}\right] \qquad t \ge 0,$$

and the marginal p.f. of J is

$$f_J(j) = \begin{cases} \displaystyle\int_0^\infty f_{T,J}(t, 1)\,dt & j = 1 \\ \displaystyle\int_0^\infty f_{T,J}(t, 2)\,dt & j = 2. \end{cases}$$

It is somewhat easier to evaluate $f_J(2)$. In the following development, $\Phi(x)$ is the d.f. for the standard normal distribution $N(0, 1)$. By completing the square we have

$$\begin{aligned} f_J(2) &= \frac{1}{100}\,e^{0.005}\int_0^\infty \exp\left[\frac{-(t+1)^2}{200}\right] dt \\ &= \frac{1}{100}\,e^{0.005}\sqrt{2\pi}\,10\int_0^\infty \frac{1}{\sqrt{2\pi}\,10}\exp\left[\frac{-(t+1)^2}{200}\right] dt. \end{aligned}$$

We now make the change of variable $z = (t+1)/10$ and obtain

$$\begin{aligned} f_J(2) &= \frac{1}{10}\,e^{0.005}\sqrt{2\pi}\int_{0.1}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2}{2}\right) dz \\ &= \frac{1}{10}\,e^{0.005}\sqrt{2\pi}\,[1 - \Phi(0.1)] \\ &= 0.1159. \end{aligned}$$

Therefore $f_J(1) = 0.8841$. Finally, the conditional p.f. of J, given decrement at t, is derived from (10.2.15) as

$$f_{J|T}(1|t) = \frac{t}{t+1}$$

and

$$f_{J|T}(2|t) = \frac{1}{t+1}.$$

▼

Example 10.2.2

For the joint distribution of T and J specified in Example 10.2.1, calculate $\mathrm{E}[T]$ and $\mathrm{E}[T|J = 2]$.

Solution:

Using the marginal p.d.f. $f_T(t)$, we have

$$\mathrm{E}[T] = \int_0^\infty t \left\{ \frac{t+1}{100} \exp\left[\frac{-(t^2+2t)}{200}\right] \right\} dt.$$

Integration by parts, as in (3.5.1), yields

$$\begin{aligned} \mathrm{E}[T] &= -t \exp\left[\frac{-(t^2+2t)}{200}\right]\Bigg|_0^\infty \\ &\quad + \int_0^\infty \exp\left[\frac{-(t^2+2t)}{200}\right] dt \\ &= 0 + 100 f_J(2), \end{aligned}$$

hence

$$\mathrm{E}[T] = 11.59.$$

Using the conditional p.d.f. $f_{T,J}(t, 2)/f_J(2)$, we have

$$\mathrm{E}[T|J = 2] = \int_0^\infty t \left\{ 100^{-1} \exp\left[\frac{-(t^2+2t)}{200}\right] \right\} (0.1159)^{-1}\, dt.$$

This integral may be evaluated as follows:

$$\begin{aligned} \mathrm{E}[T|J = 2] &= \mathrm{E}[(T+1) - 1|J = 2] \\ &= (0.1159)^{-1} \int_0^\infty \frac{t+1}{100} \exp\left[\frac{-(t^2+2t)}{200}\right] dt - 1 \\ &= -(0.1159)^{-1} \exp\left[\frac{-(t^2+2t)}{200}\right]\Bigg|_0^\infty - 1 \\ &= 7.63. \end{aligned}$$

The point of Examples 10.2.1 and 10.2.2 is that once the joint distribution of T and J is specified, marginal and conditional distributions can be derived, and the moments of these distributions determined. ▼

In some instances, a particular application may require a modification of the above model. A continuous distribution for time-until-termination, T, is inadequate in applications where there is a time at which there is a positive probability of decrement. One example of this is a pension plan with a mandatory retirement age, an age at which all remaining active employees must retire. A second example is term life insurance in which there is typically no benefit paid on withdrawal. Thus, after a premium is paid, none of the remaining insureds withdraw until the next premium due date. Here we do not attempt to extend the notation to cover such situations. However, in Section 10.7 we describe extended models for each of these examples.

The random variable K, the curtate-future-years before decrement of (x), is defined as in Chapter 3 to be the greatest integer less than T. Using (10.2.1) and (10.2.11), we can write the joint p.f. K and J as

$$\begin{aligned}\Pr\{(K = k) \cap (J = j)\} &= \Pr\{(k < T \le k + 1) \cap (J = j)\} \\ &= \int_k^{k+1} {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt. \qquad (10.2.17)\end{aligned}$$

Rewriting ${}_tp_x^{(\tau)}$ of the integrand in the exponential form of (10.2.9) and factoring it into two factors changes (10.2.17) to

$$= {}_kp_x^{(\tau)} \int_k^{k+1} e^{-\int_k^t \mu_x^{(\tau)}(u)\, du}\, \mu_x^{(j)}(t)\, dt.$$

Changing the variables of the integrations by $r = u - k$ and $s = t - k$ yields

$$= {}_kp_x^{(\tau)} \int_0^1 e^{-\int_0^s \mu_x^{(\tau)}(k+r)\, dr}\, \mu_x^{(j)}(k + s)\, ds.$$

Thus far we have done manipulations that hold in all tables. If we are using an aggregate or ultimate (a nonselect) table where the forces of decrement depend on an initial age and the duration only through their sum, that is, the ***attained age***, then for τ and all j,

$$\mu_x(k + s) = \mu_{x+k}(s) \qquad \text{for all } x, k, \text{ and } s \ge 0,$$

and (10.2.17) may be written

$${}_kp_x^{(\tau)} \int_0^1 {}_sp_{x+k}^{(\tau)}\, \mu_{x+k}^{(j)}(s)\, ds = {}_kp_x^{(\tau)}\, q_{x+k}^{(j)}. \qquad (10.2.18)$$

The probability of decrement from all causes between ages $x + k$ and $x + k + 1$, given survival to age $x + k$, is denoted by $q_{x+k}^{(\tau)}$, and it follows that

$$\begin{aligned}q_{x+k}^{(\tau)} &= \int_0^1 {}_sp_{x+k}^{(\tau)}\, \mu_{x+k}^{(\tau)}(s)\, ds \\ &= \int_0^1 {}_sp_{x+k}^{(\tau)} \sum_{j=1}^{m} \mu_{x+k}^{(j)}(s)\, ds \\ &= \sum_{j=1}^{m} q_{x+k}^{(j)}. \qquad (10.2.19)\end{aligned}$$

An examination of (10.2.18) and (10.2.19) discloses why multiple decrement theory is also called the theory of competing risks. The probability of decrement between ages $x + k$ and $x + k + 1$ due to cause j depends on ${}_sp_{x+k}^{(\tau)}$, $0 \le s \le 1$, and thus on all the component forces. When the forces for other decrements are increased, ${}_sp_{x+k}^{(\tau)}$ is reduced, and then $q_{x+k}^{(j)}$ is also decreased.

10.3 Random Survivorship Group

Let us consider a group of $l_a^{(\tau)}$ lives age a years. Each life is assumed to have a distribution of time-until-decrement and cause of decrement specified by the p.d.f.

$$f_{T,J}(t, j) = {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t) \qquad t \ge 0,\ j = 1, 2, \ldots, m.$$

We denote by ${}_n\mathcal{D}_x^{(j)}$ the random variable equal to the number of lives who leave the group between ages x and $x + n$, $x \ge a$, from cause j. We denote $\mathrm{E}[{}_n\mathcal{D}_x^{(j)}]$ by ${}_nd_x^{(j)}$ and obtain

$$\begin{aligned} {}_nd_x^{(j)} &= \mathrm{E}[{}_n\mathcal{D}_x^{(j)}] \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t)\, dt. \end{aligned} \tag{10.3.1a}$$

As usual, if $n = 1$, we delete the prefixes on ${}_n\mathcal{D}_x^{(j)}$ and ${}_nd_x^{(j)}$. We note that

$${}_n\mathcal{D}_x^{(\tau)} = \sum_{j=1}^{m} {}_n\mathcal{D}_x^{(j)}$$

and define

$${}_nd_x^{(\tau)} = \mathrm{E}[{}_n\mathcal{D}_x^{(\tau)}] = \sum_{j=1}^{m} {}_nd_x^{(j)}. \tag{10.3.1b}$$

Then, using (10.3.1a), we have

$$\begin{aligned} {}_nd_x^{(\tau)} &= l_a^{(\tau)} \sum_{j=1}^{m} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t)\, dt \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(\tau)}(t)\, dt. \end{aligned} \tag{10.3.2}$$

If $\mathcal{L}^{(\tau)}(x)$ is defined as the random variable equal to the number of survivors at age x out of the $l_a^{(\tau)}$ lives in the original group at age a, then by analogy with (3.3.1) we can write

$$\begin{aligned} l_x^{(\tau)} &= \mathrm{E}[\mathcal{L}^{(\tau)}(x)] \\ &= l_a^{(\tau)}\, {}_{x-a}p_a^{(\tau)}. \end{aligned} \tag{10.3.3}$$

We recognize the integral of (10.3.1a) with $n = 1$ as the integral of (10.2.17) with $x = a$ and $k = x - a$. Thus for a nonselect table, we have from (10.2.18)

$$d_x^{(j)} = l_a^{(\tau)}\, {}_{x-a}p_a^{(\tau)}\, q_x^{(j)} = l_x^{(\tau)}\, q_x^{(j)}. \tag{10.3.4}$$

This result lets us display a table of $p_x^{(\tau)}$ and $q_x^{(j)}$ values in a corresponding table of $l_x^{(\tau)}$ and $d_x^{(j)}$ values. Either table is called a ***multiple decrement table.***

Example 10.3.1

Construct a table of $l_x^{(\tau)}$ and $d_x^{(j)}$ values corresponding to the probabilities of decrement given below.

x	$q_x^{(1)}$	$q_x^{(2)}$
65	0.02	0.05
66	0.03	0.06
67	0.04	0.07
68	0.05	0.08
69	0.06	0.09
70	0.00	1.00

Although this display is designed for computational ease, it may be roughly suggestive of a double decrement situation with cause 1 related to death and cause 2 to retirement. It appears that, in this case, 70 is the mandatory retirement age.

Solution:

We assume the arbitrary value of $l_{65}^{(\tau)} = 1{,}000$ and use (10.3.4) as indicated below.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$	$l_x^{(\tau)} = l_{x-1}^{(\tau)}\, p_{x-1}^{(\tau)}$	$d_x^{(1)} = l_x^{(\tau)}\, q_x^{(1)}$	$d_x^{(2)} = l_x^{(\tau)}\, q_x^{(2)}$
65	0.02	0.05	0.07	0.93	1 000.00	20.00	50.00
66	0.03	0.06	0.09	0.91	930.00	27.90	55.80
67	0.04	0.07	0.11	0.89	846.30	33.85	59.24
68	0.05	0.08	0.13	0.87	753.21	37.66	60.26
69	0.06	0.09	0.15	0.85	655.29	39.32	58.98
70	0.00	1.00	1.00	0.00	557.00	0.00	557.00

Note, as a check on the calculations, that $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$, except for rounding error.

We continue this example with the evaluation, from first principles, of several probabilities:

$$_2p_{65}^{(\tau)} = p_{65}^{(\tau)}\, p_{66}^{(\tau)} = (0.93)(0.91) = 0.8463,$$

$$_{2|}q_{66}^{(1)} = p_{66}^{(\tau)}\, p_{67}^{(\tau)}\, q_{68}^{(1)} = (0.91)(0.89)(0.05) = 0.0405,$$

$$_2q_{67}^{(2)} = q_{67}^{(2)} + p_{67}^{(\tau)}\, q_{68}^{(2)} = 0.07 + (0.89)(0.08) = 0.1412.$$

The last three columns of the above table may be used to obtain the same probabilities. The answers agree to four decimal places:

$$ {}_2p_{65}^{(\tau)} = \frac{l_{67}^{(\tau)}}{l_{65}^{(\tau)}} = \frac{846.30}{1{,}000.00} = 0.8463, $$

$$ {}_{2|}q_{66}^{(1)} = \frac{d_{68}^{(1)}}{l_{66}^{(\tau)}} = \frac{37.66}{930.00} = 0.0405, $$

$$ {}_2q_{67}^{(2)} = \frac{d_{67}^{(2)} + d_{68}^{(2)}}{l_{67}^{(\tau)}} = \frac{59.24 + 60.26}{846.30} = 0.1412. $$

▼

10.4 Deterministic Survivorship Group

The total force of decrement can also be viewed as a total (nominal annual) rate of decrement rather than as a conditional probability density. In this view, where we assume a continuous model, a group of $l_a^{(\tau)}$ lives advance through age subject to deterministic forces of decrement $\mu_a^{(\tau)}(y - a)$, $y \ge a$. The number of survivors to age x from the original group of $l_a^{(\tau)}$ lives at age a is given by

$$ l_x^{(\tau)} = l_a^{(\tau)} \exp\left[-\int_a^x \mu_a^{(\tau)}(y - a)\, dy\right], \tag{10.4.1} $$

and the total decrement between ages x and $x + 1$ is

$$ \begin{aligned} d_x^{(\tau)} &= l_x^{(\tau)} - l_{x+1}^{(\tau)} \\ &= l_x^{(\tau)} \left(1 - \frac{l_{x+1}^{(\tau)}}{l_x^{(\tau)}}\right) \\ &= l_x^{(\tau)} \left\{1 - \exp\left[-\int_x^{x+1} \mu_a^{(\tau)}(y - a)\, dy\right]\right\} \\ &= l_x^{(\tau)}(1 - p_x^{(\tau)}) \\ &= l_x^{(\tau)} q_x^{(\tau)}. \end{aligned} \tag{10.4.2} $$

Further, by definition or from differentiating (10.4.1), we have

$$ \mu_a^{(\tau)}(x - a) = -\frac{1}{l_x^{(\tau)}} \frac{dl_x^{(\tau)}}{dx}. \tag{10.4.3} $$

These formulas are analogous to those for life tables in Section 3.4. Here $q_x^{(\tau)}$ is the effective annual total rate of decrement for the year of age x to $x + 1$ equivalent to the forces $\mu_a^{(\tau)}(y - a)$, $x \le y < x + 1$.

Consider next m causes of decrement and assume that the $l_x^{(\tau)}$ survivors to age x will, at future ages, be fully depleted by these m forms of decrement. Then the $l_x^{(\tau)}$ survivors can be visualized as falling into distinct subgroups $l_x^{(j)}$, $j = 1, 2, \ldots, m$, where $l_x^{(j)}$ denotes the number from the $l_x^{(\tau)}$ survivors who will terminate at future ages due to cause j, so that

$$ l_x^{(\tau)} = \sum_{j=1}^{m} l_x^{(j)}. \tag{10.4.4} $$

We define the force of decrement at age x due to cause j by

$$\mu_a^{(j)}(x-a) = \lim_{h\to 0} \frac{l_x^{(j)} - l_{x+h}^{(j)}}{h l_x^{(\tau)}}$$

where $l_x^{(\tau)}$, not $l_x^{(j)}$, appears in the denominator. This yields

$$\mu_a^{(j)}(x-a) = -\frac{1}{l_x^{(\tau)}}\frac{dl_x^{(j)}}{dx}. \tag{10.4.5}$$

From (10.4.3)–(10.4.5) it follows that

$$\mu_a^{(\tau)}(x-a) = -\frac{1}{l_x^{(\tau)}}\frac{d}{dx}\sum_{j=1}^{m} l_x^{(j)} = \sum_{j=1}^{m} \mu_a^{(j)}(x-a). \tag{10.4.6}$$

Formula (10.4.5), substituting y for x, can be written as

$$-dl_y^{(j)} = l_y^{(\tau)}\,\mu_a^{(j)}(y-a)\,dy,$$

and integration from $y = x$ to $y = x + 1$ gives

$$l_x^{(j)} - l_{x+1}^{(j)} = d_x^{(j)} = \int_x^{x+1} l_y^{(\tau)}\,\mu_a^{(j)}(y-a)\,dy. \tag{10.4.7}$$

Summation over $j = 1, 2, \ldots, m$ yields

$$l_x^{(\tau)} - l_{x+1}^{(\tau)} = d_x^{(\tau)} = \int_x^{x+1} l_y^{(\tau)}\,\mu_a^{(\tau)}(y-a)\,dy. \tag{10.4.8}$$

Further, from division of formula (10.4.7) by $l_x^{(\tau)}$, we have

$$\frac{d_x^{(j)}}{l_x^{(\tau)}} = \int_x^{x+1} {}_{y-x}p_x^{(\tau)}\,\mu_a^{(j)}(y-a)\,dy = q_x^{(j)}. \tag{10.4.9}$$

Here $q_x^{(j)}$ is defined as the proportion of the $l_x^{(\tau)}$ survivors to age x who terminate due to cause j before age $x + 1$ when all m causes of decrement are operating.

As was the case for life tables, the deterministic model provides an alternative language and conceptual framework for multiple decrement theory.

10.5 Associated Single Decrement Tables

For each of the causes of decrement recognized in a multiple decrement model, it is possible to define a single decrement model that depends only on the particular cause of decrement. We define the ***associated single decrement model*** functions as follows:

$$\begin{aligned} {}_tp_x'^{(j)} &= \exp\left[-\int_0^t \mu_x^{(j)}(s)\,ds\right], \\ {}_tq_x'^{(j)} &= 1 - {}_tp_x'^{(j)}. \end{aligned} \tag{10.5.1}$$

Quantities such as ${}_tq_x'^{(j)}$ are called ***net probabilities of decrement*** in biostatistics because they are net of other causes of decrement. However, many other names have been given to the same quantity. One is ***independent rate of decrement,*** chosen because cause j does not compete with other causes in determining ${}_tq_x'^{(j)}$. The term we use for ${}_tq_x'^{(j)}$ is ***absolute rate of decrement.*** The use of the word ***rate*** in describing ${}_tq_x'^{(j)}$ stems from a desire to distinguish between q and q'. The symbol ${}_tq_x^{(j)}$ denotes a probability of decrement for cause j between ages x and $x + t$, and we will show that it differs from ${}_tq_x'^{(j)}$. In addition, ${}_tp_x'^{(j)}$, unlike ${}_tp_x^{(\tau)}$, is not necessarily a survivorship function, because it is not required that $\lim_{t\to\infty} {}_tp_x'^{(j)} = 0$.

While

$$\int_0^\infty \mu_x^{(\tau)}(t)\, dt = \infty,$$

we can conclude from (10.2.14) only that

$$\int_0^\infty \mu_x^{(j)}(t)\, dt = \infty$$

for at least one j. There may be causes of decrement for which this integral is finite.

We seldom have an opportunity to observe the operation of a random survival system in which a single cause of decrement operates. In an employee benefit plan, retirement, disabilities, and voluntary terminations make it impossible to directly observe the operation of a single decrement model for mortality during active service. In biostatistical applications random withdrawals from observation and arbitrary ending of the period of study may prevent the observation of mortality alone operating on a group of lives.

As we see in Section 10.6, a usual first step in constructing a multiple decrement model is to select absolute rates of decrement and to make assumptions concerning the incidence of the decrements within any single year of age to obtain probabilities $q_x^{(j)}$. The converse problem of obtaining absolute rates from the probabilities also involves assumptions about the incidence of the decrements. These assumptions are implicit in statistical methods for estimating absolute rates and will be discussed in Section 10.5.5.

In the next subsection we examine a number of relationships between a multiple decrement table and its associated single decrement tables. Then we examine a number of special assumptions about incidence of decrement over the year of age and note some implied relationships. In Section 10.5.5 some of the statistical issues in estimating a multiple decrement distribution are examined.

10.5.1 Basic Relationships

First, note that since

$${}_tp_x^{(\tau)} = \exp\left\{-\int_0^t [\mu_x^{(1)}(s) + \mu_x^{(2)}(s) + \cdots + \mu_x^{(m)}(s)]\, ds\right\},$$

we have

$$ {}_tp_x^{(\tau)} = \prod_{i=1}^{m} {}_tp_x'^{(i)}. \tag{10.5.2} $$

This result does not involve any approximation. It is based on the definition of an associated single decrement table where the sole force of decrement is equal to the force for that decrement in the multiple decrement model. We require that it hold for any method used to construct a multiple decrement table from a set of absolute rates of decrement.

Now compare the size of the absolute rates and the probabilities. From (10.5.2) we see, if some cause other than j is operating, that

$$ {}_tp_x'^{(j)} \geq {}_tp_x^{(\tau)}. $$

This implies

$$ {}_tp_x'^{(j)}\, \mu_x^{(j)}(t) \geq {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t), $$

and if these functions are integrated with respect to t over the interval (0, 1), we obtain

$$ q_x'^{(j)} = \int_0^1 {}_tp_x'^{(j)}\, \mu_x^{(j)}(t)\, dt \geq \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt = q_x^{(j)}. \tag{10.5.3} $$

The magnitude of other forces of decrement can cause ${}_tp_x'^{(j)}$ to be considerably greater than ${}_tp_x^{(\tau)}$, and thus there can be corresponding differences between the absolute rates and the probabilities.

10.5.2 Central Rates of Multiple Decrement

There is one function of the multiple decrement model that is quite close to the corresponding function for an associated single decrement model. To introduce this function, we return to a mortality table and recall the central rate of mortality, or central-death-rate at age x, denoted by m_x and defined in (3.5.13) by

$$ m_x = \frac{\int_0^1 {}_tp_x\, \mu_x(t)\, dt}{\int_0^1 {}_tp_x\, dt} = \frac{\int_0^1 l_{x+t}\, \mu_x(t)\, dt}{\int_0^1 l_{x+t}\, dt} = \frac{d_x}{L_x}. \tag{10.5.4} $$

Thus, m_x is a weighted average of the force of mortality between ages x and $x + 1$, and this justifies the term ***central rate.***

Such central rates can be defined in a multiple decrement context. The ***central rate of decrement from all causes*** is defined by

$$ m_x^{(\tau)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)\, dt}{\int_0^1 {}_tp_x^{(\tau)}\, dt} \tag{10.5.5} $$

and is a weighted average of $\mu_x^{(\tau)}(t)$, $0 \le t < 1$. Similarly, the ***central rate of decrement from cause j*** is

$$m_x^{(j)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt}{\int_0^1 {}_tp_x^{(\tau)}\, dt} \tag{10.5.6}$$

and is a weighted average of $\mu_x^{(j)}(t)$, $0 \le t < 1$. Clearly,

$$m_x^{(\tau)} = \sum_{j=1}^{m} m_x^{(j)}.$$

The corresponding central rate for the associated single decrement table is given by

$$m_x'^{(j)} = \frac{\int_0^1 {}_tp_x'^{(j)}\, \mu_x^{(j)}(t)\, dt}{\int_0^1 {}_tp_x'^{(j)}\, dt}. \tag{10.5.7}$$

This is again a weighted average of $\mu_x^{(j)}(t)$ over the same age range, the weight function now ${}_tp_x'^{(j)}$ rather than ${}_tp_x^{(\tau)}$. If the force $\mu_x^{(j)}(t)$ is constant for $0 \le t < 1$, we have $m_x^{(j)} = m_x'^{(j)} = \mu_x^{(j)}(0)$. If $\mu_x^{(j)}(t)$ is an increasing function of t, then ${}_tp_x'^{(j)}$ gives more weight to higher values than does ${}_tp_x^{(\tau)}$, and $m_x'^{(j)} > m_x^{(j)}$. If $\mu_x^{(j)}(t)$ is a decreasing function of t, then $m_x'^{(j)} < m_x^{(j)}$. See Exercise 10.33 for a more formal treatment of these statements.

Central rates provide a convenient but approximate means of proceeding from the $q_x'^{(j)}$ to the $q_x^{(j)}$, $j = 1, 2, \ldots, m$, and vice versa. This is illustrated in Exercise 10.18.

10.5.3 Constant Force Assumption for Multiple Decrements

Let us examine specific assumptions concerning the incidence of decrements. First, let us use an assumption of a constant force for decrement j and for the total decrement over the interval $(x, x + 1)$. This implies

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(0)$$

and

$$\mu_x^{(\tau)}(t) = \mu_x^{(\tau)}(0) \qquad 0 \le t < 1.$$

Then, for $0 \le s \le 1$, we have

$$\begin{aligned} {}_sq_x^{(j)} &= \int_0^s {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt \\ &= \frac{\mu_x^{(j)}(0)}{\mu_x^{(\tau)}(0)} \int_0^s {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)\, dt = \frac{\mu_x^{(j)}(0)}{\mu_x^{(\tau)}(0)}\, {}_sq_x^{(\tau)}. \end{aligned} \tag{10.5.8}$$

But also for any r in (0, 1), under the constant force assumption,

$$r\mu_x^{(\tau)}(0) = -\log {}_rp_x^{(\tau)}$$

and

$$r\mu_x^{(j)}(0) = -\log {}_rp_x'^{(j)},$$

so that from (10.5.8),

$${}_sq_x^{(j)} = \frac{\log {}_rp_x'^{(j)}}{\log {}_rp_x^{(\tau)}}\, {}_sq_x^{(\tau)}. \tag{10.5.9}$$

Equation (10.5.9) can be rearranged as

$${}_rp_x'^{(j)} = ({}_rp_x^{(\tau)})^{{}_sq_x^{(j)}/{}_sq_x^{(\tau)}}$$

and then in the limit as r goes to 1 can be solved for $q_x'^{(j)}$ to give

$$q_x'^{(j)} = 1 - (p_x^{(\tau)})^{{}_sq_x^{(j)}/{}_sq_x^{(\tau)}}. \tag{10.5.10}$$

If the constant force assumption holds for all decrements (and then automatically for the total decrement), (10.5.9), as r and s approach 1, together with (10.5.2), can be used for calculating $q_x^{(j)}$ from given values of $q_x'^{(j)}$, $j = 1, 2, \ldots, m$. Also, (10.5.10) is useful for obtaining absolute rates from a set of probabilities of decrement. Note that for (10.5.9) and (10.5.10) special treatment is required if $p_x'^{(j)}$ or $p_x^{(\tau)}$ equals 0.

10.5.4 Uniform Distribution Assumption for Multiple Decrements

Formula (10.5.10) holds under alternative assumptions. One of these is that both decrement j and total decrement, in the multiple decrement context, have a uniform distribution of decrement over the interval $(x, x + 1)$. Thus we assume that

$${}_tq_x^{(j)} = t\, q_x^{(j)}$$

and

$${}_tq_x^{(\tau)} = t\, q_x^{(\tau)}.$$

Also under the given assumption, we see from (10.2.12) that

$${}_tp_x^{(\tau)}\, \mu_x^{(j)}(t) = q_x^{(j)} \tag{10.5.11}$$

and

$$\mu_x^{(j)}(t) = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - t\, q_x^{(\tau)}}.$$

Then

$$\begin{aligned}
{}_sp_x'^{(j)} &= \exp\left[-\int_0^s \mu_x^{(j)}(t)\, dt\right] \\
&= \exp\left(-\int_0^s \frac{q_x^{(j)}}{1 - t\, q_x^{(\tau)}}\, dt\right) \\
&= \exp\left[\frac{q_x^{(j)}}{q_x^{(\tau)}} \log\,(1 - sq_x^{(\tau)})\right] \\
&= ({}_sp_x^{(\tau)})^{q_x^{(j)}/q_x^{(\tau)}}.
\end{aligned} \tag{10.5.12}$$

At $s = 1$, (10.5.10) and (10.5.12) yield the same equation relating $q_x'^{(j)}$ with $q_x^{(j)}$ and $q_x^{(\tau)}$. As a result, (10.5.9) with $r = 1$ can be used to obtain $q_x^{(j)}$. Exercise 10.22 provides additional insights into the connection between the developments in Sections 10.5.3 and 10.5.4.

Example 10.5.1

Continue Example 10.3.1 evaluating $q_x'^{(1)}$ and $q_x'^{(2)}$ by (10.5.10).

Solution:

By (10.5.10), the following results are obtained.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x'^{(1)}$	$q_x'^{(2)}$
65	0.02	0.05	0.02052	0.05052
66	0.03	0.06	0.03095	0.06094
67	0.04	0.07	0.04149	0.07147
68	0.05	0.08	0.05215	0.08213
69	0.06	0.09	0.06294	0.09291
70	0.00	1.00	—	—

At age 70, the rates depend on mandatory retirement, and there is no particular need for $q_{70}'^{(1)}$, $q_{70}'^{(2)}$ although they could be identified, respectively, using $q_{70}^{(1)}$ and $q_{70}^{(2)}$. ▼

10.5.5 Estimation Issues

The definition of the absolute rate of decrement given in (10.5.1) depends on the force of decrement in multiple decrement theory as defined in (10.2.11). From definition (10.5.1), the developments in this section and their application in constructing multiple decrement distributions from assumed absolute rates of decrement follow. Questions remain, however, about the interpretation and estimation of ${}_tp_x'^{(j)}$.

If the joint p.d.f. $f_{T,J}(t, j)$ is known, then the survival function and forces of decrement are determined using the formulas of Section 10.2. For example, (10.2.13) follows as a consequence of the assumption that decrements occur from m mutually exclusive causes. The issue is, under what conditions can information obtained in a single decrement environment be used to construct the distribution of (T, J)?

We will illustrate this issue by considering two causes of decrement. Each associated single decrement environment has its time-until-decrement random variable $T_j(x)$ and its survival function $s_{T_j}(t) = \Pr\{T_j(x) > t\}$, $j = 1, 2$. The joint survival function of $T_1(x)$ and $T_2(x)$ is given by

$$s_{T_1,T_2}(t_1, t_2) = \Pr\{(T_1(x) > t_1) \cap (T_2(x) > t_2)\}.$$

In this context, the time-until-decrement random variable T equals the minimum of $T_1(x)$ and $T_2(x)$ and, in accordance with Section 9.3, (9.3.1), its survival function is

$$s_T(t) = s_{T_1,T_2}(t, t).$$

If $T_1(x)$ and $T_2(x)$ are independent,

$$s_T(t) = s_{T_1}(t)s_{T_2}(t) = s_{T_1,T_2}(t, 0)s_{T_1,T_2}(0, t),$$

and

$$\mu_x^{(\tau)}(t) = -\frac{d}{dt}\log s_{T_1}(t)s_{T_2}(t)$$

$$= \mu_x^{(1)}(t) + \mu_x^{(2)}(t). \qquad (10.5.13)$$

On the other hand, if $T_1(x)$ and $T_2(x)$ are dependent,

$$\mu_x^{(\tau)}(t) = -\frac{d}{dt}\log s_{T_1,T_2}(t, t)$$

$$\neq -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t). \qquad (10.5.14)$$

The two terms on the right-hand side of (10.5.13) are called ***marginal forces of decrement*** associated, in order, with $T_1(x)$ and $T_2(x)$.

If $T_1(x)$ and $T_2(x)$ are independent, then the marginal forces of decrement from a single decrement environment can be used with (10.5.2) to determine ${}_tp_x^{(\tau)}$. If $T_1(x)$ and $T_2(x)$ are dependent, we have no assurance that assuming (10.5.2) yields the survival function of time-until-decrement in a multiple decrement environment.

Example 10.5.2

This example builds on Examples 9.2.1, 9.2.2, and 9.3.1. The dependent random variables $T_1(x)$ and $T_2(x)$ have a joint p.d.f. given by

$$f_{T_1,T_2}(s, t) = 0.0006(s - t)^2 \qquad 0 < s < 10,\ 0 < t < 10$$
$$= 0 \qquad \text{elsewhere.}$$

The joint survival function is exhibited in Example 9.2.2, and the survival function of $T = \min[T_1, T_2]$ is exhibited in Example 9.3.1.

Show that

$$-\frac{d}{dt}\log s_T(t) \neq -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t).$$

Solution:

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{(10 - t)} \qquad 0 < t < 10,$$

and

$$-\frac{d}{dt}\log s_{T_1,T_2}(t,0) = \frac{4{,}000 - 1{,}200t + 120t^2}{20{,}000 - 4{,}000t + 600t^2 - 40t^3}$$
$$= \frac{100 - 30t + 3t^2}{500 - 100t + 15t^2 - t^3}$$
$$= \frac{100 - 30t + 3t^2}{(10 - t)(50 - 5t + t^2)},$$

which by symmetry is also equal to $-\frac{d}{dt}\log s_{T_1,T_2}(0,t)$. Therefore

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{10-t} \neq \frac{1}{10-t}\left(\frac{200 - 60t + 6t^2}{50 - 5t + t^2}\right).$$ ▼

Example 10.5.3

This example builds on Examples 9.2.3 and 9.3.2. The independent random variables $T_1(x)$ and $T_2(x)$ have a joint p.d.f. given by

$$f_{T_1,T_2}(s,t) = [0.02(10 - s)][0.02(10 - t)] \qquad \begin{array}{l} 0 < s < 10 \\ 0 < t < 10. \end{array}$$

Show that

$$-\frac{d}{dt}\log s_T(t) = -\frac{d}{dt}\log s_{T_1,T_2}(t,0) - \frac{d}{dt}\log s_{T_1,T_2}(0,t).$$

Solution:

The survival function of $T = \min[T_1(x), T_2(x)]$ is displayed in Example 9.3.2. Therefore,

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{10-t} \qquad 0 < t < 10,$$

$$-\frac{d}{dt}\log s_{T_1,T_2}(t,0) = \frac{2}{10-t} \qquad 0 < t < 10,$$

and by symmetry

$$-\frac{d}{dt}\log s_{T_1,T_2}(0,t) = \frac{2}{10-t} \qquad 0 < t < 10.$$

As a result,

$$-\frac{d}{dt}\log s_T(t) = -\frac{d}{dt}\log s_{T_1T_2}(t,0) - \frac{d}{dt}\log s_{T_1T_2}(0,t).$$ ▼

An interesting but distressing aspect of Examples 10.5.2 and 10.5.3 is that two dependent time-until-decrement random variables and two independent time-until-decrement random variables yield the same distribution of $T = \min[T_1(x), T_2(x)]$. Values of T can be observed, but without additional information it is impossible to select between the two models that may be generating the data. This, as in Section 9.3, is an example of nonidentifiability. Henceforth, in this chapter when constructing multiple decrement distributions from associated single decrement distributions, we assume that the component random variables are independent.

Remark:

The correspondence between the theory for the joint life model and the theory for the multiple decrement model can provide insights, but it is not complete. The difference between the two models centers on two facts that were identified in the discussion of (10.5.2) and Example 10.5.2. Realizations of both $T(x)$ and $T(y)$ can, at least in theory, be observed, while only the minimum of $T_1(x)$ and $T_2(x)$ and which one is the minimum can be observed. The corresponding problem in estimating joint life models was mentioned in Section 9.3. In addition, $\lim_{t\to\infty} {}_tp_x = \lim_{t\to\infty} {}_tp_y = 0$, whereas there is no assurance that

$$\lim_{t\to\infty} {}_tp_x'^{(j)} = 0 \qquad j = 1, 2.$$

10.6 Construction of a Multiple Decrement Table

In building a multiple decrement model it is best if data, including that on age and cause of decrement for the population under study, can be used to estimate directly the probabilities $q_x^{(j)}$. Large, well-established employee benefit plans may have such data. For other plans, such data are frequently not available. An alternative is to construct the model from associated single decrement rates assumed appropriate for the population under study. The adequacy of the model should then be tested by reviewing data as they become available.

Once satisfactory associated single decrement tables are selected, the results of Section 10.5 can be used to complete the construction of the multiple decrement table. The availability of a set of $p_x'^{(j)}$, for $j = 1, 2, \ldots, m$ and all values of x, will permit the computation of $p_x^{(\tau)}$ by (10.5.2) and of $q_x^{(\tau)}$ by $q_x^{(\tau)} = 1 - p_x^{(\tau)}$. The remaining step is to break $q_x^{(\tau)}$ into its components $q_x^{(j)}$ for $j = 1, 2, \ldots, m$. If either the constant force or the uniform distribution of decrement assumption is adopted in the model, (10.5.9) can be used for the calculation of the $q_x^{(j)}$.

Example 10.6.1

Use (10.5.2) and (10.5.9) to obtain the multiple decrement table corresponding to absolute rates of decrement given below. Presumably the actuary has examined the characteristics of the participant group and has decided that associated single decrement tables yielding these rates are appropriate for the group under study. It is also assumed that cause 3 is retirement that can occur between ages 65 and 70 and is mandatory at 70.

x	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{\prime(3)}$
65	0.020	0.02	0.04
66	0.025	0.02	0.06
67	0.030	0.02	0.08
68	0.035	0.02	0.10
69	0.040	0.02	0.12

Solution:

The table below contains the results of the calculation of the probabilities of decrement. Formula (10.5.2) can be rewritten as

$$q_x^{(\tau)} = 1 - \prod_{j=1}^{3} (1 - q_x^{\prime(j)}).$$

In this equation the assumed independence among the three causes of decrement is apparent. Formula (10.5.9), and the mandatory retirement condition, yield the multiple decrement probabilities. The multiple decrement table is constructed as in Example 10.3.1.

x	$q_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
65	0.07802	0.01940	0.01940	0.03921	1 000.00	19.40	19.40	39.21
66	0.10183	0.02401	0.01916	0.05867	921.99	22.14	17.67	54.09
67	0.12545	0.02851	0.01891	0.07803	828.09	23.61	15.66	64.62
68	0.14887	0.03290	0.01866	0.09731	724.20	23.83	13.51	70.47
69	0.17210	0.03720	0.01841	0.11649	616.39	22.93	11.35	71.80
70	1.00000	0.00000	0.00000	1.00000	510.31	0.00	0.00	510.31

▼

It has been noted that (10.5.9) and (10.5.10) will not be used if $p_x^{\prime(j)}$ or $p_x^{(\tau)} = 0$; some alternative device will be necessary. One such method, which handles this indeterminacy and lends itself to special adjustments, is based on assumed distributions of decrement in the associated single decrement tables rather than on assumptions about multiple decrement probabilities as in Section 10.5. We first examine an assumption of uniform distribution of decrement (in each year of age) in the associated single decrement tables. We restrict our attention to situations with three decrements, but the method and formulas easily extend for $m > 3$. Under the stated assumption,

$$_tp_x^{\prime(j)} = 1 - t\, q_x^{\prime(j)} \qquad j = 1, 2, 3;\ 0 \le t \le 1 \tag{10.6.1}$$

and

$$_tp_x^{\prime(j)}\, \mu_x^{(j)}(t) = \frac{d}{dt}(-{}_tp_x^{\prime(j)}) = q_x^{\prime(j)}. \tag{10.6.2}$$

It follows that

$$
\begin{aligned}
q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt \\
&= \int_0^1 {}_tp_x^{\prime(1)}\, \mu_x^{(1)}(t)\, {}_tp_x^{\prime(2)}\, {}_tp_x^{\prime(3)}\, dt \\
&= q_x^{\prime(1)} \int_0^1 (1 - t\, q_x^{\prime(2)})(1 - t\, q_x^{\prime(3)})\, dt \\
&= q_x^{\prime(1)} \left[1 - \frac{1}{2}(q_x^{\prime(2)} + q_x^{\prime(3)}) + \frac{1}{3} q_x^{\prime(2)} q_x^{\prime(3)} \right].
\end{aligned} \tag{10.6.3}
$$

Similar formulas hold for $q_x^{(2)}$, $q_x^{(3)}$, and it can be verified that

$$
\begin{aligned}
q_x^{(1)} + q_x^{(2)} + q_x^{(3)} &= q_x^{\prime(1)} + q_x^{\prime(2)} + q_x^{\prime(3)} \\
&\quad - (q_x^{\prime(1)} q_x^{\prime(2)} + q_x^{\prime(1)} q_x^{\prime(3)} + q_x^{\prime(2)} q_x^{\prime(3)}) \\
&\quad + q_x^{\prime(1)} q_x^{\prime(2)} q_x^{\prime(3)} \\
&= 1 - (1 - q_x^{\prime(1)})(1 - q_x^{\prime(2)})(1 - q_x^{\prime(3)}) = q_x^{(\tau)}.
\end{aligned} \tag{10.6.4}
$$

Example 10.6.2

Obtain the probabilities of decrement for ages 65–69 from the data in Example 10.6.1, under the assumption of a uniform distribution of decrement in each year of age in each of the associated single decrement tables.

Solution:

This is an application of (10.6.3).

x	$q_x^{\prime(1)}$	$q_x^{\prime(2)}$	$q_x^{\prime(3)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
65	0.020	0.02	0.04	0.01941	0.01941	0.03921
66	0.025	0.02	0.06	0.02401	0.01916	0.05866
67	0.030	0.02	0.08	0.02852	0.01892	0.07802
68	0.035	0.02	0.10	0.03292	0.01867	0.09727
69	0.040	0.02	0.12	0.03723	0.01843	0.11643

These probabilities are close to those obtained by (10.5.9), displayed in Example 10.6.1. ▼

We conclude this section with another example illustrating the use of a special distribution for one of the decrements. Special distributions are sometimes required by the facts of the situation being modeled.

Example 10.6.3

Consider a situation with three causes of decrement: mortality, disability, and withdrawal. Assume mortality and disability are uniformly distributed in each year of age in the associated single decrement tables with absolute rates of $q_x^{\prime(1)}$ and

$q_x'^{(2)}$, respectively. Also assume that withdrawals occur only at the end of the year with an absolute rate of $q_x'^{(3)}$.

a. Give formulas for the probabilities of decrement in the year of age x to $x+1$ for the three causes.
b. Reformulate the probabilities under the assumptions that
   (i) In the associated single decrement model, withdrawals occur only at the age's midyear or year end, and
   (ii) Equal proportions, namely $(1/2)\, q_x'^{(3)}$, of those beginning the year withdraw at the midyear and at the year end.

Remark:

Until now our multiple decrement models have been fully continuous, except possibly to recognize a mandatory retirement age. Moreover, our theory began with a multiple decrement model and after defining the forces $\mu_x^{(j)}(t)$, $j = 1, 2, \ldots, m$ proceeded to the associated single decrement tables. In this example we start with the single decrement tables, and in one of these tables the decrement takes place discretely at the ends of stated intervals. We do not attempt to define a force of decrement for this discrete case but proceed by direct methods to build, from the single decrement tables, a multiple decrement model possessing the relationships (10.2.19) and (10.5.2) established in our prior theory.

Solution:

a. Figure 10.6.1 displays survival factors for the given single decrement tables and for a multiple decrement table where

$$ {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)} $$

for nonintegral $t \geq 0$. At $t = 1$, ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$ are discontinuous, so we consider

$$ \lim_{t \to 1-} {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, 1 $$

and

$$ p_x^{(\tau)} = p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). $$

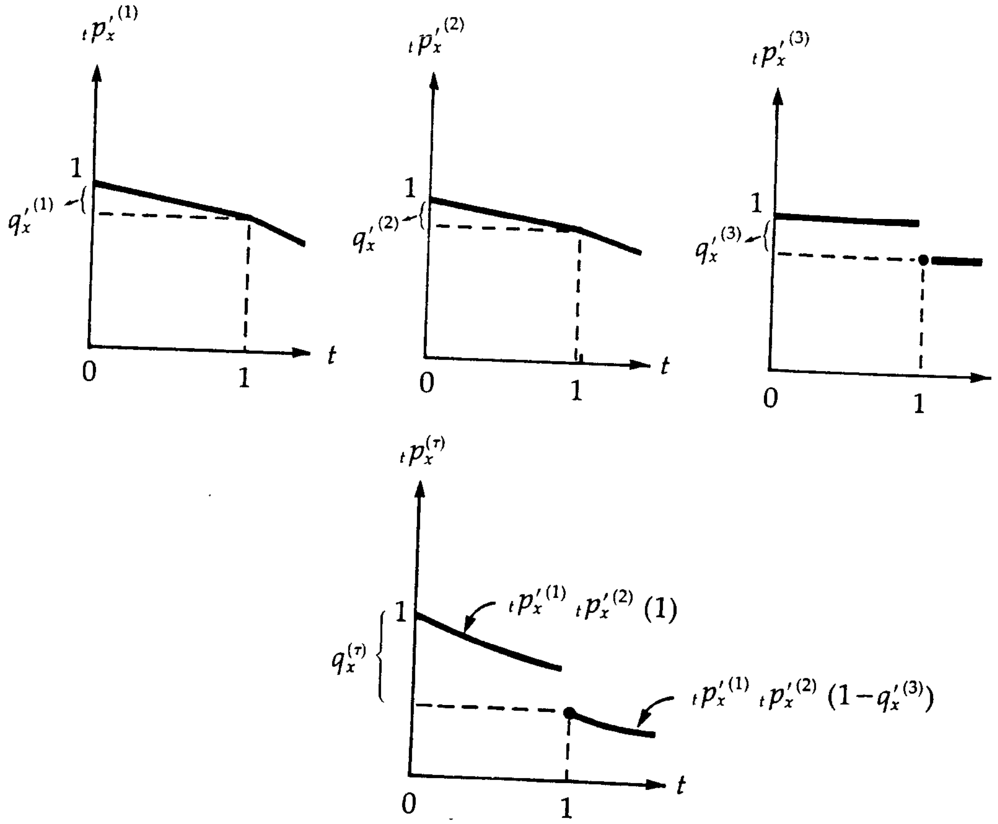
We also require that, for our multiple decrement table,

$$ q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = 1 - p_x^{(\tau)} = 1 - p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). $$

We set

$$ \begin{aligned} q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt \\ &= \int_0^1 {}_tp_x'^{(1)}\, {}_tp_x'^{(2)} (1) \mu_x^{(1)}(t)\, dt \\ &= q_x'^{(1)} \int_0^1 (1 - t q_x'^{(2)})\, dt \\ &= q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(2)}\right). \end{aligned} $$

Survival Factors, ${}_tp_x'^{(j)}$, j = 1, 2, 3, and ${}_tp_x^{(\tau)}$



Similarly, we set

$$q_x^{(2)} = q_x'^{(2)}\left(1 - \frac{1}{2}\,q_x'^{(1)}\right).$$

Then

$$\begin{aligned} q_x^{(3)} &= q_x^{(\tau)} - (q_x^{(1)} + q_x^{(2)}) \\ &= 1 - p_x'^{(1)}p_x'^{(2)}(1 - q_x'^{(3)}) - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)}, \end{aligned}$$

and, since

$$1 - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)} = p_x'^{(1)}p_x'^{(2)},$$

$$q_x^{(3)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)}.$$

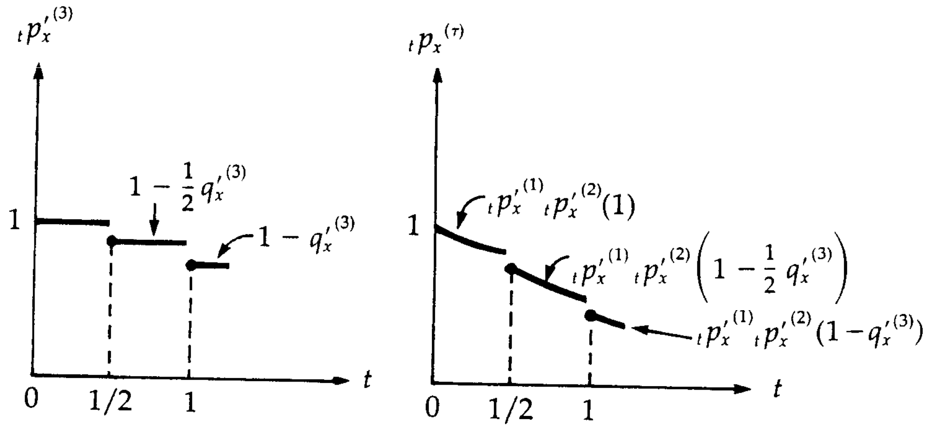
Note that

$$\lim_{t\to 1-} {}_tp_x^{(\tau)} - \lim_{t\to 1+} {}_tp_x^{(\tau)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)} = q_x^{(3)}\ ;$$

that is, the discontinuity at $t = 1$ equals $q_x^{(3)}$.

b. Here ${}_tp_x'^{(1)}$ and ${}_tp_x'^{(2)}$ are as in Figure 10.6.1, but ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$ now have discontinuities at $t = 1/2$ and $t = 1$, as shown in Figure 10.6.2.

Survival Factors, ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$



Proceeding as in (a), but taking account of the intervals [0, 1 / 2) and [1 / 2, 1), we set

$$
\begin{aligned}
q_x^{(1)} &= q_x'^{(1)} \int_0^{1/2} (1 - t\, q_x'^{(2)})\, dt \\
&\quad + q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(3)}\right) \int_{1/2}^{1} (1 - t\, q_x'^{(2)})\, dt \\
&= q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(2)} - \frac{1}{4} q_x'^{(3)} + \frac{3}{16} q_x'^{(2)} q_x'^{(3)}\right).
\end{aligned}
$$

Similarly, we set

$$
q_x^{(2)} = q_x'^{(2)} \left(1 - \frac{1}{2} q_x'^{(1)} - \frac{1}{4} q_x'^{(3)} + \frac{3}{16} q_x'^{(1)} q_x'^{(3)}\right).
$$

Then,

$$
\begin{aligned}
q_x^{(3)} &= 1 - p_x^{(\tau)} - q_x^{(1)} - q_x^{(2)} \\
&= 1 - p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}) - q_x^{(1)} - q_x^{(2)},
\end{aligned}
$$

which reduces to

$$
q_x^{(3)} = q_x'^{(3)} \left(1 - \frac{3}{4} q_x'^{(1)} - \frac{3}{4} q_x'^{(2)} + \frac{5}{8} q_x'^{(1)} q_x'^{(2)}\right).
$$

▼

10.7 Notes and References

The history of multiple decrement theory was reviewed by Seal (1977). Chiang (1968) developed the theory using the language of competing risks. The foundation for the actuarial theory of multiple decrement models was built by Makeham (1874). Menge (1932) and Nesbitt and Van Eenam (1948) provided insight into the deterministic interpretation of forces of decrement and of increment. Bicknell and

Nesbitt (1956) developed a very general theory for individual insurances using a deterministic multiple decrement model. Hickman (1964) redeveloped this theory using the language of the stochastic model, and this redevelopment is the basis for much of this chapter. The analysis of life tables by cause of death is the subject of papers by Greville (1948) and Preston, Keyfitz, and Schoen (1973).

The perplexing estimation issues that arise when the times-until-decrement are not independent are discussed by Elandt-Johnson and Johnson (1980). Promislow (1991b) makes the excellent point that in practice multiple decrement models should be select in the sense of Chapter 3. He developed a theory and associated notation for select multiple decrement models. Exercises 10.3 and 10.24 are built on a discussion by Robinson (1984).

Carriere (1994) applied copulas, discussed in Section 9.6.2, to create multiple decrement distributions that incorporate dependent component random variables. Carriere also discusses the problem of identifiability and reviews the conditions under which it is possible to identify a unique joint survival function $s_{T_1(x)\ T_2(x)}(t, t)$.

Exercises

Section 10.2

10.1. Let $\mu_x^{(j)}(t) = \mu_x^{(j)}(0)$, $j = 1, 2, \ldots, m$, $t \geq 0$. Obtain expressions for
a. $f_{T,J}(t, j)$ b. $f_J(j)$ c. $f_T(t)$.
The functions called for in (a) and (c) are p.d.f.'s, and the function in (b) is a p.f. Show that T and J are independent random variables.

10.2. A multiple decrement model with two causes of decrement has forces of decrement given by

$$\mu_x^{(1)}(t) = \frac{1}{100 - (x + t)}$$

and

$$\mu_x^{(2)}(t) = \frac{2}{100 - (x + t)} \qquad t < 100 - x.$$

If $x = 50$, obtain expressions for
a. $f_{T,J}(t, j)$ b. $f_T(t)$ c. $f_J(j)$ d. $f_{J|T}(j|t)$.

10.3. Given the joint p.d.f.

$$\begin{aligned} f_{T,J}(t, j) &= pu_1\, e^{-(u_1+v_1)t} + (1 - p)u_2\, e^{-(u_2+v_2)t} \qquad 0 \leq t,\ j = 1 \\ &= pv_1\, e^{-(u_1+v_1)t} + (1 - p)v_2\, e^{-(u_2+v_2)t} \qquad 0 \leq t,\ j = 2 \end{aligned}$$

where $0 < p < 1$ and $0 < u_1, u_2, v_1, v_2,$

find

a. The marginal p.d.f.s $f_T(t)$ and $f_J(j)$
b. The survival function $s_T(t)$.

Section 10.3

10.4. Using the multiple decrement probabilities given in Example 10.3.1, evaluate the following:

a. ${}_3p_{65}^{(\tau)}$ b. ${}_{3|}q_{65}^{(1)}$ c. ${}_3q_{65}^{(2)}$.

10.5. The following multiple decrement probabilities apply to students entering a 4-year college.

Curtate Duration, at Beginning of Academic Year	Probability of		
	Academic Failure, $j = 1$	Withdrawal for All Other Reasons, $j = 2$	Survival through the Academic Year
0	0.15	0.25	0.60
1	0.10	0.20	0.70
2	0.05	0.15	0.80
3	0.00	0.10	0.90

An entering class has 1,000 members.

a. What is the expectation of the number of graduates? What is the variance?
b. What is the expected number of those who will fail sometime during the 4-year program? What is the variance of the number of students who will fail?

10.6. Construct a multiple decrement table on the basis of the data in Exercise 10.5 and use it to exhibit

a. The marginal distribution of the random variable J (mode of exit), which takes on values for academic failure, withdrawal, and graduation
b. The conditional distribution of the mode of termination, given that a student has terminated in the third year.

Section 10.4

10.7. Given that $\mu^{(1)}(x) = 1/(a - x)$, $0 \le x < a$, and $\mu^{(2)}(x) = 1$, derive expressions for

a. $l_x^{(\tau)}$ b. $d_x^{(1)}$ c. $d_x^{(2)}$.

Assume $l_0^{(\tau)} = a$.

10.8. Given $\mu^{(1)}(x) = 2x/(a - x^2)$, $0 \le x \le \sqrt{a}$, and $\mu^{(2)}(x) = c$, $c > 0$, and $l_0^{(\tau)} = 1{,}000$, derive an expression for $l_x^{(\tau)}$.

10.9. Derive expressions for the following derivatives:

a. $\dfrac{d}{dx}\, {}_tq_x^{(\tau)}$ b. $\dfrac{d}{dx}\, {}_tq_x^{(j)}$ c. $\dfrac{d}{dt}\, {}_tq_x^{(j)}$.

10.10. Using the data in Exercise 10.5, and assuming a uniform distribution of all decrements in the multiple decrement model, calculate a table of $q_k'^{(j)}$, $j = 1, 2$, $k = 0, 1, 2, 3$ (where k is the curtate duration).

10.11. If $\mu_x^{(1)}(t)$ is a constant c for $0 \le t \le 1$, derive expressions in terms of c and ${}_tp_x^{(\tau)}$ for
a. $q_x'^{(1)}$ b. $m_x^{(1)}$ c. $q_x^{(1)}$.

10.12. Show that under appropriate assumptions of a uniform distribution of decrements

a. $m_x^{(\tau)} = \dfrac{q_x^{(\tau)}}{1 - (1/2)\, q_x^{(\tau)}}$ b. $m_x^{(j)} = \dfrac{q_x^{(j)}}{1 - (1/2)\, q_x^{(\tau)}}$ c. $m_x'^{(j)} = \dfrac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}}$

and, conversely,

d. $q_x^{(\tau)} = \dfrac{m_x^{(\tau)}}{1 + (1/2)\, m_x^{(\tau)}}$ e. $q_x^{(j)} = \dfrac{m_x^{(j)}}{1 + (1/2)\, m_x^{(\tau)}}$ f. $q_x'^{(j)} = \dfrac{m_x'^{(j)}}{1 + (1/2)\, m_x'^{(j)}}$.

10.13. Order the following in terms of magnitude and state your reasons:

$$q_x'^{(j)},\quad q_x^{(j)},\quad m_x'^{(j)}.$$

10.14. Given, for a double decrement table, that $q_{40}'^{(1)} = 0.02$ and $q_{40}'^{(2)} = 0.04$, calculate $q_{40}^{(\tau)}$ to four decimal places.

10.15. For a double decrement table you are given that $m_{40}^{(\tau)} = 0.2$ and $q_{40}'^{(1)} = 0.1$. Calculate $q_{40}'^{(2)}$ to four decimal places assuming
a. Uniform distribution of decrements in the multiple decrement model
b. Uniform distribution of decrements in the associated single decrement tables.

10.16. Using the data in Exercise 10.5 and assuming a uniform distribution of decrements in the multiple decrement model, construct a table of $m_k^{(j)}$, $j = 1, 2$, $k = 0, 1, 2, 3$ (where k is the curtate duration). Calculate each result to five decimal places.

10.17. Given that decrement may be due to death, 1, disability, 2, or retirement, 3, use (10.5.9) to construct a multiple decrement table based on the following absolute rates.

Age x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
62	0.020	0.030	0.200
63	0.022	0.034	0.100
64	0.028	0.040	0.120

10.18. Recalculate the multiple decrement table from the absolute rates of decrement in Exercise 10.17 by means of the ***central rate bridge.*** [Hint: To use the central rate bridge, first calculate $m_x'^{(j)}$ by the formula

$$m_x'^{(j)} \cong \frac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}} \qquad j = 1, 2, 3,$$

which holds if there is a uniform distribution of decrement in the associated single decrement tables. Next, assume $m_x^{(j)} \cong m_x'^{(j)}$, $j = 1, 2, 3$, and proceed to $q_x^{(j)}$ by

$$q_x^{(j)} = \frac{d_x^{(j)}}{l_x^{(\tau)}} = \frac{d_x^{(j)}}{l_x^{(\tau)} - (1/2)\, d_x^{(\tau)} + (1/2)\, d_x^{(\tau)}} = \frac{m_x^{(j)}}{1 + (1/2)\, m_x^{(\tau)}}.$$

This second relation holds if there is a uniform distribution of total decrement in the multiple decrement table. But then

$$\begin{aligned} {}_tp_x^{(\tau)} = 1 - t q_x^{(\tau)} &\neq {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)} \\ &= (1 - t q_x'^{(1)})(1 - t q_x'^{(2)})(1 - t q_x'^{(3)}) \end{aligned}$$

under the condition of a uniform distribution in the associated single decrement tables. Thus there is an inconsistency in the stated conditions, but the calculations may be accurate enough for this purpose.]

10.19. Indicate arguments for the following relations:

a. $m_x'^{(j)} \cong m_x^{(j)}$

b. $\dfrac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}} \cong \dfrac{q_x^{(j)}}{1 - (1/2)\, q_x^{(\tau)}}.$

Show that these lead to

c. $q_x^{(j)} \cong \dfrac{q_x'^{(j)}\, [1 - (1/2)\, q_x^{(\tau)}]}{1 - (1/2)\, q_x'^{(j)}}$

d. $q_x'^{(j)} \cong \dfrac{q_x^{(j)}}{1 - (1/2)\, (q_x^{(\tau)} - q_x^{(j)})}.$

Compare (c) and (d) to (10.5.9) and (10.5.10).

10.20. Use the values of $q_x^{(j)}$, $q_x'^{(j)}$ from Example 10.5.1 to calculate values of $m_x^{(j)}$, $m_x'^{(j)}$, $j = 1, 2$, $x = 65, \ldots, 69$, under appropriate assumptions of uniform distribution of decrements (see Exercise 10.12).

10.21. Which of the following statements would you accept? Revise where necessary.

a. $q_x^{(j)} \cong \dfrac{m_x^{(j)}}{1 + (1/2)\, m_x^{(j)}}$

b. $\displaystyle\int_0^1 l_{x+t}^{(\tau)}\, dt \cong \frac{l_x^{(\tau)}}{1 + (1/2)\, m_x^{(\tau)}}$

c. $q_x^{(1)} = q_x'^{(1)}[1 - (1/2) q_x'^{(2)}]$ in a double decrement table where there is a uniform distribution of decrement for the year of age x to $x + 1$ in each of the associated single decrement tables.

10.22. a. For a certain age x, particular cause of decrement j, and constant K_j, show that the following conditions are equivalent:

(i) ${}_tq_x^{(j)} = K_j \, {}_tq_x^{(\tau)} \qquad 0 \le t \le 1$

(ii) $\mu_x^{(j)}(t) = K_j \, \mu_x^{(\tau)}(t) \qquad 0 \le t \le 1$

(iii) $1 - {}_tq_x'^{(j)} = (1 - {}_tq_x^{(\tau)})^{K_j} \qquad 0 \le t \le 1.$

[Hint: Show (i) ⇒ (ii) ⇒ (iii) ⇒ (ii) ⇒ (i).]

b. Verify that, in a multiple decrement table, where either

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(0) \qquad 0 \le t \le 1, j = 1, 2, \ldots, m$$

(the constant force assumption for each cause of decrement) or

$${}_tq_x^{(j)} = t \, q_x^{(j)} \qquad 0 \le t \le 1, j = 1, 2, \ldots, m$$

(the uniform distribution for each cause of decrement), then

$${}_tq_x^{(j)} = K_j \, {}_tq_x^{(\tau)} \qquad 0 \le t \le 1, j = 1, 2, \ldots, m.$$

c. Assume that in part (a), condition (ii), $\mu_x^{(\tau)}(t)$, $0 \le t \le 1$, is given by

(i) $kt^n \qquad k > 0, n > 0 \qquad$ (Weibull)

(ii) $Bc^t \qquad B > 0, c > 1 \qquad$ (Gompertz)

and for each example find the corresponding expressions for ${}_tq_x^{(j)}$ and $1 - {}_tq_x'^{(j)}$.

10.23. a. Prove that

$$\mu_x^{(j)}(t) = K_j \, \mu_x^{(\tau)}(t) \qquad 0 \le t, j = 1, 2$$

where

$$K_j = \int_0^\infty {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt \qquad j = 1, 2,$$

if and only if the random variables T and J are independent.

b. If $T_1(x)$ and $T_2(x)$ are independent and J and T are independent, show that

$${}_tp_x'^{(j)} = ({}_tp_x^{(\tau)})^{K_j} \qquad j = 1, 2.$$

[Remark: Note that $K_j = f_J(j)$.]

10.24. This exercise is a continuation of Exercise 10.3 and uses the notation of Section 10.5.5. The joint survival function is given by

$$s_{T_1, T_2}(t_1, t_2) = pe^{-u_1t_1 - v_1t_2} + (1 - p)e^{-u_2t_1 - v_2t_2}$$
$$0 \le t_1, t_2, u_1, u_2, v_1, v_2$$
$$0 < p < 1.$$

Confirm that

$$s_{T_1, T_2}(t, t) \ne S_{T_1, T_2}(t, 0) \, S_{T_1, T_2}(0, t)$$

and

$$-\left.\frac{\partial \log s_{T_1,T_2}(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=t} \neq -\frac{d \log s_{T_1,T_2}(t, 0)}{dt}.$$

Section 10.6

10.25. Redo Exercise 10.10 by use of the formula for $q_x'^{(j)}$ in Exercise 10.19.

10.26. Show that $\mu_x^{(j)}(1/2) = m_x^{(j)}$, under the assumption of a uniform distribution of each decrement in each year of age in a multiple decrement context.

10.27. How would you proceed to construct the multiple decrement table if the given rates were those given below?
a. $q_x'^{(1)}, q_x'^{(2)}, q_x^{(3)}$
b. $q_x'^{(1)}, q_x^{(2)}, q_x^{(3)}$

10.28. In Example 10.6.2 suppose that decrement 3 at age 69 is not uniformly distributed but follows the pattern

$${}_tp_{69}'^{(3)} = \begin{cases} 1 - 0.12t & 0 < t < 1 \\ 0 & t = 1. \end{cases}$$

In words, the cause 3 absolute rate is 0.12 during the year. Then, just before age 70, all remaining survivors terminate due to cause 3. This is consistent with an assumption that $q_{69}'^{(3)} = 1$. What then is the value of $q_{69}^{(3)}$?

10.29. In a double decrement table where cause 1 is death and cause 2 is withdrawal, it is assumed that
- Deaths in the year from age h to $h + 1$ are uniformly distributed,
- Withdrawals in the year from age h to age $h + 1$ occur immediately after the attainment of age h.

From this table it is noted that, at age 50, $l_{50}^{(\tau)} = 1{,}000$, $q_{50}^{(2)} = 0.2$, and $d_{50}^{(1)} = 0.06\ d_{50}^{(2)}$. Determine $q_{50}'^{(1)}$.

Miscellaneous

10.30. On the basis of a triple decrement table, display an expression for the probability that (20) will not terminate before age 65 for cause 2.

10.31. a. You are given $q_x'^{(1)}, q_x'^{(2)}, m_x^{(3)}, m_x^{(4)}$. How would you proceed to construct a multiple decrement table where active service of an employee group is subject to decrement from death, 1, withdrawal, 2, disability, 3, and retirement, 4?
b. On the basis of the table in (a), give an expression for the probability that, in the future, an active member age y will not retire but will terminate from service for some other cause.

10.32. Prove and interpret the relation

$$q_x^{(j)} = q_x'^{(j)} - \sum_{k \neq j} \int_0^1 {}_tp_x^{(\tau)}\ \mu_x^{(k)}(t)\ {}_{1-t}q_{x+t}'^{(j)}\ dt.$$

10.33. Let

$$w^{(\tau)}(t) = \frac{{}_tp_x^{(\tau)}}{\int_0^1 {}_tp_x^{(\tau)}\,dt}$$

and

$$w^{(j)}(t) = \frac{{}_tp_x'^{(j)}}{\int_0^1 {}_tp_x'^{(j)}\,dt} \qquad 0 \le t \le 1.$$

Assume that j and at least one other cause have positive forces of decrement on the interval $0 \le t \le 1$.

a. Show that
   (i) $w^{(\tau)}(0) > w^{(j)}(0)$
   (ii) $w^{(\tau)}(1) < w^{(j)}(1)$
   (iii) There exists a unique number r, $0 < r < 1$, such that $w^{(\tau)}(r) = w^{(j)}(r)$.

b. Let

$$-I = \int_0^r [w^{(j)}(t) - w^{(\tau)}(t)]dt.$$

   Show that

$$I = \int_r^1 [w^{(j)}(t) - w^{(\tau)}(t)]dt.$$

c. Assume that $\mu_x^{(j)}(t)$ is an increasing function on the interval $0 \le t \le 1$. Use the mean value theorem for integrals to establish the following inequalities:

$$\begin{aligned} m_x'^{(j)} - m_x^{(j)} &= \int_0^1 [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\,dt \\ &= \int_0^r [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\,dt \\ &\quad + \int_r^1 [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\,dt \\ &= -\mu_x^{(j)}(t_0)\,I + \mu_x^{(j)}(t_1)\,I \qquad 0 < t_0 < r < t_1 < 1 \\ &= I\,[\mu_x^{(j)}(t_1) - \mu_x^{(j)}(t_0)] > 0. \end{aligned}$$

10.34. The joint distribution of T and J is specified by

$$\left.\begin{aligned} \mu_x^{(1)}(t) &= \frac{\theta t^{\alpha-1}\, e^{-\beta t}}{\displaystyle\int_t^{\infty} s^{\alpha-1}\, e^{-\beta s} ds} \\ \mu_x^{(2)}(t) &= \frac{(1-\theta) t^{\alpha-1}\, e^{-\beta t}}{\displaystyle\int_t^{\infty} s^{\alpha-1}\, e^{-\beta s}\, ds} \end{aligned}\right\} \begin{aligned} &0 < \theta < 1 \\ &\alpha > 0 \\ &\beta > 0 \\ &t \geq 0. \end{aligned}$$

a. Obtain expressions for $f_{T,J}(t, j)$, $f_J(j)$, and $f_T(t)$.
b. Express $E[T]$ and $Var(T)$ in terms of α and β.
c. Confirm that J and T are independent.

11

APPLICATIONS OF MULTIPLE DECREMENT THEORY

11.1 Introduction

The multiple decrement model developed in Chapter 10 provides a framework for studying many financial security systems. For example, life insurance policies frequently provide for special benefits if death occurs by accidental means or if the insured becomes disabled. The single decrement model, the subject of Chapters 3 through 9, does not provide a mathematical model for policies with such multiple benefits. In addition, there may be nonforfeiture benefits that are paid when the insured withdraws from the set of premium-paying policyholders. The determination of the amount of these nonforfeiture benefits, and related public policy issues, are discussed in Chapter 16. The basic models associated with these multiple benefits are developed in this chapter.

Another major application of multiple decrement models is in pension plans. In this chapter we consider basic methods used in calculating the actuarial present values of benefits and contributions for a participant in a pension plan. The participants of a plan may be a group of employees of a single employer, or they may be the employees of a group of employers engaged in similar activities. A plan, upon a participant's retirement, typically provides pensions for age and service or for disability. In case of withdrawal from employment, there can be a return of accumulated participant contributions or a deferred pension. For death occurring before the other contingencies, there can be a lump sum or income payable to a beneficiary. Payments to meet the costs of the benefits are referred to as contributions, not premiums as for insurance, and are payable in various proportions by the participants and the plan sponsor.

A pension plan can be regarded as a system for purchasing deferred life annuities (payable during retirement) and certain ancillary benefits with a temporary annuity of contributions during active service. The balancing of the actuarial present values of benefits and contributions may be on an individual basis, but frequently it is on some aggregate basis for the whole group of participants. Methods to accomplish this balance comprise the theory of pension funding. Here we are concerned with only the separate valuation of the pension plan's actuarial present value of benefits and contributions with respect to a typical participant. Aggregate values can then be obtained by summation over all the participants. The basic tools for valuing the benefits of, and the contributions to, a pension plan are presented here, but their application to the possible funding methods for a plan is deferred to Chapter 20.

In Section 11.6 we study disability benefits commonly found in conjunction with individual life insurance. The benefits include those for waiver of premium and for disability income. There is a discussion of a widely used single decrement approximation for calculating benefit premiums and benefit reserves for these disability coverages.

11.2 Actuarial Present Values and Their Numerical Evaluation

Actuarial applications of multiple decrement models arise when the amount of benefit payment depends on the mode of exit from the group of active lives. We let $B_{x+t}^{(j)}$ denote the value of a benefit at age $x + t$ incurred by a decrement at that age by cause j. Then the actuarial present value of the benefits, denoted in general by $\bar{A}$, will be given by

$$\bar{A} = \sum_{j=1}^{m} \int_0^{\infty} B_{x+t}^{(j)} \, v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt. \tag{11.2.1}$$

If $m = 1$ and $B_{x+t}^{(1)} = 1$, $\bar{A}$ reduces to $\bar{A}_x$, the actuarial present value for a unit of whole life insurance with immediate payment of claims.

More appropriate for this chapter is the example of a ***double indemnity provision,*** which provides for the death benefit to be doubled when death is caused by accidental means. Let $J = 1$ for death by accidental means and $J = 2$ for death by other means, and take $B_{x+t}^{(1)} = 2$ and $B_{x+t}^{(2)} = 1$. The actuarial present value for an n-year term insurance is given by

$$\bar{A} = 2 \int_0^n v^t \, {}_tp_x^{(\tau)} \mu_x^{(1)}(t) \, dt + \int_0^n v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(2)}(t) \, dt. \tag{11.2.2}$$

For numerical evaluation, the first step is to break the expression into a set of integrals, one for each of the years involved. For the first integral,

$$2 \int_0^n v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t) \, dt = 2 \sum_{k=0}^{n-1} v^k \, {}_kp_x^{(\tau)} \int_0^1 v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(1)}(k + s) \, ds.$$

If now we assume, as for (10.5.11), that each decrement in the multiple decrement context has a uniform distribution in each year of age, we have

$$2\int_0^n v^t\,{}_tp_x^{(\tau)}\,\mu_x^{(1)}(t)\,dt = 2\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}\int_0^1 (1+i)^{1-s}\,ds$$
$$= \frac{2i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}.$$

Applying a similar argument for the second integral and combining, we get

$$\bar{A} = \frac{i}{\delta}\left[\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}(2q_{x+k}^{(1)} + q_{x+k}^{(2)})\right]$$
$$= \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}\,(q_{x+k}^{(1)} + q_{x+k}^{(\tau)})$$
$$= \bar{A}^{1\,(1)}_{x:\overline{n}|} + \bar{A}^{1}_{x:\overline{n}|}, \tag{11.2.3}$$

where $\bar{A}^{1\,(1)}_{x:\overline{n}|}$ is the actuarial present value of term insurance benefits of 1 covering death from accidental means and $\bar{A}^{1}_{x:\overline{n}|}$ is the actuarial present value for term insurance benefits of 1 covering death from all causes. Here ${}_kp_x^{(\tau)}$ could be taken as the survival function from a mortality table. If values of $q_{x+k}^{(1)}$ are available, it would be unnecessary to develop the full double decrement table in order to calculate (11.2.3) under the assumption that each decrement has a uniform distribution in each year of age.

This example is simple because the benefit amount does not change as a function of age at decrement, and, in particular, it does not change within a year of age. For a contrasting example, we take $B_{x+t}^{(1)} = t$ and $B_{x+t}^{(2)} = 0$ for $t > 0$. In this case,

$$\bar{A} = \int_0^\infty t\,v^t\,{}_tp_x^{(\tau)}\,\mu_x^{(1)}(t)\,dt = \sum_{k=0}^{\infty} v^k\,{}_kp_x^{(\tau)}\int_0^1 (k+s)v^s\,{}_sp_{x+k}^{(\tau)}\,\mu_x^{(1)}(k+s)\,ds.$$

We again make the assumption that each decrement in the multiple decrement context has a uniform distribution in each year of age, and we obtain

$$\bar{A} = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}\int_0^1 (k+s)(1+i)^{1-s}\,ds$$
$$= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(1)}\,\frac{i}{\delta}\left(k + \frac{1}{\delta} - \frac{1}{i}\right). \tag{11.2.4}$$

In practice, $B_{x+t}^{(j)}$ is often a complicated function, possibly requiring some degree of approximation. For such a case, if we apply the uniform distribution assumption to the j-th integral in (11.2.1), we obtain

$$\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(j)}\int_0^1 B_{x+k+s}^{(j)}\,(1+i)^{1-s}\,ds.$$

Then, use of the midpoint integration rule yields

$$\sum_{k=0}^{\infty} v^{k+1/2}\,{}_kp_x^{(\tau)}\,q_{x+k}^{(j)}\,B_{x+k+1/2}^{(j)} \tag{11.2.5}$$

as a practical formula for the evaluation of the integral.

As an example, we return to (11.2.4) where the quantity

$$k + \frac{1}{\delta} - \frac{1}{i}$$

can be viewed as an effective mean benefit amount for the year $k + 1$, and the familiar i/δ term can be viewed as the correction needed to provide immediate payment of claims. The value given by (11.2.4) is closely approximated by

$$\sum_{k=0}^{\infty} v^{k+1/2} \, {}_kp_x^{(\tau)} \, q_{x+k}^{(j)} \left(k + \frac{1}{2}\right), \tag{11.2.6}$$

which makes use of the midpoint rule for approximate integration to evaluate

$$\int_0^1 (k + s)(1 + i)^{1-s} \, ds.$$

In Section 10.6 we discussed situations where a uniform distribution of decrement assumption was not appropriate. For such situations, special adjustments to the actuarial present value should be made. We reexamine Example 10.6.3 where, in the associated single decrement model for decrement (3), one-half the expected withdrawals occur at midyear and the other half occur at year end. The actuarial present value for withdrawal benefits is given by

$$\begin{aligned} \bar{A} = \sum_{k=0}^{\infty} v^k \, {}_kp_x^{(\tau)} \bigg[& \frac{1}{2} q'^{(3)}_{x+k} \, v^{1/2} \, B^{(3)}_{x+k+1/2} \left(1 - \frac{1}{2} q'^{(1)}_{x+k}\right)\left(1 - \frac{1}{2} q'^{(2)}_{x+k}\right) \\ & + \frac{1}{2} q'^{(3)}_{x+k} \, v \, B^{(3)}_{x+k+1} \, (1 - q'^{(1)}_{x+k})(1 - q'^{(2)}_{x+k}) \bigg]. \end{aligned}$$

Here we are dealing with the distribution of decrement in the context of the associated single decrement tables, rather than in the multiple decrement context. A possible approximation would be to use a geometric average value of the interest factor in the year of withdrawal, such as $v^{3/4}$, and the arithmetic average value of the withdrawal benefit, such as

$$\hat{B}^{(3)}_{x+k} = \frac{1}{2}\left(B^{(3)}_{x+k+1/2} + B^{(3)}_{x+k+1}\right).$$

Thus,

$$\begin{aligned} \bar{A} & \cong \sum_{k=0}^{\infty} v^{k+3/4} \, {}_kp_x^{(\tau)} \, \hat{B}^{(3)}_{x+k} \bigg[\frac{1}{2} q'^{(3)}_{x+k} \left(1 - \frac{1}{2} q'^{(1)}_{x+k}\right)\left(1 - \frac{1}{2} q'^{(2)}_{x+k}\right) \\ & \qquad + \frac{1}{2} q'^{(3)}_{x+k} \, (1 - q'^{(1)}_{x+k})(1 - q'^{(2)}_{x+k}) \bigg] \\ & = \sum_{k=0}^{\infty} v^{k+3/4} \, {}_kp_x^{(\tau)} \, q^{(3)}_{x+k} \, \hat{B}^{(3)}_{x+k}. \end{aligned}$$

Remark:

In this section we have not used the format employed in Chapter 6 to state premium determination problems. This was done to achieve brevity. The premium problems of this section could have been approached by formulating a loss function and invoking the equivalence principle or some other premium principle.

Assume, for example, an insurance to (x) paying

a. $2B$ upon death due to an accident before age r

b. B upon death due to all other causes before age r, and

c. B upon death after age r.

d. a single premium payment

Two causes of decrement are recognized, $J = 1$, the accidental cause, and $J = 2$, the nonaccidental cause. The loss function is

$$L = \begin{cases} 2Bv^T - \pi & J = 1 \quad 0 < T \le r - x \\ Bv^T - \pi & J = 2 \quad 0 < T \le r - x \\ Bv^T - \pi & J = 1, 2 \quad T > r - x. \end{cases}$$

The equivalence principle requires that $E[L] = 0$, or

$$\pi = B\left[\int_0^{r-x} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t) \, dt + \int_0^{\infty} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t) \, dt\right].$$

A measure of the dispersion due to the random natures of time and cause of death is provided by $\mathrm{Var}(L) = E[L^2]$. One can verify that, for this case,

$$\mathrm{Var}(L) = B^2\left[3\int_0^{r-x} v^{2t} \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t) \, dt + \int_0^{\infty} v^{2t} \, {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t) \, dt\right] - \pi^2.$$

In the general case, with actuarial present value given by (11.2.1), we have

$$\mathrm{Var}(L) = E[L^2] = \sum_{j=1}^{m} \int_0^{\infty} (B_{x+t}^{(j)} \, v^t - \bar{A})^2 \, {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt,$$

which can be reduced to

$$\mathrm{Var}(L) = \sum_{j=1}^{m} \int_0^{\infty} (B_{x+t}^{(j)} \, v^t)^2 \, {}_tp_x^{(\tau)} \, \mu_{x(t)}^{(j)} \, dt - (\bar{A})^2. \tag{11.2.7}$$

11.3 Benefit Premiums and Reserves

We examine, in this section, a method of paying for benefits included in a life insurance policy in a multiple decrement setting. Often these extra benefits are included in life insurance contracts on a policy rider basis; that is, a specified extra premium is charged for the extra benefit, and a separate reserve is held for this benefit. The extra premium is payable only for as long as the benefit has value. In the case of double indemnity it is common to pay the extra amount only for death from accidental means before a specific age, such as 65, and thus the specified extra premiums would be payable only until that age.

We henceforth consider the double indemnity benefit as such a benefit. The model is not complete because the possibility of withdrawal, with a corresponding withdrawal benefit, is not included. We consider withdrawal benefits in Section 11.4, but most of this subject is discussed in Chapters 15 and 16.

Consider the fully discrete model for a whole life policy to a person age 30 with a double indemnity rider. The benefit amount is one for nonaccidental death, decrement $J = 1$, and is two for death by accidental means, decrement $J = 2$. The principle of equivalence is now applied twice, once for the premium payable for life for the policy without the rider and once for the premium payable to age 65 for the extra benefit payable on accidental death before age 65.

For the policy without the rider the benefit level is one under either decrement 1 or 2 and the premium is payable for life. Thus

$$P_{30}^{(\tau)} = \frac{\sum_{k=0}^{\infty} v^{k+1}\ {}_kp_{30}^{(\tau)}\ q_{30+k}^{(\tau)}}{\sum_{k=0}^{\infty} v^k\ {}_kp_{30}^{(\tau)}}. \tag{11.3.1}$$

The benefit premium for the rider reflects that the premium is payable through age 64, and its benefit amount, payable under decrement 2 only, is unity. It is given by

$$ {}_{35}P_{30}^{(2)} = \frac{\sum_{k=0}^{34} v^{k+1}\ {}_kp_{30}^{(\tau)}\ q_{30+k}^{(2)}}{\sum_{k=0}^{34} v^k\ {}_kp_{30}^{(\tau)}}. \tag{11.3.2}$$

We now display the benefit reserve for the policy with the rider for years prior to attaining age 65:

$$\begin{aligned} {}_kV = {} & \sum_{h=0}^{\infty} v^{h+1}\ {}_hp_{30+k}^{(\tau)}\ q_{30+k+h}^{(\tau)} + \sum_{h=0}^{34-k} v^{h+1}\ {}_hp_{30+k}^{(\tau)}\ q_{30+k+h}^{(2)} \\ & - \left(P_{30}^{(\tau)} \sum_{h=0}^{\infty} v^h\ {}_hp_{30+k}^{(\tau)} + {}_{35}P_{30}^{(2)} \sum_{h=0}^{34-k} v^h\ {}_hp_{30+k}^{(\tau)} \right). \end{aligned}$$

This reserve is the sum of a reserve on the base policy plus a reserve on a policy that pays only on failure through decrement 2. The reserve on the base policy is a Chapter 7 benefit reserve for a fully discrete whole life insurance with $q_x = q_x^{(\tau)} =$ $\cancel{q_x^{(1)} + p_x^{(2)}}$. $q_x^{(1)} + q_x^{(1)}$.

11.4 Withdrawal Benefit Patterns That Can Be Ignored in Evaluating Premiums and Reserves

A single decrement model for an individual life insurance benefit with annual premiums and reserves was built in Chapters 6 through 8. In that model the timing and, perhaps, the amount of benefit payments are determined by the time of death of the insured, and premiums are paid until death or the end of the premium period as specified in the policy. In practice, there is no way to prevent the cessation of premium payments by the policyholder before death or the end of the premium period. In this situation an issue arises about how to reconcile the interests of the parties to the policy for which a model derived from multiple decrement theory is appropriate. Public policy considerations that should guide the reconciliation of the interests of the insurance system and the terminating insured have been subject to discussion since the early days of insurance.

Before premiums and reserves can be determined, a guiding principle must be adopted. A guiding principle is required as well in the determination of ***nonforfeiture benefits,*** those benefits that will not be lost because of the premature cessation of premium payments. In this section we adopt a simple operational principle, one that is, in effect, close to that adopted in U.S. insurance regulation. The principle is that the withdrawing insured receives a value such that the benefit, premium, and reserve structure, built using the single decrement model, remains appropriate in the multiple decrement context.

This principle is motivated by a particular concept of equity about the treatment of the two classes of policyholders, those who terminate before the basic insurance contract is fulfilled and those who continue. Clearly several concepts of what constitutes equity are possible, ranging from the view that terminating policyholders have not fulfilled the contract, and are therefore not entitled to nonforfeiture benefits, to the view that a terminating policyholder should be returned to his original position by the return of the accumulated value of all premiums, perhaps less an insurance charge. The concept of equity, which is the foundation of the principle adopted in the United States, is an intermediate one; that is, withdrawing life insurance policyholders are entitled to nonforfeiture benefits, but these benefits should not force a change in the price-benefit structure for continuing policyholders.

To illustrate some of the implications of this principle, we will develop a model for a whole life policy on a fully continuous payment basis with death and withdrawal benefits. The force of withdrawal is denoted by $\mu_x^{(2)}(t)$ with $\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t)$. For multiple decrement models, it is required that

$$\int_0^\infty \mu_{x+t}^{(\tau)}\, dt = \infty$$

so that

$$\lim_{t\to\infty} {}_tp_x^{(\tau)} = 0,$$

but it is not necessary for $\mu_x^{(2)}(t)$ and the derived ${}_tp_x^{\prime(2)}$ to have these properties.

We assume that the introduction of withdrawals into the model does not change the force of mortality, which is labeled for this development $\mu_x^{(1)}(t)$ in both the single and double decrement models. In other words, time-until-death and time-until-withdrawal will be assumed to be independent, but this assumption may not be realized in practice. This issue was discussed in Chapter 10.

We start our model by specializing (8.6.4) to the case of a whole life insurance and single decrement premiums and reserves:

$$\frac{d}{dt}\,{}_t\bar{V}(\bar{A}_x) = \bar{P}(\bar{A}_x) + \delta\,{}_t\bar{V}(\bar{A}_x) - \mu_x^{(1)}(t)\,[1 - {}_t\bar{V}(\bar{A}_x)]. \tag{11.4.1}$$

Recalling from Section 10.2 that

$$\frac{d}{dt}\,{}_tp_x^{(\tau)} = -{}_tp_x^{(\tau)}[\mu_x^{(1)}(t) + \mu_x^{(2)}(t)],$$

we can express the following derivative as

$$\begin{aligned}\frac{d}{dt}\,[v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)] &= v^t\,{}_tp_x^{(\tau)}\{\bar{P}(\bar{A}_x) + \delta\,{}_t\bar{V}(\bar{A}_x) - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)]\} \\ &\quad - v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)[\delta + \mu_x^{(1)}(t) + \mu_x^{(2)}(t)] \\ &= v^t\,{}_tp_x^{(\tau)}[\bar{P}(\bar{A}_x) - \mu_x^{(1)}(t) - \mu_x^{(2)}(t)\,{}_t\bar{V}(\bar{A}_x)]. \end{aligned} \tag{11.4.2}$$

The progress of the reserves for a whole life insurance that includes withdrawal benefit ${}_tV(A_x)$ using premiums and reserves derived from a double decrement model is analogous to (11.4.1) and is shown in (11.4.3). In this expression, the superscript $\underline{2}$ denotes premiums and reserves based on the double decrement model:

$$\begin{aligned}\frac{d}{dt}\,[{}_t\bar{V}(\bar{A}_x)^{\underline{2}}] &= \bar{P}(\bar{A}_x)^{\underline{2}} + \delta\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}} \\ &\quad - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}] - \mu_x^{(2)}(t)[{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}]. \end{aligned} \tag{11.4.3}$$

The last term in (11.4.3) is the net cost of withdrawal when the reserve ${}_tV(A_x)^{\underline{2}}$ is treated as a savings fund available to offset benefits [see (8.4.5)]. Thus,

$$\begin{aligned}\frac{d}{dt}\,[v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}}] &= v^t\,{}_tp_x^{(\tau)}\,\{\bar{P}(\bar{A}_x)^{\underline{2}} + \delta\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}} - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}] \\ &\quad - \mu_{x+t}^{(2)}[{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^{\underline{2}}]\} \\ &\quad - v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)^{\underline{2}}\,[\delta + \mu_x^{(1)}(t) + \mu_x^{(2)}(t)] \\ &= v^t\,{}_tp_x^{(\tau)}[\bar{P}(\bar{A}_x)^{\underline{2}} - \mu_x^{(1)}(t) - \mu_x^{(2)}(t)\,{}_t\bar{V}(\bar{A}_x)]. \end{aligned} \tag{11.4.4}$$

Combining (11.4.2) and (11.4.4), we obtain

$$\frac{d}{dt}\,\{v^t\,{}_tp_x^{(\tau)}[{}_t\bar{V}(\bar{A}_x)^{\underline{2}} - {}_t\bar{V}(\bar{A}_x)]\} = v^t\,{}_tp_x^{(\tau)}[\bar{P}(A_x)^{\underline{2}} - \bar{P}(\bar{A}_x)]. \tag{11.4.5}$$

We now integrate (11.4.5) from $t = 0$ to $t = \infty$ to obtain

$$0 = \bar{a}_x^{(\tau)}[\bar{P}(\bar{A}_x)^{\underline{2}} - \bar{P}(\bar{A}_x)], \tag{11.4.6}$$

which implies that

$$\bar{P}(\bar{A}_x)^2 = \bar{P}(\bar{A}_x).$$

Thus (11.4.5) reduces to

$$\frac{d}{dt}\{v^t\,{}_tp_x^{(\tau)}[{}_t\bar{V}(\bar{A}_x)^2 - {}_t\bar{V}(\bar{A}_x)]\} = 0,$$

which, with the initial condition that

$${}_0\bar{V}(\bar{A}_x)^2 = {}_0\bar{V}(\bar{A}_x),$$

implies that

$${}_t\bar{V}(\bar{A}_x)^2 = {}_t\bar{V}(\bar{A}_x) \qquad \text{for all } t \ge 0. \tag{11.4.7}$$

Therefore, if the withdrawal benefit in a double decrement model whole life insurance, fully continuous payment basis, is equal to the reserve under the single decrement model, the premium and reserves under the double decrement model are equal to the premium and reserves under the single decrement model. This result is not directly applied to the practical problem of defining nonforfeiture benefits. However, it does suggest the basic idea of how to minimize the impact of withdrawal or nonforfeiture benefits on premiums and reserves (determined under a single decrement model). These ideas are developed further in Chapter 16.

The ideas of this section are closely related to Example 6.6.2, where it was demonstrated that if the death benefit during the premium-paying period for a deferred life annuity is the accumulated value of the premiums, then the premium does not depend on the mortality assumption during the deferral period. This idea is elaborated in Example 11.4.1.

Example 11.4.1

A continuously paid life annuity issued on (x) provides an income benefit commencing at age $x + n$ at an annual rate of 1. The benefit for death (decrement $J = 1$) or withdrawal (decrement $J = 2$) during the n-year deferral period, paid at the moment of death, will be the accumulated benefit premiums with interest at the rate used in the premium calculation. Premiums are paid continuously from age x to $x + n$ or to the age of decrement, if less than $x + n$.

a. Formulate a loss variable.
b. Determine the annual benefit premium rate π using the principle of equivalence.
c. Determine the benefit reserve at time t, $0 \le t \le n$.

Solution:

a.
$$L = \begin{cases} \pi\, v^T\, \bar{s}_{\overline{T}|} - \pi\, \bar{a}_{\overline{T}|} & 0 \le T \le n, \quad J = 1, 2 \\ v^n\, \bar{a}_{\overline{T-n}|} - \pi\, \bar{a}_{\overline{n}|} & T > n, \qquad\quad J = 1. \end{cases}$$

b. Applying the principle of equivalence, we obtain

$$\mathrm{E}[L] = \int_n^\infty (v^n\, \bar{a}_{\overline{t-n}|} - \pi\, \bar{a}_{\overline{n}|})\, {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt.$$

This yields

$$v^n\,{}_np_x^{(\tau)}\,\bar{a}_{x+n} = \pi\,\bar{a}_{\overline{n}|}\,{}_np_x^{(\tau)} \quad\text{and}\quad \pi = \frac{v^n\,\bar{a}_{x+n}}{\bar{a}_{\overline{n}|}} = \frac{\bar{a}_{x+n}}{\bar{s}_{\overline{n}|}}.$$

c. The reserve at time t, $t \le n$, viewed prospectively, is given by

$$\int_0^{n-t} (\pi\,v^s\,\bar{s}_{\overline{t+s}|} - \pi\,\bar{a}_{\overline{s}|})\,{}_sp_{x+t}^{(\tau)}\,\mu_x^{(\tau)}(t+s)\,ds$$

$$+ \int_{n-t}^{\infty} (v^{n-t}\,\bar{a}_{\overline{s-(n-t)}|} - \pi\,\bar{a}_{\overline{n-t}|})\,{}_sp_{x+t}^{(\tau)}\,\mu_x^{(1)}(t+s)\,ds$$

$$= \pi\,\bar{s}_{\overline{t}|}(1 - {}_{n-t}p_{x+t}^{(\tau)}) + {}_{n-t|}\bar{a}_{x+t} - \pi\,\bar{a}_{\overline{n-t}|}\,{}_{n-t}p_{x+t}^{(\tau)}$$

$$= \pi\,\bar{s}_{\overline{t}|}.$$

The simplification of the last term comes from the definition of π in part (b). The benefit premium and reserve during the deferred period can be viewed as derived from a zero decrement model. ▼

11.5 Valuation of Pension Plans

Two sets of assumptions are needed to determine the actuarial present values of pension plan benefits and of contributions to support these benefits. These sets can be identified as demographic (the service table and survival functions for retired lives, disabled lives, and perhaps lives who have withdrawn) and economic (investment return and salary scale) assumptions.

11.5.1 Demographic Assumptions

A starting point for the valuation of pension plan benefits is a multiple decrement (service) table constructed to represent a survivorship group of participants subject, in the various years of active service, to given probabilities of

- Withdrawal from service
- Death in service
- Retirement for disability, and
- Retirement for age-service.

The notations for these probabilities for the year of age x to $x+1$ are $q_x^{(w)}$, $q_x^{(d)}$, $q_x^{(i)}$, and $q_x^{(r)}$, respectively. These are consistent with the notations developed in Chapter 10. Also, we use the survivorship function $l_x^{(\tau)}$ from Chapter 10, which satisfies

$$l_{x+1}^{(\tau)} = l_x^{(\tau)}[1 - (q_x^{(w)} + q_x^{(d)} + q_x^{(i)} + q_x^{(r)})] = l_x^{(\tau)}\,p_x^{(\tau)}.$$

This function can be used to evaluate such expression as ${}_kp_x^{(\tau)}$, thus,

$${}_kp_x^{(\tau)} = \frac{l_{x+k}^{(\tau)}}{l_x^{(\tau)}}.$$

One can also proceed by direct recursion, namely,

$$_{k}p_x^{(\tau)} = {}_{k-1}p_x^{(\tau)}\, p_{x+k-1}^{(\tau)}.$$

The forces of decrement related to a service table will be continuous at most ages. They will be denoted by $\mu_x^{(w)}(t)$, $\mu_x^{(d)}(t)$, $\mu_x^{(i)}(t)$, and $\mu_x^{(r)}(t)$. At some ages, discontinuities may occur. This occurs most frequently at age α, the first eligible age for retirement. We generally assume that decrements are spread across each year of age.

In the early years of service, withdrawal rates tend to be high, and the benefit for withdrawal may be only the participant's contributions, if any, possibly accumulated with interest. After a period of time, for example, 5 years, withdrawal rates will be somewhat lower, and the withdrawing participant may be eligible for a deferred pension. If these conditions hold, it may be necessary to use select rates of withdrawal for an appropriate number of years. Conditions for disability retirement may also indicate a need for a select basis. The mathematical modifications to a select basis are relatively easy to make, and the theory is more adaptable if select functions are used. In this chapter we denote the age of entry by x, but we do not otherwise indicate whether an aggregate table, select table, or select-and-ultimate table is intended.

The Illustrative Service Table in Appendix 2B illustrates a service table for entry age 30, earliest age for retirement $\alpha = 60$, and with no probability of active service beyond age 71. Here $l_{71}^{(\tau)} = 0$.

As noted earlier, the principal benefits under a pension plan are annuities to eligible beneficiaries. For the valuation of such annuity benefits, it is necessary to adopt appropriate mortality tables that will differ if retirement is for disability, for age-service, or perhaps withdrawal. The corresponding annuity values will be indicated by post-fixed superscripts. The continuous annuity value is used as a convenient means of approximating the actual form of pension payment that usually is monthly, but may have particular conditions as to initial and final payments.

11.5.2 Projecting Benefit Payment and Contribution Rates

A common form of pension plan is one that defines the rate of retirement income by formula. These plans are called ***defined benefit plans***. Some pension plans define benefit income rates as a function of the level of compensation at or near retirement. In these cases, it is necessary to estimate future salaries to value the benefits. Sponsor contributions are also often expressed as a percentage of salary, so here too estimation of future salaries is important. To accomplish these estimations, we define the following salary functions:

$(AS)_{x+h}$ is the actual annual salary rate at age $x + h$, for a participant who entered at age x and is now at attained age $x + h$,

$(ES)_{x+h+t}$ is the projected (estimated) annual salary rate at age $x + h + t$.

Further, we assume that we have a salary scale function S_y to use for these projections, such that

$$(ES)_{x+h+t} = (AS)_{x+h} \frac{S_{x+h+t}}{S_{x+h}}. \tag{11.5.1}$$

The salary functions S_y may reflect merit and seniority increases in salary as well as those caused by inflation. For example, in the Illustrative Service Table, $S_y = (1.06)^{y-30} s_y$, where the s_y factor represents the progression of salary due to individual merit and experience increases, and the 6% accumulation factor is to allow for long-term effects of inflation and of increases in productivity of all members of the plan. As was the case of the $l_x^{(\tau)}$ function, one of the values of S_y can be chosen arbitrarily. For instance, in the Illustrative Service Table, S_{30} is taken as unity. The S_y function is usually assumed to be a step function, with constant level throughout any given year of age.

We now move to the problem of estimating the benefit level for a pension plan. For this purpose, we introduce the function $R(x, h, t)$ to denote the projected annual income benefit rate to commence at age $x + h + t$ for a participant, who entered h years ago at age x. Both x and h are assumed to be integers. We assume that the income benefit rate remains level during payout so that when we come to expressing the actuarial present value of the benefit at time of retirement, it will simply be $R(x, h, t)\bar{a}^r_{x+h+t}$. As stated in the previous section, the post-fixed superscript r indicates that a mortality table appropriate for retired lives should be used.

We now consider several common types of income benefit rate functions $R(x, h, t)$. The estimation procedure falls into two groups. First, there are functions that do not depend on salary levels. For other types of benefit formulas, which depend on future salaries, the projected annual income rate must be estimated. There are those that depend on either the final salary rate or on an average salary rate over the last several years prior to retirement. There are also formulas that depend on the average salary over the career with the plan sponsor. The following are examples of the more common types of benefit formula together with their estimation.

a. Consider an income benefit rate that is a fraction d of the final salary rate. Thus $R(x, h, t) = d(ES)_{x+h+t}$. Here we estimate the final salary from the current salary at age $x + h$ by $(ES)_{x+h+t} = (AS)_{x+h}(S_{x+h+t} / S_{x+h})$ so that $R(x, h, t) = d(AS)_{x+h}(S_{x+h+t} / S_{x+h})$.
b. A ***final m-year average salary benefit*** rate is a fraction d of the average salary rate over the last m years prior to retirement. We illustrate this in the common case where $m = 5$. In this case, if $t > 5$, an estimate of the average salary over the last 5 years is given as

$$(AS)_{x+h} \frac{0.5\, S_{x+h+k-5} + S_{x+h+k-4} + S_{x+h+k-3} + S_{x+h+k-2} + S_{x+h+k-1} + 0.5\, S_{x+h+k}}{5\, S_{x+h}}$$

where k is the greatest integer in t.

The thinking behind this expression is that if retirement occurs at midyear, the current year's salary is earned only for the last half year of service. A notation in common usage for the above average is ${}_5Z_{x+h+k} / S_{x+h}$. If the participant is within 5 years of possible retirement, account could be taken of actual rather than projected salaries.

The above formulas do not reflect the amount of service of the participant at retirement. We now look at three formulas where the benefits are proportional to the number of years of service at retirement.

c. Consider an income benefit that is d times the total number of years of service, including any fraction in the final year of employment. In this case $R(x, h, t) = d(h + t)$. If only whole years of service are to be counted, then $R(x, h, t) = d(h + k)$, where k is the greatest integer in t.
d. Consider an income benefit rate that is the product of a fraction d of the final 5-year average salary and the number of years of service at retirement. A typical formula would be, where d is a designated fraction,

$$= d(h + t)(AS)_{x+h} \frac{{}_5Z_{x+h+k}}{S_{x+h}}.$$

Again, if the participant is within 5 years of possible retirement, account could be taken of actual rather than projected salaries.
e. Consider an income benefit rate that is d times the number of years of service times the average salary over the entire career. Such a benefit formula is called a ***career average benefit.*** This formula is equivalent to a benefit rate of a fraction d of the entire career earnings of the retiree.

The analysis of career average retirement benefits breaks naturally into two parts, one for past service for which the salary information is known and one for future service where salaries must be estimated. Here past salaries enter into the valuation of benefits for all participants and not just for those participants very near to retirement age. If the total of past salaries for a participant at age $x + h$ is denoted by $(TPS)_{x+h}$, the benefit rate attributed to past service is $d(TPS)_{x+h}$. The retirement income benefit rate based on future service is given by

$$d(AS)_{x+h} \frac{S_{x+h} + S_{x+h+1} + \cdots + S_{x+h+k-1} + 0.5\, S_{x+h+k}}{S_{x+h}},$$

where k is the greatest integer in t and retirements are assumed to occur at midyear.

Finally, we display one benefit formula where the service component for participants with a large number of years of service at retirement is modified.

f. Consider an income benefit rate that is the product of the 3-year final average salary and 0.02 times the number of years of service at retirement for the first 30 years of service with an additional 0.01 per year of service above 30 years. Following (d),

$$\begin{aligned} &= 0.02\,(h + t)\,(AS)_{x+h} \frac{{}_3Z_{x+h+k}}{S_{x+h}} && h + t \le 30 \\ &= [0.30 + 0.01\,(h + t)]\,(AS)_{x+h} \frac{{}_3Z_{x+h+k}}{S_{x+h}} && h + t > 30. \end{aligned}$$

11.5.3 Defined-Benefit Plans

We now seek to develop formulas for actuarial present values of such benefits and of the contributions expected to be used to fund the promised benefits. We do so first for a general case of a defined-benefit plan and then examine a specific example that includes a typical pattern of defined benefits.

Let us first look at the evaluation of, and approximation to, the actuarial present value for an age-retirement benefit. Assume that the benefit rate function has been found as $R(x, h, t)$ and that the benefit involves life annuities with no certain period. We can then write an integral expression for the actuarial present value of the retirement benefit as

$$\text{APV} = \int_{\alpha-x-h}^{\infty} v^t \, {}_tp_{x+h}^{(\tau)} \, \mu_x^{(r)}(h + t) \, R(x, h, t) \, \bar{a}_{x+h+t}^{r} \, dt. \tag{11.5.2}$$

As in Section 11.2, we approximate the integral for practical calculation of the actuarial present value. To do so, we write

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^k \, {}_kp_{x+h}^{(\tau)} \int_0^1 v^s \, {}_sp_{x+h+k}^{(\tau)} \, \mu_x^{(r)}(h + k + s) \, R(x, h, k + s) \, \bar{a}_{x+h+k+s}^{r} \, ds.$$

By assuming a uniform distribution of retirements in each year of age, we can rewrite this as

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^k \, {}_kp_{x+h}^{(\tau)} \, q_{x+h+k}^{(r)} \int_0^1 v^s \, R(x, h, k + s) \, \bar{a}_{x+h+k+s}^{r} \, ds.$$

Using the midpoint approximation for the remaining integrals gives

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^{k+1/2} \, {}_kp_{x+h}^{(\tau)} \, q_{x+h+k}^{(r)} \, R(x, h, k + 1/2) \, \bar{a}_{x+h+k+1/2}^{r}. \tag{11.5.3}$$

Formula (11.5.3) is the general means by which we calculate the actuarial present value of retirement and, by extension, other benefits of a pension plan.

We now present an example that shows the types of calculations that might be used for the valuation of the several benefits of a hypothetical defined-benefit pension plan.

Example 11.5.1

Find the actuarial present values of the following benefits for a participant who was hired 3 years ago at age 30 and who currently has a salary of \$45,000.

a. Retirement income for any participant of at least age 65 or whenever the sum of the attained age and the number of years of service exceeds a total of 90. The benefit is in the form of a 10-year certain and life annuity, payable monthly, at an annual rate of 0.02 times the final 5-year average salary times the total number of years of service, including any final fraction.

b. Retirement income for any participant with at least 5 years of service upon withdrawal. The benefit and income benefit rate formula is as for age retirement. However, the initial payment of the annuity is deferred until the earliest possible date of age retirement had the participant continued in the active status.
c. Retirement income for those disabled participants too young for age retirement. The income benefit rate is the larger of 50% or the percentage based on years of service, uses the average salary over the preceding 5 years, and is for a 10-year certain and life annuity.
d. Lump sum benefit for those participants who die while still in active status. The benefit amount is two times the salary rate at the time of death.

Solution:

The participant was hired at age 30 and so is eligible for retirement at age 60 [60 + (60 − 30) = 90]. Assuming midyear retirements, the income benefit rate is given by

$$= 0.02\,(h + t)\,(AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}}$$

$$= 0.02\,(3 + k + 0.5)\,(45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}$$

for retirements starting in the year following age 33 + k. The actuarial present value of age-retirement benefits is approximated by

$$\text{APV} = 900 \sum_{k=27}^{\infty} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(r)}$$

$$\times\,(3.5 + k)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}\,\bar{a}^{r}_{\overline{33+k+1/2:\overline{10}|}}.$$

Benefits are paid for those withdrawing from active status between ages 35 and 60. After attaining age 60, the withdrawals are classified as retirements. The income benefit rate is again given by

$$= 0.02\,(h + t)\,(AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}}$$

$$= 0.02\,(3 + k + 0.5)\,(45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}.$$

For withdrawals at ages 35 through 37, adjustments using actual wage data rather than the Z function could be made. The actuarial present value of the withdrawal benefits is approximated by

$$\text{APV} = 900 \sum_{k=2}^{26} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(w)}$$

$$\times\,(3.5 + k)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}\,{}_{27-k-1/2|}\bar{a}^{w}_{\overline{33+k+1/2:\overline{10}|}}.$$

For the disability benefit the income benefit rate function makes a distinction between disabilities starting before and after the participant has worked for 25 years. Thus the income benefit rate is

$$= 0.5\,(AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}} = 0.5\,(45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}} \qquad \text{for } 0 \le k \le 21$$

$$= 0.02\,(h+t)\,(AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}} = 0.02\,(3+k+0.5)\,(45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}$$

$$\text{for } 22 \le k \le 26.$$

Thus, the actuarial present value of the disability benefits is approximated by

$$\text{APV} = 22{,}500 \sum_{k=0}^{21} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(i)}\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}\,\bar{a}^{\,i}_{33+k+1/2:\overline{10}|}$$

$$+\,900 \sum_{k=22}^{26} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(i)}\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}\,(3.5+k)\,\bar{a}^{\,i}_{33+k+1/2:\overline{10}|}.$$

For the death benefit, the projected lump sum benefit amount is

$$= 2\,(AS)_{x+h}\,\frac{S_{x+h+k}}{S_{x+h}}$$

$$= 2\,(45{,}000)\,\frac{S_{30+3+k}}{S_{30+3}}.$$

Thus, the actuarial present value of the death benefits is approximated by

$$\text{APV} = 90{,}000 \sum_{k=0}^{\infty} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(d)}\,\frac{S_{30+3+k}}{S_{30+3}}.$$

▼

There are many funding or budgeting methods available to assure that contributions are made to the plan in an orderly and appropriate manner. An overview of these methods is presented in Chapter 20.

11.5.4 Defined-Contribution Plans

The principal benefit under a pension plan is normally the deferred annuity for age-service retirement. In ***defined-contribution plans,*** the actuarial present value is simply the accumulation under interest of contributions made by or for the participant, and the benefit is an annuity that can be purchased by such accumulation. The accumulated amount is typically available upon death and, under certain conditions, upon withdrawal before retirement. We examine the interplay between the rate of contribution and the rate of income provided to the participant in the following example. The defined-contribution rate can be determined with a retirement income goal. The risk that the goal will not be achieved is held by the participant of the plan, not the sponsor. Budget constraints on the sponsor may, of course, restrict the amount of the contributions.

Example 11.5.2

Find the contribution level for the sponsor to provide for age-retirement at age 65 that has as its objective a 10-year certain and life annuity with an initial benefit rate of 50% of the average salary over the 5 years between ages 60 and 65. The contribution rate, which is to be applied as a proportion of salary, is calculated for a new participant at age 30. Assume that there are no withdrawal benefits for the first 5 years but after that the accumulated contributions are vested, that is, become the property of the withdrawing participant, and will be applied toward an annuity to start no earlier than age 60. (An active participant who becomes disabled is treated as a withdrawal and is covered by a separate disability income coverage for the period between the date of disability and age 65 at which time a regular age retirement commences.) Upon death after the end of the 5-year vesting period but before retirement income has commenced, the accumulated contributions are paid out.

Solution:

During the first 5 years our model contains both mortality and withdrawal decrements. The scheme of benefits in this plan after the 5-year vesting period are similar to those discussed in Example 11.4.1. Thus for the period after time 5 until the participant attains age 65, we use a zero decrement model.

Let us start by calculating the actuarial present value of a contribution rate of c times the annual salary rate, assumed for convenience to be 1, at age 30:

$$\text{APV} = c \left[\sum_{k=0}^{4} v^{k+1/2} \frac{S_{30+k}}{S_{30}} \, {}_kp_{30}^{(\tau)} \left(1 - \frac{1}{2}\, q_{30+k}^{(\tau)}\right) + {}_5p_{30}^{(\tau)} \sum_{k=5}^{34} v^{k+1/2} \frac{S_{30+k}}{S_{30}} \right]. \tag{11.5.4}$$

In (11.5.4) contributions are assumed to occur at midyear, and the projected salary rate at age $30 + k$ is (S_{30+k} / S_{30}) times 1, the initial salary at age 30. For $k = 0$ to 4, we need survivorship to time $k + (1/2)$. Since the model has no decrements after time 5, survivorship to time 5 is all that is needed for $k = 5$ to 34.

We now estimate the desired benefit payment rate at age 65 as the first step in estimating the actuarial present value of the target benefits. The average salary projected to be earned between the ages of 60 and 65 is $(S_{60} + S_{61} + S_{62} + S_{63} + S_{64}) / (5\, S_{30})$. The desired benefit rate is one-half of this, and the actuarial present value of the target benefit is given by

$$\text{APV} = {}_5p_{30}^{(\tau)}\, v^{35}\, (0.5)\, \frac{S_{60} + S_{61} + S_{62} + S_{63} + S_{64}}{5\, S_{30}}\, \bar{a}^{r}_{\overline{65:\overline{10|}}}. \tag{11.5.5}$$

In this expression, survivorship is required only to duration 5, but we must discount for interest for the full 35 years.

We now equate the two actuarial present values to solve for c, the sponsor's contribution rate, which will be applied to all future salary payment in accordance with the plan to achieve the stated retirement income goal. ▼

Some plans of this type have both sponsor and participant contributions. It is common here for some kind of matching between the size of the sponsor contribution and the size of the participant contribution.

11.6 Disability Benefits with Individual Life Insurance

In Section 11.5.3 we discuss disability benefits included in pension plans. We now turn to disability benefits commonly found with individual life insurance. Provision can be made for the waiver of life insurance premiums during periods of disability. Alternatively, policies can contain a provision for a monthly income, sometimes related to the face amount, if disability occurs. The multiple decrement model is appropriate for studying these provisions.

The usual disability clause provides a benefit for total disability. Total disability can require a disability severe enough to prevent engaging in any gainful occupation, or it can require only the inability to engage in one's own occupation. Total disability that has been continuous for a period of time specified in the policy, called the ***waiting or elimination period,*** qualifies the policyholder to receive benefit payments. The waiting period can be 1, 3, 6, or 12 months. In policies with waiver of premium, it is common to make the benefits ***retroactive,*** that is, to refund any premiums paid by the insured during the waiting periods. Coverage is only for disabilities that occur prior to a disability benefit expiry age, typically 60 or 65. However, benefits in the form of an annuity, either as disability income or as waiver or premiums, will often continue to a higher age, typically the maturity date or paid-up date of the life insurance policy.

11.6.1 Disability Income Benefits

Let us start by expressing the actuarial present value of a disability income benefit of 1,000 per month issued to (x) under coverage expiring at age y and with income running to age u. We assume that the waiting period is m months. Using notation from Chapter 10 and earlier sections of Chapter 11, we can express the actuarial present value as a definite integral as

$$\bar{A} = \int_0^{y-x} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(i)}(t) \, v^{m/12} \, {}_{m/12}p^i_{[x+t]} \, (12{,}000 \, \ddot{a}^{(12)i}_{[x+t]:\overline{u-x-t-m/12}|}) \, dt. \quad (11.6.1)$$

The i superscript on ${}_{m/12}p^i_{[x+t]}$ indicates a survival probability for a disabled life. We now break up the integral into separate integrals for each year. Upon making the assumption of uniform distribution for the disability decrement within each year of age and replacing t by $k + s$, we obtain an expression for the actuarial present value much like (11.2.5):

$$\bar{A} = 12{,}000 \sum_{k=0}^{y-x-1} v^k \; {}_kp_x^{(\tau)} \; q_{x+k}^{(i)} \; v^{m/12}$$

$$\times \int_0^1 v^s \; {}_{m/12}p^i_{[x+k+s]} \; \ddot{a}^{(12)i}_{[x+k+s]+m/12:\; \overline{u-x-k-s-m/12}|} \, ds. \tag{11.6.2}$$

A simplification of this formula occurs when the decrement i (disability) is defined to occur only if the person who is disabled survives to the end of the waiting period of m months. If death occurs during the waiting period, the decrement is regarded as death. This means that the disabled life survivorship factor, ${}_{m/12}p^i_{[x+k+s]}$, is unnecessary as it has been taken into account in the definition of $q^{(i)}_{x+k}$. We note that it also means that the attained age at entry into the disabled life state is reached at the completion of the waiting period and is so indicated in the select age of the disabled life annuity function.

By the midpoint method the integrals in (11.6.2) are evaluated as

$$\int_0^1 v^s \; \ddot{a}^{(12)i}_{[x+k+s+m/12]:\overline{u-x-k-s-m/12}|} \, ds = v^{1/2} \; \ddot{a}^{(12)i}_{[x+k+1/2+m/12]:\overline{u-x-k-1/2-m/12}|} \, . \tag{11.6.3}$$

With these two changes (11.6.2) can be written as

$$A = 12{,}000 \sum_{k=0}^{y-x-1} v^{k+1/2} \; {}_kp_x^{(\tau)} q_{x+k}^{(i)} v^{m/12} \ddot{a}^{(12)i}_{[x+k+1/2+m/12]:\overline{u-x-k-1/2-m/12}|} \, . \tag{11.6.4}$$

11.6.2 Waiver-of-Premium Benefits

Let us go through the same process for a waiver of premium benefit. We assume that the premium, P, to be waived is payable g times per year for life. The primary difference between this and the disability income benefit is that the waiver benefit is a g-thly payment annuity starting at the first premium due date after the end of the waiting period.

We start with a special case with its actuarial present value written with the definite integrals already broken down to individual years of time of disablement. The case chosen is the waiver of semiannual premiums, payable for life, in the event of a disability occurring prior to age y and continuing through a 4-month waiting period. We further assume that the benefits are retroactive by which we mean that for a premium paid to the insurer on a due date during the waiting period, reimbursement with interest at the valuation rate will be made. This will increase the number of integrals within each year of age because disabilities that start within the first 2 months of each half year do not have premiums due during the waiting period:

$$
\begin{aligned}
\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \Bigg[& \int_0^{1/6} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \, {}_{1/2-s|}\ddot{a}_{[x+k+s]}^{(2)i} \, ds \\
& + \int_{1/6}^{1/2} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \left({}_{1-s|}\ddot{a}_{[x+k+s]}^{(2)i} + {}_{4/12}p_{[x+k+s]}^{i} \, v^{1/2-s} \frac{1}{2} \right) ds \\
& + \int_{1/2}^{2/3} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \, {}_{1-s|}\ddot{a}_{[x+k+s]}^{(2)i} \, ds \\
& + \int_{2/3}^{1} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \left({}_{3/2-s|}\ddot{a}_{[x+k+s]}^{(2)i} + {}_{4/12}p_{[x+k+s]}^{i} \, v^{1-s} \frac{1}{2} \right) ds \Bigg]. \qquad (11.6.5)
\end{aligned}
$$

If we incorporate the assumption of a uniform distribution of disability within each year of age and include only those disabilities that survive the waiting period as disabilities, (11.6.5) becomes

$$
\begin{aligned}
\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} q_{x+k}^{(i)} \Bigg[& \int_0^{1/6} v^{s+4/12} \, {}_{1/2-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} \, ds \\
& + \int_{1/6}^{1/2} v^{s+4/12} \, {}_{1-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} + v^{1/2} \frac{1}{2} \, ds \\
& + \int_{1/2}^{2/3} v^{s+4/12} \, {}_{1-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} \, ds \\
& + \int_{2/3}^{1} v^{s+4/12} \, {}_{3/2-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} + v \frac{1}{2} \, ds \Bigg]. \qquad (11.6.6)
\end{aligned}
$$

We now use the midpoint approximate integration method for each of the several integrals within each year of age to obtain

$$
\begin{aligned}
\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \, q_{x+k}^{(i)} \Bigg[& \frac{1}{6} v^{5/12} \, {}_{1/12|}\ddot{a}_{[x+k+5/12]}^{(2)i} \\
& + \frac{1}{3}\left(v^{2/3} \, {}_{1/3|}\ddot{a}_{[x+k+2/3]}^{(2)i} + v^{1/2} \frac{1}{2} \right) + \frac{1}{6} v^{11/12} \, {}_{1/12|}\ddot{a}_{[x+k+11/12]}^{(2)i} \\
& + \frac{1}{3}\left(v^{7/6} \, {}_{1/3|}\ddot{a}_{[x+k+7/6]}^{(2)i} + v \frac{1}{2} \right) \Bigg]. \qquad (11.6.7)
\end{aligned}
$$

11.6.3 Benefit Premiums and Reserves

Equivalence principle benefit premiums for disability income and waiver of premium benefits are found by equating the actuarial present value of benefits to the actuarial present value of premiums. For the waiver benefit discussed in Section 11.6.2, the annual benefit premium, ${}_{y-x}\Pi_x$, equals $\bar{A}$ of (11.6.7) divided by $\ddot{a}_{x:\overline{y-x}|}^{(\tau)}$.

Active life benefit reserves, that is, the reserve when premiums are not being waived, are most conveniently expressed by a premium difference formula:

$$
{}_kV = ({}_{y-x-k}\Pi_{x+k} - {}_{y-x}\Pi_x) \, \ddot{a}_{x+k:\overline{y-x-k}|}^{(\tau)}. \qquad (11.6.8)
$$

The terminal reserve for a disabled life is the actuarial present value of future disability benefits, calculated on the assumption that the insured has incurred a disability. The amount of premium waived, or disability income rate, is multiplied by the actuarial present value of an appropriate disabled life annuity. This value takes into account the age at disablement, the duration since disablement, and the terminal age for benefits.

11.7 Notes and References

We have not defined insurer's losses and studied their variances in this chapter. Formula (11.2.7) gave a means of doing so if we consider the total benefits for all causes of decrements. If we consider only a single benefit, such as the retirement benefit, there is more than one way of defining losses. The usual concept is that premiums and reserves, for a benefit in regard to a particular cause of decrement, apply only to that decrement. Thus, if decrement due to a second cause occurs, then, with respect to the first cause, there is zero benefit and a gain emerges. An insurer's loss based on this concept would lead, for example, to (11.4.3). However, losses defined in this way may have nonzero covariances, so that the loss variance for all benefits is not the sum of the loss variances for the individual benefits.

Alternatively, one may consider that when a particular cause of decrement occurs, the reserves accumulated for the benefits in regard to all the other causes are released to offset the benefit outgo for the given cause. In this case, the loss random variables defined for the benefits for the several causes of decrement have zero-valued covariances, and the loss variance for all benefits is the sum of the loss variances for the individual benefits. However, the premiums and reserves for the individual benefits are more difficult to compute on this second basis and individually differ significantly from those on the usual basis. For insights into these matters, see Hickman (1964).

The result stated in Section 11.4 concerning the neutral impact on premiums and reserves when a withdrawal benefit equals the reserve on the death benefit does not hold for fully discrete insurances. This was pointed out by Nesbitt (1964), who reported on work by Schuette. The problem results from the fact that, in the discrete model, the probability of withdrawal

$$q_{x+k}^{(w)} = \int_0^1 \exp\left[-\int_0^t \mu_x^{(\tau)}(k+s)\,ds\right] \mu_x^{(w)}(k+t)\,dt$$

depends on the force of mortality.

While there are many papers and a number of books dealing with pension fund mathematics, it seems useful for the purposes of this introductory treatment to refer only to other actuarial texts with similar chapters; see, for example, Hooker and Longley-Cooke (1957), Jordan (1967), and Neill (1977). These authors stress the formulation of actuarial present values in terms of pension commutation functions and the use of tables of such functions to carry out computations.

In contrast, a major portion of our presentation has been in terms of integrals and approximating sums, with the integrands or summands expressed in terms of basic functions. These approximating sums can be computed by various processes that may or may not make use of commutation functions. For pension benefits determined by complex eligibility or income conditions, it can be more flexible and efficient to calculate by processes not requiring extensive formulation by commutation functions. An opposing view, indicating the power of commutation functions for expressing actuarial present values and controlling their computation, is given by Chamberlain (1982).

There is an alternative foundation for constructing a model for disability insurance. In Section 11.6 we used a multiple decrement model that did not explicitly provide for recovery from disability. Models with several states of disability, with provision for transition from state to state, have been developed. These models are frequently the foundation of long-term care insurance. Hoem (1988) provides an introduction with valuable references to these ideas.

The multiple decrement model developed in Chapter 10 and applied in this chapter can be viewed as being made up of $m + 1$ states; m are called absorbing states in that it is impossible to return from them to the active state. These m states are associated with the m causes of decrement, and the remaining state is associated with continuing survival. If some of the m decrements are not absorbing, but are such that transition to the active state or one of the other nonabsorbing states is possible, a more complex but possibly more realistic model results. Estimation of the probabilities of transition among the states can be difficult because the probabilities can depend on the path followed to the current state.

Exercises

Section 11.2

11.1. Employees enter a benefit plan at age 30. If an employee remains in service until retirement, the employee receives an annual pension of 300 times years of service. If the employee dies in service before retirement, the beneficiary is paid 20,000 immediately. If the employee withdraws before age 70 for any reason except death, the member receives a deferred (to age 70) life annuity of 300 times years of service. Give an expression, in terms of integrals and continuous annuities, for the actuarial present value of these benefits at age 30.

11.2. Let $J = 1$ represent death by accidental means and $J = 2$ represent death by other means. You are given that
(i) $\delta = 0.05$
(ii) $\mu_x^{(1)}(t) = 0.005$ for $t \geq 0$ where $\mu_x^{(1)}(t)$ is the force of decrement for death by accidental means.
(iii) $\mu_x^{(2)}(t) = 0.020$ for $t \geq 0$.
A 20-year term insurance policy, payable at the moment of death, is issued to a life age (x) providing a benefit of 2 if death is by accidental means and

providing a benefit of 1 for other deaths. Find the expectation and variance of the present value of benefits random variable.

Section 11.4

11.3. A double decrement model is defined by $\mu_x^{(1)}(t) = \mu_x^{(2)}(t) = 1/(a - t)$, $0 \le t < a$.

a. In the single decrement model with decrement (1) only, the prospective loss variable at duration t is given by

$$_tL^1_{\,} = v^{T(x)-t} \qquad 0 < t \le T(x),\ J = 1.$$

Confirm that

$$_tV^1_{\,} = \frac{1 - e^{-\delta(a-t)}}{\delta(a - t)} = \bar{A}^{\,1}_{x+t:\,}.$$

b. In the double decrement model, the prospective loss variable at duration t is given by

$$\begin{aligned} {}_tL^2 &= v^{T(x)-t} && 0 < t \le T(x),\ J = 1 \\ &= v^{T(x)-t}\ {}_{T(x)}V^1 && 0 < t \le T(x),\ J = 2. \end{aligned}$$

Confirm that $\mathrm{E}[{}_tL^2 | T \ge t] = {}_tV^1$.

Section 11.5

11.4. A pension plan valuation assumes a linear salary scale function satisfying $S_{20} = 1$. If $(ES)_{45} = 2(AS)_{25}$, find S_x for $x \ge 20$.

11.5. A new pension plan with two participants, (35) and (40), provides annual income at retirement equal to 2% of salary at the final rate times the number of years of service, including any fraction of a year. If

(i) Salary increases occur continuously,
(ii) For (40), $(AS)_{40} = 50{,}000$ and $S_{40+t} = 1 + 0.06t$, and
(iii) For (35), $(AS)_{35} = 35{,}000$ and $S_{35+t} = 1 + 0.10t$,

calculate the maximum value of $[R(40, 0, t) - R(35, 0, t)]$ for $t \ge 0$.

11.6. It is assumed that, for a new participant entering at age 30, there will be annual increases in salary at the rate of 5% per year to take care of the effects of inflation and increases in productivity. In addition, it is assumed that promotion raises of 10% of the existing salary will occur at ages 40, 50, and 60.

a. Construct a salary scale function, S_{30+k}, to express these assumptions.
b. Write an expression for the actuarial present value of contributions of 10% of future salary for a new entrant with annual salary 24,000 and with increases in salary according to the scale constructed in (a).

11.7. Every year, a plan sponsor contributes 10% of that portion of each participant's salary in excess of a certain amount. That amount is 15,000 this year and will increase by 5% annually. Express the actuarial present value of the sponsor's contribution for a participant entering now at age 35 with a salary of 40,000.

11.8. A plan provides for an income benefit rate, payable from retirement to age 65, of 2% of the final 3-year average salary for each year of service. After age 65 the income benefit rate is 1-1/3% of the final 3-year average salary for each year of service.

a. For a participant age 50, who entered service at age 30 and currently has a salary of 48,000, express the actuarial present value of the participant's benefit if the earliest retirement age is 55 and there is no mandatory retirement age.

b. If the maximum number of years to be credited in the plan is 35, express the actuarial present value of the benefit for the above participant.

c. Give an expression for the actuarial present value of the income benefit associated with past service for the above participant.

11.9. A career average plan provides a retirement income of 2% of aggregate salary during a participant's years of service. The earliest age of retirement is 58, and all retirements are completed by age 68. For a participant age 50 who entered service at age 30 and has 450,000 total of past salaries with a current salary of 42,000, write expressions for

a. The participant's total income benefit rate in case of retirement at exact age 65

b. The participant's midyear total income benefit rate in case of retirement between ages 65 and 66

c. The actuarial present value of this participant's retirement benefit for past service

d. The actuarial present value of this participant's retirement benefit for future service.

11.10. A new participant in a pension plan, age 45, has a choice of two benefit options:

(1) A defined-contribution plan with contributions of 20% of salary each year. Contributions are made at the beginning of each year and earn 5% per year. Accumulated contributions are used to purchase a monthly life annuity-due.

(2) A defined-benefit plan with an annual benefit, payable monthly, of 40% of the final 2-year average salary.

You are given that (a) $\ddot{a}_{65}^{(12)} = 10$ and (b) $S_{45+k} = (1.05)^k$ for $k \geq 0$ where S_y is a step function, constant over each year of age. Assuming that retirement occurs at exact age 65 and that the participant survives to retirement, calculate the ratio of the expected monthly payment under the defined-contribution plan to that under the defined-benefit plan.

11.11. Display definite integrals for the actuarial present values of the following possible benefits of an employee benefit package. Assume that the employee is currently age 40 and earning 40,000 annually. This employee was hired at age 25 and has received a total of 320,000 in salary since hire. Retirement benefits are available only after age 55, and withdrawal benefits are only available before age 55 in the form of an annuity with payments deferred until the employee reaches age 55.

a. A retirement benefit at the annual benefit rate of 50% of the final salary.
b. A retirement benefit at the rate of 0.015 times the product of the final salary rate multiplied by the exact number of years (including fractions) of service at the moment of retirement.
c. A withdrawal benefit using the benefit income formula in (b).
d. A retirement benefit at the rate of 0.025 times the total salary paid over the whole career to the employee.
e. A withdrawal benefit using the benefit income formula in (d).

11.12. A retirement benefit consisting of a continuous annuity, payable for life, is part of an employer's benefit package. The annual benefit income rate is 60% of the salary rate applicable at the moment of retirement for retirements between ages 60 and 70. For retirements after attaining age 70, the benefit rate is 60% of the salary rate applicable between ages 69 and 70. Give an approximating sum for the actuarial present value of this benefit for a person age 30 who has just been hired at a salary of 35,000.

Section 11.6

11.13. a. Give an expression for the annual benefit premium, payable to age 60, for a disability income insurance issued to (35) of 2,000 per month payable to age 65 in case (35) becomes disabled before age 60 and survives a waiting period of 6 months.
b. Give an expression for the active life benefit reserve at the end of 10 years for the insurance in (a).

Miscellaneous

11.14. The Hattendorf theorem for the fully continuous model as stated in Exercise 8.24 can be restated in the definitions and notation of this chapter for the fully continuous multiple decrement model:

$$\text{Var}({}_0L^{\underline{m}}) = \sum_{j=1}^{m} \int_0^\infty [v^t\,(B_{x+t}^{(j)} - {}_t\bar{V}^{\underline{m}})]^2\,{}_tp_x^{(\tau)}\,\mu_x^{(j)}(t)\,dt.$$

Confirm that this result holds for the fully continuous whole life insurance discussed in Section 11.4.

Outline of solution:

a. Confirm that the loss random variable for this insurance is

$${}_0L^{\underline{2}} = \begin{cases} v^T - \bar{P}(\bar{A}_x)^{\underline{2}}\,\bar{a}_{\overline{T}|} & 0 \le T,\ J = 1 \\ v^T\,{}_T\bar{V}(\bar{A}_x) - \bar{P}(\bar{A}_x)^{\underline{2}}\,\bar{a}_{\overline{T}|} & 0 \le T,\ J = 2. \end{cases}$$

b. Use (11.4.6) and (11.4.7) to rewrite the differential equation (11.4.3) and then employ the integrating factor $e^{-\delta t}$ to obtain the solution

$$v^t\, {}_t\bar{V}(\bar{A}_x) = \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{t|}} - \int_0^t e^{-\delta s}\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds.$$

c. Use the result of part (b) to modify both lines of the definition of ${}_0L_=^2$ in part (a) and then show that

$$\begin{aligned}\mathrm{Var}({}_0L_=^2) = \int_0^\infty \Bigg\{ & v^t[1 - {}_t\bar{V}(\bar{A}_x)] \\ & - \int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds \Bigg\}^2 {}_tp_x^{(\tau)} \mu_x^{(1)}(t)\, dt \\ & + \int_0^\infty \left\{ \int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds \right\}^2 {}_tp_x^{(\tau)}\, \mu_x^{(2)}(t)\, dt.\end{aligned}$$

d. Perform the indicated squaring operation on the factor in the integrand of the first integral in part (c) and combine the two integrals that include $\{\int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds\}^2$ as a component of the integrand. Then use integration by parts to obtain

$$\mathrm{Var}({}_0L_=^2) = \int_0^\infty \{v^t\, [1 - {}_t\bar{V}(\bar{A}_x)]\}^2\, {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt.$$

This result provides a reduction of variance argument for establishing the withdrawal benefit as ${}_t\bar{V}(\bar{A}_x)$.

12

COLLECTIVE RISK MODELS FOR A SINGLE PERIOD

12.1 Introduction

In Chapters 3 through 11 we considered models for long-term insurances. The inclusion of interest in these models was essential. In this chapter we return to a topic introduced in Chapter 2, namely, short-term insurance policies. Consequently interest will be ignored. The purpose of this chapter is to present an alternative to the individual policy model discussed in Chapter 2.

The individual risk model of Chapter 2 considers individual policies and the claims produced by each policy. Then aggregate claims are obtained by summing over all the policies in the portfolio.

For the ***collective risk model*** we assume a random process that generates claims for a portfolio of policies. This process is characterized in terms of the portfolio as a whole rather than in terms of the individual policies comprising the portfolio. The mathematical formulation is as follows: Let N denote the number of claims produced by a portfolio of policies in a given time period. Let X_1 denote the amount of the first claim, X_2 the amount of the second claim, and so on. Then,

$$S = X_1 + X_2 + \cdots + X_N \tag{12.1.1}$$

represents the aggregate claims generated by the portfolio for the period under study. The number of claims, N, is a random variable and is associated with the frequency of claim. The individual claim amounts $X_1, X_2, \ldots$ are also random variables and are said to measure the severity of claims.

In order to make the model tractable, we usually make two fundamental assumptions:
1. $X_1, X_2, \ldots$ are identically distributed random variables.
2. The random variables $N, X_1, X_2, \ldots$ are mutually independent.

Expression (12.1.1) will be called a random sum, and unless stated otherwise, assumptions (1) and (2) will be made concerning its components.

A principal tool for developing the theory of this chapter is the moment generating function (m.g.f.). These functions provide a simple but powerful means for the reader to gain a working knowledge of the collective theory of risk. A reader who has not worked with them recently would do well to review the m.g.f.'s, means, and variances of the widely used probability distributions summarized in Appendix 5.

12.2 The Distribution of Aggregate Claims

In this section we see how the distribution of aggregate claims in a fixed time period can be obtained from the distribution of the number of claims and the distribution of individual claim amounts.

Let $P(x)$ denote the common d.f. of the independent and identically distributed X_i's. Let X be a random variable with this d.f. Then let

$$p_k = \mathrm{E}[X^k] \tag{12.2.1}$$

denote the k-th moment about the origin, and

$$M_X(t) = \mathrm{E}[e^{tX}] \tag{12.2.2}$$

denote the m.g.f. of X. In addition, let

$$M_N(t) = \mathrm{E}[e^{tN}] \tag{12.2.3}$$

denote the m.g.f. of the number of claims, and let

$$M_S(t) = \mathrm{E}[e^{tS}] \tag{12.2.4}$$

denote the m.g.f. of aggregate claims. The d.f. of aggregate claims will be denoted by $F_S(s)$.

Using (2.2.10) and (2.2.11), in conjunction with assumptions (1) and (2) of Section 12.1, we obtain

$$\mathrm{E}[S] = \mathrm{E}[\mathrm{E}[S|N]] = \mathrm{E}[p_1\, N] = p_1\, \mathrm{E}[N] \tag{12.2.5}$$

and

$$\begin{aligned} \mathrm{Var}(S) &= \mathrm{E}[\mathrm{Var}(S|N)] + \mathrm{Var}(\mathrm{E}[S|N]) \\ &= \mathrm{E}[N\, \mathrm{Var}(X)] + \mathrm{Var}(p_1 N) \\ &= \mathrm{E}[N]\, \mathrm{Var}(X) + p_1^2\, \mathrm{Var}(N) \end{aligned} \tag{12.2.6}$$

where $\mathrm{Var}(X) = p_2 - p_1^2$.

The result stated in (12.2.5), that the expected value of aggregate claims is the product of the expected individual claim amount and the expected number of claims, is not surprising. Expression (12.2.6) for the variance of aggregate claims also has a natural interpretation. The variance of aggregate claims is the sum of two components where the first is attributed to the variability of individual claim amounts and the second to the variability of the number of claims.

In a similar fashion we derive an expression for the m.g.f. of S:

$$\begin{aligned} M_S(t) &= \mathrm{E}[e^{tS}] = \mathrm{E}[\mathrm{E}[e^{tS}|N]] \\ &= \mathrm{E}[M_X(t)^N] = \mathrm{E}[e^{N \log M_X(t)}] \\ &= M_N[\log M_X(t)]. \end{aligned} \tag{12.2.7}$$

Example 12.2.1

Assume that N has a geometric distribution; that is, the p.f. of N is given by

$$\Pr(N = n) = pq^n \qquad n = 0, 1, 2, \ldots \tag{12.2.8}$$

where $0 < q < 1$ and $p = 1 - q$. Determine $M_S(t)$ in terms of $M_X(t)$.

Solution:

Since

$$M_N(t) = \mathrm{E}[e^{tN}] = \sum_{n=0}^{\infty} p(qe^t)^n = \frac{p}{1 - qe^t},$$

(12.2.7) tells us that

$$M_S(t) = \frac{p}{1 - qM_X(t)}. \tag{12.2.9}$$

▼

To derive the d.f. of S, we distinguish according to how many claims occur and use the law of total probability,

$$\begin{aligned} F_S(x) = \Pr(S \le x) &= \sum_{n=0}^{\infty} \Pr(S \le x|N = n) \Pr(N = n) \\ &= \sum_{n=0}^{\infty} \Pr(X_1 + X_2 + \cdots + X_n \le x) \Pr(N = n). \end{aligned} \tag{12.2.10}$$

In terms of the convolution defined in Section 2.3, we can write

$$\begin{aligned} \Pr(X_1 + X_2 + \cdots + X_n \le x) &= P{*}P{*}P{*} \cdots {*}P(x) \\ &= P^{*n}(x), \end{aligned} \tag{12.2.11}$$

which is the n-th convolution of P defined in Chapter 2. Recall that

$$P^{*0}(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0. \end{cases}$$

Thus (12.2.10) becomes

$$F_S(x) = \sum_{n=0}^{\infty} P^{*n}(x) \Pr(N = n). \tag{12.2.12}$$

If the individual claim amount distribution is discrete with p.f. $p(x) = \Pr(X = x)$, the distribution of aggregate claims is also discrete. By analogy with the above derivation, the p.f. of S can be obtained directly as

$$f_S(x) = \sum_{n=0}^{\infty} p^{*n}(x) \Pr(N = n) \tag{12.2.13}$$

where

$$p^{*n}(x) = p*p* \cdots * p(x) = \Pr(X_1 + X_2 + \cdots + X_n = x) \tag{12.2.11A}$$

$$\text{and } p^{*0}(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Here the inequality sign in the probability symbol in (12.2.11) has been replaced by the equal sign.

Example 12.2.2

Consider an insurance portfolio that will produce zero, one, two, or three claims in a fixed time period with probabilities 0.1, 0.3, 0.4, and 0.2, respectively. An individual claim will be of amount 1, 2, or 3 with probabilities 0.5, 0.4, and 0.1, respectively. Calculate the p.f. and d.f. of the aggregate claims.

Solution:

The calculations are summarized below. Only nonzero entries are exhibited.

(1) x	(2) $p^{*0}(x)$	(3) $p^{*1}(x) = p(x)$	(4) $p^{*2}(x)$	(5) $p^{*3}(x)$	(6) $f_S(x)$	(7) $F_S(x)$
0	1.0	—	—	—	0.1000	0.1000
1	—	0.5	—	—	0.1500	0.2500
2	—	0.4	0.25	—	0.2200	0.4700
3	—	0.1	0.40	0.125	0.2150	0.6850
4	—	—	0.26	0.300	0.1640	0.8490
5	—	—	0.08	0.315	0.0950	0.9440
6	—	—	0.01	0.184	0.0408	0.9848
7	—	—	—	0.063	0.0126	0.9974
8	—	—	—	0.012	0.0024	0.9998
9	—	—	—	0.001	0.0002	1.0000
n	0	1	2	3	—	—
$\Pr(N = n)$	0.1	0.3	0.4	0.2	—	—

Since there are at most three claims and each produces a claim amount of at most 3, we can limit the calculations to $x = 0, 1, 2, \ldots, 9$.

Column (2) lists the p.f. of a degenerate distribution with all the probability mass at 0. Column (3) lists the p.f. of the individual claim amount random variable. Columns (4) and (5) are obtained recursively by applying

$$\begin{aligned} p^{*(n+1)}(x) &= \Pr(X_1 + X_2 + \cdots + X_{n+1} = x) \\ &= \sum_y \Pr(X_{n+1} = y) \Pr(X_1 + X_2 + \cdots + X_n = x - y) \\ &= \sum_y p(y)\, p^{*n}(x - y). \end{aligned} \tag{12.2.14}$$

Since only three different claim amounts are possible, the evaluation of (12.2.14) will involve a sum of three or fewer terms. Next, (12.2.13) is used to compute the p.f. displayed in column (6). For this step, it is convenient to record the p.f. of N in the last row of the results. Finally, the elements of column (7) are obtained as partial sums of column (6). An alternative approach, not illustrated here, would be to perform the convolutions in terms of the d.f.'s, obtain $F_S(x)$ from (12.2.12), and calculate $f_S(x) = F_S(x) - F_S(x - 1)$. ▼

If the claim amount distribution is continuous, it cannot be concluded that the distribution of S is continuous. If $\Pr(N = 0) > 0$, the distribution of S will be of the mixed type; that is, it will have a mass of probability at 0 and be continuous elsewhere. This idea is illustrated in the following example.

Example 12.2.3

In Example 12.2.1, add the assumption that

$$P(x) = 1 - e^{-x} \qquad x > 0;$$

that is, the individual claim amount distribution is exponential with mean 1. Then show that

$$M_S(t) = p + q\,\frac{p}{p - t} \tag{12.2.15}$$

and interpret the formula.

Solution:

First, we rewrite (12.2.9) as follows:

$$M_S(t) = p + q\,\frac{pM_X(t)}{1 - qM_X(t)}\,.$$

Then we substitute

$$M_X(t) = \int_0^\infty e^{tx}\, e^{-x}\, dx = (1 - t)^{-1}$$

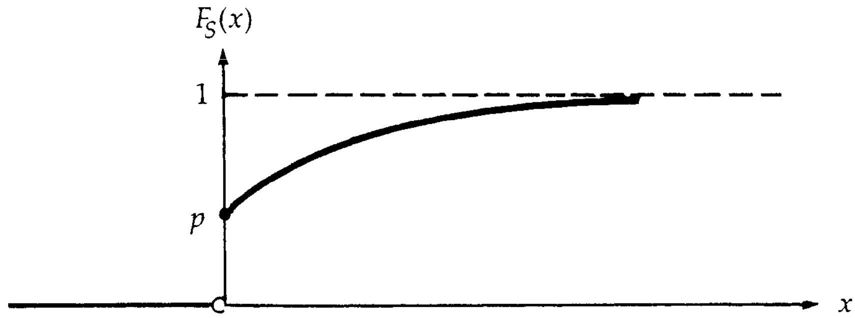
to obtain (12.2.15).

Since 1 is the m.g.f. of the constant 0 and $p/(p - t)$ is the m.g.f. of the exponential distribution with d.f. $1 - e^{-px}$, $x > 0$, (12.2.15) can be interpreted as a weighted average (with weights p and q, respectively). It follows that the d.f. of S is the corresponding weighted average of distributions. Thus, for $x > 0$

$$F_S(x) = p(1) + q(1 - e^{-px}) = 1 - qe^{-px}. \tag{12.2.16}$$

This distribution is of the mixed type. Its d.f. is shown in Figure 12.2.1. ▼

Graph of $F_S(x)$



12.3 Selection of Basic Distributions

In this section we discuss some issues in selecting the distribution of the number of claims N and the common distribution of the X_i's. As different considerations apply to these two selections, a separate subsection will be devoted to each.

12.3.1 The Distribution of *N*

One choice for the distribution of N is the Poisson with p.f. given by

$$\Pr(N = n) = \frac{\lambda^n e^{-\lambda}}{n!} \qquad n = 0, 1, 2, \ldots \tag{12.3.1}$$

where $\lambda > 0$. For the Poisson distribution, $\mathrm{E}[N] = \mathrm{Var}(N) = \lambda$. With this choice for the distribution of N, the distribution of S is called a ***compound Poisson distribution.*** Using (12.2.5) and (12.2.6), we have that

$$\mathrm{E}[S] = \lambda p_1 \tag{12.3.2}$$

and

$$\mathrm{Var}(S) = \lambda p_2 \,. \tag{12.3.3}$$

Substituting the m.g.f. of the Poisson distribution

$$M_N(t) = e^{\lambda(e^t-1)} \tag{12.3.4}$$

into (12.2.7), we obtain the m.g.f. of the compound Poisson distribution,

$$M_S(t) = e^{\lambda[M_X(t)-1]} \,. \tag{12.3.5}$$

The compound Poisson distribution has many attractive features, some of which are discussed in Section 12.4.

When the variance of the number of claims exceeds its mean, the Poisson distribution is not appropriate. In this situation, use of the negative binomial distribution has been suggested. The negative binomial distribution has a p.f. given by

$$\Pr(N = n) = \binom{r + n - 1}{n} p^r q^n \qquad n = 0, 1, 2, \ldots. \tag{12.3.6}$$

This distribution has two parameters: $r > 0$ and $0 < p < 1$; $q = 1 - p$. For this distribution, we have

$$M_N(t) = \left(\frac{p}{1 - q e^t}\right)^r, \tag{12.3.7}$$

$$\mathrm{E}[N] = \frac{rq}{p}, \tag{12.3.8}$$

and

$$\mathrm{Var}(N) = \frac{rq}{p^2}. \tag{12.3.9}$$

When a negative binomial distribution is chosen for N, the distribution of S is called a ***compound negative binomial distribution.*** Substituting from (12.3.8) and (12.3.9) into (12.2.5) and (12.2.6), we have

$$\mathrm{E}[S] = \frac{rq}{p} p_1 \tag{12.3.10}$$

and

$$\mathrm{Var}(S) = \frac{rq}{p} p_2 + \frac{rq^2}{p^2} p_1^2. \tag{12.3.11}$$

Substituting from (12.3.7) into (12.2.7), we obtain

$$M_S(t) = \left[\frac{p}{1 - qM_X(t)}\right]^r. \tag{12.3.12}$$

We observe that the family of geometric distributions used in Examples 12.2.1 and 12.2.3 is contained as a special case ($r = 1$) of the two-parameter family of negative binomial distributions.

A family of distributions for the number of claims can be generated by assuming that the Poisson parameter Λ is a random variable with p.d.f. $u(\lambda)$, $\lambda > 0$, and that the conditional distribution of N, given $\Lambda = \lambda$, is Poisson with parameter λ. There are several situations in which this might be a useful way to consider the distribution of N. For example, consider a population of insureds where various classes of insureds within the population generate numbers of claims according to Poisson distributions with different values of λ for the various classes. If the relative frequency of the values of λ is denoted by $u(\lambda)$, we can use the law of total probability to obtain

$$\begin{aligned}\Pr(N = n) &= \int_0^\infty \Pr(N = n | \Lambda = \lambda)\, u(\lambda) d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} u(\lambda) d\lambda. \end{aligned} \tag{12.3.13}$$

Furthermore, using (2.2.10) and (2.2.11), we have

$$\mathrm{E}[N] = \mathrm{E}[\mathrm{E}[N|\Lambda]] = \mathrm{E}[\Lambda] \tag{12.3.14}$$

and

$$\begin{aligned}\mathrm{Var}(N) &= \mathrm{E}[\mathrm{Var}(N|\Lambda)] + \mathrm{Var}(\mathrm{E}[N|\Lambda]) \\ &= \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda).\end{aligned} \tag{12.3.15}$$

Also,

$$M_N(t) = \mathrm{E}[e^{tN}] = \mathrm{E}[\mathrm{E}[e^{tN}|\Lambda]] = \mathrm{E}[e^{\Lambda(e^t-1)}] = M_\Lambda(e^t - 1). \tag{12.3.16}$$

The equality,

$$\mathrm{E}[e^{tN}|\Lambda] = e^{\Lambda(e^t-1)},$$

follows from the hypothesis that the conditional distribution of N, given Λ, is Poisson with parameter Λ.

A comparison of (12.3.14) and (12.3.15) shows that, as in the case of the negative binomial distribution, $\mathrm{E}[N] < \mathrm{Var}(N)$. In fact, the negative binomial distribution can be derived in this fashion, which will be shown in the following example.

Example 12.3.1

Assume that $u(\lambda)$ is the gamma p.d.f. with parameters α and β,

$$u(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \qquad \lambda > 0 \tag{12.3.17}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy.$$

a. Show that the marginal distribution of N is negative binomial with parameters

$$r = \alpha, \qquad p = \frac{\beta}{1+\beta}. \tag{12.3.18}$$

b. By substituting $\mathrm{E}[\Lambda] = \alpha / \beta$ and $\mathrm{Var}(\Lambda) = \alpha / \beta^2$ into (12.3.14) and (12.3.15), verify (12.3.8) and (12.3.9).

Solution:

a. Substituting

$$M_\Lambda(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \tag{12.3.19}$$

into (12.3.16), we have

$$\begin{aligned}M_N(t) = M_\Lambda(e^t - 1) &= \left[\frac{\beta}{\beta - (e^t - 1)}\right]^\alpha \\ &= \left\{\frac{\beta / (\beta + 1)}{1 - [1 - \beta / (\beta + 1)]e^t}\right\}^\alpha.\end{aligned} \tag{12.3.20}$$

Comparison of (12.3 20) with (12.3.7) confirms that this distribution for N is negative binomial with parameters $r = \alpha$,

$$p = \frac{\beta}{1+\beta}, \tag{12.3.21}$$

$$q = 1 - p = \frac{1}{1+\beta}.$$

b. The suggested substitutions into (12.3.14) and (12.3.15) yield

$$\mathrm{E}[N] = \frac{\alpha}{\beta} = \frac{rq}{p}$$

as in (12.3.8) and

$$\mathrm{Var}(N) = \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{rq}{p}\left(1 + \frac{q}{p}\right) = \frac{rq}{p^2}$$

as in (12.3.9). ▼

The following is another example of a distribution for N that is obtained by mixing Poisson distributions.

Example 12.3.2

Assume that $u(\lambda)$ is the inverse Gaussian p.d.f. with parameters α and β. Exhibit the moment generating function of N, E[N], and Var(N).

Solution:

Example 2.3.5 contains the basic facts about the inverse Gaussian distribution.

Applying (12.3.16) yields

$$M_N(t) = M_\Lambda(e^t - 1) = e^{\alpha\{1-[1-2(e^t-1)/\beta]^{1/2}\}},$$

and from (12.3.14) and (12.3.15) we obtain

$$\mathrm{E}[N] = \mathrm{E}[\Lambda] = \frac{\alpha}{\beta}$$

and

$$\begin{aligned}\mathrm{Var}(N) &= \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda)\\ &= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{\alpha(\beta+1)}{\beta^2}.\end{aligned}$$

This distribution is called the ***Poisson inverse Gaussian distribution.*** ▼

Table 12.3.1 summarizes pertinent information on the compound distributions resulting from the selections for N discussed here.

Compound Distributions of S

$$S = \sum_{j=1}^{N} X_j$$

$N, X_1, X_2, \ldots$ are independent random variables.
Each X_j has d.f. $P(x)$,
m.g.f. $M_X(t)$, and
$p_k = E[X^k] \quad k = 1, 2, \ldots$

$$P^{*0}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P^{*n}(x) = \begin{cases} \sum_{j=0}^{x} p(x-j)P^{*(n-1)}(j), \text{ or} \\ \int_0^x p(x-y)P^{*(n-1)}(y)dy \end{cases}$$

Definitions	Distribution Function, $F_S(x)$	Restrictions on Parameters	Moment Generating Function, $M_S(t)$	Mean	Variance
General	$\sum_{n=0}^{\infty} \Pr(N = n)P^{*n}(x)$	—	$M_N[\log M_X(t)]$	$p_1 \mathrm{E}[N]$	$\mathrm{E}[N](p_2 - p_1^2) + p_1^2 \mathrm{Var}(N)$
Compound Poisson	$\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} P^{*n}(x)$	$\lambda > 0$	$e^{\lambda[M_X(t)-1]}$	λp_1	λp_2
Compound Negative Binomial	$\sum_{n=0}^{\infty} \binom{r+n-1}{n} p^r q^n P^{*n}(x)$	$0 < p < 1$, $q = 1 - p$, $r > 0$	$\left[\frac{p}{1 - qM_X(t)}\right]^r \quad qM_X(t) < 1$	$\frac{rqp_1}{p}$	$\frac{rqp_2}{p} + \frac{rq^2p_1^2}{p^2}$
Compound Poisson Inverse Gaussian	no known closed form	$\alpha > 0$, $\beta > 0$	$\exp\left\{-\alpha\left[1 - \left\{1 - \frac{2[M_X(t) - 1]}{\beta}\right\}^{1/2}\right]\right\}$	$\frac{\alpha}{\beta} p_1$	$\frac{\alpha}{\beta}\left(p_2 + \frac{p_1^2}{\beta}\right)$

12.3.2 The Individual Claim Amount Distribution

On the basis of (12.2.12) we see that convolutions of the individual claim amount distribution may be required. Thus, when possible, it is convenient to select that distribution from a family of distributions for which convolutions can be calculated easily either by formula or numerically. For example, if the claim amount has the normal distribution with mean μ and variance σ^2, then its n-th convolution is the normal distribution with mean $n\mu$ and variance $n\sigma^2$. For many types of insurance, the claim amount random variable is only positive, and its distribution is skewed to the right. For these insurances we might choose a gamma distribution that also has these properties. The n-th convolution of a gamma distribution with parameters α and β is also a gamma distribution but with parameters $n\alpha$ and β. This can be confirmed by noting from (12.3.19) that $M_X(t) = [\beta / (\beta - t)]^\alpha$, and hence the m.g.f. associated with $P^{*n}(x)$ is

$$M_X(t)^n = \left(\frac{\beta}{\beta - t}\right)^{n\alpha} \qquad t < \beta. \tag{12.3.22}$$

If the claim amounts have an exponential distribution with parameter 1, the p.d.f. is given by

$$p(x) = e^{-x} \qquad x > 0.$$

This is a gamma distribution with $\alpha = \beta = 1$. Then, by using (12.3.19), we conclude that the n-th convolution is a gamma distribution with parameters $\alpha = n$, $\beta = 1$; that is,

$$p^{*n}(x) = \frac{x^{n-1}\,e^{-x}}{(n-1)!} \qquad x > 0. \tag{12.3.23}$$

To obtain an expression for $P^{*n}(x)$, we perform integration by parts n times as follows:

$$\begin{aligned}
1 - P^{*n}(x) &= \int_x^\infty \frac{y^{n-1}\,e^{-y}}{(n-1)!}\,dy \\
&= -\frac{y^{n-1}}{(n-1)!}\,e^{-y}\Bigg|_x^\infty + \int_x^\infty \frac{y^{n-2}\,e^{-y}}{(n-2)!}\,dy \\
&= \frac{x^{n-1}}{(n-1)!}\,e^{-y} + [1 - P^{*(n-1)}(x)] \\
&= e^{-y}\sum_{i=0}^{n-1}\frac{x^i}{i!}\,.
\end{aligned} \tag{12.3.24}$$

Then, using (12.2.12), we have

$$1 - F_S(x) = \sum_{n=1}^{\infty} \Pr(N = n)e^{-y}\sum_{i=0}^{n-1}\frac{x^i}{i!} \qquad x > 0. \tag{12.3.25}$$

This exponential distribution case shows that even with simple assumed distributions, the distribution of aggregate claims may not have a simple form. Therefore, it may be more practical to select a discrete claim amount distribution and calculate the required convolutions numerically. For compound Poisson distributions it has been established that the convolution method can be shortened or, alternatively, that it can be bypassed by use of a recursive formula for directly calculating the distribution function of S. These computational shortcuts are discussed in the following section.

12.4 Properties of Certain Compound Distributions

In this section we discuss some mathematical properties of certain compound distributions. Two theorems concerning the compound Poisson are presented.

The first shows that the sum of independent random variables, each having a compound Poisson distribution, also has a compound Poisson distribution.

Theorem 12.4.1

If $S_1, S_2, \ldots, S_m$ are mutually independent random variables such that S_i has a compound Poisson distribution with parameter λ_i and d.f. of claim amount $P_i(x)$, $i = 1, 2, \ldots, m$, then $S = S_1 + S_2 + \cdots + S_m$ has a compound Poisson distribution with

$$\lambda = \sum_{i=1}^{m} \lambda_i \tag{12.4.1}$$

and

$$P(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} P_i(x). \tag{12.4.2}$$

Proof:

We let $M_i(t)$ denote the m.g.f. of $P_i(x)$. According to (12.3.5), the m.g.f. of S_i is

$$M_{S_i}(t) = \exp\{\lambda_i[M_i(t) - 1]\}.$$

By the assumed independence of $S_1, \ldots, S_m$, the m.g.f. of their sum is

$$M_S(t) = \prod_{i=1}^{m} M_{S_i}(t) = \exp\left\{\sum_{i=1}^{m} \lambda_i[M_i(t) - 1]\right\}.$$

Finally, we rewrite the exponent to obtain

$$M_S(t) = \exp\left\{\lambda\left[\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} M_i(t) - 1\right]\right\}. \tag{12.4.3}$$

Since this is the m.g.f. of the compound Poisson distribution, specified by (12.4.1) and (12.4.2), the theorem follows. ■

This result has two important consequences for building insurance models. First, if we combine m insurance portfolios, where the aggregate claims of each of the portfolios have compound Poisson distributions and are mutually independent, then the aggregate claims for the combined portfolio will also have a compound Poisson distribution. Second, we can consider a single insurance portfolio for a period of m years. Here we assume independence among the annual aggregate claims for the m years and that the aggregate claims for each year have a compound Poisson distribution. It is not necessary that the annual aggregate claims distributions be identical. Then it follows from Theorem 12.4.1 that the total claims for the m-year period will have a compound Poisson distribution.

Example 12.4.1

Let $x_1, x_2, \ldots, x_m$ be m different numbers and suppose that $N_1, N_2, \ldots, N_m$ are mutually independent random variables. Further, suppose that N_i ($i = 1, 2, \ldots, m$) has a Poisson distribution with parameter λ_i. What is the distribution of

$$x_1 N_1 + x_2 N_2 + \cdots + x_m N_m? \tag{12.4.4}$$

Solution:

By interpreting $x_i N_i$ to have a compound Poisson distribution with Poisson parameter λ_i and a degenerate claim amount distribution at x_i, we can apply Theorem 12.4.1 to establish that the sum in (12.4.4) has a compound Poisson distribution with

$$\lambda = \sum_{i=1}^{m} \lambda_i$$

and p.f. of claim amount $p(x)$ where

$$p(x) = \begin{cases} \dfrac{\lambda_i}{\lambda} & x = x_i, \quad i = 1, 2, \ldots, m \\ 0 & \text{elsewhere.} \end{cases} \tag{12.4.5}$$

▼

We show in Theorem 12.4.2 that the construction in Example 12.4.1 is reversible: that is, every compound Poisson distribution with a discrete claim amount distribution can be represented as a sum of the form (12.4.4). We let $x_1, x_2, \ldots, x_m$ denote the discrete values for individual claim amounts and let

$$\pi_i = p(x_i) \qquad i = 1, 2, \ldots, m \tag{12.4.6}$$

denote their respective probabilities. Let N_i be the number of terms in (12.1.1) that are equal to x_i. Then, by collecting terms, we see that

$$S = x_1 N_1 + x_2 N_2 + \cdots + x_m N_m\,. \tag{12.4.7}$$

In general, the N_i's of (12.4.7) are dependent random variables. However, in the special case of a compound Poisson distribution for S, they are independent, as is shown in Theorem 12.4.2.

Before stating Theorem 12.4.2, we cite some properties of the ***multinomial distribution*** that are used in the proof. For the multinomial, each of n independent trials result in one of m different outcomes. The probability that a trial ends in outcome i is denoted by π_i. We denote the random variable that counts the number of outcomes i in n trials by N_i. Then

$$1 = \sum_{i=1}^{m} \pi_i, \qquad n = \sum_{i=1}^{m} N_i,$$

and the joint p.f. of $N_1, N_2, \ldots, N_m$ is given by

$$\Pr(N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!} \pi_1^{n_1} \pi_2^{n_2} \cdots \pi_m^{n_m}. \qquad (12.4.8)$$

By using (12.4.8) we obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] = (\pi_1 e^{t_1} + \pi_2 e^{t_2} + \cdots + \pi_m e^{t_m})^n. \qquad (12.4.9)$$

The multivariate discrete distribution with p.f. given by (12.4.8) and m.g.f. given by (12.4.9) is called a multinomial distribution with parameters $n, \pi_1, \ldots, \pi_m$.

Theorem 12.4.2

If S, as given in (12.4.7), has a compound Poisson distribution with parameter λ and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6), then

a. $N_1, N_2, \ldots, N_m$ are mutually independent.

b. N_i has a Poisson distribution with parameter $\lambda_i = \lambda \pi_i$, $i = 1, 2, \ldots, m$.

Proof:

We start by defining the m.g.f. of the joint distribution of $N_1, N_2, \ldots, N_m$ by use of (2.2.10) for conditional expectations. Note that for a fixed number of independent claims (trials) where each claim results in one of m claim amounts, the numbers of claims of each amount have a multinomial distribution with parameters $n, \pi_1, \pi_2, \ldots, \pi_m$. Hence, given

$$N = \sum_{i=1}^{m} N_i = n,$$

the conditional distribution of $N_1, N_2, \ldots, N_m$ is this multinomial distribution. For this case, we use (12.4.9) to obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] = \sum_{n=0}^{\infty} \mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right) \middle| N = n\right] \Pr(N = n)$$

$$= \sum_{n=0}^{\infty} (\pi_1 e^{t_1} + \cdots + \pi_m e^{t_m})^n \frac{e^{-\lambda} \lambda^n}{n!}. \qquad (12.4.10)$$

We now perform the required summation by recognizing (12.4.10) as a Taylor series expansion of an exponential function. We obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] = \exp(-\lambda)\exp\left(\lambda \sum_{i=1}^{m} \pi_i e^{t_i}\right)$$

$$= \prod_{i=1}^{m} \exp[\lambda \pi_i(e^{t_i} - 1)]. \qquad (12.4.11)$$

Since this is the product of m functions each of a single variable t_i, (12.4.11) shows the mutual independence of the N_i's. Furthermore, if we set $t_i = t$, and $t_j = 0$ for $j \neq i$ in (12.4.11), we obtain

$$\mathrm{E}[\exp(tN_i)] = \exp[\lambda \pi_i(e^t - 1)], \qquad (12.4.12)$$

which is the m.g.f. of the Poisson distribution with parameter $\lambda \pi_i$. This proves statement (b). ■

Formula (12.4.7) and Theorem 12.4.2 provide an alternative method for tabulating a compound Poisson distribution with a discrete claim amount distribution. First, we compute the p.f.'s of $x_1 N_1, x_2 N_2, \ldots, x_m N_m$. Since the nonzero entries for the p.f. of $x_i N_i$ are at multiples of x_i and are Poisson probabilities, this is an easy task. Then the convolution of these m distributions is calculated to obtain the p.f. of S. This method is particularly convenient if m, the number of different claim amounts, is small. Even if a continuous distribution has been selected for the individual claim amounts, a discrete approximation can sometimes be used with this alternative method to produce a satisfactory approximation to the distribution of S. The basic and the alternative methods for tabulating the distribution of S are compared in the following example.

Example 12.4.2

Suppose that S has a compound Poisson distribution with $\lambda = 0.8$ and individual claim amounts that are 1, 2, or 3 with probabilities 0.25, 0.375, and 0.375, respectively. Compute $f_S(x) = \Pr(S = x)$ for $x = 0, 1, \ldots, 6$.

Solution:

For the basic method, the calculations parallel those in Example 12.2.2 and are summarized below.

Basic Method Calculations

(1) x	(2) $p^{*0}(x)$	(3) $p(x)$	(4) $p^{*2}(x)$	(5) $p^{*3}(x)$	(6) $p^{*4}(x)$	(7) $p^{*5}(x)$	(8) $p^{*6}(x)$	(9) $f_S(x)$
0	1	—	—	—	—	—	—	0.449329
1	—	0.250000	—	—	—	—	—	0.089866
2	—	0.375000	0.062500	—	—	—	—	0.143785
3	—	0.375000	0.187500	0.015625	—	—	—	0.162358
4	—	—	0.328125	0.070313	0.003906	—	—	0.049905
5	—	—	0.281250	0.175781	0.023438	0.000977	—	0.047360
6	—	—	0.140625	0.263672	0.076172	0.007324	0.000244	0.030923
n	0	1	2	3	4	5	6	
$e^{-0.8}\dfrac{(0.8)^n}{n!}$	0.449329	0.359463	0.143785	0.038343	0.007669	0.001227	0.000164	

For the alternative method outlined in this section, the calculations are displayed below.

Alternative Method Calculations

(1)	(2)	(3)	(4)	(5)	(6)
(x)	$\Pr(1N_1 = x)$	$\Pr(2N_2 = x)$	$\Pr(3N_3 = x)$	$\Pr(N_1 + 2N_2 = x)$ $= (2)*(3)$	$\Pr(N_1 + 2N_2 + 3N_3 = x)$ $= (4)*(5) = f_S(x)$
0	0.818731	0.740818	0.740818	0.606531	0.449329
1	0.163746	—	—	0.121306	0.089866
2	0.016375	0.222245	—	0.194090	0.143785
3	0.001092	—	0.222245	0.037201	0.162358
4	0.000055	0.033337	—	0.030973	0.049905
5	0.000002	—	—	0.005703	0.047360
6	0.000000	0.003334	0.033337	0.003287	0.030923
i	1	2	3		
λ_i	0.2	0.3	0.3		
	$\dfrac{e^{-0.2}\,(0.2)^x}{x!}$	$\dfrac{e^{-0.3}\,(0.3)^{x/2}}{(x/2)!}$	$\dfrac{e^{-0.3}\,(0.3)^{x/3}}{(x/3)!}$		

For the application of the formulas of this section, we note that $m = 3$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $\lambda_1 = \lambda\, p(1) = 0.2$, $\lambda_2 = \lambda\, p(2) = 0.3$, and $\lambda_3 = \lambda\, p(3) = 0.3$. First, we compute columns (2), (3), and (4). The nonzero entries are Poisson probabilities. Then we obtain the convolution of the p.f.'s in columns (2) and (3) and record the result in column (5). Finally, we convolute the p.f.'s displayed in columns (4) and (5) and record the result in column (6).

Remember that the complete p.f. is not displayed in either set of calculations. The example required probabilities for only $x = 0, 1, \ldots, 6$. However, $\Pr(S \le 6) = f_S(0) + f_S(1) + \cdots + f_S(6) = 0.973526$. ▼

Formula (12.4.7) and Theorem 12.4.2 have another implication. Instead of defining a compound Poisson distribution of S by specifying the parameter λ and the d.f. $P(x)$ of the discrete individual claim amounts, we can define the distribution in terms of the possible individual claim amounts $x_1, x_2, \ldots, x_m$ and the parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ of the associated Poisson distributions described in part (b) of Theorem 12.4.2. Thus for x_i there is an associated Poisson distribution of N_i with parameter λ_i. In terms of this new definition of the distribution of S, we have from $E[N_i] = \mathrm{Var}(N_i) = \lambda_i$ and the independence of the N_i's that

$$E[S] = E\left[\sum_{i=1}^{m} x_i N_i\right] = \sum_{i=1}^{m} x_i \lambda_i \tag{12.4.13}$$

and

$$\mathrm{Var}(S) = \mathrm{Var}\left(\sum_{i=1}^{m} x_i N_i\right) = \sum_{i=1}^{m} x_i^2 \lambda_i . \tag{12.4.14}$$

Formula (12.4.13) could be obtained by starting from (12.3.2) and noting that

$$\lambda p_1 = \lambda \sum_{i=1}^{m} x_i \pi_i = \sum_{i=1}^{m} x_i \lambda_i .$$

Similarly we can obtain (12.4.14) from (12.3.3).

In some cases it is useful, as in Example 12.4.1, to regard S as a sum of mutually independent random variables $x_i N_i$, $i = 1, 2, \ldots, m$, where $x_i N_i$ has a compound Poisson distribution with parameter λ_i and degenerate claim amount distribution at x_i. This interpretation follows from Theorem 12.4.2 and underlies the alternative method illustrated in Example 12.4.2.

There is a third way, the ***recursive method,*** for evaluating certain compound distributions for which the only possible claim amounts are positive integers. It is based on the recursive formula of the following theorem.

Theorem 12.4.3

For compound distributions where the probability distribution for N, the number of claims, satisfies the condition $\Pr(N = n)/\Pr(N = n - 1) = a + (b/n)$, for $n = 1, 2, \ldots$, and where the distribution of claim amounts is restricted to the positive integers,

$$f_S(x) = \sum_{i=1}^{x} \left[a + \left(\frac{bi}{x} \right) \right] p(i) f_S(x - i) \qquad x = 1, 2, \ldots \qquad (12.4.15)$$

with the starting value given by $f_S(0) = \Pr(N = 0)$.

We establish the following lemma to be used in the proof of the theorem.

Lemma

For $X_1, X_2, X_3, \ldots, X_n$, which are independent and identically distributed random variables taking on values restricted to the positive integers, we have, for positive integer values of x,

(i) $$p^{*n}(x) = \sum_{i=1}^{x} p(i) p^{*(n-1)}(x - i)$$

(ii) $$p^{*n}(x) = \frac{n}{x} \sum_{i=1}^{x} i\, p(i) p^{*(n-1)}(x - i).$$

Proof of the Lemma:

For $n = 1$, both (i) and (ii) reduce to $p^{*1}(x) = p(x) \times p^{*(0)}(0)$. For $n > 1$ we establish (i) by using the Law of Total Probability to evaluate $\Pr(X_1 + X_2 + \cdots + X_n = x)$ by conditioning on the value taken by X_1 as

$$\sum_{i=1}^{x} \Pr(X_1 = i) \Pr(X_2 + X_3 + \cdots + X_n = x - i).$$

We then note that $\Pr(X_2+X_3+\cdots+X_n = x - i)$ and $\Pr(X_1+X_2+\cdots+X_n = x)$ can be evaluated by using $(n-1)$-fold and n-fold convolutions, respectively, of $p(i)$; see (2.3.4).

For $n > 1$, we establish (ii) by considering the conditional expectations $E[X_k | X_1+X_2+X_3+\cdots+X_n = x]$ for $k = 1, 2, 3, \ldots, n$. From reasons of symmetry, these quantities are the same for all such k. Since their sum is x, each is equal to x/n. The conditional expectation $E[X_1 | X_1+X_2+X_3+\cdots+X_n = x]$ is evaluated as

$$\sum_{i=1}^{x} i \Pr(X_1 = i) \Pr(X_2+X_3+\cdots+X_n = x - i) / \Pr(X_1 + X_2 + X_3+\cdots+X_n = x).$$

We then note that $\Pr(X_2+X_3+\cdots+X_n = x - i)$ and that $\Pr(X_1+X_2+\cdots+X_n = x)$ can be evaluated by using $(n-1)$-fold and n-fold convolutions, respectively, of $p(i)$. Solving for $p^{*n}(x)$ completes the proof. ■

Proof of the Theorem:

First,

$$f_S(x) = \sum_{n=1}^{\infty} \Pr(N = n)\, p^{*n}(x).$$

With $\Pr(N = n) = [a + (b/n)] \Pr(N = n - 1)$, we have

$$f_S(x) = a \sum_{n=1}^{\infty} \Pr(N = n - 1)\, p^{*n}(x) + \sum_{n=1}^{\infty} \frac{b}{n} \Pr(N = n - 1)\, p^{*n}(x),$$

and by the two parts of the lemma

$$f_S(x) = a \sum_{n=1}^{\infty} \Pr(N = n - 1) \sum_{i=1}^{x} p(i)\, p^{*(n-1)}(x - i)$$
$$+ \sum_{n=1}^{\infty} \frac{b}{n} \Pr(N = n - 1) \frac{n}{x} \sum_{i=1}^{x} i\, p(i)\, p^{*(n-1)}(x - i).$$

Interchanging the order of summation, we get

$$f_S(x) = a \sum_{i=1}^{x} p(i) \sum_{n=1}^{\infty} \Pr(N = n - 1)\, p^{*(n-1)}(x - i)$$
$$+ \frac{b}{x} \sum_{i=1}^{x} ip(i) \sum_{n=1}^{\infty} \Pr(N = n - 1)\, p^{*(n-1)}(x - i)$$
$$= a \sum_{i=1}^{x} p(i)\, f_S(x - i) + \frac{b}{x} \sum_{i=1}^{x} i\, p(i)\, f_S(x - i)$$
$$= \sum_{i=1}^{x} \left(a + \frac{b\,i}{x}\right) p(i)\, f_S(x - i).$$ ■

We now examine the only three distributions that satisfy the required relationship between successive values of $\Pr(N = n)$.

a. Poisson: $[\Pr(N = n) / \Pr(N = n - 1)] = \lambda / n$. The recursion formula for the compound Poisson is

$$f_S(x) = \frac{\lambda}{x} \sum_{i=1}^{x} i p(i) f_S(x - i) \quad \text{with} \quad f_S(0) = e^{-\lambda}. \tag{12.4.16}$$

b. Negative binomial: $[\Pr(N = n) / \Pr(N = n - 1)] = (1 - p)[(n + r - 1) / n]$ so that $a = (1 - p)$ and $b = (1 - p)(r - 1)$. The recursion formula for the compound negative binomial is

$$f_S(x) = (1 - p) \sum_{i=1}^{x} \left[(r - 1) \frac{i}{x} + 1 \right] p(i) f_S(x - i) \tag{12.4.17}$$

with $f_S(0) = p^r$.

c. Binomial with parameters m and p:

$$\frac{\Pr(N = n)}{\Pr(N = n - 1)} = \frac{m + 1 - n}{n} \frac{p}{1 - p}$$

so that $a = -[p / (1 - p)]$ and $b = (m + 1)[p / (1 - p)]$. The recursion formula in this case is

$$f_S(x) = \left(\frac{p}{1 - p} \right) \sum_{i=1}^{x} \left[(m + 1) \frac{i}{x} - 1 \right] p(i) f_S(x - i) \tag{12.4.18}$$

with $f_S(0) = (1 - p)^m$.

Example 12.4.2 (recomputed)

For the compound Poisson distribution of this example, compute $f_S(x) = \Pr(S = x)$ by the recursive method.

Solution:

Substituting the values used for the alternative method of calculation into (12.4.16) yields

$$f_S(x) = \frac{1}{x} [0.2 f_S(x - 1) + 0.6 f_S(x - 2) + 0.9 f_S(x - 3)] \qquad x = 1, 2, \ldots.$$

Recalling that $f_S(x) = 0$, $x < 0$, and $f_S(0) = e^{-\lambda} = 0.449329$, we readily reproduce the values of $f_S(x)$ given in the basic method calculations. ▼

12.5 Approximations to the Distribution of Aggregate Claims

In Section 2.4 the normal distribution was employed as an approximation to the distribution of aggregate claims in the individual model. The normal approximation is the first developed here for use with the collective model.

For the compound Poisson distribution, the two parameters of the normal approximation are given by (12.3.2) and (12.3.3). For the compound negative binomial distribution the parameters are given by (12.3.10) and (12.3.11). In each of the two cases the approximation is better when the expected number of claims is large, or, in other words, when λ is large for the compound Poisson case and when r is large for the negative binomial. These two results are contained in Theorem 12.5.1, which may be interpreted as a version of the central limit theorem.

Theorem 12.5.1

a. If S has a compound Poisson distribution, specified by λ and $P(x)$, then the distribution of

$$Z = \frac{S - \lambda p_1}{\sqrt{\lambda p_2}} \tag{12.5.1}$$

converges to the standard normal distribution as $\lambda \rightarrow \infty$.

b. If S has a compound negative binomial distribution, specified by r, p and $P(x)$, then the distribution of

$$Z = \frac{S - r(q/p)p_1}{\sqrt{r(q/p)p_2 + r(q^2/p^2)p_1^2}} \tag{12.5.2}$$

converges to the standard normal distribution as $r \rightarrow \infty$.

Proof:

We shall prove statement (a) by showing that

$$\lim_{\lambda \rightarrow \infty} M_Z(t) = e^{t^2/2}.$$

Statement (b) can be proved using a similar strategy, but the proof involves additional steps.

From (12.5.1), it follows that

$$M_Z(t) = M_S\left(\frac{t}{\sqrt{\lambda p_2}}\right) \exp\left(-\frac{\lambda p_1\, t}{\sqrt{\lambda p_2}}\right).$$

Now we use (12.3.5) to obtain

$$M_Z(t) = \exp\left\{\lambda\left[M_X\left(\frac{t}{\sqrt{\lambda p_2}}\right) - 1\right] - \frac{\lambda p_1\, t}{\sqrt{\lambda p_2}}\right\}, \tag{12.5.3}$$

and then substitute the expansion

$$M_X(t) = 1 + \frac{p_1 t}{1!} + \frac{p_2 t^2}{2!} + \cdots, \tag{12.5.4}$$

with $t/\sqrt{\lambda p_2}$ in place of t, into (12.5.3) to obtain

$$M_Z(t) = \exp\left(\frac{1}{2}\, t^2 + \frac{1}{6}\,\frac{1}{\sqrt{\lambda}}\,\frac{p_3}{p_2^{3/2}}\, t^3 + \cdots\right). \tag{12.5.5}$$

Then as $\lambda \rightarrow \infty$, $M_Z(t)$ approaches $e^{t^2/2}$, the m.g.f. of the standard normal distribution. ■

A normal distribution may not be the best approximation to the aggregate claim distribution because the normal distribution is symmetric and the distribution of aggregate claims is often skewed. This skewness is evident in Table 12.5.1, which shows that the third central moment of S under each of the compound Poisson and compound negative binomial distributions is not 0. For positive claim amount distributions, $P(0) = 0$, and the third central moment of S is positive in each case.

Calculation of Third Central Moment of S

Step	Distribution of S: Compound Poisson	Compound Negative Binomial
$M_S(t)$	$\exp\{\lambda[M_X(t) - 1]\}$	$\left[\dfrac{p}{1 - qM_X(t)}\right]^r$
$\log M_S(t)$	$\lambda[M_X(t) - 1]$	$r \log p - r \log[1 - qM_X(t)]$
$\dfrac{d^3}{dt^3} \log M_S(t)$	$\lambda M_X'''(t)$	$\dfrac{rqM_X'''(t)}{1 - qM_X(t)} + \dfrac{3rq^2M_X'(t)M_X''(t)}{[1 - qM_X(t)]^2} + \dfrac{2rq^3M_X'(t)^3}{[1 - qM_X(t)]^3}$
$\mathrm{E}[(S - \mathrm{E}[S])^3] = \left.\dfrac{d^3}{dt^3} \log M_S(t)\right\|_{t=0}$ *	λp_3	$\dfrac{rqp_3}{p} + \dfrac{3rq^2p_1p_2}{p^2} + \dfrac{2rq^3p_1^3}{p^3}$

*For $k \ge 4$, $\left.\dfrac{d^k}{dt^k} \log M(t)\right|_{t=0}$ is not the k-th central moment.

In completing Table 12.5.1 we use properties of the logarithm of a m.g.f., for example,

$$\left.\frac{d}{dt} \log M_X(t)\right|_{t=0} = \frac{M_X'(0)}{M_X(0)} = \mu$$

and

$$\frac{d^2}{dt^2} \log M_X(t)\Big|_{t=0} = \frac{M_X''(0)M_X(0) - M_X'(0)^2}{M_X(0)^2} = \sigma^2.$$

In Exercise 12.23(a), the reader is asked to confirm the relation used in the last row of Table 12.5.1.

Because of this skewness we seek a more general approximation to the distribution of aggregate claims, one that accommodates skewness. For this second approximation, we begin with a gamma distribution. This choice is motivated by the fact that the gamma distribution has a positive third central moment as do the compound Poisson and compound negative binomial distributions with positive claim amounts. We let $G(x{:}\alpha, \beta)$ denote the d.f. of the gamma distribution with parameters α and β; that is,

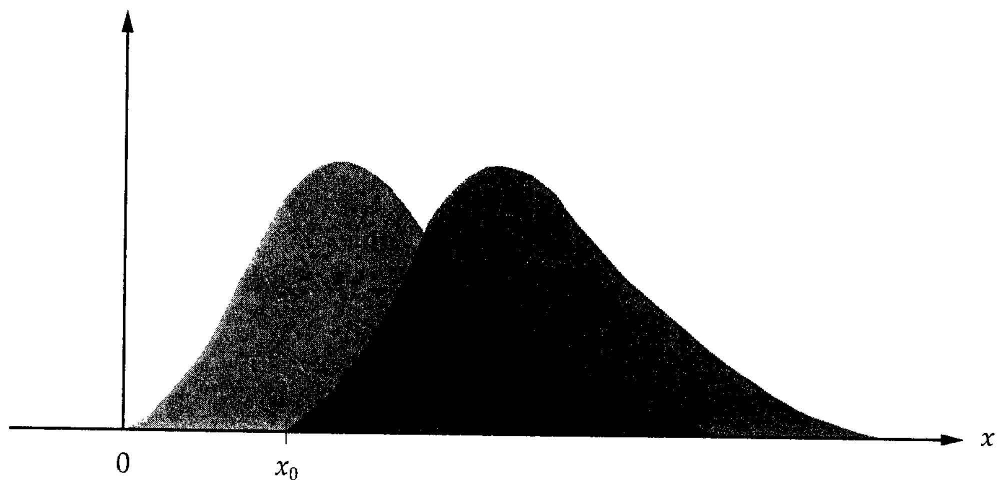
$$G(x{:}\alpha, \beta) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}\, dt. \tag{12.5.6}$$

Then for any x_0 we define a new d.f., denoted by $H(x{:}\alpha, \beta, x_0)$, as

$$H(x{:}\alpha, \beta, x_0) = G(x - x_0{:}\alpha, \beta). \tag{12.5.7}$$

This amounts to a translation of the distribution $G(x{:}\alpha, \beta)$ by x_0. Figure 12.5.1 illustrates this for the case $x_0 > 0$ where $g(x)$, $x \geq 0$, and $h(x)$, $x \geq x_0$, denote, respectively, the p.d.f.'s associated with $G(x{:}\alpha, \beta)$ and $H(x{:}\alpha, \beta, x_0)$.

Translated Gamma Distribution



We approximate the distribution of aggregate claims S by a translated gamma distribution where the parameters α, β, and x_0 are selected by equating the first moment and second and third central moments of S with the corresponding characteristics of the translated gamma distribution. Since central moments of the translated gamma are the same as for the basic gamma distribution, this procedure imposes the requirements

$$\mathrm{E}[S] = x_0 + \frac{\alpha}{\beta}, \tag{12.5.8}$$

$$\mathrm{Var}(S) = \frac{\alpha}{\beta^2}, \tag{12.5.9}$$

$$\mathrm{E}[(S - \mathrm{E}[S])^3] = \frac{2\alpha}{\beta^3}. \tag{12.5.10}$$

From these we obtain

$$\beta = 2\,\frac{\mathrm{Var}(S)}{\mathrm{E}[(S - \mathrm{E}[S])^3]}, \tag{12.5.11}$$

$$\alpha = 4\,\frac{[\mathrm{Var}(S)]^3}{\mathrm{E}[(S - \mathrm{E}[S])^3]^2}, \tag{12.5.12}$$

$$x_0 = \mathrm{E}[S] - 2\,\frac{[\mathrm{Var}(S)]^2}{\mathrm{E}[(S - \mathrm{E}[S])^3]}. \tag{12.5.13}$$

For a compound Poisson distribution this procedure leads to

$$\alpha = 4\lambda \frac{p_2^3}{p_3^2}, \tag{12.5.14}$$

$$\beta = 2 \frac{p_2}{p_3}, \tag{12.5.15}$$

$$x_0 = \lambda\, p_1 - 2\lambda \frac{p_2^2}{p_3}. \tag{12.5.16}$$

Remark:

We can show that if $\alpha \to \infty$, $\beta \to \infty$, and $x_0 \to -\infty$ such that

$$\begin{aligned} x_0 + \frac{\alpha}{\beta} &= \mu \text{ (constant)}, \\ \frac{\alpha}{\beta^2} &= \sigma^2 \text{ (constant)}, \end{aligned} \tag{12.5.17}$$

the distribution $H(x{:}\alpha, \beta, x_0)$ converges to the $N(\mu, \sigma^2)$ distribution. Therefore, the family of normal distributions is contained, as limiting distributions, within this family of three-parameter gamma distributions. In this sense, this approximation is a generalization of the normal approximation.

Example 12.5.1

Consider the Poisson distribution with parameter $\lambda = 16$. This is the same as the compound Poisson distribution with $\lambda = 16$ and a degenerate claim amount distribution at 1. Compare this distribution with approximations by

a. A translated gamma distribution
b. A normal distribution.

Solution:

a. Here $p_k = 1$, $k = 1, 2, 3$, and from (12.5.14)–(12.5.16), we have $\alpha = 64$, $\beta = 2$, $x_0 = -16$. Note that, unlike the case in Figure 12.5.1, x_0 is negative.
b. For the normal approximation, we use $\mu = 16$ and $\sigma = 4$.

The results given below compare the three distributions. In the approximations, the half-integer discontinuity correction was used to approximate $F_S(x)$ for $x = 5, 10, \ldots, 40$.

	Exact	Approximations	
x	$\sum_{y=0}^{x} \frac{e^{-16}(16)^y}{y!}$	$G(x+16.5:64, 2)$	$\Phi\left(\frac{x+0.5-16}{4}\right)$
5	0.001384	0.001636	0.004332
10	0.077396	0.077739	0.084566
15	0.466745	0.466560	0.450262
20	0.868168	0.868093	0.869705
25	0.986881	0.986604	0.991226
30	0.999433	0.999378	0.999856
35	0.999988	0.999985	0.999999
40	1.000000	1.000000	1.000000

▼

In the case of the compound negative binomial distribution there is an additional argument that supports the use of a gamma approximation. The argument is outlined in the Appendix to this chapter.

12.6 Notes and References

Chapter 2 of Seal (1969) contains an extensive survey of the literature on collective risk models, including the pioneering work of Lundberg on the compound Poisson distribution. Several authors, for example, Dropkin (1959) and Simon (1960), have used the negative binomial distribution to model the number of automobile accidents by a collection of policyholders in a fixed period.

In Example 12.3.1 we derived the negative binomial distribution by assuming that the unknown Poisson parameter has a gamma distribution. This idea goes back at least as far as Greenwood and Yule's work on accident proneness (1920). An alternative derivation in terms of a contagion model is due to Polya and Eggenberger and may be found in Chapter 2 of Bühlmann (1970). In the special case where r is an integer, the negative binomial can be obtained as the distribution of the number of Bernoulli trials that end in failure prior to the r-th success. This development may be found in most probability texts, but has little relevance to the subject of this chapter.

Theorem 12.4.2 has been known for some time and can be studied in Section 2, Chapter 2 of Feller (1968). The alternative method for computing probabilities for a compound Poisson distribution, which is based on Theorem 12.4.2, was suggested by Pesonen (1967) and implemented by Halmstad (1976) in the calculation of stop-loss premiums. Theorem 12.4.2 has a converse, which was not stated in Section 12.4. Renyi (1962) shows that the mutual independence of $N_1, N_2, \ldots, N_m$ implies that N has a Poisson distribution. Hence the alternative method of computing will work only for the compound Poisson distribution.

The alternative method of computation may also be adopted to build a simulation model for aggregate claims. Instead of determining the individual claim amounts, one simulates $N_1, N_2, \ldots, N_m$ and obtains a realization of S directly from (12.4.7). For one determination of S, the expected number of random numbers required is $1 + \lambda$ under the basic method. If the alternative method is used, exactly m random numbers are required for each determination of a value of S.

There are several more elaborate methods of approximating the distribution of aggregate claims. The normal power and Esscher approximations are described in Beard, Pesonen, and Pentikäinen (1984). Several of the approximation methods have been compared by Bohman and Esscher (1963, 1964). Seal (1978a) presents the case for the translated gamma approximation and illustrates its excellent performance. Bowers (1966) approximated the distribution of aggregate claims by a sum of orthogonal functions, the first term of which is the gamma distribution. The result, stated in the Appendix to this chapter, that the gamma distribution can be obtained as a limit from the compound negative binomial is due to Lundberg (1940).

Sometimes the distribution of aggregate claims can be obtained from a numerical inversion of its m.g.f.; this is developed in Chapter 3 of Seal (1978).

A monograph by Panjer and Willmot (1992) develops more completely the ideas of this chapter with particular emphasis on recursive calculation and discrete approximations.

Appendix

Theorem 12.A.1

If the random variables S_k, $k = 0, 1, 2, \ldots$, have compound negative binomial distributions with parameters r and $q(k)$ and claim amount d.f. $P(x)$, and if the parameters of the negative binomial distributions are such that

$$\frac{q(k)}{p(k)} = k\frac{q}{p}$$

for $k = 1, 2, 3, \ldots$, where $q = 1 - p$ is a constant, then the distribution of

$$\frac{S_k}{\mathrm{E}[S_k]}$$

approaches $G(x; r, r)$ as $k \to \infty$.

Proof:

Using (12.3.12), we find the m.g.f. of $S_k / \mathrm{E}[S_k]$ to be

$$\left[\frac{p(k)}{1 - q(k)M_X(t / \mathrm{E}[S_k])}\right]^r. \tag{12.A.1}$$

We also have

$$M_X\left(\frac{t}{\mathrm{E}[S_k]}\right) = 1 + \frac{p_1}{\mathrm{E}[S_k]}\, t + \frac{p_2}{2\mathrm{E}[S_k]^2}\, t^2 + \cdots. \tag{12.A.2}$$

If (12.A.2) is substituted into (12.A.1), we obtain

$$\left\{\frac{p(k)}{1 - q(k) - [q(k)p_1 / \mathrm{E}[S_k]]t - [q(k)p_2 / 2\mathrm{E}[S_k]^2]t^2 - \cdots}\right\}^r. \tag{12.A.3}$$

Now, since

$$\mathrm{E}[S_k] = r\,\frac{q(k)p_1}{p(k)} = r\,\frac{kqp_1}{p},$$

we see that the m.g.f. of $S_k / \mathrm{E}[S_k]$ is

$$\left[1 - \frac{1}{r}t - \frac{p_2}{2r^2p_1^2(q/p)k}t^2 - \cdots\right]^{-r} = \left[1 - \frac{1}{r}t - R(k)\right]^{-r}$$

where the remainder term $R(k)$ is such that $\lim_{k\to\infty} R(k) = 0$. Therefore,

$$\lim_{k\to\infty} \mathrm{E}\left[\exp\left(t\frac{S_k}{\mathrm{E}[S_k]}\right)\right] = \left(\frac{r}{r-t}\right)^r, \tag{12.A.4}$$

which is the m.g.f. of a $G(x{:}r, r)$ distribution. ■

It follows from (12.A.4) that the m.g.f. of S_k itself is approximately

$$\left(\frac{r}{r - \mathrm{E}[S_k]t}\right)^r = \left\{\frac{r}{r - [rq(k)/p(k)]p_1t}\right\}^r = \left\{\frac{p(k)/[q(k)p_1]}{p(k)/[q(k)p_1] - t}\right\}^r,$$

which is the m.g.f. of $G\{x{:}r, [p(k)/q(k)p_1]\}$. Thus, when k is large, which under the hypothesis of Theorem 12.A.1 implies that the expected number of claims, $rq(k)/p(k) = rk(q/p)$, is large, the distribution of aggregate claims is approximately a gamma distribution.

Theorem 12.A.1 is presented to provide an argument supporting the use of gamma distributions to approximate the distribution of aggregate claims. Comparison of the main ideas in Theorem 12.5.1(b) and Theorem 12.A.1 leads to insights. Theorem 12.5.1(b) follows closely the pattern of the central limit theorem. If in (12.5.2) one writes

$$Z = \frac{S/r - (q/p)p_1}{\sqrt{(q/p)p_2 + (q^2/p^2)p_1^2}/\sqrt{r}},$$

the correspondence is clear, with the parameter r playing the role of n in the central limit theorem.

In Theorem 12.A.1 the parameter r of the negative binomial distribution remains fixed. The expected number of claims changes in proportion to a size parameter k, by compensating changes in $q(k)$ and $p(k) = 1 - q(k)$. Under the hypothesis of Theorem 12.A.1,

$$\mathrm{Var}(S_k) = \frac{rkq}{p}p_2 + r\frac{k^2q^2}{p^2}p_1^2$$

and

$$\mathrm{Var}\left(\frac{S_k}{\mathrm{E}[S_k]}\right) = \frac{p}{rkqp_1^2}p_2 + \frac{1}{r}.$$

As the size parameter $k \to \infty$,

$$\mathrm{Var}\left(\frac{S_k}{\mathrm{E}[S_k]}\right) \to \frac{1}{r},$$

as indicated by Theorem 12.A.1. Thus the gamma approximation may be considered in the negative binomial case when the expected number of claims is large and the claim amount distribution has relatively small dispersion.

Exercises

Section 12.1

12.1. Let S denote the number of people crossing a certain intersection by car in a given hour. How would you model S as a random sum?

12.2. Let S denote the total amount of rain that falls at a weather station in a given month. How would you model S as a random sum?

Section 12.2

12.3. Suppose N has a binomial distribution with parameters n and p. Express each of the following in terms of n, p, p_1, p_2, and $M_X(t)$:
a. E[S] b. Var(S) c. $M_S(t)$.

12.4. For the distribution specified in Example 12.2.2, calculate
a. E[N] b. Var(N) c. E[X]
d. Var(X) e. E[S] f. Var(S).

Section 12.3

12.5. Suppose that the claim amount distribution is the same as in Example 12.2.2, but that N has a Poisson distribution with E[N] = 1.7. Calculate
a. E[S] b. Var(S).

12.6. Suppose that S has a compound Poisson distribution with $\lambda = 2$ and $p(x) = 0.1x$, $x = 1, 2, 3, 4$. Calculate probabilities that aggregate claims equal 0, 1, 2, 3, and 4.

12.7. Consider the family of negative binomial distributions with parameters r and p. Let $r \to \infty$ and $p \to 1$ such that $r(1 - p) = \lambda$ remains constant. Show that the limit obtained is the Poisson distribution with parameter λ. [Hint: Note that $p^r = [1 - (\lambda / r)]^r \to e^{-\lambda}$ as $r \to \infty$, and consider the convergence of the m.g.f.]

12.8. Suppose that S has a compound Poisson distribution with Poisson parameter λ and claim amount p.f.

$$p(x) = [-\log(1 - c)]^{-1} \frac{c^x}{x} \qquad x = 1, 2, 3, \ldots, \qquad 0 < c < 1.$$

Consider the m.g.f. of S and show that S has a negative binomial distribution with parameters p and r. Express p and r in terms of c and λ.

12.9. Let

$$g(x) = 3^{18} x^{17} \frac{e^{-3x}}{17!}$$

and

$$h(x) = 3^6\, x^5 \frac{e^{-3x}}{5!} \qquad x > 0$$

be two p.d.f.'s. Write the convolution of these two distributions, that is, exhibit $g*h(x)$. [Hint: Proceed directly from the definition of convolution in Section 2.3, or make use of (12.3.19).]

12.10. Suppose that the number of accidents incurred by an insured driver in a single year has a Poisson distribution with parameter λ. If an accident happens, the probability is p that the damage amount will exceed a deductible amount. On the assumption that the number of accidents is independent of the severity of the accidents, derive the distribution of the number of accidents that result in a claim payment.

12.11. The m.g.f. of the Poisson inverse Gaussian distribution is given in the solution to Example 12.3.2. Replace the α parameter by $\lambda\beta$ so that the mean is now λ and the variance is $\lambda + \lambda\beta$. Show that

$$\lim_{\beta\to\infty} M_N(t) = e^{\lambda(e^t-1)}.$$

This confirms that the Poisson inverse Gaussian distribution approaches the Poisson distribution as $\beta \to \infty$ and the mean remains constant.

Section 12.4

12.12. Suppose that S_1 has a compound Poisson distribution with Poisson parameter $\lambda = 2$ and claim amounts that are 1, 2, or 3 with probabilities 0.2, 0.6, and 0.2, respectively. In addition, S_2 has a compound Poisson distribution with Poisson parameter $\lambda = 6$ and claim amounts that are either 3 or 4 with probability 0.5 for each. If S_1 and S_2 are independent, what is the distribution of $S_1 + S_2$?

12.13. Suppose that N_1, N_2, N_3 are mutually independent and that N_i has a Poisson distribution with $\mathrm{E}[N_i] = i^2$, $i = 1, 2, 3$. What is the distribution of $S = -2N_1 + N_2 + 3N_3$?

12.14. If N has a Poisson distribution with parameter λ, express $\Pr(N = n + 1)$ in terms of $\Pr(N = n)$.

Note that this recursive formula may be useful in calculations such as those for successive entries in columns (2), (3), and (4) of the alternate method calculations of Example 12.6.

12.15. Suppose that S has a compound Poisson distribution with parameter λ and discrete p.f. $p(x)$, $x > 0$. Let $0 < \alpha < 1$.

Consider $\tilde{S}$ with a distribution that is compound Poisson with Poisson parameter $\tilde{\lambda} = \lambda / \alpha$ and claim amount p.f. $\tilde{p}(x)$ where

$$\tilde{p}(x) = \begin{cases} \alpha p(x) & x > 0 \\ 1 - \alpha & x = 0. \end{cases}$$

This means we are allowing for claim amounts of 0 (as could happen if there is a deductible) and are modifying the distributions accordingly. Show that S and $\tilde{S}$ have the same distribution by

a. Comparing the m.g.f.'s of S and $\tilde{S}$
b. Comparing the definition of the distribution of S and $\tilde{S}$ in terms of possible claim amounts and the Poisson parameters of the distributions of their frequencies.

12.16. In Example 12.2.2, let N_1 be the random number of claims of amount 1 and N_2 the random number of claims of amount 2. Compute

a. $\Pr(N_1 = 1)$ b. $\Pr(N_2 = 1)$ c. $\Pr(N_1 = 1, N_2 = 1)$.

Are N_1 and N_2 independent?

12.17. Compute $f_S(x)$ for $x = 0, 1, 2, \ldots, 5$ for the following three compound distributions, each with claim amount distribution given by $p(1) = 0.7$ and $p(2) = 0.3$:

a. Poisson with $\lambda = 4.5$
b. Negative binomial with $r = 4.5$ and $p = 0.5$
c. Binomial with $m = 9$ and $p = 0.5$
d. For each of the distributions of (a), (b), and (c) calculate the mean and variance of the number of claims.

12.18. Let S, as given in (12.4.7), have a compound negative binomial distribution with parameters r and p and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6).

a. Show that N_i has a negative binomial distribution with parameters r and $p/(p + q\pi_i)$.
b. Show that, in general, N_1 and N_2 are not independent.
[Hint: Use the joint m.g.f. of $N_1, N_2, \ldots, N_m$ as in the proof of Theorem 12.4.2.]

12.19. Show that the compound distribution of Example 12.2.2 does not satisfy the hypotheses of Theorem 12.4.3.

Section 12.5

12.20. Show that if N has a Poisson distribution with parameter λ, the distribution of

$$Z = \frac{N - \lambda}{\sqrt{\lambda}}$$

approaches a $N(0, 1)$ distribution as $\lambda \rightarrow \infty$.

12.21. Use $\log M_S(t)$ as given in Table 12.5.1 to verify (12.3.3) and (12.3.11).

12.22. Suppose that the d.f. of S is $G(x{:}\alpha, \beta)$. Use the m.g.f. [see (12.3.19)] to show that

$$E[S^h] = \frac{\alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + h - 1)}{\beta^h} \qquad h = 1, 2, 3, \ldots .$$

12.23. a. Verify that

$$\left.\frac{d^3}{dt^3} \log M_X(t)\right|_{t=0} = E[(X - E[X])^3].$$

b. Use (a) to show that if S has a $G(x{:}\alpha, \beta)$ distribution, then

$$E[(S - E[S])^3] = \frac{2\alpha}{\beta^3}.$$

12.24. a. For a given α, determine β and x_0 so that $H(x{:}\alpha, \beta, x_0)$ has mean 0 and variance 1.

b. What is the limit of $H(x{:}\alpha, \sqrt{\alpha}, -\sqrt{\alpha})$ as $\alpha \to \infty$?

12.25. Suppose that S has a compound Poisson distribution with $\lambda = 12$ and claim amounts that are uniformly distributed between 0 and 1. Approximate $\Pr(S < 10)$ using

a. The normal approximation

b. The translated gamma approximation.

Miscellaneous

12.26. The loss ratio for a collection of insurance policies over a single premium period is defined as $R = S / G$ where S is aggregate claims and G is aggregate premiums. Assume that $G = p_1 E[N](1 + \theta)$, $\theta > 0$.

a. Show that

$$E[R] = (1 + \theta)^{-1}$$

and that

$$\operatorname{Var}(R) = \frac{E[N] \operatorname{Var}(X) + p_1^2 \operatorname{Var}(N)}{[p_1 E[N](1 + \theta)]^2}.$$

b. Develop an expression for $\operatorname{Var}(R)$ if

(i) N has a Poisson distribution

(ii) N has a negative binomial distribution.

12.27. Suppose that the distribution of S_1 is compound Poisson, given by λ and $P_1(x)$, and that the distribution of S_2 is compound negative binomial, given by r, p with $q = 1 - p$, and $P_2(x)$. Show that S_1 and S_2 have the same distribution provided that $\lambda = -r \log p$ and

$$P_1(x) = \frac{\sum_{k=1}^{\infty} (q^k / k)\, P_2^{*k}(x)}{-\log p}.$$

[Hint: Show equality of the m.g.f.'s.]
Note that, in the sense of this exercise, every compound negative binomial distribution can be considered as compound Poisson.

12.28. Let S, as given in (12.4.7), have a compound Poisson inverse Gaussian distribution with parameters α and β and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6).
a. Show that N_i has a Poisson inverse Gaussian distribution and determine its parameter values.
b. Show that, in general, N_1 and N_2 are not independent.

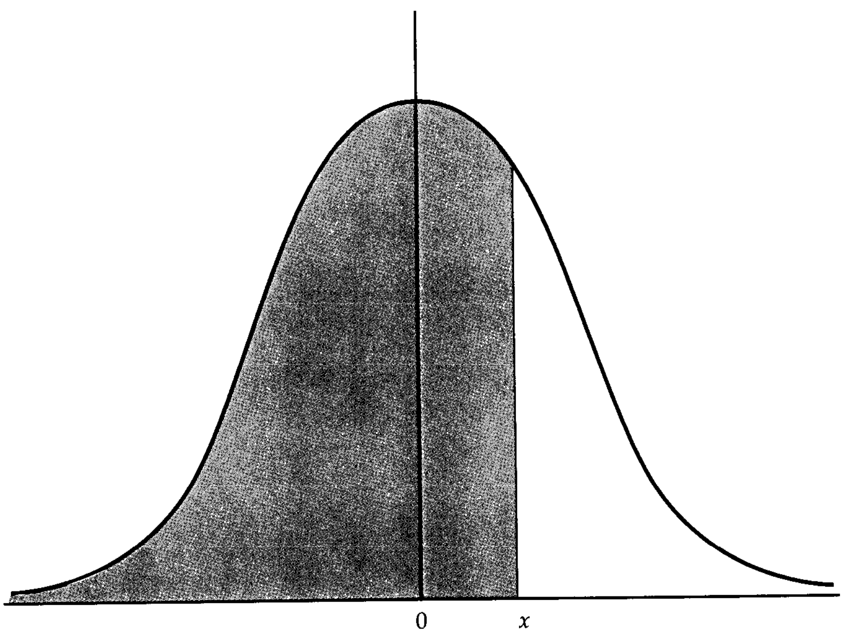
12.29. Follow the steps displayed in Table 12.5.1 to show that the third central moment of S, when it has a compound Poisson inverse Gaussian distribution, can be expressed in the parameters of its distribution as

$$\frac{\alpha}{\beta} p_3 + \frac{3\alpha}{\beta^2} p_1 p_2 + \frac{3\alpha}{\beta^3} p_1^3.$$

12.30. a. Verify that the extension of (2.2.10) for the mean and (2.2.11) for the variance to the third central moment, $\mu_3(W) = \mathrm{E}[\{W - \mathrm{E}[W]\}^3]$, is $\mathrm{E}[\mu_3(W|V)] + 3\,\mathrm{Cov}(\mathrm{Var}(W|V), \mathrm{E}[W|V]) + \mu_3(\mathrm{E}[W|V])$.
[Hint: Write $\mathrm{E}[W - \mathrm{E}[W]] = \mathrm{E}[(W - \mathrm{E}[W|V]) + (\mathrm{E}[W|V] - \mathrm{E}[W])]$, expand the third power, and take expectations termwise.]
b. Apply the result of (a) to S of (12.1.1) to express its third central moment in terms of the parameters of its distribution. [Hint: Refer to (12.2.5) and (12.2.6).]
c. Apply the formula of (b) to the compound distributions of Table 12.5.1 to confirm the third central moments shown there.
d. Apply the formula of (b) to the compound Poisson inverse Gaussian distribution to confirm the formula of the previous exercise.

Appendix 1

NORMAL DISTRIBUTION TABLE



Normal Distribution

The table on page 674 gives the value of

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2}\, dw$$

for certain values of x. The integer of x is given in the top row, and the first decimal place of x is given in the left column. Since the density function of x is symmetric, the value of the cumulative distribution function for negative x can be obtained by subtracting from unity the value of the cumulative distribution function for x.

x	0	1	2	3
0.0	0.5000	0.8413	0.9772	0.9987
0.1	0.5398	0.8643	0.9821	0.9990
0.2	0.5793	0.8849	0.9861	0.9993
0.3	0.6179	0.9032	0.9893	0.9995
0.4	0.6554	0.9192	0.9918	0.9997
0.5	0.6915	0.9332	0.9938	0.9998
0.6	0.7257	0.9452	0.9953	0.9998
0.7	0.7580	0.9554	0.9965	0.9999
0.8	0.7881	0.9641	0.9974	0.9999
0.9	0.8159	0.9713	0.9981	1.0000

Selected Points of the Normal Distribution

$\Phi(x)$	x
0.800	0.842
0.850	1.036
0.900	1.282
0.950	1.645
0.975	1.960
0.990	2.326
0.995	2.576

Appendix 2A

ILLUSTRATIVE LIFE TABLE

Illustrative Life Table: Basic Functions

Age	l_x	d_x	$1{,}000\ q_x$
0	100 000.00	2 042.1700	20.4217
1	97 957.83	131.5672	1.3431
2	97 826.26	119.7100	1.2237
3	97 706.55	109.8124	1.1239
4	97 596.74	101.7056	1.0421
5	97 495.03	95.2526	0.9770
6	97 399.78	90.2799	0.9269
7	97 309.50	86.6444	0.8904
8	97 222.86	84.1950	0.8660
9	97 138.66	82.7816	0.8522
10	97 055.88	82.2549	0.8475
11	96 973.63	82.4664	0.8504
12	96 891.16	83.2842	0.8594
13	96 807.88	84.5180	0.8730
14	96 723.36	86.0611	0.8898
15	96 637.30	87.7559	0.9081
16	96 549.54	89.6167	0.9282
17	96 459.92	91.6592	0.9502
18	96 368.27	93.9005	0.9744
19	96 274.36	96.3596	1.0009
20	96 178.01	99.0569	1.0299
21	96 078.95	102.0149	1.0618
22	95 976.93	105.2582	1.0967
23	95 871.68	108.8135	1.1350
24	95 762.86	112.7102	1.1770
25	95 650.15	116.9802	1.2330
26	95 533.17	121.6585	1.2735
27	95 411.51	126.7830	1.3288
28	95 284.73	132.3953	1.3895
29	95 152.33	138.5406	1.4560

Illustrative Life Table: Basic Functions

Age	l_x	d_x	$1{,}000\ q_x$
30	95 013.79	145.2682	1.5289
31	94 868.53	152.6317	1.6089
32	94 715.89	160.6896	1.6965
33	94 555.20	169.5052	1.7927
34	94 385.70	179.1475	1.8980
35	94 206.55	189.6914	2.0136
36	94 016.86	201.2179	2.1402
37	93 815.64	213.8149	2.2791
38	93 601.83	227.5775	2.4313
39	93 374.25	242.6085	2.5982
40	93 131.64	259.0186	2.7812
41	92 872.62	276.9271	2.9818
42	92 595.70	296.4623	3.2017
43	92 299.23	317.7619	3.4427
44	91 981.47	340.9730	3.7070
45	91 640.50	366.2529	3.9966
46	91 274.25	393.7687	4.3141
47	90 880.48	423.6978	4.6621
48	90 456.78	456.2274	5.0436
49	90 000.55	491.5543	5.4617
50	89 509.00	529.8844	5.9199
51	88 979.11	571.4316	6.4221
52	88 407.68	616.4165	6.9724
53	87 791.26	665.0646	7.5755
54	87 126.20	717.6041	8.2364
55	86 408.60	774.2626	8.9605
56	85 634.33	835.2636	9.7538
57	84 799.07	900.8215	10.6230
58	83 898.25	971.1358	11.5752
59	82 927.11	1 046.3843	12.6181
60	81 880.73	1 126.7146	13.7604
61	80 754.01	1 212.2343	15.0114
62	79 541.78	1 302.9994	16.3813
63	78 238.78	1 399.0010	17.8812
64	76 839.78	1 500.1504	19.5231
65	75 339.63	1 606.2618	21.3203
66	73 733.37	1 717.0334	23.2871
67	72 016.33	1 832.0273	25.4391
68	70 184.31	1 950.6476	27.7932
69	68 233.66	2 072.1177	30.3680

Illustrative Life Table: Basic Functions

Age	l_x	d_x	$1{,}000\ q_x$
70	66 161.54	2 195.4578	33.1833
71	63 966.08	2 319.4639	36.2608
72	61 646.62	2 442.6884	39.6240
73	59 203.93	2 563.4258	43.2982
74	56 640.51	2 679.7050	47.3108
75	53 960.80	2 789.2905	51.6911
76	51 171.51	2 889.6965	56.4708
77	48 281.81	2 978.2164	61.6840
78	45 303.60	3 051.9717	67.3671
79	42 251.62	3 107.9833	73.5589
80	39 143.64	3 143.2679	80.3009
81	36 000.37	3 154.9603	87.6369
82	32 845.41	3 140.4624	95.6134
83	29 704.95	3 097.6146	104.2794
84	26 607.34	3 024.8830	113.6860
85	23 582.45	2 921.5530	123.8867
86	20 660.90	2 787.9129	134.9367
87	17 872.99	2 625.4088	146.8926
88	15 247.58	2 436.7474	159.8121
89	12 810.83	2 225.9244	173.7533
90	10 584.91	1 998.1533	188.7738
91	8 586.75	1 759.6818	204.9298
92	6 827.07	1 517.4869	222.2749
93	5 309.58	1 278.8606	240.8589
94	4 030.72	1 050.9136	260.7257
95	2 979.81	840.0452	281.9122
96	2 139.77	651.4422	304.4456
97	1 488.32	488.6776	328.3410
98	999.65	353.4741	353.5993
99	646.17	245.6772	380.2041
100	400.49	163.4494	408.1188
101	237.05	103.6560	437.2837
102	133.39	62.3746	467.6133
103	71.01	35.4358	498.9935
104	35.58	18.9023	531.2793
105	16.68	9.4105	564.2937
106	7.27	4.3438	597.8266
107	2.92	1.8458	631.6360
108	1.08	0.7163	665.4495
109	0.36	0.2517	698.9685
110	0.11	0.0793	731.8742

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	$1{,}000\ A_x$	$1{,}000\ (^2A_x)$
0	16.80096	49.0025	25.9210
1	17.09819	32.1781	8.8845
2	17.08703	32.8097	8.6512
3	17.07314	33.5957	8.5072
4	17.05670	34.5264	8.4443
5	17.03786	35.5930	8.4547
6	17.01675	36.7875	8.5310
7	16.99351	38.1031	8.6666
8	16.96823	39.5341	8.8553
9	16.94100	41.0757	9.0917
10	16.91187	42.7245	9.3712
11	16.88089	44.4782	9.6902
12	16.84807	46.3359	10.0460
13	16.81340	48.2981	10.4373
14	16.77685	50.3669	10.8638
15	16.73836	52.5459	11.3268
16	16.69782	54.8404	11.8295
17	16.65515	57.2558	12.3749
18	16.61024	59.7977	12.9665
19	16.56299	62.4720	13.6080
20	16.51330	65.2848	14.3034
21	16.46105	68.2423	15.0569
22	16.40614	71.3508	15.8730
23	16.34843	74.6170	16.7566
24	16.28783	78.0476	17.7128
25	16.22419	81.6496	18.7472
26	16.15740	85.4300	19.8657
27	16.08733	89.3962	21.0744
28	16.01385	93.5555	22.3802
29	15.93683	97.9154	23.7900
30	15.85612	102.4835	25.3113
31	15.77161	107.2676	26.9520
32	15.68313	112.2754	28.7206
33	15.59057	117.5148	30.6259
34	15.49378	122.9935	32.6772
35	15.39262	128.7194	34.8843
36	15.28696	134.7002	37.2574
37	15.17666	140.9437	39.8074
38	15.06159	147.4572	42.5455
39	14.94161	154.2484	45.4833

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	$1{,}000\ A_x$	$1{,}000\ ({}^2A_x)$
40	14.81661	161.3242	48.6332
41	14.68645	168.6916	52.0077
42	14.55102	176.3572	55.6199
43	14.41022	184.3271	59.4833
44	14.26394	192.6071	63.6117
45	14.11209	201.2024	68.0193
46	13.95459	210.1176	72.7205
47	13.79136	219.3569	77.7299
48	13.62235	228.9234	83.0624
49	13.44752	238.8198	88.7329
50	13.26683	249.0475	94.7561
51	13.08027	259.6073	101.1469
52	12.88785	270.4988	107.9196
53	12.68960	281.7206	115.0885
54	12.48556	293.2700	122.6672
55	12.27581	305.1431	130.6687
56	12.06042	317.3346	139.1053
57	11.83953	329.8381	147.9883
58	11.61327	342.6452	157.3280
59	11.38181	355.7466	167.1332
60	11.14535	369.1310	177.4113
61	10.90412	382.7858	188.1682
62	10.65836	396.6965	199.4077
63	10.40837	410.8471	211.1318
64	10.15444	425.2202	223.3401
65	9.89693	439.7965	236.0299
66	9.63619	454.5553	249.1958
67	9.37262	469.4742	262.8299
68	9.10664	484.5296	276.9212
69	8.83870	499.6963	291.4559
70	8.56925	514.9481	306.4172
71	8.29879	530.2574	321.7850
72	8.02781	545.5957	337.5361
73	7.75683	560.9339	353.6443
74	7.48639	576.2419	370.0803
75	7.21702	591.4895	386.8119
76	6.94925	606.6460	403.8038
77	6.68364	621.6808	421.0184
78	6.42071	636.5634	438.4155
79	6.16101	651.2639	455.9527

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	1,000 A_x	1,000 $({}^2A_x)$
80	5.90503	665.7528	473.5861
81	5.65330	680.0019	491.2698
82	5.40629	693.9837	508.9574
83	5.16446	707.6723	526.6012
84	4.92824	721.0431	544.1537
85	4.69803	734.0736	561.5675
86	4.47421	746.7428	578.7956
87	4.25710	759.0320	595.7923
88	4.04700	770.9244	612.5133
89	3.84417	782.4056	628.9163
90	3.64881	793.4636	644.9611
91	3.46110	804.0884	660.6105
92	3.28118	814.2726	675.8298
93	3.10914	824.0111	690.5878
94	2.94502	833.3007	704.8565
95	2.78885	842.1408	718.6115
96	2.64059	850.5325	731.8321
97	2.50020	858.4791	744.5010
98	2.36759	865.9853	756.6047
99	2.24265	873.0577	768.1330
100	2.12522	879.7043	779.0793
101	2.01517	885.9341	789.4400
102	1.91229	891.7573	799.2147
103	1.81639	897.1852	808.4054
104	1.72728	902.2295	817.0170
105	1.64472	906.9025	825.0563
106	1.56850	911.2170	832.5324
107	1.49838	915.1860	839.4558
108	1.43414	918.8224	845.8386
109	1.37553	922.1396	851.6944
110	1.32234	925.1507	857.0377

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	1,000 A_{xx}	1,000 $(^2A_{xx})$	$\ddot{a}_{x:x+10}$	1,000 $A_{x:x+10}$	1,000 $(^2A_{x:x+10})$
0	16.13448	86.7274	50.8875	16.28443	78.2400	34.7076
1	16.71842	53.6745	17.4565	16.55328	63.0218	18.1309
2	16.70637	54.3565	16.9753	16.52270	64.7527	18.2195
3	16.68957	55.3072	16.6683	16.48839	66.6947	18.4277
4	16.66839	56.5060	16.5191	16.45053	68.8378	18.7468
5	16.64317	57.9339	16.5121	16.40925	71.1745	19.1700
6	16.61421	59.5733	16.6324	16.36464	73.6996	19.6923
7	16.58178	61.4085	16.8664	16.31677	76.4091	20.3096
8	16.54614	63.4258	17.2017	16.26571	79.2997	21.0188
9	16.50749	65.6137	17.6271	16.21147	82.3696	21.8172
10	16.46599	67.9626	18.1330	16.15408	85.6181	22.7036
11	16.42178	70.4655	18.7116	16.09353	89.0457	23.6776
12	16.37492	73.1176	19.3572	16.02977	92.6543	24.7402
13	16.32547	75.9170	20.0661	15.96277	96.4469	25.8935
14	16.27340	78.8643	20.8373	15.89244	100.4282	27.1413
15	16.21865	81.9632	21.6726	15.81866	104.6042	28.4891
16	16.16111	85.2203	22.5769	15.74131	108.9826	29.9441
17	16.10065	88.6424	23.5556	15.66025	113.5710	31.5141
18	16.03715	92.2366	24.6142	15.57534	118.3771	33.2071
19	15.97049	96.0099	25.7588	15.48645	123.4087	35.0317
20	15.90053	99.9697	26.9958	15.39343	128.6737	36.9970
21	15.82715	104.1234	28.3320	15.29615	134.1800	39.1126
22	15.75021	108.4786	29.7746	15.19448	139.9353	41.3884
23	15.66958	113.0429	31.3311	15.08826	145.9474	43.8349
24	15.58511	117.8241	33.0098	14.97738	152.2240	46.4632
25	15.49667	122.8299	34.8192	14.86169	158.7725	49.2847
26	15.40413	128.0682	36.7681	14.74106	165.6003	52.3114
27	15.30734	133.5468	38.8662	14.61538	172.7144	55.5555
28	15.20617	139.2737	41.1234	14.48452	180.1217	59.0301
29	15.10047	145.2564	43.5502	14.34836	187.8286	62.7483
30	14.99012	151.5028	46.1574	14.20681	195.8411	66.7238
31	14.87498	158.0203	48.9566	14.05976	204.1648	70.9706
32	14.75491	164.8162	51.9595	13.90712	212.8047	75.5028
33	14.62981	171.8977	55.1785	13.74882	221.7652	80.3352
34	14.44953	179.2716	58.6264	13.58478	231.0501	85.4824
35	14.36398	186.9444	62.3164	13.41497	240.6623	90.9593
36	14.22304	194.9221	66.2622	13.23933	250.6040	96.7805
37	14.07662	203.2104	70.4777	13.05785	260.8765	102.9610
38	13.92461	211.8144	74.9770	12.87052	271.4799	109.5154
39	13.76695	220.7386	79.7749	12.67736	282.4136	116.4579

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	$1{,}000\ A_{xx}$	$1{,}000\ (^2A_{xx})$	$\ddot{a}_{x:x+10}$	$1{,}000\ A_{x:x+10}$	$1{,}000\ (^2A_{x:x+10})$
40	13.60357	229.9867	84.8858	12.47840	293.6755	123.8024
41	13.43441	239.5619	90.3247	12.27370	305.2625	131.5623
42	13.25943	249.4664	96.1064	12.06333	317.1700	139.7502
43	13.07861	259.7015	102.2457	11.84740	329.3924	148.3778
44	12.89194	270.2677	108.7571	11.62604	341.9222	157.4559
45	12.69943	281.1642	115.6552	11.39940	354.7507	166.9939
46	12.50112	292.3892	122.9537	11.16767	367.8678	177.0001
47	12.29706	303.9398	130.6661	10.93105	381.2615	187.4810
48	12.08733	315.8114	138.8051	10.68978	394.9184	198.4414
49	11.87202	327.9986	147.3826	10.44412	408.8233	209.8841
50	11.65127	340.4941	156.4093	10.19438	422.9597	221.8099
51	11.42522	353.2895	165.8951	9.94087	437.3092	234.2171
52	11.19405	366.3746	175.8482	9.68395	451.8518	247.1016
53	10.95797	379.7377	186.2752	9.42400	466.5661	260.4567
54	10.71721	393.3656	197.1814	9.16142	481.4292	274.2728
55	10.47203	407.2435	208.5696	8.89664	496.4168	288.5375
56	10.22273	421.3546	220.4410	8.63011	511.5030	303.2353
57	9.96964	435.6810	232.7940	8.36232	526.6612	318.3475
58	9.71308	450.2029	245.6250	8.09375	541.8633	333.8526
59	9.45345	464.8990	258.9275	7.82491	557.0805	349.7258
60	9.19114	479.7465	272.6922	7.55633	572.2833	365.9390
61	8.92659	494.7213	286.9070	7.28853	587.4417	382.4614
62	8.66024	509.7977	301.5568	7.02206	602.5251	399.2593
63	8.39257	524.9491	316.6234	6.75745	617.5030	416.2961
64	8.12406	540.1477	332.0853	6.49524	632.3449	433.5327
65	7.85522	555.3647	347.9183	6.23597	647.0206	450.9279
66	7.58658	570.5707	364.0947	5.98016	661.5006	468.4383
67	7.31867	585.7356	380.5839	5.72831	675.7560	486.0192
68	7.05202	600.8289	397.3525	5.48092	689.7590	503.6243
69	6.78718	615.8203	414.3642	5.23847	703.4830	521.2065
70	6.52467	630.6790	431.5803	5.00138	716.9030	538.7185
71	6.26504	645.3750	448.9598	4.77008	729.9954	556.1128
72	6.00881	659.8785	466.4595	4.54495	742.7386	573.3422
73	5.75650	674.1606	484.0346	4.32634	755.1127	590.3606
74	5.50858	688.1934	501.6393	4.11456	767.1002	607.1233
75	5.26555	701.9503	519.2266	3.90989	778.6857	623.5869
76	5.02783	715.4057	536.7489	3.71254	789.8559	639.7107
77	4.79586	728.5362	554.1588	3.52273	800.6001	655.4561
78	4.57002	741.3197	571.4091	3.34060	810.9096	670.7874
79	4.35066	753.7364	588.4536	3.16625	820.7782	685.6720

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	1,000 A_{xx}	1,000 $(^2A_{xx})$	$\ddot{a}_{x:x+10}$	1,000 $A_{x:x+10}$	1,000 $(^2A_{x:x+10})$
80	4.13809	765.7683	605.2473	2.99977	830.2020	700.0806
81	3.93260	777.3999	621.7467	2.84117	839.1791	713.9874
82	3.73442	788.6175	637.9108	2.69046	847.7098	727.3701
83	3.54375	799.4102	653.7007	2.54760	855.7965	740.2101
84	3.36075	809.7690	669.0804	2.41251	863.4431	752.4921
85	3.18552	819.6876	684.0169	2.28509	870.6554	764.2049
86	3.01814	829.1617	698.4806	2.16521	877.4407	775.3401
87	2.85866	838.1892	712.4451	2.05273	883.8075	785.8931
88	2.70706	846.7701	725.8879	1.94748	889.7655	795.8619
89	2.56332	854.9067	738.7899	1.84925	895.3253	805.2478
90	2.42735	862.6027	751.1355	1.75786	900.4984	814.0543
91	2.29908	869.8636	762.9129	1.67309	905.2969	822.2875
92	2.17836	876.6967	774.1136	1.59471	909.7331	829.9554
93	2.06505	883.1102	784.7323	1.52251	913.8199	837.0680
94	1.95899	889.1137	794.7670	1.45626	917.5703	843.6367
95	1.85998	894.7179	804.2185	1.39571	920.9973	849.6744
96	1.76783	899.9341	813.0901	1.34065	924.1140	855.1951
97	1.68232	904.7742	821.3876	1.29084	926.9335	860.2140
98	1.60324	909.2506	829.1188	1.24605	929.4689	864.7475
99	1.53035	913.3762	836.2934	1.20604	931.7333	868.8126
100	1.46344	917.1638	842.9228	1.17060	933.7399	872.4279
101	1.40226	920.6266	849.0197	1.13946	935.5020	875.6129
102	1.34659	923.7777	854.5980	1.11241	937.0336	878.3888
103	1.29620	926.6301	859.6727	1.08917	938.3489	880.7785
104	1.25086	929.1969	864.2600	1.06949	939.4630	882.8066
105	1.21032	931.4911	868.3771	1.05308	940.3917	884.5002
106	1.17437	933.5261	872.0421	1.03965	941.1518	885.8881
107	1.14277	935.3151	875.2746	1.02889	941.7609	887.0017
108	1.11526	936.8720	878.0956	1.02047	942.2374	887.8735
109	1.09161	938.2110	880.5276	1.01406	942.6001	888.5376
110	1.07154	939.3470	882.5952	1.00934	942.8678	889.0280

Appendix 2B

ILLUSTRATIVE SERVICE TABLE

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	S_x
30	100 000	100	19 900	—	—	1.00
31	79 910	80	14 376	—	—	1.06
32	65 454	72	9 858	—	—	1.13
33	55 524	61	5 702	—	—	1.20
34	49 761	60	3 971	—	—	1.28
35	45 730	64	2 693	46	—	1.36
36	42 927	64	1 927	43	—	1.44
37	40 893	65	1 431	45	—	1.54
38	39 352	71	1 181	47	—	1.63
39	38 053	72	989	49	—	1.74
40	36 943	78	813	52	—	1.85
41	36 000	83	720	54	—	1.96
42	35 143	91	633	56	—	2.09
43	34 363	96	550	58	—	2.22
44	33 659	104	505	61	—	2.36
45	32 989	112	462	66	—	2.51
46	32 349	123	421	71	—	2.67
47	31 734	133	413	79	—	2.84
48	31 109	143	373	87	—	3.02
49	30 506	156	336	95	—	3.21
50	29 919	168	299	102	—	3.41
51	29 350	182	293	112	—	3.63
52	28 763	198	259	121	—	3.86
53	28 185	209	251	132	—	4.10
54	27 593	226	218	143	—	4.35
55	27 006	240	213	157	—	4.62
56	26 396	259	182	169	—	4.91
57	25 786	276	178	183	—	5.21
58	25 149	297	148	199	—	5.53
59	24 505	316	120	213	—	5.86

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	S_x
60	23 856	313	—	—	3 552	6.21
61	19 991	298	—	—	1 587	6.56
62	18 106	284	—	—	2 692	6.93
63	15 130	271	—	—	1 350	7.31
64	13 509	257	—	—	2 006	7.70
65	11 246	204	—	—	4 448	8.08
66	6 594	147	—	—	1 302	8.48
67	5 145	119	—	—	1 522	8.91
68	3 504	83	—	—	1 381	9.35
69	2 040	49	—	—	1 004	9.82
70	987	17	—	—	970	10.31

Appendix 3

SYMBOL INDEX

Symbol	Page
${}_kW_x$	503
${}_kW_{x:\overline{n}\rvert}$	503
${}^h_kW_x$	503
$(Wa)_t$	627
${}_k\bar{W}(\bar{A}_x)$	503
${}_k\bar{W}(\bar{A}_{x:\overline{n}\rvert})$	503
${}^h_k\bar{W}(\bar{A}_x)$	503
(x)	52
$(x_1x_2\cdots x_m)$	263
$(\overline{x_1x_2\cdots x_m})$	268
$\dfrac{k}{x_1x_2\cdots x_m}$	556
$\dfrac{[k]}{x_1x_2\cdots x_m}$	556
X_i	27, 367
$X(\theta)$	617
Y	134
$y(s,s+m)$	657
$Y(t,n)$	659
z_t	94
Z	94
${}_mZ_y$	352
α	294, 519
$\alpha(m)$	152
$\bar{\alpha}$	520
α^{CRVM}	522
β	519, 610
$\beta(m)$	152
$\bar{\beta}$	520
β^{CRVM}	522
$\beta(x,u)$	600
$\Gamma(\alpha)$	374

Symbol	Page
δ	96
δ_t	96
θ	41, 617
$\lambda(t)$	625
$\lambda(t,n)$	659
Λ	373
Λ_h	242
$\mu(x)$	55
$\mu_x(t)$	79
$\mu_x^{(d)}$	351
$\mu_x^{(i)}$	351
$\mu_x^{(w)}$	351
$\mu_x^{(j)}(t)$	311
$\mu_x^{(\tau)}(t)$	311
$\mu_{xy}(t)$	266
$\mu_{\overline{xy}}(t)$	270
$\mu(x,u)$	589
π_h	230
π_t	597
ρ	610
τ	310, 608
$\phi(x)$	600
$\phi(x,u)$	600
$\psi(u)$	400
$\tilde{\psi}(u)$	401
$\psi(u,t)$	400
$\psi(u;w)$	427
$\tilde{\psi}(u,w)$	404
ω	63

Appendix 4

GENERAL RULES FOR SYMBOLS OF ACTUARIAL FUNCTIONS

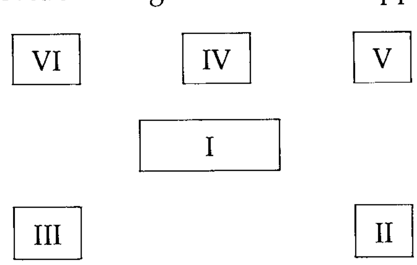
An actuarial function is represented by a principal symbol and a combination of auxiliary symbols such as letters, numerals, double dots, circles, hats, and horizontal and vertical bars. The principal symbol expresses the general definition of the function; choice and placement of the auxiliary symbols at the top and corners give precise meaning. We review the rules for selecting and placing the symbols and show one or more functional forms in common application areas.

This notation is based upon the system of International Actuarial Notation (IAN) that was originally adopted by the Second International Congress of Actuaries in London in 1898 and is modified periodically under the guidance of the Permanent Committee of Actuarial Notations of the International Actuarial Association. IAN is a basic system of principles that does not cover all areas of actuarial applications. In this text these principles have been followed, and sometimes extended, to construct consistent notation where needed.

This Appendix is meant to provide the reader with an overview of basic patterns for expressing the symbols appearing in this book. Although it is a good introduction to IAN, it is not exhaustive. Authoritative sources for further reference are:

- Actuarial Society of America, "International Actuarial Notation," *Transactions,* XLVIII, 1947: 166–176.
- Faculty of Actuaries, *Transactions,* XIX, 1950: 89.
- *Journal of the Institute of Actuaries,* LXXV, 1949: 121.

An actuarial symbol can be viewed as illustrated below. Box I represents the principal symbol, the others subscripts or superscripts. The roman numerals in the boxes correspond to the section designations of this Appendix.



Section I. Center

Principal Symbol	Description	Topic
i	Effective rate of interest for a time period, usually 1 year, or with a superscript in Position V, a nominal rate.	Interest
v	Present value of 1 due at the end of the effective interest period, usually 1 year.	
δ	Force of interest, usually stated as an annual rate.	
d	Effective rate of interest-in-advance, or discount rate, for a time period of usually 1 year, or with a superscript in Position V, a nominal rate. This symbol never has a subscript in Position II.	
l	Expected number, or number, of survivors at a given age.	Life Tables
d	Expected number, or number, of those dying within a given time period. This symbol always has a subscript in Position II.	
p	Probability of surviving for a given time period.	
q	Probability of dying within a given time period.	
μ	Force of mortality, usually stated on an annual basis.	
m	Central death rate for a given time period.	
L	Expected number, or number, of years lived within a time period by the survivors at the beginning of the period.	
T	Expected total, or total, future lifetime of the survivors at a given age. (The above are survivorship group definitions of the life table functions denoted by l, d, L, and T. For the alternative stationary population definitions, see Chapter 19.)	
A	Actuarial present value (net single premium) of an insurance or pure endowment of 1.	Life Insurance and Pure Endowments

Principal Symbol	*Description*	*Topic*
(IA)	Actuarial present value (net single premium) of an insurance with a benefit amount of 1 at the end of the first year, increasing linearly at a rate of 1 per year.	
(DA)	Actuarial present value (net single premium) of a term insurance with an initial benefit amount equal to the term and decreasing linearly at the rate of 1 per year.	
E	Actuarial present value of a pure endowment of 1.	
a	Actuarial present value of an annuity of 1 per time period, usually 1 year.	Annuities
s	Actuarial accumulated value of an annuity of 1 per time period, usually 1 year.	
(Ia)	Actuarial present value of an annuity payable at the rate of 1 per year at the end of the first year and increasing linearly at a rate of 1 per year.	
(Da)	Actuarial present value of a temporary annuity with an initial payment rate equal to the term and decreasing linearly at a rate of 1 per period.	
P	Level annual premium rate to cover only benefits, usually determined by the equivalence principle.	Premiums
V	Reserve to cover future benefits in excess of future benefit premiums.	Reserves
W	Face amount of a paid-up policy purchased with a cash value equal to the reserve. (Principle symbols for benefit premiums, reserves, and amounts of reduced paid-up insurance, P, V, and W, are combined with benefit symbols unless the benefit is a level unit insurance payable at the end of the year of death.)	[Examples: $\bar{P}_x$; $P(\bar{A}_x)$; ${}_{10}V^{(4)}\,(\bar{A}_{x:\overline{n}|})$; $P^{(12)}({}_{30|}\ddot{a}_{35}^{(12)})$]
S	Salary scale function used to project salaries.	Pensions
Z	Average of a given number of salary scale function values, usually at unit intervals in the independent variable.	

Section II. Lower Space to the Right

Auxiliary Symbol	*Description*	*Examples*
x; 10	A single letter or numeral is the individual's age at the commencement of the overall time period implied by the principal symbol.	a_x; $\bar{a}_{10}$ q_x; ${}_5q_{10}$ $\bar{A}_x$; A_{10}
$\overline{n}\rvert$; $\overline{10}\rvert$	A term certain is indicated by a single letter or numeral under an angle.	$A_{x:\overline{n}\rvert}$; $\ddot{a}_{\overline{10}\rvert}$
$[x]$; $[35]$ $[x] + t$; $[35 - n]$ $+ n$	Alphanumeric expressions enclosed by brackets indicate the age at which the life was selected. A term, representing duration since selection, may be added to the bracketed expression to express the attained age of the life.	$l_{[x]}$; $l_{[x]+10}$ $A_{[35]}$; $\ddot{a}^{i}_{[35-n]+n}$
xyz or $x{:}y{:}z$ $25{:}\overline{10}\rvert$	Two or more alphanumeric characters indicate a joint status that survives until the first death or expiration of the indicated lives and terms certain.	l_{xyz}; $A_{x:y:z}$ $\ddot{a}_{25:\overline{10}\rvert}$; $P_{25:\overline{10}\rvert}$
$/\overline{\quad}\backslash$	This symbol emphasizes the joint status when ambiguity is possible.	$A^{1}_{/\overline{xy}\backslash :z}$
$\overset{1}{x}{:}\overline{10}\rvert$; $\overset{2}{x}\underset{1}{y}z$	Numerals can be placed above or below the individual statuses of a collection of alphanumeric characters to show the order in which the units are to fail for an (insurable) event to occur. Benefits are payable upon the failure of the status with a numeral above it.	$\bar{A}^{1}_{x:\overline{n}\rvert}$; ${}_{\infty}q_{\underset{1}{x}\underset{2}{y}\overset{3}{z}}$
$\overline{xyz}$; $\overline{65{:}60{:}\overline{10}\rvert}$	A horizontal bar over a collection of alphanumeric characters defines a status that survives until the last survivor of the individual statuses fails.	$a_{\overline{xyz}}$; $\bar{A}_{\overline{xy:\overline{n}\rvert}}$
$\overline{xyz}^{\,r}$; $\overline{x{:}y{:}10}^{\,[r]}$	A single alphanumeric character, say, r, above the right end of the bar over the set of alphanumeric characters defines a status that survives as long as at least r of the individual statuses survive. If the r is enclosed in brackets, the status exists only while exactly r of the individual statuses survive.	$\bar{a}_{\overline{xyz}}^{[2]}$; $\bar{A}_{\overline{xyz}}^{2}$

Auxiliary Symbol	Description	Examples
$y\|x$; $60\|55$; $\overset{1}{y}z\|x$	A vertical bar separating the alphanumeric characters indicates that the income or coverage of the principal symbol commences upon the failure, as specified, of the status before the bar and continues until the failure of the status following the bar, providing the statuses fail in that order.	$a_{y\|x}$; $a_{\underset{1}{w}y\overset{3}{z}\|x}$

Section III. Lower Space to the Left

Auxiliary Symbol	Description	Examples
n; 15	A single alphanumeric character shows the time for which the principal symbol is evaluated. For an annual premium, P, this position shows the maximum number of years for which the premiums are paid if this is less than the period of coverage of an insurance or the period of deferral for a deferred annuity.	${}_{n}p_{x}$; ${}_{15}E_{30}$; ${}_{20}P_{25}$; ${}_{20}V_{40:\overline{30}\|}$
$n\|m$; $n\|$	An alphanumeric pair separated by a vertical bar indicates a period of deferment (left of the bar) and a period following deferment (right of the bar). In some cases, when either is equal to 1 or infinity, it can be omitted.	${}_{n\|m}q_{x}$; ${}_{n\|}\bar{a}_{x}$

Section IV. Top Center

Auxiliary Symbol	Description	Examples
$\ddot{}$	The double dot (dieresis) on an annuity symbol indicates that the payments are at the beginning of the periods, that is, an annuity-due. Without the dieresis, the annuity is an annuity-immediate with payments at the ends of the periods.	$\ddot{a}_x$; $\ddot{s}_{\overline{40}\vert}$
$\bar{}$	A horizontal bar indicates that the frequency of events is infinite. For annuities the payments are considered to be made continuously, and for insurances the benefit is paid at the moment of failure.	$\bar{a}_x$; $\bar{A}_x$ ${}_3\bar{V}_x$; $\bar{P}(\bar{A}_x)$
$\circ$	A circle (degree sign) means that the benefit or lifetime is complete, that is, credited up to the time of death.	$\mathring{a}_x$; $\mathring{e}_x$

Section V. Upper Space to the Right

Auxiliary Symbol	Description	Examples
(m); (12)	An alphanumeric character in parentheses shows the number of annuity payments in an interest period, usually 1 year. For an insurance it is the number of periods in a year at the end of which the death benefit can be paid. On multiple decrement symbols it indicates the cause of decrement to be used or that the total of all decrements is to be used.	$s_{\overline{10}\vert}^{(12)}$; $A_x^{(m)}$ $q_x^{(2)}$; ${}_tp_x^{(\tau)}$
$\{m\}$; $\{12\}$	An alphanumeric character in braces shows the number of apportionable annuity-due payments in a time period, usually 1 year. On a principal symbol of a premium or a reserve, it shows that premiums are paid on this basis.	$\ddot{a}_{30:\overline{20}\vert}^{\{12\}}$; $P_{30}^{\{1\}}$ ${}_tV^{\{2\}}(\bar{A}_x)$
r; i	An alphabetic character indicates the special basis used for the actuarial present value.	$\ddot{a}_{65}^r$; $\ddot{a}_{[x]}^i$

Section VI. Upper Space to the Left

Auxiliary Symbol	*Description*	*Examples*
h; 2	Alphanumeric character indicating the number of years during which premiums are paid if this is less than the coverage period of the insurance or the deferral period of the deferred annuity. This is used only on the principal symbols V or W where Position III is used for the time for which the function is evaluated.	${}^{h}_{5}V_{30}$
	In this text a new use for this position is to show that the actuarial present value of an annuity or an insurance is calculated at a multiple of the assumed force of interest.	${}^{2}\bar{A}_{x}$; ${}^{2}\ddot{a}_{20:\overline{10}\vert}$

Appendix 5

SOME MATHEMATICAL FORMULAS USEFUL IN ACTUARIAL MATHEMATICS

The purpose here is not to recall familiar standard formulas and techniques, but to indicate some that may be less familiar to actuarial students.

Calculus

If

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t)\, dx,$$

then

$$\frac{dF(t)}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x, t)\, dx + f(\beta(t), t) \frac{d}{dt} \beta(t)$$
$$- f(\alpha(t), t) \frac{d}{dt} \alpha(t).$$

Calculus of Finite Differences

Operators

a. Shift:

$$\mathrm{E}[f(x)] = f(x + 1)$$

b. Difference:

$$\Delta f(x) = f(x + 1) - f(x) = (\mathrm{E} - 1) f(x)$$

c. Repeated differences:

$$\begin{aligned} \Delta^n f(x) &= \Delta[\Delta^{n-1} f(x)] \\ &= (\mathrm{E} - 1)^n f(x) \\ &= \sum_{k=0}^{n} \tbinom{n}{k} (-1)^{n-k} f(x + k) \end{aligned}$$

d. Difference of a product:

$$\Delta[f(x)\, g(x)] = f(x+1)\, \Delta g(x) + g(x)\, \Delta f(x)$$

e. Antidifference:

If

$$\Delta f(x) = g(x),$$

then

$$\Delta^{-1} g(x) = f(x) + w(x)$$

where

$$w(x) = w(x+1).$$

Applications

a. Representation of a polynomial (Newton's formula): Let $p_n(x)$ be a polynomial of degree n; then

$$p_n(x) = \sum_{k=0}^{n} \binom{x-a}{k} \Delta^k p_n(a).$$

b. Summation of series:

If

$$\Delta F(x) = f(x),$$

then

$$\begin{aligned} f(1) &= F(2) - F(1) \\ f(2) &= F(3) - F(2) \\ &\;\;\vdots \\ f(n) &= F(n+1) - F(n) \\ \sum_{x=1}^{n} f(x) &= F(n+1) - F(1) = \Delta^{-1} f(x)\Big|_{1}^{n+1}. \end{aligned}$$

c. Summation by parts:

$$\sum_{x=1}^{n} g(x)\, \Delta f(x) = f(x)\, g(x)\Big|_{1}^{n+1} - \Delta^{-1}[f(x+1)\, \Delta g(x)]\Big|_{1}^{n+1}$$

[Proof: Sum each side of the equation for $\Delta[f(x)g(x)]$ from $x = 1$ to $x = n$.]

Probability Distributions

Discrete Distributions	p.f.	Restrictions on Parameters	Moment Generating Function, $M(s)$	Moments	
				Mean	Variance
Binomial	$\binom{n}{x}p^x q^{n-x},\ x = 0, 1, \ldots n$	$0 < p < 1$ $q = 1 - p$	$(pe^s + q)^n$	np	npq
Bernoulli	Special case $n = 1$				
Negative Binomial	$\binom{r+x-1}{x}p^r q^x,\ x = 0, 1, 2, \ldots$	$0 < p < 1$ $q = 1 - p$ $r > 0$	$\left(\frac{p}{1 - qe^s}\right)^r,\ qe^s < 1$	$\frac{rq}{p}$	$\frac{rq}{p^2}$
Geometric	Special case $r = 1$				
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, 2, \ldots$	$\lambda > 0$	$e^{\lambda(e^s-1)}$	λ	λ
Uniform	$\frac{1}{n},\ x = 1, \ldots, n$	n, a positive integer	$\frac{e^s(1 - e^{sn})}{n(1 - e^s)},\ s \neq 0$ $1,\ s = 0$	$\frac{n + 1}{2}$	$\frac{n^2 - 1}{12}$

Continuous Distributions	p.d.f.	Restrictions on Parameters	Moment Generating Function, $M(s)$	Moments	
				Mean	Variance
Uniform	$\frac{1}{b-a}, \quad a < x < b$	—	$\frac{e^{bs} - e^{as}}{(b-a)s}, s \neq 0$ $1, s = 0$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} \exp[-(x-\mu)^2/2\sigma^2]$, $-\infty < x < \infty$	$\sigma > 0$	$\exp(\mu s + \sigma^2 s^2/2)$	μ	σ^2
Gamma	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$	$\alpha > 0, \beta > 0$	$\left(\frac{\beta}{\beta - s}\right)^\alpha, s < \beta$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Exponential	Special case $\alpha = 1$				
Chi-square	Special case $\alpha = \frac{k}{2}, \beta = \frac{1}{2}$	k, a positive integer			
Inverse Gaussian	$\frac{\alpha}{\sqrt{2\pi\beta}} x^{-3/2} \exp\left[-\frac{(\beta x - \alpha)^2}{2\beta x}\right]$, $x > 0$	$\alpha > 0, \beta > 0$	$\exp\left[\alpha\left(1 - \sqrt{1 - \frac{2s}{\beta}}\right)\right]$, $s < \frac{\beta}{2}$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Pareto	$\alpha x_0^\alpha / x^{\alpha+1}, x > x_0$	$x_0 > 0, \alpha > 0$		$\frac{\alpha x_0}{\alpha - 1}$ $\alpha > 1$	$\frac{\alpha x_0^2}{(\alpha-2)(\alpha-1)^2}$ $\alpha > 2$
Lognormal	$\frac{1}{x\sigma\sqrt{2\pi}} \exp[-(\log x - m)^2/2\sigma^2]$, $x > 0$	$-\infty < m < \infty$ $\sigma > 0$		$e^{m+\sigma^2/2}$	$(e^{\sigma^2} - 1)e^{2m+\sigma^2}$

Appendix 7

ANSWERS TO EXERCISES

Chapter 1

1.1. a. and b.

w	$u(w)$	$u(w_1,w_2)$	$u(w_1,w_2,w_3)$
0	-1.00	125×10^{-6}	-48×10^{-10}
4 000	-0.500	93×10^{-6}	-34×10^{-10}
6 700	-0.250	78×10^{-6}	-14×10^{-10}
8 300	-0.125	74×10^{-6}	—
10 000	0.000	—	—

1.2. b. 2, 2 d. 2 log 2

1.3. c. Var(X)

1.7. a. Yes, for all w b. $(90 < w < 100) \cup (w > 110)$

1.11. $-\dfrac{n}{2\alpha} \log (1 - 2\alpha)$

1.12. a. $G = 400 \log \dfrac{13}{12} = 32.02$ b. $G = 150 \log \dfrac{3}{2} = 60.82$

1.13. a. 30 b. 26

1.14. Complete insurance

1.17. a. $-[1 - F(d)]$

1.18. a. 10, 100

1.19. a. 50, $\dfrac{2{,}500}{3}$ b. $k = 0.25$, $d = 50$

c. $\mathrm{Var}[X - I_1(X)] = 468.75$, $\mathrm{Var}[X - I_2(X)] = 260.42$

Chapter 2

2.1. $\dfrac{1}{2}, \dfrac{19}{4}$

2.2. $\dfrac{1}{2}, \dfrac{77}{12}$

2.3. $\frac{35}{4}, \frac{1{,}085}{48}$

2.4. $\frac{7}{4}, \frac{77}{48}$

2.5. $\frac{49}{4}, \frac{735}{16}$

2.6. $\frac{a}{100}$ and $a^2\left(\frac{197}{30{,}000}\right)$

2.7.

x	$F_S(x)$
0	0.2268
1	0.2916
2	0.4374
3	0.6210
4	0.7434
5	0.8586
6	0.9018
7	0.9582
8	0.9762
9	0.9918
10	0.9948
11	0.9988
12	0.9996
13	1.0000

2.8. c. $\frac{1}{48}, \frac{1}{6}, \frac{1}{2}$

2.9. $E[X] = \frac{\alpha}{\beta}$, $\mathrm{Var}(X) = \frac{\alpha}{\beta^2}$

2.10. $E[X] = 1$, $\mathrm{Var}(X) = \frac{1}{3}$; $E[Y] = \frac{3}{2}$, $\mathrm{Var}(Y) = \frac{3}{4}$; $\Pr(X + Y > 4) \cong 0.0748$;
$\Pr(X + Y > 4) = 0.0833$

2.11. a. $b = -1, c = 1, d = a$ or $b = 1, c = 0, d = -a$
b. 0.0228, 0.1587, 0.5000

2.12. a. 18, 36 b. 27.8713, 31.9607

2.13. a. 0.0041 b. 0.0045

2.14. 3.56; that is, 35,600

2.15. a. 6.4, 6.144 b. $7(10^4)$, $17.072(10^8)$ c. 1.37341

2.16. 0.0062

Chapter 3

3.1.

$s(x)$	$F(x)$	$f(x)$	$\mu(x)$
$\cos x$	$1 - \cos x$	$\sin x$	—
—	$1 - e^{-x}$	e^{-x}	1
$\frac{1}{1+x}$	—	$\frac{1}{(1+x)^2}$	$\frac{1}{1+x}$

3.2. a. $\exp\left[\frac{-B}{\log c}(c^x - 1)\right]$ b. $\exp(-ux^{n+1})$ $u = \frac{k}{n+1}$

c. $\left(1 + \frac{x}{b}\right)^{-a}$

3.3. $\mu(x) = \frac{x^2}{4}$, $f(x) = \frac{x^2}{4} e^{-x^3/12}$, $F(x) = 1 - e^{-x^3/12}$

3.4. a. $\int_0^\infty \mu(x)\, dx < \infty$ b. $s'(x) > 0$ for some x including $x = 1, 2$

c. $\int_0^\infty f_X(x)\, dx = 2^n\Gamma(n) > 1$ for $n \geq 1$

3.5. a. $\frac{1}{100 - x}$ b. $\frac{x}{100}$ c. $\frac{1}{100}$ d. $\frac{3}{10}$

3.6. a. $1 - \frac{t}{60}$ b. $\frac{1}{60 - t}$ c. $\frac{1}{60}$

3.7. a. $\frac{8}{9}$ b. $\frac{1}{8}$ c. $\frac{1}{8}$ d. $\frac{1}{128}$ e. $\frac{128}{3}$

3.9. 0.001994

3.10. $f_X(x) = \binom{10}{x}(0.77107)^x(0.22893)^{10-x}$ $x = 0, 1, 2, \ldots, 10$

$E[\mathcal{L}(x)] = 7.7107$, $\mathrm{Var}[\mathcal{L}(x)] = 1.765211$

3.11. a. $\frac{9}{4}$ for each b. $\frac{27}{16}$ for each c. $-\frac{1}{3}$

3.12. a. ${}_5q_0 = 0.01505$ is more than 10 times ${}_5q_5 = 0.001503$
b. ${}_{55|5}q_{25} = 0.156729$

3.15. 1,436.19

3.18. a. $\frac{1}{c}$ b. $\frac{1}{c^2}$ c. $\frac{1}{c}\log 2$ d. 0

3.19. a. $te^{-t^2/2}$ b. $\sqrt{\frac{\pi}{2}}$

3.20. a. $\frac{(100 - x)}{2}$ b. $\frac{(100 - x)^2}{12}$ c. $\frac{(100 - x)}{2}$

3.23. a. $\mathring{e}_x = \dfrac{10 - x}{2} \quad x = 0, 1, 2, \ldots, 9$

$e_x = \dfrac{9 - x}{2} \quad x = 0, 1, 2, \ldots, 9$

3.24. a. $u(0) = e^{-\lambda}, \quad -\dfrac{c(x)}{d(x)} = 0, \quad \dfrac{1}{d(x)} = \dfrac{\lambda}{x + 1}$

b. $u(0) = (1 - p)^n, \quad -\dfrac{c(x)}{d(x)} = 0, \quad \dfrac{1}{d(x)} = \dfrac{(n - x)p}{(x + 1)(1 - p)}$

3.25. a. $u(0) = 0 \quad -\dfrac{c(x)}{d(x)} = 1 \quad \dfrac{1}{d(x)} = v$

b. $u(0) = 0 \quad -\dfrac{c(x)}{d(x)} = 1 + i \quad \dfrac{1}{d(x)} = 1 + i$

3.28. Uniform distribution: 0.989709
Constant force: 0.989656
Balducci: 0.989602

3.29. a. 77.59 b. 29.11

3.30. a. 0.044 b. 0.04421

3.31.

	Uniform Distribution	Constant Force	Balducci
a.	0.012696	0.012616	0.012537
b.	0.013676	0.013770	0.013865
c.	0.013770	0.013770	0.013770

3.35. a. $\dfrac{\alpha}{\omega - x}$ b. $\dfrac{\omega - x}{\alpha + 1}$

3.36. a. 0.00142 b. 0.99867

3.37. a. 0.317 b. 0.140

3.39. 0.97920

3.40. $\log\left(1 - \dfrac{q_{[x]}}{2}\right) - \log(1 - q_{[x]})$

3.41. $q'_x < 2q_x$

3.43. a. $\left(\dfrac{1 + Bc^x}{1 + B}\right)^{-A/(B \log c)}$

3.44. a. $\dfrac{5^7}{4^{18}}$ b. 77.2105

3.45. b. $-\log(1 - q_x)$ c. $\dfrac{-q_x^2}{(1 - q_x)\log(1 - q_x)}$ d. $\dfrac{1}{45}$

3.49. $q_{40}^1 = 0.0055547, \quad q_{40} = 0.0027812$

3.50. Check value $e_{40} = 35.367$

3.51. $e_{20} = 46.038$, $e_{40} = 28.366$, $e_{60} = 13.264$, $e_{80} = 3.889$, $e_{100} = 0.503$

3.52. Check value $\mathring{e}_{40} = 35.867$

3.53. Starting value $e_{y:\overline{0}|} = 0$, Check value $e_{25:\overline{20}|} = 19.369$

3.54. Starting value $e_{\omega:\overline{10}|} = 0$, Check value $e_{40:\overline{10}|} = 9.809$

3.55. $e_{15:\overline{25}|} = 24.610$

Chapter 4

4.5. b. $\bar{A}^{1}_{x:\overline{n}|} = \dfrac{\mu_{x+n}}{\delta + \mu_{x+n}} A_{x:\overline{n}|}^{\ \ 1}$ c. $-\dfrac{\mu_{x+n}}{\delta + \mu_{x+n}} (A_{x:\overline{n}|}^{\ \ 1})^2$, where n satisfies (b)

d. $n = \dfrac{\log 2}{\mu + \delta}$, min $\mathrm{Cov}(Z_1, Z_2) = -\dfrac{\mu}{4(\mu + \delta)}$

4.6. a. 0.237832 b. 0.416667

4.7. a. 0.092099 b. 0.055321

4.8. a. $\dfrac{20}{3(100 - x)}\left[1 - \left(\dfrac{20}{120 - x}\right)^3\right]$,

$$\frac{20}{7(100 - x)}\left[1 - \left(\frac{20}{120 - x}\right)^7\right] - \left\{\frac{20}{3(100 - x)}\left[1 - \left(\frac{20}{120 - x}\right)^3\right]\right\}^2$$

b. $\dfrac{20}{3(100 - x)}\left[10 - 10\left(\dfrac{20}{120 - x}\right)^2 - (100 - x)\left(\dfrac{20}{120 - x}\right)^3\right]$

4.10. a. $\dfrac{\mu}{(\mu + \delta)^2}$ b. $\mu\left[\dfrac{2}{(\mu + 2\delta)^3} - \dfrac{\mu}{(\mu + \delta)^4}\right]$

4.11. a. 0.407159 b. 5.554541

4.13. a. 0.5 b. 0.05

4.14. b. $(IA)_{x:\overline{m}|} = (IA)^{1}_{x:\overline{m}|} + mA_{x:\overline{m}|}^{\ \ 1}$

4.15. a. $v^{[k+(j+1)/m]}$ b. $A_x^{(m)} = \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x \sum_{j=0}^{m-1} {}_{j/m|1/m}q_{x+k}\, (1 + i)^{[1-(j+1)/m]}$

4.23. $A_x + A_{x:\overline{65-x}|}^{\ \ 1}$

4.24. 4,007.85

4.26. a. $\dfrac{9{,}100}{14 - k}$

b. $1{,}000{,}000[{}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2] + (k\pi)^2[{}^2\bar{A}^{1}_{x:\overline{n}|} - (\bar{A}^{1}_{x:\overline{n}|})^2] - 2{,}000\, k\pi\, \bar{A}^{1}_{x:\overline{n}|} A_{x:\overline{n}|}^{\ \ 1}$

where π is the net single premium in (a)

4.27. a. 0.307215

4.34. a. $A_{20:\overline{20}|}^{\ \ 1} = 0.01827$

${}^2A_{20:\overline{20}|}^{\ \ 1} = 0.01143$

b. 110,933,839

4.35. b. $A_{\omega:\overline{0}|} = 0.0$

4.36. a. 3.06569 b. $(I\bar{A})_x = (\bar{A}^1_{x:\overline{1}|} + vp_x\,\bar{A}_{x+1}) + vp_x\,(I\bar{A})_{x+1}, \quad (I\bar{A})_\omega = 0$

c. $(\bar{I}\bar{A})_x = [(\bar{I}\bar{A})^1_{x:\overline{1}|} + vp_x\,\bar{A}_{x+1}] + vp_x\,(\bar{I}\bar{A})_{x+1}, \quad (\bar{I}\bar{A})_\omega = 0$

d. $(I\bar{A})_x = (d/\delta)[q_x + p_x\,\bar{A}_{x+1}] + vp_x\,(I\bar{A})_{x+1},$

$$(\bar{I}\bar{A})_x = \frac{d}{\delta}\left[\left(\frac{i-\delta}{i\delta}\right) q_x + p_x\,\bar{A}_{x+1}\right] + vp_x\,(\bar{I}\bar{A})_{x+1}$$

4.38. b. $\bar{A}_{y:\overline{0}|} = 1$ c. $\bar{A}_{45:\overline{65-45}|} = 0.34743$

4.39. mean = 38,056.82, variance = 42,337,224.63

Chapter 5

5.1. a. 16.008, 12.761, 5.397 b. 3.137, 10.230, 9.523

5.2. a. 0.111, 0.251, 0.572 b. 0.0251

5.4. $-\mathrm{Var}(v^T) = -({}^2\bar{A}_x - \bar{A}_x^2)$

5.6. a. $$F_y(y) = \begin{cases} 1 - (1-\delta y)^{\mu/\delta} & 0 \le y < \dfrac{1}{\delta} \\ 1 & \dfrac{1}{\delta} \le y \end{cases}$$

b. $$F_y(y) = \begin{cases} 1 - (1-\delta y)^{\mu/\delta} & 0 \le y < \bar{a}_{\overline{n}|} \\ 1 & \bar{a}_{\overline{n}|} \le y \end{cases}$$

c. $$F_y(y) = \begin{cases} 1 - (v^n-\delta y)^{\mu/\delta} & 0 \le y < \dfrac{v^n}{\delta} \\ 1 & \dfrac{v^n}{\delta} \le y \end{cases}$$

d. $$F_y(y) = \begin{cases} 0 & 0 < y \le \bar{a}_{\overline{n}|} \\ 1 - (1-\delta y)^{\mu/\delta} & \bar{a}_{\overline{n}|} < y < \dfrac{1}{\delta} \\ 1 & \dfrac{1}{\delta} \le y \end{cases}$$

5.7. $\ddot{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{1}|} - v^n\,{}_np_x\,\ddot{a}_{x+n:\overline{1}|} + vp_x\,\ddot{a}_{x+1:\overline{n}|} \qquad \begin{matrix} n = 1, 2, \ldots \\ x = 0, 1, \ldots \end{matrix}$

$\ddot{a}_{\omega:\overline{n}|} = \ddot{a}_{\omega:\overline{1}|}$ which equals 1/2 by the trapezoidal rule

5.8. ${}_{n|}\bar{a}_x = v^n\,{}_np_x\,\bar{a}_{x+n:\overline{1}|} + vp_x\,{}_{n|}\bar{a}_{x+1} \qquad \begin{matrix} x = 0, 1, \ldots \\ n = 0, 1, 2, \ldots \end{matrix}$

${}_{n|}\bar{a}_\omega = 0$

5.9. $\bar{a}_{\overline{x:\overline{n}|}} = v^n\,{}_np_x\,\bar{a}_{x+n:\overline{1}|} + \bar{a}_{\overline{n}|}(1 - vp_x) + vp_x\,\bar{a}_{\overline{x+1:\overline{n}|}}$

$\bar{a}_{\overline{\omega:\overline{n}|}} = \bar{a}_{\overline{n}|}$

5.14. $\dfrac{2}{i}\,(a_{x:\overline{n}|} - {}^2a_{x:\overline{n}|}) - {}^2a_{x:\overline{n}|} - (a_{x:\overline{n}|})^2$

5.15. $\dfrac{s^{(m)}_{\overline{1}|} - 1}{d^{(m)}}\,A_x$

5.16. a. $\frac{1}{m}\sum_{h=0}^{m-1} v^{h/m}\ {}_{h/m}p_x + \frac{1}{m}\sum_{h=m}^{(y-x)m-1} v^{h/m}\ {}_{h/m}p_x$

b. $\alpha(m) - \beta(m)(1 - vp_x)$

c. $c(x) = \alpha(m) - \beta(m)(1 - vp_x),\quad d(x) = vp_x,\quad \ddot{a}_{y:\overline{0}|}^{(m)} = 0$

5.17. $\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x),\quad {}_{n|}\ddot{a}_x - \frac{m-1}{2m}\,{}_nE_x$

5.22. a. $\alpha(m)\,\ddot{s}_{25:\overline{40}|} - \beta(m)\left(\frac{1}{{}_{40}E_{25}} - 1\right)$ b. (i) 15.038 (ii) 196.380

5.23. a. $Y = \begin{cases} (I\ddot{a})^{(m)}_{\overline{K+J+1/m}|} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ (I\ddot{a})^{(m)}_{\overline{n}|} & K = n, n+1, \ldots \end{cases}$

5.24. a. $Y = \begin{cases} (D\ddot{a})^{(m)}_{\overline{K+J+1/m}|} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ (D\ddot{a})^{(m)}_{\overline{n}|} & K = n, n+1, \ldots \end{cases}$

5.25. a. $Y = \begin{cases} (I\ddot{a})^{(m)}_{\overline{K+J+1/m}|} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ n\ \ddot{a}^{(m)}_{\overline{K+J+1/m}|} & K = n, n+1, \ldots, \quad J = 0, 1, \ldots, m-1 \end{cases}$

5.31. a. $\ddot{a}_x + 0.03(Ia)_x$

b. $\sum_{k=0}^{\infty}(1.03)^k v^k\ {}_kp_x = \ddot{a}'_x$ evaluated at interest rate $i' = \frac{i - 0.03}{1.03}$

5.32. $\int_0^n (n - t)\, v^t\ {}_tp_x\, dt$

5.33. $1{,}200\left(\frac{a_{30}^{(12)} + {}_{10|}a_{30}^{(12)} + 3\ {}_{20|}a_{30}^{(12)} + 5\ {}_{30|}a_{30}^{(12)} - 10\ {}_{40|}a_{30}^{(12)}}{{}_{40}E_{30}}\right)$

5.34. $\bar{a}_{35:\overline{25}|} - {}_{25}p_{35}\ \bar{a}_{\overline{25}|}$

5.35. $\ddot{a}_{x:\overline{n}|} - {}_np_x\ \ddot{a}_{\overline{n}|}$

5.36. $\frac{1}{12}\ddot{a}_{x:\overline{25}|} - \frac{25}{12}\,{}_{25}E_x$

5.38. $v^{2n}\ {}_np_x\,(1 - {}_np_x)\,\ddot{a}^2_{x+n} + v^{2n}\ {}_np_x\,\frac{{}^2A_{x+n} - A^2_{x+n}}{d^2}$

5.41. a. $\alpha(m) = 1 + \frac{m^2 - 1}{12m^2}\delta^2 + \frac{2m^4 - 5m^2 + 3}{720m^4}\delta^4 + \cdots$

$$\beta(m) = \frac{m-1}{2m}\left[1 + \frac{m+1}{3m}\delta + \frac{m(m+1)}{12m^2}\delta^2 + \frac{(m+1)(6m^2-4)}{360m^3}\delta^3 + \cdots\right]$$

b. $\alpha(\infty) = 1 + \frac{1}{12}\delta^2 + \frac{1}{360}\delta^4 + \cdots$

$$\beta(\infty) = \frac{1}{2}\left[1 + \frac{1}{3}\delta + \frac{1}{12}\delta^2 + \frac{1}{60}\delta^3 + \cdots\right]$$

5.44. $\frac{I}{\delta} + \left(J - \frac{I}{\delta}\right) v^T, \frac{I}{\delta} + \left(J - \frac{I}{\delta}\right) \bar{A}_x, \left(J - \frac{I}{\delta}\right)^2 ({}^2\bar{A}_x - \bar{A}_x^2)$

5.45. a. 14.353 b. 13.350 c. 1.002

5.51. a. 488.23 b. 700.48 c. 531.77

5.53. $\ddot{a}_{55:\overline{10}|} = 7.45735$

5.54. $\bar{a}_{55:\overline{10}|} = 7.19783$

5.56. $\bar{a}_{55:\overline{10}|} = 7.19783$

5.57. $\bar{a}_{60:\overline{10}|} = 6.46348, \quad \text{Var}(X) = 1.82621$

5.58. $\ddot{a}_{65}^{(12)} = 10.13343, \quad \text{Var}(Y) = 16.87662$

5.59. 10.40189

Chapter 6

6.1. 0, 0.1779

6.3. 0.303598

6.4. a. 0.02 b. 0.00857 c. 0.02885

6.6. $\dfrac{\mu}{\mu + 2\delta} = {}^2\bar{A}_x$

6.10.

Insurance	Annual Premiums for (35) Fully Continuous	Semicontinuous	Fully Discrete
10-Year endowment	0.075128	0.072885	0.072810
30-Year endowment	0.015371	0.014894	0.014751
60-Year endowment	0.008913	0.008621	0.008374
Whole life	0.008903	0.008611	0.008362
30-Year term	0.005117	0.004958	0.004815
10-Year term	0.002669	0.002589	0.002514

6.12. $A_x = \dfrac{1 - r}{1 + i - r}, \; P_x = \dfrac{1 - r}{1 + i},$

$\ddot{a}_x = \dfrac{1 + i}{1 + i - r},$

$\dfrac{{}^2A_x - A_x^2}{(d\ddot{a}_x)^2} = \dfrac{(1 - r)r}{1 + 2i + i^2 - r}$

6.13. 0.019139

6.15. 0.032868

6.16. 0.0413

6.17. With the common $(\bar{A}_{40:\overline{25}|})$ omitted from the premium symbols,
$P \leq P^{(2)} \leq P^{\{4\}} \leq P^{\{12\}} \leq \bar{P}$

6.18. $\dfrac{100}{99}$

6.19. 740.93

6.21. $P(A'^{\,1}_{45:\overline{20|}})$ where $A'^{\,1}_{45:\overline{20|}}$ is the actuarial present value of a 20-year term insurance on (45) under which $b_{k+1} = \ddot{s}_{\overline{k+1|}}$

6.22. a. 11.5451, 20.4106 b. 6.3099, 25.6458

6.24. ${}_{25}P_{60}$

6.25. b. $P^{(12)}(A^{(12)}_{\overset{1}{65}:\overline{10|}}) + d^{(12)}$

6.26. $\dfrac{100{,}000}{(1.1\ \ddot{s}_{\overline{30|}} - 0.1\ \ddot{s}_{35:\overline{30|}})}$

6.27. 0.008

6.28. $\dfrac{11{,}000\ A_x + 25\ \ddot{a}_{x:\overline{20|}}}{\ddot{a}_{x:\overline{20|}} - 1.1(I_{\overline{20|}}A)_x}$

6.29. $\dfrac{2\ A_{25} - A^{\ 1}_{25:\overline{10|}}}{2\ \ddot{a}_{25:\overline{40|}} - \ddot{a}_{25:\overline{10|}}}$

6.30. $L_1 = v^T - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T|}} \equiv 1 - \left(\dfrac{1}{\bar{a}_x}\right)\bar{a}_{\overline{T|}} = L_2$

6.31. a. -0.08 b. 0.1296 c. 0.1587

6.32. $\dfrac{\bar{A}_x}{2\ddot{a}_x - \ddot{a}_{x:\overline{5|}}}$

6.33. ${}_{20}P^{\{m\}}(\bar{A}_x) - {}_{20}P^{(m)}(\bar{A}_x) = {}_{20}\bar{P}(\bar{A}_x)\left(\dfrac{\bar{A}^{\,1}_{x:\overline{20|}} - A^{1\,(m)}_{x:\overline{20|}}}{\delta\ddot{a}^{(m)}_{x:\overline{20|}}}\right)$

6.35. a. $\dfrac{\mu}{\delta u + \bar{P}}\left(\dfrac{\delta u + \bar{P}}{\delta + \bar{P}}\right)^{\mu/\delta} \qquad -\dfrac{\bar{P}}{\delta} < u < 1$

$0 \qquad$ elsewhere

6.36. a. $\dfrac{1}{\sqrt{3}}$

b. 0.02

Chapter 7

7.1. ${}_1V = 0.15111$ ${}_2V = 0.30809$ ${}_3V = 0.47118$ ${}_4V = 0.64067$

7.2. ${}_1V = 0.14925$ ${}_2V = 0.30492$ ${}_3V = 0.46741$ ${}_4V = 0.63712$

7.3. ${}_1V = 1.2871$ ${}_2V = 2.6996$ ${}_3V = 4.2553$ ${}_4V = 5.9748$

7.4. ${}_1V = 0.15064$ ${}_2V = 0.30730$ ${}_3V = 0.47025$ ${}_4V = 0.63980$

7.5. a. $1 = \frac{1}{5}\int_0^5 e^{0.1[1.06^{-t}(1+P/\delta)-P/\delta]}\,dt$ where $\delta = \log(1.06)$

$$ {}_1V = 10\log\left(\frac{1}{4}\int_0^4 e^{0.1[1.06^{-t}(1+P/\delta)-P/\delta]}\,dt\right) \quad \text{where } P \text{ and } \delta \text{ are } \infty \text{ in (a)}$$

b. $P = 0.388380, \quad {}_1V = 0.182825$

7.6. $${}_tL = \begin{cases} v^U - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{U}|} & U < n-t \\ v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|} & U \ge n-t \end{cases}$$

7.7. $\mathrm{E}[{}_tL] = \bar{a}_{x+t:\overline{n-t}|}$, $\mathrm{Var}({}_tL) = \dfrac{{}^2\bar{A}_{x+t:\overline{n-t}|} - \bar{A}^2_{x+t:\overline{n-t}|}}{\delta^2}$

7.8. a. $\bar{A}_{45:\overline{20}|} - {}_{20}\bar{P}(\bar{A}_{35:\overline{30}|})\,\bar{a}_{45:\overline{10}|}$ b. $\bar{A}^{\;\;1}_{50:\overline{5}|}$

7.9. a. $u_0 = \dfrac{-\log(\bar{A}_x)}{\delta}$ b. 23.2476

7.10. 41.7524

7.11. $$F_{{}_tL}(y) = 0 \qquad y < v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|}$$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}\left(t - \dfrac{1}{\delta}\log\dfrac{\delta y + \bar{P}(\bar{A}_{x:\overline{n}|})}{\delta + \bar{P}(\bar{A}_{x:\overline{n}|})}\right)}{1 - F_{T(x)}(t)} \qquad v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|} \le y < 1$$

$$F_{{}_tL}(y) = 1 \qquad y \ge 1$$

7.12. $$F_{{}_tL}(y) = 0 \qquad y < \bar{P}(\bar{A}^{\;1}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|}$$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}(n)}{1 - F_{T(x)}(t)} = \frac{{}_np_x}{{}_tp_x} \qquad -\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|} \le y < v^{n-t} - \bar{P}(\bar{A}^{\;1}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|}$$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}\left(t - \dfrac{1}{\delta}\log\dfrac{\delta y + \bar{P}(\bar{A}^{\;1}_{x:\overline{n}|})}{\delta + \bar{P}(\bar{A}^{\;1}_{x:\overline{n}|})}\right)}{1 - F_{T(x)}(t)} \qquad v^{n-t} - \bar{P}(\bar{A}^{\;1}_{x:\overline{n}|}) \le y < 1$$

7.14. $\bar{A}_{50} - {}_{20}\bar{P}(\bar{A}_{40})\,\bar{a}_{50:\overline{10}|}$, $[{}_{10}\bar{P}(\bar{A}_{50}) - {}_{20}\bar{P}(\bar{A}_{40})]\,\bar{a}_{50:\overline{10}|}$,

$\left[1 - \dfrac{{}_{20}\bar{P}(\bar{A}_{40})}{{}_{10}\bar{P}(\bar{A}_{50})}\right]\bar{A}_{50}$, ${}_{20}\bar{P}(\bar{A}_{40})\,\bar{s}_{40:\overline{10}|} - {}_{10}\bar{k}_{40}$

7.15. $\bar{A}_{50:\overline{10}|} - \bar{P}(\bar{A}_{40:\overline{20}|})\,\bar{a}_{50:\overline{10}|}$, $[\bar{P}(\bar{A}_{50:\overline{10}|}) - \bar{P}(\bar{A}_{40:\overline{20}|})]\,\bar{a}_{50:\overline{10}|}$,

$\left[1 - \dfrac{\bar{P}(\bar{A}_{40:\overline{20}|})}{\bar{P}(\bar{A}_{50:\overline{10}|})}\right]\bar{A}_{50:\overline{10}|}$, $\bar{P}(\bar{A}_{40:\overline{20}|})\,\bar{s}_{40:\overline{10}|} - {}_{10}\bar{k}_{40}$,

$1 - \dfrac{\bar{a}_{50:\overline{10}|}}{\bar{a}_{40:\overline{20}|}}$, $\dfrac{\bar{P}(\bar{A}_{50:\overline{10}|}) - \bar{P}(\bar{A}_{40:\overline{20}|})}{\bar{P}(\bar{A}_{50:\overline{10}|}) + \delta}$, $\dfrac{\bar{A}_{50:\overline{10}|} - \bar{A}_{40:\overline{20}|}}{1 - \bar{A}_{40:\overline{20}|}}$

7.16. $\bar{P}({}_{30|}\bar{a}_{35})\,\bar{s}_{35:\overline{20}|}$

7.18. (7.3.3)

7.19. $A_{50} - {}_{20}P_{40}\,\ddot{a}_{50:\overline{10}|}$, $({}_{10}P_{50} - {}_{20}P_{40})\,\ddot{a}_{50:\overline{10}|}$, $\left(1 - \dfrac{{}_{20}P_{40}}{{}_{10}P_{50}}\right)A_{50}$,

${}_{20}P_{40}\,\ddot{s}_{40:\overline{10}|} - {}_{10}k_{40}$

7.20. $A_{50:\overline{10}|} - P_{40:\overline{20}|}\, \ddot{a}_{50:\overline{10}|}$, $(P_{50:\overline{10}|} - P_{40:\overline{20}|})\, \ddot{a}_{50:\overline{10}|}$,

$\left(1 - \dfrac{P_{40:\overline{20}|}}{P_{50:\overline{10}|}}\right) A_{50:\overline{10}|}$, $P_{40:\overline{20}|}\, \ddot{s}_{40:\overline{10}|} - {}_{10}k_{40}$,

$1 - \dfrac{\ddot{a}_{50:\overline{10}|}}{\ddot{a}_{40:\overline{20}|}}$, $\dfrac{P_{50:\overline{10}|} - P_{40:\overline{20}|}}{P_{50:\overline{10}|} + d}$, $\dfrac{A_{50:\overline{10}|} - A_{40:\overline{20}|}}{1 - A_{40:\overline{20}|}}$

7.22. $\dfrac{1}{5}$

7.23.

Insurance	Fully Continuous	Semicontinuous	Fully Discrete
30-Year endowment	0.17530	0.17504	0.17407
Whole life	0.08604	0.08566	0.08319
30-Year term	0.03379	0.03370	0.03273

7.24. (b) and (c)

7.26. All but (d)

7.27. All

7.29. 0.008

7.30. 0.240

7.31. a. 0.005527
b. 0.051255
c. 0.946122
d. 0.132109

7.32. a. 0.0241821 b. 0.0189660

Chapter 8

8.1. a. $\dfrac{1 - r}{1 + i}$

b. $\dfrac{(1 - r)(1 + i + r)}{(1 + i)(1 + i - r)}$

8.2. $$\frac{\int_0^\infty b_t\, v^t\, {}_tp_x\, \mu_x(t)\, dt}{\int_0^\infty w(t)\, v^t\, {}_tp_x\, dt}$$

8.3. a. $\dfrac{\mu}{\delta + \mu}$

b. $\dfrac{\mu t}{\delta + \mu}$

8.5. a. $(P_{x+1} - vq_{x+h})\ {}_hp_x\ vq_x$
b. $(P_{x+1} - vq_{x+h})\ {}_hp_x(v\ {}_jp_x\ q_{x+j} + {}_jq_x\ P_{x+1})$
c. If $P_{x+1} - vq_{x+n} < 0$, then $\mathrm{Cov}(C_j, C_h) < 0$ for all $j < h$

8.6. If $1 - vq_{x+h}\ \ddot{s}_{\overline{h+1}|} < 0$, then $\mathrm{Cov}(C_j, C_h) < 0$ for all $j < h$

8.13. $(\bar{A}_{x:\overline{40}|})$ is omitted from the reserve and premium symbols
a. $\frac{1}{2}\ {}_{20}V + \frac{1}{2}\ {}_{21}V + \frac{1}{2}\ P$ b. $\frac{1}{2}\ {}_{20}\bar{V} + \frac{1}{2}\ {}_{21}\bar{V}$
c. $\frac{1}{2}\ {}_{20}V^{(2)} + \frac{1}{2}\ {}_{21}V^{(2)}$ d. $\frac{1}{3}\ {}_{20}V^{(2)} + \frac{2}{3}\ {}_{21}V^{(2)} + \frac{1}{3}\ P^{(2)}$
e. Same as (b) f. $\frac{1}{3}\ {}_{20}\bar{V} + \frac{2}{3}\ {}_{21}\bar{V} + \frac{1}{3}\ P^{\{2\}}$

8.14. 0.05448

8.17. b. $\mathrm{Var}(L) = 0.076090$

8.18. a. 0.0067994 b. 0.1858077 c. 0.2012024
d. 0.0275369 e. 0.0255406

8.21. $-{}_tp_x[\delta\ {}_t\bar{V}(\bar{A}_x) + \bar{P}(\bar{A}_x)]$

8.22. a. ${}_tp_x\ [\pi_t + \delta_t \bar{V} - b_t\ \mu_x(t)]$
b. $v^t[\pi_t + \mu_x(t)\ {}_t\bar{V} - b_t\ \mu_x(t)]$
c. $v^t\ {}_tp_x\ [\pi_t - b_t\ \mu_x(t)]$

8.26. a. and b. 1,491.03 c. 343.84 d. 0

8.27. a. 1,490,915
b. 6,450,962; 1,495,093, which is 1.00280 times the reserve
c. 5,311,375; supplement is 3,791, which is 0.00254 times the reserve
d. For b.: 645,096,250; 149,133,281, which is 1.00028 times the reserve
For c.: 531,137,500; supplement is 37,911, which is 0.00025 times the reserve

8.28. a. 1,104,260 is the reserve for these policies
b. 6,450,962; 1,108,438, which is 1.00378 times the reserve
c. 5,311,375; supplement is 3,791, which is 0.00343 times the reserve
d. For b.: 645,096,250; 110,467,781, which is 1.00038 times the reserve
For c.: 531,137,500; supplement is 37,911, which is 0.00034 times the reserve

8.29. $5{,}000[{}_{10}\bar{V}(\bar{A}_{30}) + P^{\{1\}}(\bar{A}_{30}) + {}_{11}\bar{V}(\bar{A}_{30})]$

8.30. a. 0.2 b. 0.25 c. 0.7584 d. 0.27

8.32. 0.081467

8.34. a. 355.6563
b. 2,614.2511

保险精算丛书

精算数学

N. L. 鲍尔斯等 著 余跃年 郑韫瑜 译

上海科学技术出版社

内 容 简 介

本书介绍了生命表，人寿保险及生存年金的趸缴保费与期缴保费计算，责任准备金及现金价值（解约金）的计算。此外，还介绍了多重生命理论，人口理论，退休金计划及养老金累积理论。各章还配有大量习题，书后附有参考文献，可供进一步学习研究之用。本书可供保险业精算人员、高校师生及其他对保险精算有兴趣的读者阅读和参考。

N. L. Bowers
Actuarial Mathematics
Society of Actuaries
1986

《保险精算丛书》
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N.L.鲍尔斯 等著
余跃年 郑韫瑜 译
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《保险精算丛书》编委会

《保险精算丛书》前言

保险，作为商品社会中处理风险的一种有效方法，已被全世界所普遍采纳。在现代保险业蓬勃发展的进程中，科学的理论和方法，特别是精确的定量计算，起着十分重要的作用。保险业运营中的一些重要环节，如新险种的设计、保险费率和责任准备金的计算、分保额的确定、养老金等社会保障计划的制定等等，都需要由精算师 (Actuary) 依据精算学 (Actuarial Science) 原理来分析和处理。有鉴于此，许多发达国家都以法律形式规定，保险公司的营业报告必须由精算师签字方为有效。这也是国家对保险业进行调控管理的一种手段。

所谓精算学，实际上是将数学方法应用于金融保险所形成的一套理论体系。它的基础包括精算数学、利息理论、风险理论、人口数学、修匀数学、生存模型和生命表构造等等，还包括一些更专门的内容。这一套理论的重要性和正确性，已经得到国际社会的公认。

在我国，虽然早在 1949 年就由中央人民政府批准成立了中国人民保险公司，但是，由于种种历史原因，在相当长一段时间内我国的保险业发展缓慢，人才培养远不能适应实际需要。特别是精算学的研究和精算人才的培养，未得到应有的重视。在保险业的实际运作中，也很少严格按照精算学的原理办事。这一切都影响了我国保险业的进一步发展及与国际接轨。这种情况已引起保险界、教育界和学术界的注意，正在采取积极措施改变现状。刚刚颁布的《保险法》更明确规定："经营人身保险业务的保险公司，必须聘用经金融监督管理部门认可的精算专业人员，建立精算报告制度。"在此情况下，迫切需要引进国际上先进的精算学

理论，并结合我国的实际加以应用，本丛书就是在这样的背景下翻译出版的。

《保险精算丛书》(第一辑) 是由复旦大学数学系、中国人民保险公司上海市分公司 (以下简称人保上海分公司) 合作翻译的，由上海科学技术出版社出版。全国政协副主席、中科院院士苏步青为丛书题写书名；复旦大学研究生院院长、中科院院士李大潜担任丛书主编；中国人民保险公司上海市分公司总经理何静芝、副总经理钱建中，上海市新闻出版局局长徐福生担任丛书总顾问。上海是我国保险业的发源地之一，历来是保险业的中心。成立于 1950 年的人保上海分公司，经过 45 年艰难曲折的发展，业务有了很大开拓，1994 年已实现业务收入 30 亿元人民币，占上海保险市场的 80%。根据市场的需要，公司已开办了财产、人身、责任、信用四大类约 200 多个险种。特别是作为公司主要业务之一的国内人身保险业务，1994 年的业务收入已近 12 亿元。公司所开设的人身险种类也从 1982 年时的一种，扩展到各种形态的医疗保险、定期和终身保险及责任不同的各种人身意外伤害保险等多个品种，并逐步形成系列化。上海保险市场虽然在不断扩大，但竞争也日趋激烈。特别是一些实力雄厚的国际著名大保险公司的进入，促使国内各保险公司采取有力措施不断提高从业人员的业务素质，包括学习精算知识和培养精算人才。正是由于这样的需要，人保上海分公司决定与复旦大学数学系联手，在上海科学技术出版社的积极支持下，翻译了这套《保险精算丛书》。

复旦大学数学系不仅在数学的基础理论研究方面成就卓著，而且历来重视数学在国民经济中的应用，并取得多项重大研究成果。近年来，他们为了拓宽数学应用的领域，又开辟了精算学研究的新方向，并进行了大量的实际工作。他们在数学系研究生和本科生中开设了有关精算的课程和专题讨论，努力培养精算人才；他们还与各大保险公司合作，从事保险精算实际课题的研究，招收应用数学 (保险) 大专班，举办面向社会的保险精算培训班，培

训了一批人员参加 A.S.A (北美精算师学会准会员) 资格考试 (该项考试的上海考点就设在复旦大学内), 并于第一期考试中取得通过率超过 90% 的优异成绩。与人保上海分公司合作翻译这套《保险精算丛书》，不仅是复旦数学系理论和实践相结合的一项新的举措，也是他们面向社会培养国家急需的精算人才的重要措施。

"保险精算丛书" (第一辑) 共六本，分别为：

《利息理论》, S.G. 凯利森著, 尚汉冀译;

《风险理论》, N.L. 鲍尔斯著, 郑韫瑜、余跃年译;

《精算数学》, N.L. 鲍尔斯著, 余跃年、郑韫瑜译;

《人口数学》, R.L. 布朗著, 郑培明译;

《修匀数学》, D. 伦敦著, 徐诚浩译;

《生存模型》, D. 伦敦著, 陈子毅译。

所依据的原书均是北美精算师学会 (Society of Actuaries) 为其准会员 (A.S.A) 资格考试所指定的教材和参考书，具有一定的权威性。阅读这套丛书，不论对读者了解和掌握精算学基本原理并应用于保险业实践，还是对读者准备参加 A.S.A 资格考试 (该项考试在中国的北京、上海、天津、长沙等地已设有考点), 均会有很大帮助。

保险精算在我国是一项刚刚起步的新事物，这套丛书是高等院校、保险公司和出版社三方共同合作，编写翻译出版学术水平较高、填补国家缺门的专业书籍的一种有益的探索。我们热诚希望广大读者提出宝贵意见，以利于我们改进工作，做好这套丛书的出版工作，促进保险精算事业在中国的发展。

编者谨识

1995 年 11 月于上海

目　　录

第一章　生存分布与生命表

§1.1　引　言

在风险理论中，我们阐明了保险如何能增进面临随机损失的个体的期望效用，并建立了若干保险的简单模型。这些模型的基础是 Bernoulli 随机变量。在某些场合，还出现另一个有关损失额的随机变量。在精算数学里，将要建立的模型主要涉及与个人生存多久相联系的随机损失，其中，剩余寿命(time-until- death) 随机变量 $T(x)$ 是基本的营造材料。这一章先建立若干描述并使用剩余寿命分布及相应的死亡年龄(age-at -death) 分布的概念。

在精算学的很多模型中，生命表(life table) 是不可分割的组成部分。用生命表可得出死亡年龄的分布。除保险领域以外，生命表在人口学、生物医学统计乃至可靠性研究中都有应用。

某些学者将精算学的起源定在 1693 年，那一年 Edmund Halley(译注：哈雷彗星以其姓氏命名) 发表“根据 Breslau 城出生与下葬统计表对人类死亡程度的估计”，包含在这篇论文里被称为 Breslau 表的生命表因其令人惊异的现代记号与观念而引起人们注意。

§1.2　与死亡年龄有关的概率

一、生存函数

对于新生儿，其死亡年龄 X 是一个连续型随机变量。用 $F(x)$ 记 X 的分布函数，

$$F(x) = Pr(X \leq x) \qquad x \geq 0, \tag{1.2.1}$$

并置

$$s(x)=1-F(x)=Pr(X>x) \qquad x\geq 0. \tag{1.2.2}$$

对任何正数 x, 值 $F(x)$ 等于新生儿在 x 岁或之前死亡的概率，而 $s(x)$ 等于新生儿活到 x 岁 (即 x 岁以后死亡) 的概率。因 $X\geq 0$, 故 $F(0)=0$, 从而 $s(0)=1$ 。函数 $s(x)$ 称为 *生存函数*(survival function) 。在精算学及人口学中，生存函数是传统上的出发点，相当于概率统计学中分布函数所起的作用。由两者的关系 $s(x)=1-F(x)$, 无论使用哪一个都是等价的，例如从分布函数所具有的性质就可得出生存函数的相应性质。

与死亡年龄有关的概率，既可用生存函数也可用分布函数来表述，例如，新生儿在年龄 x 与 z $(x<z)$ 之间死亡的概率为

$$\begin{aligned} Pr(x<X\leq z) &= F(z)-F(x) \\ &= s(x)-s(z). \end{aligned}$$

二、 x 岁人的剩余寿命

新生儿在生存到 x 岁的条件下于年龄 x 与 z 之间死亡的条件概率为

$$\begin{aligned} Pr(x<X\leq z|X>x) &= \frac{F(z)-F(x)}{1-F(x)} \\ &= \frac{s(x)-s(z)}{s(x)}. \end{aligned} \tag{1.2.3}$$

以后将用 (x) 来表示 *年龄 x 的生命*(life-aged-x), (x) 的剩余寿命记为 $T(x)$ 。

在精算学中，常常需要作出有关 $T(x)$ 的概率陈述。为促进研究交流， 1898 年的国际精算大会采用了作为国际精算符号规则一部分的一组符号，确立了通用精算函数符号以及采用新符号的原则。此后，该体系由国际精算学会的常设符号规则委员会根据需要修订扩充。本书将尽可能遵从这些符号的约定。

精算学中的符号与概率论中使用的有所不同，读者也许还不太熟悉。譬如，概率论中单变量函数写成 $q(x)$, 而在精算符号体系中则写成 q_x 。类似地，多变量函数在精算符号中用上标、下标或其它标号的混合来表示。附录 4 给出了精算函数符号的一般规则，读者在继续阅读之前最好浏览一下这些符号的形式。

我们有符号

$$_tq_x = Pr[T(x) \le t] \qquad t \ge 0, \tag{1.2.4}$$

$$_tp_x = 1 - {}_tq_x = Pr[T(x) > t] \qquad t \ge 0. \tag{1.2.5}$$

符号 $_tq_x$ 可解释为 (x) 将在 t 年内死去的概率，即 $_tq_x$ 关于变量 t 是 $T(x)$ 的分布函数。另一方面， $_tp_x$ 可解释为 (x) 将活到 * $x+t$ 岁的概率，即 $_tp_x$ 是 (x) 的生存函数。在年龄 $x=0$ 的特别情形下， $T(0)=X$ 且

$$_xp_0 = s(x) \qquad x \ge 0. \tag{1.2.6}$$

如果 $t=1$, 约定允许省略 (1.2.4) 及 (1.2.5) 所定义符号中的前缀，即有

$$q_x = Pr[(x)\text{将在 1 年内死去}],$$

$$p_x = Pr[(x)\text{将至少活到}x+1\text{岁}].$$

对于 (x) 将生存 t 年并在其后的 u 年内死去这一更为一般的事件，即 (x) 将在 $x+t$ 岁与 $x+t+u$ 岁之间死去这个事件，有一个特殊的符号

$$\begin{aligned} {}_{t|u}q_x &= Pr[t < T(x) \le t+u] \\ &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x - {}_{t+u}p_x. \end{aligned} \tag{1.2.7}$$

---

* 这里及以后凡涉及“活到某某岁”是指至少活到某某岁，不是恰好活到某某岁就死去。

与前面一样，当 $u=1$ 时，${}_{t|u}q_x$ 中的 u 可省略而成为 ${}_{t|}q_x$。

至此，(x) 将在年龄 x 与 $x+u$ 之间死去的概率好象有两种表达式，式 (1.2.7) 在 $t=0$ 时为其一，(1.2.3) 在 $z=x+u$ 为其二。这两个概率是否不同？式 (1.2.3) 可解释为一个新生儿在年龄 x 与 $z=x+u$ 之间死去的条件概率。有关这个现龄 x 的新生儿的仅有信息是其活到这个年龄，所以概率陈述是建立在新生儿生存的条件分布上的。

另一方面，当 $t=0$ 时 (1. 2. 7) 则给定了现龄 x 的 **被观察生命** (life observed) 在 x 与 $x+u$ 岁之间死去的概率。对现龄 x 的生命进行观察，可能获得比新生儿单纯活到 x 岁更多的信息，譬如刚通过保险体检或刚开始对疾病进行治疗等。与此有关的生命表将在 §1.8 讨论。

这一节对以上两种情形不加区别，认为现龄 x 人的生存与新生儿在活到年龄 x 条件下的条件生存分布相同，即

$$
{}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0} = \frac{s(x+t)}{s(x)}, \tag{1.2.8}
$$

$$
{}_tq_x = 1 - \frac{s(x+t)}{s(x)}. \tag{1.2.9}
$$

按这种方式，(1.2.7) 及其许多特殊情形可表示成

$$
\begin{aligned}
{}_{t|u}q_x &= \frac{s(x+t)-s(x+t+u)}{s(x)} \\
&= \frac{s(x+t)}{s(x)}\,\frac{s(x+t)-s(x+t+u)}{s(x+t)} \\
&= {}_tp_x\,{}_uq_{x+t}.
\end{aligned} \tag{1.2.10}
$$

三、整值剩余寿命

在某些精算模型中，使用一个离散型的剩余寿命随机变量，定义为

$$
K(x) = k \quad 当 k \le T(x) < k+1 时
$$
$$
k = 0, 1, 2, \cdots
$$

显然，$K(x)$ 是 $T(x)$ 的整数部分。由于 $T(x)$ 是连续型随机变量，$Pr[T(x)=k]=Pr[T(x)=k+1]=0$, 从而 $K(x)$ 的概率函数为

$$\begin{aligned}
Pr[K(x) &= k] = Pr[k \le T(x) < k+1] \\
&= {}_{k+1}q_x - {}_kq_x = {}_kp_x - {}_{k+1}p_x \\
&= {}_kp_x q_{x+k} = {}_{k|}q_x \qquad k=0,1,2,\cdots
\end{aligned} \tag{1.2.11}$$

随机变量 $K(x)$ 可解释成 (x) 未来存活的完整年数．称为 (x) 的*整值剩余寿命*(curtate-future-lifetime)。

在根据上下文可以明确为 (x) 的整值剩余寿命时，$K(x)$ 有时可简写成 K。同样，(完全) 剩余寿命 $T(x)$ 也时常简写成 T。

四、死亡效力

式 (1.2.3) 分别以分布函数与生存函数表示 (0) 在活到 x 岁条件下于年龄 x 与 z 之间死亡的条件概率。令其中 $z=x+\Delta x$, 则该条件概率等于现龄 x 的生命在今后 Δx 年内死亡的概率

$$\begin{aligned}
{}_{\Delta x}q_x &= Pr[x < X \le x+\Delta x | X > x] \\
&= \frac{F(x+\Delta x)-F(x)}{1-F(x)} \cong \frac{f(x)\Delta x}{1-F(x)},
\end{aligned} \tag{1.2.12}$$

其中 $f(x)=F'(x)$ 是死亡年龄 (连续型随机变量) 的概率密度函数。式 (1.2.12) 中的量

$$\frac{f(x)}{1-F(x)} = \lim_{\Delta x \to 0^+} \frac{Pr[x < X \le x+\Delta x | X > x]}{\Delta x}$$

称为*死亡效力*(force of mortality) 或瞬时死亡率，记作 μ_x, 即

$$\mu_x = \frac{F'(x)}{1-F(x)} = -\frac{s'(x)}{s(x)}. \tag{1.2.13}$$

显然 $\mu_x \geq 0$ 。在研究部件及系统寿命的可靠性理论中， μ_x 称为 失效率(failure rate or hazard rate) 或故障率，更完整的名称为 失效率函数(hazard rate function) 。

就如生存函数一样，死亡效力也可用来确定 X 的分布。为此，将 (1.2.13) 中 x 改为 y 并加以整理得

$$-\mu_y dy = d\log s(y).$$

对这个表达式从 x 到 $x+t$ 积分，有

$$-\int_x^{x+t} \mu_y dy = \log\left[\frac{s(x+t)}{s(x)}\right] = \log {}_tp_x,$$

即

$$ {}_tp_x = \exp(-\int_x^{x+t} \mu_y dy). \tag{1.2.14}$$

有时，作代换 $s = y - x$ 可将 (1.2.14) 改写成

$$ {}_tp_x = \exp(-\int_0^t \mu_{x+s} ds). \tag{1.2.15}$$

特别是，为了与 (1.2.6) 一致而改变符号后有

$$ {}_xp_0 = s(x) = \exp(-\int_0^x \mu_s ds), \tag{1.2.16}$$

并且

$$F(x) = 1 - s(x) = 1 - \exp(-\int_0^x \mu_s ds), \tag{1.2.17}$$

$$\begin{aligned} F'(x) = f(x) &= \mu_x \exp(-\int_0^x \mu_s ds) \\ &= {}_xp_0 \mu_x. \end{aligned} \tag{1.2.18}$$

以 $G(t)$ 与 $g(t)$ 分别记 (x) 剩余寿命 $T(x)$ 的分布函数与概率密度函数，从 (1.2.4) 知 $G(t)={}_tq_x$，于是

$$\begin{aligned} g(t) &= \frac{d}{dt}{}_tq_x \\ &= \frac{d}{dt}\left[1-\frac{s(x+t)}{s(x)}\right] \\ &= \frac{s(x+t)}{s(x)}\left[-\frac{s'(x+t)}{s(x+t)}\right] \\ &= {}_tp_x\mu_{x+t} \qquad t \geq 0. \end{aligned} \tag{1.2.19}$$

这样，${}_tp_x\mu_{x+t}dt$ 是 (x) 在年龄 t 与 $t+dt$ 之间死去的概率，且

$$\int_0^{\infty} {}_tp_x\mu_{x+t}dt = 1.$$

由 (1.2.19) 可得

$$\frac{d}{dt}(1-{}_tp_x) = -\frac{d}{dt}{}_tp_x = {}_tp_x\mu_{x+t}. \tag{1.2.20}$$

这一等价形式在精算数学若干方面有用。由

$$\lim_{n\to\infty} {}_np_x = 0$$

还可得

$$\lim_{n\to\infty}(-\log {}_np_x) = \infty,$$

即

$$\lim_{n\to\infty}\int_x^{x+n} \mu_y dy = \infty.$$

这一节的内容概括于下列表 1.2.1 及表 1.2.2 中：

表 1.2.1　定义

概念名称	符号
死亡年龄随机变量	x
现龄 x 的生命	(x)
(x) 的剩余寿命随机变量	$T(x)$ 或 T

表 1.2.2　死亡年龄的概率论函数

	分布函数 $F(x)$	概率密度函数 $f(x)$	生存函数 $s(x)$	死亡效力 μ_x
		要求		
当 $x<0$	$F(x)=0$	$f(x)=0$	$s(x)=1$	$\mu_x=0$
当 $x\geq 0$	$F(x)\geq 0$	$f(x)\geq 0$	$s(x)\geq 0$	$\mu_x\geq 0$
$\lim\limits_{x\to\infty}$	$F(\infty)=1$	$\int_0^\infty f(x)=1$	$s(\infty)=0$	$\int_0^\infty \mu_x dx=\infty$
		关系		
表示方式				
$F(x)$	$F(x)$	$F'(x)$	$1-F(x)$	$\frac{F'(x)}{1-F(x)}$
$f(x)$	$\int_0^x f(s)ds$	$f(x)$	$1-\int_0^x f(s)ds$	$\frac{f(x)}{\int_x^\infty f(s)ds}$
$s(x)$	$1-s(x)$	$-s'(x)$	$s(x)$	$\frac{-s'(x)}{s(x)}$
μ_x	$1-e^{-\int_0^x \mu_s ds}$	$e^{-\int_0^x \mu_s ds}\mu_x$	$e^{-\int_0^x \mu_s ds}$	μ_x

表 1.2.2 的下半部分概括了一般概率论中的函数关系及其对死亡年龄的特殊应用。有关死亡年龄的许多问题可按一般概率论的方式形成，以下即为一例。

例 1.2.1: 如以 $\bar{A}$ 表示事件 A 的补事件且 $Pr(\bar{A})\neq 0$, 则在概率论中成立恒等式

$$Pr(A\cup B)=Pr(A)+Pr(\bar{A})Pr(B|\bar{A}).$$

对事件 $A=[T(x)\leq t]$ 及 $B=[t<T(x)\leq 1]$, $0<t<1$, 用精算术语来改写这一恒等式。

解：$Pr(A\cup B)$ 成为 $Pr[T(X)\leq 1]=q_x$, $Pr(A)$ 为 ${}_tq_x$, 而 $Pr(B|\bar{A})$ 则是 ${}_{1-t}q_{x+t}$, 所以

$$q_x={}_tq_x+{}_tp_x\,{}_{1-t}q_{x+t}.$$

§1.3 生命表

正式发表的生命表通常包含基本函数 q_x, l_x, d_x 的数值，按各年龄列成表格，可能的话还会增加一些衍生函数。在展示生命表之前，先解释一下与前节的概率函数直接相关的这些函数。

一、生命表函数与生存函数的关系

在 (1.2.9) 中，(x) 将在 t 年内死亡的条件概率表示为

$$_tq_x = 1 - \frac{s(x+t)}{s(x)},$$

特别是有

$$q_x = 1 - \frac{s(x+1)}{s(x)}.$$

我们现在考虑由 l_0 个新生生命组成的群体，$l_0 = 100,000$ 。每个新生者的死亡年龄分布由生存函数 $s(x)$ 确定。同时，以 $\mathcal{L}(x)$ 记群体中生存到年龄 x 的个体数，按 $j = 1, 2, \cdots, l_0$ 对个体标记，则有

$$\mathcal{L}(x) = \sum_{j=1}^{l_0} I_j,$$

其中 I_j 是个体 j 生存的一个指标，即

$$I_j = \begin{cases} 1 & \text{当个体}j\text{生存到年龄}x, \\ 0 & \text{其它}. \end{cases}$$

由 $E[I_j] = s(x)$ 得

$$E[\mathcal{L}(x)] = \sum_{j=1}^{l_0} E[I_j] = l_0 s(x).$$

将 $E[\mathcal{L}(x)]$ 记为 l_x, 则 l_x 表示 l_0 个新生生命中能生存到年龄 x 的期望个数，且

$$l_x = l_0 s(x). \tag{1.3.1}$$

进一步，在指标 I_j 相互独立的假定下，$\mathcal{L}(x)$ 服从参数 $n = l_0$ 及 $p = s(x)$ 的二项分布。注意，(1.3.1) 并不要求独立性假定。

类似地，用 ${}_n\mathcal{D}_x$ 记初始 l_0 个生命中在年龄 x 与 $x+n$ 之间死去的个数。将 $E[{}_n\mathcal{D}_x]$ 记为 ${}_nd_x$ 。由于新生个体在年龄 x 与 $x+n$ 之间死亡的概率为 $s(x)-s(x+n)$, 用类似于得出 l_x 的论证可获表示式

$$\begin{aligned} {}_nd_x = E[{}_n\mathcal{D}_x] &= l_0[s(x)-s(x+n)] \\ &= l_x - l_{x+n}. \end{aligned} \tag{1.3.2}$$

当 $n=1$ 时，省略 ${}_n\mathcal{D}_x$ 与 ${}_nd_x$ 的前缀 n 而成为 $\mathcal{D}_x$ 与 d_x 。

从 (1.3.1) 可以看出

$$-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{s'(x)}{s(x)} = \mu_x (s' = \frac{ds}{dx}), \tag{1.3.3}$$

$$-dl_x = l_x \mu_x dx. \tag{1.3.4}$$

鉴于

$$l_x \mu_x = l_{0\,x}p_0 \mu_x,$$

(1.3.4) 中的因子 $l_x\mu_x$ 可解释成年龄区间 $(x, x+dx)$ 内的期望死亡密度。此外，还有

$$l_x = l_0 \exp(-\int_0^x \mu_y dy), \tag{1.3.5}$$

$$l_{x+n} = l_x \exp(-\int_x^{x+n} \mu_y dy), \tag{1.3.6}$$

$$l_x - l_{x+n} = \int_x^{x+n} l_y \mu_y dy. \tag{1.3.7}$$

对于人类生命，死亡年龄超过 110 岁的很少见，所以通常假定，存在一个年龄 ω，使得当 $x < \omega$ 时 $s(x) > 0$，而当 $x \geq \omega$ 时 $s(x) = 0$ 。这样一种年龄 ω 称为 *最终年龄*(limiting age) 。

为引用方便，以后将生存函数都是 $s(x)$ 的 l_0 个新生生命所组成的群体称为 *随机生存组*(random survivorship group) 。

二、生命表实例

在“美国 1979–1981 年全体人口生命表”（表 1.3.1）中，列出了函数 ${}_tq_x, l_x$ 及 ${}_td_x$，其中 $l_0 = 100\,000$ 。除生命的第一年外，表中函数 ${}_tq_x$ 与 ${}_td_x$ 的 t 取值 1，表中出现的其它函数将在 §1.5 中讨论。

1979–1981 年美国生命表并不是从观察 100000 个新生儿直至最后生存者死亡而建立起来的，而是基于不同年龄者的死亡概率得出的。在按这个生命表使用随机生存组的概念时，我们假定了从中导出的概率适合生存组中个体的寿命。

表 1.3.1 美国 1979–1981 年全体人口生命表

年龄区间	死亡比例	1000000 个出生者中		静止人口		平均余命
两个年龄间的时期	期初生存者在期间的死亡比例	期初生存数	期间死亡数	在年龄区间中	在这个及以后所有年龄区间中	期初存活者的平均剩余年数
(1)	(2)	(3)	(4)	(5)	(6)	(7)
x 至 $x+t$	${}_tq_x$	l_x	${}_td_x$	${}_tL_x$	T_x	e_x
天						
0–1	0.00463	100000	463	273	7387758	73.88
1–7	0.00246	99537	245	1635	7387485	74.22
7–28	0.00139	99292	138	5708	7385850	74.38
28–365	0.00418	99154	414	91357	7380142	74.43
年						
0–1	0.01260	100000	1260	98973	7387758	73.88
1–2	0.00093	98740	92	98694	7288785	73.82
2–3	0.00065	98648	64	98617	7190091	72.89
3–4	0.00050	98584	49	98560	7091474	71.93
4–5	0.00040	98535	40	98515	6992914	70.97

年龄区间	死亡比例	1000000 个出生者中		静止人口		平均余命
5–6	0.00037	98495	36	98477	6894399	70.00
6–7	0.00033	98459	33	98442	6795 922	69.02
7–8	0.00030	98426	30	98412	6697480	68.05
8–9	0.00027	98396	26	98383	6599068	67.07
9–10	0.00023	98370	23	98358	6500685	66.08
10–11	0.00020	98347	19	98338	6402327	65.10
11–12	0.00019	98328	19	98319	6303989	64.11
12–13	0.00025	98309	24	98297	6205670	63.12
13–14	0.00037	98285	37	98266	6107373	62.14
14–15	0.00053	98248	52	98222	6009107	61.16
15–16	0.00069	98196	67	98163	5910885	60.19
16–17	0.00083	98129	82	98087	5812722	59.24
17–18	0.00095	98047	94	98000	5714635	58.28
18–19	0.00105	97953	102	97902	5616635	57.34
19–20	0.00112	97851	110	97796	5518733	56.40
20–21	0.00120	97741	118	97682	5420973.	55.46
21–22	0.00127	97623	124	97561	5323255	54.53
22–23	0.00132	97499	129	97435	5225694	63.60
23–24	0.00134	97370	130	97306	5128259	52.67
24–25	0.00133	97.240	130	97175	5030953	51.74
25–26	0.00132	97110	128	97046	4933778	50.81
26–27	0.00131	96982	126	96919	4836733	49.87
27–28	0.00130	96856	126	96793	4739813	48.94
28–29	0.00130	96730	126	96667	4643020	48.00
29–30	0.00131	96604	127	96541	4546353	47.06
30–31	0.00133	96477	127	96414	4449812	46.12
31–32	0.00134	96350	130	96284	4353398	45.18
32–33	0.00137	96220	132	96155	4257114	44.24
33–34	0.00142	96088	137	96019	4160959	43.30
34–35	0.00150	95951	143	95880	4064940	42.36
35–36	0.00159	95808	153	95731	3969060	41.43
36–37	0.00170	95655	163	95574	3873329	40.49

年龄区间	死亡比例	1000000 个出生者中		静止人口		平均余命
37–38	0.00188	95492	175	95404	3777755	39.56
38–39	0.00197	95317	188	95224	3682351	38.63
39–40	0.00213	95129	203	95027	3587127	37.71
40–41	0.0232	94926	220	94817	3492100	36.79
41–42	0.00254	94706	241	94585	3397283	35.87
42-43	0.00279	94465	264	94334	3302698	34.96
43–44	0.00306	94201	288	94057	3208364	34.06
44–45	0.00335	93913	314	93759	3114307	33.16
45–46	0.0366	94599	343	93427	3020551	32.27
46–47	0.00401	93256	374	93069	2927124	31.39
47–48	0.00442	92882	410	92677	2834055	30.51
48–49	0.00488	92472	451	92246	2741378	29.65
49–50	0.00830	92021	495	91773	2649132	28.79
50–51	0.00589	91526	540	91256	2557359	27.94
51–52	0.00642	90986	584	90695	2466103	27.10
52–53	0.00699	90402	631	90086	2375408	26.28
53–54	0.00761	89771	684	89430	2285322	25.46
54–55	0.00830	89087	739	88717	2195892	24.65
55–56	0.00902	88348	797	87950	2107175	23.85
56–57	0.00978	87551	856	87122	2019225	23.06
57–58	0.01059	86695	919	86236	1932103	22.29
58–59	0.01151	85776	987	85283	1845867	21.52
59–60	0.01254	84789	1063	84258	1760584	20.76
60–61	0.01368	83726	1145	83153	1676326	20.02
61–62	0.01493	82581	1233	81965	1593173	19.29
62-63	0.01628	81348	1324	80686	1511208	18.58
63–64	0.01767	80024	1415	79316	1430522	17.88
64–65	0.01911	78609	1502	77859	1351206	17.19
65–66	0.02059	77107	1587	76314	1273347	16.51
66–67	0.02216	75520	1674	74683	1197033	15.85
67–68	0.02389	73846	1764	72964	1122350	15.20
68–69	0.02585	72082	1864	71150	1049386	14.56

年龄区间	死亡比例	1000000 个出生者中		静止人口		平均余命
69–70	0.02806	70218	1970	69233	978236	13.93
70–71	0.03052	68248	2083	67206	909003	13.32
71–72	0.03315	66165	2193	65069	841797	12.74
72–73	0.03593	63972	2299	62823	776728	12.14
73–74	0.03882	61673	2394	60476	713905	11.58
74–75	0.04184	59279	2480	58039	653429	11.02
75–76	0.04507	56799	2560	55520	595390	10.48
76–77	0.04867	54239	2640	52919	539870	9.95
77–78	0.05274	51599	2721	50238	486951	9.44
78–79	0.05742	48878	2807	47475	436713	8.93
79–80	0.06277	46071	2891	44626	389238	8.45
80–81	0.06882	43180	2972	41694	344612	7.98
81–82	0.07552	40208	3036	38689	302918	7.53
82–83	0.08278	37172	3077	35634	264229	7.11
83–84	0.09041	34095	3083	32553	228595	6.70
84–85	0.09842	31012	3052	29486	196042	6.32
85–86	0.10725	27960	2999	26461	166556	5.96
86–87	0.11712	24961	2923	23500	140095	5.61
87–88	0.12717	22038	2803	20636	116595	5.29
88–89	0.13708	19235	2637	17917	95959	4.99
89–90	0.14728	16598	2444	15376	78042	4.70
90–91	0.15868	14154	2246	13031	62666	4.43
91–92	0.17169	11908	2045	10886	49635	4.17
92–93	0.18570	9863	1831	8948	38749	3.93
93–94	0.20023	8032	1608	7228	29801	3.71
94–95	0.21495	6424	1381	5733	22573	3.51
95–96	0.22976	5043	1159	4463	16840	3.34
96–97	0.24338	3884	945	3412	12377	3.19
97–98	0.25637	2939	754	2562	8965	3.05
98–99	0.26868	2185	587	1892	6403	2.93
99–100	0.28030	1598	448	1374	4511	2.82
100–101	0.29120	1150	335	983	3137	2.73

年龄区间	死亡比例	1000000 个出生者中		静止人口		平均余命
101–102	0.30139	815	245	692	2154	2.64
102–103	0.31089	570	177	481	1462	2.57
103–104	0.31970	393	126	330	981	2.50
104–105	0.32786	267	88	223	651	2.44
105–106	0.33539	179	60	150	428	2.38
106–107	0.34233	119	41	99	278	2.33
107–108	0.34870	78	27	64	179	2.29
108–109	0.35453	51	18	424	115	2.24
109–110	0.35988	33	12	27	73	2.20

对 1979–1981 美国生命表作以下评注将会是有益的：

1. 新生儿生存组中大约 1% 预期在生命的第一年内死去；
2. 新生儿组中约 77% 期望可活到 65 岁；
3. 新生儿生存组中死亡人数最多的预期发生在 83 岁时；
4. 预期死亡人数的局部极小值发生在 11 岁及 27 岁附近；
5. 表中没有给出最终年龄 ω;
6. 虽然 l_x 的数值被舍入为整数，但是根据 (1.3.1) 式，并不存在非这样做的理由。

表 1.3.1 的形式是描述死亡年龄分布的常规方法。另一种可以替代的方法是用解析形式来表示生存函数，诸如 $s(x)=e^{-cx}, c>0, x\geq 0$ 等。绝大多数为保险而对人类死亡进行的研究都使用表 1.3.1 所隐含的 $s(x)=l_x/l_0$, 但生命表只对整数 x 陈列了 $l_0 s(x)$, 对于非整数 x, 则需要用适当的插值办法来估计 $s(x)$, 具体细节将在 §1.6 讨论。

例 1.3.1: 根据表 1.3.1 估计关于 (20) 的以下事件概率。

(1) 至少活到 60 岁；

(2) 在 70–80 岁之间死亡。

解：

(1) $\frac{s(60)}{s(20)}=\frac{l_{60}}{l_{20}}=\frac{83726}{97741}=0.8566.$

(2) $\frac{s(70)-s(80)}{s(20)}=\frac{l_{70}-l_{80}}{l_{20}}=\frac{68248-43180}{97{,}741}=0.2565$

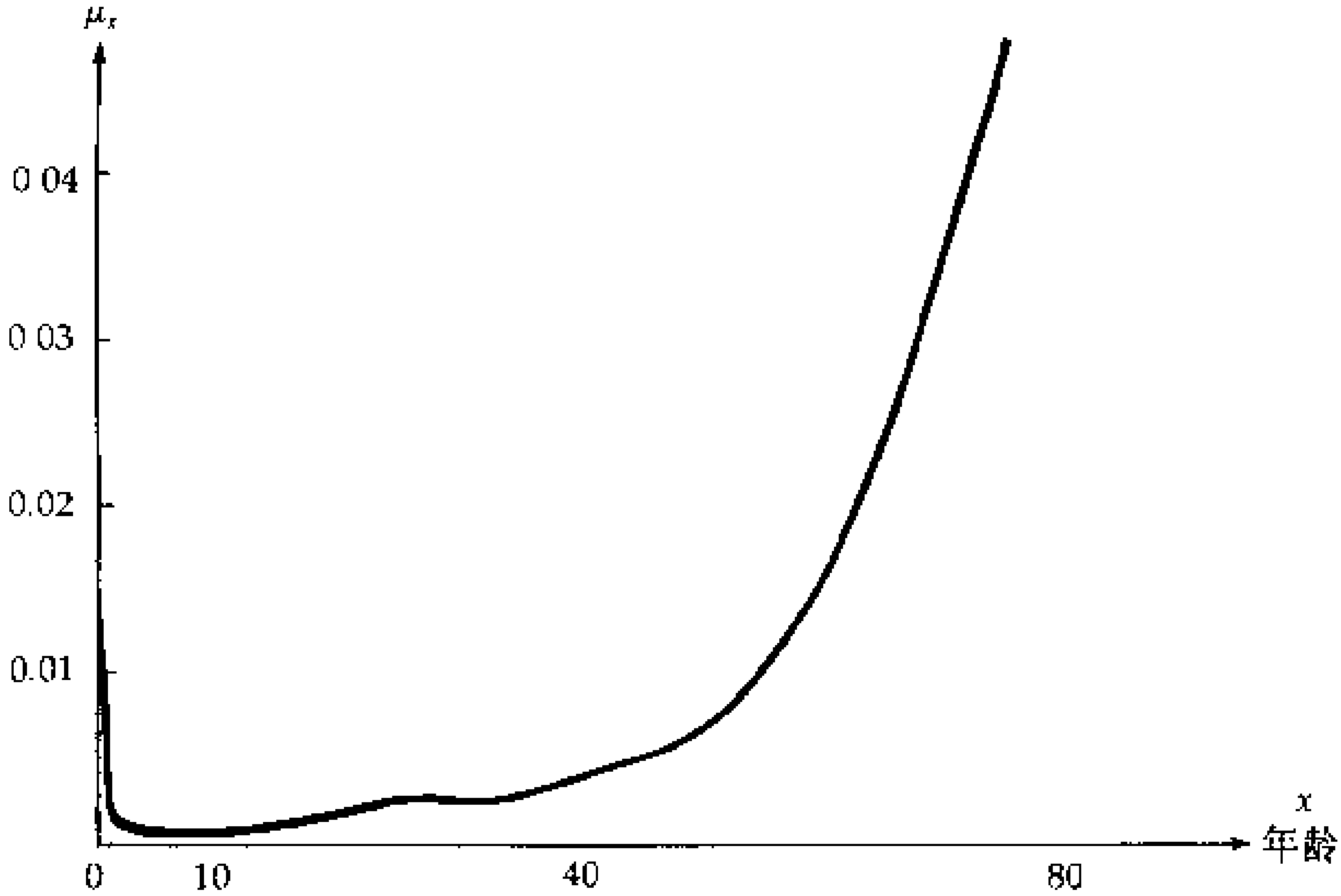

图 1.3.1　死亡效力

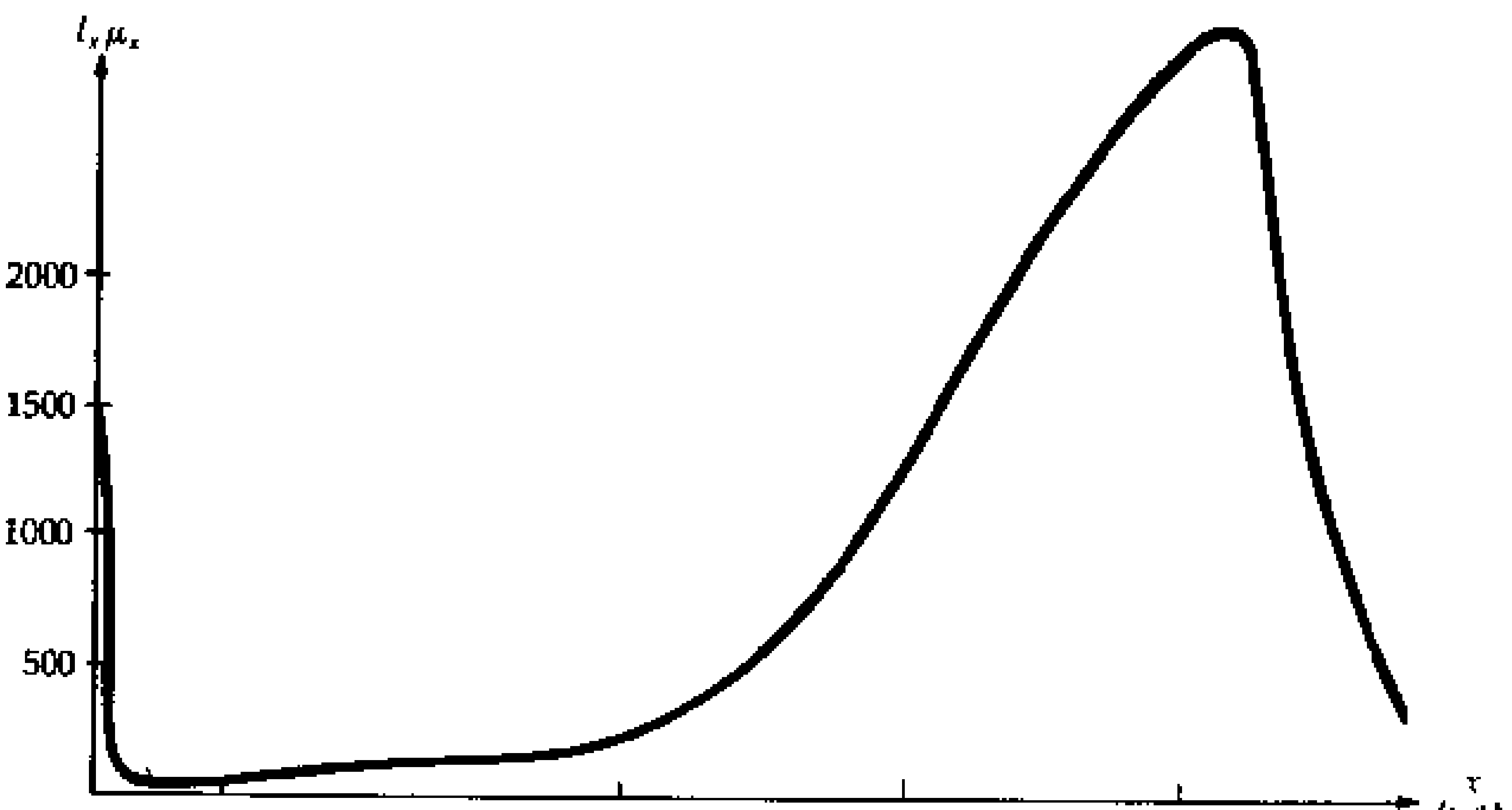

图 1.3.2　$l_x\mu_x$ 的图形

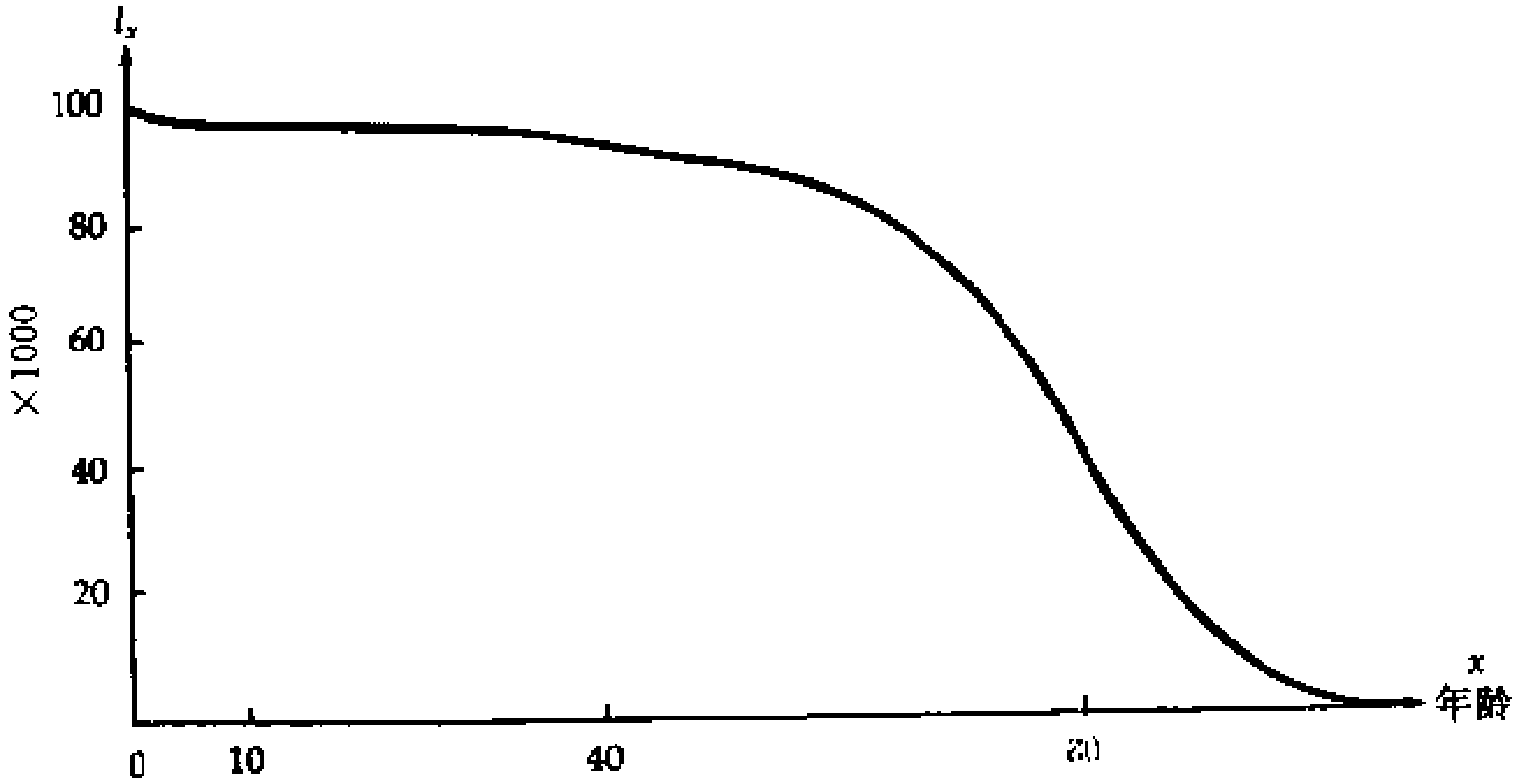

图 1.3.3　l_x 的图形

通过对以上图 1.3.1–1.3.3 的研究可以加深生命表函数的印象，这些图并不是根据表 1.3.1 画出来的，而是反映了新近的死亡状况。

注意，在图 1.3.1 中，$\mu_x > 0$ 及 $\int_0^\infty \mu_x dx = \infty$ 看来满足 (参见表 1.2.2)，且死亡效力开始时较大，在 10 岁左右降到最低点。在图 1.3.2 及 1.3.3 中，函数 $l_x\mu_x$ 与新生儿死亡年龄的概率密度函数成比例 (比例因子为 l_0)，$l_x\mu_x$ 的图形称为*死亡曲线*(the curve of deaths)。死亡曲线 $l_x\mu_x$ 在 10 岁附近有一个局部极小值，最大值在 80 岁附近取得。函数 l_x 与生存函数 $s(x)$ 成比例，其拐点对应 $l_x\mu_x$ 的局部极值点，这是因为

$$\frac{d}{dx}l_x\mu_x = \frac{d}{dx}(-\frac{d}{dx}l_x) = -\frac{d^2}{dx^2}l_x.$$

§1.4 决定性生存组

接下去将对生命表作出第二种非概率性的解释，这种解释在数学上起源于减率（负增长率）概念，且跟生物学与经济学中增长率应用有关。这在本质上是非随机性的，并引出 决定性生存组(deterministic survivorship group) 或 群(cohort) 的概念。

生命表所表示的决定性生存组具有以下特征：(1) 初始由 l_0 个 0 岁生命组成。(2) 在每一年龄 x, 有效年死亡率 (减量) 由生命表中 q_x 确定。(3) 封闭性，即除了开始 l_0 个外无新成员加入，成员减少系有效年死亡率 (减量) 的结果。

从这些特征可得出决定性生存组的发展：

$$
\begin{aligned}
l_1 &= l_0(1-q_0) = l_0 - d_0 \\
l_2 &= l_1(1-q_1) = l_1 - d_1 = l_0 - (d_0+d_1) \\
\vdots & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
l_x &= l_{x-1}(1-q_{x-1}) = l_{x-1} - d_{x-1} = l_0 - \sum_{y=0}^{x-1} d_y \\
&= l_0\Big(1 - \frac{\sum\limits_{y=0}^{x-1} d_y}{l_0}\Big) = l_0(1 - {}_xq_0),
\end{aligned}
\tag{1.4.1}
$$

其中 l_x 是生存组中活到 x 岁的生命数。由基数l_0 及一系列值 q_x 生成的等式链 (1.4.1) 可改写成

$$
\begin{aligned}
l_1 &= l_0p_0 \\
l_2 &= l_1p_1 = (l_0p_0)p_1 \\
\vdots & \quad \vdots \qquad\qquad \vdots \\
l_x &= l_{x-1}p_{x-1} = l_0\Big(\prod_{y=0}^{x-1} p_y\Big) = l_0\,{}_xp_0.
\end{aligned}
\tag{1.4.2}
$$

决定性生存组与金融数学中的复利模型可以类比，表 1.4.1 概括了它们之间的一些类似之处。

表 1.4.1　复利数学与决定性生存组的相关概念

复利	生存组
$A(t)=$ 时刻 t(年) 的资金总额	$l_x=$ 年龄 x(岁) 的组员总数
有效年利率 (增量) $i_t=\frac{A(t+1)-A(t)}{A(t)}$	有效年死亡率 (减量) $q_x=\frac{l_x-l_{x+1}}{l_x}$
有效 n 年利率 (从时刻 t 开始) ${}_ni_t=\frac{A(t+n)-A(t)}{A(t)}$	有效 n 年死亡率 (从年龄 x 开始) ${}_nq_x=\frac{l_x-l_{x+n}}{l_x}$
利息效力 (在时刻 t) $\delta_t=\lim_{\Delta t\to 0}[\frac{A(t+\Delta t)-A(t)}{A(t)\Delta t}]=\frac{1}{A(t)}\frac{dA(t)}{dt}$	死亡效力 $\mu_x=\lim_{\Delta x\to 0}[\frac{l_x-l_{x+\Delta x}}{l_x\Delta x}]=-\frac{1}{l_x}\frac{dl_x}{dx}$

表 1.3.1 中的小标题对应于决定性生存组解释。尽管随机生存组与决定性生存组的数学基础不同，但是引出的函数 q_x,l_x,d_x 具有同样的性质，可同样用来作进一步分析。随机生存组概念具有充分运用概率论的优势，而决定性生存组在概念上较简单且便于应用，但它忽视了生存数的随机波动。

§1.5　其它生命表函数

在导出 $T(x)$ 分布的矩的表达式前，先证明一个有用的定理。

定理 1.5.1: 设 T 是一个连续型随机变量，分布函数为 $G(t)$，概率密度函数为 $g(t)=G'(t)$ 。如果 $G(0)=0,z(t)$ 是非负单调可微函数，且使得 $E[z(T)]$ 存在，那么

$$\begin{aligned}E[z(T)] &= \int_0^\infty z(t)g(t)dt\\ &= z(0)+\int_0^\infty z'(t)[1-G(t)]dt.\end{aligned}$$

证：分部积分，

$$\int_0^t z(s)g(s)ds = -\int_0^t z(s)d[1-G(s)]$$

$$= \quad -z(s)[1-G(s)]|_0^t + \int_0^t [1-G(s)]z'(s)ds.$$

由此可见，只须证明 $\lim_{t\to\infty} z(t)[1-G(t)] = 0$ 。因 $z(t)$ 非负，当 $z(t)$ 单调减少时，$z(t)$ 必有界，由分布函数性质 $\lim_{t\to\infty} G(t) = 1$ 可得 $\lim_{t\to\infty} z(t)[1-G(t)] = 0$ 。而当 $z(t)$ 单调增加时，

$$0 \le z(t)[1-G(t)] = z(t)\int_t^\infty g(s)ds \le \int_t^\infty z(s)g(s)ds.$$

根据 $E[z(T)]$ 存在性可知 $\lim_{t\to\infty} \int_t^\infty z(s)g(s)ds = 0$，于是也得出

$$\lim_{t\to\infty} z(t)[1-G(t)] = 0.$$

定理从而证毕。

$E[T(x)]$ 称为**期望剩余寿命**(complete-expectation-of-life)，记作 $\overset{\circ}{e}_x$ 。根据这一定义

$$\overset{\circ}{e}_x = E[T(x)] = \int_0^\infty t\,{}_tp_x \mu_{x+t}dt. \tag{1.5.1}$$

按 $z(t) = t$ 及 $G(t) = 1 - {}_tp_x$，用定理 1.5.1 可得

$$\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt. \tag{1.5.2}$$

按 $z(t) = t^2$ 用定理 1.5.1 可得

$$E[T(x)^2] = \int_0^\infty t^2\,{}_tp_x\mu_{x+t}dt = 2\int_0^\infty t\,{}_tp_x dt,$$

从而可将方差 $\mathrm{Var}[T(x)]$ 表示成

$$\mathrm{Var}[T(x)] = E[T(x)^2] - (E[T(x)])^2 = 2\int_0^\infty t\,{}_tp_x dt - \overset{\circ}{e}{}_x^2.$$

以上应用定理 1.5.1 时，假定 $E[T(x)]$ 及 $E[T(x)^2]$ 存在，譬如当生存函数 $s(x) = (1+x)^{-1}$ 时，这些假定就不成立。

(x) 的中位剩余寿命(median future lifetime)$m(x)$ 由以下等式决定：

$$Pr[T(x) > m(x)] = \frac{1}{2},$$

即通过

$$\frac{s(x + m(x))}{s(x)} = \frac{1}{2} \tag{1.5.3}$$

解出 $m(x)$ 。特别地， $m(0)$ 由 $s[m(0)] = \frac{1}{2}$ 解出。

对于离散随机变量，可用类似方法证明一个与定理 1.5.1 平行的定理。

定理 1.5.2: 设 K 是取值于非负整数的离散型随机变量，分布函数为 $G(k)$, 概率函数 $g(k) = \Delta G(k-1)$, 如果 $z(k)$ 是非负单调函数且使得 $E[z(K)]$ 存在，那么

$$\begin{aligned} E[z(K)] &= \sum_{k=0}^{\infty} z(k)g(k) \\ &= z(0) + \sum_{k=0}^{\infty}[1 - G(k)]\Delta z(k), \end{aligned}$$

这里，差分记号 Δ 含义如下： $\Delta h(k) = h(k+1) - h(k)$ 。

对 (x) 的整值剩余寿命 K, 按 $G(k) = 1 - {}_{k+1}p_x$, 应用定理 1.5.2 可得

$$E[z(k)] = z(0) + \sum_{k=0}^{\infty} \Delta z(k)_{k+1}p_x. \tag{1.5.4}$$

特别有

$$E[K] = \sum_{k=0}^{\infty} k_k p_x q_{x+k} = \sum_{k=0}^{\infty} {}_{k+1}p_x, \tag{1.5.5}$$

称为期望整值剩余寿命(curtate-expectation-of-life), 符号为 e_x 。同理有

$$E[K^2] = \sum_{k=0}^{\infty} k^2{}_k p_x q_{x+k} = \sum_{k=0}^{\infty}(2k+1)_{k+1}p_x, \tag{1.5.6}$$

从而

$$\mathrm{Var}[K] = E[K^2] - (E[K])^2 = \sum_{k=0}^{\infty}(2k+1)_{k+1}p_x - e_x^2.$$

下面给出生命表 1.3.1 中另外一些函数的定义。符号 L_x 表示初始 l_0 个生命的生存组在年龄 x 与 $x+1$ 之间生存总年数的期望值，即

$$L_x = \int_0^1 t l_{x+t}\mu_{x+t}dt + l_{x+1}, \tag{1.5.7}$$

其中 l_{x+1} 项代表活到 $x+1$ 岁的所有生存者在 1 年中的生存年数和，积分项则代表所有在年龄 x 与 $x+1$ 之间死亡者在这一年中的生存年数和。对 (1.5.7) 行使分部积分得

$$\begin{aligned} L_x &= -\int_0^1 t dl_{x+t} + l_{x+1} \\ &= -tl_{x+t}\big|_0^1 + \int_0^1 l_{x+t}dt + l_{x+1} = \int_0^1 l_{x+1}dt. \end{aligned} \tag{1.5.8}$$

函数 L_x 可用来定义 x 岁中位死亡率(central-death-rate at age x)m_x，其定义为

$$m_x = \frac{\int_0^1 l_{x+t}\mu_{x+t}dt}{\int_0^1 l_{x+1}dt} = \frac{l_x - l_{x+1}}{L_x}. \tag{1.5.9}$$

符号 T_x 表示初始 l_0 个成员的生存组在年龄 x 以后生存总年数的期望值，即

$$\begin{aligned} T_x &= \int_0^{\infty} t l_{x+t}\mu_{x+t}dt \\ &= -\int_0^{\infty} t dl_{x+t} = \int_0^{\infty} l_{x+t}dt, \end{aligned} \tag{1.5.10}$$

其中运用了定理 1.4.1($g(t) = \frac{l_{x+t}\mu_{x+t}}{l_x} = {}_tp_x\mu_{x+t}$)。生存组在年龄 x 时的 l_x 个生存者以后的平均生存年数为

$$\frac{T_x}{l_x} = \frac{\int_0^{\infty} l_{x+t}dt}{l_x} = \int_0^{\infty} {}_tp_x dt = \mathring{e}_x$$

与 (1.5.1) 及 (1.5.2) 相同。

最后一个函数是，在年龄 x 与 $x+1$ 之间，死亡者在这一年中的平均生存年数 $a(x)$，其定义为

$$a(x)=\frac{\int_0^1 tl_{x+t}\mu_{x+t}dt}{\int_0^1 l_{x+t}\mu_{x+t}dt}. \tag{1.5.11}$$

按概率观点

$$a(x)=\frac{\int_0^1 t_tp_x\mu_{x+t}dt}{\int_0^1 {}_tp_x\mu_{x+t}dt}=E[T|T<1].$$

如果假定

$$l_{x+t}\mu_{x+t}dt=d_xdt \qquad 0\le t\le 1,$$

也就是说一年中死亡系均匀分布，那么

$$a(x)=\int_0^1 tdt=\frac{1}{2}.$$

除了年轻与年老的场合，可用 $\frac{1}{2}$ 作为 $a(x)$ 近似值。

例 1.5.1: 证明

$$L_x=a(x)l_x+[1-a(x)]l_{x+1}$$

及 $L_x\cong\frac{l_x+l_{x+1}}{2}$.

解：从 (1.5.7) 及 (1.5.11) 可得

$$a(x)=\frac{L_x-l_{x+1}}{l_x-l_{x+1}},$$

即 $L_x=a(x)l_x+[1-a(x)]l_{x+1}$. 近似公式

$$L_x\cong\frac{l_x+l_{x+1}}{2}$$

可通过对 (1.5.8) 中积分运用梯形规则近似得出。

§1.3–1.5 中，有关生命表的一些关键术语概括列在表 1.5.1 之中。

表 1.5.1　符号

概念名称	符号
(x) 的整值剩余寿命随机变量	$K(x)$ 或 K
(x) 的期望整值剩余寿命	$e_x = E[K(x)] = E[K]$
(x) 的期望剩余寿命	$\mathring{e}_x = E[T(x)] = E[T]$
(x) 的剩余寿命随机变量	$T(x)$ 或 T
初始 l_0 个成员的生存组在年龄 x 以后生存总年数	T_x
生存组中活到 x 岁的成员数随机变量	$\mathcal{L}(x)$
生存组中活到 x 岁的期望数	$l_x = E[\mathcal{L}(x)]$
在年龄 x 与 $x+n$ 之间死亡数随机变量	${}_n\mathcal{D}_x$
在年龄 x 与 $x+n$ 之间的期望死亡数	${}_nd_x = E[{}_n\mathcal{D}_x]$
(x) 的中位剩余寿命	$m(x)$
(x) 岁中位死亡率	m_x

§1.6　关于分数年龄的假设

§1.3 的生命表完全确定了整值剩余寿命 K 的概率分布，但要想得出 T 的分布，还须对整数点之间的分布作适当假定。我们将考察三种精算学中广泛使用的假设，这些假设都以生存函数来表达。以下 k 是整数， $0 \leq t \leq 1$, 三种假设分别为

(1) 死亡是均匀分布的： $s(x+t) = (1-t)s(x) + ts(x+1)$ 。

(2) 死亡效力是常数： $s(x+t) = s(x)e^{-\mu t}$, 其中 $\mu = -\log p_x$ 。

(3) Balducci 假设： $\frac{1}{s(x+t)} = \frac{(1-t)}{s(x)} + \frac{t}{s(x+1)}$ 。

我们也可等价地用概率密度函数、分布函数或死亡效力等来表达以上各种假设，详见表 1.6.1 。

本书将主要采用死亡为均匀分布的假设，以下对表 1.6.1 中假设 (1) 各栏进行推导，其它假设下的情形可作为练习，由读者自行推导。将以上假设 (1) 中 $s(x+t)$ 的表达式代入

表 1.6.1　分数年龄的概率论函数 *

假设 函数	(1) 均匀分布	(2) 常数死亡效力	(3) Balducci 假设
${}_tq_x$	tq_x	$1-e^{-\mu t}$	$\frac{tq_x}{1-(1-t)q_x}$
${}_tp_x$	$1-tq_x$	$e^{-\mu t}$	$\frac{p_x}{1-(1-t)q_x}$
${}_yq_{x+t}$	$\frac{yq_x}{1-tq_x}$	$1-e^{-\mu y}$	$\frac{yq_x}{1-(1-y-t)q_x}$
μ_{x+t}	$\frac{q_x}{1-tq_x}$	μ	$\frac{q_x}{1-(1-t)q_x}$
${}_tp_x\mu_{x+t}$	q_x	$e^{-\mu t}\mu$	$\frac{p_xq_x}{[1-(1-t)q_x]^2}$

$$ {}_tq_x = \frac{s(x)-s(x+t)}{s(x)} \qquad 0 \le t \le 1, $$

得出第一个栏目

$$ {}_tq_x = \frac{t[s(x)-s(x+1)]}{s(x)} = tq_x, $$

第二个栏目

$$ {}_tp_x = 1 - {}_tq_x = 1 - tq_x, $$

而第三个栏目

$$ \begin{aligned} {}_yq_{x+t} &= \frac{s(x+t)-s(x+t+y)}{s(x+t)} \\ &= \frac{y[s(x)-s(x+1)]/s(x)}{\{s(x)-t[s(x)-s(x+1)]\}/s(x)} \\ &= \frac{yq_x}{1-tq_x}. \end{aligned} $$

对于第四个栏目，由 $s'(x+t) = \frac{d}{dt}s(x+t) = -s(x)+s(x+1)$ 得

$$ \begin{aligned} \mu_{x+t} &= -\frac{s'(x+t)}{s(x+t)} = \frac{s(x)-s(x+1)}{[(1-t)s(x)+ts(x+1)]} \\ &= \frac{[s(x)-s(x+1)]/s(x)}{\{s(x)-t[s(x)-s(x+1)]\}/s(x)} = \frac{q_x}{1-tq_x}. \end{aligned} $$

* 表中 x 是整数，$0 < t < 1$(前三行对 $t=0$ 及 $t=1$ 也成立), $0 \le y \le 1, y+t \le 1, \mu = -\log p_x$.

最后一个栏目系第二与第四栏目的乘积。

对于整数 x, 按下式

$$T = K + S \tag{1.6.1}$$

定义随机变量 $S = S(x)$, 它是死亡年的生存时间 (即小数部分)。于是，对于 $0 < s \leq 1$,

$$\begin{aligned} Pr[k < T \leq k+s] &= Pr[K = k \cap S \leq s] \\ &= {}_{k|s}q_x = {}_kp_x\, {}_sq_{x+k}. \end{aligned}$$

如果利用表 1.6.1 中均匀分布假设下的 ${}_sp_{x+k}$, 那么

$$\begin{aligned} Pr[K = k \cap S \leq s] &= {}_kp_x s q_{x+k} \\ = {}_{k|}q_x s &= Pr[K = k]Pr[S \leq s]. \end{aligned} \tag{1.6.2}$$

换言之，在死亡为均匀分布的假设下，随机变量 K 与 S 相互独立，而且 S 服从区间 $(0,1)$ 上的均匀分布。

例 1.6.1: 在常数死亡效力假设下，随机变量 K 与 S 是否也独立？

解: 用表 1.6.1 中常数死亡效力假设下的栏目可得到

$$\begin{aligned} P_r[K = k \cap S \leq s] &= {}_kp_x\, {}_sq_{x+k} \\ &= {}_kp_x[1 - (p_{x+k})^s]. \end{aligned}$$

当 p_{x+k} 不是与 k 无关时，我们无法将 K 与 S 的联合概率分解成两个分别只依赖 k 与只依赖 s 的概率乘积；在 $p_{x+k} = p$, 与 k 无关这一特殊情形，

$$Pr[K = k \cap S \leq s] = p^k(1 - p^s)$$

$$= (1-p)p^k\frac{1-p^s}{1-p} = Pr[K = k]Pr[S \leq s],$$

K 与 S 独立。

例 1.6.2: 在死亡为均匀分布的假设下，证明 (1) $\mathring{e}_x = e_x + \frac{1}{2}$, (2) $\mathrm{Var}[T] = \mathrm{Var}[K] + \frac{1}{12}$.

解：(1) $\mathring{e}_x = E[T] = E[K+S] = E[K] + E[S] = e_x + \frac{1}{2}$, 这里使用了区间 $(0,1)$ 上均匀分布的数学期望为 $\frac{1}{2}$ 。

(2) 根据 K 与 S 的独立性及 $(0,1)$ 上均匀分布的方差为 $\frac{1}{12}$ 可得出

$$\begin{aligned} \mathrm{Var}[T] &= \mathrm{Var}[K+S] \\ &= \mathrm{Var}[K] + \mathrm{Var}[S] = \mathrm{Var}[K] + \frac{1}{12} \end{aligned}$$

§1.7 某些关于死亡的解析规律

对死亡或生存函数作出解析形式假定的理由有三个。第一是哲学上的，一些作者基于生物学论据提出，人类生存服从与物理定律同样简单的定律。第二是出于实际考虑，与一个只含几个参数的函数打交道总比或许有 100 个参数或死亡概率的生命表容易些。第三个理由在于从死亡数据出发估计参数的便利性。

对生存函数的简单解析形式的支持近年已经下降，许多人觉得，相信死亡服从某种普遍定律是天真烂漫的。而且随着高速计算机的出现，某些解析形式在计算上的优势也不再重要。尽管如此，有关死亡的解析定律的生物学论据，在一些令人感兴趣的研究中又旧事重提。

表 1.7.1 展示了对应于不同假说的解析死亡效力与生存函数，并给出了发现者与发表年份。注意表中特殊符号

$$m = \frac{B}{\log c}, u = \frac{k}{(n+1)}.$$

表中 Gompertz 律是 Makeham 律当 $A = 0$ 时的特殊情形，两者在 $c = 1$ 时导致指数分布 (常数死亡效力) 。

表 1.7.1　各种假设下的死亡与生存函数

发现者	μ_x	$s(x)$	限制
de Moivre (1729)	$(\omega - x)^{-1}$	$1 - \frac{x}{\omega}$	$0 \le x < \omega$
Gompertz (1825)	Bc^x	$\exp[-m(c^x - 1)]$	$B > 0, c \ge 1,$ $x \ge 0$
Makeham (1860)	$A + Bc^x$	$\exp[-Ax - m(c^x - 1)]$	$B > 0, A \ge -B,$ $c \ge 1, x \ge 0$
Weibull (1939)	kx^n	$\exp(-ux^{n+1})$	$k > 0, n > 0,$ $x \ge 0$

表 1.7.1 中 $s(x)$ 一列系代入式 (1.2.16) 而得，例如按 Makeham 死亡律，

$$\begin{aligned} s(x) &= \exp\left[-\int_0^x (A + Bc^x)dx\right] \\ &= \exp\left[-Ax - \frac{B(c^x - 1)}{\log c}\right] \\ &= \exp[-Ax - m(c^x - 1)], \end{aligned}$$

其中 $m = \frac{B}{\log c}$。

附录 2A 的示例生命表在年龄 13–110 之间是按 Makeham 律编制的，那里

$$1000\mu_x = 0.7 + 0.05(10^{0.04})^x. \tag{1.7.1}$$

请务必牢记，该示例生命表仅仅起示例作用。

§1.8　选择与终极生命表

在 §1.2 曾经提到，(x) 活到 $x + t$ 岁的概率 ${}_tp_x$ 可有两种解释。第一种解释是在仅有的新生儿活到 x 岁假定下的概率，此时可用生存函数求出这个概率；第二种解释是在关于 x 岁生命进一步信息可获得情况下的概率，这时用原始生存函数计算有关

(x) 的剩余寿命概率就显得不妥。譬如，一个年龄 x 的人可能已被接受人寿保险，这一信息促使我们认为其剩余寿命分布不同于我们本来可能会假设的。在这种情况下，需要能反映新获得信息的特殊生存函数。换言之，关于这种生命的完整模型是一族生存函数，对每一个诸如被接受保险或致残年龄，都有一个相应的生存函数。这一族生存函数可看成一个二元函数，其中一个变量是获得新信息的年龄 $[x]$，另一个变量是此后的生命延续时间 t。这样，与这个二元生存函数相联系的每个生命表函数都是按 $[x]$ 与 t 排列的二元阵列。

图 1.8.1 中的简略阵列说明了以上概念。例如，已知一组 30 岁人群的特殊信息，就可为他们建立一个特殊生命表。在 30 岁后 i 年至 $i+1$ 年内死亡的条件概率记为 $q_{[30]+i}$, $i=0,1,2,\cdots$，出现在图 1.8.1 的第一行。下标中的 [30] 表明与图中第一行相联系的是在 30 岁获得特殊信息后的生存函数。图中第二行是在 31 岁获得特殊信息后的死亡概率。在精算学中，这样的生存表称为**选择生命表**(select life table)。

选择对于剩余寿命 T 的分布的影响，会在选择之后逐渐消失。经过一段时间，不管曾经在哪一个年龄选择，活到相同岁数人的死亡概率将基本上相等。更精确地说，如果存在一个最小整数 r，使得对所有选择年龄 $[x]$ 及所有 $j>0$，$|q_{[x]+r}-q_{[x-j]+r+j}|$ 小于某个正数，那么可以通过截断二维阵列的 $r+1$ 列以后各列而建立一组所谓**选择与终极生命表**(select-and-ultimate tables)。对于延续时间超过 r 的可使用近似

$$q_{[x-j]+r+j} \cong q_{[x]+r} \quad j>0.$$

开始的 r 年延续构成了**选择期**(select period)。

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年选择期，即

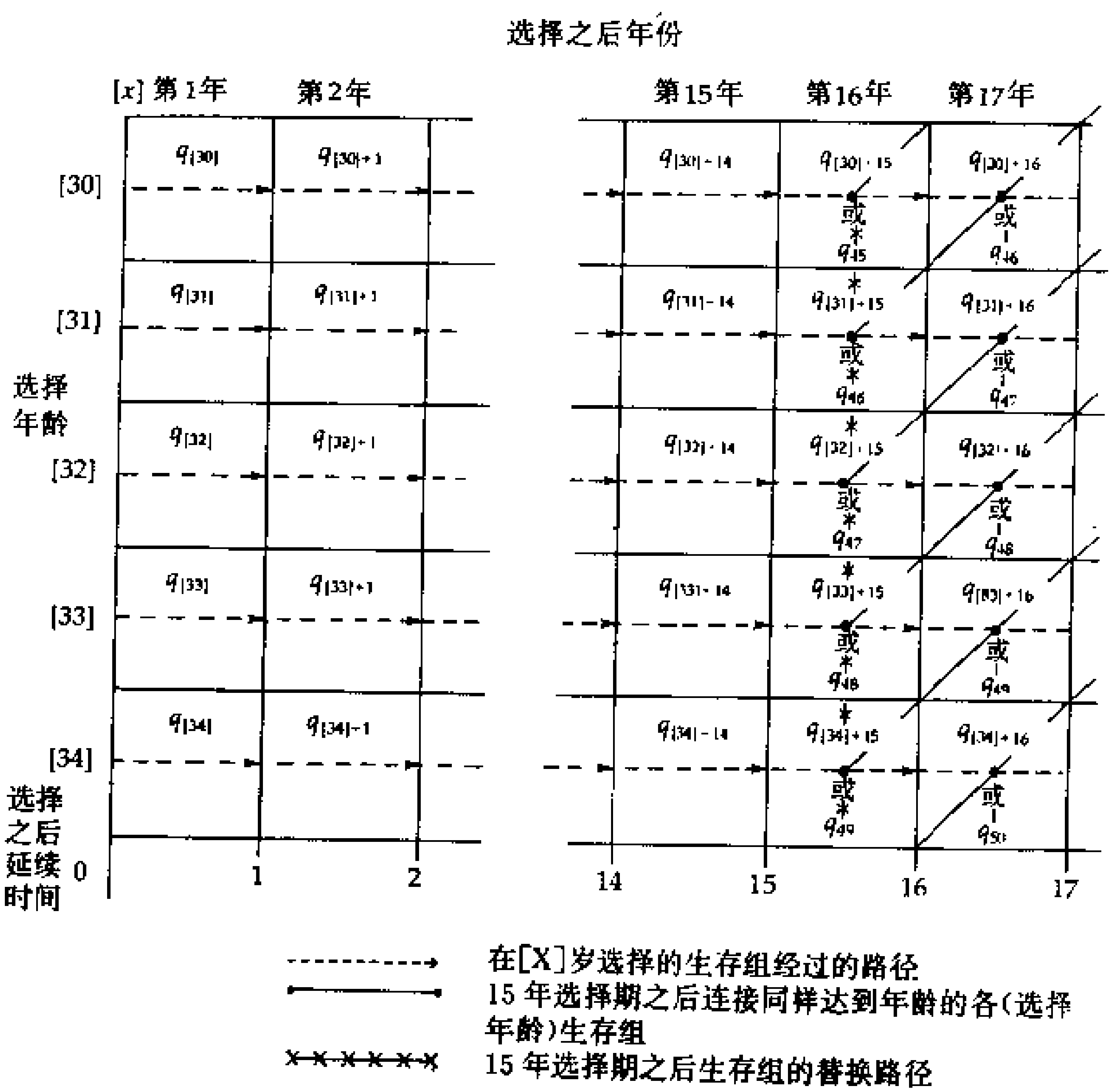


图 1.8.1　选择期为 15 年的选择、终极与综合死亡表

$$q_{[x-j]+15+j} \cong q_{[x]+15} \qquad j>0.$$

超过选择期的有关死亡概率只以达到的年龄作为下标，例如 $q_{[30]+15}$ 与 $q_{[25]+20}$ 都写成 q_{45} 。

一个生命表，其中函数只对达到年龄给出取值的，称为综合表，例如表 1.3.1 。选择与终极表的最后一列是一个特殊的综合表，通常称为终极表 。

表 1.8.1 包含死亡概率及相应的函数 $l_{[x]+k}$ 取值，摘自英国精算学会发表的英国生命保险表 A1967–1970 。这个表的选择期为 2 年，作为示例，当然要比选择期为 15 年的表容易使用。

表 1.8.1　**英国** A1967–1970 **选择与终极表摘录**

$[x]$	(1) $1000q_{[x]}$	(2) $1000q_{[x]+1}$	(3) $1000q_{x+2}$	(4) $l_{[x]}$	(5) $l_{[x]+1}$	(6) l_{x+2}	(7) $x+2$
30	0.43767	0.57371	0.69882	33829	33814	33795	32
31	0.45326	0.59924	0.73813	33807	33791	33771	33
32	0.47711	0.63446	0.79004	33784	33767	33746	34
33	0.50961	0.68001	0.85577	33760	33742	33719	35
34	0.55117	0.73655	0.93663	33734	33715	33690	36

在表 1.8.1 中，我们注意到 32 岁人的三个死亡概率满足

$$q_{[32]} = 0.00047711 < q_{[31]+1} = 0.00059924 < q_{32} = 0.00069882.$$

这些概率的大小顺序看上去好象是有道理的，因为刚被接受人寿保险者的死亡率应该更低些。表中列 (3) 可视为终极死亡概率。

为建立一个选择与终极生命表，一般先建立终极部分。这一过程可使用诸如 (1.4.1) 的等式产生一组数值 $l_{x+r} = l_{[x]+r}$，其中 r 是选择期长度。然后根据关系式

$$l_{[x]+r-k-1} = \frac{l_{[x]+r-k}}{p_{[x]+r-k-1}} \qquad k = 0, 1, 2, \cdots, r-1$$

完成选择部分。

例 1.8.1: 用表 1.8.1 估计

(1) ${}_2p_{[20]}$，　(2) ${}_{1|}q_{[31]}$，　(3) ${}_3q_{[31]+1}$.

解：将这一章前面建立的公式运用于选择与终极表，于是

(1)

$$
\begin{aligned}
{}_2p_{[30]} &= \frac{l_{[30]+2}}{l_{[30]}} = \frac{l_{32}}{l_{[30]}} \\
&= \frac{33795}{33829} = 0.99899.
\end{aligned}
$$

(2)

$$
\begin{aligned}
{}_{1|}q_{[31]} &= \frac{l_{[31]+1} - l_{[31]+2}}{l_{[31]}} = \frac{l_{[31]+1} - l_{33}}{l_{[31]}} \\
&= \frac{33791 - 33771}{33807} = 0.00059.
\end{aligned}
$$

(3)

$$
\begin{aligned}
{}_3q_{[31]+1} &= \frac{l_{[31]+1} - l_{[31]+4}}{l_{[31]+1}} = \frac{l_{[31]+1} - l_{35}}{l_{[31]+1}} \\
&= \frac{33791 - 33719}{33791} = 0.00213.
\end{aligned}
$$

习　题

§1.2

1. 按表 1.2.2 完成下表空缺的栏目。

$s(x)$	$F(x)$	$f(x)$	μ_x
			$\mathrm{tg}\ x,\ 0 \le x \le \frac{\pi}{2}$
$e^{-x},\ x \ge 0$			
	$1 - \frac{1}{1+x},\ x \ge 0$		

2. 验证下列每一个函数可作为死亡效力，并写出相应的生存函数 (以下 $x \ge 0$) 。

(1)	Bc^x	$B > 0 \quad c > 1$	(Gompertz)
(2)	kx^n	$n > 0 \quad k > 0$	(Weibull)
(3)	$a(b + x)^{-1}$	$a > 0 \quad b > 0$	(Pareto)

3. 验证下列函数可作为生存函数，并给出相应的 μ_x, $f(x)$ 与 $F(x)$ 。

$$s(x)=e^{-x^3/12}\ x\geq 0.$$

4. 说明下列函数为什么不能作为相应符号所指明的函数。

(1). $\mu_x=(1+x)^{-3}$, $x\geq 0$.

(2). $s(x)=1-\frac{22x}{12}+\frac{11x^2}{8}-\frac{7x^3}{24}$, $0\leq x\leq 3$.

(3). $f(x)=x^{n-1}e^{-x/2}$, $x\geq 0$, $n\geq 1$.

5. 设 $s(x)=1-\frac{x}{100}$, $0\leq x\leq 100$ 。计算

(1) μ_x.　　(2) $F(x)$.

(3) $f(x)$.　　(4) $Pr[10<X<40]$.

6. 验证 ${}_{k|}q_0=-\Delta s(k)$ 以及 $\sum_{k=0}^{\infty}{}_{k|}q_0=1$ 。

§1.3　§1.4

7. 如果当 $20\leq x\leq 25$ 时 $\mu_x=0.001$，估计 ${}_{2|2}q_{20}$ 。

8. 如果生存组中 10 个生命的生存时间相互独立并由表 1.3.1 给出，求 $\mathcal{L}(65)$ 的概率函数、均值及方差。

9. 设 $s(x)=1-\frac{x}{12}$; $0\leq x\leq 12$, $l_0=9$, 并且生存时间相互独立，则 $({}_3\mathcal{D}_0,{}_3\mathcal{D}_3,{}_3\mathcal{D}_6,{}_3\mathcal{D}_9)$ 服从多项分布。计算

(1) 每个随机变量的期望值。

(2) 每个随机变量的方差。

(3) 每对随机变量的相关系数。

10. 以表 1.3.1 为基础，

(1) 比较 ${}_5q_0$ 与 ${}_5q_5$ 的值。

(2) 估计 (25) 在 80 岁与 85 岁之间死去的概率。

11. 设 l_{x+t} 在区间 $0\leq t\leq 1$ 内严格递减，证明

(1) 如 l_{x+t} 下凹，则 $q_x>\mu_x$ 。

(2) 如 l_{x+t} 上凹，则 $q_x<\mu_x$ 。

12. 证明

(1) 当 $\frac{d}{dx}\mu_x<\mu_x^2$ 时，　$\frac{d}{dx}(l_x\mu_x)<0$.

(2) 当 $\frac{d}{dx}\mu_x=\mu_x^2$ 时，　$\frac{d}{dx}(l_x\mu_x)=0$.

(3) 当 $\frac{d}{dx}\mu_x > \mu_x^2$ 时， $\frac{d}{dx}(l_x\mu_x) > 0$.

13. 设一个随机生存组由两个子生存组构成：(1) 1600 个新生儿生存者；(2)540 个 10 年后加入的 10 岁生存者。适合两者的死亡表摘录如下：

x	l_x
0	40
10	39
70	26

如 Y_1 与 Y_2 分别是子生存组 (1) 与 (2) 中活到 70 岁的生存者人数，在各生命独立性的假定下估计常数 c, 使得 $Pr(Y_1+Y_2 > c) = 0.05$ 。

§1.5

14. 以 $\mathring{e}_{x:\overline{n}|}$ 记 (x) 在年龄 x 与 $x+n$ 之间的期望剩余寿命。证明

$$\begin{aligned}\mathring{e}_{x:\overline{n}|} &= \int_0^n t\,{}_tp_x\mu_{x+t}dt + n\,{}_np_x \\ &= \int_0^n {}_tp_x dt.\end{aligned}$$

这个量称为部分期望剩余寿命。

15. 设随机变量 T 的概率密度函数为 $f(t) = ce^{-ct}$, $t \geq 0$, 其中常数 $c > 0$ 。计算

(1)$\mathring{e}_x = E[T]$. (2)Var$[T]$. (3) median $[T]$ (中位值) 。

16. 设 $\mu_{x+t} = t$, $t \geq 0$, 计算

(1) ${}_tp_x\mu_{x+t}$. (2) $\mathring{e}_x$.

17. 设随机变量 T 的分布函数为

$$F(t) = \begin{cases} \frac{t}{100-x} & 0 \leq t < 100-x \\ 1 & t \geq 100-x. \end{cases}$$

计算

(1) $\overset{\circ}{e}_x$.　　(2) $\mathrm{Var}[T]$.　　(3) median $[T]$.

18. 证明

(1) $\frac{\partial}{\partial x}{}_tp_x = {}_tp_x(\mu_x - \mu_{x+t})$.

(2) $\frac{d}{dx}\ \overset{\circ}{e}_x = \overset{\circ}{e}_x\ \mu_x - 1$.

(3) $\Delta e_x = q_x e_{x+1} - p_x$.

19. 设 $s(x) = \frac{\sqrt{100-x}}{10}$, $0 \le x \le 100$, 求

(1) ${}_{17}p_{19}$;　　(2) ${}_{15}q_{36}$;　　(3) ${}_{15|13}q_{36}$;

(4) μ_{36};　　(5) e_{36}.

20. 验证以下断语：

(1) $a(x)d_x = L_x - L_{x+1}$

(2) 例 1.5.1 中的近似在表 1.3.1 中用于计算 L_0, 但并未用于计算 L_1 。

(3) $T_x = \sum_{k=0}^{\infty} L_{x+k}$.

§1.6

21. 验证表 1.6.1 中常数死亡效力与 Balducci 假设下有关栏目内的各表达式。

22. 在表 1.6.1 的三种假设下分别画出 μ_{x+t}, $0 < t < 1$ 的图形，并画出相应的生存函数图形。

23. 用表 1.3.1 中 l_x 栏目，在表 1.6.1 的三种假设下分别计算 ${}_{1/2}p_{65}$ 。

24. 用表 1.3.1 以及每一年龄死亡均匀分布的假设, 求 median$[T]$, 其中 T 是

(1) 0 岁;　　(2) 50 岁

人的剩余寿命。

25. 设 $q_{70} = 0.04$, $q_{71} = 0.05$ 。计算 (70) 在年龄 $70\frac{1}{2}$ 与 $71\frac{1}{2}$ 之间死亡的概率，其中假定

(1) 每一年龄死亡均匀分布；

(2) 每一年龄死亡服从 Balducci 假设。

26. 用表 1.3.1 中 l_x 栏目及表 1.6.1 中每个假设分别计算

(1)$\lim_{x\to 60^-}\mu_x$;　　(2) $\lim_{x\to 60^-}\mu_x$;　　(3) $\mu_{60\frac{1}{2}}$.

27. 如采取常数死亡效力假定，证明

(1) $a(x)=\frac{(1-e^{-\mu})/\mu-e^{-\mu}}{1-e^{-\mu}}$.　　(2) $a(x)\cong\frac{1}{2}-\frac{q_x}{12}$.

28. 如采取 Balducci 假设，证明

(1)$a(x)=-\frac{p_x}{q_x^2}(q_x+\log p_x)$.　　(2) $a(x)\cong\frac{1}{2}-\frac{q_x}{6}$.

§1.7

29. 验证表 1.7.1 中 de Moivre 律及 Weibull 律有关栏目内的表达式。

30. 考虑由下式给出的 de Moivre 死亡律的修正：

$$s(x)=(1-\frac{x}{\omega})^{\alpha},\ 0\le x<\omega,\ \alpha>0.$$

计算

(1)μ_x.　　(2) $\mathring{e}_x$.

31. 用表 1.8.1 计算

(1) ${}_2q_{[32]+1}$.　　(2) ${}_2p_{[31]+1}$.

32. 量

$$1-\frac{q_{[x]+k}}{q_{x+k}}=I(x,k)$$

称为 选择指数 。当它接近于 0 时，表明选择的作用消失。由表 1.8.1 计算 $x=32$, $k=1$ 及 2 时的选择指数。

综合题

33. 设一个 50 岁生命在 50 至 51 岁间面临额外危险。如果正常情况下在 50 至 51 岁间死亡概率为 0.006, 而额外危险可表示成附加一个年初值为 0.03 并均匀递减到年末值为 0 的死亡效力，计算该生命活到 51 岁的概率。

34. 求常数 $c>0$, 使得当死亡效力由 μ_{x+t}, $0\le t\le 1$ 改变成 $\mu_{x+t}-c$ 时，(x) 在 1 年内死亡的概率减半。答案用 q_x 表示。

35. 从一个标准死亡表，将其死亡效力翻倍导出第二个死亡表。对任意给定年龄 x, 新表的死亡率 $q'x$ 与标准表的死亡率 q_x

相比，是高于两倍、等于两倍还是低于两倍？

36. 如 $\mu_x = Bc^x$，则函数 $l_x\mu_x$ 在满足 $\mu_{x_0} = \log c$ 的年龄 x_0 取得最大值。[提示：利用习题 12]

37. 设 $\mu_x = \frac{Ac^x}{1+Bc^x}$，$x > 0$。

(1) 计算生存函数 $s(x)$。

(2) 验证死亡年龄 X 分布的众数（使概率密度函数取得最大值的数）为

$$x_0 = \frac{\log(\log c) - \log A}{\log c}.$$

38. 设对于 $40 < x < 100$，$\mu_x = \frac{3}{100-x} - \frac{10}{250-x}$，计算

(1) ${}_{40}p_{50}$。　　(2) 死亡年龄 X 分布的众数。

39. (1) 在死亡均匀分布的假定下证明

$$m_x = \frac{q_x}{1-\frac{1}{2}q_x} \text{ 与 } q_x = \frac{m_x}{1+\frac{1}{2}m_x}.$$

(2) 在常数死亡效力假设下计算 m_x，结果用 q_x 表示。

(3) 在 Balducci 假设下计算 m_x，结果用 q_x 表示。

(4) 设 $l_x = 100 - x$，$0 \le x \le 100$，根据定义

$${}_n m_x = \frac{\int_0^n l_{x+t}\mu_{x+t}dt}{\int_0^n l_{x+t}dt}$$

计算 ${}_{10}m_{50}$。

40. 证明 K 与 S 独立当且仅当表达式

$$\frac{{}_s q_{x+k}}{q_{x+k}}$$

对于 $0 \le s \le 1$ 不依赖于 k。

第二章　人寿保险

§2.1　引　言

这一章将建立为减轻死亡所引起不利影响而设计的人寿保险的模型。与风险理论中讨论过的短期模型不同，这些保险的长期性使得从投保到赔付期间的投资收益 (利息) 成为不可忽视的因素。在这里所考察的人寿保险中，赔付金额与时间只依赖于被保险人的死亡时间，换言之，模型将以被保险人的剩余寿命随机变量为依据。

尽管这一章只提及人寿保险，但这些想法也适用于其它对象的保险，如机器设备，贷款，商业冒险等。实际上，只要财政影响的金额与时间可以仅仅以某随机事件的发生时间来表示的话，这一章的一般模型就有用武之地。

§2.2　死亡即刻赔付保险

设 t 是从保单签发到投保人死亡这段时间区间的长度。在这一节亦将考虑的两全保险中，t 可能比保单签发到赔付时间更长。v_t 是从赔付时刻回溯至保单签发时的利息贴现因子，称为贴现函数。b_t 是赔付的受益金额，称为受益函数。

对于贴现函数，我们假定利息效力是决定性的，即模型不包含利息效力的概率分布，而且通常在利息效力为常数的假定下给出比较简单的公式。

受益赔付额在保单发行时的现值

$$z_t = b_t v_t, \tag{2.2.1}$$

称为现值函数。从保单签发到被保险人死亡所经历的时间就是被保险人（投保时年龄为 x）的剩余寿命随机变量 $T=T(x)$。于是保单在发行时的现值是随机变量 z_T。除非根据上下文需要更周到的符号，我们将这个随机变量记为 Z，即

$$Z = b_T v_T \tag{2.2.2}$$

下面将就各种情形建立 Z 的概率模型。对具体人寿保险的第一步分析是定出 b_t 与 v_t，随后下一步根据 T 的分布得出 Z 的某些特征。以下对几种常见保险分别按步就班进行讨论，其结果总结在这一节最后的表 2.2.1 中。

一、定额受益保险

n 年期人寿保险只有当被保险人在 n 年内死亡时提供赔付。如果当 (x) 在 n 年内死亡时应付受益人金额为 1 个单位，那么

$$b_t = \begin{cases} 1 & t \le n \\ 0 & t > n \end{cases}$$

$$v_t = v^t \qquad t \ge 0$$

$$Z = \begin{cases} v^T & T \le n \\ 0 & T > n, \end{cases}$$

其中 $v = e^{-\delta} = (1+i)^{-1}$，$\delta$ 是利息效力，i 是有效年利率。这里有两个约定：其一是因为剩余寿命乃非负变量，b_t 及 v_t 等只对非负值 t 定义；其二是对于使 b_t 为 0 的 t 值，v_t 取值多少无关紧要，可根据方便来写出 v_t 的定义。

对于人寿保险，现值随机变量 Z 的期望值称为*净趸缴保费*或*趸缴纯保费*(net single premium)，它不包含附加保费。趸缴意味着一次性缴付而不是按年、半年、季、月或人寿保险实践中接受的其它方式分期缴付。

读者可能发现，与随机事件发生相关的赔付额现值的期望值在不同场合有不同名称。在风险理论中，期望损失称为纯保费。

这一词汇普遍用于财产与责任保险中，在第三章里与生存相关联的支付额现值的期望值称为 **精算现值** (actuarial present value)，这与退休计划中的术语一致。虽然以上三者都合适，但在讨论人寿保险时，我们采用净趸缴保费这一名称。更准确的称呼应该是 **期望赔付现值** 。我们将按国际精算符号规则 (见附录 4) 来记净趸缴保费。

在 (x) 死亡即刻应付 1 个单位金额的 n 年期保险的净趸缴保费为 $E[Z]$, 记作 $\bar{A}^{1}_{x:\overline{n}|}$ 。根据 (1.2.19) 给出的 T 的概率密度函数可以计算

$$\begin{aligned}\bar{A}^{1}_{x:\overline{n}|} &= E[Z] = E[z_T] = \int_0^{\infty} z_t g(t)dt \\ &= \int_0^n v^t {}_tp_x \mu_{x+t} dt. \end{aligned} \tag{2.2.3}$$

Z 的分布的 j 阶矩为

$$\begin{aligned}E[Z^j] &= \int_0^n (v^t)^j {}_tp_x \mu_{x+t} dt \\ &= \int_0^n e^{-(\delta j)t} {}_tp_x \mu_{x+t} dt,\end{aligned}$$

其中 δ 是利息效力，即 $v = e^{-\delta}$ 。以上公式表明， Z 的 j 阶矩相当于利息效力换成 $j\delta$ 的期望值。这一高阶矩性质在一般情况下也成立。

定理 2.2.1: 对于 (x) 的有关人寿保险，设时刻 s(从投保之时算起) 的利息效力为 δ_s, 受益函数与贴现函数分别为 b_t 与 v_t 。如果对所有 t 都有 $b_t^j = b_t$, 那么按利息效力 δ_s 计算的 $E[Z^j]$ 等于按利息效力 $j\delta_s$ 计算的 $E[Z]$, 即 $E[Z^j]@\delta_s = E[Z]@j\delta_s$ 。

证：根据利息效力定义，有

$$v_t = \exp(-\int_0^t \delta_s ds), \tag{2.2.4}$$

而

$$v_t^j = \exp(-\int_0^t j\delta_s ds)$$

是相应于利息效力 $j\delta_s$ 的贴现函数。

$$\begin{aligned} E[Z^j] &= E[(b_T v_T)^j] = E[b_T^j v_T^j] \\ &= E[b_T v_T^j]. \end{aligned}$$

从定理 2.2.1 可得出

$$\mathrm{Var}[Z] = {}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2, \tag{2.2.5}$$

其中 ${}^2\bar{A}^1_{x:\overline{n}|}$ 是 1 个单位 n 年期保险按利息效力为 2δ 计算的净趸缴保费。

终身人寿保险(whole life insurance) 在被保险人未来任何时候死亡时都提供赔付。如果 (x) 死亡时应付金额为 1 个单位，那么

$$b_t = 1,\ t \geq 0,$$

$$v_t = v^t,\ t \geq 0,$$

$$Z = v^T,\ (T \geq 0);$$

这里利息效力为常数。净趸缴保费为

$$\bar{A}_x = E[Z] = \int_0^\infty v^t\, {}_tp_x \mu_{x+t} dt. \tag{2.2.6}$$

终身人寿保险是 n 年期保险当 $n \to \infty$ 的极限情形。

例 2.2.1：设 (x) 的剩余寿命 $T(x)$ 的概率密度函数为

$$g(t) = \begin{cases} \frac{1}{80} & 0 < t < 80, \\ 0 & \text{其它}. \end{cases}$$

按利息效力 δ 计算 1 个单位金额终身保险的现值随机变量 Z 的

(1) 净趸缴保费。

(2) 方差。

(3) 第 90 个百分位数 $\xi_{0.9}$ 。

解：

(1) $\bar{A}_x = E[Z] = \int_0^{\infty} v^t g(t)dt = \int_0^{80} e^{-\delta t}\frac{1}{80}dt = \frac{1-e^{-80\delta}}{80\delta}$, $\delta \neq 0$.

(2) $\mathrm{Var}[Z] = {}^2\bar{A}_x - (\bar{A}_x)^2 = \frac{1-e^{-160\delta}}{160\delta} - (\frac{1-e^{-80\delta}}{80\delta})^2$, $\delta \neq 0$.

(3) 根据定义， $Pr[Z \leq \xi_{0.9}] = 0.9$ 。由 $Z = v^T$ 及 $v = e^{-\delta} < 1$ 可知 (参见图 2.2.1), $Z \leq \xi_{0.9}$ 等价于 $T \geq h$, 其中 h 满足 $v^h = \xi_{0.9}$ 。从

$$Pr[T \geq h] = \int_h^{80} \frac{1}{80}dt = 0.9$$

解得 $h = 8$, 于是

$$\xi_{0.9} = v^8 = e^{-8\delta}.$$

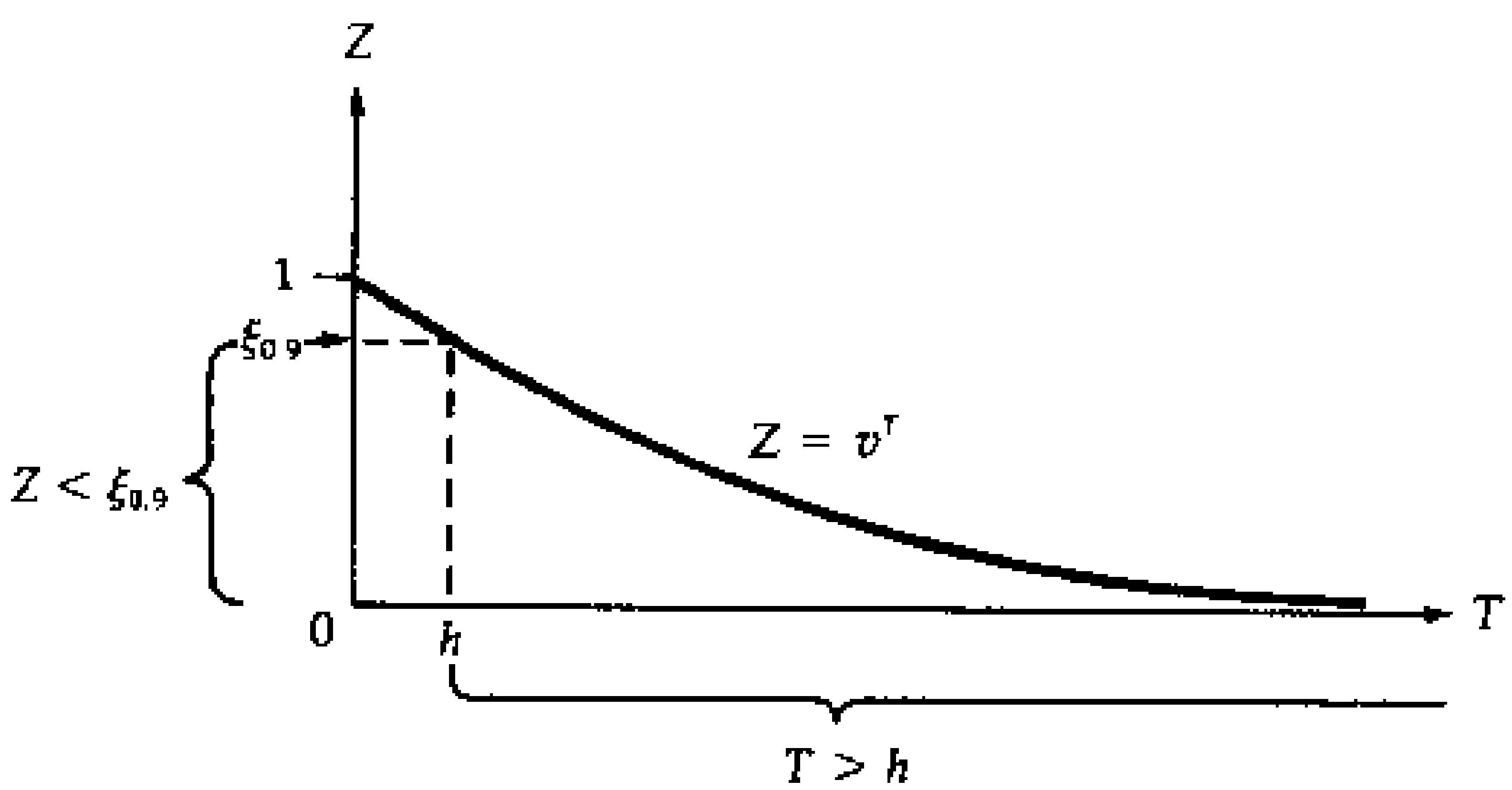


图 2.2.1　$\xi_{0.9}$ 之决定

例 2.2.2: 设 100 个相互独立的 x 岁生命的死亡效力 $\mu = 0.04$(相同), 投保死亡时赔付受益金额为 10 单位的终身人寿保险。受益金将从一个瞬时收益率 (即利息效力) $\delta = 0.06$ 的投资基金提取。计算 $t = 0$ 时的最低投资金额, 使得大致以概率 0.95 保证有充足的基金可供支付死亡受益。

解: 对每个生命,

$$b_t = 10 \quad t \geq 0, \quad v_t = v^t \quad t \geq 0, \quad Z = 10v^T \quad T \geq 0$$

将这 100 个生命 (譬如按保单顺序) 编号, 总赔付额的现值为

$$S = \sum_{1}^{100} Z_j$$

其中 Z_j 相互独立, 与 Z 有相同的分布。由

$$E[Z] = 10\bar{A}_x = 10\int_0^\infty e^{-\mu t}\mu dt = 10\frac{\mu}{\mu+\delta} = 4,$$

$$\mathrm{Var}[Z] = 10^2[{}^2\bar{A}_x - (\bar{A}_x)^2] = 100(0.25 - 0.16) = 9,$$

可得

$$E[S] = 100 \times 4 = 400,$$

$$\mathrm{Var}[S] = 100 \times 9 = 900.$$

所求最低金额 h 应满足

$$Pr(S \leq h) = 0.95,$$

即

$$Pr\left(\frac{S - E[S]}{\sqrt{\mathrm{Var}[S]}} \leq \frac{h - 400}{30}\right) = 0.95.$$

按中心极限定理作正态近似可得

$$\frac{h-400}{30} = 1.645,$$

$$h = 449.35.$$

初始基金 449.35 与赔付额现值的期望值 400 的差异 49.35 乃风险附加费 (即安全附加费)，是净趸缴保费的 12.34% 。

这个例子使用了风险理论中的个体风险模型，并对 S 的概率分布作正态近似。与短期情形的主要区别在于，这里保费的利息收入也用来提供赔付受益金。图 2.2.2 画出了假如在时刻 1/8, 7/8, 9/8, 13/8, 15/8 分别有一个死亡及时刻 10/8 有两个死亡时的基金在开始两年内的变化情况，在表示支付受益金的间断之间，是瞬时增长率 (利息效力) 为 $\delta = 0.06$ 的指数曲线弧段。

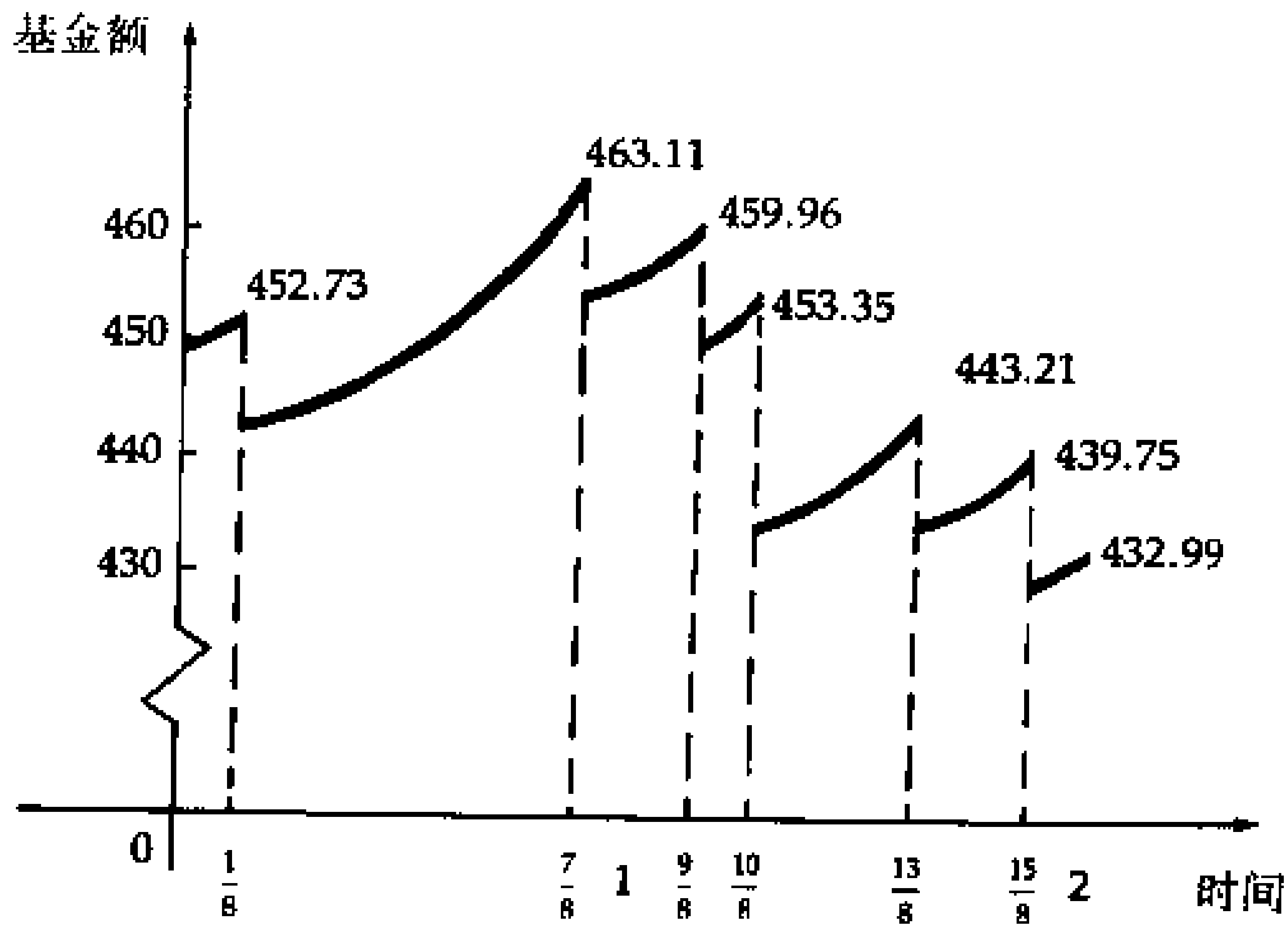


图 2.2.2　基金变化情况示例

二、两全保险

n 年期生存保险(纯生存保险) 只有当被保险人从保单生效起至少活 n 年时才提供支付。如果应付额为 1 个单位，那么

$$b_t = \begin{cases} 0 & t \le n \\ 1 & t > n, \end{cases}$$

$$v_t = v^n \qquad t \geq 0$$

$$Z = \begin{cases} 0 & T \leq n \\ v^n & T > n. \end{cases}$$

在生存保险中唯一不确定的因素是理赔是否发生。若理赔发生，则赔付金额与时间是预先确定的。n 年期生存保险的净趸缴保费记为 $A_{x:\overline{n}|}^{\ \ 1}$ 。设 Y 是活到 $x+n$ 岁这一事件的指示变量，即被保险人活到 $x+n$ 岁时取值 1, 否则取值 0, 于是 $Z = v^n Y$,

$$\begin{aligned} A_{x:\overline{n}|}^{\ \ 1} &= E[Z] = v^n E[Y] = v^n {}_n p_x, \\ \mathrm{Var}[Z] &= v^{2n}\mathrm{Var}[Y] = v^{2n}[E[Y^2] - (E[Y])^2] \\ &= v^{2n}[{}_n p_x - ({}_n p_x)^2] = v^{2n}{}_n p_x {}_n q_x \\ &= {}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2. \end{aligned} \tag{2.2.7}$$

n 年期两全保险(亦称养老保险) 不管被保险人在 n 年内死亡还是生存到 n 年期末都提供支付。如果保险金额为 1 个单位且死亡受益在死亡即刻赔付，那么

$$b_t = 1 \qquad t \geq 0,$$

$$v_t = \begin{cases} v^t & t \leq n \\ v^n & t > n, \end{cases}$$

$$Z = \begin{cases} v^T & T \leq n \\ v^n & T > n. \end{cases}$$

其净趸缴保费记为 $\bar{A}_{x:\overline{n}|}$ 。

这种保险可看作 n 年定期人寿保险与 n 年期生存保险的混合。设定期保险，生存保险与两全保险的现值随机变量分别为 Z_1，Z_2 与 Z_3 。显然有

$$Z_3 = Z_1 + Z_2, \tag{2.2.8}$$

两边取数学期望得

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ 1}. \tag{2.2.9}$$

根据定理 2.2.1 可得 $E[Z_3^j]@\delta = E[Z_3]@j\delta$

$$\mathrm{Var}[Z_3] = {}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2. \tag{2.2.10}$$

方差 $\mathrm{Var}[Z_3]$ 也可通过 (2.2.8) 得出，

$$\mathrm{Var}[Z_3] = \mathrm{Var}[Z_1] + \mathrm{Var}[Z_2] + 2\mathrm{Cov}[Z_1, Z_2]. \tag{2.2.11}$$

按公式

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y], \tag{2.2.12}$$

并注意到 $Z_1Z_2 = 0$, 可知

$$\mathrm{Cov}[Z_1, Z_2] = -E[Z_1]E[Z_2] = -\bar{A}^{1}_{x:\overline{n}|}A_{x:\overline{n}|}^{\ \ \ 1}. \tag{2.2.13}$$

将 (2.2.5),(2.2.7) 及 (2.2.13) 代入 (2.2.11) 可获得用 n 年定期人寿保险与生存保险的净趸缴保费表示的 $\mathrm{Var}[Z_3]$ 。

由于净趸缴保费是正的， $\mathrm{Cov}[Z_1, Z_2]$ 必小于零，也就是说 Z_1 与 Z_2 负相关。不过两者的相关系数并非 -1 。

三、延期保险

m 年递延保险(延期保险) 只有当被保险人在保单生效的 m 年之后死亡才提供受益支付，其方式与期限可以是以上讨论过的任何一种。例如 m 年递延终身保险，当死亡时应付金额为 1 个单位时，

$$b_t = \begin{cases} 1 & t > m \\ 0 & t \le m, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T & T > m \\ 0 & T \le m. \end{cases}$$

其净趸缴保费记为 ${}_{m|}\bar{A}_x$:

$$ {}_{m|}\bar{A}_x = \int_m^\infty v^t {}_tp_x \mu_{x+t} dt. \tag{2.2.14}$$

例 2.2.3: 考虑赔付金额为 1 个单位的 5 年递延终身保险。设被保险人 (x) 的死亡效力为常数 $\mu = 0.04$, 按 $\delta = 0.10$ 计算受益赔付额现值分布的：(1) 期望值，(2) 方差，(3) 中位数 $\xi_{0.5}$ 。

解：(1) 对常数效力 μ 及 δ,

$$ {}_{5|}\bar{A}_x = \int_5^\infty e^{-\delta t} e^{-\mu t} \mu dt = \frac{\mu}{\mu+\delta} e^{-5(\mu+\delta)}.$$

将 $\mu = 0.04$ 及 $\delta = 0.10$ 代入得

$$ {}_{5|}\bar{A}_x = \frac{2}{7} e^{-0.7} = 0.1419.$$

(2) 根据定理 2.2.1,

$$\mathrm{Var}[Z] = \frac{\mu}{\mu+2\delta} e^{-5(\mu+2\delta)} - [\frac{\mu}{\mu+\delta} e^{-5(\mu+\delta)}]^2 = 0.0301.$$

(3) Z 与 T 的关系在图 2.2.3 中给出。虽然 T 是连续型随机变量，但 Z 却是混合型的，在 $Z = 0$ 处有集中的概率。当 $Pr(Z = 0) = Pr(T \le m) \ge 0.5$ 时，$\xi_{0.5} = 0$; 否则是下式的解。

$$Pr(Z \le \xi_{0.5}) = 0.5. \tag{2.2.15}$$

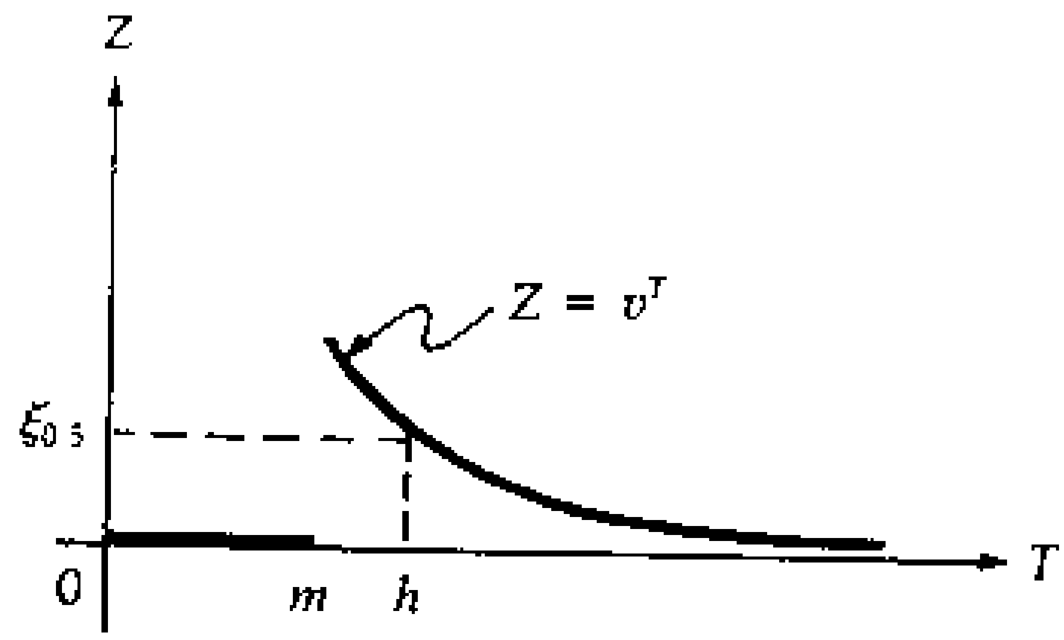


图 2.2.3　Z 与 T 的关系图

在本例中

$$
\begin{aligned}
Pr(Z=0) &= Pr(T \le 5) = \int_0^5 e^{-0.04t} 0.04 dt = 1 - e^{-0.2} \\
&= 0.1813 < 0.5.
\end{aligned}
$$

于是 $\xi_{0.5}$ 由 (2.2.15) 决定，即

$$Pr(Z=0) + Pr(0 < Z \le \xi_{0.5}) = 0.5,$$

$$Pr(0 < Z \le \xi_{0.5}) = 0.3187.$$

这等价于

$$Pr(v^T < \xi_{0.5}) = 0.3187.$$

也就是

$$Pr(T > \log_v \xi_{0.5}) = 0.3187.$$

若记 $h = \log_v \xi_{0.5}$，则 h 满足

$${}_hp_x = 0.3187,$$

利用 ${}_hp_x = e^{-0.04h}$ 可得

$$h = \frac{\log 0.3187}{-0.04},$$

于是

$$\begin{aligned}\xi_{0.5} &= v^h = e^{-\delta\frac{\log 0.3187}{-0.04}} \\ &= (0.3187)^{\frac{0.10}{0.04}} = 0.0573.\end{aligned}$$

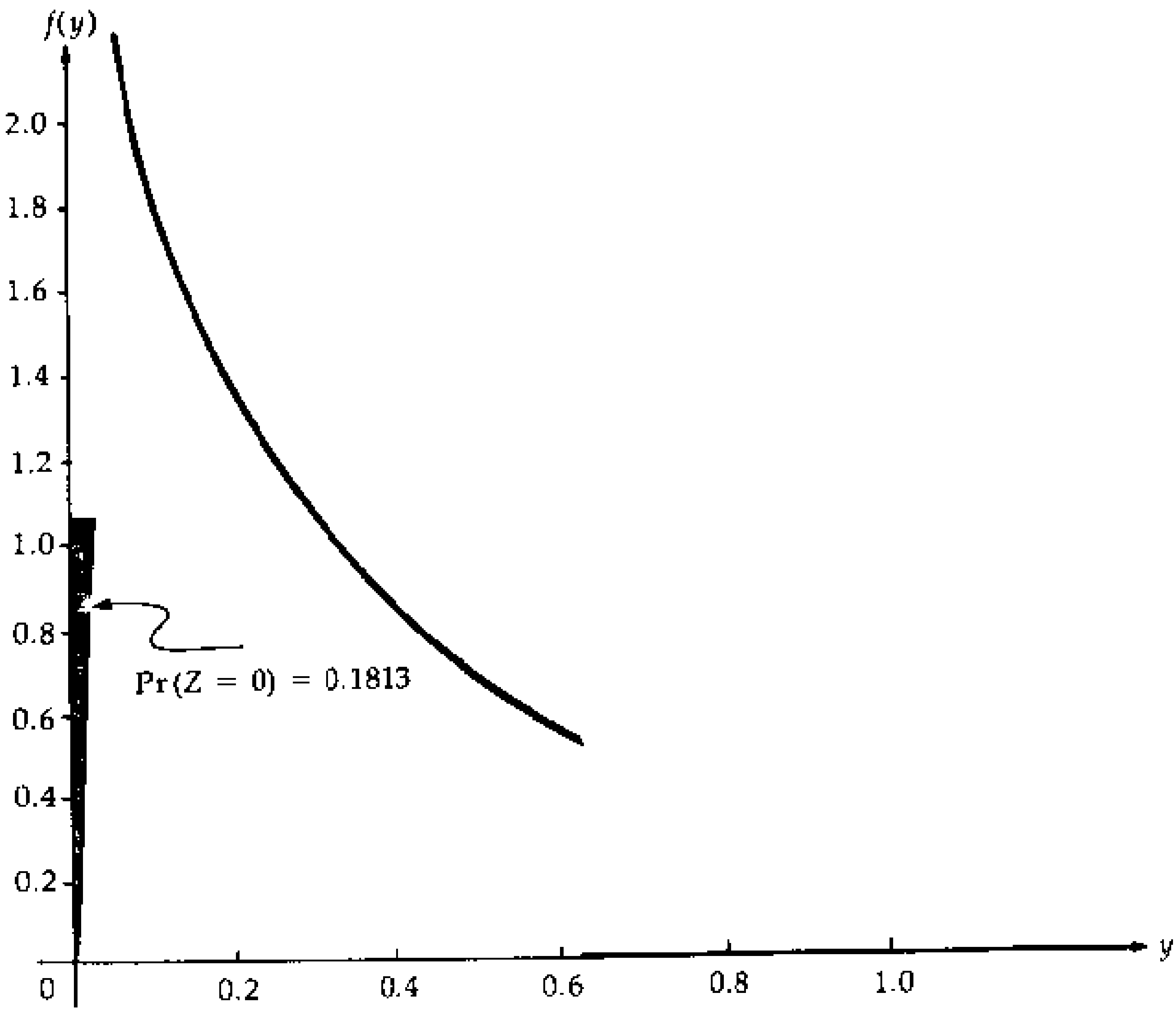


图 2.2.4　　Z 的分布

这个例题中具有非零概率密度的 Z 的最大值为 $e^{-0.1\times5}=0.6065$, 对应于 $T=5$ 。用类似于求中位值的方式可以得出

$$F(y)=Pr(Z\leq y)=0.1813+y^{0.4}.$$

从而概率密度函数为 $0.4y^{-0.6}, 0<y\leq0.6065$ 。 Z 的分布略图在图 2.2.4 中给出，其中的尖状阴影标记代表在 $Z=0$ 的集中概率，值为 0.1813, 它不能从纵坐标轴的刻度读出，因为后者只适用于 $y>0$ 时的概率密度函数。

Z 的分布向右高度偏斜,尽管其总的分布落在区间 $(0,0.6065)$ 里，其均值为 0.1419 ，而它的中位值只有 0.0573. 这种向大的正值方向偏斜的特征是所有保险领域中的许多理赔分布所共有的。

四、变额受益保险

由 (2.2.1) 给出的一般模型也可应用于分析死亡受益金按算术级数递增或递减的保险，这种保险常作为附加受益出售。

递增终身人寿保险(increasing whole life insurance) 当 (x) 在第 1 年死亡时赔付 1 个单位受益金额，第 2 年死亡时则赔付 2 个单位受益金，并以此类推。其受益函数为

$$b_t=[t+1] \qquad t\geq0,$$

其中方括号表示最大整数函数

$$[t]=k \quad k\leq t<k+1,\ k=0,\pm1,\cdots.$$

由贴现函数

$$v_t=v^t \qquad t\geq0$$

可写出赔付额现值随机变量

$$Z=b_Tv_T=[T+1]v_T.$$

这种保险的净趸缴保费为

$$\begin{aligned}(I\bar{A})_x &= E[Z]=\int_0^{\infty}[t+1]v^t{}_tp_x\mu_{x+t}dt \\ &= \sum_{k=1}^{\infty}k\int_{k-1}^{k}v^t{}_tp_x\mu_{x+t}dt.\end{aligned}$$

与前面 1 个单位定额受益保险不同，高阶矩并不等于按调整利息效力计算的净趸缴保费，只能按定义直接计算。

保险受益金的递增可以比一年一次更频繁，或者相反。对于年递增 m 次的终身人寿保险：当被保险人 (x) 在第 1 个 $\frac{1}{m}$ 年内死亡时，受益金额为 $\frac{1}{m}$，在第 2 个 $\frac{1}{m}$ 年 (即 $\frac{1}{m}$ 年至 $\frac{2}{m}$ 年) 内死亡时，受益金额为 $\frac{2}{m}$；以此类推。其受益函数可表示成

$$b_t=\frac{[tm+1]}{m}\quad t\geq 0,$$

由

$$v_t=v^t\qquad t\geq 0$$

得

$$Z=\frac{v^T[mT+1]}{m}.$$

其净趸缴保费为

$$(I^{(m)}\bar{A})_x=E[Z]=\int_0^{\infty}\frac{v^t}{m}[mt+1]{}_tp_x\mu_{x+t}dt.$$

在 $m\to\infty$ 的极限情形是在时间 t 死亡时赔付受益金额 t 的保险，其有关函数为

$$b_t=t\quad t\geq 0,$$

$$v_t=v^t\quad t\geq 0,$$

$$Z=Tv^T.$$

净趸缴保费记号为 $(\bar{I}\bar{A})_x$ 。

$$(\bar{I}\bar{A})_x = \int_0^\infty tv^t{}_tp_x\mu_{x+t}dt.$$

将此式写成

$$(\bar{I}\bar{A})_x = \int_0^\infty (\int_0^t ds)v^t{}_tp_x\mu_{x+t}dt.$$

交换积分次序并根据 (2.2.14) 可得

$$\begin{aligned}(\bar{I}\bar{A})_x &= \int_0^\infty \int_s^\infty v^t{}_tp_x\mu_{x+t}dtds \\ &= \int_0^\infty {}_{s|}\bar{A}_x ds,\end{aligned}$$

这表明，一个保额连续递增终身人寿保险等价于一系列递延定额终身人寿保险。

如果只有当死亡发生在 n 年期限内时才支付受益金的话，这种保险称为递增 n 年期人寿保险。

与递增 n 年期人寿保险互补的是 递减 n 年期人寿保险(decreasing $n-$ year term life insurance), 在第 1 年死亡时赔付 n ，第 2 年死亡时赔付 $n-1$, 以此类推，在最后第 n 年死亡时赔付 1 。这种保险的有关函数为

$$b_t = \begin{cases} n-[t] & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \quad t > 0,$$

$$Z = \begin{cases} v^T(n-[T]) & T \le n \\ 0 & T > n. \end{cases}$$

其净趸缴保费为

$$(D\bar{A})^1_{x:\overline{n}|} = \int_0^n v^t(n-[t])_tp_x\mu_{x+t}dt.$$

这一节模型的概要列于表 2.2.1 中，其最后一列数字根据表后附注指明是否可根据定理 2.2.1 计算高阶矩。

2.2.1 死亡即刻赔付保险概要

(1) 保险名称	(2) 受益函数 b_t $t \geq 0$	(3) 贴现函数 v_t $t \geq 0$
终身人寿	1	v^t
n 年期人寿	$1 \quad t \leq n$ $0 \quad t > n$	v^t
n 年期生存	$0 \quad t \leq n$ $1 \quad t > n$	v^n
n 年期两全	1	$v^t \quad t \leq n$ $v^n \quad t > n$
递延 m 年的 n 年期人寿	$1 \quad m < t \leq m+n$ $0 \quad t \leq m, t > m+n$	v^t
按年递增 n 年期人寿	$[t+1] \quad t < n$ $0 \quad t \geq n$	v^t
按年递减 n 年期人寿	$n-[t] \quad t < n$ $0 \quad t \geq n$	v^t
年 m 次递增终身人寿	$\frac{[tm+1]}{m}$	v^t

	(4) 现值函数 z_t $(t \geq 0)$	(5) 净趸缴保费	(6) 高阶矩
终身人寿	v^t	$\bar{A}_x$	
n 年期人寿	v^t	$\bar{A}^1_{x:\overline{n}\rvert}$	1.
n 年期生存	$0 \quad t \leq n$ $v^n \quad t > n$	$A_{x:\overline{n}\rvert}^{\ \ 1}$	1.
n 年期两全	$v^t \quad t \leq n$ $v^n \quad t > n$	$\bar{A}_{x:\overline{n}\rvert}$	1.
递延 m 年的 n 年期人寿	$v^t \quad m < t \leq m+n$ $0 \quad t \leq m, t > m+n$	${}_{m\|n}\bar{A}_x$	1.
按年递增 n 年期人寿	$[t+1]v^t \quad t < n$ $0 \quad t \geq n$	$(I\bar{A})^1_{x:\overline{n}\rvert}$	2.
按年递减 n 年期人寿	$(n-[t])v^t \quad t < n$ $0 \quad t \geq n$	$(D\bar{A})^1_{x:\overline{n}\rvert}$	2.
年 m 次递增终身人寿	$\frac{v^t[tm+1]}{m}$	$(I^{(m)}\bar{A})_x$	2.

1. j 阶矩等于按 j 倍利息效力计算的净趸缴保费，记为 ${}^jA, j > 1$ 。

2. 直接按定义计算 $E[Z^j]$ 。

§2.3 死亡年末赔付保险

上一节建立的人寿保险模型中，受益金是在死亡后立刻赔付的。在实践中，几乎所有的保险都是如此。这些模型建立在保单发行时被保险人的剩余寿命 T 的基础上，然而有关 T 之概率分布的最佳信息来自离散形式的生命表，也就是说，现成的是整值剩余寿命 K 的概率分布。这一节与下一节建立的模型将弥合这个差异，在那些模型中，受益金与赔付时间只依赖于被保险人存活的完整年数，即所谓*死亡年末赔付*(payable at the end of the year of death) 。

所要建立的模型将以被保险人 (x) 的整值剩余寿命为基础。受益函数 b_{k+1} 与贴现函数 v_{k+1} 分别是受益金额与从赔付时刻回溯至保单签发时的贴现因子。当被保险人的整值剩余寿命取值为 k 时，其死亡时间是在第 $k+1$ 年。受益赔付额在保单发行时的现值函数为

$$z_{k+1} = b_{k+1}v_{k+1}. \tag{2.3.1}$$

类似地， $Z = z_{K+1}$ 是现值随机变量。

对于死亡年末赔付 1 个单位金额的 n 年定期保险，有

$$b_{k+1} = \begin{cases} 1 & k = 0, 1, \cdots, n-1 \\ 0 & \text{其它}, \end{cases}$$

$$v_{k+1} = v^{k+1},$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \cdots, n-1 \\ 0 & \text{其它}. \end{cases}$$

这种保险的净趸缴保费为

$$A_{x:\overline{n}|}^{1}=E[Z]=\sum_{k=0}^{n-1} v^{k+1}{}_{k}p_{x}q_{x+k}. \tag{2.3.2}$$

定理 2.2.1 经适当改变符号后，对死亡年末赔付保险亦成立。例如对以上 n 年定期保险

$$\operatorname{Var}[Z]={}^{2}A_{x:\overline{n}|}^{1}-(A_{x:\overline{n}|}^{1})^{2},$$

其中

$${}^{2}A_{x:\overline{n}|}^{1}=\sum_{k=0}^{n-1} e^{-2\delta(k+1)}{}_{k}p_{x}q_{x+k}.$$

对于 (x) 的终身人寿保险模型，可在 n 年定期保险模型中令 $n\to\infty$, 得净趸缴保费

$$A_{x}=\sum_{k=0}^{\infty} v^{k+1}{}_{k}p_{x}q_{x+k}. \tag{2.3.3}$$

两端乘以 l_x, 得

$$l_{x}A_{x}=\sum_{k=0}^{\infty} v^{k+1}d_{x+k}, \tag{2.3.4}$$

这一等式是保单发行时 l_x 个 x 岁被保险人的净趸缴保费基金总和与按死亡预期流出的资金量现值之间的平衡关系。

表达式

$$\sum_{k=r}^{\infty} v^{k+1}d_{x+k} \tag{2.3.5}$$

是保单发行时相应于 r 年之后按死亡预期提供赔付的那部分资金，它按所假定的利率经过 r 年之后成为

$$v^{-r}\sum_{k=r}^{\infty} v^{k+1}d_{x+k}=\sum_{k=r}^{\infty} v^{k-r+1}d_{x+k}. \tag{2.3.6}$$

与 (2.3.4) 比较可见，表达式 (2.3.6) 等于 $l_{x+r}A_{x+r}$ 。这一数额与经过 r 年赔付支出与利息收入的实际基金之间的差异有两方面来源：其一是按所采用的生命表预期的死亡与实际死亡的偏差，其二是按假定利率计算的利息收入与实际利息收入的偏差。

例 2.3.1: 一组 30 岁男性建立的基金设定在每个成员死亡时赔付其指定人 1000 元，约定提供给基金的金额等于按美国 1979–1981 年全体男性人口生命表及年利率 6% 计算的净趸缴保费。这个基金运行的实际结果是，第 2 与第 5 年分别有 1 人死亡，第 1 年利息收入的年利率为 6%，第 2 与第 3 年利率都是 6.5%，第 4 与第 5 年是 7% 。试问第 5 年末该基金按计划之初决定的期望值与实际基金的差异是多少？

解：将 $x = 30, v = \frac{1}{1.06}$ 及生命表数据代入 (2.3.4) 可算得 $A_{30} = 0.11518$, 同理可得 $A_{35} = 0.1445842$, 又 $\frac{l_{35}}{l_{30}} = 0.9902582$ 。100 人的基金开始值为

$$100 \times 1000A_{30} = 11518,$$

5 年后的期望值应该是

$$\begin{aligned} 100 \times 1000\frac{l_{35}}{l_{30}}A_{35} &= 10^5 \times 0.9902582 \times 0.1445842 \\ &= 14317.57. \end{aligned}$$

以 F_k 记 k 年末的基金值，其实际结果是

$$F_0 = 11518,$$

$$F_1 = 11518 \times 1.06 = 12209.08,$$

$$F_2 = 12209.08 \times 1.065 - 1000 = 12002.67,$$

$$F_3 = 12002.67 \times 1.065 = 12782.84,$$

$$F_4 = 12782.84 \times 1.07 = 13677.64,$$

$$F_5 = 13677.64 \times 1.07 - 1000 = 13635.07.$$

所求差额为 $14317.57 - 13635.07 = 682.50$ 。这一结果综合了 5 年期间的投资与死亡经验，实际投资收益超过了假定的 6% 年收益率，而另一方面， 2 人死亡却比期望数 0.9742 要多。将这种综合结果按诸如投资收益、死亡理赔等不同来源进行解释是保险公司精算师的职责之一。

受益金额为 1 个单位的 n 年期两全保险是 1 个单位受益金的 n 年期保险与上一节 n 年期生存保险的混合，有关函数为

$$b_{k+1} = 1 \quad k = 0, 1, \cdots$$

$$v_{k+1} = \begin{cases} v^{k+1} & k = 0, 1, \cdots, n-1 \\ v^n & k = n, n+1, \cdots \end{cases}$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \cdots, n-1 \\ v^n & K = n, n+1, \cdots. \end{cases}$$

其净趸缴保费是

$$A_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}{}_kp_x q_{x+k} + v^n{}_np_x. \qquad (2.3.7)$$

递增终身保险在被保险人存活 k 个完整年份后死亡的第 $k+1$ 年末赔付 $k+1$ 个单位，其受益，贴现及现值随机变量为

$$b_{K+1} = K + 1, K = 0, 1, 2, \cdots,$$

$$v_{K+1} = v^{K+1}, K = 0, 1, 2, \cdots,$$

$$Z = (K+1)v^{K+1}, K = 0, 1, 2, \cdots.$$

净趸缴保费记为 $(IA)_x$ 。

递减 n 年定期保险在存活 k 个完整年份后死亡的第 $k+1$ 年末赔付 $n-k$ 个单位，有关函数及随机变量为

$$b_{k+1}=\begin{cases} n-k & k=0,1,\cdots,n-1 \\ 0 & k=n,n+1,\cdots \end{cases}$$

$$v_{k+1}=v^{k+1}, \qquad k=0,1,\cdots$$

$$Z=\begin{cases} (n-K)v^{K+1} & K=0,1,\cdots,n-1 \\ 0 & K=n,n+1,\cdots. \end{cases}$$

这种保险的净趸缴保费符号是 $(DA)^{1}_{x:\overline{n}|}$。

与死亡即刻赔付的情形类似，年末赔付的递增保险等价于一系列保额为 1 个单位的递延定额保险。同样，递减定期保险等价于一组不同保险期限的定额保险之联合。图 2.3.1 对递减 8 年期保险画出了函数 b_{k+1} 的图形，从图中可以看出，它等于保险额为 1 个单位的 1 年， 2 年， $\cdots$ 8 年定期保险的受益函数之和，也可看作保额为 8 单位的一年定期保险，保额为 7 单位递延 1 年的一年定期保险， $\cdots$，保额为 1 单位的递延 7 年的一年定期保险之和。

净趸缴保费的有关等式可从分析上予以证实。按照定义，

$$\begin{aligned} (DA)^{1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1}(n-k)v^{k+1}{}_{k}p_{x}q_{x+k} \\ &= \sum_{k=0}^{n-1}(n-k)_{k|}A^{1}_{x:\overline{1}|}, \end{aligned} \tag{2.3.8}$$

即递减 n 年期保险的净缴趸保费等于一组一年期的延期保险的净趸缴保费之和。将

$$n-k=\sum_{j=0}^{n-k-1}1$$

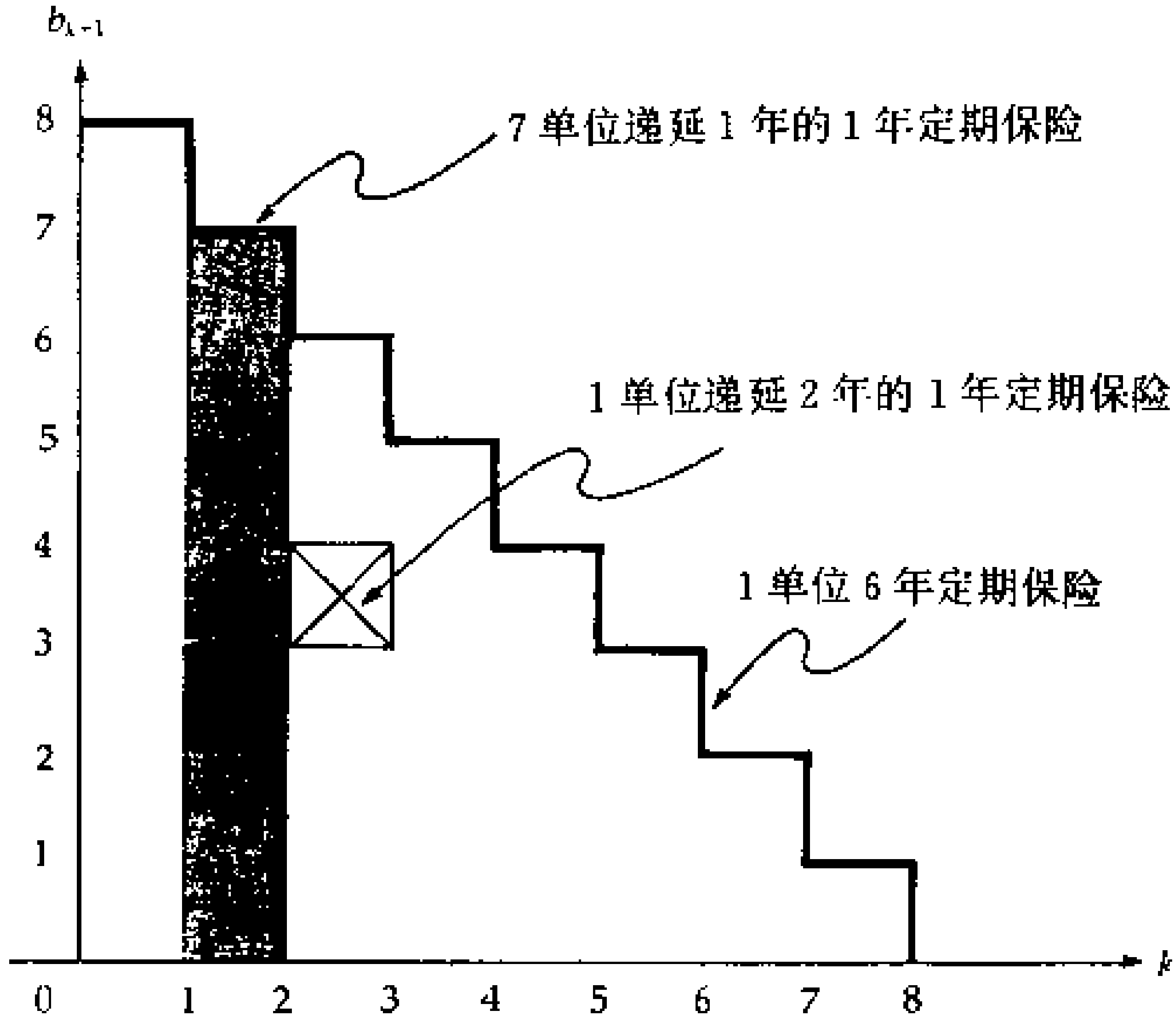


图 2.3.1　递减 8 年期保险

代入 (2.3.8) 式并交换求和次序可得

$$
\begin{aligned}
(DA)^1_{x:\overline{n}|} &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1} 1\cdot v^{k+1}{}_kp_x q_{x+k} \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1} v^{k+1}{}_kp_x q_{x+k} \\
&= \sum_{j=0}^{n-1} A^1_{x:\overline{n-j}|},
\end{aligned}
$$

即递减 n 年期保险的净趸缴保费也等于一组定期保险的净趸缴保费之和。

最后，表 2.3.1 给出了本节讨论的死亡年末赔付保险的有关函数及符号概要。

表 2.3.1 死亡年末赔付保险概要

(1) 保险名称	(2) 受益函数 b_{k+1} $(k=0,1,\cdots)$	(3) 贴现函数 v_{k+1} $(k=0,1,\cdots)$
终身人寿	1	v^{k+1}
n 年期人寿	1 $\quad k=0,1,\cdots,n-1$ 0 $\quad k=n,n+1,\cdots$	v^{k+1}
n 年期两全	1	$v^{k+1} \quad k=0,1,\cdots,n-1$ $v^n \quad k=n,n+1,\cdots$
递延 m 年的 n 年期人寿	1 $\quad k=m,\cdots,m+n-1$ 0 $\quad k=0,\cdots,m-1,m+n,\cdots$	v^{k+1}
按年递增 n 年期人寿	$k+1 \quad k=0,1,\cdots,n-1$ 0 $\quad k=n,n+1,\cdots$	v^{k+1}
按年递减 n 年期人寿	$n-k \quad k=0,1,\cdots,n-1$ 0 $\quad k=n,n+1,\cdots$	v^{k+1}
按年递增终身人寿	$k+1$	v^{k+1}

	(4) 现值函数 z_{k+1} $(k=0,1,\cdots)$	(5) 净趸缴保费	(6) 高阶矩
终身人寿	v^{k+1}	A_x	1.
n 年期人寿	$v^{k+1} \quad k=0,1,\cdots,n-1$ 0 $\quad k=n,n+1,\cdots$	$A^1_{x:\overline{n}\vert}$	1.
n 年期两全	$v^{k+1} \quad k=0,1,\cdots,n-1$ $v^n \quad k=n,n+1,\cdots$	$A_{x:\overline{n}\vert}$	1.
递延 m 年的 n 年期人寿	$v^{k+1} \quad k=m,\cdots,m+n-1$ 0 $\quad k=0,\cdots,m-1,m+n,\cdots$	${}_{m\vert n}A_x$	1.
按年递增 n 年期人寿	$(k+1)v^{k+1} \quad k=0,1,\cdots,n-1$ 0 $\quad k=n,n+1,\cdots$	$(IA)^1_{x:\overline{n}\vert}$	2.
按年递减 n 年期人寿	$(n-k)v^{k+1} \quad k=0,1,\cdots,n-1$ 0 $\quad k=n,n+1,\cdots$	$(DA)^1_{x:\overline{n}\vert}$	2.
按年递增终身人寿	$(k+1)v^{k+1}$	$(IA)_x$	2.

1. 定理 2.2.1 成立，形式上有 $\mathrm{Var}[Z] = {}^2A - A^2$ 。

2. 定理 2.2.1 不成立，需直接计算 $\mathrm{Var}[Z]$ 。

§2.4 死亡即刻赔付与年末赔付关系

我们从终身人寿保险开始探讨死亡即刻赔付与死亡年末赔付的保险之间关系。对于受益金额为 1 个单位的死亡即刻赔付终身人寿保险，从 (2.2.6) 可知其净趸缴保费为

$$\begin{aligned}
\bar{A}_x &= \int_0^{\infty} v^t {}_tp_x \mu_{x+t} dt \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} v^t {}_tp_x \mu_{x+t} dt \\
&= \sum_{k=0}^{\infty} \int_0^1 v^{k+s} {}_{k+s}p_x \mu_{k+s+1} ds \\
&= \sum_{k=0}^{\infty} v^{k+1} {}_kp_x \int_0^1 v^{s-1} {}_sp_{x+k} \mu_{x+k+s} ds. \qquad (2.4.1)
\end{aligned}$$

采用 §1.6 讨论的整数年龄之间死亡函数形式的假定，式 (2.4.1) 中最后的积分可以用离散生命表函数来表示。

在 1 年中死亡是均匀分布的假设下 (参见表 1.6.1)

$${}_sp_{x+k}\mu_{x+k+s} = q_{x+k} \; 0 \le s \le 1,$$

代入 (2.4.1) 得

$$\begin{aligned}
\bar{A}_x &= \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k} \int_0^1 e^{-\delta(s-1)} ds \\
&= \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k} \frac{e^{\delta}-1}{\delta} = \frac{i}{\delta} A_x, \qquad (2.4.2)
\end{aligned}$$

其中 $i = e^{\delta} - 1$ 为有效年利率。

在死亡均匀分布的假定下，以上等式应在情理之中。1 个单位的金额在 1 年中均匀地连续支付的话，按利息效力 δ, 等价于年末一次性支付 $\frac{i}{\delta}$, 所以理所当然应该有 $\bar{A}_x = \frac{i}{\delta}A_x$ 。

式 (2.4.2) 也可利用均匀死亡假设下剩余寿命随机变量的性质得出。由 (1.6.1), $T = K + S$, 其中 S 是 死亡之年小数生存部分随机变量。在死亡均匀分布假设下， K 与 S 独立且 S 服从 $(0,1)$ 区间上的均匀分布。据此可得

$$\begin{aligned} \bar{A}_x &= E[v^T] = E[v^{K+1}v^{S-1}] = E[v^{K+1}]E[v^{S-1}] \\ &= A_x \int_0^1 v^{s-1}ds = \frac{i}{\delta}A_x \end{aligned} \tag{2.4.3}$$

§1.6 也讨论过整数年龄之间死亡效力为常数的假设，此时死亡即刻赔付与死亡年末赔付保险的净趸缴保费关系可作为习题请读者自行推导。至于 Balducci 假设，所得的死亡效力在一个年岁中递减与人类生命的现实不符，而且导出的净趸缴保费关系也更为复杂。

接下来分析按年递增的 n 年期人寿保险。对于死亡即刻赔付的这种保险，其现值随机变量为

$$Z = \begin{cases} [T+1]v^T & T < n \\ 0 & T \geq n. \end{cases}$$

由于 $[T+1] = K+1$, 利用 $T = K + S$ 得

$$Z = \begin{cases} (K+1)v^{K+1}v^{S-1} & T < n \\ 0 & T \geq n. \end{cases}$$

将年末赔付的按年递增 n 年期保险的现值随机变量记为 W, 则

$$W = \begin{cases} (K+1)v^{K+1} & K = 0, 1, \cdots, n-1 \\ 0 & K = n, n+1, \cdots, \end{cases}$$

$$Z = Wv^{S-1}.$$

于是在死亡均匀分布的假设下，

$$\begin{aligned} E[Z] &= E[Wv^{S-1}] = E[W]E[v^{S-1}] \\ &= (IA)^1_{x:\overline{n}|}\frac{i}{\delta}. \end{aligned}$$

注意到以上两个结果的相似性：

$$\bar{A}_x = \frac{i}{\delta}A_x,$$

$$(I\bar{A})^1_{x:\overline{n}|} = \frac{i}{\delta}(IA)^1_{x:\overline{n}|}.$$

对于一般的死亡即刻赔付模型

$$Z = b_T v_T. \tag{2.4.4}$$

以上两种保险用到的条件为

(1) $v_T = v^T$, 其中 $v = e^{-\delta} = (1+i)^{-1}$ 。

(2) b_T 只依赖于 T 的整数部分，即可表示成 $b_T = b^*_{k+1}$ 。

在这两个条件之下有

$$E[Z] = E[b^*_{K+1}v^{K+1}v^{S-1}]. \tag{2.4.5}$$

如果假定整数年龄间死亡是均匀分布的话，那么 K 与 S 独立且 S 服从 $(0,1)$ 区间上均匀分布，(2.4.5) 成为

$$\begin{aligned} E[Z] &= E[b^*_{K+1}v^{K+1}]E[v^{S-1}] \\ &= \frac{i}{\delta}E[b^*_{K+1}v^{K+1}], \end{aligned} \tag{2.4.6}$$

上式最后 $E[b^*_{K+1}v^{K+1}]$ 是受益函数与贴现函数分别为 b^*_{k+1} 与 v^{k+1} 的死亡年末赔付保险的净趸缴保费。

例 2.4.1: 对现龄 35 岁男性的 30 年期死亡即刻赔付保额为 10000 的两全保险，根据附录 2A 的生命表以及死亡均匀分布的假设，按年利率 $i=0.06$ 计算净趸缴保费与方差。

解: 对两全保险，$v_T \neq v^T$, 所以不能直接应用 (2.4.6)。回顾 (2.2.8), 两全保险可看作定期人寿保险与生存保险之和，这样就能将 (2.4.6) 分别应用于定期保险及生存保险部分。由 (2.2.9) 得

$$
\begin{aligned}
\bar{A}_{35:\overline{30|}} &= \bar{A}^{1}_{35:\overline{30|}} + A_{35:\overset{1}{\overline{30|}}} \\
&= \frac{i}{\delta} A^{1}_{35:\overline{30|}} + A_{35:\overset{1}{\overline{30|}}} \\
&= \frac{0.06}{\log(1.06)}\left[A_{35} - (1.06)^{-30}\frac{l_{65}}{l_{35}}A_{65}\right] + (1.06)^{-30}\frac{l_{65}}{l_{35}} \\
&= 1.0297087 \times \left[0.1287194 - (1.06)^{-30} \times \frac{75339.63}{94206.55}\right. \\
&\quad \left. \times 0.4397965\right] + (1.06)^{-30} \times \frac{75339.63}{94206.56} \\
&= 0.208727.
\end{aligned}
$$

方差

$$
\begin{aligned}
\mathrm{Var}[Z] &= {}^2\bar{A}_{35:\overline{30|}} - (\bar{A}_{35:\overline{30|}})^2 \\
&= {}^2\bar{A}^{1}_{35:\overline{30|}} + {}^2A_{35:\overset{1}{\overline{30|}}} - (\bar{A}_{35:\overline{30|}})^2 \\
&= 0.0309294 + ((1.06)^2)^{-30}\frac{l_{65}}{l_{35}} - (0.208727)^2 \\
&= 0.011606.
\end{aligned}
$$

对于 10000 个单位的保单，$10000\bar{A}_{35:\overline{30|}} = 2087.27$, $10000^2 \mathrm{Var}[Z] = 1160600$。

例 2.4.2: 对现龄 50 岁男性的第 1 年死亡即刻赔付 5000, 第 2 年死亡即刻赔付 4000 并以此类推的按年递减 5 年期人寿保险，

根据附录 2A 生命表以及死亡均匀分布假设，按年利率 6% 计算净趸缴保费。

解：考虑表 2.2.1 可知 (对 1 个单位保额)

$$b_t = \begin{cases} 5-[t] & t \le 5 \\ 0 & t > 5. \end{cases}$$

对于 t 的整数部分 $[t]=k$，显然

$$b_t = b_{k+1}^* = \begin{cases} 5-k & k=0,1,2,3,4 \\ 0 & k=5,6,\cdots \end{cases}$$

而 b_{k+1}^* 是死亡年末赔付的递减 5 年期保险 (1 个单位保额) 的受益函数，于是由 (2.4.6)

$$\begin{aligned} (D\bar{A})_{50:\overline{5}|}^{1} &= \frac{i}{\delta}(DA)_{50:\overline{5}|}^{1} \\ &= 1.0297087\sum_{k=0}^{4}(5-k)v^{k+1}{}_kp_{50}q_{50+k} \\ &= 1.0297087\sum_{k=0}^{4}(5-k)v^{k+1}\frac{d_{50+k}}{l_{50}} \\ &= 0.088307. \end{aligned}$$

所求保费为 $1000(D\bar{A})_{50:\overline{5}|}^{1} = 88.307$ 。

如果死亡即刻赔付保险的受益函数不能表示为 K 的函数，那么它与年末赔付保险的净趸缴保费之间关系就需要直接进行分析。例如，考察死亡即刻赔付的连续递增终身保险，其受益函数为

$$b_t = t \qquad t > 0,$$

贴现函数与现值函数为

$$\begin{aligned} v_t &= v^t \quad t > 0, \\ z_t &= tv^t \quad t > 0. \end{aligned}$$

赔付额现值随机变量

$$
\begin{aligned}
Z &= Tv^T = (K+S)v^{K+S} \\
&= (K+1)v^{K+S} + (S-1)v^{K+S} \\
&= (K+1)v^{K+1}v^{S-1} + v^{K+1}(S-1)v^{S-1} \\
&= (K+1)v^{K+1}e^{\delta(1-S)} - v^{K+1}(1-S)e^{\delta(1-S)}.
\end{aligned}
$$

在死亡均匀分布的假设之下， K 与 S 独立， S 服从 $(0,1)$ 区间上的均匀分布， $1-S$ 亦然，

$$
\begin{aligned}
E[Z] &= E[(K+1)v^{K+1}]E[e^{\delta(1-S)}] \\
&\quad -E[v^{K+1}]E[(1-S)e^{\delta(1-S)}] \\
&= (IA)_x\frac{i}{\delta} - A_x\int_0^1 ue^{\delta u}du \\
&= \frac{i}{\delta}(IA)_x - (\frac{e^\delta}{\delta} - \frac{e^\delta-1}{\delta^2})A_x \\
&= \frac{i}{\delta}[(IA)_x - (\frac{i+1}{i} - \frac{1}{\delta})A_x],
\end{aligned}
$$

即

$$
(\bar{I}\bar{A})_x = \frac{i}{\delta}[(IA)_x - (\frac{1}{d} - \frac{1}{\delta})A_x],
$$

其中 $d=\frac{i}{1+i}$ 是 (银行) 贴现率。

§2.5 递归方程

保险模型价值 (净趸缴保费) 的递归方程可从前几节的有关表达式直接导出，譬如

$$
A_x = \sum_{k=0}^{\infty} v^{k+1}{}_kp_xq_{x+k}
$$

$$
\begin{aligned}
&= vq_x + \sum_{k=1}^{\infty} v^{k+1} {}_k p_x q_{x+k} \\
&= vq_x + vp_x \sum_{k=1}^{\infty} v^k {}_{k-1} p_{x+1} q_{x+k} \\
&= vq_x + vp_x \sum_{j=0}^{\infty} v^{j+1} {}_j p_{x+1} q_{x+1+j} \\
&= vq_x + vp_x A_{x+1}.
\end{aligned}
$$

这一等式的意义是明显的，(x) 的单位金额终身人寿保险在第 1 年年末的价值相当于 (x) 在 1 年中死亡情况下的 1 个单位赔付额或生存满 1 年情况下的净趸缴保费 A_{x+1}，即期望值 $1 \cdot q_x + A_{x+1} p_x$ 。

以上等式也可以用另一种方法得出。由定义以及 (x) 的剩余寿命 K 取值为非负整数这个事实可知

$$
\begin{aligned}
A_x &= E[Z] = E[v^{K+1} | K \geq 0]. \\
E[Z] &= E[v^{K+1} | K = 0] Pr(K = 0) \\
&\quad + E[v^{K+1} | K \geq 1] Pr(K \geq 1) \\
&= vq_x + vE[v^{(K-1)+1} | K - 1 \geq 0] p_x. \qquad (2.5.1)
\end{aligned}
$$

显然，在 $K \geq 1$ 的条件下，$K - 1$ 是 $(x+1)$ 的剩余寿命，假定其概率分布就是在 $K \geq 1$ 条件下 $K - 1$ 的条件分布，于是

$$
E[v^{(K-1)+1} | K - 1 \geq 0] = A_{x+1}, \qquad (2.5.2)
$$

代入 (2.5.1) 得出

$$
A_x = vq_x + vA_{x+1} p_x. \qquad (2.5.3)
$$

以上推导过程中用到假定：年龄 $x+1$ 新参加保险者的剩余寿命概率分布与现龄 $x+1$ 在 1 年前参加保险者的剩余寿命概率

分布相同。这在 §1.8 中曾经讨论，用那里的选择表术语，(2.5.2) 右端的 A_{x+1} 应该是 $A_{[x]+1}$，而 (2.5.3) 则成为

$$A_{[x]} = vq_{[x]} + vA_{[x]+1}p_{[x]}.$$

在 (2.5.3) 中以 $1-q_x$ 代 p_x 得

$$A_x = vq_x + vA_{x+1}(1-q_x).$$

两端乘 $l_xv^{-1} = l_x(1+i)$ 得

$$l_x(1+i)A_x = l_xA_{x+1} + d_x(1-A_{x+1}).$$

按随机生存组解释，这个方程的含义是： A_x 按年利率 i 经历 1 年后，可为所有人提供 A_{x+1}，并为预期在这一年中死亡者提供额外的 $1-A_{x+1}$ 。

在等式 $A_x = v[q_x(1-A_{x+1}) + A_{x+1}]$ 两端乘 $v^{-1} = (1+i)$ 并适当移项可得

$$A_{x+1} - A_x = iA_x - q_x(1-A_{x+1}). \tag{2.5.4}$$

这个等式说明，年龄 x 的投保者在活到 $x+1$ 岁时的净趸缴保费与当初 x 岁时的 (净趸缴) 保费之差额等于保费的 1 年利息减去提供 1 年保险的成本。

在等式 $A_x - vA_{x+1} = vq_x(1-A_{x+1})$ 两端乘 v^{x-1} 得

$$v^{x-1}A_x - v^xA_{x+1} = v^xq_x(1-A_{x-1}). \tag{2.5.5}$$

对 x 从 $x=y$ 加到 ∞,

$$v^{y-1}A_y = \sum_{x=y}^{\infty} v^xq_x(1-A_{x+1}),$$

于是

$$A_y = \sum_{x=y}^{\infty} v^{x-y+1} q_x (1 - A_{x+1}).$$

这表明，(y) 的净趸缴保费等于其未来所有年份的保险成本现值之和。

对于死亡即刻赔付保险也可建立类似的表达式，但需使用微积分，导出的是微分方程。

譬如 (x) 的终身人寿保险，与 (2.5.4) 相似的连续情形等式为

$$\frac{d}{dx}\bar{A}_x = -\mu_x + \bar{A}_x(\delta + \mu_x) = \delta \bar{A}_x - \mu_x(1 - \bar{A}_x), \qquad (2.5.6)$$

其推导过程如下：对 (2.2.6) 的积分作变量代换 $y = t + x$ 得

$$\begin{aligned} \bar{A}_x &= \int_0^{\infty} v^t {}_t p_x \mu_{x+t} dt \\ &= \int_x^{\infty} v^{y-x} {}_{y-x} p_x \mu_y dy, \end{aligned}$$

利用 ${}_y p_0 = {}_x p_0 {}_{y-x} p_x$ 得

$$\bar{A}_x = \frac{v^{-x}}{{}_x p_0} \int_x^{\infty} v^y {}_y p_0 \mu_y dy,$$

由 $v^{-x} = e^{\delta x}$ 可知 $\frac{d}{dx} v^{-x} = \delta v^{-x}$，又根据 $\frac{d}{dx} {}_x p_0 = -{}_x p_0 \mu_x$，按微分演算规则可算得

$$\begin{aligned} \frac{d\bar{A}_x}{dx} &= [\delta v^{-x} \frac{1}{{}_x p_0} - v^{-x} \frac{(-{}_x p_0 \mu_x)}{({}_x p_0)^2}] \int_x^{\infty} v^y {}_y p_0 \mu_y dy \\ &\quad - \frac{v^{-x}}{{}_x p_0} v^x {}_x p_0 \mu_x \\ &= (\delta + \mu_x) \bar{A}_x - \mu_x. \end{aligned}$$

方程 (2.5.6) 也可用另一种方法得出，其方法与前面推导有关 A_x 的等式类似，也使用条件数学期望：

$$\begin{aligned}\bar{A}_x &= E[v^T] \\ &= E[v^T|T \le h]Pr(T \le h) + E[v^T|T > h]Pr(T > h).\end{aligned} \tag{2.5.7}$$

由于

$$Pr(T \le h) = {}_hq_x,\ Pr(T > h) = {}_hp_x. \tag{2.5.8}$$

在给定 $T \le h$ 条件下 T 的条件概率密度函数为

$$f(t|T \le h) = \begin{cases} \frac{f(t)}{F(h)} = \frac{{}_tp_x\mu_{x+t}}{{}_hq_x} & 0 \le t \le h \\ 0 & \text{其它。} \end{cases}$$

于是

$$E[v^T|T \le h] = \int_0^h v^t \frac{{}_tp_x\mu_{x+t}}{{}_hq_x} dt. \tag{2.5.9}$$

与 (2.5.2) 类似，有

$$\begin{aligned}E[v^T|T > h] &= v^h E[v^{T-h}|T - h > 0] \\ &= v^h \bar{A}_{x+h}.\end{aligned} \tag{2.5.10}$$

将 (2.5.8)–(2.5.10) 代入 (2.5.7) 得

$$\bar{A}_x = (\int_0^h v^t \frac{{}_tp_x\mu_{x+t}}{{}_hq_x}) dt\,{}_hq_x + v^h \bar{A}_{x+h}\,{}_hp_x. \tag{2.5.11}$$

由此可见

$$\frac{\bar{A}_{x+h} - \bar{A}_x}{h} = -\frac{1}{h}\int_0^h v^t\,{}_tp_x\mu_{x+t}dt + \frac{1 - v^h\,{}_hp_x}{h}\bar{A}_{x+h}. \tag{2.5.12}$$

注意到

$$\lim_{h\to 0} \frac{1 - v^h\,{}_hp_x}{h} = -\frac{d}{dt}(v^t\,{}_tp_x)|_{t=0} = \delta + \mu_x.$$

在 (2.5.12) 中令 $h \to 0$ 得出

$$\begin{aligned}\frac{d}{dx}\bar{A}_x &= -v^0{}_0p_x\mu_{x+0} + (\delta + \mu_x)\bar{A}_{x+0}\\ &= -\mu_x + (\delta + \mu_x)\bar{A}_x.\end{aligned}$$

§2.6 计算基数

净趸缴保费的计算过程往往繁琐，为简明起见，引入一些计算中间出现的函数，叫做计算基数 (或换算函数)，其中有：

$$D_x = v^x l_x,$$

$$C_x = v^{x+1}d_x = D_x v q_x,$$

$$M_x = \sum_{k=0}^{\infty} C_{x+k} = \sum_{j=x}^{\infty} C_j,$$

$$\begin{aligned}R_x &= \sum_{k=0}^{\infty} M_{x+k} = \sum_{j=x}^{\infty} M_j,\\ &= \sum_{k=0}^{\infty}(k+1)C_{x+k}.\end{aligned}$$

利用这些计算基数，譬如对于 (x) 的保额为 1 个单位在年龄 y 与 z 之间死亡时年末赔付的延期人寿保险，其净趸缴保费可表示成

$$\begin{aligned}{}_{y-x|z-x}A_x &= \sum_{k=y-x}^{z-x-1} {}_kp_x q_{x+k} = \sum_{k=y-x}^{z-x-1} v^{k+1}\frac{d_{x+k}}{l_x}\\ &= \frac{1}{v^x l_x}\sum_{k=y-x}^{z-x-1} v^{x+k+1}d_{x+k} = \frac{1}{v^x l_x}\sum_{j=y}^{z-1} v^{j+1}d_j\\ &= \frac{1}{D_x}\sum_{j=y}^{z-1} C_j = \frac{M_y - M_z}{D_x}.\end{aligned}$$

对于表 2.3.1 所列在 (x) 死亡年末赔付的 n 年期递增人寿保险，其净趸缴保费可用计算基数表示成

$$
\begin{aligned}
(IA)^{1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1}(k+1)v^{k+1}{}_{k|}q_x = \sum_{k=0}^{n-1}(k+1)v^{k+1}\frac{d_{x+k}}{l_x} \\
&= \frac{1}{v^x l_x}\sum_{k=0}^{n-1}(k+1)C_{x+k} = \frac{1}{D_x}\sum_{j=0}^{n-1}\sum_{k=j}^{n-1}C_{x+k} \\
&= \frac{1}{D_x}\sum_{j=0}^{n-1}(M_{x+j}-M_{x+n}) \\
&= \frac{R_x - R_{x+n} - nM_{x+n}}{D_x}.
\end{aligned}
\tag{2.6.1}
$$

m 年期生存保险的净趸缴保费可写成

$$
A_{x:\overline{n}|}^{\ \ 1} = \frac{D_{x+n}}{D_x}. \tag{2.6.2}
$$

n 年期两全保险可写成

$$
A_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{2.6.3}
$$

对于死亡即刻赔付保险，有关计算基数定义为

$$
\begin{aligned}
\overline{C}_x &= \int_0^1 v^{x+t} l_{x+t}\mu_{x+t}dt \\
&= \int_0^1 D_{x+t}\mu_{x+t}dt.
\end{aligned}
\tag{2.6.4}
$$

$$
\overline{M}_x = \sum_{y=x}^{\infty}\overline{C}_y = \int_x^{\infty} D_y\mu_y dy
$$

$$
\overline{R}_x = \sum_{y=x}^{\infty}\overline{M}_y.
$$

例 2.6.1：考虑在第 1 年死亡时即刻赔付 10000 、在第 2 年死亡即刻赔付 9000 并以此类推的递减人寿保险。按附录 2 生命表以及 $i=0.06$ 计算 30 岁投保人的净趸缴保费：

(1) 保障期限至第 10 年底。

(2) 保障期限至第 5 年底。

解：(1) 按例 2.4.2 同样道理，有

$$(D\bar{A})^{1}_{30:\overline{10}|}=\frac{i}{\delta}(DA)^{1}_{30:\overline{10}|}.$$

为了用计算基数来表示 $(DA)^{1}_{30:\overline{10}|}$，我们可将这个递减保险看成十个期限分别为 1 年，2 年，$\cdots$，10 年的人寿保险之和 (参考图 2.3.1)。于是

$$\begin{aligned}(DA)^{1}_{30:\overline{10}|} &= \frac{\sum_{k=1}^{10}(M_{30}-M_{30+k})}{D_{30}}\\ &= \frac{10M_{30}-(R_{31}-R_{41})}{D_{30}}.\end{aligned}$$

查附录 2A 表可算出所求净缴趸保费为

$$\begin{aligned}1000(D\bar{A})^{1}_{30:\overline{10}|} &= 1000\frac{i}{\delta}(DA)^{1}_{30:\overline{10}|}\\ &= 1000\times 1.02971\times 0.07816499=80.49.\end{aligned}$$

(2) 可看作 5 单位的 5 年期人寿保险与 5 年递减人寿保险之混合，所求净趸缴保费为

$$\begin{aligned}&1000[5\bar{A}^{1}_{30:\overline{5}|}+(D\bar{A})^{1}_{30:\overline{5}|}]\\ =\ &1000[5\frac{i}{\delta}A^{1}_{30:\overline{5}|}+\frac{i}{\delta}(DA)^{1}_{30:\overline{5}|}]\\ =\ &1000\times 1.02971\times\frac{5(M_{30}-M_{35})+5M_{30}-(R_{31}-R_{36})}{D_{30}}\\ =\ &1029.71\times\frac{10M_{30}-5M_{35}-R_{31}+R_{36}}{D_{30}}\\ =\ &58.69.\end{aligned}$$

习 题

除非特别说明，以下保险系死亡即刻赔付，利息效力为常数 δ，其等价的利率与 (银行) 贴现率分别为 i 与 d 。

§2.2

1. 如对所有 $x>0$, $\mu_x=\mu$ 为常数，证明 $\bar{A}_x=\frac{\mu}{(\mu+\delta)}$ 。

2. 设 $\mu_x=\frac{1}{1+x}$, $x>0$ 。

(1) 通过分部积分证明

$$\bar{A}_x=1-\delta\int_0^{\infty}e^{-\delta t}\frac{1+x}{1+x+t}dt.$$

(2) 用 (1) 中的表达式证明对所有 $x>0$, $\frac{d\bar{A}_x}{dx}<0$ 。

3. 证明 $\frac{d\bar{A}_x}{di}=-v(\bar{I}\bar{A})_x$ 。

4. 证明由 (2.2.10) 与 (2.2.11) 给出的 2 个单位受益金额 n 年期两全保险现值的方差表达式是相等的。

5. 设 Z_1 与 Z_2 由方程 (2.2.8) 定义。

(1) 证明 $\lim_{n\to 0}\mathrm{Cov}[Z_1,Z_2]=\lim_{n\to\infty}\mathrm{Cov}[Z_1,Z_2]=0.$

(2) 建立 $\mathrm{Cov}[Z_1,Z_2]$ 最小时两全保险的期限所满足的隐式方程。

(3) 给出 (2) 中最小值的公式。

(4) 当死亡效力 μ 为常数时，化简 (2) 中的方程与 (3) 中的公式。

6. 设死亡由 $l_x=100-x$, $0\le x\le 100$ 所描述，利息效力 $\delta=0.05$ 。

(1) 计算 $\bar{A}^1_{40:\overline{25}|}$.

(2) 对于保单生效起时间 t 死亡时受益金额为 $e^{0.05t}$ 的 25 年期人寿保险，决定 40 岁投保人的净趸缴保费。

7. 按 $\omega=100$ 的 de Moivre 生存函数，计算 $i=0.10$ 时的

(1) $\bar{A}^1_{30:\overline{10}|}$.

(2) 以上 (1) 中所表示的保险在保单签发时的现值方差。

8. 设 $\delta_t=\frac{0.2}{1+0.05t}$, 且 $l_x=100-x$, $0\le x\le 100$ 。计算

(1) x 岁生效的终身人寿保险的净趸缴保费与受益现值方差。

(2) $(\bar{I}\bar{A})_x$.

9. (1) 证明 $\bar{A}_x$ 是 (x) 的剩余寿命 T 的矩母函数在 $-\delta$ 的取值。

(2) 然后证明：当 T 服从参数为 α 与 β 的 Γ 分布时，$\bar{A}_x=(1+\frac{\delta}{\beta})^{-\alpha}$.

10. 设对所有 $t>0$, $b_t=t$, $\mu_{x+t}=\mu$, $\delta_t=\delta$, 导出

(1) $(\bar{I}\bar{A})_x=E[b_Tv^T]$;　　　(2) $\mathrm{Var}[b_Tv^T]$

的表达式。

§2.3

11. 设 $l_x=100-x$, $0\le x\le 100$, 且 $i=0.05$ 。求

(1) $A_{40:\overline{25}|}$.　　　(2) $(IA)_{40}$.

12. 证明：对于 $m<n$, $A_{x:\overline{n}|}=A^1_{x:\overline{m}|}+v^m{}_mp_xA_{x+m:\overline{n-m}|}$ 。并解释这一结果。

13. 设 $A_x=0.25$, $A_{x+20}=0.40$, $A_{x:\overline{20}|}=0.55$, 计算

(1) $A_{x:\overset{1}{\overline{20}|}}$.　　　(2) $A^1_{x:\overline{20}|}$.

14. (1) 描述净趸缴保费符号为 $(IA)_{x:\overline{m}|}$ 的保险受益。

(2) 将 (1) 中的净趸缴保费用表 2.2.1 与 2.3.1 中给出的符号表示。

§2.4

15. 考虑以长度为 $\frac{1}{m}$ 年的时段衡量的时间尺度。对于在死亡发生时的那个 m 分之一年时段末赔付 1 个单位的终身保险，设 k 是自保单生效起存活的完整年数，j 是死亡那年存活的完整 m 分之一年时段数。

(1) 这类保险的现值函数是什么？

(2) 建立与 (2.4.1) 类似的上述保险的净趸缴保费 $A_x^{(m)}$ 。

(3) 在 1 年中死亡均匀分布的假设下，证明

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

16. 在整数年龄间死亡效力为常数的假设下，证明 (2.4.1) 可写成

$$\bar{A}_x = \sum_{k=0}^{\infty} v^{k+1}{}_k p_x \mu_{x+k} \frac{i + q_{x+k}}{\delta + \mu_{x+k}},$$

其中 $\mu_{x+k} = -\log p_{x+k}$.

§2.5

17. 用类似于推导 (2.5.6) 的方法 (第一种方法), 证明

$$\frac{d\bar{A}_{x:\overline{n}|}^{1}}{dx} = \bar{A}_{x:\overline{n}|}^{1}(\mu_x + \delta) + A_{x:\overline{n}|}^{\ \ 1}\mu_{x+n} - \mu_x \qquad x \geq 0.$$

18. 按以下步骤解微分方程 (2.5.6):

(1) 用积分因子

$$\exp[-\int_y^x (\delta + \mu_z) dz]$$

得出

$$\bar{A}_y = \int_y^{\infty} \mu_x \exp[-\int_y^x (\delta + \mu_z) dz] dx.$$

(2) 用积分因子 $e^{-\delta x}$ 得出

$$\bar{A}_y = \int_y^{\infty} \mu_x v^{x-y} (1 - \bar{A}_x) dx.$$

19. 证明

$$(IA)_x = vq_x + v[A_{x+1} + (IA)_{x+1}]p_x.$$

建立这个等式时用到什么假定？

§2.6

20. 证明并解释

$$\frac{1}{D_x}\left[\sum_{k=0}^{n-1} C_{x+k} v^{n-k-1} + D_{x+n}\right] = v^n.$$

21. 对于在 65 岁之前死亡提供 2 个单位受益而在 65 岁之后死亡提供 1 个单位受益的双倍保障至 65 岁的保单，用计算基数表示净趸缴保费。假定受益金在死亡年末支付。

22. 一种在 0 岁投保的死亡即刻赔付保单其分段受益金列于下表

年龄	死亡受益
0	1000
1	2000
2	4000
3	6000
4	8000
5–20	10000
21 及以后	50000

用计算基数写出净趸缴保费。

23. 在死亡均匀分布的假设下，用 D_x, C_x, M_x 及 R_x 表示以下净趸缴保费：

(1) $(I\bar{A})_{30:\overline{35|}}$.　　(2) $(\bar{I}\bar{A})_{30:\overline{35|}}$.　　(3)$(\bar{I}_{\overline{10|}}\bar{A})_{30:\overline{35|}}$.

综合题

24.

(1) 回答死亡效力增加一个常数与利息效力增加同一个常数是否对 A_x 产生相同影响。

(2) 证明：如果单个死亡概率 q_{x+n} 增加到 $q_{x+n}+c$, 那么 A_x 将增加

$$cv^{n+1}{}_np_x(1-A_{x+n+1}).$$

25. 一种 x 岁人 n 年期的修正生存保险当被保险人在 n 年期限内死亡时退还净趸缴保费。当受益金额为 1000 时，该保险的净趸缴保费为 700 。如保费不退还，则净趸缴保费只需 650 。

(1) 计算以上 x 岁人的 n 年期修正生存保险当受益金额为 1000 并且在期间死亡时保费 $100k\%$ 退还时的净趸缴保费。

(2) 对于 (1) 中的修正生存保险，用生存保险与定期人寿保险的净趸缴保费表示其现值方差。

26. 某器具制造商销售其产品时提供一种 5 年期保障。在 5 年内失效时按比例返还部分购款。譬如购买后 3 又 $\frac{3}{4}$ 年失效，则退还 25% 的购款。根据统计分析，新的产品在第 1 年内失效的概率估计为 0.2, 在第 2, 第 3, 第 4 年每 1 年内失效的概率都是 0.1, 第 5 年失效的概率为 0.2 。

(1) 假定购买后一年内的失效均匀分布。当 $i = 0.10$ 时计算购买价格中的净趸缴保费比例。

(2) 如果退款改成购买附带 5 年保障新的产品时的价格折扣， (1) 的答案会不会改变？

27. (1) 证明

$$\mu_x \cong \frac{\frac{i}{\delta}(M_{x-1} - M_{x+1})}{2D_x}$$

(2) 用附录 2A 示例生命表按 $i = 0.06$ 近似估计 μ_{30} 。

(3) 按 $i = 0$ 重做 (2) 。

第三章　生存年金

§3.1　引　言

前一章主要研究各种形式人寿保险提供的与死亡相关联的赔付，而这一章则主要考察各种形式生存年金所提供的与生存相关联的支付。*生存年金*（life annuity）是间隔相等时期（如按月、季度、半年或一年）绵延不断的一系列支付，但这些支付是以指定领取人活着为条件的，一旦领取人死亡，支付即告结束。生存年金可以是定期的（限于指定年限），也可以是终身的；首次支付可以是即期的，也可以是延期的；每期支付时间可以是期初（*期初年金*annuities-due），也可以是期末（*期末年金*annuities-immediate）。

在利息理论中讨论的是*确定性年金*(annuities-certain)，那里给出了有关年金的各种术语及符号。生存年金除了引入生存作为支付条件外，与确定性年金基本相似，在第二章生存保险及两全保险中，已遇到过生存年金情形。

生存年金在人寿保险经营中起主要作用，通常保费以年金方式分期缴付，理赔时支付的保险金可籍选择权转成受益人的生存年金。某些险种干脆用指定的收入受益形式代替死亡时的一次性给付，譬如向活着的配偶每月支付收入 1000 元。

生存年金在退休金体系中则更为重要，实际上，退休计划可看作在职时以某种定期年金方式购买延期生存年金（退休后提供支付）的一种体系，这种定期年金可能包含变动的分担，其估价不仅应考虑到利息与死亡因素，还应考虑诸如工资增长及死亡以外其它原因终止参与等因素。在残疾保险及抚恤保险中生存年金

也起作用。在残疾保险场合，需要考虑到由于致残的被保险人恢复健康而导致年金受益终止的情况。至于抚恤保险中向活着的配偶支付的年金受益，也可能因再婚而终止。

在这一章建立的理论的大多数应用中，在有关个人处于某种特定状况时年金支付将持续下去。然而，这一章建立的理论还可应用于更广泛的场合，只要涉及的是不确定的周期性支付，这些应用的例子将在以后讨论多重生命及多重损因的章节里看到。

我们将应用*当期支付技巧*(current payment technique) 来估价生存年金，这个方法与复利理论中从累积值及未来单次支付的现值出发用求和或积分方式推广到未来一系列支付的估价方法相似。另一种可采用的方法称为*综合支付技巧*(aggregate payment technique)，着眼于考虑年金在因死亡或到期而结束时的总值。每一种技巧都有其优越性并提供不同侧面的见识，两种方法产生的公式之等价性乃定理 1.5.1 与 1.5.2 的直接结果。

§3.2 与生存相联的一次性支付

与前一章人寿保险一样，除非特别说明，有效年利率 i 为常数 (或等价的利息效力 δ 为常数) 。

考虑向现龄 x 的生命在存活 n 年后的 n 年末支付 1 个单位的情形，这种受益在第二章里称为 n 年期生存保险。与保险相联系，很自然地用术语净趸缴保费及符号 $A_{x:\overset{\,1}{\overline{n|}}}$ 来表示单位生存保险的期望现值。而与生存年金尤其是养老金积累相联系，经常使用术语精算现值及符号 ${}_nE_x$ 。这里将采用后者，精算一词意味着除利息以外还有期望或其它因素在其中。于是，当 n 年末 (x) 仍活着时 1 单位支付的精算现值为

$$ {}_nE_x = A_{x:\overset{\,1}{\overline{n|}}} = v^n\,{}_np_x. \tag{3.2.1}$$

例 3.2.1: 用附录 2A 示例生命表并按有效年利率 6%, 计算当 25 岁男性存活 40 年后支付 10000 的精算现值。

解: 所求值为

$$\begin{aligned}10000{}_{40}E_{25} &= 10000v^{40}{}_{40}p_{25}\\ &= 10000\times 0.09722219\times 0.78765825\\ &= 765.78.\end{aligned}$$

显然，这个例子利息贴现因素的作用远大于生存因素。

公式 (3.2.1) 可重写成形式

$$l_{x\,n}E_x(1+i)^n = l_{x+n}. \tag{3.2.2}$$

按决定性生存组概念，l_x 个 x 岁生存者每人存入金额 ${}_nE_x$，所得基金按年利率 i 累积 n 年后，在 l_{x+n} 个 $x+n$ 岁生存者中分享，每人正好可得 1 单位。这里假定，现龄 x 岁群体的人数严格地按生命表给定的方式递减。

例 3.2.1 中 $l_{25}=95650.15$ 个 25 岁生存者每人提供 765. 78 构成初始基金总额为 73246971.87, 按年利率 6%, 40 年后累积达 $73246971.878(1.06)^{40}=753397692.3$, 只有 $l_{65}=75339.63$ 个 65 岁生存者，每人可分享 10000 。单纯按年利率 6%, 765.78 在 40 年后仅有 7876.59, 期间 $l_{25}-l_{65}$ 个死亡者的积累贡献使每个 65 岁生存者的份额增加到 10000. 这个数额 10000, 是在利息因素使基金增值与生存因素使基金受益人数减少的双重作用下，从 25 岁时的 765.78 到 65 岁时的积累，称为精算积累值。

一般地，x 岁时提供 1 单位在 n 年末时的*精算积累值*(actuarial accumulated value) 定义为精算现值等于 1 的数额 S, 即 $S_nE_x=1$ 或

$$S=\frac{1}{{}_nE_x}=\frac{1}{v^n{}_np_x}=(1+i)^n\frac{l_x}{l_{x+n}}. \tag{3.2.3}$$

公式 (3.2.3) 表明，精算积累因子 $\frac{1}{{}_nE_x}$ 是利息积累因子 $(1+i)^n$ 与生存积累因子 $\frac{1}{{}_np_x}=\frac{l_x}{l_{x+n}}$ 的乘积。

例 3.2.2: 根据示例生命表及年利率 6% 求 25 岁时提供的 1000 在 65 岁时的精算积累值.

解: 所求值为

$$1000\frac{1}{{}_{40}E_{25}} = 1000\times(1.06)^{40}\frac{l_{25}}{l_{65}} = 13058.60.$$

在这个例子中，年死亡率范围从 25 岁时的 0.0012230 到 64 岁时的 0.0195231, 与年利率 0.06 相比，影响甚小。

例 3.2.3: 导出

$$(1)\ \frac{\partial}{\partial x}{}_nE_x, \qquad (2)\ \frac{\partial}{\partial n}{}_nE_x$$

的表达式，并观察当 n 给定 x 变动时以及 x 给定 n 变动时 ${}_nE_x$ 如何变化。

解: (1) 由

$${}_np_x = \exp(-\int_x^{x+n}\mu_y dy)$$

可得

$$\begin{aligned}\frac{\partial}{\partial x}{}_nE_x &= v^n\frac{\partial}{\partial x}{}_np_x = v^n{}_np_x(\mu_x-\mu_{x+n})\\ &= {}_nE_x(\mu_x-\mu_{x+n}).\end{aligned}$$

如果当 $x\le y\le x+n$ 时 μ_y 增加，那么 $\frac{\partial}{\partial x}{}_nE_x<0$, ${}_nE_x$ 随年龄减少；如果当 $x\le y\le x+n$ 时 μ_y 为常数，那么 $\frac{\partial}{\partial x}{}_nE_x$=0; 如果当 $x\le y\le x+n$ 时 μ_y 减少，例如早年时，那么 ${}_nE_x$ 随年龄增加。

(2) 由 ${}_nE_x = e^{-\delta n}{}_np_x = \exp[-\int_x^{x+n}(\mu_y+\delta)dy]$ 得

$$\frac{\partial}{\partial n}{}_nE_x = -{}_nE_x(\mu_{x+n}+\delta),$$

这里 $\partial_nE_x/\partial n<0$, ${}_nE_x$ 是 n 的递减函数。

例 3.2.4: 对 $n>t$, 证明并解释以下关系式：

(1) ${}_nE_x = {}_tE_{x\,n-t}E_{x+t}$.　　　(2) $\frac{{}_tE_x}{{}_nE_x} = \frac{1}{{}_{n-t}E_{x+t}}$.

解:

(1) ${}_nE_x = v^n{}_np_x = v^tv^{n-t}{}_tp_{x\,n-t}p_{x+t} = {}_tE_{x\,n-t}E_{x+t}$. 当 (x) 活到 n 年末时 1 单位支付的精算现值可通过按 $x+n$ 岁时的支付在 $x+t$ 岁时的精算现值再取其 x 岁时的精算现值获得。

(2) 这个式子无非是 (1) 的改写，其含义是， $x+t$ 岁时提供的 1 单位在 $x+n$ 岁时的精算积累值等于 $x+t$ 岁时的 1 单位支付在 x 岁时的精算现值再到 $x+n$ 岁时的精算积累值。

§3.3　连续生存年金

当决定生存年金的精算现值时，我们既可使用综合支付技巧，也可使用当期支付技巧。前者的步骤如下：

(1) 记录下年金在死亡发生于时间 t 的所有支付仅按利息折算成的现值。

(2) 将以上求得的现值乘以在时间 t 死亡的概率或概率密度。

(3) 按所有可能的死亡时间 t 将 (2) 中结果相加 (积分) 。

当期支付技巧的步骤为：

(1) 记下时间 t 的支付额。

(2) 决定以上数额的精算现值。

(3) 按所有可能的支付时间 t 将 (2) 的结果相加 (积分) 。

第一种技巧导致剩余寿命随机变量的解释，其步骤最终产生一个期望值。当期支付技巧也可建立在概率模型之上。决定性解释对两种方法都可行，但在决定性模型中通常使用当期支付技巧。

这些技巧可从以下确定在 (x) 活着时每年 1 单位连续 * 支付的终身年金精算现值过程中予以说明，这个值记为 $\bar{a}_x$ 。

用 T 表示 (x) 的剩余寿命，终身年金支付的现值为 $Y = \bar{a}_{\overline{T|}}$,

* 这里及以后，连续支付的含义是指每时每刻连续不断地支付。

按综合支付技巧确定的终身年金精算现值为

$$\bar{a}_x = E[Y] = E[\bar{a}_{\overline{T}|}]. \tag{3.3.1}$$

由于 T 的概率密度函数为 ${}_tp_x\mu_{x+t}$, 我们有

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu_{x+t}dt. \tag{3.3.2A}$$

换一种方法，与复利公式

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt$$

类似，按当期支付技巧考虑时间 t 的瞬时支付 dt 的精算现值 $v^t{}_tp_xdt$, 并将所有这些瞬间值积聚起来得出

$$\bar{a}_x = \int_0^\infty v^t{}_tp_xdt. \tag{3.3.2B}$$

对 $z(t) = a_{\overline{t}|}$ 及 $g(t) = {}_tp_x\mu_{x+t}$ 运用定理 1.5.1, 式 (3.3.2A) 就归结为 (3.3.2B) 。

进一步，对 $z(t) = v^t$ 及 $g(t) = {}_tp_x\mu_{x+t}$, 运用定理 1.5.1 于

$$\bar{A}_x = \int_0^\infty v^t{}_tp_x\mu_{x+t}dt,$$

得出

$$\bar{A}_x = 1 + \int_0^\infty {}_tp_xdv^t = 1 - \delta\bar{a}_x \tag{3.3.3}$$,

或

$$1 = \delta\bar{a}_x + \bar{A}_x. \tag{3.3.4}$$

公式 (3.3.4) 与利息理论中的关系

$$1 = \delta\bar{a}_{\overline{t}|} + v^t$$

相似，它表明，现在投资 1 个单位可在 (x) 活着时连续支付年息 δ 并在 (x) 死亡时偿还。

$\overline{a}_x$ 与 $\overline{A}_x$ 的关系也可从以下表达式获得：

$$Y = \overline{a}_{\overline{T}|} = \frac{1 - v^T}{\delta} = \frac{1 - Z}{\delta}, \tag{3.3.5}$$

其中 $Z = v^T$ 是终身人寿保险的现值随机变量。将 (3.3.5) 代入 (3.3.1)，得

$$\overline{a}_x = E[\frac{1 - Z}{\delta}] = \frac{1 - \overline{A}_x}{\delta}, \tag{3.3.6}$$

与 (3.3.3) 及 (3.3.4) 相当。由 $1 = \delta\overline{a}_{\overline{\infty}|} + 0$ 还可将 (3.3.6) 写成

$$\overline{a}_x = \overline{a}_{\overline{\infty}|} - \overline{a}_{\overline{\infty}|} A_x. \tag{3.3.7}$$

这个公式表明，终身年金等价于一个连续支付的永久年金减掉一个 (x) 死亡时开始的永久年金 (其结果终止了生存年金) 。

在我们的模型假设基础上，为衡量连续生存年金的死亡风险，需计算 $\mathrm{Var}[\overline{a}_{\overline{T}|}]$:

$$\begin{aligned} \mathrm{Var}[\overline{a}_{\overline{T}|}] &= \mathrm{Var}[\frac{1 - v^T}{\delta}] = \frac{1}{\delta^2}\mathrm{Var}[v^T] \\ &= \frac{1}{\delta^2}[{}^2\overline{A}_x - (\overline{A}_x)^2], \end{aligned} \tag{3.3.8}$$

其中 ${}^2\overline{A}_x$ 是按利息效力为 2δ 计算的终身人寿保险净趸缴保费 (参见第二章) 。

从等式

$$\delta\overline{a}_{\overline{T}|} + v^T = 1 \tag{3.3.9}$$

也可得出 (3.3.4)

$$E[\delta\overline{a}_{\overline{T}|} + v^T] = \delta\overline{a}_x + \overline{A}_x = 1$$

以及

$$\mathrm{Var}[\delta\bar{a}_{\overline{T|}}+v^T]=0.$$

公式 (3.3.9) 显示，对于每年支付 δ 的连续生存年金与赔付 1 的终身人寿保险之联合，不存在死亡风险。

例 3.3.1: 在死亡效力为常数 μ=0.04 以及利息效力 δ=0.06 的假定下求

(1) $\bar{a}_x$.

(2) $\bar{a}_{\overline{T|}}$ 的标准差。

(3) $\bar{a}_{\overline{T|}}$ 超过 $\bar{a}_x$ 的概率。

解:

(1) 直接计算

$$\begin{aligned}\bar{a}_x &= \int_0^\infty v^t{}_tp_x dt=\int_0^\infty e^{-0.06t}e^{-0.04t}dt\\ &= \int_0^\infty e^{-0.10t}dt=10.\end{aligned}$$

(2) 先计算

$$\begin{aligned}\bar{A}_x &= E[e^{-0.06T}]=\int_0^\infty e^{-0.06t}e^{-0.04t}(0.04)dt=0.4,\\ {}^2\bar{A}_x &= \int_0^\infty e^{-0.12}e^{-0.04t}(0.04)dt=0.25,\end{aligned}$$

于是

$$\mathrm{Var}[\bar{a}_{\overline{T|}}]=\frac{1}{(0.06)^2}[0.25-(0.4)^2]=25.$$

标准差 $\sqrt{\mathrm{Var}[\bar{a}_{\overline{T|}}]}=5$

(3) 所求概率

$$Pr[\bar{a}_{\overline{T|}}>\bar{a}_x]=Pr[\bar{a}_{\overline{T|}}>10]=Pr[\frac{1-v^T}{0.06}>10]$$

$$= \quad Pr[0.4 > e^{-0.06T}] = Pr[T > -\frac{log 0.4}{0.06}]$$
$$= \quad Pr[T > 15.27] = \int_{15.27}^{\infty} e^{-0.04t}(0.04)dt = 0.54.$$

在所述假定下，有 54% 的可能 $\overline{a}_x$ 不敷提供单位生存年金。

接下去转而讨论定期年金及延期年金 (延付年金)。每年 1 单位连续支付的 n 年定期生存年金当 (x) 在其后 n 年内活着时提供支付，其精算现值记为 $\overline{a}_{x:\overline{n}|}$, 按当期支付技巧

$$\overline{a}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x dt. \tag{3.3.10}$$

现在对

$$\overline{A}^{1}_{x:\overline{n}|} = \int_0^n v^t {}_tp_x \mu_{x+t} dt = \int_0^n v^t(-d_t p_x)$$

运用分部积分，得

$$\overline{A}^{1}_{x:\overline{n}|} = 1 - v^n {}_np_x - \delta \overline{a}_{x:\overline{n}|},$$

即

$$1 = \delta \overline{a}_{x:\overline{n}|} + \overline{A}_{x:\overline{n}|}. \tag{3.3.11}$$

请读者与 (3.3.4) 对照并作出解释。

综合支付技巧从现值随机变量

$$Y = \begin{cases} \overline{a}_{\overline{T}|} & 0 \le T < n \\ \overline{a}_{\overline{n}|} & T \ge n \end{cases} \tag{3.3.12}$$

出发，确定

$$\overline{a}_{x:\overline{n}|} = E[Y] = \int_0^n \overline{a}_{\overline{t}|} {}_tp_x \mu_{x+t} dt + \overline{a}_{\overline{n}|} {}_np_x.$$

经分部积分，上式就成为 (3.3.10)。将 $(1 - v^T)/\delta$ 代入 $\overline{a}_{\overline{T}|}$,$(1 - v^n)/\delta$ 代入 $\overline{a}_{\overline{n}|}$, 从 (3.3.12) 可以看出 $Y = (1 - Z)/\delta$, 其中

$$Z = \begin{cases} v^T & 0 \le T < n \\ v^n & T \ge n \end{cases}$$

是 n 年期两全保险的现值随机变量 [参见表 2.2.1 并与 (3.3.5) 对照], 于是

$$\overline{a}_{x:\overline{n}|} = E[Y] = \frac{1}{\delta}(1 - E[Z]) = \frac{1}{\delta}(1 - \overline{A}_{x:\overline{n}|}), \tag{3.3.13}$$

与 (3.3.11) 相同。

为计算方差，可根据 $Y = \frac{1-Z}{\delta}$ 及 (2.2.10) 得

$$\text{Var}[Y] = \frac{1}{\delta^2}\text{Var}[Z] = \frac{1}{\delta^2}[{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2]. \tag{3.3.14}$$

用年金值来表示，公式 (3.3.14) 成为

$$\begin{aligned}\text{Var}[Y] &= \frac{1}{\delta^2}[1 - 2\delta^2\overline{a}_{x:\overline{n}|} - (1 - \delta\overline{a}_{x:\overline{n}|})^2] \\ &= \frac{2}{\delta}(\overline{a}_{x:\overline{n}|} - {}^2\overline{a}_{x:\overline{n}|}) - (\overline{a}_{x:\overline{n}|})^2. \end{aligned} \tag{3.3.15}$$

对于当 (x) 活到 $x+n$ 岁之后每年 1 单位连续支付的延期年金，其精算现值记作 ${}_{n|}\overline{a}_x$ 。按当期支付技巧，有

$$ {}_{n|}\overline{a}_x = \int_n^{\infty} v^t {}_tp_x dt \tag{3.3.16}$$

以及关系

$$\begin{aligned} {}_{n|}\overline{a}_x &= \int_0^{\infty} v^t {}_tp_x dt - \int_0^n v^t {}_tp_x dt \\ &= \overline{a}_x - \overline{a}_{x:\overline{n}|} \qquad (3.3.17) \\ &= \frac{\overline{A}_{x:\overline{n}|} - \overline{A}_x}{\delta}. \qquad (3.3.18) \end{aligned}$$

如应用综合支付技巧，从现值随机变量

$$Y = \begin{cases} 0 = \overline{a}_{\overline{T}|} - \overline{a}_{\overline{T}|} & 0 \le T < n \\ v^n\overline{a}_{\overline{T-n}|} = \overline{a}_{\overline{T}|} - \overline{a}_{\overline{n}|} & T \ge n \end{cases}$$

出发，

$$\begin{aligned}
{}_{n|}\overline{a}_x &= E[Y] = \int_n^\infty v^n \overline{a}_{\overline{t-n|}}\, {}_tp_x \mu_{x+t} dt \\
&= \int_0^\infty v^n \overline{a}_{\overline{s|}}\, {}_{n+s}p_x \mu_{x+n+s} ds \\
&= v^n\, {}_np_x \int_0^\infty \overline{a}_{\overline{s|}}\, {}_sp_{x+n} \mu_{x+n+s} ds,
\end{aligned}$$

这表明

$$ {}_{n|}\overline{a}_x = {}_nE_x \overline{a}_{x+n}. \tag{3.3.19}$$

这个公式也可通过对 (3.3.16) 作代换 $t=n+s$ 得到。从 Y 的定义也可看出：递延 n 年的终身年金现值随机变量 (Y) 等于终身年金的现值减去 n 年期生存年金的现值，对此取期望值也可得出 (3.3.17).

延期年金的现值随机变量 Y 的方差可根据定理 1.2.1 按以下方式计算：

$$\begin{aligned}
\mathrm{Var}[Y] &= \int_n^\infty v^{2n} (\overline{a}_{\overline{t-n|}})^2\, {}_tp_x \mu_{x+t} dt - ({}_{n|}\overline{a}_x)^2 \\
&= v^{2n}\, {}_np_x \int_0^\infty (\overline{a}_{\overline{s|}})^2\, {}_sp_{x+n} \mu_{x+n+s} ds - ({}_{n|}\overline{a}_x)^2 \\
&= v^{2n}\, {}_np_x \int_0^\infty 2\overline{a}_{\overline{s|}} v^s\, {}_sp_{x+n} ds - ({}_{n|}\overline{a}_x)^2 \\
&= \frac{2}{\delta} v^{2n}\, {}_np_x \int_0^\infty (v^s - v^{2s})\, {}_sp_{x+n} ds - ({}_{n|}\overline{a}_x)^2 \\
&= \frac{2}{\delta} v^{2n}\, {}_np_x (\overline{a}_{x+n} - {}^2\overline{a}_{x+n}) - ({}_{n|}\overline{a}_x)^2.
\end{aligned} \tag{3.3.20}$$

这一公式的另一种建立方法作为习题解答。

对于当 (x) 在 $x+m$ 岁与 $x+m+n$ 岁之间活着时每年 1 单位连续支付的延付定期年金，其精算现值记作 ${}_{m|n}\overline{a}_x$，于是

$$ {}_{m|n}\overline{a}_x = \int_m^{m+n} v^t\, {}_tp_x dt \tag{3.3.21}$$

$$= \overline{a}_{x:\overline{m+n}|} - \overline{a}_{x:\overline{m}|} \tag{3.3.22}$$

$$= \frac{\overline{A}_{x:\overline{m}|} - \overline{A}_{x:\overline{m+n}|}}{\delta} \tag{3.3.23}$$

$$= {}_mE_x \overline{a}_{x+m:\overline{n}|}. \tag{3.3.24}$$

与利息理论中的函数

$$\overline{s}_{\overline{n}|} = \int_0^n (1+i)^{n-t} dt$$

类似有

$$\overline{s}_{x:\overline{n}|} = \frac{1}{{}_nE_x}\overline{a}_{x:\overline{n}|} = \int_0^n \frac{{}_tE_x}{{}_nE_x} dt = \int_0^n \frac{1}{{}_{n-t}E_{x+t}} dt, \tag{3.3.25}$$

它表示当 (x) 活着时每年 1 单位连续支付的 n 年定期生存年金在期末的精算积累值。

最后，通过对式 (3.3.2B) 的积分求导可得 $d\overline{a}_x/dx$ 的表达式

$$\begin{aligned}\frac{d\overline{a}_x}{dx} &= \int_0^\infty v^t(\frac{\partial}{\partial x}{}_tp_x)dt = \int_0^\infty v^t{}_tp_x(\mu_x - \mu_{x+t})dt \\ &= \mu_x\overline{a}_x - \overline{A}_x = \mu_x\overline{a}_x - (1-\delta\overline{a}_x),\end{aligned}$$

即

$$\frac{d\overline{a}_x}{dx} = (\mu_x + \delta)\overline{a}_x - 1. \tag{3.3.26}$$

公式 (3.3.26) 可解释为，生存年金精算现值关于年龄的变化率由利息收入率 $\delta\overline{a}_x$，生存受益率 $\mu_x\overline{a}_x$ 与支付支出率 -1 合成。

例 3.3.2: 导出

(1) $\frac{\partial}{\partial x}\overline{a}_{x:\overline{n}|}$.　　　(2) $\frac{\partial}{\partial n}{}_{n|}\overline{a}_x$ 的表达式。

解:

(1) 类似于推导 (3.3.26),

$$\begin{aligned}\frac{\partial}{\partial x}\overline{a}_{x:\overline{n}|} &= \mu_x\overline{a}_{x:\overline{n}|} - \overline{A}^{1}_{x:\overline{n}|} \\ &= \mu_x\overline{a}_{x:\overline{n}|} - (1 - \delta\overline{a}_{x:\overline{n}|} - {}_nE_x) \\ &= (\mu_x+\delta)\overline{a}_{x:\overline{n}|} - (1 - {}_nE_x).\end{aligned}$$

(2) 直接可得

$$\frac{\partial}{\partial n}{}_{n|}\overline{a}_x = \frac{\partial}{\partial n}\int_n^\infty v^t{}_tp_x dt = -v^n{}_np_x.$$

表 3.3.1 总结了连续生存年金的各种概念。

表 3.3.1　连续生存年金 (每年 1 单位连续支付) 概要

年金名称	现值随机变量	精算现值 $E[Y]$ 等于
终身生存年金	$\overline{a}_{\overline{T}\rvert}$　$T \geq 0$	$\overline{a}_x = \int_0^\infty v^t_t p_x dt$
n 年定期 生存年金	$\overline{a}_{\overline{T}\rvert}$　$0 \leq T < n$ $\overline{a}_{\overline{n}\rvert}$　$T \geq n$	$\overline{a}_{x:\overline{n}\rvert} = \int_0^n v^t{}_tp_x dt$
递延 n 年的 终身生存年金	0　$0 \leq T < n$ $\overline{a}_{\overline{T}\rvert} - \overline{a}_{\overline{n}\rvert}$　$T \geq n$	${}_{n\rvert}\overline{a}_x = \int_n^\infty v^t{}_tp_x dt$
递延 m 年的 n年定期 生存年金	0　$0 \leq T < m$ $\overline{a}_{\overline{T}\rvert} - \overline{a}_{\overline{m}\rvert}$　$m \leq T < m+n$ $\overline{a}_{\overline{m+n}\rvert} - \overline{a}_{\overline{m}\rvert}$　$T \geq m+n$	${}_{m\rvert n}\overline{a}_x = \int_m^{m+n} v^t{}_tp_x dt$

附加关系

$1 = \delta\overline{a}_x + \overline{A}_x.$

$1 = \delta\overline{a}_{x:\overline{n}\rvert} + \overline{A}_{x:\overline{n}\rvert}.$

${}_n\overline{a}_x = \overline{a}_x - \overline{a}_{x:\overline{n}\rvert}.$

$\overline{s}_{x:\overline{n}\rvert} = \frac{\overline{a}_{x:\overline{n}\rvert}}{{}_nE_x} = \int_0^n (1+i)^{n-t}\frac{l_{x+t}}{l_{x+n}}dt.$

§3.4　离散生存年金

离散生存年金理论与对应的连续生存年金理论几乎完全相似，只是积分改成求和，微分改成差分。对于连续年金，不存在期初付还是期末付的问题，但对于离散年金两者是有区别的。我们将从精算应用中具有更突出作用的期初年金开始，例如大多数个人人寿保险都是通过分期缴费的期初年金方式购买的.

考虑当 (x) 活着时每年年初支付 1 单位的终身年金，其精算现值为 $\ddot{a}_x$ 。鉴于在时间 k 支付 1 的精算现值为

$${}_kE_x = v^k{}_kp_x,$$

由当期支付技巧可得

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x. \tag{3.4.1}$$

用生存组函数 l_x 来表示，式 (3.4.1) 成为

$$\ddot{a}_x = \frac{1}{l_x}\sum_{k=0}^{\infty} v^k l_{x+k}. \tag{3.4.2}$$

按生命表的生存组解释，为了从一个基金连同利息中使年龄 $x+k$ 的 l_{x+k} 个生存者每人获得 1 单位， $k=0,1,2,\cdots$，需要 l_x 个 x 岁生存者向该基金提供的数额就是 $\ddot{a}_x$ 。

为运用综合支付技巧，考虑年金支付的现值随机变量 $Y=\ddot{a}_{\overline{K+1|}}$，其中随机变量 K 是 (x) 的整值随机变量。于是

$$\begin{aligned}\ddot{a}_x &= E[Y]=E[\ddot{a}_{\overline{K+1|}}]=\sum_{k=0}^{\infty}\ddot{a}_{\overline{k+1|}}Pr[K=k] \\ &= \sum_{k=0}^{\infty}\ddot{a}_{\overline{k+1|}}\,{}_{k|}q_x.\end{aligned} \tag{3.4.3}$$

利用定理 1.5.2 以及关系

$$\Delta\ddot{a}_{\overline{k+1|}} = v^{k+1},$$

式 (3.4.3) 可转化为

$$\ddot{a}_x = 1+\sum_{k=0}^{\infty} v^{k+1}{}_{k+1}p_x,$$

与 (3.4.1) 相同。

由 (3.4.3) 可得出

$$\ddot{a}_x = E\left[\frac{1-v^{K+1}}{d}\right] = \frac{1}{d}(1-A_x). \tag{3.4.4}$$

根据 $\ddot{a}_{\overline{\infty}|}=1/d$, 可将上式写成

$$\ddot{a}_x=\ddot{a}_{\overline{\infty}|}-\ddot{a}_{\overline{\infty}|}A_x. \tag{3.4.5}$$

将 (3.4.4) 改写一下成为

$$1=d\ddot{a}_x+A_x. \tag{3.4.6}$$

可将这些等式与连续情形的对应式 (3.3.6), (3.3.7) 及 (3.3.4) 对照。公式 (3.4.6) 表明，现在投资一个单位可在 (x) 活着时提供每年的先付利息 d 并在 (x) 死亡年末偿还。

方差公式为 [参见 (3.3.8)]

$$\begin{aligned}\operatorname{Var}[\ddot{a}_{\overline{K+1}|}] &= \operatorname{Var}[\frac{1-v^{K+1}}{d}] \\ &= \frac{1}{d^2}\operatorname{Var}[v^{K+1}]=\frac{1}{d^2}[{}^2A_x-(A_x)^2]. \end{aligned} \tag{3.4.7}$$

对于当 (x) 活着时每年年初支付 1 的 n 年定期生存年金，其精算现值记作 $\ddot{a}_{x:\overline{n}|}$, 当期支付技巧导致公式

$$\ddot{a}_{x:\overline{n}|}=\sum_{k=0}^{n-1}{}_kE_x=\sum_{k=0}^{n-1}v^k{}_kp_x. \tag{3.4.8}$$

如用综合支付技巧，定义

$$Y=\begin{cases}\ddot{a}_{\overline{K+1}|} & 0\le K<n\\ \ddot{a}_{\overline{n}|} & K\ge n.\end{cases} \tag{3.4.9}$$

以及

$$Z=\begin{cases}v^{K+1} & 0\le K<n\\ v^n & K\ge n.\end{cases}$$

显然 $Y=(1-Z)/d$, 并且

$$\ddot{a}_{x:\overline{n}|}=\frac{1}{d}(1-E[Z])=\frac{1}{d}(1-A_{x:\overline{n}|}), \tag{3.4.10}$$

[参见 (3.3.13)] 。整理后得出

$$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}, \tag{3.4.11}$$

[参见 (3.3.11)] 。至于方差，有

$$\mathrm{Var}[Y] = \frac{1}{d^2}\mathrm{Var}[Z] = \frac{1}{d^2}[{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2]. \tag{3.4.12}$$

对于当 (x) 活着时从 $x+n$ 岁起每年年初支付 1 单位的延期生存年金，其精算现值记作 ${}_{n|}\ddot{a}_x$ 。这里

$$\begin{aligned} {}_{n|}\ddot{a}_x &= \sum_{k=n}^{\infty} v^k {}_kp_x & (3.4.13)\\ &= \ddot{a}_x - \ddot{a}_{x:\overline{n}|} & (3.4.14)\\ &= \frac{A_{x:\overline{n}|} - A_x}{d} & (3.4.15)\\ &= {}_nE_x\ddot{a}_{x+n}, & (3.4.16) \end{aligned}$$

[参见 (3.3.16)-(3.3.19)] 。

对于当 (x) 活着时每年 1 单位年初支付的 n 年定期期初生存年金，其 n 年期满时的精算积累值记作 $\ddot{s}_{x:\overline{n}|}$, 该函数公式为

$$\begin{aligned} \ddot{s}_{x:\overline{n}|} &= \frac{1}{{}_nE_x}\ddot{a}_{x:\overline{n}|} & (3.4.17)\\ &= \sum_{k=0}^{n-1}\frac{{}_kE_x}{{}_nE_x} = \sum_{k=0}^{n-1}\frac{1}{{}_{n-k}E_{x+k}}, & (3.4.18) \end{aligned}$$

与利息理论中有关 $\ddot{s}_{\overline{n}|}$ 的公式类似。

至于每个支付周期末支付的期末年金，符号 $\ddot{a}$ 与 $\ddot{s}$ 分别改成 a 与 s, 也就是说， a_x 表示当 (x) 活着时每年年末支付 1 的生存年金的精算现值。有关 a_x 的公式可由期初年金中使用过的类似

方法得出，也可从两种年金的关系中得出。因为期末年金与期初年金的差异只是初次支付，所以

$$a_x = \ddot{a}_x - 1 \tag{3.4.19}$$

$$= \sum_{k=1}^{\infty} v^k {}_k p_x. \tag{3.4.20}$$

换一种方法，

$$a_x = E[a_{\overline{K}|}] \tag{3.4.21}$$

$$= \sum_{k=1}^{\infty} a_{\overline{k}|} \, {}_{k|}q_x.$$

由此得出

$$\begin{aligned} a_x &= E\left[\frac{1 - v^K}{i}\right] \\ &= E\left[\frac{1 - (1+i)v^{K+1}}{i}\right] \\ &= \frac{1}{i}[1 - (1+i)A_x]. \end{aligned}$$

根据 $a_{\overline{\infty}|} = \frac{1}{i}$ 及 $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$, 可将上式改写成

$$a_x = a_{\overline{\infty}|} - \ddot{a}_{\overline{\infty}|} A_x \tag{3.4.22}$$

或者

$$1 = i a_x + (1+i)A_x. \tag{3.4.23}$$

对于当 (x) 活着时每年年末支付 1 的 n 年定期年金，其精算现值记作 $a_{x:\overline{n}|}$, 它可表示成

$$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k {}_k p_x \tag{3.4.24}$$

或

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 +_n E_x. \tag{3.4.25}$$

在后一公式中，${}_nE_x$ 是 n 年末支付的精算现值，在 n 年期的期初年金中不存在。由 $a_{x:\overline{n}|} - {}_nE_x = a_{x:\overline{n-1}|}$，以上公式还可写成

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}. \tag{3.4.26}$$

对于在 (x) 活到 $x+n$ 岁以后每年年末支付 1 的延期生存年金，其精算现值记作 ${}_{n|}a_x$，公式为

$$\begin{aligned} {}_{n|}a_x &= \sum_{k=n+1}^{\infty} v^k {}_kp_x & (3.4.27)\\ &= a_x - a_{x:\overline{n}|} & (3.4.28)\\ &= {}_nE_x a_{x+n}. & (3.4.29) \end{aligned}$$

最后，我们导出联系函数 $\ddot{a}$, a 及 A 的公式。首先，

$$\begin{aligned} A_x &= E[v^{K+1}] = E[a_{\overline{K+1}|} - a_{\overline{K}|}] \\ &= E[v\ddot{a}_{\overline{K+1}|} - a_{\overline{K}|}] \\ &= v\ddot{a}_x - a_x. & (3.4.30) \end{aligned}$$

作为对 (3.4.30) 的解释，注意到由 $v\ddot{a}_x$ 提供的每年年初支付 v, 除死亡年之外将被每年年末等价的支付 1 所抵消。因此 (3.4.30) 右端等价于 (x) 死亡年末的 1 单位支付，也就是 A_x 。

对于 n 年定期保险，相应的关系是

$$A^1_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}. \tag{3.4.31}$$

至于 n 年期两全保险

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + {}_nE_x,$$

将 (3.4.31) 代入并利用关系

$$a_{x:\overline{n}|} = a_{x:\overline{n-1}|} + {}_nE_x,$$

可导出

$$A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}. \tag{3.4.32}$$

表 3.4.1 总结了离散生存年金的各种概念。

表 3.4.1　离散生存年金 (期初付，期末付) 概要

年金名称	现值随机变量 Y	精算现值 $E[Y]$ 等于
终身生存年金		
—— 期初	$\ddot{a}_{\overline{K+1}|}$　$K \geq 0$	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k_k p_x$
—— 期末	$a_{\overline{K}|}$　$K \geq 0$	$a_x = \sum_{k=1}^{\infty} v^k_k p_x$
n 年定期生存年金		
—— 期初	$\ddot{a}_{\overline{K+1}|}$　$0 \leq K < n$ $\ddot{a}_{\overline{n}|}$　$K \geq n$	$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k_k p_x$
—— 期末	$\ddot{a}_{\overline{K}|}$　$0 \leq K < n$ $a_{\overline{n}|}$　$K \geq n$	$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k{}_k p_x$
递延 n 年的终身生存年金		
– 期初	0　$0 \leq K < n$ $\ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|}$　$K \geq n$	${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k{}_k p_x$
– 期末	0　$0 \leq K < n$ $a_{\overline{K}|} - a_{\overline{n}|}$　$K \geq n$	${}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k_k p_x$

关系:

$1 = d\ddot{a}_x + A_x$

$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$

$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$

${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$

$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} = \sum_{k=0}^{n-1}(1+i)^{n-k}\frac{l_{x+k}}{l_{x+n}}$

$A_x = v\ddot{a}_x - a_x$

$A^1_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$

$A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$

§3.5　年 m 次支付生存年金

在实践中，生存年金常常按月、按季或每半年支付一次。与确定性年金的符号类似，当 (x) 活着时一年 1 单位分 m 期期初支付的生存年金其精算现值记为 $\ddot{a}_x^{(m)}$ 。由当期支付技巧，

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{k=0}^{\infty} v^{k/m}{}_{k/m}p_x. \tag{3.5.1}$$

以下关系式更便于应用：

$$1 = d\ddot{a}_x + A_x = d^{(m)}\ddot{a}_x^{(m)} + A_x^{(m)}. \tag{3.5.2}$$

其根据在于，1 单位投资将提供每一计息周期的先付利息，并在死亡发生的期末偿还。有关 (3.5.2) 的验证，可参见 (3.4.4),(3.4.6) 以及习题 14 。

由 (3.5.2) 可得

$$\begin{aligned}\ddot{a}_x^{(m)} &= \frac{d}{d^{(m)}}\ddot{a}_x - \frac{1}{d^{(m)}}(A_x^{(m)} - A_x)\\ &= \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x - \ddot{a}_{\overline{\infty}|}^{(m)}(A_x^{(m)} - A_x).\end{aligned} \tag{3.5.3}$$

它可以这样解释：年 m 次支付生存年金等价于每年支付一系列 1 年确定性年金并消去死亡之年在死后分期支付的部分，而消去的部分等于始于死亡之 $1/m$ 年期末的年 m 次永久年金减掉始于死亡年末的年 m 次永久年金。

从 (3.5.2) 还可以写出

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \ddot{a}_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)}A_x^{(m)}, \tag{3.5.4}$$

请读者对此作出解释。

现在假定每一年龄的死亡均匀分布，在此假设下，

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x = s_{\overline{1}|}^{(m)} A_x$$

(作为习题)。于是 (3.5.3) 成为

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} \ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} A_x, \tag{3.5.5}$$

其中用标准函数 A_x 表示死亡年的消去项。

用 $1 - d\ddot{a}_x$ 取代 (3.5.5) 中的 A_x, 并注意到 $d^{(m)}\ddot{a}_{\overline{1}|}^{(m)} = d$, 可得出一个只含年金的公式

$$\begin{aligned}\ddot{a}_x^{(m)} &= \frac{1 - s_{\overline{1}|}^{(m)}(1 - d\ddot{a}_x)}{d^{(m)}} \\ &= s_{\overline{1}|}^{(m)} \ddot{a}_{\overline{1}|}^{(m)} \ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}}.\end{aligned} \tag{3.5.6}$$

公式 (3.5.6) 可与传统的 $\ddot{a}_x^{(m)}$ 近似公式相对照。传统的 $\ddot{a}_x^{(m)}$ 近似公式可通过对 (3.5.1) 右端应用 Woolhouse 求和公式得出，

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_x + \delta). \tag{3.5.7}$$

在实践中，通常截断成

$$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{m-1}{2m}, \tag{3.5.8}$$

它也可以从假定计算基数

$$D_{x+h/m} = v^{x+h/m} l_{x+h/m}$$

在每一年龄中是线性函数

$$D_x - \frac{h}{m}(D_x - D_{x+1})$$

得出 (参见习题 15) 。值得注意，一般来说 $D_{x+h/m}$ 在每个年龄中的线性性并非就是 $l_{x+h/m}$ 在每个年龄中的线性性，不过在死亡均匀分布的假设可得出精确成立的关系式

$$1 = d^{(m)}\ddot{a}_x^{(m)} + A_x^{(m)}.$$

从习题中可以发现，在高利率与低死亡效力情况下，(3.5.8) 产生扭曲的年金值 $\ddot{a}_{x:\overline{1}|}^{(12)} > \ddot{a}_{\overline{1}|}^{(12)}$ 。由于这些原因，才提出 (3.5.5) 及等价的 (3.5.6) 取代传统近似公式 (3.5.8) 。

为方便起见，(3.5.6) 可表示成如下形式

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m), \tag{3.5.9}$$

其中

$$\alpha(m) = s_{\overline{1}|}^{(m)}\ddot{a}_{\overline{1}|}^{(m)} = \frac{id}{i^{(m)}d^{(m)}}, \tag{3.5.10}$$

$$\beta(m) = \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}. \tag{3.5.11}$$

注意，$\alpha(m)$ 与 $\beta(m)$ 只依赖于 m 与利率，与年龄无关。当 m=1 时，$\alpha(1)$=1, $\beta(1)$=0, (3.5.9) 成为恒等式，而且，$\beta(m)$ 是 (3.5.5) 中消去项的系数，即 (3.5.5) 可写成

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x - \beta(m)A_x. \tag{3.5.12}$$

有关 $\alpha(m)$ 与 $\beta(m)$ 的级数展开可作为习题。

例 3.5.1: 根据附录 2A 示例生命表，按每年有效利率 6% 计算 65 岁退休者每月 1000 期初终身年金的精算现值。

解: 这里

$$\alpha(12) = s_{\overline{1}|}^{(12)}\ddot{a}_{\overline{1}|}^{(12)} = 1.02721070 \times 0.97378368 = 1.0002810,$$

$$\beta(12) = \frac{s_{\overline{1}|}^{(12)} - 1}{\ddot{a}_{\overline{1}|}^{(12)}} = 0.46811951,$$

$$\frac{11}{24} = 0.45833333.$$

$\alpha(12)$ 及 $\beta(12)$ 与传统近似公式中的 1 及 $\frac{11}{24}$ 很接近。

由示例生命表及利率 6%,

$$\ddot{a}_{65} = 9.89693,$$

$$A_{65} = 1 - d\ddot{a}_{65} = 0.4397965,$$

$$1000\mu_{65} = 0.7 + 0.05(10^{0.04})^{65} = 20.605359.$$

于是所求值 $12000\ddot{a}_{65}^{(12)}$ 可计算如下：由 (3.5.12)

$$\begin{aligned}&12000[\ddot{a}_{\overline{1}|}^{(12)}\ddot{a}_{65} - \beta(12)A_{65}]\\&= 12000(0.97378368 \times 9.89693 - 0.46811951 \times 0.4397965)\\&= 113179.\end{aligned}$$

或由 (3.5.9),

$$\begin{aligned}&12000[\alpha(12)\ddot{a}_{65} - \beta(12)]\\&= 12000(1.0002810 \times 9.89693 - 0.46811951) = 113179.\end{aligned}$$

或由 (3.5.8),

$$12000(\ddot{a}_{65} - \frac{11}{24}) = 113263.$$

或由 (3.5.7),

$$12000[\ddot{a}_{65} - \frac{11}{24} - \frac{143}{1728}(\mu_{65} + \delta)] = 113185.$$

公式 (3.5.12) 及等价的 (3.5.9) 基于死亡均匀分布假设，而 (3.5.7) 及其简化 (3.5.8) 则基于 Woolhouse 公式，没有理由期待两者得出的结果会相同。不过这个例示显示，差异相对较小。

既然已对年 m 次支付的终身年金建立起有关公式，很容易对定期年金及延期年金建立有关公式。从 (3.5.12), 有

$$
\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} &= \ddot{a}_x^{(m)} - {}_nE_x\ddot{a}_{x+n}^{(m)} \\
&= \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_x - \beta(m)A_x - {}_nE_x[\ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_{x+n} - \beta(m)A_{x+n}] \\
&= \ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_{x:\overline{n}|} - \beta(m)A_{x:\overline{n}|}^{1}. \qquad (3.5.13)
\end{aligned}
$$

类似有

$$ {}_{n|}\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)}{}_{n|}\ddot{a}_x - \beta(m)_{n|}A_x. \qquad (3.5.14)$$

根据 (3.5.9) 可得

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x). \qquad (3.5.15)$$

$${}_{n|}\ddot{a}_x^{(m)} = \alpha(m)_{n|}\ddot{a}_x - \beta(m)_nE_x. \qquad (3.5.16)$$

年支付 m 次的期末年金可通过调整相应的期初生存年金值而获得，例如，

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$$

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}(1 - {}_nE_x).$$

或者可建立一年 m 次支付期末年金与按年支付期末年金的关系式；诸如

$$1 = ia_x + (1+i)A_x = i^{(m)}a_x^{(m)} + \left(1 + \frac{i^{(m)}}{m}\right)A_x^{(m)}. \quad (3.5.17)$$

[与关系 (3.5.2) 相似]。其含义是，1 单位投资将在每个计息期末产生利息并在死亡发生的 $1/m$ 年期末偿还本息。

§3.6 等额年金计算基数公式

第二章为估价保险引入了函数 $D_x = v^x l_x$, 我们将重新考察其在生存年金估价中的作用。在 (2.6.2) 中, $A_{x:\overset{\ 1}{\overline{n|}}} = {}_nE_x$ 表示成 D_{x+n}/D_x, 从而 (3.1.3) 可写成

$$\frac{1}{{}_nE_x} = \frac{v^x l_x}{v^{x+n} l_{x+n}} = \frac{D_x}{D_{x+n}}. \tag{3.6.1}$$

更一般地有, y 岁时支付 b 在年龄 x 时的精算值为

$$\frac{bD_y}{D_x}. \tag{3.6.1}$$

当 $x < y$ 时它表示精算现值, 而当 $x > y$ 时它表示精算积累值。

考虑 (x) 的某种生存年金, 其支付方式如图 3.5.1 所示。由 (3.5.2), y 岁, $y+1$ 岁, $\cdots$, $z-1$ 岁分别支付 b 的生存年金在年龄 x 岁时的精算现值为

$$\frac{b}{D_x}\sum_{u=y}^{z-1} D_u.$$

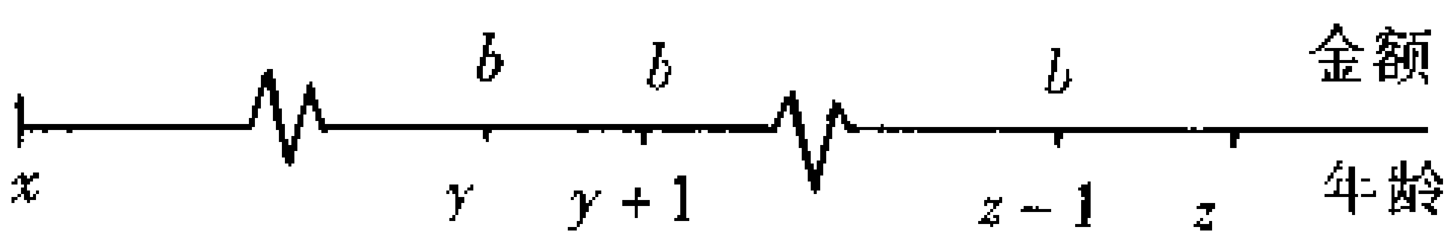


图 3.6.1　等额年金直线图

引入函数 $N_x = \sum_{u=x}^{\infty} D_u$, 则以上年金精算值可表示为

$$\frac{b}{D_x}(N_y - N_z). \tag{3.6.3}$$

这里 y 是年金的第一次支付年龄, z 是最后一次支付后 1 年的年龄, 数额都是 b 。x 既可以小于也可以等于或大于 y 以及 z 。

计算基数的用途几乎完全局限于利率为常数的情形，只要有现成关于函数 D_x 及 N_x 的表格，利用 (3.6.3) 计算生存年金就非常灵便。在使用计算基数估价按年支付生存年金的大多数场合，一般公式 (3.6.2) 与 (3.6.3) 可应付自如，这里不再以 D_x 及 N_x 重写各种形式生存年金的有关公式。

至于每年支付 m 次的生存年金估价，注意到 (3.5.1) 可写成

$$\ddot{a}_x^{(m)} = \frac{1}{mD_x}\sum_{h=0}^{\infty} D_{x+h/m}, \tag{3.6.4}$$

它提示引入函数

$$N_x^{(m)} = \frac{1}{m}\sum_{h=0}^{\infty} D_{x+h/m}. \tag{3.6.5}$$

于是

$$\ddot{a}_x^{(m)} = \frac{N_x^{(m)}}{D_x}.$$

如假定每一年龄中死亡均匀分布，则

$$\begin{aligned}\ddot{a}_x^{(m)} &= \alpha(m)\ddot{a}_x - \beta(m)\\ &= \frac{\alpha(m)N_x - \beta(m)D_x}{D_x}.\end{aligned}$$

比较以上关于 $\ddot{a}_x^{(m)}$ 的两个公式得出

$$N_x^{(m)} = \alpha(m)N_x - \beta(m)D_x. \tag{3.6.6}$$

根据 (3.5.8) 的近似公式

$$N_x^{(m)} \cong N_x - \frac{m-1}{2m}D_x \tag{3.6.7}$$

也在实践中被使用。

当知道函数 $N_x^{(m)}$ 时，对一年分 m 期支付的年金，其一般估价公式与 (3.6.3) 相似，为

$$\frac{b}{D_x}(N_y^{(m)} - N_z^{(m)}), \tag{3.6.8}$$

其中 b 是定额年收入，$\frac{b}{m}$ 是一年分 m 期每次的支付数额。

例 3.6.1: 对于 (25) 的每月 1000 第一次支付始于 65 岁的延期生存年金，用计算基数表示其精算现值。

解: 每月 1000, 岁入为 12000, 所求值为

$$\frac{12000N_{65}^{(12)}}{D_{25}},$$

其中 $N_{65}^{(12)} = \alpha(12)N_{65} - \beta(12)D_{65}$(在死亡均匀分布假设下)。

例 3.6.2: 对于 (25) 在 25 岁至 65 岁间活着时每月初支付 100, 用计算基数表示 65 岁时的精算积累值。

解: 应用 (3.6.8) 得

$$\frac{1200}{D_{65}}(N_{25}^{(12)} - N_{65}^{(12)}).$$

回到 (3.6.4), 令 $m \to \infty$, 得

$$\overline{a}_x = \frac{1}{D_x}\int_0^\infty D_{x+t}dt, \tag{3.6.9}$$

它提示定义

$$\overline{N_x} = \int_0^\infty D_{x+t}dt = \int_x^\infty D_y dy. \tag{3.6.10}$$

进一步，在每个年龄死亡均匀分布的假设下，如 (3.6.6) 有

$$\overline{N}_x = \alpha(\infty)N_x - \beta(\infty)D_x,$$

其中 [参见 (3.5.10) 与 (3.5.11)]

$$\alpha(\infty) = \bar{s}_{\overline{1}|}\bar{a}_{\overline{1}|} = \frac{id}{\delta^2} \tag{3.6.11}$$

$$\beta(\infty) = \frac{i-\delta}{\delta^2}. \tag{3.6.12}$$

有关 $\alpha(\infty)$ 及 $\beta(\infty)$ 的级数展开可作为习题。

实践中常使用近似公式

$$\overline{N}_x \cong N_x - \frac{1}{2}D_x,$$

它可通过对 (3.6.9) 应用梯形规则估计积分得出，或在 (3.6.7) 中令 $m \to \infty$ 得到。用这些函数表示的定额连续生存年金的估价公式为

$$\frac{b}{D_x}(\overline{N}_y - \overline{N}_z). \tag{3.6.14}$$

§3.7　变额年金

这一节考虑每年分 m 次支付的金额按年变动的生存年金估价问题。设从 x 岁开始终止于 $x+n$ 岁的年支付额序列为 b_x, $b_{x+1}, \cdots, b_y, \cdots, b_{x+n-1}$，每年分 m 次期初支付。对于在年龄区间 $(y, y+1)$ 内的 m 次支付，在年龄 y 的精算现值为 $b_y\ddot{a}^{(m)}_{y:\overline{1}|}$，从而整个年金在 x 岁时的精算现值 $(apv)_x$ 可表示成

$$(apv)_x = \sum_{y=x}^{x+n-1} b_y \ddot{a}^{(m)}_{y:\overline{1}|}\,{}_{y-x}E_x. \tag{3.7.1}$$

如假定在每个年龄中死亡均匀分布，由 (3.5.15) 可得

$$(apv)_x = \sum_{y=x}^{x+n-1} b_y[\alpha(m) - \beta(m)(1 - {}_1E_y)]\,{}_{y-x}E_x. \tag{3.7.2}$$

用计算基数可改写成

$$(apv)_x = \frac{1}{D_x}\sum_{y=x}^{x+n-1} b_y[\alpha(m)D_y - \beta(m)(D_y - D_{y+1})]. \quad (3.7.3)$$

定义

$$D_y^{(m)} = \frac{1}{m}\sum_{h=0}^{m-1} D_{y+h/m} = N_y^{(m)} - N_{y+1}^{(m)},$$

由 (3.6.6) 可知

$$D_y^{(m)} = \alpha(m)D_y - \beta(m)(D_y - D_{y+1}), \quad (3.7.4)$$

于是 (3.7.3) 成为

$$(apv)_x = \frac{1}{D_x}\sum_{y=x}^{x+n-1} b_y D_y^{(m)}. \quad (3.7.5)$$

以下概述期末支付变额年金平行的公式。与 (3.7.1) 对应的是

$$(apv)_x = \sum_{y=x}^{x+n-1} b_y a_{y:\overline{1}|}^{(m)}\,{}_{y-x}E_x. \quad (3.7.6)$$

在每个年龄死亡均匀分布的假设下

$$a_{y:\overline{1}|}^{(m)} = \ddot{a}_{y:\overline{1}|}^{(m)} - \frac{1}{m}(1 - {}_1E_y) = \alpha(m) - [\beta(m) + \frac{1}{m}](1 - {}_1E_y).$$

与 (3.7.3) 相对应的是

$$(apv)_x = \frac{1}{D_x}\sum_{y=x}^{x+n-1} b_y\{\alpha(m)D_y - [\beta(m) + \frac{1}{m}](D_y - D_{y-1})\}, \quad (3.7.7)$$

它提示定义

$$\tilde{D}_y^{(m)} = \alpha(m)D_y - [\beta(m) + \frac{1}{m}](D_y - D_{y+1}). \quad (3.7.8)$$

于是 (3.7.7) 成为

$$(apv)_x = \frac{1}{D_x}\sum_{y=x}^{x+n-1} b_y \tilde{D}_y^{(m)}. \tag{3.7.9}$$

当每年支付额同等，即 $b_y = b$(常数) 时，(3.7.5) 简化成 $(\frac{b}{D_x})(N_x^{(m)} - N_{x+n}^{(m)})$ 。在其它特殊情形下，如 b_y 是 y 的线性函数时，可引入计算基数

$$S_x^{(m)} = \sum_{y=x}^{\infty} N_y^{(m)} \tag{3.7.10}$$

或者

$$N_y^{(m)} = S_y^{(m)} - S_{y+1}^{(m)}.$$

这些特殊情形的某些公式可参见习题 25−27 。

例 3.7.1: 根据附录 2 示例生命表，按有效利率 6% 计算

(1) $N_{70}^{(12)}$ 与 $S_{70}^{(12)}$.

(2) 对于 (70) 的一种年金：第 1 年每月支付 100, 第 2 年每月支付 110, 以此类推，每年的月收入增加 10, 计算该生存年金的精算现值。

解:

(1) 由 (3.6.6),

$$\begin{aligned} N_{70}^{(12)} &= \alpha(12)N_{70} - \beta(12)D_{70} \\ &= 1.0002810 \times 9597.05 - 0.46811951 \times 1119.14 \\ &= 9075.48. \end{aligned}$$

类似地,

$$S_{70}^{(12)} = \alpha(12)S_{70} - \beta(12)N_{70} = 62643.28.$$

(2) 所求精算现值为

$$\frac{12(90N_{70}^{(12)}+10S_{70}^{(12)})}{D_{70}}=15464.$$

§3.8 递归方程

仍考虑年支付额序列为 $b_x, b_{x+1}, \cdots, b_{x+n-1}$ 的变额年 m 次期初支付生存年金。设 $(apv)_y$ 是从 y 岁支付到 $x+n$ 岁的相应年金在 y 岁时的精算现值，由 (3.7.1),

$$(apv)_y = b_y\ddot{a}_{y:\overline{1}|}^{(m)} + {}_1E_y(apv)_{y+1}. \tag{3.8.1}$$

从 $(apv)_{x+n}=0$ 开始，可根据以上递归方程依次算出 $(apv)_{x+n-1}$, $(apv)_{x+n-2}, \cdots, (apv)_x$ 。这种递归计算也许比按公式 (3.7.1) 或 (3.7.5) 计算单个值 $(apv)_x$ 更可取，不过后者可用来检查递归计算过程的误差累积。

举例来说，如对 $x=c, c+1, \cdots, \omega-1$ 需要 $\ddot{a}_x^{(m)}$ 值的一张表，就可利用递归方程

$$\ddot{a}_y^{(m)} = \ddot{a}_{y:\overline{1}|}^{(m)} + {}_1E_y\ddot{a}_{y+1}^{(m)}. \tag{3.8.2}$$

在每个年龄的死亡均匀分布假设下，

$$\ddot{a}_{y:\overline{1}|}^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} - \beta(m)vq_y$$

[参见 (3.5.13)], 于是

$$\ddot{a}_y^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} - \beta(m)vq_y + vp_y\ddot{a}_{y+1}^{(m)}. \tag{3.8.3}$$

从 $\ddot{a}_\omega^{(m)}=0$ 出发，按上式可递归计算出 $\ddot{a}_{\omega-1}^{(m)}, \ddot{a}_{\omega-2}^{(m)}, \cdots, \ddot{a}_c^{(m)}$ 。当 $m=1$ 时，公式简化为

$$\ddot{a}_\omega = 0, \quad \ddot{a}_y = 1 + vp_y\ddot{a}_{y+1}. \tag{3.8.4}$$

例 3.8.1: 证明 (3.8.3) 可整理成

$$\ddot{a}_y^{(m)}(1+i)-\ddot{s}_{\overline{1}|}^{(m)}+q_y[\beta(m)+\ddot{a}_{y+1}^{(m)}]=\ddot{a}_{y+1}^{(m)}, \qquad (3.8.5)$$

并对这一公式作一解释。

解: 对 (3.8.3) 乘 $1+i$, 以 $1-q_y$ 替代 p_y, 移项并注意到 $(1+i)\ddot{a}_{\overline{1}|}^{(m)}=\ddot{s}_{\overline{1}|}^{(m)}$, 就可得出 (3.8.5) 。

如使用传统近似

$$\ddot{a}_y^{(m)}\cong\ddot{a}_y-\frac{m-1}{2m},$$

与 (3.8.5) 可比较的公式为

$$\ddot{a}_y^{(m)}(1+i)-\left(1+\frac{m+1}{2m}i\right)+q_y\left(\frac{m-1}{2m}+\ddot{a}_{y+1}^{(m)}\right)\cong\ddot{a}_{y+1}^{(m)}. \qquad (3.8.6)$$

在养老金体系中，这一公式用于损益分析。公式 (3.8.5) 意味着：精算现值 $\ddot{a}_y^{(m)}$ 经过 1 年按利息累积，减去 1 年中 m 次分期支付的累积，再加上因当前年死亡而期望被消去的值后，等于 $y+1$ 岁时年金的精算现值。

公式 (3.8.5) 是建立在死亡均匀分布基础上的，而公式 (3.8.6) 在假设

$$D_{x+t}=(1-t)D_x+tD_{x+1} \qquad 0\le t\le 1$$

之下也有类似的解释。

§3.9 完全期末年金与比例期初年金

在连续年金场合，支付连续进行直至死亡，不存在调整最后支付问题，但对于离散年金，尤其是按年支付的年金，会提出根据死亡日期按比例调整的问题。譬如期末按年提供支付 5000 的

生存年金，当年金领取者在支付日前 1 个月死去时，可按从上次领取日算起活着的 11 个月比例加付最后不满 5000 的零头数额。又譬如，以按年支付保费 1000 的年金方式购买的人寿保险，当被保险人在周年缴费日 1 个月后死去时，可退还该年度剩下 11 个月的已缴保费。

每年 1 单位按 $1/m$ 年期末支付，再加上根据从上个 $1/m$ 年期末到死亡日这段时间调整的零数支付，这种完全生存年金的精算现值记为 $\mathring{a}_x^{(m)}$。因 $1/m$ 年期末支付 $1/m$ 等价于按年支付额

$$\frac{1}{m}\frac{1}{\bar{s}_{\overline{1/m}|}}$$

在 $1/m$ 年内连续支付，在 $1/m$ 年内的时刻 $t, 0<t<1/m$，连续支付的积累值就是

$$\frac{1}{m}\frac{\bar{s}_{\overline{t}|}}{\bar{s}_{\overline{1/m}|}}, \tag{3.9.1}$$

它可作为死亡发生在时刻 t 情况下的调整支付额。根据这一调整支付定义，完全期末生存年金恰好等价于年支付额为

$$\frac{1}{m}\frac{1}{\bar{s}_{\overline{1/m}|}}=\frac{1}{m}\frac{\delta}{(1+i)^{1/m}-1}=\frac{\delta}{i^{(m)}} \tag{3.9.2}$$

的连续生存年金。

这一点可以这样来看，在任何一个 $1/m$ 年中，如果 (x) 始终活着，那么连续生存年金提供的支付在该 $1/m$ 年末的值为

$$\frac{1}{m\bar{s}_{\overline{1/m}|}}\bar{s}_{\overline{1/m}|}=\frac{1}{m};$$

如果 (x) 在该 $1/m$ 年内死亡，那么连续生存年金提供的支付在死亡时等价于

$$\frac{1}{m\bar{s}_{\overline{1/m}|}}\bar{s}_{\overline{t}|},$$

与完全年金的调整支付相当。因此我们有

$$\mathring{a}_x^{(m)} = \frac{\delta}{i^{(m)}} \overline{a}_x, \tag{3.9.3}$$

它与复利理论中的关系

$$\mathring{a}_{\overline{n}|}^{(m)} = \frac{\delta}{i^{(m)}} \overline{a}_{\overline{n}|}$$

相似。

实践中调整支付额可取 (3.9.1) 的近似值 t, 然而包含利息因素的 (3.9.1) 导致更为简单的理论，例如此时有

$$1 - i^{(m)} \mathring{a}_x^{(m)} = 1 - \delta \overline{a}_x = \overline{A}_x,$$

$$1 = i^{(m)} \mathring{a}_x^{(m)} + \overline{A}_x. \tag{3.9.4}$$

对 (3.9.4) 的解释是，1 单位投资在 (x) 活着时每个 $1/m$ 年期末产生利息 $i^{(m)}/m$, 加上一个死亡时 $1/m$ 年内的利息调整

$$i^{(m)} \frac{\overline{s}_{\overline{t}|}}{m \overline{s}_{\overline{1/m}|}} = \delta \overline{s}_{\overline{t}|} = (1+i)^t - 1,$$

再加上死亡时偿还的 1 单位。

下面对期初年金考察平行的理论。每年 1 单位按 $1/m$ 年期初支付，并根据从死亡日到下个 $1/m$ 年期初这段时间长度退还付款者部分已付款，这种比例期初生存年金的精算现值记为 $\ddot{a}_x^{\{m\}}$ 。因为 $1/m$ 年期初支付 $1/m$ 等价于按年支付额

$$\frac{1}{m} \frac{1}{\overline{a}_{\overline{1/m}|}}$$

在 $1/m$ 年内连续支付，所以当死亡发生在时刻 t 时，$0 < t < 1/m$, 该 $1/m$ 年的退款取为

$$\frac{1}{m} \frac{\overline{a}_{\overline{1/m-t}|}}{\overline{a}_{\overline{1/m}|}}. \tag{3.9.5}$$

按这样定义的退款，比例期初生存年金恰好等价于年支付额为

$$\frac{1}{m}\frac{1}{\bar{a}_{\overline{1/m}|}}=\frac{1}{m}\frac{\delta}{1-v^{1/m}}=\frac{\delta}{d^{(m)}} \tag{3.9.6}$$

的连续生存年金，即

$$\ddot{a}_x^{\{m\}}=\frac{\delta}{d^{(m)}}\bar{a}_x. \tag{3.9.7}$$

而且，

$$1-d^{(m)}\ddot{a}_x^{\{m\}}=1-\delta\bar{a}_x=\overline{A}_x$$

或

$$1=d^{(m)}\ddot{a}_x^{\{m\}}+\overline{A}_x, \tag{3.9.8}$$

请读者对上式作出解释。

例 3.9.1: 建立以下公式：

(1) $\mathring{a}_{x:\overline{n}|}^{(m)}=\frac{\delta}{i^{(m)}}\bar{a}_{x:\overline{n}|}$.　　(2) $\ddot{a}_{x:\overline{n}|}^{\{m\}}=\frac{\delta}{d^{(m)}}\bar{a}_{x:\overline{n}|}$.

(3) $\ddot{a}_{x:\overline{n}|}^{\{m\}}=(1+i)^{1/m}\,\mathring{a}_{x:\overline{n}|}^{(m)}$.

解: (1)

$$\mathring{a}_{x:\overline{n}|}^{(m)}=\mathring{a}_x^{(m)}-{}_nE_x\,\mathring{a}_{x+n}^{(m)}\ \frac{\delta}{i^{(m)}}(\bar{a}_x-{}_nE_x\bar{a}_{x+n})=\frac{\delta}{i^{(m)}}\bar{a}_{x:\overline{n}|}.$$

(2)

$$\ddot{a}_{x:\overline{n}|}^{\{m\}}=\ddot{a}_x^{\{m\}}-{}_n\,E_x\ddot{a}_{x+n}^{\{m\}}=\frac{\delta}{d^{\{m\}}}(\bar{a}_x-{}_nE_x\bar{a}_{x+n})=\frac{\delta}{d^{(m)}}\bar{a}_{x:\overline{n}|}.$$

(3) $$\ddot{a}_{x:\overline{n}|}^{\{m\}}=\frac{\delta}{d^{(m)}}\bar{a}_{x:\overline{n}|}=\frac{\delta}{v^{1/m}i^{(m)}}\bar{a}_{x:\overline{n}|}=(1+i)^{1/m}\,\mathring{a}_{x:\overline{n}|}^{(m)}.$$

以上 (3) 的解释是，$\ddot{a}_{x:\overline{n}|}^{\{m\}}$ 与 $\mathring{a}_{x:\overline{n}|}^{(m)}$ 同样提供年支付 1, 分 m 次支付并按死亡日期调整，只不过 $\ddot{a}_{x:\overline{n}|}^{\{m\}}$ 在期初支付而 $\mathring{a}_{x:\overline{n}|}^{(m)}$ 在期末支付罢了。

由 (3.9.3),(1.4.2) 以及 $\lim_{\delta\to 0}(\delta/i)=1$, 可得出当 $\delta=0$ 时,

$$\mathring{a}_x=\overline{a}_x=\mathring{e}_x=\overline{a}_{\overline{\mathring{e}_x}|}.$$

在这一特殊情形中, x 岁终身年金的精算现值等于期限为期望剩余寿命的确定性年金之现值。很多人误认为这一相等关系在 $\delta>0$ 时也成立, 以下例子就显示并非如此。

例 3.9.2: 对于 $\delta>0$, 证明

(1) $\overline{a}_x<\overline{a}_{\overline{\mathring{e}_x}|}$.　　　　(2) $a_x<a_{\overline{e_x}|}$　　$x<\omega-1$.

解: (1) 注意到当 $\delta>0$ 时,

$$\frac{d^2}{dt^2}\overline{a}_{\overline{t}|}=-\delta e^{-\delta t}<0.$$

根据 Jensen 不等式 (参见《风险理论》第一章) 以及 (3.3.1), (1.5.1), 有

$$\overline{a}_x=E[\overline{a}_{\overline{T}|}]<\overline{a}_{\overline{E[T]}|}=\overline{a}_{\overline{\mathring{e}_x}|}.$$

离散生存年金基于整数值剩余寿命变量 K 以及函数 $a_{\overline{K}|}$, 而 Jensen 不等式的证明与概率分布的形式无关(不管是连续型还是离散型或者混合型) 。现在函数

$$\begin{aligned}a_{\overline{t}|}&=(1-e^{-\delta t})/i,\\ \frac{d^2}{dt^2}a_{\overline{t}|}&=-\frac{\delta^2}{i}v^t<0\qquad\delta>0.\end{aligned}$$

于是根据 Jensen 不等式,

$$a_x=E[a_{\overline{K}|}]\leq a_{\overline{E[K]}|}=a_{\overline{e_x}|}.$$

除非 K 为常数, 以上不等式是严格的, 譬如当 $x=\omega-1$ 时, K 为 0, 此时不等式成为等式。当 $x<\omega-1$ 且 $\delta>0$ 时, 有 $a_x<a_{\overline{e_x}|}$ 。

习 题

§3.2

1. 用附录 2 示例生命表及有效年利率 6%, 计算现龄 50 岁人在 20 年后活着时应付额 1000 的精算现值。

2. 在上题条件下，计算 50 岁时 1000 到 70 岁时的精算积累值。

3. 证明并解释以下关系式

$$
{}_nE_x + {}_nE_x[(1+i)^n - 1] + {}_nE_x(1+i)^n \frac{l_x - l_{x+n}}{l_{x+n}} = 1.
$$

§3.3

4. 在每一年龄死亡均匀分布的假设下，用示例生命表及有效年利率 6% 计算

(1) $\bar{a}_{20}, \bar{a}_{50}, \bar{a}_{80}$.

(2) x=20,50,80 时的 $\mathrm{Var}[\bar{a}_{\overline{T}|}]$. [提示：用 (3.3 .6) 及 (2.3.2)]

5. 用第 4 题得到的值计算下列现值随机变量的标准差与变差系数 σ/μ 。

(1) 在 20 岁，50 岁，80 岁生效的个人终身生存年金，每年数额 1000 连续支付。

(2) 一组生存年金共 100 份，每份都在 50 岁生效，年金额 1000 连续支付。

6. 证明 $\mathrm{Var}[\bar{a}_{\overline{T}|}]$ 可表示成

$$
\frac{2}{\delta}(\bar{a}_x - {}^2\bar{a}_x) - (\bar{a}_x)^2,
$$

其中 ${}^2\bar{a}_x$ 基于利息效力 2δ 。

7. 计算 $\mathrm{Cov}[\delta\bar{a}_{\overline{T}|}, v^T]$ 。

8. 如采用决定性 (比率函数) 观点，连续生存年金可从 (3.3.26) 出发：

$$\frac{d\bar{a}_y}{dy} = (\mu_y + \delta)\bar{a}_y - 1 \qquad x \le y < \omega,$$

$$\bar{a}_y = 0 \qquad \omega \le y.$$

(1) 用积分因子 $\exp[-\int_0^y (\mu_z + \delta) dz]$ 解以上微分方程式得出 (3 .2.1) 。

(2) 用积分因子 $e^{-\delta y}$ 得出方程

$$\bar{a}_x = \bar{a}_{\overline{\omega - x}|} - \int_x^\omega e^{-\delta(y-x)} \bar{a}_y \mu_y dy$$

并作一个解释。

§3.4

9. 定义 ${}_{m|n}\ddot{a}_x$ 并写出与 (3.3.21)~(3.3.24) 相似的有关 ${}_{m|n}\ddot{a}_x$ 公式。

10. 证明

$$\text{Var}[a_{\overline{K}|}] = \text{Var}[\ddot{a}_{\overline{K+1}|}] = \frac{1}{d^2}\text{Var}[v^{K+1}].$$

11. 证明并解释以下关系

(1) $a_{x:\overline{n}|} = {}_1E_x \ddot{a}_{x+1:\overline{n}|}$.

(2) ${}_{n|}a_x = \frac{A_{x:\overline{n}|} - A_x}{d} - {}_nE_x$.

12. 从 (3.4.11) 出发导出 (3.4.32) 。

13. 公式

$$1 = i a_{x:\overline{n}|} + (1+i) A_{x:\overline{n}|}$$

是否正确？如不正确，请予以纠正。

§3.5

14. 考虑

$$\ddot{a}_x^{(m)} = E[\ddot{a}^{(m)}_{\overline{K+J_m}|}],$$

其中 K 是 (x) 的整值剩余寿命，

$$J_m = \frac{j+1}{m} \qquad 当\frac{j}{m} < S \leq \frac{j+1}{m}, \qquad j = 0, 1, \cdots, m-1,$$

$S = T - K$, 用第二章习题 15 证明

(1) $1 = d^{(m)}\ddot{a}_x^{(m)} + A_x^{(m)}$.

(2) 除 $S = (j+1)/m$ 情况 (概率为 0) 外，$J_m = [mS+1]/m$. 这里方括号 [ ] 表示最大整数部分。

15. 在每个年龄中假定

$$D_{y+h/m} = D_y - \frac{h}{m}(D_y - D_{y+1}) \quad h = 0, 1, \cdots, m-1,$$

验证

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}.$$

16. 写出 (3.5.15) 及 (3.5.16) 的传统近似公式，并用 (3.5.8) 予以验证。

17. 证明与 (3.5.3) 相似的期末年金公式

$$a_x^{(m)} = s_{\overline{1}|}^{(m)} a_x + \frac{1}{i^{(m)}}[(1+i)A_x - (1+\frac{i^{(m)}}{m})A_x^{(m)}],$$

并在每一个年龄死亡均匀分布的假设下，得出

$$a_x^{(m)} = s_{\overline{1}|}^{(m)} a_x + (1+i)\frac{1-\ddot{a}_{\overline{1}|}^{(m)}}{i^{(m)}} A_x.$$

18. 证明与 (3.5.4) 相似的期末年金公式

$$a_x^{(m)} = \frac{1-(1+i^{(m)}/m)A_x^{(m)}}{i^{(m)}} = a_{\overline{\infty}|}^{(m)} - \ddot{a}_{\overline{\infty}|}^{(m)} A_x^{(m)}.$$

并在每个年龄死亡均匀分布的假设下，得出

$$a_x^{(m)} = \alpha(m)a_x + \gamma(m),$$

其中 $\gamma(m) = (1 - \ddot{a}_{\overline{1|}}^{(m)})/i^{(m)}$.

19. 在每个年龄死亡均匀分布的假设下，

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m),$$

$$a_x^{(m)} = \alpha(m)a_x - \gamma(m),$$

又根据

$$\ddot{a}_x^{(m)} - a_x^{(m)} = \frac{1}{m},$$
$$\ddot{a}_x - a_x = 1,$$

可得出

$$\gamma(m) = \alpha(m) - \beta(m) - \frac{1}{m}.$$

请直接用 $\alpha(m)$, $\beta(m)$ 的表达式验证上述关系。

20.

(1) 从 (3.5.1) 出发验证

$$\lim_{m\to\infty} \ddot{a}_x^{(m)} = \overline{a}_x.$$

(2) 用 (3.5.8) 以及以上 (1) 中结果证明

$$\overline{a} \cong a_x + \frac{1}{2}.$$

(3) 从 (3.3.2B) 出发，对积分用梯形规则近似，也得出 (2) 中结果。

21. 用 (3.5.8) 给出的传统近似公式，建立

(1) $a_x^{(m)} \cong a_x + \frac{m-1}{2m}$.

(2) $a_{x:\overline{n|}}^{(m)} \cong a_{x:\overline{n|}} + \frac{m-1}{2m}(1 - {}_nE_x)$.

(3) ${}_{n|}a_x^{(m)} \cong {}_{n|}a_x + \frac{m-1}{2m}\,{}_nE_x$.

22.

(1) 建立用 $\ddot{s}_{25:\overline{40|}}$ 表示 $\ddot{s}_{25:\overline{40|}}^{(m)}$ 的公式。

(2) 根据附录中示例生命表，按年利率 6% 计算：① $\ddot{a}^{(12)}_{25:\overline{40|}}$.
② $\ddot{s}^{(12)}_{25:\overline{40|}}$.

§3.6

23. 给出用计算基数表示的公式：

(1) $\ddot{a}_x$.　　(2) a_x.　　(3) $\ddot{a}_{x:\overline{n|}}$.

(4) $a_{x:\overline{n|}}$.　　(5) ${}_{n|}\ddot{a}_x$.　　(6) ${}_{n|}a_x$

(7) $\ddot{a}^{(m)}_{x:\overline{n|}}$.　　(8) ${}_{n|}\ddot{a}^{(m)}_x$.

24. 为估价岁入为 b 的期初年金，可使用特殊计算基数

$$\tilde{N}^{(m)}_x = \alpha(m)N_{x+1} - \gamma(m)D_x = \alpha(m)N_x - [\beta(m) + \frac{1}{m}]D_x$$

及一般公式

$$\frac{b}{D_x}[\tilde{N}^{(m)}_y - \tilde{N}^{(m)}_z].$$

用 $\tilde{N}^{(12)}_x$ 写出下列三种情况的公式

(1) $a^{(12)}_{60}$.　　(2) $a^{(12)}_{40:\overline{25|}}$.　　(3) ${}_{30|}a^{(12)}_{40}$.

§3.7

25. 考虑 (x) 的标准递增定期生存年金：第 1 年岁入 1, 第 2 年岁入 2, 以此类推到第 n 年岁入 n 为止，且每年分 m 次期初支付，其精算现值记作 $(I\ddot{a})^{(m)}_{x:\overline{n|}}$, 证明 $(I\ddot{a})^{(m)}_{x:\overline{n|}}$ 可按以下方式表示：

(1) $\sum_{k=0}^{n-1} {}_{k|n-k}\ddot{a}^{(m)}_x$.

(2) $\frac{1}{D_x}\sum_{k=0}^{n-1}(k+1)D^{(m)}_{x+k}$.

(3) $\frac{1}{D_x}\sum_{k=0}^{n-1}(N^{(m)}_{x+k} - N^{(m)}_{x+n})$.

(4) $\frac{1}{D_x}(S^{(m)}_x - S^{(m)}_{x+n} - nN^{(m)}_{x+n})$.

26. 考虑 (x) 的递减定期生存年金：第 1 年岁入 n, 第 2 年 $n-1$, 以此类推到第 n 年岁入 1 为止，且每年分 m 次期初支付，其精算现值计作 $(D\ddot{a})^{(m)}_{x:\overline{n|}}$. 证明 $(D\ddot{a})^{(m)}_{x:\overline{n|}}$ 可按以下方式表示：

(1) $\sum_{k=1}^{n} \ddot{a}_{x:\overline{k|}}^{(m)}$.

(2) $\frac{1}{D_x} \sum_{k=0}^{n-1} (n-k) D_{x+k}^{(m)}$.

(3) $\frac{1}{D_x} \sum_{k=1}^{n} (N_x^{(m)} - N_{x+k}^{(m)})$.

(4) $\frac{1}{D_x} [n N_x^{(m)} - (S_{x+1}^{(m)} - S_{x+n+1}^{(m)})]$.

27. 在习题 25 中，如果岁入并不在 $x+n$ 岁终止，而是当 (x) 继续活着时保持定额岁入 n，这种生存年金的精算现值记为 $(I_{\overline{n|}}\ddot{a})_x^{(m)}$，证明 $(I_{\overline{n|}}\ddot{a})_x^{(m)}$ 的以下表达式成立：

1) $\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_x^{(m)}$.

(2) $\frac{1}{D_x} [\sum_{k=0}^{n-1} (k+1) D_{x+k}^{(m)} + n N_{x+n}^{(m)}]$.

(3) $\frac{1}{D_x} \sum_{k=0}^{n-1} N_{x+k}^{(m)}$.

(4) $\frac{1}{D_x} (S_x^{(m)} - S_{x+n}^{(m)})$.

28. 验证公式

$$\delta(\bar{I}\bar{a})_{\overline{T|}} + T v^T = \bar{a}_{\overline{T|}},$$

其中 T 是 (x) 的剩余寿命。用这个公式证明

$$\delta(\bar{I}\bar{a})_x + (\bar{I}\bar{A})_x = \bar{a}_x,$$

这里 $(\bar{I}\bar{a})_x$ 是 t 年时支付率为 t 连续支付终生年金的精算现值。

§3.8

29. (1) 证明当 $m=1$ 时，公式 (3.8.6) 成为

$$\ddot{a}_y(1+i) - (1+i) + q_y \ddot{a}_{y+1} = \ddot{a}_{y+1}.$$

(2) 将 (1) 中公式用期末年金值来表示，得出

① $a_y(1+i) + q_y(1+a_{y+1}) = 1 + a_{y+1}$;

② $1 = ia_y + q_y(1 + a_{y+1}) + a_y - a_{y+1}$，
并给出解释。

30. 对给定的 n 与 x, 证明以下递归公式对 $h = 0, 1, \cdots, n-1$ 成立：

(1) $(D\ddot{a})^{(12)}_{x+h:\overline{n-h}|} = (n-h)\ddot{a}^{(12)}_{x+h:\overline{1}|} + {}_1E_{x+h}(D\ddot{a})^{(12)}_{x+h+1:\overline{n-h-1}|}$.

(2) $(apv)_{x+h} = (h+1)\ddot{a}^{(12)}_{x+h:\overline{1}|} + {}_1E_{x+h}(apv)_{x+h+1}$，其中 $(apv)_{x+h} = h\ddot{a}^{(12)}_{x+h:\overline{n-h}|} + (I\ddot{a})^{(12)}_{x+h:\overline{n-h}|}$.

§3.9

31. (1) 在用综合支付技巧估计 $\mathring{a}_x$ 时，说明现值随机变量为

$$a_{\overline{K}|} + \frac{v^T \bar{s}_{\overline{T-K}|}}{\bar{s}_{\overline{1}|}} = a_{\overline{K}|} + v^T \frac{\delta}{i} \bar{s}_{\overline{T-K}|},$$

其中 K, T 分别是 (x) 的整值与完全剩余寿命，并说明可约化成

$$\frac{1 - v^T}{i} = a_{\overline{T}|}.$$

(2) 证明

$$\mathring{a}_x = E\left[\frac{1-v^T}{i}\right] = \frac{\delta}{i}\bar{a}_x,$$

并得出

$$\mathrm{Var}\left[\frac{1-v^T}{i}\right]$$

的一个表达式。

32. (1) 说明 $\ddot{a}^{\{1\}}_x$ 的现值随机变量为

$$\ddot{a}_{\overline{K+1}|} - v^T \frac{\delta}{d} \bar{a}_{\overline{K+1-T}|},$$

并可约化成

$$\frac{1-v^T}{d} = \ddot{a}_{\overline{T}|}.$$

(2) 证明

$$\ddot{a}_x^{\{1\}} = \frac{\delta}{d}\overline{a}_x.$$

综合题

33. 在每一年龄的死亡均匀分布假设下，对 $0 \le t \le 1$ 证明

(1) $\ddot{a}_{x+t} = \frac{(1+it)\ddot{a}_x - t(1+i)}{1-tq_x}$.

(2) ${}_{t|}\ddot{a}_x = v^t[(1+it)\ddot{a}_x - t(1+i)]$.

(3) ${}_{1-t|}\ddot{a}_{x+t} = \frac{(1+i)^2}{1-tq_x}(\ddot{a}_x - 1)$.

(4) $A_{x+t} = \frac{1+it}{1-tq_x}A_x - \frac{tq_x}{1-tq_x}$.

34. 得出下列 (x) 的期初生存年金求值公式，初始支付额为 1, 此后每年递增额为

(1) 初始年支付额的 3%.

(2) 前一年支付额的 3%.

35. 用积分表示 $(\overline{D}\overline{a})_{x:\overline{n}|}$ 并证明

$$\frac{\partial}{\partial n}(\overline{D}\overline{a})_{x:\overline{n}|} = \overline{a}_{x:\overline{n}|}.$$

36. 给出下列按月支付的生存年金在 70 岁时的精算现值：

从 30 到 40 岁每月底 100;　　从 40 到 50 岁每月底 200;

从 50 到 60 岁每月底 500;　　从 60 到 70 岁每月底 1000.

37. 对于 (35) 的死亡即刻赔付的 25 年定期寿险，受益额在 $35+t$ 岁死亡情况下为 $\bar{s}_{\overline{t}|}, 0 \le t \le 25$, 导出净趸缴保费的简化表达式，并解释所得结果。

38. 对于 (x) 的死亡年末赔付的 n 年定期寿险，受益额在第 $k+1$ 年死亡的情况下为 $\ddot{s}_{\overline{k+1}|}, 0 \le k < n$, 导出净趸缴保费的简化表达式，并解释所得结果。

39. 导出下式的简化表达式：

$$(I\ddot{a})^{(12)}_{x:\overline{25}|} - (Ia)^{(12)}_{x\overline{25}|}.$$

40. 将延期 n 年的年支付额为 1 的连续生存年金看作理赔概率为 ${}_nP_x$ 并具有随机理赔额 $v^n\overline{a}_{\overline{T}|}$ 的保险，这里 T 的概率密度函数为 ${}_tp_{x+n}\mu_{x+n+t}$. 运用《风险理论》中式 (2.2.13) 证明该保险的方差等于

$$v^{2n}{}_np_x(1-{}_np_x)\overline{a}_{x+n}^2+v^{2n}{}_np_x\frac{{}^2\overline{A}_{x+n}-\overline{A}_{x+n}^2}{\delta^2},$$

并验证它可以化成 (3.3.20).

41. 写出习题 40 中方差公式在离散情形的类似公式。

42. 设 I_k 是指示随机变量， $Pr(I_k=1)={}_kp_x, Pr(I_k=0)={}_kq_x$. 验证：

(1) 在 (x) 活到 $x+k$ 岁时年支付 $b_k(k=0,1,2,\cdots)$ 的生存年金的精算现值可写成

$$E[\sum_{k=0}^{\infty}v^kb_kI_k].$$

(2) $E[I_jI_k]={}_kp_x \qquad j\leq k.$

$\mathrm{Cov}[I_j,I_k]={}_kp_x{}_jq_x \qquad j\leq k.$

(3) $$\mathrm{Var}[\sum_{k=0}^{\infty}v^kb_kI_k]=\sum_{k=0}^{\infty}v^{2k}b_k^2{}_kp_x{}_kq_x+2\sum_{k=0}^{\infty}\sum_{j<k}v^{j+k}b_jb_k{}_hp_x{}_jq_x.$$

43. 用左上标 2 表示利息效力为 2δ, 证明

(1) ${}^2A_x=1-(2d-d^2)^2\ddot{a}_x.$

(2) $\mathrm{Var}[v^{K+1}]=2d(\ddot{a}_x-{}^2\ddot{a}_x)-d^2(\ddot{a}_x^2-{}^2\ddot{a}_x).$

44.

(1) 将年金系数 $\alpha(m),\beta(m)$ 展开成 δ 的幂级数 (只求三项) 。

(2) 当 $m=\infty$ 时以上所得展开式变成什么？

45. 设 $g(x)$ 是非负数， X 是概率密度函数为 $f(x)$ 的随机变量，验证不等式

$$E[g(X)]=\int_{-\infty}^{\infty}g(x)f(x)dx\geq kPr[g(X)\geq k] \qquad k>0,$$

并利用它证明

$$\bar{a}_x \geq \bar{a}_{\overline{\mathring{e}_x}|} Pr[\bar{a}_{\overline{T}|} \geq \bar{a}_{\overline{\mathring{e}_x}|}] = \bar{a}_{\overline{\mathring{e}_x}|} Pr[T \geq \mathring{e}_x].$$

46. 1 单位金额用于购买某种受益组合，包括当 (x) 活着时每年 I 的生存收入及 (x) 死亡时即刻支付 J 的保险。写出这种组合的现值随机变量并给出其均值与方差。

47. 根据每一年死亡均匀分布假设，按示例生命表及实质年利率 6% 计算

(1) $\ddot{a}_{40}^{(12)}$.　　(2) $\ddot{a}_{40:\overline{30}|}^{(12)}$.　　(3) ${}_{30|}\ddot{a}_{40}^{(12)}$.

48. 当 $q_x < (i^{(2)}/2)^2$ 时，验证传统近似

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$$

在 $n = 1$ 及 $m = 2$ 的特殊情形导致

$$\ddot{a}_{x:\overline{1}|}^{(2)} > \ddot{a}_{\overline{1}|}^{(2)}.$$

49. 设精算现值 $A''_{x:\overline{m}|}$ 与 $\ddot{a}''_{x:\overline{m}|}$ 是根据以下假定计算的：

在前 n 年中利率为 $i, n < m$; 在剩余 $m - n$ 年中利率为 i'. 用代数方法证明并解释

(1) $A''_{x:\overline{m}|} = 1 - d\ddot{a}_{x:\overline{n}|} - v^n {}_np_x d' \ddot{a}'_{x+n:\overline{m-n}|}$.

(2) $A''_{x:\overline{m}|} = 1 - d'\ddot{a}''_{x:\overline{m}|} + (d' - d)\ddot{a}_{x:\overline{n}|}$.

50. 验证

$$\frac{d\ddot{a}_x}{di} = -v(Ia)_x, \quad \text{其中} \quad (Ia)_x = \sum_{t=1}^{\infty} tv^t {}_tp_x.$$

解释以上关系式。

51. 如果例 3.7.1 中的年金支付在

(1) 10 次递增

(2) 20 次递增

之后保持固定，求年金的精算现值。

52. 验证死亡效力增加一个常数对 $\ddot{a}_x$ 的影响与利息效力增加一个常数的影响相同，但对按照 $\alpha(m)\ddot{a}_x-\beta(m)$ 计算的 $\ddot{a}_x^{(m)}$ 则不然。

53. 验证：

(1) $\alpha(m)-\beta(m)d=\ddot{a}_{\overline{1}|}^{(m)}$.

(2) 递归公式 (3.8.3) 可写成

$$\ddot{a}_y^{(m)}=\alpha(m)-\beta(m)(1-vp_y)+vp_y\ddot{a}_{y+1}^{(m)}.$$

54. 考虑从退休基金资产中支付的以下期初年金组合：

年龄	年金领取人数
65	30
75	20
85	10

只要年金领取人活着，每个年金的年支付额都是 1 。设利率为 6%，死亡率由示例生命表给出。对于退休基金的这些负担现值，计算

(1) 期望值。

(2) 方差。

(3) 分布的 95% 分位数。

对于第 (2),(3) 小题，假定各生命相互独立。

第四章　净保费

§4.1　引　言

前面两章分别讨论了各种人寿保险与生存年金的精算现值，由于实践中人寿保险通常以分期付费的毛保费的生存年金方式购买，这一章将结合两者讨论。毛保费除了提供人寿保险的受益以外，还提供保险的起始与维持费用，提供利润及抵消可能的不利经验的保障。这一章讨论只提供受益支付的净年缴保费，这种净年缴保费形成从保险合同成立时开始的生存年金。

以下例 4.1.1 说明两种决定保费原则的应用，其中之一根据保险受益的期望现值决定保费。

例 4.1.1: 某保险公司打算发行一种 0 岁人的寿险保单，0 岁人的整值剩余寿命 K 的概率函数为

$$_{k|}q_0 = \frac{1}{4} \qquad k = 0, 1, 2, 3.$$

该保单在被保险人死亡年末支付 1, 保费 P 在被保险人活着时每年初缴付。设年利率 $i = 6\%$, 根据下列两个原则分别决定年缴保费 P.

(1) 原则 I： P 使得在保单签发时亏损现值的期望值为 0.

(2) 原则 II： P 是使得亏损为正的概率不超过 1/4 的最低额。

解: 对于 $K = k$ 及任何保费 P, 在保单签发时的亏损现值为

$$v^{k+1} - P\ddot{a}_{\overline{k+1|}}.$$

(1) 根据原则 I， P 应该满足

$$\sum_{k=0}^{3}(v^{k+1}-P\ddot{a}_{\overline{k+1}|})Pr[K=k]=0, \tag{4.1.1}$$

据此可得， $P=0.3667$.

(2) 鉴于 $v^j-P\ddot{a}_{\overline{j}|}$ 随着 j 增加而递减，原则 II 当 P 满足 $v^2-P\ddot{a}_{\overline{2}|}=0$ 时得以成立，届时只有当 $K=0$ 时亏损才是正的。于是根据这个原则， $P=0.4580$.

以上结果概括如下：

结果	概率	亏损现值		
		一般公式	保费原则	
k	$_{k\|}q_0$		I	II
0	1/4	$v-P\ddot{a}_{\overline{1}\|}$	0.5767	0.4854
1	1/4	$v^2-P\ddot{a}_{\overline{2}\|}$	0.1774	0.0000
2	1/4	$v^3-P\ddot{a}_{\overline{3}\|}$	-0.1993	-0.4580
3	1/4	$v^4-P\ddot{a}_{\overline{4}\|}$	-0.5547	-0.8900
		保费为	0.3667	0.4580

我们将根据称为*平衡原理*(equivalence principle) 的原则 I 来决定净保费。保险人的 (潜在) 亏损 L 定义为，受益额现值随机变量与保费 (年金) 现值随机变量之差。平衡原理可表示成

$$E[L]=0.$$

据此得出的保费称为净保费。等式等价于

$$E[\text{受益现值}]=E[\text{ 净保费现值}],$$

也就是说，净保费的精算现值等于受益的精算现值。第二章与第三章建立起来的计算这些精算现值的方法可用来化简以上等式，

并从而解出这里的净保费。譬如像例 4.1.1 当受益及保费是常数时，方程 (4.1.1) 可改写成 $A_0 = P\ddot{a}_0$, 而 $\ddot{a}_0$ 可以这样计算：

$$\sum_{k=0}^{3} v^k {}_kp_0.$$

当平衡原理用于决定一次性缴付的人寿保险或生存年金的保费时，所得出的净保费就是支付的精算现值，即净趸缴保费。

§4.2 完全连续保费

首先考虑在 (x) 死亡即刻赔付 1 单位的终身人寿保险，将阐明如何根据平衡原理决定完全连续净均衡年保费 $\overline{P}$ 。死亡发生在时间 t(从保单签发算起) 情况下保险人的亏损现值为

$$l(t) = v^t - \overline{P}\bar{a}_{\overline{t}|}. \tag{4.2.1}$$

由 $\bar{a}_{\overline{t}|} = (1-v^t)/\delta$ 可见， $l(0) = 1$, $l(t)$ 是 t 的递减函数，当 $t \to \infty$ 时 $l(t)$ 趋于 $-\overline{P}/\delta$ 。如 t_0 使得 $l(t_0) = 0$, 则在 t_0 之前死亡导致损失，而在 t_0 之后死亡则产生得益 (负损失) 。

现考虑亏损随机变量

$$L = l(T) = v^T - \overline{P}\bar{a}_{\overline{T}|}. \tag{4.2.2}$$

如果保险人根据平衡原理决定净保费，那么

$$E[L] = 0, \tag{4.2.3}$$

由此得出的净保费记作 $\overline{P}(\overline{A}_x)$ 。根据 (2.2.6) 与 (3 .3.2) 可得

$$\overline{A}_x - \overline{P}(\overline{A}_x)\bar{a}_x = 0$$

或者

$$\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\bar{a}_x}. \tag{4.2.4}$$

L 的方差可用来衡量亏损的变易程度，因 $E[L]=0$, 可知

$$\mathrm{Var}[L]=E[L^2]. \tag{4.2.5}$$

对于 (4.2.2) 中的亏损变量，

$$\begin{aligned}
\mathrm{Var}[v^T-\overline{P}\overline{a}_{\overline{T}|}] &= \mathrm{Var}[v^T(1+\frac{\overline{P}}{\delta})-\frac{\overline{P}}{\delta}] \\
&= \mathrm{Var}[v^T(1+\frac{\overline{P}}{\delta})] \\
&= \mathrm{Var}[v^T](1+\frac{\overline{P}}{\delta})^2 \\
&= [{}^2\overline{A}_x-(\overline{A}_x)^2](1+\frac{\overline{P}}{\delta})^2. \qquad (4.2.6)
\end{aligned}$$

利用 (4.2.4) 及 $\delta\overline{a}_x+\overline{A}_x=1$[见 (3.3.4)], 可将 (4 .2.6) 改写成

$$\mathrm{Var}[L]=\frac{{}^2\overline{A}_x-(\overline{A}_x)^2}{(\delta\overline{a}_x)^2}. \tag{4.2.7}$$

例 4.2.1: 按例 3.3.1 条件计算 $\overline{P}(\overline{A}_x)$ 与 $\mathrm{Var}[L]$ 。

解: 按例 3.3.1 中条件 $\mu=0.04$ 与 $\delta=0.06$ 可算得 $\overline{a}_x=10$, $\overline{A}_x=0.4$, ${}^2\overline{A}_x=0.25$, 于是

$$\overline{P}(\overline{A}_x)=\frac{\overline{A}_x}{\overline{a}_x}=0.04,$$

$$\mathrm{Var}[L]=\frac{0.25-0.16}{(0.6)^2}=0.25.$$

参见 (4.2.6) 可知，式 (4.2.7) 中分子是趸缴保费情况下提供终身人寿保险的保险人亏损 $v^T-\overline{A}_x$ 的方差，例 4.2.1 中数值为 0.09, 在那里，提供年保费保险的亏损标准差是趸缴保费保险亏损标准差的 $\sqrt{0.25/0.09}=5/3$ 倍。由死亡时间随机性引起的净年保费现值的不确定性，增加了亏损的变易程度。

例 4.2.1 算得 $\overline{P}(\overline{A}_x)=0.04$, 正好等于死亡效力 (常数), 这决非巧合。在死亡效力为常数 μ 的情况下，根据 (2.2.6) 及 (3.3.2B) 可算出

$$\overline{A}_x=\int_0^{\infty}e^{-\delta t}\mu e^{-\mu t}dt=\frac{\mu}{\mu+\delta},$$

$$\overline{a}_x=\int_0^{\infty}e^{-\delta t}e^{-\mu t}dt=\frac{1}{\mu+\delta},$$

于是

$$\overline{P}(\overline{A}_x)=\frac{\mu(\mu+\delta)^{-1}}{(\mu+\delta)^{-1}}=\mu.$$

用平衡原理可以对各种完全连续人寿保险得出决定净年保费的公式。亏损可一般地表示成

$$b_Tv_T-\overline{P}Y=Z-\overline{P}Y, \tag{4.2.8}$$

其中 b_t 与 v_t 分别是与 (2.2.1) 相联系的受益函数与贴现函数，$\overline{P}$ 是完全连续净年保费的一般符号， Y 是连续年金现值随机变量 [例如 (3.3.12)] ,Z 按 (2.2.2) 定义。

应用平衡原理，有

$$E[b_Tv_T-\overline{P}Y]=0$$

或者

$$\overline{P}=\frac{E[b_Tv_T]}{E[Y]},$$

由此可得出表 4.2.1 中净年保费公式。

注意一下延期 n 年的生存年金，在这种情形下，

$$b_Tv_T=\begin{cases}0 & T\le n\\ \overline{a}_{\overline{T-n|}}v^n & T>n\end{cases},$$

于是

$$\begin{aligned} E[b_T v_T] &= {}_n p_x E[\bar{a}_{\overline{T-n}|} v^n | T > n] \\ &= v^n {}_n p_x \bar{a}_{x+n} = A_{x:\overline{n}|}^{\ \ 1} \bar{a}_{x+n}. \end{aligned}$$

在保险实践中，递延生存年金在递延期内往往也提供某种形式的受益，这类契约的一个例子将在例 4.7.1 中考察。

表 4.2.1　完全连续净年保费

种类	亏损成份 $b_T v_T$	$\overline{P}Y$ 中的 Y	保费公式 $\overline{P} = E[b_T v_T]/E[Y]$
终身人寿保险	$1v^T$	$\bar{a}_{\overline{T}\mid}$	$\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\bar{a}_x}$
n 年定期保险	$1v^T$	$\bar{a}_{\overline{T}\mid}$, $T \le n$	$\overline{P}(\overline{A}_{x:\overline{n}\mid}^{1}) = \frac{\overline{A}_{x:\overline{n}\mid}^{1}}{\bar{a}_{x:\overline{n}\mid}}$
	0	$\bar{a}_{\overline{n}\mid}, T > n$	
n 年期两全保险	$1v^T$	$\bar{a}_{\overline{T}\mid}$, $T \le n$	$\overline{P}(\overline{A}_{x:\overline{n}\mid}) = \frac{\overline{A}_{x:\overline{n}\mid}}{\bar{a}_{x:\overline{n}\mid}}$
	$1v^n$	$\bar{a}_{\overline{n}\mid}, T > n$	
h 年缴费终身	$1v^T$	$\bar{a}_{\overline{T}\mid}$, $T \le h$	${}_h\overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\bar{a}_{x:\overline{h}\mid}}$
人寿保险	$1v^T$	$\bar{a}_{\overline{h}\mid}, T > h$	
h 年缴费 n 年期	$1v^T$	$\bar{a}_{\overline{T}\mid}$, $T \le h$	${}_h\overline{P}(\overline{A}_{x:\overline{n}\mid}) = \frac{\overline{A}_{x:\overline{n}\mid}}{\bar{a}_{x:\overline{h}\mid}}$
两全保险	$1v^T$	$\bar{a}_{\overline{h}\mid}, h < T \le n$	
	$1v^n$	$\bar{a}_{\overline{h}\mid}, T > n$	
n 年期生存保险	0	$\bar{a}_{\overline{T}\mid}$, $T \le n$	$\overline{P}(A_{x:\overline{n}\mid}^{\ \ 1}) = \frac{A_{x:\overline{n}\mid}^{\ \ 1}}{\bar{a}_{x:\overline{n}\mid}}$
	$1v^n$	$\bar{a}_{\overline{n}\mid}, T > n$	
n 年递延终身	0	$\bar{a}_{\overline{T}\mid}$, $T \le n$	$\overline{P}({}_{n\mid}\bar{a}_x) = \frac{A_{x:\overline{n}\mid}^{\ \ 1}\bar{a}_{x+n}}{\bar{a}_{x:\overline{n}\mid}}$
生存年金	$\bar{a}_{\overline{T-n}\mid} v^n$	$\bar{a}_{\overline{n}\mid}, T > n$	

例 4.2.2: 对完全连续 n 年期两全保险的保险人亏损 L, 用净趸缴保费来表示 L 的方差 (参见表 4.2.1 第三行)。

解: 用 (2.2.8) 符号，有

$$\mathrm{Var}[L] = \mathrm{Var}\left[Z_3\Big(1 + \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\delta}\Big) - \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\delta}\right].$$

利用 (2.2.10) 得

$$\mathrm{Var}[L] = \left(1 + \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\delta}\right)^2 [{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2].$$

公式 (3.3.11) 可写成

$$(\delta\overline{a}_{x:\overline{n}|})^{-1} = 1 + \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\delta}.$$

由此得出

$$\mathrm{Var}[L] = \frac{{}^2\overline{A}_{x:\overline{n}|} - (\overline{A}_{x:\overline{n}|})^2}{(\delta\overline{a}_{x:\overline{n}|})^2}.$$

等式 (3.3.4.) 与 (3.3.11) 可用来导出连续净保费之间的关系，例如从 (3.3.4)

$$\delta\overline{a}_x + A_x = 1$$

出发，有

$$\delta + \overline{P}(\overline{A}_x) = \frac{1}{\overline{a}_x},$$

$$\begin{aligned} \overline{P}(\overline{A}_x) &= \frac{1}{\overline{a}_x} - \delta \\ &= \frac{1 - \delta\overline{a}_x}{\overline{a}_x} \\ &= \frac{\delta\overline{A}_x}{1 - \overline{A}_x}. \end{aligned} \tag{4.2.9}$$

从 (3.3.11)

$$\delta\overline{a}_{x:\overline{n}|} + \overline{A}_{x:\overline{n}|} = 1$$

出发，有

$$\delta + \overline{P}(\overline{A}_{x:\overline{n}|}) = \frac{1}{\overline{a}_{x:\overline{n}|}},$$

$$
\begin{aligned}
\overline{P}(\overline{A}_{x:\overline{n}|}) &= \frac{1}{\overline{a}_{x:\overline{n}|}} - \delta \qquad (4.2.10) \\
&= \frac{1-\delta\overline{a}_{x:\overline{n}|}}{\overline{a}_{x:\overline{n}|}} \\
&= \frac{\delta\overline{A}_{x:\overline{n}|}}{1-\overline{A}_{x:\overline{n}|}}.
\end{aligned}
$$

对 (4.2.9) 及 (4.2.10) 之离散形式的文字解释将在下一节例 4.3.4 中给出。

§4.3 完全离散保费

这一节考虑的保险受益金在保单有效期内死亡年末支付，而保费则从保单生效起按年在期初缴付，年缴保费构成期初生存年金。这个模型并不符合保险实践，但在精算理论发展史上有其意义。

在以上所述场合，对 (x) 的一单位终身人寿保险，其净均衡年缴保费记作 P_x，与前一节完全连续净年保费符号比较，省去 $(\overline{A})$ 的含义是保险赔付在死亡年末支付。这种保险的保险人亏损为

$$L = v^{K+1} - P_x\ddot{a}_{\overline{K+1}|}, \qquad K = 0,1,2,\cdots \tag{4.3.1}$$

其中 K 是 (x) 的整值剩余寿命。平衡原理要求 $E[L]=0$，或者

$$E[v^{K+1}] - P_xE[\ddot{a}_{\overline{K+1}|}] = 0.$$

于是得出

$$P_x = \frac{A_x}{\ddot{a}_x}, \tag{4.3.2}$$

这是 (4.2.4) 的离散形式。

按上节得出公式 (4.2.6) 同样的步骤，只是其中用 (3.4.6) 代替 (3.3.4) 所起的作用，可得

$$\mathrm{Var}[L] = \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2}. \tag{4.3.3}$$

例 4.3.1: 设

$$_{k|}q_x = c(0.96)^{k+1}, \qquad k = 0, 1, 2, \cdots$$

其中 c=0.04/0.96, 按 i=0.06 计算 P_x 与 Var[L] 。

解: 首先计算

$$A_x = c\sum_{k=0}^{\infty}(1.06)^{-k-1}(0.96)^{k+1} = 0.4,$$

$$\ddot{a}_x = \frac{1 - A_x}{d} = 10.6.$$

然后由 (4.3.2) 得出

$$P_x = \frac{A_x}{\ddot{a}_x} = 0.0377.$$

至于 Var[L], 先计算

$$^2A_x = c\sum_{k=0}^{\infty}[(1.06)^2]^{-k-1}(0.96)^{k+1} = 0.2445,$$

于是

$$\mathrm{Var}[L] = \frac{0.2445 - 0.16}{((0.06/1.06) \times 10.6)^2} = 0.2347.$$

例 4.3.1 与 4.2.1 有联系，其桥梁为

$$_{k|}q_x = \int_k^{k+1} {_tp_x}\mu_{x+t}dt \qquad k = 0, 1, 2, \cdots. \tag{4.3.4}$$

当死亡效力为常数 μ 时， $T(x)$ 服从密度函数为

$$_tp_x\mu_{x+t} = \mu e^{-\mu t} \qquad t \geq 0$$

的指数分布，根据式 (4.3.4), 其离散形式则是概率函数为

$$_{k|}q_x = \int_k^{k+1} \mu e^{-\mu t}dt = (e^{\mu} - 1)e^{-\mu(k+1)} \qquad k = 0, 1, 2, \cdots$$

的几何分布。在与例 4.3.1 对应的连续情形中， $e^{-\mu}=0.96$, 或 $\mu=0.0408$, 也就是说，与例 4.3.1 中 $P_x=0.0377$ 对应的完全连续净年缴保费 $\overline{P}(\overline{A}_x)=\mu=0.0408$ 。

用平衡原理可对各种完全离散人寿保险得出决定净年缴保费的公式。保险人亏损可一般地表示成

$$b_{K+1}v_{K+1}-PY,$$

其中 b_{k+1} 与 v_{k+1} 分别是在 (2.3.1) 中定义的受益金额与贴现因子， P 是期初支付的完全离散净年缴保费的一般符号， Y 是离散年金现值随机变量 [例如 (3.4.9)] 。

应用平衡原理，有

$$E[b_{K+1}v_{K+1}-PY]=0$$

或者

$$P=\frac{E[b_{K+1}v_{K+1}]}{E[Y]}.$$

由此可得出表 4.3.1 中有关完全离散保险的保费公式。

例 4.3.2: 对完全离散 n 年期两全保险的保险人亏损 L, 用净趸缴保费来表示 L 的方差 (参见表 4.3.1 第三行) 。

解: 根据表 4.3.1, 令

$$Z=\begin{cases} v^{K+1} & K=0,1,\cdots,n-1 \\ v^n & K=n,n+1,\cdots \end{cases},$$

亏损 L 可写成

$$L=Z-P_{x:\overline{n|}}\frac{1-Z}{d}.$$

于是

$$\mathrm{Var}[L]=\mathrm{Var}\left[Z(1+\frac{P_{x:\overline{n|}}}{d})-\frac{P_{x:\overline{n|}}}{d}\right].$$

按表 2.3.1 所指示的定理 2.2.1,

表 4.3.1　完全离散净年缴保费

种类	损失成份		保费公式
	$b_{K+1}v_{K+1}$	PY 中的 Y	$P=E[b_{K+1}v_{K+1}]/E[Y]$
终身人寿保险	$1v^{K+1}$	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,2,\cdots$	$P_x=\frac{A_x}{\ddot{a}_x}$
n 年定期保险	$1v^{K+1}$ 0	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,2,\cdots,n-1$ $\ddot{a}_{\overline{n}\mid}, K=n,n+1,\cdots$	$P^1_{x:\overline{n}\mid}=\frac{A^1_{x:\overline{n}\mid}}{\ddot{a}_{x:\overline{n}\mid}}$
n 年期两全保险	1^{K+1} $1v^n$	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,\cdots,n-1$ $\ddot{a}_{\overline{n}\mid}, K=n,n+1,\cdots$	$P_{x:\overline{n}\mid}=\frac{A_{x:\overline{n}\mid}}{\ddot{a}_{x:\overline{n}\mid}}$
h 年缴费终身人寿保险	$1v^{K+1}$ $1v^{K+1}$	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,\cdots,h-1$ $\ddot{a}_{\overline{h}\mid}, K=h,h+1,\cdots$	${}_hP_x=\frac{A_x}{\ddot{a}_{x:\overline{h}\mid}}$
h 年缴费 n 年期两全保险	$1v^{K+1}$ $1v^{K+1}$ $1v^n$	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,\cdots,h-1$ $\ddot{a}_{\overline{h}\mid}, K=h,\cdots,n-1$ $\ddot{a}_{\overline{h}\mid}, K=n,n+1,\cdots$	${}_hP_{x:\overline{n}\mid}=\frac{A_{x:\overline{n}\mid}}{\ddot{a}_{x:\overline{h}}}$
n 年期生存保险	0 $1v^n$	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,\cdots,n-1$ $\ddot{a}_{\overline{n}\mid}, K=n,n+1,\cdots$	$P_{x:\overline{n}\mid}^{\ \ 1}=\frac{A_{x:\overline{n}\mid}^{\ \ 1}}{\ddot{a}_{x:\overline{h}\mid}}$
n 年递延终身生存年金	0 $\ddot{a}_{\overline{K+1-n}\mid}v^n$	$\ddot{a}_{\overline{K+1}\mid}, K=0,1,\cdots,n-1$ $\ddot{a}_{\overline{n}\mid}, K=n,n+1,\cdots$	$P({}_{n\mid}\ddot{a}_x)=\frac{A_{x:\overline{n}\mid}^{\ \ 1}\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\mid}}$

$$\mathrm{Var}[L]=\left(1+\frac{P_{x:\overline{n}\mid}}{d}\right)^2[{}^2A_{x:\overline{n}\mid}-(A_{x:\overline{n}\mid})^2].$$

公式 (3.4.11)

$$d\ddot{a}_{x:\overline{n}\mid}+A_{x:\overline{n}\mid}=1.$$

与表 4.3.1 中第三行结合起来可得

$$1+\frac{P_{x:\overline{n}\mid}}{d}=\frac{1}{d\ddot{a}_{x:\overline{n}\mid}}.$$

于是所求方差为

$$\frac{{}^2A_{x:\overline{n}\mid}-(A_{x:\overline{n}\mid})^2}{(d\ddot{a}_{x:\overline{n}\mid})^2}. \tag{4.3.5}$$

例 4.3.3: 考虑保险金为 10000 的完全离散终身人寿保险。用 π 记该保单的年缴保费，$L(\pi)$ 记这种保单签发时的 (潜在) 亏损随机变量，其中投保年龄为 35 岁，年利率为 6%, 并以附录中的示例生命表为计算依据。

(1) 决定保费 π_a, 使得 $L(\pi_a)$ 的均值为 0, 并计算 $L(\pi_a)$ 的方差。

(2) 求使得亏损 $L(\pi_b)$ 为正的概率小于 0.5 的最低保费 π_b 的近似值，并求 $L(\pi_b)$ 的方差。

(3) 用正态近似决定保费 π_c, 使得 100 份这种相互独立保单的总亏损为正的概率等于 0.05 。

解:

(1) π_a 就是按平衡原理决定的年缴保费

$$\begin{aligned}\pi_a &= 10000P_{35} = 10000\frac{A_{35}}{\ddot{a}_{35}} \\ &= \frac{1287.194}{15.39262} = 83.62.\end{aligned}$$

由 (4.3.3),

$$\begin{aligned}\mathrm{Var}[L(\pi_a)] &= (10000)^2\frac{{}^2A_{35}-(A_{35})^2}{(d\ddot{a}_{35})^2} \\ &= 10^8\frac{0.0348843-(0.1287194)^2}{[(0.06/1.06)\times 15.39262]^2} \\ &= \frac{1831562}{0.7591295} = 2412713.\end{aligned}$$

(2) π_b 应使得

$$Pr[L(\pi_b)>0]<0.5.$$

用整值剩余寿命 K 来表示，

$$Pr[10000v^{K+1}-\pi_b\ddot{a}_{\overline{K+1}|}>0]<0.5.$$

根据示例生命表， ${}_{42}p_{35} = 0.512510$, ${}_{43}p_{35} = 0.4808965$, 于是 $Pr(K < 42) < 0.5$, 取 π_b 使得

$$10000v^{43} - \pi_b \ddot{a}_{\overline{43}|} = 0,$$

则因 $10000v^{k+1} - \pi_b \ddot{a}_{\overline{k+1}|}$ 关于 k 递增，而有

$$Pr[L(\pi_b) > 0] = Pr[K < 42] < 0.5,$$

这样

$$\pi_b = \frac{10000}{\ddot{s}_{\overline{43}|}} = 50.31.$$

与连续情形 (4.2.6) 类似，

$$\begin{aligned}
\mathrm{Var}[L(\pi_b)] &= (10000)^2 \mathrm{Var}[v^{K+1} - \frac{\pi_b}{10000} \ddot{a}_{\overline{K+1}|}] \\
&= 10^8 \mathrm{Var}[v^{K+1}(1 + \frac{\pi_b}{10000} \frac{1}{d}) - \frac{\pi_b}{10000} \frac{1}{d}] \\
&= 10^8 \mathrm{Var}[v^{K+1}](1 + \frac{\pi_b}{10000d})^2 \\
&= 10^8 [{}^2A_{35} - (A_{35})^2](1 + \frac{\pi_b}{10000d})^2 \\
&= 1831562 \times 1.18567 \\
&= 2171630.
\end{aligned}$$

(3) 在保费为 π_c 时， 1 个保单的亏损为

$$L(\pi_c) = 10000v^{K+1} - \pi_c \ddot{a}_{\overline{K+1}|} = (10000 + \frac{\pi_c}{d})v^{K+1} - \frac{\pi_c}{d}$$

它的期望值与方差为

$$\begin{aligned}
E[L(\pi_c)] &= (10000 + \frac{\pi_c}{d})A_{35} - \frac{\pi_c}{d} \\
&= 0.1287194(10000 + \frac{\pi_c}{d}) - \frac{\pi_c}{d}, \\
\mathrm{Var}[L(\pi_c)] &= (10000 + \frac{\pi_c}{d})^2[{}^2A_{35} - (A_{35}^2)] \\
&= (10000 + \frac{\pi_c}{d})^2 \times 0.01831562.
\end{aligned}$$

对 100 份这种保单，每一份的亏损 $L_i(\pi_c)=L(\pi_c)$，$i=1,2,\cdots,100$. 总亏损 $S=\sum\limits_{i=1}^{100}L_i(\pi_c)$ 的期望值为

$$E[S]=100E[L(\pi_c)].$$

方差根据独立性可得出为

$$\mathrm{Var}[S]=100\mathrm{Var}[L(\pi_c)].$$

π_c 由 $Pr(S>0)=0.05$ 决定，按正态近似可得

$$\frac{0-E[S]}{\sqrt{\mathrm{Var}[S]}}=1.645,$$

$$\frac{-100E[L(\pi_c)]}{\sqrt{100\mathrm{Var}[L(\pi_c)]}}=1.645,$$

$$\frac{-A_{35}(10000+\frac{\pi_c}{d})+\frac{\pi_c}{d}}{(10000+\frac{\pi_c}{d})\sqrt{{}^2A_{35}-(A_{35})^2}}=0.1645.$$

于是

$$\pi_c=10000d\frac{0.1645\sqrt{{}^2A_{35}-(A_{35})^2}+A_{35}}{1-[A_{35}+0.1645\sqrt{{}^2A_{35}-(A_{35})^2}]}=100.66.$$

等式 (3.4.6) 与 (3.4.11) 可用来导出离散保费之间的关系. 例如，从 (3.4.6) 出发，有

$$d\ddot{a}_x+A_x=1,$$

$$P_x=\frac{1}{\ddot{a}_x}-d=\frac{1-d\ddot{a}_x}{\ddot{a}_x}=\frac{dA_x}{1-A_x}. \tag{4.3.6}$$

从 (3.4.11) 出发，有

$$d\ddot{a}_{x:\overline{n}|}+A_{x:\overline{n}|}=1,$$

$$P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d = \frac{1 - d\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{dA_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}. \tag{4.3.7}$$

例 4.3.4: 给出从 (4.3.6) 得出的以下方程

$$\frac{1}{\ddot{a}_x} = P_x + d \tag{4.3.8}$$

与

$$P_x = \frac{dA_x}{1 - A_x} \tag{4.3.9}$$

的文字解释。

解: 对 (4.3.8), 现时 1 单位金额等价于 (x) 活着时每年年初支付 $(\ddot{a}_x)^{-1}$ 的生存年金，也等价于每年年初先付利息 d 并在 (x) 死亡年末偿还 1 单位，而死亡年末偿还 1 单位又等价于每年支付 P_x 的期初生存年金。这样，现时 1 单位也就等价于 (x) 活着时每年年初支付 $P_x + d$, 所以，等式 $(\ddot{a}_x)^{-1} = P_x + d$ 两端都代表了现时 1 单位所提供的期初生存年金的年支付额。

对 (4.3.9), 考虑以下情况，被保险人 (x) 借入净趸缴保费 A_x 购买 1 单位终身人寿保险，同时每年年初支付先付利息 dA_x, 并在死亡年末从 1 单位受益金中扣除 A_x 用于还贷。实际上，(x) 以每年年初缴付保费 dA_x 购买了 $1 - A_x$ 单位终身人寿保险，所以 1 单位终身人寿保险的净年缴保费 $P_x = dA_x/(1 - A_x)$ 。

对涉及两全保险的相应关系 (4.3.7), 也存在类似的解释。与 (4.3.8) 对应的涉及两全保险的等式为

$$(\ddot{a}_{x:\overline{n}|})^{-1} = P_{x:\overline{n}|} + d,$$

只涉及利息的公式为

$$(\ddot{a}_{\overline{n}|})^{-1} = (\ddot{s}_{\overline{n}|})^{-1} + d.$$

例 4.3.5: 证明并解释公式

$$P_{x:\overline{n}|} = {}_nP_x + P_{x:\overline{n}|}^{\ \ 1}(1 - A_{x+n}). \tag{4.3.10}$$

解: 根据表 4.3.1,

$$P_{x:\overline{n}|}\ddot{a}_{x:\overline{n}|}=A_{x:\overline{n}|}=A^{1}_{x:\overline{n}|}+A_{x:\overset{1}{\overline{n}|}},$$

$$_{n}P_{x}\ddot{a}_{x:\overline{n}|}=A_{x}=A^{1}_{x:\overline{n}|}+A_{x:\overset{1}{\overline{n}|}}A_{x+n},$$

相减得

$$(P_{x:\overline{n}|}-{_n}P_{x})\ddot{a}_{x:\overline{n}|}=A_{x:\overset{1}{\overline{n}|}}(1-A_{x+n}).$$

由此可得出 (4.3.10) 。

至于解释, $P_{x:\overline{n}|}$ 与 $_{n}P_{x}$ 两者都在 (x) 活着时支付, 最多 n 年。在这期间, 两者保险都在 (x) 死亡年末提供 1 单位受益, 如 (x) 活到 n 年以上, $P_{x:\overline{n}|}$ 提供到期的 1 单位受益, 而 $_{n}P_{x}$ 则提供以后的终身保险, 其 (n 年末) 精算现值为 A_{x+n}, 所以差 $P_{x:\overline{n}|}-{_n}P_{x}$ 相当于 $1-A_{x+n}$ 个单位生存保险的均衡年缴保费。

在保险实践中, 人寿保险的受益金一般在死亡后即刻赔付, 而不是在死亡发生的保单年度末赔付的, 因此, 有必要考虑半连续净年缴保费, 这种保费按表 4.2.1 与 4.3.1 的顺序分别记成 $P(\overline{A}_{x})$, $P(\overline{A}^{1}_{x:\overline{n}|})$, $P(\overline{A}_{x:\overline{n}|})$, $_{h}P(\overline{A}_{x})$, $_{h}P(\overline{A}_{x:\overline{n}|})$ 。由于生存保险不涉及死亡受益, 对此就不需要半连续保费。平衡原理可用来导出与表 4.3.1 中类似的保费公式, 其中符号 A 改成 $\overline{A}$ 。例如,

$$P(\overline{A}_{x})=\frac{\overline{A}_{x}}{\ddot{a}_{x}}. \tag{4.3.11}$$

注意, 这一死亡即刻赔付终身人寿保险的年初支付的净年缴保费符号不是 $\overline{P}_{x}$, $\overline{P}_{x}$ 是死亡年末赔付终身人寿保险的连续支付年保费, 等于 $A_{x}/\overline{a}_{x}$ 。如果每一年龄死亡均匀分布, 那么根据 §2.4 可得出

$$P(\overline{A}_{x})=\frac{i}{\delta}\frac{A_{x}}{\ddot{a}_{x}}=\frac{i}{\delta}P_{x},$$

$$P(\overline{A}^{1}_{x:\overline{n}|})=\frac{i}{\delta}P^{1}_{x:\overline{n}|}, \tag{4.3.12}$$

$$P(\overline{A}_{x:\overline{n}|}) = \frac{i}{\delta} P^1_{x:\overline{n}|} + P_{x:\overset{1}{\overline{n}|}}.$$

§4.4 真正年缴 m 次保费

如在每个保单年度内，保费分 m 次缴付，并且死亡受益不作调整，那么其年保费称为真正分数保费(true fractional premiums) 。对于 1 单位死亡年末赔付的终身人寿保险，每年 m 次分期缴付的年保费记作 $P_x^{(m)}$，称为真正净均衡年保费(true net level annual premium) 。注意，每次缴付的数额为 $P_x^{(m)}/m$ 。符号 $P^{(m)}(\overline{A}_x)$ 则是死亡即刻赔付终身人寿保险的真正净均衡年保费。

这一节着眼于死亡年末赔付受益金的保险。表 4.4.1 给出了各种保险的真正分数保费符号与公式，这些保费公式可通过平衡原理获得。

在某些应用场合，将年缴 m 次的年保费表示成年缴保费的倍数是有用的。以下对较一般的保费 ${}_hP^{(m)}_{x:\overline{n}|}$ 作一下说明，所得公式经适当修改后可获得其它种类保险的相应公式。从表 4.4.1 的最后一行可得

$$ {}_hP^{(m)}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{h}|}}. \tag{4.4.1}$$

由

$$A_{x:\overline{n}|} = {}_hP_{x:\overline{n}|}\ddot{a}_{x:\overline{h}|},$$

式 (4.4.1) 可改写成

$$ {}_hP^{(m)}_{x:\overline{n}|} = \frac{{}_hP_{x:\overline{n}|}\ddot{a}_{x:\overline{h}|}}{\ddot{a}^{(m)}_{x:\overline{h}|}}. \tag{4.4.2}$$

公式 (4.4.2) 将在下一章中用到，它把年缴 m 次的年保费表示成相应的年保费与一个年金比值的乘积。

表 4.4.1　真正分数保费 *

种类	受益金支付方式	
	保单年度末支付	死亡即刻支付
终身人寿保险	$P_x^{(m)}=\frac{A_x}{\ddot{a}_x^{(m)}}$	$P^{(m)}(\overline{A}_x)=\frac{\overline{A}_x}{\ddot{a}_x^{(m)}}$
n 年定期保险	$P^{1(m)}_{x:\overline{n}\vert}=\frac{A^1_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$	$P^{(m)}(\overline{A}^1_{x:\overline{n}\vert})=\frac{\overline{A}^1_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$
n 年期两全保险	$P^{(m)}_{x:\overline{n}\vert}=\frac{A_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$	$P^{(m)}(\overline{A}_{x:\overline{n}\vert})=\frac{\overline{A}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{n}\vert}}$
h 年缴费终身人寿保险	${}_hP_x^{(m)}=\frac{A_x}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$	${}_hP^{(m)}(\overline{A}_x)=\frac{\overline{A}_x}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$
h 年缴费 n 年期两全保险	${}_hP^{(m)}_{x:\overline{n}\vert}=\frac{A_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$	${}_hP^{(m)}(\overline{A}_{x:\overline{n}\vert})=\frac{\overline{A}_{x:\overline{n}\vert}}{\ddot{a}^{(m)}_{x:\overline{h}\vert}}$

* 在保费缴付期内 (x) 活着时，每个保单年度分 m 次缴付的分数保费，每次实际支付额为 $P^{(m)}/m$ 。

例 4.4.1: 设每一年龄中死亡均匀分布，

(1) 对于 (50) 的 10000 个单位死亡年末支付的 20 年期两全保险，计算按半年分期缴付的净均衡年保费，计算时以附录中的示例生命表为依据，年利率为 6% 。

(2) 决定相应的死亡即刻赔付保险的净均衡年保费 (半连续情形) 。

解: (1) 计算 $10000P^{(2)}_{50:\overline{(2)}\vert}$ 的步骤如下，

$$
\begin{aligned}
d &= 0.056603774,\\
i^{(2)} &= 0.059126028,\\
d^{(2)} &= 0.057428275,\\
\ddot{a}^{(2)}_{\overline{1}\vert} &= 0.98564294,\\
s^{(2)}_{\overline{1}\vert} &= 1.01478151,\\
\alpha(2) &= s^{(2)}_{\overline{1}\vert}\ddot{a}^{(2)}_{\overline{1}\vert} = 1.0002122,
\end{aligned}
$$

$$
\begin{aligned}
\beta(2) &= \frac{s_{\overline{1}|}^{(2)}-1}{d^{(2)}} = 0.25739081,\\
\ddot{a}_{50:\overline{20}|} &= 11.291832,\\
A^{1}_{50:\overline{20}|} &= 0.13036536,\\
P^{1}_{50:\overline{20}|} &= 0.01154510,\\
{}_{20}E_{50} &= 0.23047353,\\
A_{50:\overline{20}|} &= 0.36083889,\\
P_{50:\overline{20}|} &= 0.03195574.
\end{aligned}
$$

在死亡均匀分布假设下，

$$
\ddot{a}^{(2)}_{50:\overline{20}|} = \alpha(2)\ddot{a}_{50:\overline{20}|} - \beta(2)(1-{}_{20}E_{50}) = 11.096159.
$$

对 x=50, n=20, m=2 用公式 (4.4.1) 可得

$$
10000P^{(2)}_{50:\overline{20}|} = 10000\frac{A_{50:\overline{20}|}}{\ddot{a}^{(2)}_{50:\overline{20}|}} = 325.19.
$$

(2) 相应的半连续年保费为

$$
\begin{aligned}
10000P^{(2)}(\overline{A}_{50:\overline{20}|}) &= 10000P^{(2)}_{50:\overline{20}|}\frac{\overline{A}_{50:\overline{20}|}}{A_{50:\overline{20}|}}\\
&= 10000P^{(2)}_{50:\overline{20}|}\frac{P(\overline{A}_{50:\overline{20}|})}{P_{50:\overline{20}|}}.
\end{aligned}
$$

在死亡均匀分布假设下，

$$
\frac{P(\overline{A}_{50:\overline{20}|})}{P_{50:\overline{20}|}} = \frac{(i/\delta)P^{1}_{50:\overline{20}|} + P_{50:\overline{20}|}^{\ \ \ 1}}{P_{50:\overline{20}|}}, \tag{4.4.3}
$$

于是可算得

$$
10000P^{(2)}(\overline{A}_{50:\overline{20}|}) = 328.68.
$$

§4.5 比例保费

另一种类型的分数保费乃比例保费(apportionable premium)，此时根据死亡时间与下一次预定缴费时间间隔长短退还部分保费。在实践中，一般按比例计算，并不计利息。这一节中，我们将考虑利息，并将年缴 m 次的保费序列看作 §3.9 的比例期初生存年金。这些年缴 m 次的净均衡比例年保费符号与半连续情形的真正分数保费符号相似，只不过上标 m 放在花括号内，如 $P^{\{m\}}(\overline{A}_x)$ 。鉴于退还部分保费的特征，自然设受益金在死亡即刻赔付。

我们仍以 h 年缴费 n 年期两全保险为例，说明年缴 m 次的比例保费公式之建立过程。从平衡原理导出

$$
{}_hP^{\{m\}}(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{\ddot{a}^{\{m\}}_{x:\overline{h}|}}. \tag{4.5.1}
$$

利用例 3.9.1(2) 可得

$$
{}_hP^{\{m\}}(\overline{A}_{x:\overline{n}|}) = \frac{\overline{A}_{x:\overline{n}|}}{(\delta/d^{(m)})\overline{a}_{x:\overline{h}|}} = \frac{d^{(m)}}{\delta}\,{}_h\overline{P}(\overline{A}_{x:\overline{n}|}). \tag{4.5.2}
$$

这意味着，年缴 m 次保费的每次摊付额为

$$
\frac{1}{m}\,{}_hP^{\{m\}}(\overline{A}_{x:\overline{n}|}) = {}_h\overline{P}(\overline{A}_{x:\overline{n}|})\frac{1-v^{1/m}}{\delta} = {}_h\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{\overline{1/m}|}. \tag{4.5.3}
$$

特别是，当 m=1 时，有

$$
{}_hP^{\{1\}}(\overline{A}_{x:\overline{n}|}) = {}_h\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{\overline{1}|}. \tag{4.5.4}
$$

公式 (4.5.3) 与 (4.5.4) 表明，这些比例保费等价于按利息贴现至每一缴费周期之初的完全连续保费。对其它种类保费也成立

类似的公式。例如，令 $h, n \to \infty$, (4.5.4) 成为

$$P^{\{1\}}(\overline{A}_x) = \overline{P}(\overline{A}_x)\overline{a}_{\overline{1}|}. \tag{4.5.5}$$

比例净保费 $P^{\{1\}}(\overline{A}_x)$ 与半连续净保费 $P(\overline{A}_x)$ 都在 (x) 活着时每年年初缴费，在 (x) 死亡时都提供 1 单位受益，所不同的只是 $P^{\{1\}}(\overline{A}_x)$ 还提供保费退款。于是差额

$$P^{\{1\}}(\overline{A}_x) - P(\overline{A}_x) \tag{4.5.6}$$

是为保费退款受益而按年在年初缴付的净均衡年保费。

从第三章习题 32 可知，保费退款受益的现值随机变量为

$$\frac{P^{\{1\}}(\overline{A}_x)v^T\overline{a}_{\overline{K+1-T}|}}{\overline{a}_{\overline{1}|}},$$

根据平衡原理，为此受益而支付的净趸缴保费为

$$\overline{A}_x^{PR} = P^{\{1\}}(\overline{A}_x)E[v^T\frac{\overline{a}_{\overline{K+1-T}|}}{\overline{a}_{\overline{1}|}}].$$

利用 (4.5.5) 可得

$$\overline{A}_x^{PR} = \overline{P}(\overline{A}_x)E[\frac{v^T - v^{K+1}}{\delta}] = \overline{P}(\overline{A}_x)(\frac{\overline{A}_x - A_x}{\delta}), \tag{4.5.7}$$

从而相应的净均衡年缴保费为

$$P(\overline{A}_x^{PR}) = \frac{\overline{P}(\overline{A}_x)(\overline{A}_x - A_x)}{\delta\ddot{a}_x}. \tag{4.5.8}$$

公式 (4.5.7) 有以下解释：保费退款受益的精算现值等于，(x) 死亡时开始的与 (x) 死亡年末开始的年率 (年支付额) 为 $\overline{P}(\overline{A}_x)$ 的连续永久年金值之差。

回到 (4.5.6), 由 (4.5.5) 可得

$$
\begin{aligned}
& P^{\{1\}}(\overline{A}_x) - P(\overline{A}_x) = \overline{P}(\overline{A}_x)\frac{d}{\delta} - \frac{\overline{A}_x}{\ddot{a}_x} \\
= \quad & \overline{P}(\overline{A}_x)(\frac{d}{\delta} - \frac{\overline{a}_x}{\ddot{a}_x}) = \overline{P}(\overline{A}_x)\frac{d\ddot{a}_x - \delta\overline{a}_x}{\delta\ddot{a}_x} \qquad (4.5.9) \\
= \quad & \overline{P}(\overline{A}_x)\frac{\overline{A}_x - A_x}{\delta\ddot{a}_x} = P(\overline{A}_x^{PR}),
\end{aligned}
$$

这就是证实了 (4.5.6) 是保险退款受益的净年缴保费。

以上分析可推广到年缴 m 次保费及其它人寿保险，一般而言，

$$P^{\{m\}}(\overline{A}) - P^{(m)}(\overline{A})$$

是退款受益的年缴 m 次保费。

例 4.5.1: 如果例 4.4.1(2) 的保单系比例保费，那么净年保费增加多少？

解: 每单位保险的比例年保费由 (4.5.2) 给出，

$$P^{\{2\}}(\overline{A}_{50:\overline{20|}}) = \overline{P}(\overline{A}_{50:\overline{20|}})\frac{d^{(2)}}{\delta} = \frac{\overline{A}_{50:\overline{20|}}}{\overline{a}_{50:\overline{20|}}}\frac{d^{(2)}}{\delta}.$$

根据死亡均匀分布假设，上式成为

$$
\begin{aligned}
P^{\{2\}}(\overline{A}_{50:\overline{20|}}) \quad &= \quad \frac{(i/\delta)A^{1}_{50:\overline{20|}} + A_{50:\overline{20|}}^{\;\;\;\;\;1}}{\alpha(\infty)\ddot{a}_{50:\overline{20|}} - \beta(\infty)(1 - {}_{20}E_{50})}\frac{d^{(2)}}{\delta} \\
&= \quad \frac{(i/\delta)P^{1}_{50:\overline{20|}} + P_{50:\overline{20|}}^{\;\;\;\;\;1}}{\alpha(\infty) - \beta(\infty)(P^{1}_{50:\overline{20|}} + d)}\frac{d^{(2)}}{\delta},
\end{aligned}
$$

这里 $\alpha(\infty) = \overline{s}_{\overline{1|}}\overline{a}_{\overline{1|}} = id/\delta^2 = 1.00028$, $\beta(\infty) = (\overline{s}_{\overline{1|}} - 1)/\delta = 0.50985$, 再加上例 4.3.1 中算得的值，可得出

$$10000P^{\{2\}}(\overline{A}_{50:\overline{20|}}) = 329.69.$$

年保费增加额为

$$10000[P^{\{2\}}(\overline{A}_{50:\overline{20|}}) - P^{(2)}(\overline{A}_{50:\overline{20|}})] = 1.01.$$

这是退款受益的 (半年缴一次) 净均衡年保费。

§4.6 计算基数

我们已经知道，净年缴保费可通过人寿保险的净趸缴保费与生存年金的精算现值来表示。在前两章中，这些保费及年金值都可用计算基数表示，从而年缴保费亦然。对于 (x) 的 n 年期两全保险，其 h 年付费的完全连续净年保费可表示成

$$_h\overline{P}(\overline{A}_{x:\overline{n|}}) = \frac{\overline{M}_x - \overline{M}_{x+n} + D_{x+n}}{\overline{N}_x - \overline{N}_{x+h}}. \tag{4.6.1}$$

上式的特殊情形包括，当 $n = \omega - x$(或 ∞) 时

$$_h\overline{P}(\overline{A}_x) = \frac{\overline{M}_x}{\overline{N}_x - \overline{N}_{x+h}}. \tag{4.6.2}$$

当 $h = n = \omega - x$(或 ∞) 时，

$$\overline{P}(\overline{A}_x) = \frac{\overline{M}_x}{\overline{N}_x}. \tag{4.6.3}$$

对于定期保险，有诸如

$$\overline{P}(\overline{A}^1_{x:\overline{n|}}) = \frac{\overline{M}_x - \overline{M}_{x+n}}{\overline{N}_x - \overline{N}_{x+n}} \tag{4.6.4}$$

及

$$_h\overline{P}(D\overline{A})^1_{x:\overline{n|}} = \frac{n\overline{M}_x - \overline{R}_{x+1} + \overline{R}_{x+n+1}}{\overline{N}_x - \overline{N}_{x+h}} \tag{4.6.5}$$

等公式。

对于死亡年末赔付人寿保险的完全离散年保费，与 (4.6.1), (4.6.2) 及 (4.6.4) 对应的公式为

$$
{}_hP_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+h}}. \tag{4.6.6}
$$

$$
{}_hP_x = \frac{M_x}{N_x - N_{x+h}}. \tag{4.6.7}
$$

$$
P^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \tag{4.6.8}
$$

至于死亡年末赔付保险的真正年缴 m 次的年保费，有公式

$$
{}_hP^{(m)}_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N^{(m)}_x - N^{(m)}_{x+h}} \tag{4.6.9}
$$

[参见 (3.6.6) 及 (3.6.7)] 。

年缴 m 次比例保费的相应公式可通过 (4.5.3) 与 (4.5.4) 给出的贴现完全连续保费表示而得出。

例 4.6.1: 考虑 (x) 的 1 单位终身人寿保险，设 5 年后净年缴保费加倍，按完全离散模型将初始净年缴保费用计算基数表示。

解: 设 P 是初始净年缴保费，根据平衡原理，决定 P 的方程为

$$
P(N_x - N_{x+5}) + 2PN_{x+5} = M_x.
$$

于是

$$
P = \frac{M_x}{N_x + N_{x+5}}.
$$

§4.7 累积增额受益

这一节的分析主要针对死亡年末赔付保险的年保费，类似的分析适用于完全连续保费，稍作更改也适用于半连续保费。首先

考虑 (x) 的 n 年人寿保险，其受益金额当死亡发生在第 $k+1$ 年时为 $\ddot{s}_{\overline{k+1}|j}$，这个受益在保单签发时的现值随机变量为

$$W=\begin{cases} v^{K+1}\ddot{s}_{\overline{K+1}|j}=\frac{1}{d_{(j)}}[v^{K+1}(1+j)^{K+1}-v^{K+1}] & 0\le K<n \\ 0 & K\ge n, \end{cases}$$

其中保险人的现值按利率 i 计算，$d_{(j)}$ 是与利率 j 等价的(银行)贴现率。根据平衡原理，净趸缴保费为

$$E[W]=\frac{A'^{1}_{x:\overline{n}|}-A^{1}_{x:\overline{n}|}}{d_{(j)}}, \tag{4.7.1}$$

其中 $A'^{1}_{x:\overline{n}|}$ 按利率 $i'=(i-j)/(1+j)$ 计算。

如 $i=j$, 则 $i'=0$, 且净趸缴保费成为

$$\begin{aligned} &\frac{{}_nq_x-A^{1}_{x:\overline{n}|}}{d}=\frac{1-{}_np_x-A_{x:\overline{n}|}+v^n{}_np_x}{d}\\ =\;&\ddot{a}_{x:\overline{n}|}-{}_np_x\ddot{a}_{\overline{n}|}=\ddot{a}_{x:\overline{n}|}-{}_nE_x\ddot{s}_{\overline{n}|}. \end{aligned} \tag{4.7.2}$$

公式 (4.7.2) 表明，当 $j=i$ 时，除非 (x) 生存满 n 年，上述特殊定期保险等价于一个 n 年期期初生存年金，在生存满 n 年时，上述生存年金在时间 n 时的积累值为 $\ddot{s}_{\overline{n}|}$，那时该定期保险提供的受益为 0 。

现考虑以下情形，(x) 可以选择年缴保费 $P_{x:\overline{n}|}$ 的 n 年期单位受益两全保险，或者选择建立一个 n 年中每年年初存入 $1/\ddot{s}_{\overline{n}|}$ 的储蓄帐户并购买一个特殊的递减定期保险，该特殊保险当 (x) 在 n 年中死于第 $k+1$ 年时年末提供受益

$$1-\frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}}\qquad k=0,1,2,\cdots,n-1.$$

显然，这一受益是两全保险的单位受益与储蓄帐户的积累值之差。假定所有估价计算均使用同样的利率 i，那么两全保险提供

的受益与上述特殊保险及储蓄帐户联合提供的受益完全相当，因此，两全保险的净年缴保费 $P_{x:\overline{n}|}$ 等于特殊定期保险的净年缴保费加上储蓄帐户年存入款 $1/\ddot{s}_{\overline{n}|}$ 。

为证实以上结论，考虑以上特殊递减定期保险的现值随机变量

$$\tilde{W}=\begin{cases} v^{K+1}(1-\dfrac{\ddot{s}_{\overline{K+1}|}}{\ddot{s}_{\overline{n}|})=v^{K+1}-\dfrac{\ddot{a}_{\overline{K+1}|}}{\ddot{s}_{\overline{n}|}} & 0\le K<n \\ 0 & K\ge n \end{cases}. \tag{4.7.3}$$

其净趸缴保费记作 $\tilde{A}^1_{x:\overline{n}|}$，由平衡原理得

$$\begin{aligned}\tilde{A}^1_{x:\overline{n}|} &= E[\tilde{W}]=A^1_{x:\overline{n}|}-\frac{\ddot{a}_{x:\overline{n}|}-{}_np_x\ddot{a}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}} \\ &= A^1_{x:\overline{n}|}-\frac{\ddot{a}_{x:\overline{n}|}-{}_nE_x\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}}\end{aligned}$$

[参见 (4.7.2)] 。这一特殊定期保险的净年缴保费

$$\begin{aligned}\tilde{P}^1_{x:\overline{n}|} &= \frac{\tilde{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}=P^1_{x:\overline{n}|}-\frac{1}{\ddot{s}_{\overline{n}|}}+P_{x:\overline{n}|}^{\ \ 1} \\ &= P_{x:\overline{n}|}-\frac{1}{\ddot{s}_{\overline{n}|}}.\end{aligned}$$

于是

$$P_{x:\overline{n}|}=\tilde{P}^1_{x:\overline{n}|}+\frac{1}{\ddot{s}_{\overline{n}|}}. \tag{4.7.4}$$

我们早就有

$$P_{x:\overline{n}|}=P^1_{x:\overline{n}|}+P_{x:\overline{n}|}^{\ \ 1},$$

式 (4.7.4) 则提供 $P_{x:\overline{n}|}$ 的又一分解，其成份是以上特殊定期保险的年保费与储蓄帐户年存入额 $1/\ddot{s}_{\overline{n}|}$，后者到 n 年末的积累值为 1 。

例 4.7.1: 考虑 (x) 的金额为 5000 的 20 年定期保险，当 (x) 在 20 年内死亡时，除保险金 5000 外，同时还加上退还已付的净年缴保费，受益在死亡年末支付，按以下两种情况分别导出净年缴保费公式：

(1) 退还的净年缴保费不计利息。

(2) 退还的净年缴保费按决定保费的相同利率累积。

解: (1) 设 π_a 为所求保费，则

$$\pi_a \ddot{a}_{x:\overline{20|}} = 5000A^{1}_{x:\overline{20|}} + \pi_a (IA)^{1}_{x:\overline{20|}},$$

$$\pi_a = 5000\frac{A^{1}_{x:\overline{20|}}}{\ddot{a}_{x:\overline{20|}} - (IA)^{1}_{x:\overline{20|}}}.$$

(2) 设 π_b 为所求保费，利用 (4.7.2) 可得

$$\pi_b \ddot{a}_{x:\overline{20|}} = 5000A^{1}_{x:\overline{20|}} + \pi_b(\ddot{a}_{x:\overline{20|}} - {}_{20}E_x \ddot{s}_{\overline{20|}}),$$

$$\pi_b = 5000\frac{A^{1}_{x:\overline{20|}}}{{}_{20}E_x \ddot{s}_{\overline{20|}}} = 5000\frac{A^{1}_{x:\overline{20|}}}{{}_{20}p_x \ddot{a}_{\overline{20|}}}.$$

在实践中，毛保费会被退还，有关公式需考虑到这一因素。

例 4.7.2: 考虑 (x) 的一份从 $x+n$ 岁开始岁入为 1 的延期年金，其净年缴保费在递延期内支付，并且在缴费期内死亡时，年末归还净年缴保费的积累额，决定其净年缴保费。

解: 令净年缴保费的精算现值 π 等于受益的精算现值，有

$$\pi \ddot{a}_{x:\overline{n|}} = {}_nE_x \ddot{a}_{x+n} + \pi(\ddot{a}_{x:\overline{n|}} - {}_nE_x \ddot{s}_{\overline{n|}}),$$

其中右端的第二项来自 (4.7.2)，由此解得

$$\pi = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n|}}}.$$

习 题

§4.1

1. 设一个 0 岁生命的整值剩余寿命服从概率函数为

$$_{k|}q_0 = \frac{1}{4} \qquad k = 0, 1, 2, 3$$

的分布，在其死亡年末赔付 1 单位的保单，每年年初缴付保费 P. 当保费按平衡原理决定时，计算保险人亏损现值的期望值与方差。

§4.2

2. 如果死亡效力随年龄严格递增，证明 $\overline{P}(\overline{A}_x) > \mu_x$ 。

3. 参照例 4.2.1, 当 $\mu_{x+t} = \mu, t > 0$ 时，导出

$$\frac{{}^2\overline{A}_x - (\overline{A}_x)^2}{(\delta\overline{a}_x)^2}$$

的一般表达式，这里 δ 是利息效力。

4. 当 $\delta = 0$ 时，证明

$$\overline{P}(\overline{A}_x) = \frac{1}{\mathring{e}_x}.$$

5. 证明：与终身人寿保险净趸缴保费相联系的亏损之方差小于与终身人寿保险净年保费相联系的亏损之方差，这里假定死亡即刻赔付以及连续支付净年保费。

6. 证明

$$(1 + \frac{d\overline{a}_x}{dx})\overline{P}(\overline{A}_x) - \frac{d\overline{A}_x}{dx} = \mu_x.$$

§4.3

7. 根据附录示例生命表及利率 6% 计算以下表中各年保费值。

完全连续	半连续	完全离散
$\overline{P}(\overline{A}_{35:\overline{10}\rvert})$	$P(\overline{A}_{35:\overline{10}\rvert})$	$P_{35:\overline{10}\rvert}$
$\overline{P}(\overline{A}_{35:\overline{30}\rvert})$	$P(\overline{A}_{35:\overline{30}\rvert})$	$P_{35:\overline{30}\rvert}$
$\overline{P}(\overline{A}_{35:\overline{60}\rvert})$	$P(\overline{A}_{35:\overline{60}\rvert})$	$P_{35:\overline{60}\rvert}$
$\overline{P}(\overline{A}_{35})$	$P(\overline{A}_{35})$	P_{35}
$\overline{P}(\overline{A}^{1}_{35:\overline{30}\rvert})$	$P(\overline{A}^{1}_{35:\overline{30}\rvert})$	$P^{1}_{35\ \overline{30}\rvert}$
$\overline{P}(\overline{A}^{1}_{35:\overline{10}\rvert})$	$P(\overline{A}^{1}_{35:\overline{30}\rvert})$	$P^{1}_{35:\overline{10}\rvert}$

8. 证明

$$ {}_{20}P^{1}_{x:\overline{30}\rvert} - P^{1}_{x:\overline{20}\rvert} = {}_{20}P({}_{20|10}A_x). $$

9. 将例 4.3.1 推广到一般情况：

$$ {}_{k|}q_x = (1-r)r^k \qquad k = 0, 1, 2, \cdots, $$

即对 $A_x, \ddot{a}_x, P_x, [{}^2A_x - (A_x)^2]/(d\ddot{a}_x)^2$ 导出用 r 及 i 表示的表达式。

§4.4

10. 用例 4.4.1 中给出的信息计算 $P^{(2)}_{50}$ 。

11. 用 $\ddot{a}^{(m)}_{x:\overline{n}\rvert}$ 的各种表达式说明 (4.4.2) 中的比值

$$ \frac{\ddot{a}_{x:\overline{h}\rvert}{}^{(m)}}{\ddot{a}_{x:\overline{h}\rvert}}. $$

是以下各式的倒数：

(1) $\ddot{a}^{(m)}_{\overline{1}\rvert} - \beta(m)P^{1}_{x:\overline{h}\rvert}$.

(2) $\alpha(m) - \beta(m)(P^{1}_{x:\overline{h}\rvert} + d)$.

(3) $1 - \frac{m-1}{2m}(P^{1}_{x:\overline{h}\rvert} + d)$.

12. 在例 4.4.1(2) 中，直接计算

$$ P^{(2)}(\overline{A}_{50:\overline{20}\rvert}) = \frac{\overline{A}_{50:\overline{20}\rvert}}{\ddot{a}^{(2)}_{50:\overline{20}\rvert}}, $$

即用示例生命表计算分子两全保险的净趸缴保费与分母生存年金值。

13. 如果

$$\frac{P_{x:\overline{12}|}^{1\,(20)}}{P_{x:\overline{20}|}^{1}} = 1.032$$

且 $P_{x:\overline{20}|} = 0.040$, 那么 $P_{x:\overline{20}|}^{(12)}$ 的值是多少？

§4.5

14. 按大小顺序排列以下保费，并说明理由。

$$P^{(2)}(\overline{A}_{40:\overline{25}|}), \overline{P}(\overline{A}_{40:\overline{25}|}), P^{\{4\}}(\overline{A}_{40:\overline{25}|}), P(\overline{A}_{40:\overline{25}|}), P^{\{12\}}(\overline{A}_{40:\overline{25}|}).$$

15. 给定

$$\frac{d}{d^{(12)}} = \frac{99}{100},$$

估计

$$\frac{P^{\{12\}}(\overline{A}_x)}{P^{\{1\}}(\overline{A}_x)}.$$

16. 设 $\overline{P}(\overline{A}_x) = 0.03$, 年利率为 5%, 对于 (x) 的金额 50000 最后保费可按比例退还的终身人寿保险，计算其半年缴一次的净年保费。

17. 证明

$$P^{\{m\}}(\overline{A}_{x:\overline{n}|}) - P^{(m)}(\overline{A}_{x:\overline{n}|}) = \overline{P}(\overline{A}_{x:\overline{n}|})\frac{\overline{A}_{x:\overline{n}|} - A_{x:\overline{n}|}^{(m)}}{\delta \ddot{a}_{x:\overline{n}|}^{(m)}}.$$

§4.6

18. 用计算基数给出以下保费的表达式：

(1) ${}_{20}P^{(12)}(\overline{A}_{x:\overline{30}|})$.　　(2) ${}_{20}\overline{P}(\overline{A}_{x:\overline{30}|})$.

(3) ${}_{20}P^{\{4\}}(\overline{A}_{x:\overline{30}|})$.　　(4) ${}_{20}P({}_{40|}\ddot{a}_{25})$.

19. 考虑 30 岁人的递减定期保险，初始保险金为 200000，以后每年年末减少 5000，直至 70 岁保险期满为止。用适当的计算基数写出形如 (4.6.1) 的缴费期为 20 年的净年缴比例保费。

§4.7

20. 将

$$1-\frac{\ddot{s}_{\overline{20}|}}{\ddot{s}_{45:\overline{20}|}}$$

表示成一个年保费，并解释所得结果。

21. 根据示例生命表及利率 6%，计算以下两种分解的每个分量。

(1) $1000P_{50:\overline{20}|}=1000(P^{1}_{50:\overline{20}|}+P_{50:\overset{\ \ 1}{\overline{20}|}})$.

(2) $1000P_{50:\overline{20}|}=1000(\tilde{P}^{1}_{50:\overline{20}|}+\frac{1}{\ddot{s}_{\overline{20}|}})$.

22. 考虑与 (4.7.3) 类似的随机变量

$$\tilde{W}=\begin{cases} v^{T}(1-\frac{\bar{s}_{\overline{T}|}}{\bar{s}_{\overline{n}|}}) & 0\le T<n \\ 0 & T\ge n \end{cases}.$$

损失

$$L=\tilde{W}-\tilde{\bar{A}}^{1}{}_{x:\overline{n}|}$$

可用在平衡原理中决定这个特殊保单的净趸缴保费 $\tilde{\bar{A}}^{1}{}_{x:\overline{n}|}$，证明

(1) $\tilde{\bar{A}}^{1}{}_{x:\overline{n}|}=\bar{A}^{1}_{x:\overline{n}|}-\frac{\bar{a}_{x:\overline{n}|}-{}_np_x\bar{a}_{\overline{n}|}}{\bar{s}_{\overline{n}|}}$.

(2) $E[\tilde{W}^{2}]=\frac{(1+i)^{2n}\,{}^2\bar{A}^{1}_{x:\overline{n}|}-2(1+i)^{n}\bar{A}^{1}_{x:\overline{n}|}+(1-{}_np_x)}{[(1+i)^{n}-1]^{2}}$.

综合题

23. 将

$$A_{40}P_{40:\overline{25}|}+(1-A_{40})P_{40}$$

表示成净年缴保费，并解释所得结果。

24.

(1) 验证

$$\frac{1}{\ddot{a}_{65:\overline{10}|}} - \frac{1}{\ddot{s}_{65:\overline{10}|}} = P^{1}_{65:\overline{10}|} + d$$

(2) 与

$$\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}$$

相应的是什么？

(3) 验证由净趸缴保费 100000 提供在 10 年内 (65) 活着时每月初支付收入并在 (65) 达到 75 岁时返还趸缴保费的年收入额为

$$100000\left(\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}\right) = 100000(\beta),$$

其中 (β) 标记本题第 (2) 小题的答案。

(4) 验证第 (3) 小题中的年收入额可用计算基数表示成

$$100000\frac{D_{65} - D_{75}}{N^{(12)}_{65} - N^{(12)}_{75}}.$$

25. 向 (35) 签发的均衡保费缴至 65 岁的某种保险提供：被保险人活到 65 之时 100000，被保险人在 65 岁之前死亡的年末退还已缴的各年毛保费及其按估价利率计算的利息。如果年缴毛保费 G 是净年缴保费 π 的 1.1 倍，写出 π 的表达式。

26. 设 ${}_{15}p_{45} = 0.038$, $p_{45:\overline{15}|} = 0.056$, $A_{60} = 0.625$, 计算 $p^{1}_{45:\overline{15}|}$.

27. 某种缴费期为 20 年的寿险保单在死亡发生的情况下返还 10000 加上所有已缴的不计息毛保费。返还保费条款既适用于缴费期内也适用于缴费期之后。保费按年缴付，受益在死亡年末支付。如果年缴毛保费是净保费的 110% 加上 25，用计算基数计算向 (x) 签发的保单的年缴毛保费。

28. 用计算基数计算，根据以下条款向 (25) 签发的终身寿险的初始净年缴保费：

前 10 年面额为 1, 此后为 2;

前 10 年每次的保费是以后每次保费的 1/2;

保费按年缴付至 65 岁；

理赔在死亡年末支付。

29. 对于受益在死亡时即刻支付的情形，用计算基础重写表 4.4.1 中的保费。

30. 设 L_1 是按完全连续净保费基础向 (X) 签发 1 单位终身寿险保单的保险人亏损， L_2 是用净趸缴保费 1 购买连续支付的终身生存年金的 (x) 的亏损。证明 $L_1 \equiv L_2$ 并给出文字解释。

31. 按完全离散基础向 x 岁人签发的 1 单位普通寿险合同的年保费为 0.048. 设 $d = 0.06, A_x = 0.4, {}^2A_x = 0.2$, L 是在保单签发时保险人的亏损随机变量。

(1) 计算 $E[L]$.

(2) 计算 $\mathrm{Var}[L]$.

(3) 现考察有 100 份同类保单的业务，其面额情况如下：

面额	保单数
1	80
4	20

假定各保单的亏损独立，用正态近似计算整个业务的赢利现值超过 20 的概率。

第五章　净保费责任准备金

§5.1　引　　言

第四章引入了平衡原理,在双方相互交换一系列支付的长期契约签订日,用于建立一个平衡关系。如用在分期偿还贷款场合,则借款人的一系列等额偿付,在借款时等价于贷款人的一次性贷款额;在保险场合,投保人缴付的一系列净保费,在保单生效时等价于根据被保险人未来死亡或生存而赔付的保险金;个人购买延期生存年金而缴付的一系列净保费,在契约签订日等价于年金机构未来的一系列按期生存支付。在贷款场合,平衡关系是根据现值建立的,而在保险与年金场合,平衡关系则是根据双方支付的精算现值建立的。

经过一段时间,双方未竟责任的平衡关系会被打破。例如,借款人可能尚有若干次分期还款待偿付,而贷款人则早已履行其责任;投保人可能还需缴纳净保费,同时保险人负有支付受益金的责任;某人的年金缴费期可能已经结束,而年金机构却还需继续按期支付。

这一章将应用平衡原理于契约开始生效以后的时期,此时出现了一个平衡项,它对于其中一方是负债项,而对于另一方则是资产(权益)项。例如在贷款场合,该平衡项即未清偿贷款,它是贷款人的资产,而是借款人的负债;在保险与年金场合,该平衡项称为净保费责任准备金(net premium reserve),它是保险或年金机构在其财务报表中必须确认的负债项,对于投保或购买年金的个人而言,这也是一份资产(权益)。

这一章的不少小节与第四章有关净保费的小节平行,而且我

们将假定保单签发时决定净保费所采用的死亡率与利率仍适用于决定净保费责任准备金。

例 5.1.1：设按例 4.1.1 订立合约的被保险人在 1 年后仍然活着，按例 4.1.1 决定的年保费 0.3667，求那时的未来责任的价值。

解：整值剩余寿命 K 的概率函数等于 1/4，$k=0,1,2,3$. 在给定 $K\geqslant 1$ 条件下，K 的条件概率函数为

$$Pr[K=k\mid K\geqslant 1]=\frac{Pr(K=k)}{Pr(K\geqslant 1)}=\frac{1/4}{3/4}=\frac{1}{3}\quad k=1,2,3.$$

那时的现值列表如下：

未来责任的现值(保单签发后 1 年，$i=0.06$)

结果 k	条件概率	保险人	被保险人	保险人的前瞻亏损
1	1/3	$v=0.9434$	$P=0.3667$	0.5767
2	1/3	$v^2=0.8900$	$P\ddot{a}_{\overline{2}\mid}=0.7126$	0.1774
3	1/3	$v^3=0.8396$	$P\ddot{a}_{\overline{3}\mid}=1.0390$	-0.1994

保险人责任的精算现值为

$$\frac{1}{3}(0.9434+0.8900+0.8396)=0.8910,$$

类似地，被保险人责任的精算现值为 0.7061，平衡项

$$0.8910-0.7061=0.1849$$

就是保单签发后 1 年(时间为 1)第二次保费即将缴付前的净保费责任准备金。

换一个角度，我们可考察前瞻亏损的期望值。对 K 的每一个取值，前瞻亏损是保险人责任的现值与被保险人责任的现值之差。前瞻亏损的期望值为

$$\frac{1}{3}(0.5767+0.1774-0.1994)=0.1849$$

§5.2 完全连续净保费责任准备金

这一章考虑与 §4.2 讨论的净保费相联系的责任准备金。对于 (x) 的 1 单位终身人寿保险，其完全连续净保费为 $\overline{P}(\overline{A}_x)$ 。从保单生效算起，被保险人生存到 t 年时相应的责任准备金记为 ${}_t\overline{V}(\overline{A}_x)$ 。为根据平衡原理决定 ${}_t\overline{V}(\overline{A}_x)$, 引入随机变量 U, 它是 $(x+t)$ 的剩余寿命，其概率密度函数为

$$ {}_up_{x+t}\mu_{x+t+u} \qquad u \geq 0. $$

定义在时间 t 的 前瞻亏损(prospective loss) 变量

$$ {}_tL = v^U - \overline{P}(\overline{A}_x)\overline{a}_{\overline{U|}}. \tag{5.2.1} $$

净保费责任准备金是前瞻亏损的期望值，即

$$ \begin{aligned} {}_t\overline{V}(\overline{A}_x) &= E[v^U] - \overline{P}(\overline{A}_x)E[\overline{a}_{\overline{U|}}] \\ &= \overline{A}_{x+t} - \overline{P}(\overline{A}_x)\overline{a}_{x+t}. \end{aligned} \tag{5.2.2} $$

这个公式表明，时间 t 时的责任准备金等于，从 $x+t$ 岁开始的人寿保险之精算现值，减去 $x+t$ 岁以后年保费 $\overline{P}(\overline{A}_x)$ 的精算现值 (这里的精算现值系指时刻 t 的值) 。

当 $t=0$ 时，公式 (5.2.2) 成为 ${}_0\overline{V}(\overline{A}_x)=0$, 这是在决定净保费时应用平衡原理的结果。注意到 U 的分布是在给定 $T>t$ 条件下 $T-t$ 的条件分布， U 的分布函数因而是

$$ 1 - \frac{{}_{t+u}p_x}{{}_tp_x} = {}_uq_{x+t} \qquad u \geq 0, $$

其概率密度函数为

$$ \frac{{}_{t+u}p_x\mu_{x+t+u}}{{}_tp_x} = {}_up_{x+t}\mu_{x+t+u} \qquad u \geq 0. $$

按导出 (4.2.6) 类似的步骤可得

$$
{}_tL = v^U\left[1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right] - \frac{\overline{P}(\overline{A}_x)}{\delta}. \tag{5.2.3}
$$

于是

$$
\begin{aligned}
\mathrm{Var}[{}_tL] &= \left[1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right]^2 \mathrm{Var}[v^U] \\
&= \left[1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right]^2 [{}^2\overline{A}_{x+t} - (\overline{A}_{x+t})^2].
\end{aligned} \tag{5.2.4}
$$

例 5.2.1: 计算例 4.2.1 中相应的 ${}_t\overline{V}(\overline{A}_x)$ 与 $\mathrm{Var}[{}_tL]$ 。

解: 此时 $\overline{A}_x, \overline{a}_x$ 及 $\overline{P}(\overline{A}_x)$ 与年龄 x 无关，从而 (5.2.2) 成为

$$
{}_t\overline{V}(\overline{A}_x) = \overline{A}_x - \overline{P}(\overline{A}_x)\overline{a}_x = 0 \qquad t \geq 0.
$$

在这一例子中，未来保费始终与未来受益平衡，毋须责任准备金去平衡。

类似地，(5.2.4) 成为

$$
\mathrm{Var}[{}_tL] = \left[1 + \frac{\overline{P}(\overline{A}_x)}{\delta}\right]^2 [{}^2\overline{A}_x - (\overline{A}_x)^2] = \mathrm{Var}[L] = 0.25,
$$

与例 4.2.1 的结果相同。这里方差既不依赖于年龄 x 也不依赖于持续时间 t 。

例 5.2.2: 设死亡遵从 deMoivre 律： $l_x = 100 - x,\ 0 \leq x < 100$, 按利率 6% 计算

(1) $\overline{P}(\overline{A}_{35})$.

(2) ${}_t\overline{V}(\overline{A}_{35})$ 及 $\mathrm{Var}[{}_tL], t = 0, 10, 20, \cdots, 60$.

解: (1) 由 $l_x = 100 - x$ 可知， ${}_tp_{35} = 1 - t/65$, ${}_tp_{35}\mu_{35+t} = 1/65, 0 \leq t < 65$, 因此

$$
\overline{A}_{35} = \int_0^{65} v^t \frac{1}{65} dt = \frac{1}{65}\overline{a}_{\overline{65|}} = 0.258047,
$$

$$\overline{a}_{35}=\frac{1-\overline{A}_{35}}{log1.06}=12.7333,$$

$$\overline{P}(\overline{A}_{35})=\frac{0.258047}{12.7333}=0.020266.$$

(2) 在 $35+t$ 岁, $\overline{A}_{35+t}=\overline{a}_{\overline{65-t}|}/(65-t)$,

$$_t\overline{V}(\overline{A}_{35})=\overline{A}_{35+t}-0.020266\frac{1-\overline{A}_{35+t}}{log1.06}.$$

又

$$^2\overline{A}_{35+t}=\int_0^{65-t}v^{2u}\frac{1}{65-t}du=\frac{1}{65-t}{}^2\overline{a}_{\overline{65-t}|}.$$

由 (5.2.4),

$$\text{Var}[_tL]=(1+\frac{0.020266}{\log 1.06})^2[^2\overline{A}_{35+t}-(\overline{A}_{35+t})^2].$$

用以上公式可算得结果如下:

t	$_t\overline{V}(\overline{A}_{35})$	$\text{Var}[_tL]$
0	0.0000	0.1187
10	0.0577	0.1001
20	0.1289	0.1174
30	0.2271	0.1073
40	0.3619	0.0861
50	0.5508	0.0508
60	0.8214	0.0097

与定额受益及净保费类似, 对一般的完全连续保险, 可定义前瞻亏损为

$$_tL=b_{t+U}v^U-\int_0^U\pi_{t+s}v^sds,$$

其中 b_{t+U} 是死亡发生在 $t+U$ 时的受益金额, π_{t+s} 是在时间 $t+s$ 缴付的完全连续保费 (年) 率, 这个一般情形的净保费责任

准备金为

$$\begin{aligned}
{}_t\overline{V} &= E[{}_tL] \\
&= \int_0^{\infty}(b_{t+u}v^u - \int_0^u \pi_{t+s}v^s ds)_u p_{x+t}\mu_{x+t+u}du \\
&= \int_0^{\infty} b_{t+u}v^u{}_u p_{x+t}\mu_{x+t+u}du - \int_0^{\infty}\pi_{t+s}v^s{}_s p_{x+t}ds,
\end{aligned} \tag{5.2.5}$$

其中的第二项积分系应用定理 1.5.1 或交换积分次序而得出。这样，${}_t\overline{V}$ 可表示成时间 t 以后受益的精算现值与净保费的精算现值之差。

与表 4.2.1 相对应，表 5.2.1 概括了各种人寿保险的责任准备金，但并未详细给出前瞻亏损 ${}_tL$ 及 $\mathrm{Var}[{}_tL]$ 的公式。

表 5.2.1　完全连续净保费责任准备金
(保单生效年龄 x，持续时间 t，单位保额)

种类	责任准备金符号	前瞻公式
终身人寿保险	${}_t\overline{V}(\overline{A}_x)$	$\overline{A}_{x+t} - \overline{P}(\overline{A}_x)\overline{a}_{x+t}$
n 年定期保险	${}_t\overline{V}(\overline{A}^{\,1}_{x:\overline{n}\vert})$	$\overline{A}^{\,1}_{x+t:\overline{n-t}\vert} - \overline{P}(\overline{A}^{\,1}_{x:\overline{n}\vert})\overline{a}_{x+t:\overline{n-t}\vert}$　$t<n$
		0　$t=n$
n 年期两全保险	${}_t\overline{V}(\overline{A}_{x:\overline{n}\vert})$	$\overline{A}_{x+t:\overline{n-t}\vert} - \overline{P}(\overline{A}_{x:\overline{n}\vert})\overline{a}_{x+t:\overline{n-t}\vert}$　$t<n$
		1　$t=n$
h 年缴费终身人寿保险	${}^h_t\overline{V}(\overline{A}_x)$	$\overline{A}_{x+t} - {}_h\overline{P}(\overline{A}_x)\overline{a}_{x+t:\overline{h-t}\vert}$　$t<h$
		$\overline{A}_{x+t}$　$t\geq h$
h 年缴费 n 年期两全保险	${}^h_t\overline{V}(\overline{A}_{x:\overline{n}\vert})$	$\overline{A}_{x+t:\overline{n-t}\vert} - {}_h\overline{P}(\overline{A}_{x:\overline{n}\vert})\overline{a}_{x+t:\overline{h-t}\vert}$　$t<h$
		$\overline{A}_{x+t:\overline{n-t}\vert}$　$h\leq t<n$
		1　$t=n$
n 年期生存保险	${}_t\overline{V}(A_{x:\overline{n}\vert}^{\ \ 1})$	$A_{x+t:\overline{n-t}\vert}^{\ \ \ \ 1} - \overline{P}(A_{x:\overline{n}\vert}^{\ \ 1})\overline{a}_{x+t:\overline{n-t}\vert}$　$t<n$
		1　$t=n$
n 年递延生存年金	${}_t\overline{V}({}_{n\vert}\overline{a}_x)$	$A_{x+t:\overline{n-t}\vert}^{\ \ \ \ 1}\overline{a}_{x+n} - \overline{P}({}_{n\vert}\overline{a}_x)\overline{a}_{x+t:\overline{n-t}\vert}$　$t<n$
		$\overline{a}_{x+t}$　$t\geq n$

§5.3 完全连续责任准备金的其它公式

前一节我们只是用了一种方法来写出完全连续责任准备金的公式，即前瞻方法(prospective method)。据此，责任准备金是未竟受益与净保费的精算现值之差。从前瞻方法很容易建立均衡保费(年)率保单的另外三个公式。以下以 n 年期两全保险为例进行说明。

从表 5.2.1 的 ${}_t\overline{V}(\overline{A}_{x:\overline{n}|})$ 公式中提取因子 $\overline{a}_{x+t:\overline{n-t}|}$，得 保费差公式 (premium-difference formula):

$$\begin{aligned}{}_t\overline{V}(\overline{A}_{x:\overline{n}|}) &= [\frac{\overline{A}_{x+t:\overline{n-t}|}}{\overline{a}_{x+t:\overline{n-t}|}} - \overline{P}(\overline{A}_{x:\overline{n}|})]\overline{a}_{x+t:\overline{n-t}|} \\ &= [\overline{P}(\overline{A}_{x+t:\overline{n-t}|}) - \overline{P}(\overline{A}_{x:\overline{n}|})]\overline{a}_{x+t:\overline{n-t}|}. \quad (5.3.1)\end{aligned}$$

它将责任准备金表示成剩余缴费期内保费差的精算现值，其保费差系从 $x+t$ 岁时按剩余受益计算的等价年保费中减去原保费。

从 ${}_t\overline{V}(\overline{A}_{x:\overline{n}})$ 的前瞻公式中提取剩余受益的精算现值，可得第二个公式

$$\begin{aligned}{}_t\overline{V}(\overline{A}_{x:\overline{n}|}) &= [1 - \overline{P}(\overline{A}_{x:\overline{n}|})\frac{\overline{a}_{x+t:\overline{n-t}|}}{\overline{A}_{x+t:\overline{n-t}|}}]\overline{A}_{x+t:\overline{n-t}|} \\ &= [1 - \frac{\overline{P}(\overline{A}_{x:\overline{n}|})}{\overline{P}(\overline{A}_{x+t:\overline{n-t}|})}]\overline{A}_{x+t:\overline{n-t}|}. \quad (5.3.2)\end{aligned}$$

它将责任准备金表示成部分受益的精算现值，该部分并不能由仍将收取的净保费所提供。注意到 $\overline{P}(\overline{A}_{x+t:\overline{n-t}|})$ 是满足未来受益所应该缴付的未来净年缴保费，而 $\overline{P}(\overline{A}_{x:\overline{n}|})$ 则是实际缴付的净保费，因而 $\overline{P}(\overline{A}_{x:\overline{n}|})/\overline{P}(\overline{A}_{x+t:\overline{n-t}|})$ 是由实缴保费所提供的受益比例。以上公式称为缴清保险公式(paid-up insurance formula)，该名称来源于第十章将讨论的不没收受益缴清保险。与 (5.3.1) 及 (5.3.2) 类似的公式也对其它一大类保险责任准备金成立。

第三个表达式称为*后顾公式*(或*追溯公式*(retrospective formula))。我们从一个更一般的关系开始，根据第二章习题 12 以及式 (3.2.22),(3.2.24), 对 $t < n - s$,

$$\overline{A}_{x+s:\overline{n-s}|} = \overline{A}^{1}_{x+s:\overline{t}|} +_t E_{x+s}\overline{A}_{x+s+t:\overline{n-s-t}|},$$

$$\overline{a}_{x+s:\overline{n-s}|} = \overline{a}_{x+s:\overline{t}|} +_t E_{x+t}\overline{a}_{x+s-t:\overline{n-s-t}|}.$$

将它们代入 ${}_s\overline{V}(\overline{A}_{x:\overline{n}|})$ 的前瞻公式，可得

$$\begin{aligned}{}_s\overline{V}(\overline{A}_{x:\overline{n}|}) &= \overline{A}^{1}_{x+s:\overline{t}|} - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+s:\overline{t}|} \\ &\quad +_t E_{x+s}[\overline{A}_{x+s+t:\overline{n-s-t}|} - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+s+t:\overline{n-s-t}|}] \\ &= \overline{A}^{1}_{x+s:\overline{t}|} + {}_tE_{x+s}\,{}_{s+t}\overline{V}(\overline{A}_{x:\overline{n}|}) - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+s:\overline{t}|}.\end{aligned} \tag{5.3.3}$$

这样，在一个时间区间始末的责任准备金有以下联系：

初始时的责任准备金等于，期间应付死亡受益的精算现值，加上以期终时责任准备金为保额的 (纯) 生存保险精算现值，再减去期间净保费的精算现值。

将 (5.3.3) 整理一下，得

$${}_s\overline{V}(\overline{A}_{x:\overline{n}|}) + \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+s:\overline{t}|} = \overline{A}^{1}_{x+s:\overline{t}|} + {}_tE_{x+s}\,{}_{s+t}\overline{V}(\overline{A}_{x:\overline{n}|}), \tag{5.3.4}$$

它表明，保险人的来源与负担的精算现值相等。在式 (5.3.4) 中置 $s = 0$ 就得出后顾公式，注意到 ${}_0\overline{V}(\overline{A}_{x:\overline{n}|}) = 0$, 有

$${}_t\overline{V}(\overline{A}_{x:\overline{n}|}) = \frac{1}{{}_tE_x}[\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x:\overline{t}|} - \overline{A}^{1}_{x:\overline{t}|}].$$

又根据 $\overline{s}_{x:\overline{t}|} = \overline{a}_{x:\overline{t}|}/{}_tE_x$, 以上公式可化为

$${}_t\overline{V}(\overline{A}_{x:\overline{n}|}) = \overline{P}(\overline{A}_{x:\overline{n}|})\overline{s}_{x:\overline{t}|} - {}_t\overline{k}_x, \tag{5.3.5}$$

这里，

$$
{}_t\overline{k}_x = \frac{\overline{A}^{\ 1}_{x:\overline{t}|}}{{}_tE_x} \tag{5.3.6}
$$

称为保险成本积累值(accumulated cost of insurance)。显然

$$
\begin{aligned}
{}_t\overline{k}_x &= \int_0^t \frac{v^s {}_sp_x\mu_{x+s}}{v^t {}_tp_x} ds \\
&= \frac{\int_0^t (1+i)^{t-s} l_{x-s}\mu_{x+s} ds}{l_{x+t}}
\end{aligned} \tag{5.3.7}
$$

可解释为生存组在年龄 x 与 $x+t$ 之间死亡赔付积累值按 l_{x+t} 个生存者每人的估价。这样，责任准备金可看成净保费的精算积累值 (按利息累积并在 $x+t$ 岁生存者中间分摊) 与保险成本积累值之差。

在进行数值计算时，究竟该使用前瞻公式还是使用后顾公式，有以下两条指导原则：

(1) 在持续时间超出缴费期的场合，前瞻公式更为便利，此时，责任准备金简化为未来受益的精算现值，如对 h 年缴费的终身人寿保险保险当 $t \geq h$ 时，${}^h_t\overline{V}(\overline{A}_x) = \overline{A}_{x+t}$ 。

(2) 在尚未提供受益的递延期内，后顾公式更方便，此时，责任准备金简化为过去净保费的精算积累值，例如，当 $t < n$ 时，${}_t\overline{V}({}_{n|}\ddot{a}_x^{(12)}) = \overline{P}({}_{n|}\ddot{a}_x^{(12)})\overline{s}_{x:\overline{t}|}$ 。

最后导出终身人寿保险责任准备金的几个特殊公式，类似公式对 n 年期两全保险责任准备金也成立，但对一般保险责任准备金不然。由公式 (4.2.9),$\overline{P}(\overline{A}_x) = (1/\overline{a}_x) - \delta$, 有

$$
\begin{aligned}
{}_t\overline{V}(\overline{A}_x) &= 1 - \delta\overline{a}_{x+t} - (\frac{1}{\overline{a}_x} - \delta)\overline{a}_{x+t} \\
&= 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}.
\end{aligned} \tag{5.3.8}
$$

利用 (5.3.1) 得

$$
{}_t\overline{V}(\overline{A}_x) = [\overline{P}(\overline{A}_{x+t}) - \overline{P}(\overline{A}_x)]\overline{a}_{x+t}
$$

$$= \frac{\overline{P}(\overline{A}_{x+t}) - \overline{P}(\overline{A}_x)}{\overline{P}(\overline{A}_{x+t}) + \delta}. \tag{5.3.9}$$

用 $\overline{A}_{x+t} = 1 - \delta\overline{a}_{x+t}$ 改写 (5.3.8), 还可得

$$_t\overline{V}(\overline{A}_x) = 1 - \frac{1 - \overline{A}_{x+t}}{1 - \overline{A}_x} = \frac{\overline{A}_{x+t} - \overline{A}_x}{1 - \overline{A}_x}. \tag{5.3.10}$$

§5.4 完全离散净保费责任准备金

这一节讨论的责任准备金与 §4.3 讨论的净年缴保费相联系，即死亡年末赔付受益并且按年缴付保费。考虑净保费为 P_x 的 1 单位终身人寿保险，其 k 年末的责任准备金记为 $_kV_x$ 。按 §4.3 及 §5.2, 定义 J 为 $(x+k)$ 的整值剩余寿命随机变量，其概率函数为 $_jp_{x+k}q_{x+k+j}$, $j = 0, 1, 2, \cdots$, 前瞻亏损定义为

$$_kL = v^{J+1} - P_x\ddot{a}_{\overline{J+1|}}, \tag{5.4.1}$$

由此按定义 $_kV_x = E[_kL]$ 得出

$$_kV_x = A_{x+k} - P_x\ddot{a}_{x+k}. \tag{5.4.2}$$

这个前瞻公式表明，责任准备金 $_kV_x$ 是从 $x+k$ 岁开始的终身人寿保险精算现值与未来实缴保费 P_x 的精算现值之差。将这里与 §5.2 比较， $J = K - k$(在 $K > k$ 条件下) 的概率函数为

$$_jp_{x+k}q_{x+k+j} = \frac{_{k+j}p_x q_{x+k+j}}{_kp_x} \qquad j = 0, 1, 2, \cdots,$$

相当于 (x) 在活到 $x+k$ 岁条件下整值剩余寿命的条件分布。

与 (5.2.4) 类似，有

$$\begin{aligned}\mathrm{Var}[_kL] &= \mathrm{Var}\left[v^{J+1}(1 + \frac{P_x}{d})\right] \\ &= \left[1 + \frac{P_x}{d}\right]^2 \mathrm{Var}[v^{J+1}].\end{aligned} \tag{5.4.3}$$

例 5.4.1: 计算例 4.3.1 的 ${}_kV_x$ 与 $\mathrm{Var}[{}_kL]$ 。

解: 由于此例中 $A_x, \ddot{a}_x, P_x$ 均与年龄 x 无关，所以

$$
{}_kV_x = A_x - P_x\ddot{a}_x = 0 \qquad k = 0, 1, 2, \cdots.
$$

又由 (5.4.3) 可算出 $\mathrm{Var}[{}_kL] = \mathrm{Var}[L] = 0.2347$ 。

现考虑 (x) 的更一般完全离散保险：死亡受益在死亡的保单年度末赔付，第 j 年的受益额为 $b_j, j = 1, 2, , \cdots$; 保费在每个保单年度期初缴付，第 j 年的保费为 $\pi_{j-1}, j = 1, 2, \cdots$ 。在第 k 个保单年度末的前瞻亏损为

$$
{}_kL = b_{k+J+1}v^{J-1} - \sum_{h=0}^{J} \pi_{k+h}v^h. \tag{5.4.4}
$$

净保费责任准备金为

$$
\begin{aligned}
{}_kV &= E[{}_kL] = \sum_{j=0}^{\infty}\left[b_{k+j+1}v^{j+1} - \sum_{h=0}^{j}\pi_{k+h}v^h\right]_j p_{x+k}q_{x+k+j} \\
&= \sum_{j=0}^{\infty} b_{k+j+1}v^{j+1}{}_jp_{x+k}q_{x+k+j} - \sum_{h=0}^{\infty}\pi_{x+h}v^h{}_hp_{x+k},
\end{aligned} \tag{5.4.5}
$$

其中第二个和式通过应用定理 1.5.2 或改变求和次序得出。于是，${}_kV$ 也是未来受益精算现值与未来净保费精算现值之差。表 5.4.1 列出的责任准备金公式与净保费表 4.3.1 对应，并且与责任准备金表 5.2.1 类似。

例 5.4.2: 决定 n 年期两全保险的 $\mathrm{Var}[{}_kL]$ 公式。

解: 由

$$
{}_kL = \begin{cases} v^{J+1}[1 + \frac{P_{x:\overline{n}|}}{d}] - \frac{P_{x:\overline{n}|}}{d} & J < n-k \\ v^{n-k}[1 + \frac{P_{x:\overline{n}|}}{d}] - \frac{P_{x:\overline{n}|}}{d} & J \geq n-k, \end{cases}
$$

表 5.4.1　完全离散净保费责任准备金
(保单生效年龄 x, 持续时间 k, 单位保额)

种类	责任准备金记号	前瞻公式
终身人寿保险	${}_kV_x$	$A_{x+k}-P_x\ddot{a}_{x+k}$
n 年定期保险	${}_kV^1_{x:\overline{n}\vert}$	$A^1_{x+k:\overline{n-k}\vert}-P^1_{x:\overline{n}\vert}\ddot{a}_{x+k:\overline{n-k}\vert}\quad k<n$ $0\quad k=n$
n 年期两全保险	${}_kV_{x:\overline{n}\vert}$	$A_{x+k:\overline{n-k}\vert}-P_{x:\overline{n}\vert}\ddot{a}_{x+k:\overline{n-k}\vert}\quad k<n$ $1\quad k=n$
h 年缴费终身人寿保险	${}^h_kV_x$	$A_{x+k}-{}_hP_x\ddot{a}_{x+k:\overline{h-k}\vert}\quad k<h$ $A_{x+k}\quad k\geq h$
h 年缴费 n 年期两年保险	${}^h_kV_{x:\overline{n}\vert}$	$A_{x+k:\overline{n-k}\vert}-{}_hP_{x:\overline{n}\vert}\ddot{a}_{x+k:\overline{h-k}\vert}\quad k<h$ $A_{x+k:\overline{n-k}\vert}\quad h\leq k<n$ $1\quad k=n$
n 年生存保险	${}_kV_{x:\overline{n}\vert}^{\ \ 1}$	$A_{x+k:\overline{n-k}\vert}^{\ \ \ \ 1}-P_{x:\overline{n}\vert}^{\ \ 1}\ddot{a}_{x+k:\overline{n-k}\vert}\quad k<n$ $1\quad k=n$
n 年递延生存年金	${}_kV({}_{n\vert}\ddot{a}_x)$	$A_{x+k:\overline{n-k}\vert}^{\ \ \ \ 1}\ddot{a}_{x+n}-P({}_{n\vert}\ddot{a}_x)\ddot{a}_{x+k:\overline{n-k}\vert}\quad k<n$ $\ddot{a}_{x+k}\quad k>n$

并根据 J 的概率函数为 ${}_jp_{x+k}q_{x+k+j}, j=0,1,\cdots$, 可得

$$\mathrm{Var}[{}_kL]=\left[1+\frac{P_{x:\overline{n}\vert}}{d}\right]^2[{}^2A_{x+k:\overline{n-k}\vert}-(A_{x+k:\overline{n-k}\vert})^2].$$

对终身人寿保险与两全保险以外的在整个保险期内缴费的其它险种，损失的方差表达式中包含许多项，在这种情况下，§5.9 的结果可有助于计算。

与 §5.3 类似的公式对完全离散责任准备金也成立，以下只对 ${}_kV_{x:\overline{n}\vert}$ 作一简单说明．其解释与推导与完全连续责任准备金基本平行。

保费差公式为

$$
{}_kV_{x:\overline{n}|} = (P_{x+k:\overline{n-k}|} - P_{x:\overline{n}|})\ddot{a}_{x+k:\overline{n-k}|}, \tag{5.4.6}
$$

缴清保险公式为

$$
{}_kV_{x:\overline{n}|} = \left[1 - \frac{P_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}\right] A_{x+k:\overline{n-k}|}. \tag{5.4.7}
$$

至于后顾公式，我们应首先建立一个与 (5.3.3) 类似的结果，即对于 $h < n - j$,

$$
{}_jV_{x:\overline{n}|} = A^{1}_{x+j:\overline{h}|} - P_{x:\overline{n}|}\ddot{a}_{x+j:\overline{h}|} + {}_hE_{x+j}\,{}_{j+h}V_{x:\overline{n}|}. \tag{5.4.8}
$$

当 $j = 0$ 时，因 ${}_0V_{x:\overline{n}|} = 0$, 有

$$
\begin{aligned}
{}_hV_{x:\overline{n}|} &= \frac{1}{{}_hE_x}(P_{x:\overline{n}|}\ddot{a}_{x:\overline{h}|} - A^{1}_{x:\overline{h}|}) \\
&= P_{x:\overline{n}|}\ddot{s}_{x:\overline{h}|} - {}_hk_x,
\end{aligned} \tag{5.4.9}
$$

这里，保险成本积累值为 ${}_hk_x = A^{1}_{x:\overline{h}|}/{}_hE_x$, 并可作生存组解释。

从责任准备金的后顾公式可得出一些有趣的关系。考虑 (x) 的两个不同保单，在开始的 h 年中受益金额都是 1 个单位，这里 h 小于或等于每个保单的缴费期。两者的责任准备金后顾公式为

$$
{}_hV_1 = P_1\ddot{s}_{x:\overline{h}|} - {}_hk_x,
$$

$$
{}_hV_2 = P_2\ddot{s}_{x:\overline{h}|} - {}_hk_x.
$$

于是

$$
{}_hV_1 - {}_hV_2 = (P_1 - P_2)\ddot{s}_{x:\overline{h}|}, \tag{5.4.10}
$$

即两者的责任准备金之差等于净保费差的精算积累值。因

$$
\frac{1}{\ddot{s}_{x:\overline{h}|}} = \frac{{}_hE_x}{\ddot{a}_{x:\overline{h}|}} = P_{x:\overline{h}|}^{\ \ 1}.
$$

公式 (5.4.10) 可整理成

$$P_1 - P_2 = P_{x:\overline{h}|}^{\ \ 1}({}_hV_1 - {}_hV_2). \tag{5.4.11}$$

现在，净 (年) 保费差表示成以 h 年末责任准备金之差为保额的 h 年期生存保险的净 (年) 保费。公式 (4.3.10) 是 (5.4.11) 当 ${}_nV_{x:\overline{n}|} = 1$ 与 ${}_n^nV_x = A_{x+n}$ 的特殊情形。另外，由 ${}_nV_{x:\overline{n}|}^1 = 0$，我们还可得

$$P_x = P_{x:\overline{n}|}^1 + P_{x:\overline{n}|}^{\ \ 1}\,{}_nV_x. \tag{5.4.12}$$

与完全连续情形相仿，完全离散的终身人寿保险与两全保险的责任准备金也有一些特殊公式。根据关系式 $A_y = 1 - d\ddot{a}_y$ 及 $1/\ddot{a}_y = P_y + d$，可得出 (5.3.8) 、 (5.3.9) 、 (5.3.10) 这些公式的平行公式：

$$\begin{aligned} {}_kV_x &= 1 - d\ddot{a}_{x+k} - (\frac{1}{\ddot{a}_x} - d)\ddot{a}_{x+k} \\ &= 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}, \end{aligned} \tag{5.4.13}$$

$${}_kV_x = 1 - \frac{1 - A_{x+k}}{1 - A_x} = \frac{A_{x+k} - A_x}{1 - A_x}, \tag{5.4.14}$$

$${}_kV_x = 1 - \frac{P_x - d}{P_{x+k} + d} = \frac{P_{x+k} - P_x}{P_{x+k} + d}. \tag{5.4.15}$$

例 5.4.3: 考虑向 l_{50} 个 50 岁的人发行保额为 1000 的完全离散 5 年定期的人寿保险，根据附录的示例生命表以及 6% 的利率，追踪这组保单的预期现金流动，并附带得出净保费的责任准备金。

解: 先计算净年缴保费 $\pi = 1000P_{50:\overline{5}|}^1 = 6.55692$，与这组保单相关，通过收取保费、贷记利息、理赔支付的预期资金积累变动情况列表如下：

(1) 年 h	(2) 年初预期保费 $l_{50+h-1}\pi$	(3) 年初预期积累 $(2)_h+(6)_{h-1}$	(4) 预期利息 $0.06(3)_h$
1	586903	586903	35214
2	583429	675662	40540
3	579682	724452	43467
4	575640	727143	43629
5	571280	676987	40619

年 h	(5) 预期死亡赔付 $1000d_{50+h-1}$	(6) 年末预期积累 $(3)_h+(4)_h-(5)_h$	(7) 年末预期生存人数 l_{50+h}	(8) $1000\times{}_hV^1_{50:\overline{5}\vert}$ $(6)_h/(7)_h$
1	529884	92233	88979.11	1.04
2	571432	144770	88407.68	1.64
3	616416	151503	87791.26	1.73
4	665063	105707	87126.20	1.21
5	717606	0	86408.60	0.00

例 5.4.4: 考虑向 l_{50} 个 50 岁人发行保额为 1000 的完全离散 5 年期两全保险，也根据示例生命表以及 6% 的利率，追踪这组保单的预期现金流动，并附带得出净保费责任准备金。

解: 此时，净年缴保费 $\pi=1000P_{50:\overline{5}\vert}=170.083$，预期现金流动列表如下：

(1) 年 h	(2) 年初预期保费 $l_{50+h-1}\pi$	(3) 年初预期积累 $(2)_h+(6)_{h-1}$	(4) 预期利息 $0.06(3)_h$
1	15223954	15223954	913417
2	15133829	30741336	1844480
3	15036638	47051022	2823061
4	14931796	64189463	3851368
5	14818680	82194446	4931667

(1) 年 h	(5) 预期死亡赔付 $1000d_{50+h-1}$	(6) 年末预期积累 $(3)_h-(4)_h-(5)_h$	(7) 年末预期生存人数 l_{50+h}	(8) $1000\times$ ${}_hV_{50:\overline{5}\rvert}$ $(6)_h/(7)_h$
1	529884	15607507	88979.11	175 .41
2	571432	32014384	88407.68	362 .12
3	616416	49257667	87791.26	561 .08
4	665065	67375766	87126.20	773 .31
5	717606	86408507	86408.60	1000 .00

图 5.4.1 与 5.4.2 显示了以上两例的预期保费与预期死亡赔付。在例 5.4.3 中，开始两年的预期保费超过预期死亡赔付，随后则低于赔付，开始 2 年超出部分保费的积累在随后赔付较高时动用，在 5 年期满时预期积累耗尽。

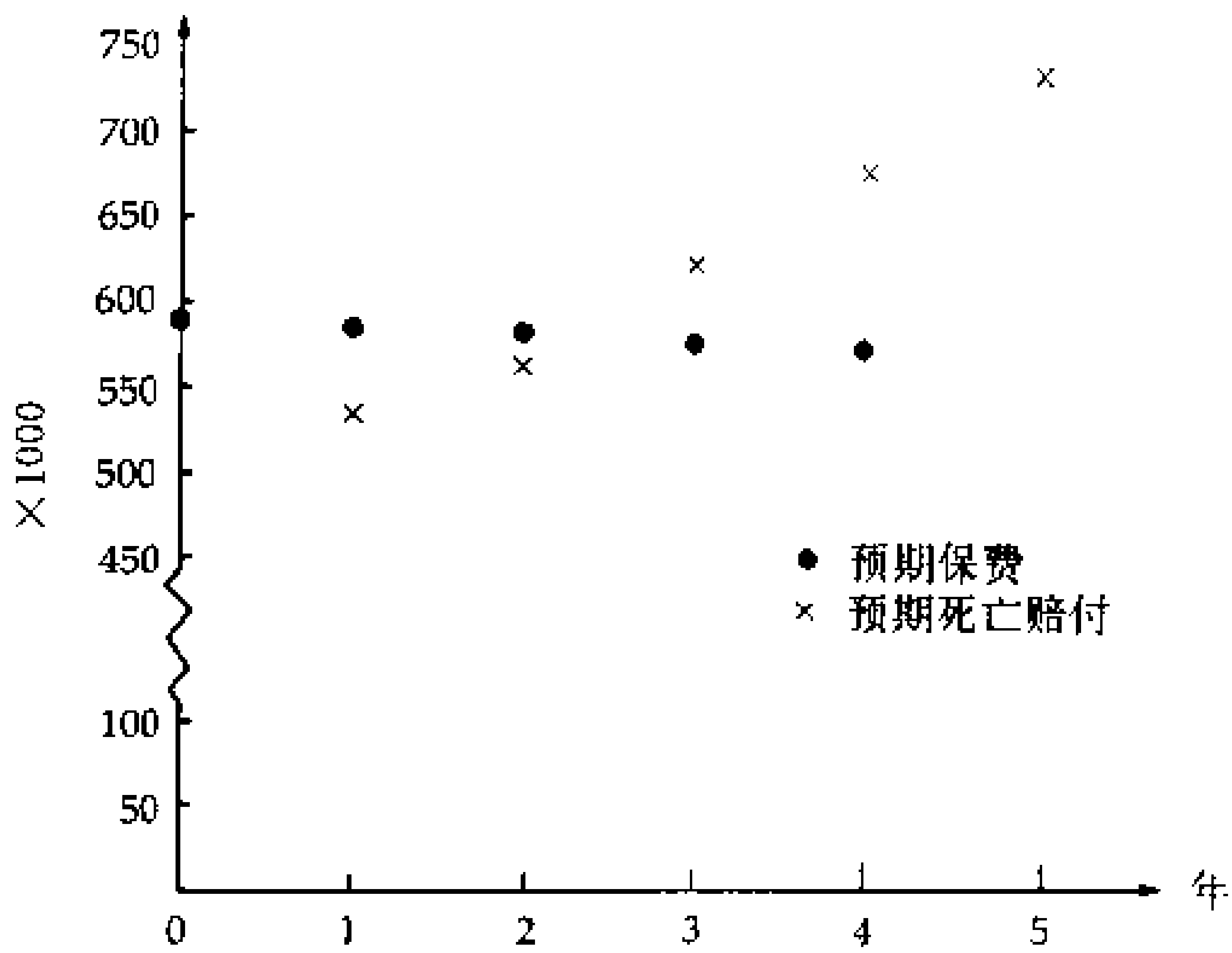


图 5.4.1　例 5.4.3

在例 5.4.4 两全保险场合，图 5.4.2 与前例截然不同，预期保费始终远远超出死亡赔付，在第 5 年末，预期积累正好可用来提

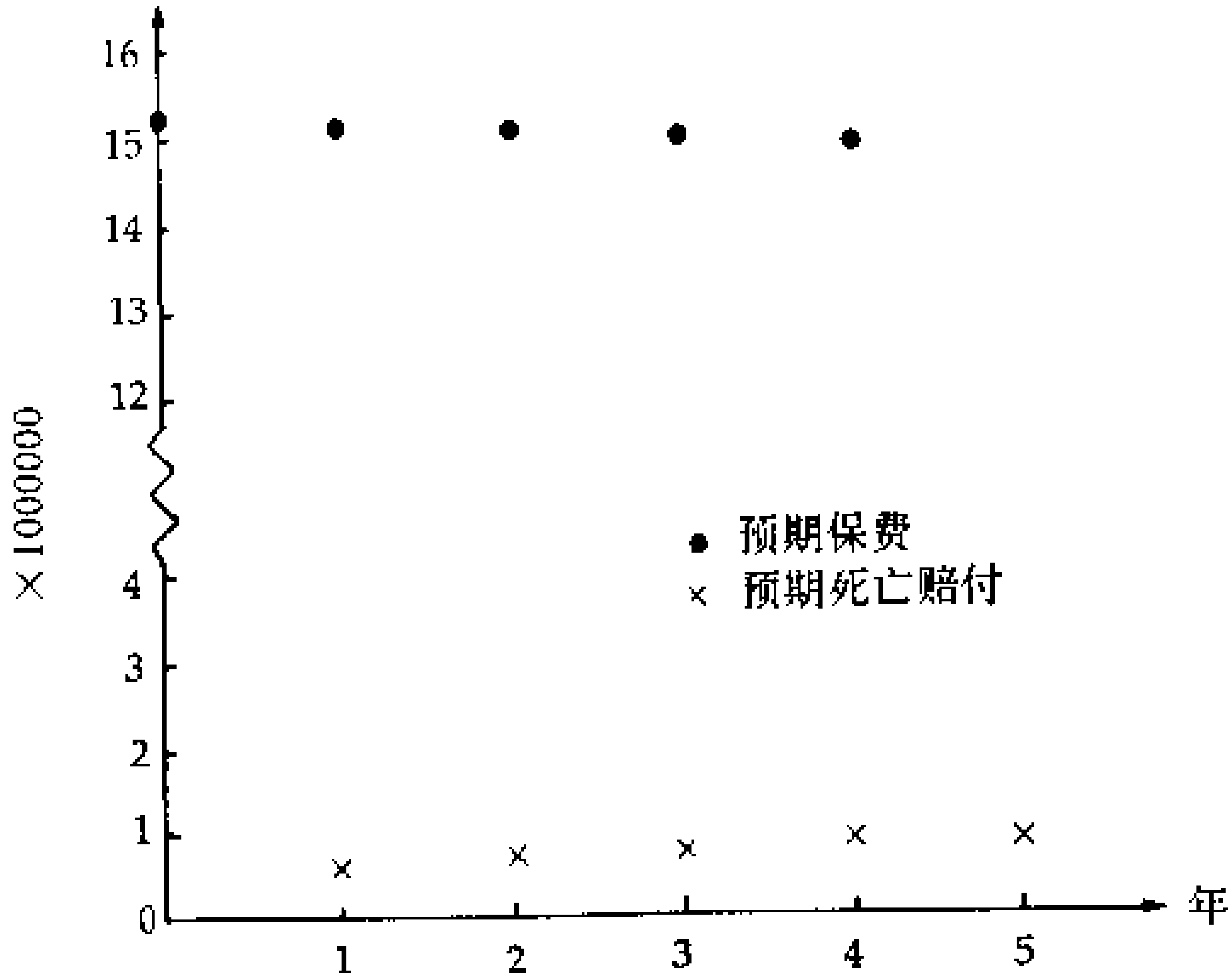


图 5.4.2　例 5.4.4

供预期生存者每人 1000 的到期支付。

以上 5 年定期保单是一个低保费低积累人寿保险的例子，而 5 年期两全保单则是一个高保费高积累的例子，大多数人寿保险介于这两个极端之间。

§5.5　半连续保费及真正 m 次保费责任准备金

根据保险实践，有必要考虑死亡即刻赔付的半连续净保费 $P(\overline{A}_x), P(\overline{A}^{\,1}_{x:\overline{n}|}), P(\overline{A}_{x:\overline{n}|}), {}_hP(\overline{A}_x), {}_hP(\overline{A}_{x:\overline{n}|})$，此时，表 5.4.1 中的责任准备金公式中，需要将 A 改为 $\overline{A}$, P 改为 $P(\overline{A})$ 。例如，h 年缴费 n 年期两全保险的半连续净保费责任准备金为

$$
{}^h_kV(\overline{A}_{x:\overline{n}|}) = \begin{cases} \overline{A}_{x+k:\overline{n-k}|} - {}_hP(\overline{A}_{x:\overline{n}|})\ddot{a}_{x+k:\overline{n-k}|} & k < h \\ \overline{A}_{x+k:\overline{n-k}|} & h \le k < n. \end{cases} \tag{5.5.1}
$$

假定在每一年中死亡均匀分布，那么由 (2.4.2) 及 (4.3.12) 可得

$$ {}_k^hV(\overline{A}_{x:\overline{n}|}) = \frac{i}{\delta}{}_k^hV_{x:\overline{n}|}^1 + {}_k^hV_{x:\overline{n}|}^{\ \ 1}. \qquad (5.5.2) $$

这种场合的半连续责任准备金很容易通过相应的完全离散责任准备金计算。

对于 §4.4 讨论的真正年缴 m 次保费，用前瞻方法可直接写出责任准备金公式

$$ {}_k^hV_{x:\overline{n}|}^{(m)} = A_{x+k:\overline{n-k}|} - {}_hP_{x:\overline{n}|}^{(m)}\ddot{a}_{x+k:\overline{h-k}|}^{(m)} \qquad k < h. \qquad (5.5.3) $$

现对 $k < h$ 考察差

$$ \begin{aligned} {}_k^hV_{x:\overline{n}|}^{(m)} - {}_k^hV_{x:\overline{n}|} &= {}_hP_{x:\overline{n}|}\ddot{a}_{x+k:\overline{h-k}|} - {}_hP_{x:\overline{n}|}^{(m)}\ddot{a}_{x+k:\overline{h-k}|}^{(m)} \\ &= {}_hP_{x:\overline{n}|}^{(m)}\frac{\ddot{a}_{x:\overline{h}|}^{(m)}}{\ddot{a}_{x:\overline{h}|}}\ddot{a}_{x+k:\overline{h-k}|} - {}_hP_{x:\overline{n}|}^{(m)}\ddot{a}_{x+k:\overline{h-k}|}^{(m)}. \qquad (5.5.4) \end{aligned} $$

在每一年中死亡均匀分布的假设下，(5.5.4) 成为

$$ \begin{aligned} {}_k^hV_{x:\overline{n}|}^{(m)} - {}_k^hV_{x:\overline{n}|} = {}_hP_{x:\overline{n}|}^{(m)}\Big\{ & \frac{\ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_{x:\overline{h}|} - \beta(m)A_{x:\overline{h}|}^1}{\ddot{a}_{x:\overline{h}|}}\ddot{a}_{x+k:\overline{h-k}|} \\ & - \left[\ddot{a}_{\overline{1}|}^{(m)}\ddot{a}_{x+k:\overline{h-k}|} - \beta(m)A_{x+k:\overline{h-k}|}^{1}\right]\Big\}, \end{aligned} $$

涉及 $\ddot{a}_{\overline{1}|}^{(m)}$ 的项抵消掉后成为

$$ \begin{aligned} {}_k^hV_{x:\overline{n}|}^{(m)} - {}_k^hV_{x:\overline{n}|} &= \beta(m){}_hP_{x:\overline{n}|}^{(m)}(A_{x+k:\overline{h-k}|}^{1} - P_{x:\overline{h}|}^1\ddot{a}_{x+k:\overline{h-k}|}) \\ &= \beta(m){}_hP_{x:\overline{n}|}^{(m)}{}_kV_{x:\overline{h}|}^1. \qquad (5.5.5) \end{aligned} $$

这表明，真正 m 次保费责任准备金等于，相应的完全离散责任准备金，加上保额为真正 m 次保费一部分 [比例 $\beta(m)$]、期限为缴费期 (h 年) 的定期保险完全离散责任准备金。

在死亡均匀分布假设下，对半连续真正 m 次保费的责任准备金，也有类似的结果。根据前瞻方法，对 $k < h$，有

$$ {}_k^hV^{(m)}(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+k:\overline{n-k}|} - {}_hP^{(m)}(\overline{A}_{x:\overline{n}|})\ddot{a}^{(m)}_{x+k:\overline{h-k}|}. \quad (5.5.6)$$

按导出 (5.5.5) 类似的步骤可得

$$ {}_k^hV^{(m)}(\overline{A}_{x:\overline{n}|}) = {}_k^hV(\overline{A}_{x:\overline{n}|}) + \beta(m)_hP^{(m)}(\overline{A}_{x:\overline{n}|})_kV^1_{x:\overline{h}|}. \quad (5.5.7)$$

令 $m \to \infty$，可得到完全连续责任准备金与其它责任准备金的一个关系：

$$ {}_k^h\overline{V}(\overline{A}_{x:\overline{n}|}) = {}_k^hV(\overline{A}_{x:\overline{n}|}) + \beta(\infty)_h\overline{P}(\overline{A}_{x:\overline{n}|})_kV^1_{x:\overline{h}|}, \quad (5.5.8)$$

这里，定期寿险责任准备金是完全离散的。

例 5.5.1: 对例 4.4.1 中的 20 年期两全保险，按真正半年付一次的保费，计算：

(1) 在完全离散基础上的 10 年末责任准备金。

(2) 在半连续基础上的相应责任准备金。

解: (1) 除在例 4.4.1 中已计算值外，还需要

$$A^1_{60:\overline{10}|} = 0.13678852,$$

$$A_{60:\overline{10}|} = 0.58798425,$$

$$\ddot{a}_{60:\overline{10}|} = 7.2789425,$$

$${}_{10}V^1_{50:\overline{20}|} = A^1_{60:\overline{10}|} - P^1_{50:\overline{20}|}\ddot{a}_{60:\overline{10}|} = 0.052752.$$

$${}_{10}V_{50:\overline{20}|} = A_{60:\overline{10}|} - P_{50:\overline{20}|}\ddot{a}_{60:\overline{10}|} = 0.355380.$$

在每一年死亡均匀分布的假设下，有

$$\ddot{a}^{(2)}_{60:\overline{10}|} = \alpha(2)\ddot{a}_{60:\overline{10}|} - \beta(2)(1 - {}_{10}E_{60}) = 7.1392299.$$

所求责任准备金 ${}_{10}V^{(2)}_{50:\overline{20|}}$, 按 (5.5.3) 计算为

$$A_{60:\overline{10|}} - P^{(2)}_{50:\overline{20|}}\ddot{a}^{(2)}_{60:\overline{10|}} = 0.355822.$$

(2) 还需计算

$\frac{i}{\delta}A^1_{50:\overline{20|}} = 0.13423835$,

$P^{(2)}(\overline{A}_{50:\overline{20|}}) = \dfrac{\overline{A}_{50:\overline{20|}}}{\ddot{a}^{(2)}_{50:\overline{20|}}} = 0.03286830$,

$\frac{i}{\delta}A^1_{60:\overline{10|}} = 0.14085233$, $\quad \overline{A}_{50:\overline{20|}} = 0.36471188$,

$P(\overline{A}_{50:\overline{20|}}) = \dfrac{\overline{A}_{50:\overline{20|}}}{\ddot{a}_{50:\overline{20|}}} = 0.03229873$. $\quad \overline{A}_{60:\overline{10|}} = 0.59204806$.

于是

$${}_{10}V^{(2)}(\overline{A}_{50:\overline{20|}}) = \overline{A}_{60:\overline{10|}} - P^{(2)}(\overline{A}_{50:\overline{20|}})\ddot{a}^{(2)}_{60:\overline{10|}} = 0.3573937,$$

或者由

$${}_{10}V(\overline{A}_{50:\overline{20|}}) = \overline{A}_{60:\overline{10|}} - P(\overline{A}_{50:\overline{20|}})\ddot{a}_{60:\overline{10|}} = 0.3569475,$$

$$\beta(2)P^{(2)}(\overline{A}_{50:\overline{20|}})_{10}V^1_{50:\overline{20|}} = 0.000466.$$

两者按 (5.5.7) 相加得 0.3573937.

§5.6 比例责任准备金

这一节讨论与 §4.5 涉及的按比例分担或连续贴现保费对应的责任准备金。对整数 k, 按前瞻方法有

$$ {}^h_kV^{\{m\}}(\overline{A}_{x:\overline{n|}}) = \overline{A}_{x+k:\overline{n-k|}} - {}_hP^{\{m\}}(\overline{A}_{x:\overline{n|}})\ddot{a}^{\{m\}}_{x+k:\overline{h-k|}}.\ k < h \tag{5.6.1}$$

根据 (4.5.2) 以及 (3.9.7),

$${}_hP^{\{m\}}(\overline{A}_{x:\overline{n|}}) = \frac{d^{(m)}}{\delta}{}_h\overline{P}(\overline{A}_{x:\overline{n|}}),$$

$$\ddot{a}^{\{m\}}_{x+k:\overline{h-k}|} = \frac{\delta}{d^{(m)}}\overline{a}_{x+k:\overline{h-k}|}.$$

代入 (5.6.1) 得

$$ {}^h_kV^{\{m\}}(\overline{A}_{x:\overline{n}|}) = \overline{A}_{x+k:\overline{n-k}|} - {}_h\overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{x+k:\overline{h-k}|} = {}^h_k\overline{V}(\overline{A}_{x:\overline{n}|}). \tag{5.6.2}$$

这意味着，不管缴费模式如何，(在每个保单年度末) 完全连续责任准备金可作为比例责任准备金。

在 §4.5, 比例分担保费可分解成

$$P^{\{1\}}(\overline{A}_x) = P(\overline{A}_x) + P(\overline{A}_x^{PR}),$$

其中上标 PR 表示保费退款受益。可以期待，对责任准备金也成立类似的分解。为证实这一点，用前瞻方法与 (4.5.7) 写出

$$\begin{aligned}{}_kV(\overline{A}_x^{PR}) &= \overline{P}(\overline{A}_x)\frac{\overline{A}_{x+k} - A_{x+k}}{\delta} - P(\overline{A}_x^{PR})\ddot{a}_{x+k}\\ &= \overline{P}(\overline{A}_x)\frac{d\ddot{a}_{x+k} - \delta\overline{a}_{x+k}}{\delta} - [P^{\{1\}}(\overline{A}_x) - P(\overline{A}_x)]\ddot{a}_{x+k}.\end{aligned}$$

由

$$\frac{d}{\delta}\overline{P}(\overline{A}_x) = P^{\{1\}}(\overline{A}_x) \tag{5.6.3}$$

可将以上表达式化成

$$\begin{aligned}{}_kV(\overline{A}_x^{PR}) &= -\overline{P}(\overline{A}_x)\overline{a}_{x+k} + P(\overline{A}_x)\ddot{a}_{x+k}\\ &= \overline{A}_{x+k} - \overline{P}(\overline{A}_x)\overline{a}_{x+k} - [\overline{A}_{x+k} - P(\overline{A}_x)\ddot{a}_{x+k}]\\ &= {}_k\overline{V}(\overline{A}_x) - {}_kV(\overline{A}_x)\\ &= {}_kV^{\{1\}}(\overline{A}_x) - {}_kV(\overline{A}_x).\end{aligned}$$

于是有

$$ {}_kV^{\{1\}}(\overline{A}_x) = {}_kV(\overline{A}_x) + {}_kV(\overline{A}_x^{PR}). \tag{5.6.4}$$

§5.7 完全离散责任准备金的递归公式

在 §5.4 曾考虑 (x) 的一般保险：第 $j+1$ 个保单年度末提供死亡受益额 b_{j+1}，年度初缴付净年保费 $\pi_j, j=0,1,\cdots$。根据 (5.4.5), 保单年度 $h-1$ 之末的责任准备金为

$$
{}_{h-1}V=\sum_{j=0}^{\infty} b_{h+j} v^{j+1}{}_{j}p_{x+h-1} q_{x+h+j-1}-\sum_{j=0}^{\infty} \pi_{h+j-1} v^{j}{}_{j}p_{x+h-1}. \tag{5.7.1}
$$

将和式中第一项 ($j=0$ 项) 分离出来，得

$$
\begin{aligned}
{}_{h-1}V = & \; b_h v q_{x+h-1}-\pi_{h-1} \\
& +v p_{x+h-1}\left\{\sum_{j=1}^{\infty} b_{h+j} v^{j}{}_{j-1}p_{x+h} q_{x+h+j-1}\right. \\
& \left.-\sum_{j=1}^{\infty} \pi_{h+j-1} v^{j-1}{}_{j-1}p_{x+h}\right\}.
\end{aligned}
$$

大括号内的项等于 ${}_hV$, 于是得

$$
{}_{h-1}V+\pi_{h-1}=b_h v q_{x+h-1}+{}_hV v p_{x+h-1}. \tag{5.7.2}
$$

用语言来表达，保单年度 h 之初的来源等于年末需求的精算现值。

公式 (5.7.2) 可整理成

$$
\pi_{h-1}=b_h v q_{x+h-1}+\left({}_hV v p_{x+h-1}-{}_{h-1}V\right), \tag{5.7.3}
$$

其中右端的第一项是保额为 b_h 的一年定期保险的净保费，第二项差所代表的金额，在年初加上 ${}_{h-1}V$ 后，经一年按利息与生存积累，在年末将成为 ${}_hV$。

为了以后与完全连续情形比较，在 (5.7.3) 两端乘 $1+i$, 并稍作整理，得

$$\pi_{h-1}+({}_{h-1}V+\pi_{h-1})i+{}_hVq_{x+h-1}=b_hq_{x+h-1}+\triangle({}_{h-1}V). \tag{5.7.4}$$

为明确 ${}_{h-1}V$ 与 ${}_hV$ 是年末责任准备金，通常称 **期末责任准备金**(terminal reserves), 而和 ${}_{h-1}V+\pi_{h-1}$ 称为保单年度 h 的 **期初责任准备金**(initial reserve)。式 (5.7.4) 左端代表保单年度 h 的来源 (从年末看), 即保费、期初责任准备金的利息、因死亡而预期释放的期末责任准备金三项之和。式 (5.7.4) 右端则由年末预期死亡受益赔付与责任准备金增量 ${}_hV-{}_{h-1}V$ 所组成。

另一种不同的分析，考虑到责任准备金 ${}_hV$ 可抵消部分死亡受益 b_h, 只有 **风险净额**(net amount at risk)b_h-V_h 在 1 年中需要保险。在 (5.7.2) 中以 $1-q_{x+h-1}$ 取代 p_{x+h-1}, 并乘 $1+i$, 得

$$ {}_hV=({}_{h-1}V+\pi_{h-1})(1+i)-(b_h-{}_hV)q_{x+h-1}. \tag{5.7.5}$$

相应于 (5.7.3) 有

$$\pi_{h-1}=(b_h-V_h)vq_{x+h-1}+(v_hV-{}_{h-1}V). \tag{5.7.6}$$

上式右端第一项是风险净额的 1 年期保险之净保费，第二项差在年初加上 ${}_{h-1}V$ 后，到年末只按利息积累将成为 ${}_hV$ 。这里 ${}_hV$ 已经部分用于抵消死亡受益，因此责任准备金只是象储蓄金一样累积。这一点还可以从以下对应于 (5.7.4) 公式看出：

$$\pi_{h-1}+({}_{h-1}V+\pi_{h-1})i=(b_h-V_h)q_{x+h-1}+\triangle({}_{h-1}V). \tag{5.7.7}$$

得出 (5.7.3) 的第一种分析，并不用责任准备金去抵消死亡受益，所以责任准备金积累按利息与生存计算 (即精算积累) 。式 (5.7.3) 右端两项都含死亡风险，而式 (5.7.6) 右端只有第一项含死亡风险。

例 5.7.1: 考虑 (x) 从 $x+n$ 岁开始岁入为 1 的延期年金，净年缴保费在递延期内支付，并且当 $x+n$ 岁之前死亡时，死亡年末赔付净保费责任准备金，决定净年缴保费以及 k 年末的净保费责任准备金。

解: 根据 (5.7.6) 以及 $b_h = {}_hV, h = 1, 2, \cdots, n$, 净年缴保费

$$\pi = v\,{}_hV - {}_{h-1}V,$$

乘 v^{h-1} 得

$$\pi v^{h-1} = v^h\,{}_hV - v^{h-1}\,{}_{h-1}V = \Delta(v^{h-1}\,{}_{h-1}V). \qquad (5.7.8)$$

对 $h = 1, 2, \cdots, n$ 相加，有

$$v^n\,{}_nV - v^0\,{}_0V = \pi \sum_{h=1}^{n} v^{h-1} = \pi \ddot{a}_{\overline{n}|}.$$

因 ${}_0V = 0, {}_nV = \ddot{a}_{x+n}$, 故

$$\pi = \frac{v^n \ddot{a}_{x+n}}{\ddot{a}_{\overline{n}|}} = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$

这一年金与例 4.7.2 相当。 k 年末责任准备金可由 (5.7.8) 对 $h = 1, 2, \cdots, k$ 相加得出：

$$v^k\,{}_kV = \pi \ddot{a}_{\overline{k}|},$$

即

$${}_kV = \pi \ddot{s}_{\overline{k}|} \qquad 1 \le k \le n.$$

例 5.7.2: (x) 的某种保险，在 n 年内 (x) 死亡时同年末赔付 1 单位再加上净保费责任准备金，导出净均衡年缴保费公式以及 k 年末净保费责任准备金，这里假定期满时责任准备金为 1 。

解: 此时 $b_h = 1 + {}_hV$, 风险净额为常数 1, 以 π 记净年缴保费, 由 (5.7.6) 得

$$v\,{}_hV - {}_{h-1}V = \pi - vq_{k+h-1},$$

乘 v^{h-1} 得

$$\triangle(v^{h-1}\,{}_{h-1}V) = \pi v^{h-1} - v^h q_{x+h-1}, \qquad (5.7.9)$$

对 $h = 1, 2, \cdots, n$ 相加得出

$$\pi = \frac{v^n + \sum\limits_{h=1}^{n} v^h q_{x+h-1}}{\ddot{a}_{\overline{n}|}}.$$

将 (5.7.9) 按 $h = 1, 2, \cdots, k$ 相加, 可解出

$${}_kV = \pi \ddot{s}_{\overline{k}|} - \sum_{h=1}^{k} (1+i)^{k-h} q_{x+h-1}.$$

§5.8 分数期责任准备金

仍考虑 (x) 的一般保险：保单年度 $j+1$ 年末死亡受益为 b_{j+1}, 年初缴保费 $\pi_j, j = 0, 1, \cdots$, 对整数 k, $0 < s < 1$ 在持续时间为 $k+s$ 时的责任准备金按前瞻方法为

$${}_{k+s}V = b_{k+1} v^{1-s}\,{}_{1-s}q_{x+k+s} + {}_{k+1}V v^{1-s}\,{}_{1-s}p_{x+k+s}. \qquad (5.8.1)$$

在每一年中死亡均匀分布的假设下,

$${}_sp_{x+k}\,{}_{1-s}q_{x+k+s} = {}_{s|1-s}q_{x+k} = (1-s)q_{x+k},$$

$${}_{1-s}q_{x+k+s} = \frac{(1-s)q_{x+k}}{1 - sq_{x+k}}, \qquad (5.8.2)$$

$${}_{1-s}p_{x+k+s} = \frac{p_{x+k}}{1 - sq_{x+k}}. \qquad (5.8.3)$$

于是 (5.8.1) 可改写成

$$ {}_{k+s}V = \frac{v^{1-s}}{1 - sq_{x+k}}[b_{k+1}(1-s)q_{x+k} + {}_{k+1}Vp_{x+k}]. $$

又由 (5.7.2),

$$ b_{k+1}q_{x+k} = ({}_kV + \pi_k)(1+i) - {}_{k+1}Vp_{x+k}, $$

责任准备金可表示成

$$ {}_{k+s}V = \frac{v^{1-s}}{1 - sq_{x+k}}[(1-s)({}_kV+\pi_k)(1+i)+s_{k+1}Vp_{x+k}]. \quad (5.8.4) $$

在实践中广泛使用这个公式的一个简单近似, 导出它的一种方式是在 (5.8.4) 中假定 i 与 q_{x+k} 都非常小, 以致 $1+i, p_{x+k}, v^{1-s}, 1-sq_{x+k}$ 都近似于 1, 于是

$$ {}_{k-s}V \cong (1-s)({}_kV + \pi_k) + s_{k+1}V. \qquad (5.8.5) $$

这个表达式是期初责任准备金 ${}_kV + \pi_k$ 与期末责任准备金 ${}_{k+1}V$ 的线性插值。另一种看待这个近似的方式是作为期末责任准备金插值

$$ (1-s)_kV + s_{k+1}V $$

与未经过净保费(unearned net premium)$(1-s)\pi_k$ 之和。一般来说, 未经过净保费等于, 该年度净保费乘以该年度未经过时间 (在一年中所占比例) 。

对于年缴一次保费, 保费已付到该年度末, 在时间 s, 未经过净保费为 $(1-s)\pi_k$ 。未经过净保费概念也将被用来讨论保费收缴更频繁场合的责任准备金近似。

我们只考虑真正半年缴一次保费情形, 受益仍在死亡年末赔付。对 $0 < s < 1/2$, 有

$$ \begin{aligned} {}_{k+s}V^{(2)} &= b_{k+1}v^{1-s}{}_{1-s}q_{x+k+s} + {}_{k+1}V^{(2)}v^{1-s}{}_{1-s}p_{x+k+s} \\ &\quad - \frac{\pi_k^{(2)}}{2}v^{1/2-s}{}_{1/2-s}p_{x+k+s}, \end{aligned} \qquad (5.8.6) $$

其中 $\pi_k^{(2)}$ 是年保费。

在每一年死亡均匀分布的假设下，可以证明，(5.8.6) 成为

$$
\begin{aligned}
{}_{k+s}V^{(2)} &= \frac{v^{1-s}}{1-sq_{x+s}}\{(1-s)_kV^{(2)}(1+i)+s_{k+1}V^{(2)}p_{x+k} \\
&\quad +\frac{\pi_k^{(2)}}{2}[(1+i)(1-s)-s(1+i)^{1/2}{}_{1/2}p_{x+k}]\},
\end{aligned}
\tag{5.8.7}
$$

它可近似为

$$
{}_{k+s}V^{(2)} \cong (1-s)_kV^{(2)}+s_{k+1}V^{(2)}+(\frac{1}{2}-s)\pi_k^{(2)}. \tag{5.8.8}
$$

这里，除了期末责任准备金插值外，还有一项等于年保费(率)乘以未经过时间 $1/2-s$ 的未经过净保费。注意，年度 k 的两次缴费每次付 $\pi_k^{(2)}/2$，包括了半年的净保费，在年初过后时间 $s(0<s<1/2)$，未经过时间的比例为 $1-s/2$，未经过净保费也等于 $(\pi_k^{(2)}/2)(1-s/2)$，为每次缴付的净保费乘以未经过时间比例。

对 $1/2<s<1$，责任准备金表达式形式与 (5.8.1) 相同，按导出 (5.8.4) 的类似步骤可得

$$
\begin{aligned}
{}_{k+s}V^{(2)} &= \frac{v^{1-s}}{1-sq_{x+k}}\Big\{(1-s)_kV^{(2)}(1+i)+s_{k+1}V^{(2)}p_{x+k} \\
&\quad +(1-s)(1+i)^{1/2}\frac{\pi_k^{(2)}}{2}[(1+i)^{1/2}+{}_{1/2}\,p_{x+k}]\Big\},
\end{aligned}
$$

它可近似为

$$
{}_{k+s}V^{(2)} \cong (1-s)_kV^{(2)}+s_{k+1}V^{(2)}+(1-s)\pi_k^{(2)}, \tag{5.8.9}
$$

也是期末责任准备金插值加上时间 s 时未经过净保费。

§5.9 亏损按各保险年度分摊

从 (5.7.6) 式可以看出，每个保单年度的净保费可分解成风险净额的 1 年定期保险之净保费与累积责任准备金的储蓄基金之存入额。后者并无死亡风险，因此可以设想，保险亏损的方差可用一年期保险的有关方差来表示。事实的确如此，附带还得出计算保险亏损方差的一种灵便方法。

对 §5.4 引入的一般完全离散保险，亏损 $L = {}_0L$ 按 (5.4.4) 可表示为

$$L = b_{K+1}v^{K+1} - \sum_{h=0}^{K} \pi_h v^h, \tag{5.9.1}$$

其中 K 是 (x) 的整值剩余寿命。这里，L 是当死亡发生在保单年度 $K+1$ 时该保险财务结果在保单生效之时的现值。我们试图将其一部分按开始 $K+1$ 年分摊，基础是 §5.7 中的第二种分析。为此引入时间 h 时由保单年度 $h+1$ 分摊的亏损

$$\Lambda_h = \begin{cases} 0 & K \le h-1, \\ vb_{h+1} - ({}_hV + \pi_h) & K = h, \\ v\,{}_{h+1}V - ({}_hV + \pi_h) & K \ge h+1. \end{cases} \tag{5.9.2}$$

显然，Λ_h 是依赖于 K 的。第一种情形代表保单持有人在第 $h+1$ 年之前死掉，第二种情形代表第 $h+1$ 年里死亡，第三种情形则表示至少活到第 $h+1$ 年末。

根据 (5.7.2)，可将 Λ_h 改写成

$$\Lambda_h = \left\{ \begin{array}{ll} 0 & K \le h-1, \\ (b_{h-1} - {}_{h+1}V)vp_{x+h} & K = h, \\ = (b_{h+1} - {}_{h+1}V)v - (b_{h+1} - {}_{h+1}V)vq_{x+h} & \\ -(b_{h+1} - {}_{h+1}V)vq_{x+h} & K \ge h+1. \\ = 0 - (b_{h+1} - {}_{h+1}V)vq_{x+h} & \end{array} \right.$$

(5.9.3)

因 $(b_{h+1} - {}_{h+1}V)vq_{x+h}$ 是风险净额 $b_{h+1} - {}_{h+1}V$ 在第 $h+1$ 年的一年期保险之净保费，我们看到，Λ_h 非零值表示的亏损与一年定期保险相联系，即时间 h 时的受益现值减去净趸缴保费。这样，从 (5.9.3) 可得

$$E[\Lambda_h] = v(b_{h+1} - {}_{h+1}V)[p_{x+h}\,{}_hp_x q_{x+h} + (-1)q_{x+h}\,{}_hp_x p_{x+h}] = 0. \tag{5.9.4}$$

于是

$$\begin{aligned} \mathrm{Var}[\Lambda_h] &= E[(\Lambda_h)^2] = v^2(b_{h+1} - {}_{h+1}V)^2[(p_{x+h})^2\,{}_hp_x q_{x+h} \\ &\qquad + (-1)^2(q_{x+h})^2\,{}_hp_x p_{x+h}] \\ &= v^2(b_{h+1} - {}_{h+1}V)^2\,{}_hp_x p_{x+h} q_{x+h}. \end{aligned} \tag{5.9.5}$$

根据一般推理，总亏损应等于每个保单年度亏损的现值之和，即

$$L = \sum_{h=0}^{\infty} v^h \Lambda_h. \tag{5.9.6}$$

其代数推导如下：

$$\begin{aligned} \sum_{h=0}^{\infty} v^h \Lambda_h &= \sum_{h=0}^{K-1} v^h \Lambda_h + v^K \Lambda_K + \sum_{h=K+1}^{\infty} v^h \Lambda_h \\ &= \sum_{h=0}^{K-1} (v^{h+1}\,{}_{h+1}V - v^h\,{}_hV - v^h \pi_h) \\ &\quad + (v^{K+1} b_{K+1} - v^K\,{}_KV - v^K \pi_K) + 0 \\ &= v^{K+1} b_{K+1} - \sum_{h=0}^{K} v^h \pi_h = L. \end{aligned}$$

类似的推导可得出

$$\sum_{h=k}^{\infty} v^h \Lambda_h = v^k\,{}_kL - v^k\,{}_kV \qquad k = 0, 1, 2, \cdots,$$

解出 ${}_kL$.

$$
{}_kL = \sum_{h=0}^{\infty} v^h \Lambda_{k+h} + {}_kV \quad k = 0, 1, 2, \cdots. \tag{5.9.7}
$$

如对 k 及 $k+j$ 利用这些关系，可得

$$
\begin{aligned}
{}_kL \quad = \quad & \sum_{h=0}^{j-1} v^h \Lambda_{k+h} + v^j {}_{k+j}L \\
& +({}_kV - v^j {}_{k+j}V) \qquad k = 0, 1, \cdots.
\end{aligned} \tag{5.9.8}
$$

从 (5.9.4) 及 (5.9.6), 可确认 $E[L] = 0$, 以下 Hattendorf 的结果提供了计算 L 的方差的一种方法，其意义在于， L 的方差可按各保险年度分摊。

定理 5.9.1(Hattendorf): 按以上所定义，有

(1) $\mathrm{Cov}[\Lambda_h, \Lambda_j] = 0 \qquad h \neq j$

(2) $\mathrm{Var}[L] = \sum\limits_{h=0}^{\infty} v^{2h} \mathrm{Var}[\Lambda_h]$.

证明: 由 (5.9.4), $E[\Lambda_h] = 0$, 所以 $\mathrm{Cov}[\Lambda_h, \Lambda_j] = E[\Lambda_h \Lambda_j]$. 不失一般性，设 $j < h$, 则 $\Lambda_h \neq 0$ 意味着 $K \geq h \geq j+1$, 从而 Λ_j 取常值 $v_{j+1}V - ({}_jV + \pi_j)$, 于是

$$
E[\Lambda_h \Lambda_j] = [v_{j+1}V - ({}_jV + \pi_j)] E[\Lambda_h] = 0,
$$

定理中第 (1) 部分得征。

至于第 (2) 部分，由 (1) 及式 (5.9.6), 可知 L 的方差等于所有 $v^h \Lambda_h$ 的方差之和。

$\Lambda_h, h = 0, 1, \cdots$ 是非独立但不相关随机变量，这使得 L 的方差可按各保单年度分摊。

由定理 5.9.1 也可得出一个计算 ${}_kL$ 方差的公式 [参见 (5.4.4)]。为此，考虑 $x+k$ 岁开始的保险，初始保费 $\pi_0' = {}_kV + \pi_k$, 随后

的保费为 $\pi'_k=\pi_{k+h}, h=1,2,\cdots$。受益 $b'_h=b_{k+h}$, 责任准备金 ${}_hV'={}_{k+h}V$, 相应的亏损变量 Λ'_h 与 $L', h=1,2,\cdots$。由 (5.4.4),

$$
\begin{aligned}
{}_kL &= b_{k+J+1}v^{J+1}-\sum_{h=0}^{J}\pi_{k+h}v^h \\
&= b'_{J+1}v^{J+1}-\sum_{h=0}^{J}\pi'_hv^h+{}_kV=L'+{}_kV,
\end{aligned}
$$

再根据 (7.9.5),

$$
\begin{aligned}
\mathrm{Var}[{}_kL] &= \mathrm{Var}[L']=\sum_{h=0}^{\infty}v^{2h}\mathrm{Var}[\Lambda'_h], \\
&= \sum_{h=0}^{\infty}v^{2h}v^2(b'_{h+1}-{}_{h+1}V')^2{}_hp_{x+k}p_{x+k+h}q_{x+k+h} \\
&= \sum_{h=0}^{\infty}v^{2h}{}_hp_{x+k}[v^2(b_{k+h+1}-{}_{k+h+1}V)^2p_{x+k+h}q_{x+k+h}]
\end{aligned}
\tag{5.9.9}
$$

方括号内的项代表未来 $k+h+1$ 年度风险净额的一年期保险之亏损方差。

为导出 $\mathrm{Var}[{}_kL]$ 与 $\mathrm{Var}[{}_{k+j}L]$ 的关系，先将 (5.9.9) 写成

$$
\begin{aligned}
\mathrm{Var}[{}_kL] &= \sum_{h=0}^{j-1}v^{2h}{}_hp_{x+k}[v^2(b_{k+h+1}-{}_{k+h-1}V)^2p_{x+k+h}q_{x-k+h}] \\
&+ \sum_{h=j}^{\infty}v^{2h}{}_hp_{x+k}[v^2(b_{k+h+1}-{}_{k+h+1}V)^2p_{x+k+h}q_{x+k+h}].
\end{aligned}
$$

在第二个和式中将求和变量 h 改为 $l+j$, 该和式成为

$$
v^{2j}{}_jp_{x+k}\sum_{l=0}^{\infty}v^{2l}{}_lp_{x+k+j}[v^2(b_{k+j+l+1}-{}_{k+j+l+1}V)^2p_{x+k+j+l}q_{x+k+j+l}],
$$

与 (5.9.9) 比较后可知以上和式等于 $\mathrm{Var}[{}_{k+j}L]$, 于是

$$\begin{aligned}\mathrm{Var}[{}_kL] &= \sum_{h=0}^{j-1} v^{2h}{}_hp_{x+k}[v^2(b_{k+h+1}-{}_{k+h+1}V)^2p_{x+k+h}q_{x+k-h}] \\ &\quad +v^{2j}{}_jp_{x+k}\mathrm{Var}[{}_{k+j}L]. \end{aligned} \tag{5.9.10}$$

正如定理 5.9.1 中 $\mathrm{Var}[L]$ 从 $L=\sum\limits_{h=0}^{\infty} v^h\wedge_h$ 导出一样， $\mathrm{Var}[{}_kL]$ 的表达式也可直接从 (5.9.7) 与 (5.9.8) 导出。

例 5.9.1: 设例 5.4.3 中的被保险人生存到第二个保单年度末，估计

(1) $\mathrm{Var}[{}_2L]$(直接计算) 。 (2) $\mathrm{Var}[{}_2L]$(用 Hattendorf 定理) 。

(3) $\mathrm{Var}[{}_3L]$ 。 (4) $\mathrm{Var}[{}_4L]$ 。

解: (1) 直接计算，将所需值列表如下：

结局	亏损	结局的条件概率
j	${}_2L$	${}_jp_{52}q_{52+j}=d_{52+j}/l_{52}$ $j=0,1,2$ ${}_3p_{52}=l_{55}/l_{52}$ $j=3,4,\cdots$
0	936.84	0.0069724
1	877.25	0.0075227
2	821.04	0.0081170
≥ 3	-18.58	0.9773879

其中

$${}_2L=\begin{cases} 1000v^{j+1}-6.55692\ddot{a}_{\overline{j+1|}} & j=0,1,2, \\ 0-6.55692\ddot{a}_{\overline{3|}} & j=3,4,\cdots. \end{cases}$$

于是 $E[{}_2L]=1.64$,

$$\begin{aligned}\mathrm{Var}[{}_2L] &= E[({}_2L)^2]-(E[{}_2L])^2 \\ &= 17717.82-(1.64)^2=17715.1. \end{aligned}$$

(2) 应用 Hattendorf 定理，可利用例 5.4.3 中责任准备金计算与 1 年期保险相联系的亏损方差：

h	q_{52+h}	$v^2(1000-1000{}_{2+h+1}V^{1}_{50:\overline{5}\rvert})^2 p_{52+h}q_{52+h}$
0	0.0069724	6140.842
1	0.0075755	6674.910
2	0.0082364	7269.991

于是由 (5.9.9),

$$\begin{aligned}\mathrm{Var}[{}_2L] &= 6140.842+(1.06)^{-2}\times 6674.910p_{52}\\ &\quad +(1.06)^{-4}\times 7269.991{}_2p_{52}=17715.1.\end{aligned}$$

值得注意，标准差为 $\sqrt{17751.1}=133.1$, 对单个保单而言，是责任准备金 $E[{}_2L]=1.69$ 的 80 多倍。

(3) 类似地用 (5.9.9) 可算出

$$\mathrm{Var}[{}_3L]=6674.910+(1.06)^{-2}\times 7269.991p_{53}=13096.2.$$

(4) 同样

$$\mathrm{Var}[{}_4L]=7269.991.$$

例 5.9.2: 考虑由例 5.4.3 以及例 5.9.1 讨论的保单 1500 份所构成的业务，所有保单的年保费即将缴付，其中 750 份保单已持续 2 年， 500 份已持续 3 年， 250 份已持续 4 年，每一组又平均分为面额 1000 与 3000 两种。

(1) 计算责任准备金总额。

(2) 在独立性假定下计算前瞻亏损的方差，并按正态分布计算，保险人能以概率 0 .95 履行这宗业务的未来责任所必须的金额。

(3) 计算与风险净额的一年期保险相联系的亏损方差，并按正态分布计算，能使保险人以概率 0.95 履行一年期责任所必须增加到责任准备金总额中去的金额。

(4) 在每组保单增加 100 倍的情况下重做 (2) 与 (3) 。

解: (1) 设 Z 是这 1500 份保单的前瞻亏损，用例 5.4.3 的结果可得责任准备金总额

$$\begin{aligned}E[Z] &= (375\times 1+375\times 3)\times 1.64+(250\times 1+250\times 3)\times 1.73\\ &\quad +(125\times 1+125\times 3)\times 1.21\\ &= 4795.\end{aligned}$$

(2) 根据例 5.9.1, 有

$$\begin{aligned}\mathrm{Var}[Z] &= (375\times 1+375\times 9)\times 17715.1\\ &\quad +(250\times 1+250\times 9)\times 13096.2\\ &\quad +(125\times 1+125\times 9)\times 7270\\ &= 1.0825962\times 10^8.\end{aligned}$$

同时， $\sigma_Z = 10404.8.$

由

$$0.05 = Pr[Z>c] = Pr\left[\frac{Z-4795}{10404.8} > \frac{c-4795}{10404.8}\right],$$

按正态近似，

$$\frac{c-4795}{10404.8} = 1.645,$$

或

$$c = 21911,$$

即为所必须金额，它是责任准备金总额 $E[Z]$ 的 4.6 倍。

(3) 这里只考虑下一年的风险，对每一保单，考虑与风险净额的一年期保险相联系的亏损变量，设 Z_1 是这些亏损变量的总和，则 $E[Z_1]=0$.

从例 5.9.1 第 (2) 部分表中的值，可知道这些一年期保险的亏损方差，于是

$$\begin{aligned}\mathrm{Var}[Z_1] &= (375\times 1+375\times 9)\times 6140.8\\ &\quad +(250\times 1+250\times 9)\times 6674.9\\ &\quad +(125\times 1+125\times 9)\times 7270\\ &= 4.880275\times 10^7.\end{aligned}$$

同时 $\sigma_{Z_1}=6985.9$.

设 c_1 是所求的增加到责任准备金总额中去的金额，则

$$0.05=Pr(Z_1>c_1)=Pr\left[\frac{Z_1-0}{6985.9}>\frac{c_1-0}{6985.9}\right],$$

按正态分布得

$$c_1=1.645\times 6985.9=11492,$$

它是责任准备金总额 4795 的 2.4 倍。

(4) 此时， $E[Z]=479500$, $\mathrm{Var}[Z]=1.0825962\times 10^{10}$, 以概率 0.95 保证履行未来责任所必须金额 c 为

$$479500+1.645\sqrt{1.0825962\times 10^{10}}=650659,$$

它是责任准备金总额 $E[Z]$ 的 1.36 倍。

$\mathrm{Var}[Z_1]$ 现在是 4.088275×10^9, 以概率 0.95 保证保险人履行下一年度责任所需增加到责任准备金总额中去的金额为 $1.645\times\sqrt{4.880275\times 10^9}=114918$, 是责任准备金总额的 24% 。

§5.10 完全连续责任准备金微分方程

这一节的结果与 §5.7 完全离散责任准备金递归公式平行。考虑 (x) 完全连续的一般保险：时间 t 死亡即刻赔付受益 b_t, 年保

费 (年率)$\pi_t, t \geq 0$ 。这样，在时间区间 $(t, t+dt)$ 内缴付的保费为 $\pi_t dt$ 。在持续时间 t 的责任准备金 ${}_t\overline{V}$ 由以下公式给出：

$$ {}_t\overline{V} = \int_0^\infty b_{t+s} v^s {}_sp_{x+t}\mu_{x+t+s} ds - \int_0^\infty \pi_{t+s} v^s {}_sp_{x+t} ds. \quad (5.10.1) $$

为简化计算，作积分变量代换 $u = t+s$, 得

$$ {}_t\overline{V} = \int_t^\infty (b_u\mu_{x+u} - \pi_u) e^{\delta(t-u)} {}_{u-t}p_{x+t} du. \quad (5.10.2) $$

由

$$ \frac{d}{dt} {}_{u-t}p_{x+t} = \frac{d}{dt} \exp[-\int_{x+t}^{x+u} \mu_y dy] = \mu_{x+t} {}_{u-t}p_{x+t} $$

可得 ${}_t\overline{V}$ 的导数为

$$ \frac{d_t\overline{V}}{dt} = -(b_t\mu_{x+t} - \pi_t) + \delta I + \mu_{x+t} I, $$

其中 I 表示 (5.10.2) 中的积分，等于 ${}_t\overline{V}$, 所以

$$ \frac{d_t\overline{V}}{dt} = \pi_t + (\delta + \mu_{x+t})_t\overline{V} - b_t\mu_{x+t}. \quad (5.10.3) $$

这里，责任准备金的变化率由三项组成：年保费，按利息及生存因素的增长率，赔付受益率支出 (减项) 。与 (5.7.4) 相应的公式为

$$ \pi_t +_t \overline{V}\delta +_t \overline{V}\mu_{x+t} = b_t\mu_{x+t} + \frac{d_t\overline{V}}{dt}. \quad (5.10.4) $$

这个关系将进项率与受益支出率及责任准备金增长率相平衡。

如果责任准备金被当作储蓄基金，可用来抵消死亡受益，那么有

$$ \pi_t +_t \overline{V}\delta = (b_t -_t \overline{V})\mu_{x+t} + \frac{d_t\overline{V}}{dt}. \quad (5.10.5) $$

这里，进项率只涉及保费和责任准备金的利息，与风险净额受益支出率及责任准备金增长率平衡。

§5.11 用计算基数表示的责任准备金公式

在前几节中，前瞻公式将责任准备金用未来受益及未来保费的精算现值来表示，后顾公式则将责任准备金用过去保费的精算积累值及受益成本积累值来表示。在第二、三章中，给出了这些保险与年金值以计算基数表示的表达式，因而，很容易写出以计算基数表示的责任准备金公式。表 5.11.1 列出了半连续情形的责任准备金公式，其中为简单起见，保费未用计算基数表示，类似的公式也可对完全连续及各种离散情况写出。

表 5.11.1 半连续净保费责任准备金
(保单生效年龄 x, 持续时间 k, 单位保额)

种类	责任准备金记号	分子（分母都是 D_{x+k}） 前瞻公式
终身人寿保险	${}_kV(\overline{A}_x)$	$\overline{M}_{x+k}-P(\overline{A}_x)N_{x+k}$
n年定期保险	${}_kV(\overline{A}^{1}_{x:\overline{n}\mid})$	$\overline{M}_{x+k}-\overline{M}_{x+n}-P(\overline{A}^{1}_{x:\overline{n}\mid})(N_{x+k}-N_{x+n})$
n年两全保险	${}_kV(\overline{A}_{x:\overline{n}\mid})$	$\overline{M}_{x+k}-\overline{M}_{x+n}+D_{x+n}-P(\overline{A}_{x:\overline{n}\mid})(N_{x+k}-N_{x+n})$
h年缴费终身寿险	${}^h_kV(\overline{A}_x)$	$\overline{M}_{x+k}-{}_hP(\overline{A}_x)(N_{x+k}-N_{x+h}),\quad k<h$ $\overline{M}_{x+k}\quad k\geq h$
h年缴费n年期两全保险	${}^h_kV(\overline{A}_{x:\overline{n}\mid})$	$\overline{M}_{x+k}-\overline{M}_{x+n}+D_{x+n}-{}_hP(\overline{A}_{x:\overline{n}\mid})(N_{x+k}-N_{x+h})\quad k<h$ $\overline{M}_{x+k}-\overline{M}_{x+n}+D_{x+n}\quad h\leq k<n$
n年生存保险	${}_kV_{x:\overline{n}\mid}^{\ \ 1}$	$D_{x+n}-P_{x:\overline{n}\mid}^{\ \ 1}(N_{x+k}-N_{x+n})$

例 5.11.1:

(1) 向 (45) 发行的一种递减定期保险，在第 1 年死亡即刻赔付 100000，以后每年递减 5000, 直至 20 年末到期。根据示例生命表并按利率 6% 计算净均衡年缴保费，同时计算第 1,2,3 年末的净保费责任准备金。

种类	责任准备金记号	分子 (分母都是 D_{x+k}) 后顾公式
终身人寿保险	${}_kV(\overline{A}_x)$	$P(\overline{A}_x)(N_x - N_{x+k}) - (\overline{M}_x - \overline{M}_{x+k})$
n年定期保险	${}_kV(\overline{A}^1_{x:\overline{n}\vert})$	$P(\overline{A}^1_{x:\overline{n}\vert})(N_x - N_{x+k}) - (\overline{M}_x - \overline{M}_{x+k})$
n年两全保险	${}_kV(\overline{A}_{x:\overline{n}\vert})$	$P(\overline{A}_{x:\overline{n}\vert})(N_x - N_{x+k}) - (\overline{M}_x - \overline{M}_{x+k})$
h年缴费终身寿险	${}^h_kV(\overline{A}_x)$	${}_hP(\overline{A}_x)(N_x - N_{x+k}) - (\overline{M}_x - \overline{M}_{x+k})$, $k < h$ 用前瞻公式, $k \geq h$
h年缴费n年期两全保险	${}^h_kV(\overline{A}_{x:\overline{n}\vert})$	${}_hP(\overline{A}_{x:\overline{n}\vert})(N_x - N_{x+k}) - (\overline{M}_x - \overline{M}_{x+k})$, $k < h$ 用前瞻公式, $h \leq k < n$
n年生存保险	${}_kV_{x:\overline{n}\vert}^{\ \ 1}$	$P_{x:\overline{n}\vert}^{\ \ 1}(N_x - N_{x+k})$

(2) 在缴费期为 15 年情况下重做 (1) 。

解: (1) 净均衡保费为

$$\frac{100000\overline{M}_{45} - 5000(\overline{R}_{46} - \overline{R}_{66})}{N_{45} - N_{65}} = 5000\frac{i}{\delta}\frac{20M_{45} - (R_{46} - R_{66})}{N_{45} - N_{65}} = 391.577.$$

用 Fackler 责任准备金积累公式 (见习题 19), 第 1 年年末的责任准备金为

$$_1V = ({}_0V + \pi_0)\frac{1+i}{p_{45}} - b_1\frac{i}{\delta}\frac{q_{45}}{p_{45}},$$

$$391.577\frac{D_{45}}{D_{46}} - 100000\frac{i}{\delta}\frac{C_{45}}{D_{46}} = 3.551.$$

第 2 年年末的责任准备金为

$$(3.551 + 391.577)\frac{D_{46}}{D_{47}} - 95000\frac{i}{\delta}\frac{C_{46}}{D_{47}} = -3.199$$

第 3 年年末的责任准备金为

$$(-3.199+391.577)\frac{D_{47}}{D_{48}}-90000\frac{i}{\delta}\frac{C_{47}}{D_{48}}=-20.473.$$

此后责任准备金始终保持非正，直至期满。

(2) 此时，净保费为

$$5000\frac{i}{\delta}\frac{20M_{45-(R_{46}-R_{66})}}{N_{45}-N_{60}}=455.221.$$

第 1,2,3 年末的责任准备金分别为 71.281, 136.664, 196.255 。现在，责任准备金始终非负。

当未来受益的精算现值小于净保费的精算现值时，会出现负的净保费责任准备金，这种场合引起被保险人终止保险而使保险人蒙受亏空。从例 5.11.1 看出，缴费期与保险期限相同的递减保险会产生负的净保费责任准备金，而适当缩短缴费期则可以使责任准备金增加到非负水平。从后顾观点看，负责任准备金源于过去的保费精算积累值低于以往受益精算积累值的情形，缩短缴费期从而提高净年缴保费可改变这种状况。

习 题

§5.2

1. 对于 (x) 的完全连续 1 单位 n 年期两全保险，${}_tL$ 为经过持续时间 t 时的前瞻亏损，验证

$$\mathrm{Var}[{}_tL]=\frac{{}^2\overline{A}_{x+t:\overline{n-t|}}-(\overline{A}_{x+t:\overline{n-t|}})^2}{(\delta\overline{a}_{x:\overline{n|}})^2}.$$

2. (x) 的趸缴保费每年 1 单位连续定期年金，在时间 t 的前瞻亏损为

$${}_tL=\begin{cases}\overline{a}_{\overline{U|}} & 0\le U<n-t,\\ \overline{a}_{\overline{n-t|}} & U\ge n-t.\end{cases}$$

计算 $E[{}_tL]$ 及 $\mathrm{Var}[{}_tL]$ 。

3. 写出前瞻公式：

(1) ${}^{20}_{10}\overline{V}(\overline{A}_{35:\overline{30|}})$

(2) (45) 的趸缴保费 1 单位 10 年定期保险在第 5 年末的责任准备金。

§5.3

4. 写出 ${}^{20}_{10}\overline{V}(\overline{A}_{40})$ 的 4 个公式。

5. 写出 ${}_{10}\overline{V}(\overline{A}_{40:\overline{20|}})$ 的 7 个公式。

6. 给出 ${}^{30}_{20}\overline{V}({}_{30|}\overline{a}_{35})$ 的后顾公式。

7. 对 $0 < t \leq m$, 证明

(1) $\overline{P}(\overline{A}_{x:\overline{m+n|}}) = \overline{P}(\overline{A}^{1}_{x:\overline{m|}}) + \overline{P}_{x:\overline{m|}}^{\ \ 1}\,{}_m\overline{V}(\overline{A}_{x:\overline{m+n|}})$

(2) ${}_t\overline{V}(\overline{A}_{x:\overline{m+n|}}) = {}_t\overline{V}(\overline{A}^{1}_{x:\overline{m|}}) + {}_t\overline{V}_{x:\overline{m|}}^{\ \ 1}\,{}_m\overline{V}(\overline{A}_{x:\overline{m+n|}})$.

8. 指出等式

$$ {}^{20}_{10}\overline{V}(\overline{A}_{30}) = \overline{A}^{1}_{40:\overline{5|}} + {}_5E_{40}\,{}^{20}_{15}\overline{V}(\overline{A}_{30}) - {}_{20}\overline{P}(\overline{A}_{30})\overline{a}_{40:\overline{5|}} $$

与 §5.3 中哪个公式相联系，并给出解释。

§5.4

9. 写出 ${}^{20}_{10}V_{40}$ 的四个公式。

10. 写出 ${}_{10}V_{40:\overline{20|}}$ 的七个公式。

11. 对 $0 < k \leq m$, 证明

$$ {}_kV_{x:\overline{m+n|}} = {}_kV^{1}_{x:\overline{m|}} + {}_kV_{x:\overline{m|}}^{\ \ 1}\,{}_mV_{x:\overline{m+n|}}. $$

12. 当 $k < n/2$ 时，${}_kV_{x:\overline{n|}} = 1/6, \ddot{a}_{x:\overline{n|}} + \ddot{a}_{x+2k:\overline{n-2k|}} = 2\ddot{a}_{x+k:\overline{n-k|}}$. 计算 ${}_kV_{x+k:\overline{n-k|}}$ 。

§5.5

13. 根据附录生命表及利率 6%, 计算下表所列责任准备金 (参见例 4.3.4) 。

完全连续	半连续	完全离散
${}_{10}\overline{V}(\overline{A}_{35:\overline{30}\vert})$	${}_{10}V(\overline{A}_{35:\overline{30}\vert})$	${}_{10}V_{35:\overline{30}\vert}$
${}_{10}\overline{V}(\overline{A}_{35})$	${}_{10}V(\overline{A}_{35})$	${}_{10}V_{35}$
${}_{10}\overline{V}(\overline{A}^{1}_{35:\overline{30}\vert})$	${}_{10}V(\overline{A}^{1}_{35:\overline{30}\vert})$	${}_{10}V^{1}_{35:\overline{30}\vert}$

14. 在每一年中死亡均匀分布的假设下，以下等式哪些是正确的？

(1) ${}_kV(\overline{A}_{x:\overline{n}\vert}) = \frac{i}{\delta}{}_kV_{x:\overline{n}\vert}$.

(2) ${}_kV(\overline{A}_x) = \frac{i}{\delta}{}_kV_x$.

(3) ${}_kV(\overline{A}^{1}_{x:\overline{n}\vert}) = \frac{i}{\delta}{}_kV^{1}_{x:\overline{n}\vert}$.

15. 在每一年中死亡均匀分布的假设下，证明

$$\frac{{}_5V^{(4)}_{30:\overline{20}\vert} - {}_5V_{30:\overline{20}\vert}}{{}^{20}_{5}V^{(4)}_{30} - {}^{20}_{5}V_{30}} = \frac{A_{30:\overline{20}\vert}}{A_{30}}.$$

16. 以下公式中哪些是 ${}_{15}V^{(m)}_{40}$ 的正确公式？

(1) $(P^{(m)}_{55} - P^{(m)}_{40})\ddot{a}^{(m)}_{55}$.　(2) $P^{(m)}_{40}\ddot{s}^{(m)}_{40:\overline{15}\vert} - {}_{15}k_{40}$.

(3) $\left[1 - \frac{P^{(m)}_{40}}{P^{(m)}_{55}}\right]A_{55}$.　(4) $1 - \frac{\ddot{a}^{(m)}_{55}}{\ddot{a}^{(m)}_{40}}$.

§5.6

17. 以下公式中哪些是 ${}_{15}V^{\{4\}}(\overline{A}_{40})$ 的正确公式？

(1) ${}_{15}\overline{V}(\overline{A}_{40})$.　(2) $[P^{\{4\}}(\overline{A}_{55}) - P^{\{4\}}(\overline{A}_{40})]\ddot{a}^{\{4\}}_{55}$.

(3) $[\overline{P}(\overline{A}_{55}) - \overline{P}(\overline{A}_{40})]\overline{a}_{55}$.　(4) $[1 - \frac{\overline{P}(\overline{A}_{40})}{\overline{P}(\overline{A}_{55})}]\overline{A}_{55}$.

(5) $1 - \frac{\overline{a}_{55}}{\overline{a}_{40}}$.　(6) $\overline{P}(\overline{A}_{40})\overline{s}_{40:\overline{15}\vert} - {}_{15}\overline{k}_{40}$.

18. 证明

(1) $P^{\{m\}}(\overline{A}_{x:\overline{n}\vert}) = {}_nP^{\{m\}}(\overline{A}_x) + (1 - \overline{A}_{x+n})P^{\{m\}}{}^{\,1}_{x:\overline{n}\vert}$.

(2) ${}_kV^{\{m\}}(\overline{A}_{x:\overline{n}\vert}) = {}^{n}_{k}V^{\{m\}}(\overline{A}_x) + (1 - \overline{A}_{x+n}){}_kV^{\{m\}}{}^{\,1}_{x:\overline{n}\vert}$.

§5.7

19. 验证经 $h+1$ 置换 h 的 (5.7.2) 可整理成

$$_{h+1}V = ({}_hV + \pi_h)\frac{1+i}{p_{x+h}} - b_{h+1}\frac{q_{x+h}}{p_{x+h}},$$

并给出解释。这个公式称为 Fackler 责任准备金积累公式。

20. 对 (x) 的 1 单位终身人寿保险，证明

(1) ${}_kV_x = \sum\limits_{h=0}^{k-1} \frac{P_x - vq_{x+h}}{{}_{k-h}E_{x+h}}$.

(2) ${}_kV_x = \sum\limits_{h=0}^{k-1} [P_x - vq_{x+h}(1 - {}_{h+1}V_x)](1+i)^{k-h}$.

给出这些公式的文字解释。

21. 如 $b_{h+1} = {}_{h+1}V, {}_0V = 0, \pi_h = \pi, h = 0, 1, \cdots, k-1$, 证明 ${}_kV = \pi\ddot{s}_{\overline{k}|}$ 。

[提示：用 (5.7.6)]

22. 对于受益 $b_n = \ddot{a}_{\overline{n-h}|}, h = 1, 2, \cdots, n, {}_0V = {}_nV = 0$ 的 n 年期保险，设 π 是净均衡年缴保费。证明

(1) $\pi = \frac{\ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$.

(2) ${}_kV = \ddot{a}_{\overline{n-k}|} - \ddot{a}_{x+k:\overline{n-k}|} - \pi\ddot{a}_{x+k:\overline{n-k}|}$.

[提示：直接证明或用 (5.7.2)]

§5.8

23. 从 (5.8.1) 出发，建立

$${}_sp_{x+k}\,{}_{k+s}V + v^{1-s}\,{}_sq_{x+k}\,b_{k+1} = (1+i)^s({}_kV + \pi_k) \qquad 0 < s < 1.$$

按一般性推理解释这个结果。

24. 解释以下公式，其中 $0 < r < 1/m$.

(1)

$$\begin{aligned} {}_{k+(h/m)+r}V^{(m)} \;\cong\; & (1 - \frac{h}{m} - r)\,{}_kV^{(m)} + (\frac{h}{m} + r)\,{}_{k+1}V^{(m)} \\ & + (\frac{1}{m} - r)P^{(m)}. \end{aligned}$$

(2)

$$
\begin{aligned}
{}_{k+(h/m)+r}V^{\{m\}} \cong & \left(1-\frac{h}{m}-r\right){}_{k}V^{\{m\}} \\
& +\left(\frac{h}{m}+r\right){}_{k+1}V^{\{m\}}+\left(\frac{1}{m}-r\right)P^{\{m\}}.
\end{aligned}
$$

25. 对下列责任准备金，导出与 (5.8.5), (5.8.8), (5.8.9) 中某些式子类似的公式。

(1) ${}_{20\frac{1}{2}}V(\overline{A}_{x:\overline{40}|})$.　　(2) ${}_{20\frac{1}{2}}\overline{V}(\overline{A}_{x:\overline{40}|})$.

(3) ${}_{20\frac{1}{2}}V^{(2)}(\overline{A}_{x:\overline{40}|})$.　　(4) ${}_{20\frac{2}{3}}V^{(2)}(\overline{A}_{x:\overline{40}|})$.

(5) ${}_{20\frac{1}{2}}V^{\{2\}}(\overline{A}_{x:\overline{40}|})$.　　(6) ${}_{20\frac{2}{3}}V^{\{2\}}(\overline{A}_{x:\overline{40}|})$.

26. 根据示例生命表以及利率 6%, 求 ${}_{10\frac{1}{6}}V^{\{4\}}(\overline{A}_{25})$ 的近似值。

27. 证明 (5.8.4) 可写成

$$
{}_{k+s}V=\frac{1-s}{1-sq_{x+k}}({}_{k}V+\pi_k)(1+i)^s+(1-\frac{1-s}{1-sq_{x+k}})_{k+1}Vv^{1-s}.
$$

§5.9

28. 对 (x) 的保额为 1 的完全离散终身人寿保险，证明

(1) $\operatorname{Var}[L]=\sum\limits_{h=0}^{\infty}(\frac{\ddot{a}_{x+h+1}}{\ddot{a}_x})^2v^{2(h+1)}{}_{h}p_xp_{x+h}q_{x+h}$.

(2) $\operatorname{Var}[{}_{k}L]=\sum\limits_{h=0}^{\infty}(\frac{\ddot{a}_{x+k+h+1}}{\ddot{a}_x})^2v^{2(h+1)}{}_{h}p_{x+k}p_{x+k+h}q_{x+k+h}$.

29. 对于每年 1 单位的期初生存年金，考虑亏损

$$
L=\ddot{a}_{\overline{K+1}|}-\ddot{a}_x \qquad K=0,1,2,\cdots,
$$

在时间 h 由年度 h 一年分摊的亏损

$$
\wedge_h=\begin{cases}0 & K\le h-1,\\ -(\ddot{a}_{x+h}-1)=-vp_{x+h}\ddot{a}_{x+h+1} & K=h,\\ v\ddot{a}_{x+h+1}-(\ddot{a}_{x+h}-1)=vq_{x+h}\ddot{a}_{x+h+1} & K\ge h+1.\end{cases}
$$

(1) 解释 $\wedge_h$ 的公式。

(2) 证明

① $L = \sum_{h=0}^{\infty} v^h \wedge_h$.　　② $E[\wedge_h] = 0$.

③ $\mathrm{Var}[\wedge_h] = v^2(\ddot{a}_{x+h+1})^2 {}_hp_x p_{x+h} q_{x+h}$.

30.

(1) 对于例 5.7.2 的保险，建立式

$$\mathrm{Var}[L] = \sum_{h=0}^{n-1} v^{2(h+1)} {}_hp_x p_{x+h} q_{x+h}.$$

(2) 如 $\delta = 0.05, n = 20, \mu_{x+t} = 0.01, t \geq 0$, 计算 (1) 中保险的 $\mathrm{Var}[L]$ 值。

31. 对于 25 岁人缴费期限 20 年的单位保额终身人寿保险，根据示例生命表以及利率 6% 计算：

(1) ${}_{20}P_{25}$.　(2) ${}^{20}_{19}V_{25}$.　(3) ${}^{20}_{20}V_{25}$.
(4) $\mathrm{Var}[{}_{20}L]$.　(5) $\mathrm{Var}[{}_{18}L]$ (用定理 5.8.1).

§5.10

32. 解释以下微分方程：

(1) $\frac{d}{dt}{}_t\overline{V} = \pi_t + (\delta + \mu_{x+t}){}_t\overline{V} - b_t\mu_{x+t}$.

(2) $\frac{d}{dt}{}_t\overline{V} = \pi_t + \delta_t\overline{V} - (b_t - {}_t\overline{V})\mu_{x+t}$.

33. 如 $b_t = {}_t\overline{V}, {}_0\overline{V} = 0$ 且 $\pi_t = \pi, t \geq 0$, 证明 ${}_t\overline{V} = \pi\bar{s}_{\overline{t|}}$ 。

34. 求 $\frac{d}{dt}\{[1 - {}_t\overline{V}(\overline{A}_x)]{}_tp_x\}$ 。

§5.11

35. 一种向 (30) 发行的死亡即刻赔付并且到 65 岁为止的定期保险，其受益如下表所示：

死亡年龄：	30-50	50-55	55-60	60-65
受益：	100000	90000	80000	60000

用计算基数写出以下公式：

(1) 半年缴一次的净比例年保费。

(2) 在以上 (1) 的情况下在 30 年末的责任准备金。

36. 一种 (35) 的趸缴保费保单，在活到 65 岁时提供 100000，在 65 岁之前死亡时于死亡年末归还不含利息的净趸缴保费。设净趸缴保费为 S，用计算基数写出

(1) S 的公式。

(2) k 年末责任准备金的前瞻公式。

(3) k 年末责任准备金的后顾公式。

37. 用 $P = {}_{20}P^{(12)}(\overline{A}_{30:\overline{35|}})$ 以及计算基数，写出以下责任准备金的前瞻公式与后顾公式。

(1) ${}^{20}_{10}V^{(12)}(\overline{A}_{30:\overline{35|}})$.　　(2) ${}^{20}_{25}V^{(12)}(\overline{A}_{30:\overline{35|}})$.

综合题

38. 设 ${}_nv_x = 0.080$, $p_x = 0.024$, $p_{x:\overline{n|}}^{\ \ 1} = 0.2$, 计算 $p_{x:\overline{n|}}^{1}$ 。

39. 设 ${}_{10}V_{35} = 0.150, {}_{20}V_{35} = 0.354$, 计算 ${}_{10}V_{45}$ 。

40. 某种向 (25) 签发的终身寿险在死亡年末支付 1 单位受益，保费按年缴付至 65 岁为止，前 10 年的净保费为 p_{25}，接下去 30 年增加到一个新的均衡年保费水平。

(1) 计算从 35 岁到 64 岁应缴净年保费。

(2) 计算第 10 年的期末责任准备金。

(3) 在 10 年末，保单持有人可选择继续按净保费 p_{25} 缴付至 65 岁为止，同时 35 岁之后的死亡受益额降为 B 。计算 B 。

(4) 如果行使了第 (3) 小题的选择权，计算第 12 年的期末责任准备金。

41. 用 (5.10.3) 写出表达式：

(1) $\frac{d}{dt}({}_tp_{x\,t}\overline{V})$　　(2) $\frac{d}{dt}(v^t{}_t\overline{V})$　　(3) $\frac{d}{dt}(v^t{}_tp_{x\,t}\overline{V})$ 并解释结果。

42. 在每年死亡的 Balducci 假设下，证明与 (5.8.1) 相当的

公式为

$$ {}_{k+s}V = v^{1-s}[(1-s)({}_kV + \pi_k)(1+i) + s_{k+1}V]. $$

43. 设 $\delta = 0.05, q_x = 0.05$, 每一年死亡均匀分布，计算
(1) $(\bar{I}\bar{A})^1_{x:\overline{1}|}$　　(2) ${}_{1/2}V(\bar{I}\bar{A})^1_{x:\overline{1}|}$

44. * 根据每一年死亡均匀分布假设以及在真正半年缴一次的保费场合的下列责任准备金演变方程

$$ \left[{}_kV^{(2)} + \frac{\pi_k^{(2)}}{2}\right](1+i) + \frac{\pi_k^{(2)}}{2}\,{}_{1/2}p_{x+k}(1+i)^{1/2} $$
$$ = p_{x+k}\,{}_{k+1}V^{(2)} + q_{x+k}b_{k+1}, $$

由 (5.8.7) 得出 (5.8.6) 。

45. * 证明

$$ \int_0^\infty (v^t - \overline{P}(\overline{A}_x)\bar{a}_{\overline{t}|})^2\,{}_tp_x\mu_{x+t}dt = \int_0^\infty [1 - {}_t\overline{V}(\overline{A}_x)]^2 v^{2t}\,{}_tp_x\mu_{x+t}dt $$

并解释这个结果。

46. * 设

$$ {}_{k,m}L = \begin{cases} b_{k+J+1}v^{J+1} - {}_kV - \sum\limits_{h=0}^{J}\pi_{k+h}v^h & 0 \le J \le m-1 \\ {}_{k+m}Vv^m - {}_kV - \sum\limits_{h=0}^{m-1}\pi_{k+h}v^h & J \ge m, \end{cases} $$

对于 $h = 0, 1, \cdots, m-1$,

$$ \Lambda_{k+h} = \begin{cases} 0 & J \le h-1 \\ vb_{k+h+1} - ({}_{k+h}V + \pi_{k+h}) & J = h \\ v_{k+h+1}V - ({}_{k+h}V + \pi_{k+h}) & J \ge h+1, \end{cases} $$

证明

(1) ${}_{k,m}L = \sum_{h=0}^{m-1} v^h \Lambda_{k+h}$.

(2) $\mathrm{Var}[{}_{k,m}L] = \sum_{h=0}^{m-1} v^{2h}\mathrm{Var}[\Lambda_{k+h}]$.

47. 按照例 5.4.4 中被保险人活到第 2 个保单年度末，重做例 5.9.1 。

48. 按照例 5.4.4 及习题 47 所描述与讨论的那种保单 1500 份重做例 5.9.2 。

49. 在习题 48 中，对于活到第 4 个保单年度末的被保险人，有关支付的额度与时间并不存在不确定性。对于持续到时间 2 及 3 的被保险人，重做习题 48 。

50. 如果

(1) 按完全连续基础，

(2) 按完全离散基础，

表 5.11.1 的第 5 行应作何种改变？

51. 对于向 (30) 签发的以比例净保费按年缴付的 10000 元普通寿险，用保费及期末责任准备金符号写出第 11 个保单年度的年中时的净保费责任准备金公式。

52. 某种 3 年期两全保单面额为 3, 死亡受益在死亡年末支付，用平衡原理决定的净年缴保费为 0.94, 按照利率 20% 产生的责任准备金如下：

年度末	责任准备金
1	0.66
2	1.56
3	3.00

计算

(1) q_x 。 (2) q_{x+1} 。

(3) 保单签发时亏损 ${}_0L$ 的方差。

(4) 第 1 年度末 (未来) 亏损 ${}_1L$ 的方差。

第六章 多重生命函数

§6.1 引 言

第一至第五章建立的理论，涉及与单个生命死亡时间相关联的受益分析。在这一章中，我们将这一理论推广到涉及多个生命的受益分析，其应用在退休金计划中常见的是连生－最后生存者年金的选择权。参加者可选择将在他活着时提供支付的特定的支付额改变为只要参加者及其受益人有一人活着时就提供支付的较低的支付额。

多重生命精算计算的应用极为常见。例如在财产及赠与税中，来自信托财产的投资收益可能向一组继承人支付，只要其中至少有一人还活着。在最后一个死亡之后，信托财产的本金捐赠给一所大学。其遗产继承税中因慈善捐赠而可以免除的税额由精算计算决定。有些家庭保单的受益因被保险人及其配偶的死亡顺序不同而有差异，有些根据连生基础的保单为财产计划提供现金。

这一章将限于讨论两个生命的情况，着重于基本受益并应用第一至第三章的概念与方法，不讨论年保费、责任准备金或其他与两重生命组合有关的问题，这些论题将在第十二章讨论。

一个应用于多个生命的有效抽象是状况(status)的概念，某种状况或者存在，或者消亡。x 岁的单个生命可定义一种 (x) 活着的状况，其剩余寿命 $T(x)$ 则是该状况存在所持续的时间，也就是直到该状况消亡所经历时间。符号 $\overline{n|}$ 则表示恰好存在 n 年并随后消亡的确定性状况。涉及多个生命时，可按不同方式定义各种各样更复杂的状况，譬如，可以是所有成员都活着的状况，也可以是至少一个成员活着的状况，等等。

在状况及其存在被定义后，我们可应用于建立年金与保险的模型。年金在状况存在的条件下提供支付，而保险则在状况消亡时提供支付。保险还可以限于这样一种场合，支付只有当有关个人按某种顺序死亡时才提供。

§6.2 连生状况

当所有成员活着时存在，而当其中有一个死去时消亡的状况称为 连生状况 (joint-life status)。连生状况记成 $(x_1x_2\cdots x_m)$，其中 m 是成员总数，x_i 代表成员 i 的 (现在) 年龄。第一至第三章引入的符号仍将继续使用，只不过单个年龄的下标现在改为多个年龄，例如，$A_{xy}, {}_tp_{xy}$ 对连生状况 (xy) 的含义与 $A_x, {}_tp_x$ 对单个生命 (x) 的含义相同：${}_tp_{xy}$ 是连生状况 (xy) 至少存在 t 年的概率，A_{xy} 是对状况 (xy) 的 1 单位终身保险 [即状况 (xy) 消亡的年度末提供一单位受益] 的净趸缴保费。

现考察状况的剩余寿命 (消亡时间) 随机变量 T。对于连生状况 $(x_1x_2\cdots x_m)$,$T(x_1x_2\cdots x_m)$=$\min[T(x_1),T(x_2),\cdots,T(x_m)]$，其中 $T(x_i)$ 是单个生命 (x_i) 的剩余寿命。对状况的剩余寿命，§1.2.2 至 §1.6(除 §1.3.2 生命表实例外) 的有关概念及关系式也照样适用，以下将不加说明予以利用。

在应用中，一般并无连生状况的死亡律或生命表可以直接利用，通常只能以单个生命的有关概率来表示连生状况存在或消亡的概率。现实地说，连生保险或年金所包含的个体之间往往有某种联系，他们的剩余寿命随机变量一般不独立。然而，这些随机变量如何相依是很难确定的，并且我们也不打算去试图确定。因此，以下将采用独立性假设，即成员组中个体的剩余寿命随机变量相互独立。

考虑两重生命的情形：$x_1 = x, x_2 = y, T = T(xy)$ 的分布

函数为

$$
\begin{aligned}
F_T(t) &= Pr(T \le t) \\
&= Pr[\min[T(x), T(y)] \le t] \\
&= 1 - Pr[T(x) > t \text{ 且 } T(y) > t]. \qquad (6.2.1)
\end{aligned}
$$

根据独立性

$$
\begin{aligned}
F_T(t) &= 1 - Pr[T(x) > t]Pr[T(y) > T] \\
&= 1 - {}_tp_x {}_tp_y, \qquad (6.2.2)
\end{aligned}
$$

即，从独立性导出，连生状况 (xy) 至少存在到时间 t 的概率 ${}_tp_{xy}$ 满足

$$
{}_tp_{xy} = {}_tp_x {}_tp_y. \qquad (6.2.3)
$$

对 $F_T(t)$ 关于 t 求导，可得 T 的概率密度函数

$$
\begin{aligned}
f_T(t) &= \frac{d}{dt}(1 - {}_t p_x {}_tp_y) \\
&= {}_tp_x {}_tp_y(\mu_{x+t} + \mu_{y+t}). \qquad (6.2.4)
\end{aligned}
$$

$T = T(xy)$ 的分布也可根据有关生命的死亡效力来确定。为此，先引入在时间 t 状况 (xy) 消亡效力符号，普通的符号为 $\mu_{x+t:y+t}$，但为便于讨论更一般状况，我们将使用符号 $\mu_{xy}(t)$ 。与 (1.2.12) 类似，

$$
\mu_{xy}(t) = \frac{f_{T(xy)}(t)}{1 - F_{T(xy)}(t)}. \qquad (6.2.5)
$$

由 $T(x)$,$T(y)$ 独立性，可从 (6.2.2) 及 (6.2.4) 得出

$$
\mu_{xy}(t) = \mu_{x+t} + \mu_{y+t}. \qquad (6.2.6)
$$

换言之，在独立性假定下，连生状况的消亡效力等于个体死亡效力之和。与第一章一样，$T(xy)$ 的分布也可以用状况 (xy) 的消亡效力来描述。

连生状况 (xy) 在时间 k 与 $k+1$ 之间消亡的概率为

$$\begin{aligned} Pr(k<T\leq k+1) &= Pr(T\leq k+1)-Pr(T\leq k) \\ &= {}_kp_{xy}-{}_{k+1}p_{xy}={}_kp_{xy}q_{x+k:y+k}, \end{aligned} \tag{6.2.7}$$

这里，连生状况 $(x+k:y+k)$ 在一年内消亡的概率可用个体死亡概率写成

$$\begin{aligned} q_{x+k:y+k} &= 1-p_{x+k:y+k}=1-p_{x+k}p_{y+k} \\ &= 1-(1-q_{x+k})(1-q_{y+k}) \\ &= q_{x+k}+q_{y+k}-q_{x+k}q_{y+k}. \end{aligned} \tag{6.2.8}$$

从 §1.2 三中对 (x) 的整值剩余寿命的讨论可知，(6.2.7) 也提供了状况 (xy) 在消亡前经历的整年数随机变量 K 的概率函数，即对 $k=0,1,2,\cdots$

$$\begin{aligned} Pr(K=k) &= Pr(k\leq T<k+1)=Pr(k<T\leq k+1) \\ &= {}_kp_{xy}q_{x+k:y+k}={}_{k|}q_{xy}. \end{aligned} \tag{6.2.9}$$

§6.3 最后生存状况

在人寿保险中，除了受益时间为一组成员中第 1 个死亡的时间外，也有受益时间为最后一个成员的死亡时间这种情况，这一节就考察最后死亡时间随机变量。

当一组成员中至少有一人活着时为存在，而当最后一个成员死去时消亡的状况称为*最后生存状况*(last-survivor status)，记成 $(\overline{x_1x_2\cdots x_m})$，其中 m 是成员总数，x_i 是成员 i 的年龄。

最后生存状况的剩余寿命 $T = \max[T(x_1), T(x_2), \cdots, T(x_m)]$，在每个个体的剩余寿命 $T(x_i)$ 相互独立的假设下，对 2 个生命有

$$\begin{aligned} F_T(t) &= Pr(T \le t) = Pr[\max[T(x), T(y)] \le t] \\ &= Pr[T(x) \le t \text{ 且 } T(y) \le t] \qquad (6.3.1) \\ &= Pr[T(x) \le t] Pr[T(y) \le t] = (1 - {}_tp_x)(1 - {}_tp_y) \\ &= 1 - {}_tp_x - {}_tp_y + {}_tp_x {}_tp_y. \qquad (6.3.2) \end{aligned}$$

于是

$$ {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_x {}_tp_y. \qquad (6.3.3)$$

对 (6.3.2) 关于 t 求导，可得 $T = T(\overline{xy})$ 的概率密度函数

$$\begin{aligned} f_T(t) &= \frac{d}{dt}[(1 - {}_tp_x)(1 - {}_tp_y)] \\ &= (1 - {}_tp_x){}_tp_y\mu_{y+t} + (1 - {}_tp_y){}_tp_x\mu_{x+t} \qquad (6.3.4A) \\ &= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x {}_tp_y(\mu_{x+t} + \mu_{y+t}). \\ & \qquad (6.3.4B) \end{aligned}$$

即使 $T(x)$ 与 $T(y)$ 并不独立，在 $T(xy), T(\overline{xy}), T(x), T(y)$ 之间也有一种一般关系。$T(\overline{xy})$ 或等于 $T(x)$ 或等于 $T(y)$，而 $T(xy)$ 必等于其中的另一个，因此总成立

$$T(xy) + T(\overline{xy}) = T(x) + T(y). \qquad (6.3.5A)$$

又由

$$\begin{aligned} & Pr[T(xy) \le t] = Pr[T(x) \le t \text{或} T(y) \le t] \\ & = Pr[T(x) \le t] + Pr[T(y) \le t] - Pr[T(x) \le t \text{且} T(y) \le t] \end{aligned}$$

以及 (6.3.1) 可得，

$$F_{T(xy)} + F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t), \qquad (6.3.5B)$$

$$f_{T(xy)}(t)+f_{T(\overline{xy})}(t)=f_{T(x)}(t)+f_{T(y)}(t), \tag{6.3.5C}$$

$$ {}_tp_{\overline{xy}}={}_tp_x+{}_tp_y-{}_tp_{xy}, \tag{6.3.5D}$$

$$f_{T(\overline{xy})}(t)={}_tp_x\mu_{x+t}+{}_tp_y\mu_{y+t}-{}_tp_{xy}\mu_{xy}(t). \tag{6.3.5E}$$

类似地，最后生存状况 $(\overline{xy})$ 在时间 t 的消亡效力

$$\mu_{\overline{xy}}(t)=\frac{f_{T(\overline{xy})}(t)}{1-F_{T(\overline{xy})}(t)}.$$

由 (6.3.5D) 及 (6.3.5E) 可得

$$\mu_{\overline{xy}}(t)=\frac{{}_tp_x\mu_{x+t}+{}_tp_y\mu_{y+t}-{}_tp_{xy}\mu_{xy}(t)}{{}_tp_x+{}_tp_x-{}_tp_{xy}} \tag{6.3.6}$$

最后生存状况 $(\overline{xy})$ 的整值剩余寿命 $K=K(\overline{xy})$ 的概率函数可从以下根据 (6.3.5B) 得出的关系式

$$Pr[K(\overline{xy})=k]+Pr[K(xy)=k]=Pr[K(x)=k]+Pr[K(y)=k]$$

导出，对 $k=0,1,2,\cdots$,

$$Pr[K(\overline{xy})=k]={}_kp_xq_{x+k}+{}_kp_yq_{y+k}-{}_kp_{xy}q_{x+k:y+k}. \tag{6.3.7}$$

对于独立的生命，按 (6.2.3) 及 (6.2.7) 可将 (6.3.7) 写成

$$\begin{aligned}Pr(K(\overline{xy})=k) &= {}_kp_xq_{x+k}+{}_kp_yq_{y+k}\\ &\quad -{}_kp_x{}_kp_y(q_{x+k}+q_{y+k}-q_{x+k}q_{y+k})\\ &= (1-{}_kp_y){}_kp_xq_{x+k}+(1-{}_kp_x){}_kp_yq_{y+k}\\ &\quad +{}_kp_x{}_kp_yq_{x+k}q_{y+k},\end{aligned} \tag{6.3.8}$$

其中前两项为第 2 个死亡发生在时间 k 与 $k+1$ 之间的概率，第 3 项则是两个都死在该年度内的概率。这一表达式与连续型情形的 (6.3.4A) 类似，不过那里相当于这里第 3 项的为 0 。

§6.4 概率与期望值

这一节将利用前两节得出的公式解决有关连生及最后生存剩余寿命的概率问题，并导出期望值、方差以及协方差的表达式。

例 6.4.1: 设 (80) 与 (85) 的剩余寿命独立，给出以下概率的表达式：

(1) 第 1 个死亡发生在未来 5 年后 10 年前。

(2) 最后 1 个死亡发生在未来 5 年后 10 年前。

解： (1) 对于 $T = T(80:85)$, 有

$$Pr(5 < T \leq 10) = Pr(T > 5) - Pr(T > 10)$$

$$= {}_5p_{80:85} - {}_{10}p_{80:85} = {}_5p_{80}\,{}_5p_{85} - {}_{10}p_{80}\,{}_{10}p_{85}.$$

(2) 对于 $T = T(\overline{80:85})$, 用 (6.3.5D) 得

$$Pr(5 < T \leq 10) = Pr(T > 5) - Pr(T > 10) = {}_5p_{\overline{80:85}} - {}_{10}p_{\overline{80:85}}$$

$$= {}_5p_{80} - {}_{10}p_{80} + {}_5p_{85} - {}_{10}p_{85} - \left({}_5p_{80:85} - {}_{10}p_{80:85}\right).$$

按独立性假定，可用 ${}_5p_{80}\,{}_5p_{85}$ 代 ${}_5p_{80:85}$，${}_{10}p_{80}\,{}_{10}p_{85}$ 代 ${}_{10}p_{80:85}$.

对于一般状况 (u) 的剩余寿命 $T = T(u)$, §1.5 有关 (x) 的剩余寿命期望值结果也同样成立。例如对一般状况 (u), 从 (1.4.2) 可得

$$\mathring{e}_u = E[T(u)] = \int_0^\infty {}_tp_u\,dt. \tag{6.4.1}$$

如 (u) 是连生状况 (xy), 则

$$\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy}\,dt, \tag{6.4.2}$$

对最后生存状况，有

$$\mathring{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}}\,dt. \tag{6.4.3}$$

由 (6.3.5D) 可知,

$$\mathring{e}_{\overline{xy}}=\mathring{e}_x+\mathring{e}_y-\mathring{e}_{xy}\ . \tag{6.4.4}$$

从 (1.4.5) 可得 $K=K(u)$ 的期望值

$$e_u=\sum_{k=1}^{\infty}{}_kp_u.$$

对特殊的状况 (xy) 及 $(\overline{xy})$, 分别有

$$e_{xy}=\sum_{k=1}^{\infty}{}_kp_{xy},$$

$$e_{\overline{xy}}=\sum_{k=1}^{\infty}{}_kp_{\overline{xy}}.$$

由 (6.3.5D) 亦可知,

$$e_{\overline{xy}}=e_x+e_y-e_{xy}.$$

§1.5 中导出的方差公式, 也能用来计算任何状况 (u) 的剩余寿命或整值剩余寿命的方差, 如

$$\mathrm{Var}[T(xy)]=2\int_0^{\infty}t\,{}_tp_{xy}dt-(\mathring{e}_{xy})^2,$$

$$\mathrm{Var}[T(\overline{xy})]=2\int_0^{\infty}t\,{}_tp_{\overline{xy}}dt-(\mathring{e}_{\overline{xy}})^2.$$

为用单重生命函数表示 $T(xy)$ 与 $T(\overline{xy})$ 的协方差, 我们从

$$\mathrm{Cov}[T(xy),T(\overline{xy})]=E[T(xy)T(\overline{xy})]-E[T(xy)]E[T(\overline{xy})]$$

开始。根据得出 (6.3.5A) 同样的理由,

$$T(xy)T(\overline{xy})=T(x)T(y).$$

于是，由独立性，

$$E[T(xy)T(\overline{xy})]=E[T(x)T(y)]=E[T(x)]E[T(y)],$$

从而

$$\mathrm{Cov}[T(xy),T(\overline{xy})]=\mathring{e}_x\mathring{e}_y-\mathring{e}_{xy}\mathring{e}_{\overline{xy}}\,. \qquad (6.4.5)$$

将 (6.4.4) 代入 (6.4.5) 可得

$$\mathrm{Cov}[T(xy),T(\overline{xy})]=(\mathring{e}_x-\mathring{e}_{xy})(\mathring{e}_y-\mathring{e}_{xy}). \qquad (6.4.6)$$

上式右端两个因子都非负，因此一般 $T(xy)$ 与 $T(\overline{xy})$ 正相关。

§6.5 人寿保险与生存年金

人寿保险与生存年金也可对一般状况 (u) 定义，第二与第三章中的模型与公式可应用于状况，只要将那里单个生命 (x) 的剩余寿命分布改为状况 (u) 的剩余寿命分布即可。利用 §6.3 中的关系式，可将涉及多重生命状况的精算现值及方差等表达式用单重生命函数表示。

对于在一般状况消亡的年末赔付 1 单位金额的保险， §2.3 的结果适用。于是对于状况 (u) 的整值剩余寿命变量 $K=K(u)$, 受益赔付的时间为 $K+1$, 在保单签发时的现值为 $Z=v^{K+1}$, 净趸缴保费 A_u 为

$$E[Z]=\sum_{k=0}^{\infty}v^{k+1}Pr(K=k), \qquad (6.5.1)$$

方差为

$$\mathrm{Var}[Z]={}^2A_u-(A_u)^2. \qquad (6.5.2)$$

作为说明，考虑在 (x) 与 (y) 最后 1 个死亡的年末赔付 1 单位的保险，根据 (6 .2.7) 及 (6.5.1), 有

$$A_{\overline{xy}}=\sum_{k=0}^{\infty}v^{k+1}({}_kp_xq_{x+k}+{}_kp_yq_{y+k}-{}_kp_{xy}q_{x+k:y+k}),$$

按利息效力 δ 与 2δ 计算后代入 (6.5.2) 可算出方差。

§3.4 中各式各样的离散年金公式对与一般状况存在相关联的年金支付也同样有效。例如对状况 (u) 的 n 年定期生存年金，有

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & 0 \le K < n \\ \ddot{a}_{\overline{n}|} & K \ge n, \end{cases} \qquad (3.4.9\ 重述)$$

$$\ddot{a}_{u:\overline{n}|} = E[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_{k|}q_u + \ddot{a}_{\overline{n}|}\, {}_np_u$$

$$\ddot{a}_{u:\overline{n}|} = \sum_{k=0}^{n-1} {}_kE_u = \sum_{k=0}^{n-1} v^k\, {}_kp_u \qquad (3.4.8\ 重述)$$

$$\ddot{a}_{u:\overline{n}|} = \frac{1}{d}(1 - A_{u:\overline{n}|}) \qquad (3.4.10\ 重述)$$

$$\mathrm{Var}[Y] = \frac{1}{d^2}[{}^2A_{u:\overline{n}|} - (A_{u:\overline{n}|})^2]. \qquad (3.4.12\ 重述)$$

作为说明，考虑在 n 年内 (x) 与 (y) 均活着时每年年初支付 1 单位的年金，这是连生状况 (xy) 的生存年金。在以上公式中，${}_tp_u$ 用 ${}_tp_{xy}$ 代替，或在独立情况下用 ${}_tp_x\,{}_tp_y$ 代替，就可获得年金的精算现值。至于 (3.4.12) 中的方差，可利用关系式

$$\begin{aligned} A_{xy:\overline{n}|} &= 1 - d\ddot{a}_{xy:\overline{n}|}, \\ {}^2A_{xy:\overline{n}|} &= 1 - (2d - d^2)\, {}^2\ddot{a}_{xy:\overline{n}|}. \end{aligned}$$

或直接计算其中的净趸缴保费。

最后在生存状况与连生状况的年金及保险的模型之间建立一些关系，与 (6.3.5A) 同样道理，$K(\overline{xy})$ 或等于 $K(x)$ 或等于 $K(y)$，而 $K(xy)$ 必等于其中的另一个，因此

$$v^{K(\overline{xy})+1} + v^{K(xy)+1} = v^{K(x)+1} + v^{K(y)+1}, \qquad (6.5.3)$$

$$\ddot{a}_{\overline{K(\overline{xy})+1}|} + \ddot{a}_{\overline{K(xy)+1}|} = \ddot{a}_{\overline{K(x)+1}|} + \ddot{a}_{\overline{K(y)+1}|}, \qquad (6.5.4)$$

$$v^{K(xy)+1}v^{K(\overline{xy})+1} = v^{K(x)+1}v^{K(y)+1}. \qquad (6.5.5)$$

在 (6.5.3) 及 (6.5.4) 两端取期望值，可得

$$A_{\overline{xy}} + A_{xy} = A_x + A_y,$$

$$\ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y.$$

这些公式使我们能用单个生命及连生状况的保险或年金来表示最后生存状况的保险或年金的精算现值。

现考虑连续情形的保险与年金。对于一般状况 (u) 及其剩余寿命随机变量 $T = T(u)$, §2.2 与 §3.3 中有关现值随机变量、精算现值以及方差等公式对状况 (u) 也同样成立。

对于在状况 (u) 消亡即刻赔付一单位的保险，其生效时的现值、净趸缴保费及方差分别为

$$\begin{aligned} Z &= v^T, \\ \overline{A}_u &= \int_0^\infty v^t {}_tp_u \mu_{u+t} dt \qquad \text{(2.1.6 重述)} \\ \mathrm{Var}[Z] &= {}^2\overline{A}_u - (\overline{A}_u)^2. \end{aligned}$$

作为说明，对于最后生存状况 $(\overline{xy})$, 有

$$\overline{A}_{\overline{xy}} = \int_0^\infty v^t {}_tp_{\overline{xy}} \mu_{\overline{xy}}(t) dt.$$

根据 (6.3.6), 上式可写成

$$\overline{A}_{\overline{xy}} = \int_0^\infty v^t [{}_tp_x \mu_{x+t} + {}_tp_y \mu_{y+t} - {}_tp_{xy} \mu_{xy}(t)] dt.$$

对于每年 1 单位连续支付直至状况 (u) 消亡的年金，有

$$\begin{aligned} Y &= \overline{a}_{\overline{T}|}, \\ \overline{a}_u &= \int_0^\infty \overline{a}_{\overline{t}|} {}_tp_u \mu_{u+t} dt, \qquad \text{(3.3.2A 重述)} \\ &= \int_0^\infty v^t {}_tp_u dt, \qquad \text{(3.3.2B 重述)} \end{aligned}$$

$$\mathrm{Var}[Y]=\frac{{}^2\overline{A}_u-(\overline{A}_u)^2}{\delta^2}. \qquad (3.3.8\ 重述)$$

利息等式

$$\delta\overline{a}_{\overline{T}|}+v^T=1 \qquad (3.3.9重述)$$

也可用来得出状况保险与年金的联系。

作为应用，考虑当 (x) 或 (y) 至少有 1 个活着时每年 1 单位连续支付年金，这是一个 $(\overline{xy})$ 年金，在以上公式中 $T=T(\overline{xy})$，于是

$$\begin{aligned}
Y &= \overline{a}_{\overline{T(\overline{xy})}|},\\
\overline{a}_{\overline{xy}} &= \int_0^\infty \overline{a}_{\overline{t}|}\left[{}_tp_x\mu_{x+t}+{}_tp_y\mu_{y+t}-{}_tp_{xy}\mu_{xy}(t)\right]dt\\
&= \int_0^\infty v^t{}_tp_{\overline{xy}}dt,\\
\mathrm{Var}[Y] &= \frac{{}^2\overline{A}_{\overline{xy}}-(\overline{A}_{\overline{xy}})^2}{\delta^2}.
\end{aligned}$$

公式 (6.5.3)-(6.5.5) 中整值剩余寿命 K 换成 (完全) 剩余寿命 T 也照样成立：

$$v^{T(\overline{xy})}+v^{T(xy)}=v^{T(x)}+v^{T(y)}, \qquad (6.5.6)$$

$$\overline{a}_{\overline{T(\overline{xy})}|}+\overline{a}_{\overline{T(xy)}|}=\overline{a}_{\overline{T(x)}|}+\overline{a}_{\overline{T(y)}|}, \qquad (6.5.7)$$

$$v^{T(\overline{xy})}v^{T(xy)}=v^{T(x)}v^{T(y)}.$$

它们可用来导出这些状况保险与年金的净趸缴保费、精算现值、方差及协方差之间的关系。例如

$$\overline{A}_{\overline{xy}}-\overline{A}_{xy}=\overline{A}_x+\overline{A}_y, \qquad (6.5.8)$$

$$\overline{a}_{\overline{xy}}+\overline{a}_{xy}=\overline{a}_x+\overline{a}_y. \qquad (6.5.9)$$

与独立情形导出 $\mathrm{Cov}[T(\overline{xy}), T(xy)]$ 表达式相同，在 $T(x)$ 与 $T(y)$ 独立时，

$$\mathrm{Cov}[v^{T(\overline{xy})}, v^{T(xy)}] = (\overline{A}_x - \overline{A}_{xy})(\overline{A}_y - \overline{A}_{xy}). \qquad (6.5.10)$$

上式右端项的两个因子皆非负，因此 (6.5.10) 中的协方差非负。

例 6.5.1: 考虑在 (x) 与 (y) 最后 1 个死亡即刻赔付 1 的 n 年期人寿保险，如至少有 1 个活到时间 n, 则没有任何受益。计算净趸缴保费。

解: 重述 (2.2.3) 并利用 (6.3.6), 有

$$\begin{aligned}
\overline{A}^{1}_{\overline{xy}:\overline{n}|} &= \int_0^n v^t {}_tp_{\overline{xy}}\mu_{\overline{xy}}(t)dt \\
&= \int_0^n v^t[{}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{xy}(t)]dt \\
&= \overline{A}^{1}_{x:\overline{n}|} + \overline{A}^{1}_{y:\overline{n}|} - \overline{A}^{1}_{\overline{xy}:\overline{n}|}
\end{aligned}$$

其中符号 $\overline{A}^{1}_{xy:\overline{n}|}$ 表示连生状况 (xy) 之 n 年定期保险的净趸缴保费。

例 6.5.2: 某种年金在 (x) 与 (y) 都活着时，每年连续支付 1, 当 (x) 或 (y) 其中之一活着而另一个已死亡时，每年连续支付 2/3 。设 $T(x)$ 与 $T(y)$ 独立，导出以下公式：

(1) 年金的现值随机变量。

(2) 年金的精算现值。

(3) 年金现值随机变量的方差。

解:

(1) 该年金可看成以下两者的组合，其一是年支付 2/3 直至 (x) 与 (y) 最后一个死亡 [时间 $T(\overline{xy})$], 另一是年支付 1/3 直至 (x) 或 (y) 至少有一个死亡 [时间 $T(xy)$]。因此所求年金的现值随机变量为

$$Z = \frac{2}{3}\overline{a}_{\overline{T(\overline{xy})}|} + \frac{1}{3}\overline{a}_{\overline{T(xy)}|}.$$

(2) 精算现值为

$$E[Z]=\frac{2}{3}\overline{a}_{\overline{xy}}+\frac{1}{3}\overline{a}_{xy}.$$

用 (6.5.9) 代 $\overline{a}_{\overline{xy}}$, 得

$$E[Z]=\frac{2}{3}\overline{a}_x+\frac{2}{3}\overline{a}_y-\frac{1}{3}\overline{a}_{xy}.$$

另一种方法由重述的 (3.2.2B), 得

$$E[Z]=\frac{2}{3}\int_0^\infty v^t{}_tp_{\overline{xy}}dt+\frac{1}{3}\int_0^\infty v^t{}_tp_{xy}dt.$$

将 $(\overline{xy})$ 生存到时间 t 考虑为三个互不相容事件之和，并利用独立性，可写出

$${}_tp_{\overline{xy}}={}_tp_{xy}+{}_tp_x(1-{}_tp_y)+{}_tp_y(1-{}_tp_x).$$

代入以上 $E[Z]$ 表达式中第一项积分，得

$$\begin{aligned}E[Z]\ &=\ \int_0^\infty v^t{}_tp_{xy}dt+\frac{2}{3}\int_0^\infty v^t{}_tp_x(1-{}_tp_y)dt\\&\quad+\frac{2}{3}\int_0^\infty v^t{}_tp_y(1-{}_tp_x)dt.\end{aligned}$$

以上表达式也可这样考虑：第一项是当 (x) 与 (y) 都活着时每年 1 单位支付的精算现值。第二项是当 (x) 活着而 (y) 已死亡时每年 2/3 单位支付的精算现值。第三项有类似解释。

(3) 方差

$$\begin{aligned}&\mathrm{Var}[Z]=\mathrm{Var}\left[\frac{2}{3}\overline{a}_{\overline{T(\overline{xy})}|}+\frac{1}{3}\overline{a}_{\overline{T(xy)}|}\right]\\&=\frac{4}{9}\mathrm{Var}[\overline{a}_{\overline{T(\overline{xy})}|}]+\frac{1}{9}\mathrm{Var}[\overline{a}_{\overline{T(xy)}|}]+2\times\frac{2}{9}\mathrm{Cov}[\overline{a}_{\overline{T(\overline{xy})}|},\overline{a}_{\overline{T(xy)}|}].\end{aligned}$$

根据习题 14,

$$\begin{aligned}\mathrm{Cov}[\overline{a}_{\overline{T(\overline{xy})}|}, \overline{a}_{T(xy)}] &= \frac{1}{\delta^2}\mathrm{Cov}[v^{T(\overline{xy})}, v^{T(xy)}] \\ &= \frac{(\overline{A}_x - \overline{A}_{xy})(\overline{A}_y - \overline{A}_{xy})}{\delta^2}.\end{aligned}$$

于是

$$\mathrm{Var}[Z] = \frac{\frac{4}{9}[{}^2\overline{A}_{\overline{xy}} - (\overline{A}_{\overline{xy}})^2] + \frac{1}{9}[{}^2\overline{A}_{xy} - (\overline{A}_{xy})^2] + \frac{4}{9}(\overline{A}_x - \overline{A}_{xy})(\overline{A}_y - \overline{A}_{xy})}{\delta^2}.$$

例 6.5.3: 计算按以下条件连续支付年金的精算现值：(1) 开始 n 年内，每年支付数额 1，为确定性年金。(2) n 年后，当 (x) 与 (y) 都活着时每年数额 1 。(3) 当 (x) 活着而 (y) 已死亡时每年数额 3/4 。(4) 当 (y) 活着而 (x) 已死亡时每年数额 1/2 。

解: 用当期支付技巧，先计算每一种情形的精算现值，相加便得所求年金的精算现值。

情形一： n 年确定性年金的精算现值 (即现值)

$$\int_0^n v^t dt = \overline{a}_{\overline{n}|}.$$

情形二： n 年之后，当 (x) 与 (y) 都活着时支付的精算现值

$$\int_n^\infty v^t {}_tp_{xy} dt = \int_0^\infty v^t {}_tp_{xy} dt - \int_0^n v^t {}_tp_{xy} dt = \overline{a}_{xy} - \overline{a}_{xy:\overline{n}|}.$$

情形三： n 年之后，当 (x) 活着而 (y) 已死亡时支付的精算现值

$$\frac{3}{4}\int_n^\infty v^t {}_tp_x(1 - {}_tp_y)dt = \frac{3}{4}(\overline{a}_x - \overline{a}_{x:\overline{n}|}) - \frac{3}{4}(\overline{a}_{xy} - \overline{a}_{xy:\overline{n}|}).$$

情形四： n 年之后，当 (y) 活着而 (x) 已死亡时支付的精算现值

$$\frac{1}{2}(\overline{a}_y - \overline{a}_{y:\overline{n}|}) - \frac{1}{2}(\overline{a}_{xy} - \overline{a}_{xy:\overline{n}|}).$$

相加，得所求精算现值

$$\overline{a}_{\overline{n}|} - \frac{3}{4}\overline{a}_x + \frac{1}{2}\overline{a}_y - \frac{1}{4}\overline{a}_{xy} - \frac{3}{4}\overline{a}_{x:\overline{n}|} - \frac{1}{2}\overline{a}_{y:\overline{n}|} + \frac{1}{4}\overline{a}_{xy:\overline{n}|}.$$

§6.6 在特殊死亡律下的求值

这一节考察，在 Makeham 死亡律及其重要特例 Gompertz 死亡律下，有关多重生命状况的净趸缴保费及精算现值的计算。

首先，假定每个生命的死亡服从 Gompertz 死亡律： $\mu_x = Bc^x$. 我们来寻求能代替连生状况 (xy) 的单重生命状况 (w), 为此考虑

$$\mu_{xy}(s) = \mu_{w+s} \qquad s \geq 0. \tag{6.6.1}$$

由 (6.2.6),

$$Bc^{x+s} + Bc^{y+s} = Bc^{w+s},$$

$$c^x + c^y = c^w. \tag{6.6.2}$$

根据 (6.6.2) 可定出所需要的 w, 于是对 $t > 0$,

$$\begin{aligned} {}_tp_w &= \exp(-\int_0^t \mu_{w+s}ds) \\ &= \exp(-\int_0^t \mu_{xy}(s)ds) = {}_tp_{xy}. \end{aligned} \tag{6.6.3}$$

因此，有关连生状况 (xy) 的概率、期望值、方差都与单重生命 (w) 的相等。但一般 w 不见得是整数，如根据有关数值的表格计算，则需要作适当插值。

在 Makeham 死亡律下，情况稍微复杂些。此时连生状况的死亡效力为

$$\mu_{xy}(s) = \mu_{x+s} + \mu_{y+s} = 2A + Bc^s(c^x + c^y). \tag{6.6.4}$$

由于 $2A$ 这一项，无法用 (服从同样参数 Makeham 死亡律的) 单个生命来代替。不过可用另一个连生状况 (ww) 来取代 (xy)，由

$$\mu_{ww}(s) = 2\mu_{w+s} = 2(A + Bc^s c^w) \tag{6.6.5}$$

选 w 满足

$$2c^w = c^x + c^y. \tag{6.6.6}$$

与 Gompertz 情形不同在于，这里计算所需要的一维阵列表格是建立在涉及两个相同年龄生命的连生状况 (ww) 函数上的。

例 6.6.1: 用 (1.7.1) 与基于附录示例生命表的 $\ddot{a}_{xx}$ 值，按利率 6% 计算 $\ddot{a}_{60:70}$ 的值。将所得值与 $\ddot{a}_{x:x+10}$ 表中值 $\ddot{a}_{60:70}$ 比较。

解: 附录 2A 示例生命表在 13-110 岁是按 Makeham 死亡律生成的：

$$1000\mu_x = 0.7 + 0.05(10^{0.04})^x,$$

即 $c = 10^{0.04}$，由 $c^{60} + c^{70} = 2c^w$ 解得 $w = 66.11276$. 用线性插值可算得

$$\begin{aligned}\ddot{a}_{60:70} &= \ddot{a}_{ww} = (67 - w)\ddot{a}_{66:66} + (w - 66)\ddot{a}_{67:67} \\ &= 0.88724 \times 7.58658 + 0.11276 \times 7.31867 = 7.55637.\end{aligned}$$

$\ddot{a}_{x:x+10}$ 表 (该表未附) 中 $\ddot{a}_{60:70}$ 的值为 7.55633.

§6.7 每年死亡均匀分布假设下的求值

这一节假定，连生状况中的每个生命在每 1 年中的死亡均匀分布，并给出这一假定下死亡即刻赔付保险的净趸缴保费求值公式，以及一年支付数次年金的精算现值求值。

回顾表 1.6.1, 在每年死亡均匀分布假设下, ${}_tp_x = 1 - tq_x$,

$$ {}_tp_x\mu_{x+t} = q_x \qquad 0 \le t \le 1. \tag{6.7.1} $$

应用于 $T(x)$ 与 $T(y)$ 独立的连生状况 (xy), 对 $0 \le t \le 1$ 有

$$
\begin{aligned}
{}_tp_{xy}\mu_{xy}(t) &= {}_tp_x{}_tp_y(\mu_{x+t} + \mu_{y+t}) \\
&= {}_tp_y({}_tp_x\mu_{x+t}) + {}_tp_x({}_tp_y\mu_{y+t}) \\
&= (1 - tq_y)q_x + (1 - tq_x)q_y \qquad (6.7.2) \\
&= q_x + q_y - q_xq_y + (1 - 2t)q_xq_y \\
&= q_{xy} + (1 - 2t)q_xq_y.
\end{aligned}
$$

根据 (2.4.1), 一般状况 (u) 保险的净趸缴保费为

$$ \overline{A}_u = \sum_{k=0}^{\infty} v^{k+1}{}_kp_u \int_0^1 e^{-\delta(s-1)} \frac{{}_{k+s}p_u}{{}_kp_u} \mu_u(k+s)ds. $$

利用 (6.7.2) 可写出连生状况 (xy) 的净趸缴保费

$$
\begin{aligned}
\overline{A}_{xy}^{\prime} &= \sum_{k=0}^{\infty} v^{k+1}{}_kp_{xy}[q_{x+k:y+k} \int_0^1 e^{-\delta(s-1)}ds \\
&\quad + q_{x+k}q_{y+k} \int_0^1 e^{-\delta(s-1)}(1-2s)ds] \\
&= \frac{i}{\delta}A_{xy} + \frac{i}{\delta}(1 - \frac{2}{\delta} + \frac{2}{i}) \sum_{k=0}^{\infty} {}_kp_{xy}q_{x+k}q_{y+k}. \qquad (6.7.3)
\end{aligned}
$$

根据 (2.4.2) 可知, 当 $T(xy)$ 在每一年中为均匀分布时, 式 (6.7.3) 右端第一项等于 $\overline{A}_{xy}$, 但即使 $T(x)$ 与 $T(y)$ 独立且分别都在每一年中均匀分布, 也不能得出 $T(xy) = \min\{T(x), T(y)\}$ 的同样结论。事实上, 当 $T(x)$ 与 $T(y)$ 在同一年龄区间的条件下, 它们的最小值并非均匀分布, 而服从略微前移的一种分布。正是

因为这种前移，才使得 (6.7.3) 需要第二项来弥补相应的更早的理赔所带来的额外成本。第二项乘积中的利息因子项 $\frac{i}{\delta}(1-\frac{2}{\delta}+\frac{2}{i})$ 很接近 $i/6$, 而两个生命在同一年死亡时才赔付受益的净趸缴保费又相当小. 所以通常忽略这一校正项，将 (6.7.3) 简化为近似式

$$\overline{A}_{xy} \cong \frac{i}{\delta} A_{xy}. \tag{6.7.4}$$

如上所述，当 $T(xy)$ 在每一年均匀分布时，(6.7.4) 成为精确等式。

为求 $\overline{a}_{xy}$, 根据 (3.3.6) 可得

$$\overline{a}_{xy} = \frac{1}{\delta}(1 - \overline{A}_{xy}).$$

将 (6.7.3) 代入，有

$$\overline{a}_{xy} = \frac{1}{\delta}\{1 - \frac{i}{\delta}[A_{xy} + (1 - \frac{2}{\delta} + \frac{2}{i})\sum_{k=0}^{\infty} v^{k+1}{}_kp_{xy}q_{x+k}q_{y+k}]\}.$$

现根据 (3.4.6), 对状况 (xy), 以 $1-d\ddot{a}_{xy}$ 替代 A_{xy}, 并利用 (3.6.11) 及 (3.6.12) 可写出

$$\overline{a}_{xy} = \alpha(\infty)\ddot{a}_{xy} - \beta(\infty) - \frac{i}{\delta^2}(1 - \frac{2}{\delta} + \frac{2}{i})\sum_{k=0}^{\infty} v^{k+1}{}_kp_{xy}q_{x+k}q_{y+k}. \tag{6.7.5}$$

这一公式是在 $T(x)$ 与 $T(y)$ 独立并且在每 1 年均匀分布的情况下得出的。如果假定 $T(xy)$ 本身在每 1 年中均匀分布，那么由 (3.5.9) 的连续情形 $(m=\infty)$ 可得

$$\overline{a}_{xy} = \alpha(\infty)\ddot{a}_{xy} - \beta(\infty). \tag{6.7.6}$$

公式 (6.7.6) 与 (6.7.5) 相差一个小量，它近似于 $i/(6\delta)$ 乘以两个个体在将来同一年内死亡时赔付的保险的净趸缴保费。

为了用同样方法求年付 m 次期初年金的精算现值，需要 $A_{xy}^{(m)}$ 的表达式。在各生命每 1 年中的死亡均匀分布假设下，与连续情形类似，

$$A_{xy}^{(m)} = \sum_{k=0}^{\infty} v^k {}_kp_{xy} \sum_{j=1}^{m} v^{j/m} ({}_{(j-1)/m}p_{x+k:y+k} - {}_{j/m}\, p_{x+k:y+k}). \tag{6.7.7}$$

在 $T(x)$ 与 $T(y)$ 还独立的情况下，上式可化成 (参见习题 29)

$$A_{xy}^{(m)} = \frac{i}{i^{(m)}} A_{xy} + \frac{i}{i^{(m)}}(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}) \sum_{k=0}^{\infty} v^{k+1} {}_kp_{xy} q_{x+k} q_{y+k}. \tag{6.7.8}$$

当 $m \to \infty$ 时，上式的极限形式即 (6.7.3) 。(6.7.8) 右端第一项是 $A_{xy}^{(m)}$ 的常规近似，并且当 $T(xy)$ 在每年均匀分布时精确相等。而

$$\frac{i}{i^{(m)}}(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}) \cong \frac{m^2 - 1}{6m^2} i,$$

其右端小于 $i/6$.

将 (6.7.8) 代入对 (xy) 重述的 (3.5.2), 并以 $1 - d\ddot{a}_{xy}$ 取代 A_{xy}, 可获得与 (6.7.5) 类似的公式。如果 (6.7.8) 右端第二项忽略不计，那么 $\ddot{a}_{xy}^{(m)}$ 的公式可简化成

$$\ddot{a}_{xy}^{(m)} = \alpha(m)\ddot{a}_{xy} - \beta(m). \tag{6.7.9}$$

根据 (3.5.9), 在 $T(xy)$ 每年均匀分布的假定下，以上等式精确成立。

§6.8 单重次顺位函数

这一节研究的保险除了与状况消亡的时间有关以外，还与其中成员的死亡顺序相关联。

我们从 (x) 在 n 年内并且在 (y) 之前死亡概率的计算开始。这个概率记为 ${}_nq_{xy}^{1}$，其中 x 上面的 1 表示概率所涉事件系 (x) 先死于 (y) 之前，n 则表示事件发生在 n 年内。从概率论中可知，${}_nq_{xy}^{1}$ 等于 $T(x)$ 与 $T(y)$ 联合概率密度函数的一个二重积分，积分区域相当于 $T(x) \le T(y)$ 且 $T(x) \le n$. 在 $T(x)$ 与 $T(y)$ 独立的假设下，有

$$\begin{aligned} {}_nq_{xy}^{1} &= \int_0^n \int_t^\infty {}_sp_y\mu_{y+s}\,{}_tp_x\mu_{x+t}\,ds\,dt, && (6.8.1)\\ &= \int_0^n {}_tp_x\mu_{x+t}\left(\int_t^\infty {}_sp_y\mu_{y+s}\,ds\right)dt, \\ &= \int_0^n {}_tp_x\mu_{x+t}\,{}_tp_y\,dt, \\ &= \int_0^n {}_tp_{xy}\mu_{x+t}\,dt. && (6.8.2)\end{aligned}$$

这个表达式的解释包含三个方面：首先，因 t 是 (x) 的死亡时间，${}_tp_{xy}$ 表示 (x) 与 (y) 都活到时间 t 的概率；其次，$\mu_{x+t}dt$ 是 (x) 在活到 $x+t$ 岁条件下于未来 dt 时间段内死亡的概率；最后，对 0 与 n 之间的所有时间求和（积分）。

我们也可求出 (y) 在 n 年内并且在 (x) 之后死亡的概率，这个概率记为 ${}_nq_{xy}^{\;2}$，其中 y 上面的 2 表示 (y) 第二次死亡，n 表示发生在 n 年内。${}_nq_{xy}^{\;2}$ 等于 $T(x)$ 与 $T(y)$ 联合概率密度函数的一个二重积分，积分区域对应于事件 $[0 \le T(x) \le T(y) \le n]$。在 $T(x)$ 与 $T(y)$ 独立的假定下，

$$\begin{aligned} {}_nq_{xy}^{\;2} &= \int_0^n \int_0^t {}_sp_x\mu_{x+s}\,{}_tp_y\mu_{y+t}\,ds\,dt \\ &= \int_0^n (1-{}_tp_x)\,{}_tp_y\mu_{y+t}\,dt && (6.8.3)\\ &= {}_nq_y - {}_nq_{xy}^{1}.\end{aligned}$$

在第二个表达式中的被积项是乘积

$$Pr[(x)\text{死于}(y)\text{之前}|(y)\text{在时刻}t\text{死亡}]Pr[(y)\text{在时刻}t\text{死亡}],$$

对顺位保险的净趸缴保费，可写出类似的积分。

交换积分次序，${}_nq_{xy}^{\ 2}$ 也可写成

$$\begin{aligned} {}_nq_{xy}^{\ 2} &= \int_0^n \int_s^n {}_sp_x\mu_{x+s}{}_tp_y\mu_{y+t}dtds \\ &= \int_0^n ({}_sp_y - {}_np_y){}_sp_x\mu_{x+s}ds \\ &= {}_nq_{xy}^{1} - {}_np_y{}_nq_x. \end{aligned} \tag{6.8.4}$$

式 (6.8.4) 中第二个积分表达式可解释成：(x) 死于时间 s 与 $s+ds$ 之间的概率，乘以 (y) 活到时间 s 但活不过时间 n 的概率，然后对 0 与 n 之间的所有时间 s 积分。现在我们从 (6.8.4) 得出

$$ {}_nq_{xy}^{1} = {}_nq_{xy}^{\ \ 2} + {}_np_y{}_nq_x, $$

从而

$$ {}_nq_{xy}^{1} \geq {}_nq_{xy}^{\ 2}. $$

对顺位保险的净趸缴保费，也可写出类似的积分。

例 6.8.1: 对于当 (y) 仍活着的情况下在 (x) 死亡之时赔付 1 单位的保险，导出净趸缴保费公式。

解: 该保险的净趸缴保费记为 $\overline{A}_{xy}^{1}$, 等于 $E[Z]$, 其中

$$ Z = \begin{cases} v^{T(x)} & T(x) \leq T(y) \\ 0 & T(x) > T(y) \end{cases}. $$

在 $T(x)$ 与 $T(y)$ 独立的假定下，

$$\begin{aligned} \overline{A}_{xy}^{1} &= \int_0^\infty \int_t^\infty v^t{}_sp_y\mu_{y+s}{}_tp_x\mu_{x+t}dsdt \\ &= \int_0^\infty v^t{}_tp_{xy}\mu_{x+t}dt. \end{aligned}$$

最后一个表达式可解释如下：当 (x) 在未来时间 t 死去而 (y) 仍活着时，需赔付现值为 v^t 的受益。

例 6.8.2: 对于当 (x) 已死去的情况下在 (y) 死亡之时赔付 1 单位的保险，导出净趸缴保费公式。

解: 该保险的净趸缴保费记为 $\overline{A}_{xy}^{\;2}$, 等于 $E[Z]$, 这里

$$Z = \begin{cases} v^{T(y)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

用 $T(x)$ 与 $T(y)$ 的联合概率密度函数 (在独立假定下) 表示 $E[Z]$, 有

$$\begin{aligned} \overline{A}_{xy}^{\;2} &= \int_0^{\infty}\int_0^{t} v^t {}_sp_x\mu_{x+s}{}_tp_y\mu_{y+t}dsdt \\ &= \int_0^{\infty} v^t(1-{}_tp_x){}_tp_y\mu_{y+t}dt \\ &= \overline{A}_y - \overline{A}_{xy}^{\;1}. \end{aligned}$$

这里，用先死赔付单重次顺位保险的净趸缴保费可表示非先死赔付单重次顺位保险的净趸缴保费，这是数值求解单重次顺位保险的第一步。

另一个表达式可通过改变二重积分次序得到，象 (6.8.4) 一样，有

$$\overline{A}_{xy}^{\;2} = \int_0^{\infty}\int_s^{\infty} v^t {}_tp_y\mu_{y+t}{}_sp_x\mu_{x+s}dtds.$$

在内层积分中以 $r+s$ 代换 t, 得

$$\begin{aligned} \overline{A}_{xy}^{\;2} &= \int_0^{\infty}\int_0^{\infty} v^{r+s}{}_{r+s}p_y\mu_{y+r+s}{}_sp_x\mu_{x+s}drds \\ &= \int_0^{\infty} v^s{}_sp_y{}_sp_x\mu_{x+s}\left(\int_0^{\infty} v^r{}_rp_{y+s}\mu_{y+s+r}dr\right)ds \\ &= \int_0^{\infty} v^s\overline{A}_{y+s}{}_sp_y{}_sp_x\mu_{x+s}ds. \end{aligned}$$

最后一个积分可看作一般结果 $E[W] = E[E[W|V]]$ 的应用，这里 $V = T(x)$, $W = Z$, 在 $T(x) = s$ 条件下 Z 的条件期望值是净保费 ${}_{s|}\overline{A}_y = v^s A_{y+s}{}_sp_y$.

这一节只讨论了顺位保险。被称为继承年金(reversionary annuities) 的顺位年金将在第十二章讨论，其支付在给定状况消亡并且第二种状况存在时开始，持续到第二种状况消亡为止。(作为一开始的例子，参见例 6.5.2 及例 6.5.3 。)

§6.9 单重次顺位函数的求值

这一节考虑单重次顺位函数及净趸缴保费的求值，并考察 Gompertz 死亡律、 Makeham 死亡律以及死亡均匀分布假设的影响。

例 6.9.1: 设死亡效力遵从 Gompertz 死亡律，计算

(1) 当 (x) 在 (y) 之前死亡之时赔付 1 单位的 n 年期顺位保险的净趸缴保险。

(2) 在 n 年内 (x) 死于 (y) 之前的概率。

解: (1) 所求净趸缴保费为

$$\overline{A}^{\,1}_{xy:\overline{n}|} = \int_0^n v^t\,{}_tp_{xy}\mu_{x+t}dt.$$

在 Gompertz 死亡律之下，

$$\begin{aligned}\overline{A}^{\,1}_{xy:\overline{n}|} &= \int_0^n v^t\,{}_tp_{xy}Bc^xc^tdt \\ &= \frac{c^x}{c^x+c^y}\int_0^n v^t\,{}_tp_{xy}B(c^x+c^y)c^tdt \\ &= \frac{c^x}{c^x+c^y}\overline{A}_{\overset{1}{\overline{xy}}:\overline{n}|}.\end{aligned}$$

进一步，如 (6.6.2) 成立，则

$$\overline{A}_{\overset{1}{\overline{xy}}:\overline{n}|} = \overline{A}^{\,1}_{w:\overline{n}|},$$

$$\overline{A}^{\,1}_{xy:\overline{n}|} = \frac{c^x}{c^w}\overline{A}^{\,1}_{w:\overline{n}|}. \tag{6.9.2}$$

(2) 根据 (6.8.2), ${}_nq_{xy}^1$ 就是当 $v=1$ 时的 $\overline{A}_{xy:\overline{n}|}^{1}$ 。于是由 (6.9.2) 可知，在 Gompertz 死亡律下，

$$ {}_nq_{xy}^1 = \frac{c^x}{c^w}{}_nq_w, \tag{6.9.3}$$

其中 $c^w = c^x + c^y$ 。

例 6.9.2: 在 Makeham 死亡律下重做例 6.9.1 。

解: (1)

$$\begin{aligned}
\overline{A}_{xy:\overline{n}|}^{1} &= \int_0^n v^t {}_tp_{xy}(A+Bc^xc^t)dt \\
&= A\int_0^n v^t {}_tp_{xy}dt + \frac{c^x}{c^x+c^y}\int_0^n v^t {}_tp_{xy}B(c^x+c^y)c^t dt \\
&= A(1-\frac{2c^x}{c^x+c^y})\int_0^n v^t {}_tp_{xy}dt \\
&\quad + \frac{c^x}{c^x+c^y}\int_0^n v^t {}_tp_{xy}[2A+B(c^x+c^y)c^t]dt \\
&= A(1-\frac{2c^x}{c^x+c^y})\overline{a}_{xy:\overline{n}|} + \frac{c^x}{c^x+c^y}\overline{A}_{\overline{xy}:\overline{n}|}^{1}.
\end{aligned}$$

利用 (6.6.6) 可得,

$$\overline{A}_{xy:\overline{n}|}^{1} = A(1-\frac{c^x}{c^w})\overline{a}_{ww:\overline{n}|} + \frac{c^x}{2c^w}\overline{A}_{\overline{ww}:\overline{n}|}^{1}. \tag{6.9.4}$$

(2) 在以上结果中置 $v=1$, 得

$$ {}_nq_{xy}^1 = A(1-\frac{c^x}{c^w})\,\mathring{e}_{ww:\overline{n}|} + \frac{c^x}{2c^w}{}_nq_{ww}. \tag{6.9.5}$$

对于在死亡年末赔付的顺位保险，其净趸缴保费为

$$A_{xy}^1 = \sum_{k=0}^{\infty} v^{k+1}{}_kp_{xy}q_{\overline{x+k}:y+k}^{\;1}. \tag{6.9.6}$$

在个体每年死亡均匀分布的假设下，

$$
\begin{aligned}
q_{\overset{1}{x+k}:y+k} &= \int_0^1 {}_sp_{x+k:y+k}\mu_{x+k+s}ds \\
&= \int_0^1 q_{x+k}(1-sq_{y+k})ds \qquad (6.9.7)\\
&= q_{x+k}(1-\frac{1}{2}q_{y+k}).
\end{aligned}
$$

据此，可用 $q_{\overset{1}{x+k}:y-k}$ 表示 ${}_sp_{x+k:y+k}\mu_{x+k+s}$，

$$
\begin{aligned}
&{}_sp_{x+k:y+k}\mu_{x+k+s} = q_{x+k}(1-{}_sq_{y+k})\\
&= q_{x+k}(1-\frac{1}{2}q_{y+k}) + (\frac{1}{2}-s)q_{x+k}q_{y+k}\\
&= q_{\overset{1}{x+k}:y+k} + (\frac{1}{2}-s)q_{x+k}q_{y+k}. \qquad (6.9.8)
\end{aligned}
$$

对于即刻赔付情形，净趸缴保费为

$$
\begin{aligned}
\overline{A}_{xy}^1 &= \sum_{k=0}^{\infty} v^k {}_kp_{xy}\int_0^1 v^s {}_sp_{x+k:y+k}\mu_{x+k+s}ds\\
&= \sum_{k=0}^{\infty} v^{k+1} {}_kp_{xy}[q_{\overset{1}{x+k}:y+k}\int_0^1 (1+i)^{1-s}ds\\
&\quad + q_{x+k}q_{y+k}\int_0^1 (1+i)^{1-s}(\frac{1}{2}-s)ds],\\
&= \frac{i}{\delta}A_{xy}^1 + \frac{1}{2}\frac{i}{\delta}(1-\frac{2}{\delta}+\frac{2}{i})\sum_{k=0}^{\infty} v^{k+1} {}_kp_{xy}q_{x+k}q_{y+k}
\end{aligned}
$$

(6.9.9)

上式右端第 2 项相对于整个保费而言非常小，它是 (6.7.3) 第 2 项的 1/2 。

习　题

§6.2

1. 用单个生命的概率 ${}_np_x$ 与 ${}_np_y$ 表示 (在独立假定下)

(1) (xy) 将生存 n 年的概率。

(2) (x) 与 (y) 中恰有 1 个生存 n 年的概率。

(3) (x) 与 (y) 中至少有 1 个生存 n 年的概率。

(4) (xy) 将在 n 年内消亡的概率。

(5) (x) 与 (y) 中至少有 1 个在 n 年内死亡的概率。

(6) 2 个生命都在 n 年内死亡的概率。

2. 验证 (x) 生存 n 年且 (y) 生存 $n-1$ 年的概率可表示成

$$\frac{{}_np_{x:y-1}}{p_{y-1}} \quad \text{或} \quad p_x\ {}_{n-1}p_{x+1:y}.$$

3. 求

$$\int_0^n {}_tp_{xx}\mu_{xx}(t)dt.$$

§6.3

4. 用代数方式以及一般推理证明

$${}_tp_{\overline{xy}} = {}_tp_{xy} + {}_tp_x(1 - {}_tp_y) + {}_tp_y(1 - {}_tp_x).$$

5. 求 (x) 与 (y) 中至少有一个在第 $n+1$ 年死亡的概率。这个概率是否就是 ${}_{n|}q_{\overline{xy}}$? 请解释。

§6.4

6. 给定 ${}_{25}p_{25:50} = 0.2$, ${}_{15}p_{25} = 0.9$, 计算 40 岁人活到 75 岁的概率。

7. 设 $\mu_x = 1/(100-x), 0 < x < 100$. 计算

(1) ${}_{10}p_{40:50}$.　(2) ${}_{10}p_{\overline{45:50}}$.　(3) $\mathring{e}_{40:50}$.

(4) $\mathring{e}_{\overline{40:50}}$.　(5) $\mathrm{Var}[T(40:50)]$.　(6) $\mathrm{Var}[T(\overline{40:50})]$.

(7) $\mathrm{Cov}[T(40:50), T(\overline{40:50})]$.

(8) $T(40:50)$ 与 $T(\overline{40:50})$ 的相关系数。

8. 求 $\frac{d\overset{\circ}{e}_{xx}}{dx}$.

9. 证明：两个生命 (30) 与 (40) 在同一年死亡的概率可表示成

$$1+e_{30:40}-p_{30}(1+e_{31:40})-p_{40}(1+e_{30:41})+p_{30:40}(1+e_{31:41}).$$

10. 证明：两个生命 (30) 与 (40) 在同一年龄死亡的概率可表示成

$$_{10}p_{30}(1+e_{40:40})-2\,_{11}p_{30}(1+e_{40:41})+p_{40}\ _{11}p_{30}(1+e_{41:41}).$$

11. 设个体 I 与 II 适用的死亡效率分别为

$$\mu_x^{\mathrm{I}}=\log\frac{10}{9},\ x\geq 0$$

与

$$\mu_x^{\mathrm{II}}=(10-x)^{-1},\ 0\leq x<10,$$

如两者都正好 1 岁，求第 1 个死亡发生在 3 到 5 岁之间的概率。

§6.5

12. 证明

$$a_{\overline{\overline{xy}:\overline{n}|}}=a_{\overline{n}|}+_{n|}a_{xy},$$

并叙述相应的受益。

13. 对于记成 $\overline{A}_{\overline{x:\overline{n}|}}$ 的净趸缴保费，叙述其受益并证明

$$\overline{A}_{\overline{x:\overline{n}|}}=\overline{A}_x-\overline{A}_{x:\overline{n}|}+v^n.$$

14. 对于独立的剩余寿命 $T(x)$ 与 $T(y)$, 证明

$$\mathrm{Cov}[v^{T(\overline{xy})},v^{T(xy)}]=(\overline{A}_x-\overline{A}_{xy})(\overline{A}_y-\overline{A}_{xy}).$$

15. 用单重及连生生存年金值表示，在 (25) 与 (30) 中至少有 1 个在不到 50 岁活着时，每年 1 单位连续支付年金的精算现值。

16. 用单重及连生生存年金值表示，在 (25) 与 (30) 中至少有 1 个在 50 岁以上活着时，每年年末支付 1 单位递延年金的精算现值。

17. 对于在 2 个生命 (x) 与 (y) 都活着时每年支付 1, 当 (x) 死后减为 1/2, 而当 (y) 死后减为 **1/3** 的 n 年定期期初年金，计算其精算现值。

18. 某种在 (x) 活着时支付的 1 单位期末年金，在 (y) 活着或者死后 n 年内支付给 (x), 但从现在起最多支付 m 年 $(m > n)$ 。证明其精算现值为

$$a_{x:\overline{n}|} + {}_nE_x a_{x+n:y:\overline{m-n}|}.$$

19. 某种每年 1 单位连续支付的年金，当 (40) 与 (55) 至少有一个活着并且超过 60 岁时支付，但当 (40) 活着并且不满 55 岁时不支付。给出其精算现值的表达式。

20. 适格连生生存者年金 (qualified joint-and-survivor annuity) 当 (x) 活着时每年支付一个初始数额，如果 (y) 在 (x) 死后仍活着，则按初始数额的一定比例 $p(1/2 \le p \le 1)$ 继续支付。

(1) 用单生及连生年金的精算现值表示每年分 m 期支付初始数额为 1 的这种期初年金的精算现值。

(2) 如果向 (x) 与 (y) 支付的适格连生生存者年金之精算现值与向 (x) 支付的某种生存年金之精算现值等价，那么两者称为精算等价的。导出适格连生生存者年金初始年支付数额，与精算等价的生存年金年支付数额之比值的表达式。

§6.6

21. 在 Makeham 死亡律下，当状况 (xy) 用 (ww) 取代时，证明

$$w - y = \frac{\log(c^{\triangle} + 1) - \log 2}{\log c},$$

其中 $\triangle = x - y \ge 0$ 。(这表明 ω 可通过对年轻的年龄 y 加

上一个 $\triangle = x - y$ 的函数。这个性质称为均匀上升法则(law of uniform seniority).)

22. 根据附录的示例生命表以及利率 6% 计算 $\ddot{a}_{50:60:\overline{10}|}$ 。求解过程中使用 $\ddot{a}_{xx}$ 表插值和 $\ddot{a}_{x:x+10}$ 表。

23. 给定遵从 Makeham 律的死亡表，年龄 x 与 y 以及等价的同龄状况 (ww) 。证明

(1) ${}_tp_w$ 是 ${}_tp_x$ 与 ${}_tp_y$ 的几何平均。

(2) ${}_tp_x + {}_tp_y > 2{}_tp_w, x \neq y$.

(3) $a_{\overline{xy}} > a_{\overline{ww}}, x \neq y$.

24. 给定遵从 Makeham 死亡律的死亡表，证明 $\bar{a}_{xy}$ 等于单重生命 (w) 的生存年金之精算现值，其中 $c^w = c^x + c^y$, 后者的利息效力 $\delta' = \delta + A$ 。进一步证明

$$\overline{A}_{xy} = \overline{A}'_w + A\bar{a}'_w,$$

这里，带撇的函数系根据利息效力 δ' 计算。

25. 考虑两个死亡表，分别适用男性与女性，死亡效力分别为

$$\mu_z^M = 3a + \frac{3bz}{2} \text{与 } \mu_z^F = a + bz.$$

我们希望用年龄都为 w 的一个男性与一个女性的连生生存年金精算现值来求一个 x 岁男性与一个 y 岁女性的连生生存年金的精算现值。用 x 与 y 表示 w 。

§6.7

26. 当 $q_x = q_y = 1$ 并且 (x) 与 (y) 在一年中死亡都均匀分布的情况下，求 $\mathring{e}_{xy}$ 。

27. 设 $T(x)$ 与 $T(y)$ 独立，并且在下一年均匀分布。给定 (x) 与 (y) 都在下一年中死去的条件下，证明 (xy) 的消亡时间在一年中并非均匀分布。[提示：验证 $Pr[T(xy) \leq t|(T(x) \leq 1) \cap (T(y) \leq 1)] = 2t - t^2$]

28. 证明

$$\begin{aligned}\frac{1}{\delta} &= \frac{1}{i}[1-(\frac{i}{2}-\frac{i^2}{3}+\frac{i^3}{4}-\frac{i^4}{5}+\cdots)]^{(-1)} \\ &= \frac{1}{i}(1+\frac{i}{2}-\frac{i^2}{12}+\frac{i^3}{24}-\frac{19i^4}{720}+\cdots),\end{aligned}$$

进而得出

$$\frac{i}{\delta}(1-\frac{2}{\delta}+\frac{2}{i}) \cong \frac{i}{6}-\frac{i^3}{360}+\cdots.$$

29. 当每一年的死亡均匀分布时，证明对任何 x, y 及 $j = 1, 2, \cdots, m$

$$_{(j-1)/m}p_{xy} - _{j/m}\,p_{xy} = \frac{1}{m}q_{xy} + \frac{m+1-2j}{m^2}q_x q_y,$$

进而验证表达式 (6.7.8) 。

§6.8

30. 根据一般推理说明

$$_nq^{1}_{xy} = {}_nq_{xy}^{\;2} + {}_nq_x\,{}_np_y.$$

当 $n \to \infty$ 时，该方程变成什么？

31. 对于在 (x) 死亡年末当 (y) 还活着情况下赔付 1 单位的保险，证明其净趸缴保费可以表示成 $vp_y\ddot{a}_{x:y+1} - a_{xy}$ 。

32. 证明 $A^1_{xy} - A_{xy}^{\;2} = A_{xy} - A_y$.

33. 对于在 (50) 死亡之时当 (20) 已死或达到 40 岁情况下赔付 1 单位的保险，用单重生命及第 1 个死亡顺位保险的净趸缴保费表示其净趸缴保费。

34. 对于在 (x) 死亡之时当 (y) 在此前 n 年间死亡的情况下赔付 1 单位的保险，用单个生命及第 1 个死亡顺位保险的净趸缴保费表示其净趸缴保费。

35. 设 $\mu_x = 1/(100-x), 0 \leq x < 100$. 计算 $_{25}q^{\;2}_{25:50}$ 。

§6.9

36. 对于遵从 Makeham 死亡律的死亡表，例如，$A=0.003$，$c^{10}=3$，

(1) 在 $\mathring{e}_{40:50}=17$ 时，计算 ${}_{\infty}q^{1}_{40:50}$。

(2) 用 $\overline{A}_{40:50}$ 与 $\overline{a}_{40:50}$ 表示 $\overline{A}^{1}_{40:50}$。

37. 设死亡遵从 Gompertz 死亡律：$\mu_x=10^{-4}2^{x/8}, x>35$. 根据 (6.9.2),

$$\overline{A}^{1}_{40:48:\overline{10|}}=f\overline{A}^{1}_{w:\overline{10|}},$$

计算 f 与 w。

综合题

38. 状况 $(\overline{n|})$ 是恰好存在 n 年的确定性状况，它可与不确定的生存状况联合使用，例如在 $\overline{A}_{x:\overline{n|}},\overline{A}^{1}_{x:\overline{n|}},A_{x:\overset{1}{\overline{n|}}},\ddot{a}_{x:\overline{n|}},A_{xy:\overline{n|}}$ 之中。简化并解释

(1) $\overline{a}_{\overline{x:\overline{n|}}}$

(2) $\overline{A}_{x:\overset{2}{\overline{n|}}}$

39. 用概率性质 $Pr(A\cup B)=Pr(A)+Pr(B)-Pr(A\cap B)$ 得出 (6.3.5D)。

40. 求 $\frac{\partial}{\partial x}\mathring{e}_{xy}$。

第七章　多重损失模型

§7.1　引　言

第六章将一至五章的理论由单个生命推广到多重生命，然而都受制于单一的死亡这个或然事件。现在我们回到单个生命，然而却受制于多重或然事件。作为这一推广的应用，我们可观察一个企业的雇员人数减少情况，其缘由可能是辞职，或者残疾，或者死亡，或者退休等等。在劳动力计划中，也许只需要估计现时在职员工将来各年份仍将继续留在企业中工作的人数，对此，第一章建立的生存模型是适用的，只要将其中的基本随机变量 — 剩余寿命 (死亡时间) — 解释成在职的终止时间即可。不过精算师所需要的雇员受益计划模型中，在终止工作时支付的受益可能依赖于终止的原因，譬如，退休受益一般不同于死亡或伤残受益。这样，适用于雇员受益体制的”生存”模型应该既包括终止时间随机变量也包括终止原因随机变量。另外，受益结构也常常依赖于收入，这是另一种不同类型的不确定性，将在第八章讨论。

作为另一个应用，大多数个人寿险在约定的缴费期结束前发生停缴保费时提供不没收受益支付。这种保险的一个合适模型应同时将终止时间与终止原因作为随机变量纳入其中。

残疾收入保险向符合保单规定的致残的被保险人提供周期性支付。在某些场合，周期性支付额可能依赖于致残是否因为疾病或意外事故。被保险人可能由于死亡、退保、致残或保险到期而改变其地位。适用于残疾保险的完整模型应将终止时间随机变量与终止原因随机变量纳入其中。

在公共健康计划中，人们感兴趣于用死亡原因对生存与死亡

进行分析，公共健康的目标可通过对死亡时间与死亡原因的联合分布的研究而设置，对心血管疾病与癌症的最先研究就是根据这种分析而建立的。

这一章的主要目的在于，研究涉及给定状况终止时间及其终止原因两个随机变量的分布，所获得的模型在以上所述的雇员受益计划、不没收受益赔付、残疾收入保险以及公共健康计划中都有应用。在精算学中，给定状况的终止称为*损失*(decrement)，这一章的内容称为*多重损失理论*(multiple decrement theory)，而在生物统计学中则称为*竞争风险理论*(theory of competing risks)。

多重损失理论也可建立在决定性比率函数的基础上，这个观点将在 §7.4 概述。

§7.2 两个随机变量

第一章讨论连续型随机变量 $T(x)$ 的方法同样适用于状况的终止时间，只需略作词汇上改变即可。实际上，我们将用同样的符号 $T(x)$ 或 T 来表示终止时间。但现在还将引入第二个随机变量，用来表示状况终止的原因，这个随机变量记为 $J(x)=J$，它是一个离散型随机变量。

在雇员受益计划应用中，随机变量 J 的取值可以是 1,2,3 或 4，分别对应于辞职、残疾、死亡或退休。在人寿保险应用中，J 的取值可以是 1 或 2，分别对应于被保险人死亡或选择终止缴付保费，等等。在残疾保险的应用中，J 的取值可以是 1,2 ,3 或 4，分别对应于被保险人死亡、退保、致残或保险到期。最后，在公共健康的应用中，可能有许多损失原因。譬如在某种研究中，J 的取值可能为 1,2,3 或 4，分别对应于死亡原因为心血管疾病、癌症、意外事故或所有其它原因。

这一节的目的是讨论 T 与 J 的联合分布及其有关的边际分布与条件分布。T 与 J 的联合概率密度函数记为 $f(t,j)$，J 的边

际概率函数记为 $h(j)$, T 的边际密度函数记为 $g(t)$ 。图 7.2.1 描绘了这些分布。注意联合分布是混合型的。

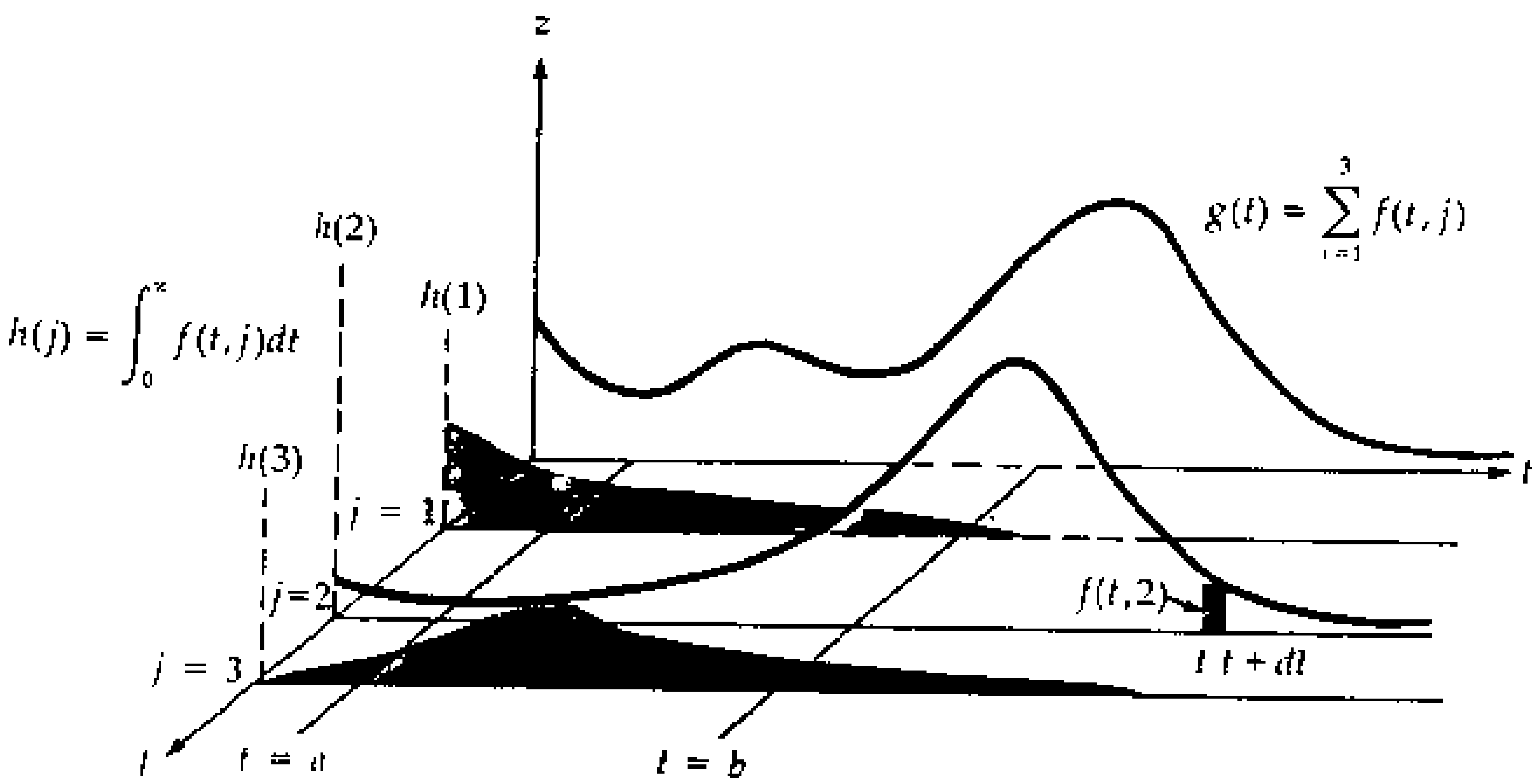


图 7.2.1　$f(t, j)$ 的图形

如果总共有 m 种原因，分别标记为 $1, 2, \cdots, m$ 。那么，

$$\sum_{j=1}^{m} h(j) = 1,$$

$$\int_0^{\infty} g(t)dt = 1.$$

由 T 与 J 定义的事件概率可以用概率密度函数来表示。例如，

$$f(t, j)dt = Pr[t < T \leq t + dt, J = j]. \tag{7.2.1}$$

$$Pr(a < T \leq b) = \sum_{j=1}^{m} \int_a^b f(t, j)dt,$$

$$Pr(0 < T \leq t, J = j) = \int_0^t f(s, j)ds. \tag{7.2.2}$$

上式给出的由原因 j 引起且损失发生在时间 t 之前的概率有特殊符号

$$ {}_tq_x^{(j)} = \int_0^t f(s,j)ds \qquad t \geq 0, j = 1,2,\cdots,m. \tag{7.2.3} $$

根据边际分布的定义

$$ h(j) = \int_0^\infty f(s,j)ds = {}_\infty q_x^{(j)} \qquad j = 1,2,\cdots,m, \tag{7.2.4} $$

这是与第一章不同的新内容。至于 $g(t)$ 及其分布函数 $G(t)$，有

$$ \begin{aligned} g(t) &= \sum_{j=1}^{m} f(t,j), \\ G(t) &= \int_0^t g(s)ds = \sum_{j=1}^{m} \int_0^t f(s,j)ds. \end{aligned} \tag{7.2.5} $$

第一章引入的符号可予以推广使之符合这里的随机变量 T。用上标 (τ) 表示一个函数涉及所有的损失效力原因，于是

$$ {}_tq_x^{(\tau)} = Pr(T \leq t) = G(t) = \int_0^t g(s)ds, \tag{7.2.6} $$

$$ {}_tp_x^{(\tau)} = Pr(T > t) = 1 - {}_tq_x^{(\tau)}, \tag{7.2.7} $$

$$ \begin{aligned} \mu_{x+t}^{(\tau)} &= \frac{g(t)}{1-G(t)} = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tq_x{}^{(\tau)}, \\ &= -\frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tp_x^{(\tau)} = -\frac{d}{dt} \log {}_tp_x^{(\tau)}. \end{aligned} \tag{7.2.8} $$

在数学上，随机变量 T 的这些函数与第一章的完全等同，所不同的是在应用中对它们的解释有差异罢了。

与前面一些章类似，式 (7.2.1) 可分析为

$$ f(t,j)dt = Pr[T > t] Pr[(t < T \leq t + dt) \cap (J = j) | T > t]. \tag{7.2.9} $$

与 (1.1.12) 相仿，这引出原因 j 的损失效力定义：

$$\mu_{x+t}^{(j)} = \frac{f(t,j)}{1-G(t)} = \frac{f(t,j)}{{}_tp_x^{(\tau)}}. \tag{7.2.10}$$

原因 j 在年龄 $x+t$ 的损失效力可作条件概率解释，它是在给定存在到 $x+t$ 岁的条件下，T 与 J 的联合条件概率密度函数在 $x+t$ 的值。式 (7.2.9) 于是可改写成

$$f(t,j)dt = {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}dt \qquad j=1,2,\cdots,m, t\geq 0. \quad (7.2.9重述)$$

用语言来表述为

原因j在时间t与$t+dt$之间的联合损失概率

= (x)在给定状况下存在到时间t的概率${}_tp_x^{(\tau)}$

× 在损失尚未在时间 t 之前发生的条件

下由于原因 j 而在时间 t 与 $t+dt$ 之间

发生损失的条件概率 $\mu_{x+t}^{(j)}dt$.

对 (7.2.3) 求导并利用 (7.2.10) 可得

$$\mu_{x+t}^{(j)} = \frac{1}{{}_tp_x^{(\tau)}}\frac{d}{dt}{}_tq_x^{(j)}, \tag{7.2.11}$$

而由 (7.2.6), (7.2.5) 及 (7.2.3),

$${}_tq_x^{(\tau)} = \sum_{j=1}^{m} {}_tq_x^{(j)}. \tag{7.2.12}$$

因此根据 (7.2.8),(7.2.12) 及 (7.2.11), 有

$$\mu_{x+1}^{(\tau)} = \sum_{j=1}^{m} \mu_{x+t}^{(j)}, \tag{7.2.13}$$

损失的总效力等于 m 种原因的损失效力之和。

联合、边际、条件概率密度函数或概率函数可用精算符号来表达，例如：

$$f(t,j) = {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}, \tag{7.2.10重述}$$

$$h(j) = {}_\infty q_x^{(j)}, \tag{7.2.4重述}$$

$$g(t) = {}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}. \tag{7.2.8重述}$$

给定损失时间 t, J 的条件概率函数为

$$h(j|T=t) = \frac{f(t,j)}{g(t)} = \frac{\mu_{x+t}^{(j)}}{\mu_{x+t}^{(\tau)}}. \tag{7.2.14}$$

最后，(7.2.3) 中的概率可写成

$${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\mu_{x+s}^{(j)}ds. \tag{7.2.3重述}$$

例 7.2.1: 考虑 2 个损失原因的多重损失模型，其损失效力分别为

$$\mu_{x+t}^{(1)} = \frac{t}{100} \qquad t \geq 0,$$

$$\mu_{x+t}^{(2)} = \frac{1}{100} \qquad t \geq 0.$$

计算这个模型的联合、边际、条件概率密度函数或概率函数。

解: 既然

$$\mu_{x+s}^{(\tau)} = \mu_{x+s}^{(1)} + \mu_{x+s}^{(2)} = \frac{s+1}{100},$$

那么生存概率

$$\begin{aligned} {}_tp_x^{(\tau)} &= \exp[-\int_0^t \frac{s+1}{100}ds] \\ &= \exp[-(t^2+2t)/200] \qquad t \geq 0. \end{aligned}$$

于是 T 与 J 的联合概率密度函数为

$$f(t,j)=\begin{cases}\frac{t}{100}\exp[-(t^2+2t)/200] & t\geq 0, j=1,\\ \frac{1}{100}\exp[-(t^2+2t)/200] & t\geq 0, j=2,\end{cases}$$

T 的边际概率密度函数为

$$g(t)=\sum_{j=1}^{2}f(t,j)=\frac{t+1}{100}\exp[-(t^2+2t)/200] \qquad t\geq 0,$$

J 的边际概率函数为

$$h(j)=\begin{cases}\int_0^\infty f(t,1)dt & j=1\\ \int_0^\infty f(t,2)dt & j=2.\end{cases}$$

$h(2)$ 较易计算：

$$\begin{aligned}h(2) &= \frac{1}{100}e^{0.005}\int_0^\infty \exp[-(t+1)^2/200]dt\\ &= \frac{1}{100}e^{0.005}\sqrt{2\pi}10\int_0^\infty \frac{1}{\sqrt{2\pi}10}\exp[-(t+1)^2/200]dt.\end{aligned}$$

作变量代换 $z=(t+1)/10$, 得

$$\begin{aligned}h(2) &= \frac{1}{10}e^{0.005}\sqrt{2\pi}\int_{0.1}^\infty \frac{1}{\sqrt{2\pi}}\exp(-z^2/2)dz\\ &= \frac{1}{10}e^{0.005}\sqrt{2\pi}[1-\Phi(0.1)]=0.1159,\end{aligned}$$

其中 $\Phi(x)$ 是标准正态分布 $N(0,1)$ 的分布函数。与此同时，有 $h(1)=1-h(2)=0.8841$ 。最后，由 (7.2.14), 在给定损失时间 t 的情况下， J 的条件概率函数为

$$h(j|t)=\begin{cases}\frac{t}{t+1} & j=1\\ \frac{1}{t+1} & j=2.\end{cases}$$

例 7.2.2: 按照例 7.2.1 中的联合分布，计算 $E[T]$ 及 $E[T|J=2]$.

解: 用边际概率密度函数 $g(t)$ 计算

$$E[T]=\int_0^\infty t\{\frac{t+1}{100}\exp[-(t^2+2t)/200\}dt.$$

根据定理 1.4.1 进行分部积分，可得

$$\begin{aligned}E[T] &= -t\exp[-(t^2+2t)/200]|_0^\infty\\ &\quad+\int_0^\infty \exp[-(t^2+2t)/200]dt\\ &= 0+100h(2)=11.59.\end{aligned}$$

用给定 $J=2$ 时 T 的条件概率密度函数 $g(t|J=2)=f(t,2)/h(2)$ 计算，

$$E[T|J=2]=\int_0^\infty t\{100^{-1}\exp[-(t^2+2t)/200]\}/h(2)dt.$$

以上积分可这样进行：

$$\begin{aligned}&E[T|J=2]=E[(T+1)-1|J=2]\\ &=\frac{1}{0.1159}\int_0^\infty \frac{t+1}{100}\exp[-(t^2+2t)/200]dt-1\\ &=-(0.1159)^{-1}\exp[-(t^2+2t)/200]|_0^\infty-1\\ &=7.63.\end{aligned}$$

例 7.2.1 与例 7.2.2 表明，一旦 T 与 J 的联合分布得到确认，边际分布与条件分布就可以导出，这些分布的矩也能够决定。

在某些特别场合的应用中，以上模型需作适当修正。当这些应用中在某个时刻的损失概率为正时，终止时间 T 的连续型分布就不合适了。

这种场合的一个例子是具有强制退休年龄的退休金计划。另一个例子是退保时不支付受益的定期人寿保险。这样，在缴付后直至下次缴费日前，被保险人不会退保。这里不打算将记号推广以便包括这种情形，在 §7.7 将对每个例子阐述推广的模型。

与第三章类似，T 的整数部分随机变量 K, 是在损失前经过的整值年数。K 与 J 的联合概率函数为

$$
\begin{aligned}
Pr[K=k, J=j] &= Pr[k < T \leq k+1, J=j] \\
&= \int_k^{k+1} {}_tp_x^{(\tau)} \mu_{x+t}^{(j)} dt \\
&= {}_kp_x^{(\tau)} \int_0^1 {}_sp_{x+k}^{(\tau)} \mu_{x+k+s}^{(j)} ds \\
&= {}_kp_x^{(\tau)} q_{x-k}^{(j)}, \qquad (7.2.15)
\end{aligned}
$$

其中，

$$
q_{x+k}^{(j)} = \int_0^1 {}_sp_{x+k}^{(\tau)} \mu_{x+k+s}^{(j)} ds \qquad (7.2.16)
$$

(比较重述的 (7.2.3)) 。在给定残存到 $x+k$ 岁情况下，不管何种原因引起发生在年龄 $x+k$ 与 $x+k+1$ 之间的损失概率记为 $q_{x+k}^{(\tau)}$,

$$
\begin{aligned}
q_{x+k}^{(\tau)} &= \int_0^1 {}_sp_{x+k}^{(\tau)} \mu_{x+k+s}^{(\tau)} ds \\
&= \int_0^1 {}_sp_{x+k}^{(\tau)} \sum_{j=1}^{m} \mu_{x-k+s}^{(j)} ds \\
&= \sum_{j=1}^{m} q_{x+k}^{(j)}. \qquad (7.2.17)
\end{aligned}
$$

式 (7.2.16) 与 (7.2.17) 揭示了为何多重损失理论也称为竞争风险理论。由原因 j 引起在年龄 $x+k$ 与 $x+k+1$ 之间损失的概率

依赖于 ${}_sp_{x+k}^{(\tau)}, 0 \le s \le 1$, 从而依赖于各成份的效力。当其它损失效力增强时， ${}_sp_{x+k}^{(\tau)}$ 减小，随之 $q_{x+k}^{(j)}$ 亦减小。

§7.3 随机残存组

考察一组 a 岁的 $l_a^{(\tau)}$ 个生命，每一个的损失 (终止) 时间与原因的分布由以下联合概率密度函数确定：

$$f(t,j) = {}_tp_a^{(\tau)}\mu_{a+t}^{(j)} \qquad t \ge 0, j = 1,2,\cdots,m.$$

在年龄 x 与 $x+n$ 之间因原因 j 而离开的成员数随机变量记为 ${}_n\mathcal{D}_x^{(j)}, x \ge a$, 期望值记为 ${}_nd_x^{(j)}$,

$$ {}_nd_x^{(j)} = E[{}_n\mathcal{D}_x^{(j)}] = l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\mu_{a+t}^{(j)}dt. \qquad (7.3.1\text{A})$$

如通常一样，当 $n=1$ 时，可省去 ${}_n\mathcal{D}_x^{(j)}$ 与 ${}_nd_x^{(j)}$ 的前缀。注意，

$$ {}_n\mathcal{D}_x^{(\tau)} = \sum_{j=1}^{m} {}_n\mathcal{D}_x^{(j)}.$$

定义

$$ {}_nd_x^{(\tau)} = E[{}_n\mathcal{D}_x^{(\tau)}] = \sum_{j=1}^{m} {}_nd_x{}^{(j)}. \qquad (7.3.1\text{B})$$

用 (7.3.1A) 可得

$$\begin{aligned} {}_nd_x^{(\tau)} &= l_a^{(\tau)} \sum_{j=1}^{m} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\mu_{a+t}^{(j)}dt \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\mu_{a+t}^{(\tau)}dt. \qquad (7.3.2) \end{aligned}$$

如果用 $\mathcal{L}^{(\tau)}(x)$ 表示原先 $l_a^{(\tau)}$ 个 a 岁成员组在 x 岁时的残存数随机变量，类似于 (1.3.1), 定义

$$l_x^{(\tau)} = E[\mathcal{L}^{(\tau)}(x)] = l_a^{(\tau)} {}_{x-a}p_a^{(\tau)}. \qquad (7.3.3)$$

对 $n=1$ 时的 (7.3.1A) 作代换 $s=t-(x-a)$, 可得

$$d_x^{(j)}=l_a^{(\tau)}\int_0^1 {}_{s+x-a}p_a^{(\tau)}\mu_{x+s}ds.$$

利用 (7.3.3) 及 (7.2.16), 有

$$\begin{aligned} d_x^{(j)} &= l_a^{(\tau)}{}_{x-a}p_a^{(\tau)}\int_0^1 {}_sp_x^{(\tau)}\mu_{x+s}^{(j)}ds \\ &= l_x^{(\tau)}q_x^{(j)}. \end{aligned} \tag{7.3.4}$$

这些结果使得我们可从 $l_x^{(\tau)}$ 与 $d_x^{(j)}$ 的数值表得出 $p_x^{(\tau)}$ 与 $q_x^{(j)}$ 的数值表，两表都称为 *多重损失表* (multiple decrement table)。

例 7.3.1: 根据以下损失概率表建立相应的 $l_x^{(\tau)}$ 与 $d_x^{(j)}$ 的数值表。

x	$q_x^{(1)}$	$q_x^{(2)}$
65	0.02	0.05
66	0.03	0.06
67	0.04	0.07
68	0.05	0.08
69	0.06	0.09
70	0.00	1.00

可以设想，这是一个二损失场合，原因 1 对应于死亡，原因 2 则对应于退休。这个例子显示出， 70 岁是强制退休年龄。

解: 任意给定数值 $l_{65}^{(\tau)}=1000$, 用 (7.3.3) 与 (7.3.4) 可得下表：

x	$q_x^{(1)}$	$q_x^{(2)}$	$p_x^{(\tau)}$	$l_x^{(\tau)}=l_{x-1}^{(\tau)}p_{x-1}^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
65	0.02	0.05	0.93	1000.00	20.00	50.00
66	0.03	0.06	0.91	930.00	27.90	55.80
67	0.04	0.07	0.89	846.30	33.85	59.24
68	0.05	0.08	0.87	753.21	37.66	60.26
69	0.06	0.09	0.85	655.29	39.32	58.98
70	0.00	1.00	0.00	557.0	0.00	557.00

其中 $d_x^{(1)} = l_x^{(\tau)} q_x^{(1)}$, $d_x^{(2)} = l_x^{(\tau)} q_x^{(2)}$ 。

作为对计算的验证，注意到 $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$，差异至多只能是舍入误差。

我们继续此例，求几个概率：

$$
\begin{aligned}
{}_2p_{65}^{(\tau)} &= p_{65}^{(\tau)} p_{66}^{(\tau)} = 0.93 \times 0.91 = 0.8463, \\
{}_{2|}q_{66}^{(1)} &= p_{66}^{(\tau)} p_{67}^{(\tau)} q_{68}^{(1)} = 0.91 \times 0.89 \times 0.05 = 0.0405. \\
{}_2q_{67}^{(2)} &= q_{67}^{(2)} + p_{67}^{(\tau)} q_{68}^{(2)} = 0.07 + 0.89 \times 0.08 = 0.1412.
\end{aligned}
$$

本例算得表格的最后 3 列可用来得出同样的概率，结果精确到 4 位小数：

$$
\begin{aligned}
{}_2p_{65}^{(\tau)} &= \frac{l_{67}^{(\tau)}}{l_{65}^{(\tau)}} = \frac{846.30}{1000.00} = 0.8463, \\
{}_{2|}q_{66}^{(1)} &= \frac{d_{68}^{(1)}}{l_{66}^{(\tau)}} = \frac{37.66}{930.00} = 0.0405, \\
{}_2q_{67}^{(2)} &= \frac{d_{67}^{(2)} + d_{68}^{(2)}}{l_{67}^{(\tau)}} = \frac{59.24 + 60.26}{846.30} = 0.1412.
\end{aligned}
$$

§7.4 决定性残存组

总的损失效力也可看作总的 (名义年) 损失率，而不作为条件概率密度。从这个观点看，一组 $l_a^{(\tau)}$ 个 a 岁生命随着年龄增加，其成员数按决定性损失效力 $\mu_y^{(\tau)}, y \geq a$ 演变。原先 $l_a^{(\tau)}$ 个 a 岁生命在 x 岁时的残存者个数为

$$
l_x^{(\tau)} = l_a^{(\tau)} \exp[-\int_a^x \mu_y^{(\tau)} dy], \tag{7.4.1}
$$

在年龄 x 与 $x+1$ 之间总的减少数为

$$
d_x^{(\tau)} = l_x^{(\tau)} - l_{x+1}^{(\tau)} = l_x^{(\tau)}[1 - \frac{l_{x+1}^{(\tau)}}{l_x^{(\tau)}}]
$$

$$
\begin{aligned}
&= l_x^{(\tau)}\{1-\exp[-\int_x^{x+1}\mu_y^{(\tau)}dy]\} \qquad (7.4.2)\\
&= l_x^{(\tau)}[1-p_x^{(\tau)}] = l_x^{(\tau)}q_x^{(\tau)}.
\end{aligned}
$$

根据定义或对 (7.4.1) 求导有

$$
\mu_x^{(\tau)} = -\frac{1}{l_x^{(\tau)}}\frac{dl_x^{(\tau)}}{dx}. \qquad (7.4.3)
$$

这些公式与 §1.4 中有关生命表的公式相似。

现考虑 m 种损失原因，并假定 x 岁时的 $l_x^{(\tau)}$ 个残存者未来终将因这些损失了结。于是 $l_x^{(\tau)}$ 个残存者可视为由 m 个不同小组构成，小组 j 有 $l_x^{(j)}$ 个成员，他们未来将因原因 j 而终结，$j=1,2,\cdots,m$. 自然，

$$
l_x^{(\tau)} = \sum_{j=1}^{m} l_x^{(j)}. \qquad (7.4.4)
$$

现在可定义原因 j 引起的损失效力

$$
\mu_x^{(j)} = \lim_{h\to 0}\frac{l_x^{(j)}-l_{x+h}^{(j)}}{hl_x^{(\tau)}} = -\frac{1}{l_x^{(\tau)}}\frac{dl_x^{(j)}}{dx}. \qquad (7.4.5)
$$

由 (7.4.3)–(7.4.5),

$$
\mu_x^{(\tau)} = -\frac{1}{l_x^{(\tau)}}\frac{d}{dx}\sum_{j=1}^{m} l_x^{(j)} = \sum_{j=1}^{m}\mu_x^{(j)}. \qquad (7.4.6)
$$

改换变量符号后，式 (7.4.5) 可写成

$$
-dl_y^{(j)} = l_y^{(\tau)}\mu_y^{(j)}dy,
$$

从 $y=x$ 到 $y=x+1$ 积分，得

$$
l_x^{(j)} - l_{x+1}^{(j)} = d_x^{(j)} = \int_x^{x+1} l_y^{(\tau)}\mu_y^{(j)}dy. \qquad (7.4.7)
$$

对 $j=1,2,\cdots,m$ 相加，得

$$l_x^{(\tau)}-l_{x+1}^{(\tau)}=d_x^{(\tau)}=\int_x^{x+1} l_y^{(\tau)}\mu_y^{(\tau)}dy. \tag{7.4.8}$$

在 (7.4.7) 两端除以 $l_x^{(\tau)}$, 还可得

$$\frac{d_x^{(j)}}{l_x^{(\tau)}}=\int_x^{x+1} {}_{y-x}p_x^{(\tau)}\mu_y^{(j)}dy=q_x^{(j)}. \tag{7.4.9}$$

这样，$q_x^{(j)}$ 等于 $l_x^{(\tau)}$ 个 x 岁残存者中因原因 j 在 $x+1$ 岁之前终结的比例。

与生命表情形一样，决定性模型提供了多重损失理论的另一种语言与概念框架。

§7.5 相应的单重损失表

对于在多重损失模型中确认的每一个损失原因，有可能建立相应的单重损失模型。定义相应的单重损失模型函数如下：

$$\begin{aligned} {}_tp'^{(j)}_x &= \exp[-\int_0^t \mu_{x+s}^{(j)}ds], \\ {}_tq'^{(j)}_x &= 1-{}_tp'^{(j)}_x. \end{aligned} \tag{7.5.1}$$

象 ${}_tq'^{(j)}_x$ 那样的量在生物统计学中称为*净损失概率*(net probabilities of decrement), 在有些场合称为*独立损失率*(independent rate of decrement) 。这里净、独立的含义在于，原因 j 在 ${}_tq'^{(j)}_x$ 的决定过程中并不与其它损失原因竞争。本书将采用的术语是*绝对损失率*(absolute rate of decrement), 而避免使用概率这样的词汇。符号 ${}_tq_x^{(j)}$ 表示在年龄 x 与 $x+j$ 之间由原因 j 引起的损失概率，它与 ${}_tq'^{(j)}_x$ 不同。而 ${}_tp'^{(j)}_x$ 与 ${}_tp_x^{(t)}$ 不同，也不一定是一个生存函数，因为并不要求 $\lim_{t\to\infty}{}_tp'^{(j)}_x=0$ 。

尽管

$$\int_0^\infty \mu_{x+t}^{(\tau)}dt = \infty.$$

但是由 (7.2.13) 只能得出，至少有 1 个 j 使得

$$\int_0^\infty \mu_{x+t}^{(j)}dt = \infty.$$

可能有些损失原因使该积分有限。

观察单重损失原因起作用的随机生存系统机会渺茫。在雇员受益计划中，退休、残疾与自愿终止使得直接观察单重损失原因的实际作用成为不可能。在生物统计应用中，任意决定的研究期与观察中止，也妨碍了观察单重损失原因。

在 §7.6 中将会看到，建立多重损失模型通常第一步是选择绝对损失率，并为获得概率 $q_x^{(j)}$ 作出每年损失的影响范围假设。

一. 基本关系

首先注意，

$$\begin{aligned} {}_tp_x^{(\tau)} &= \exp\{-\int_0^t [\mu_{x+s}^{(1)} + \mu_{x+s}^{(2)} + \cdots + \mu_{x+s}^{(m)}]ds\} \\ &= \prod_{i=1}^{m} {}_tp'^{(j)}_x. \end{aligned} \tag{7.5.2}$$

这一关系式并不涉及任何近似，它在任何用来从一组绝对损失率建立多重损失表的方法中均成立。

现在来比较绝对率与概率的大小。由 (7.5.2) 可见，

$${}_tp'^{(j)}_x \geq {}_tp_x^{(\tau)},$$

从而

$${}_tp'^{(j)}_x \mu_{x+t}^{(j)} \geq {}_t p_x^{(\tau)} \mu_{x+t}^{(j)}.$$

在区间 (0,1) 上对 t 积分，得

$$q'^{(j)}_x = \int_0^1 {}_tp'^{(j)}_x \mu_{x+t}^{(j)}dt \geq \int_0^1 {}_tp_x^{(\tau)} \mu_{x+t}^{(j)}dt = q_x^{(j)}. \tag{7.5.3}$$

其它损失原因可促使 ${}_tp'^{(j)}_x$ 远大于 ${}_tp^{(\tau)}_x$，这导致相应的绝对损失率与损失概率的差异。

二. 多重损失中位率

回忆 §1.5 中在 x 岁时的中位死亡率 m_x，根据定义式 (1.5.9)，

$$m_x = \frac{\int_0^1 {}_tp_x\mu_{x+t}dt}{\int_0^1 {}_tp_xdt} = \frac{\int_0^1 l_{x+t}\mu_{x+t}dt}{\int_0^1 l_{x+t}dt} = \frac{d_x}{L_x}, \tag{7.5.4}$$

m_x 是年龄 x 与 $x+1$ 之间死亡效力的加权平均。

在多重损失场合，也可定义类似的中位率。**总损因中位损失率**(central rate of decrement from all causes) 定义为

$$m_x^{(\tau)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}dt}{\int_0^1 {}_tp_x^{(\tau)}dt}, \tag{7.5.5}$$

它是 $\mu_{x+t}^{(\tau)}, 0 \le t < 1$ 的加权平均。类似地有，**损因 j 中位损失率**(central rate of decrement from cause j) 定义为

$$m_x^{(j)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\mu_{x+t}^{(j)}dt}{\int_0^1 {}_tp_x^{(\tau)}dt}, \tag{7.5.6}$$

它是 $\mu_{x+t}^{(j)}, 0 \le t < 1$ 的加权平均。显然

$$m_x^{(\tau)} = \sum_{j=1}^m m_x^{(j)}.$$

相应的单重损失中位率定义为

$$m'^{(j)}_x = \frac{\int_0^1 {}_tp'^{(j)}_x\mu_{x+t}^{(j)}dt}{\int_0^1 {}_tp'^{(j)}_xdt}, \tag{7.5.7}$$

这也是 $\mu_{x+t}^{(j)}$ 在同样范围内的加权平均，只不过权为 ${}_tp'^{(j)}_x$ 而不是 ${}_tp_x^{(\tau)}$ 罢了。如果效力 $\mu_{x+t}^{(j)}$ 在 $0 \le t < 1$ 为常数，那么有 $m_x^{(j)} =$

$m'^{(j)}_x = \mu^{(j)}_x$；如果 $\mu^{(j)}_{x+t}$ 是 t 的递增函数，那么 $m'^{(j)}_x > m^{(j)}_x$；如果 $\mu^{(j)}_{x+t}$ 是 t 的递减函数，那么 $m'^{(j)}_x < m^{(j)}_x$ 。

中位率可提供从 $q'^{(j)}_x$ 到 $q^{(j)}_x$ 的方便然而近似的处理办法，反之亦然。

三．常数损失效力假设

首先假设，每一年的各损失效力为常数，即

$$\mu^{(j)}_{x+t} = \mu^{(j)}_x \qquad 0 \le t < 1,$$

随之

$$\mu^{(\tau)}_{x+t} = \mu^{(\tau)}_x \qquad 0 \le t < 1.$$

于是有

$$\begin{aligned} q^{(j)}_x &= \int_0^1 {}_tp^{(\tau)}_x \mu^{(j)}_x dt \\ &= \frac{\mu^{(j)}_x}{\mu^{(\tau)}_x} \int_0^1 {}_tp^{(\tau)}_x \mu^{(\tau)}_x dt = \frac{\mu^{(j)}_x}{\mu^{(\tau)}_x} q^{(\tau)}_x. \end{aligned} \tag{7.5.8}$$

在常数损失效力假设下，还有

$$\begin{aligned} \mu^{(\tau)}_x &= -\log p^{(\tau)}_x, \\ \mu^{(j)}_x &= -\log p'^{(j)}_x \qquad j = 1, 2, \cdots, m. \end{aligned}$$

因而由 (7.5.8) 可得出

$$q^{(j)}_x = \frac{\log p'^{(j)}_x}{\log p^{(\tau)}_x} q^{(\tau)}_x. \tag{7.5.9}$$

这个公式与 (7.5.2) 一起，可用来根据 $q'^{(j)}_x, j = 1, 2, \cdots, m$ 计算 $q^{(j)}_x$ 。

从方程 (7.5.9) 可解出 $q'^{(j)}_x$,

$$q'^{(j)}_x = 1 - [1 - q^{(\tau)}_x]^{(q^{(j)}_x / q^{(\tau)}_x)}, \tag{7.5.10}$$

这个结果对于由给定损失概率得出绝对损失率有用。

注意，如果 $p'^{(j)}_x$ 或 $p^{(\tau)}_x$ 为 0, 那么 (7.5.9) 与 (7.5.10) 需特殊处理。

四．多重损失均匀分布假设

公式 (7.5.10) 在另一种假设下也成立，那就是每种损失在每一年中均匀分布假设，即

$$ {}_tq_x^{(j)} = tq_x^{(j)} \qquad j = 1,2,\cdots,m, 0 \le t < 1, $$

相加可得

$$ {}_tq_x^{(\tau)} = 1 -{}_t p_x^{(\tau)} = tq_x^{(\tau)}. $$

根据 (7.2.11), 在同样假设下有

$$ {}_tp_x^{(\tau)}\mu_{x+t}^{(j)} = q_x^{(j)} \qquad 0 \le t < 1, \qquad (7.4.11) $$

$$ \mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - tq_x^{(\tau)}}. $$

于是

$$ \begin{aligned} q'^{(j)}_x &= 1 - \exp[-\int_0^1 \mu_{x+t}^{(j)}dt] \\ &= 1 - \exp[-\int_0^1 \frac{q_x^{(j)}}{1 - tq_x^{(\tau)}}dt] \\ &= 1 - \exp[\frac{q_x^{(j)}}{q_x^{(\tau)}}log(1 - q_x^{(\tau)})], \end{aligned} $$

与公式 (7.5.10) 相同。进一步的关系可参考习题 21 。

例 7.5.1: 续例 7.3.1, 根据 (7.5.10) 计算 $q'^{(1)}_x$ 及 $q'^{(2)}_x$ 。

解: 计算结果列表如下：

x	$q_x^{(1)}$	$q_x^{(2)}$	$q'^{(1)}_x$	$q'^{(2)}_x$
65	0.02	0.05	0.02052	0.05052
66	0.03	0.06	0.03095	0.06094
67	0.04	0.07	0.04149	0.07147
68	0.05	0.08	0.05215	0.08213
69	0.06	0.09	0.06294	0.09291
70	0.00	1.00	—	—

在 70 岁时，由于强制退休，已毋须 $q'^{(1)}_{70}$ 与 $q'^{(2)}_{70}$ 。

§7.6 多重损失表的构造

在建立多重损失模型过程中，最好能有可直接用来估计概率 $q_x^{(j)}$ 的有关损失年龄与原因的数据。庞大成熟的雇员受益计划可能会有这样的数据，但对其它一些计划，这样的数据通常并非现成。一种替代方法是通过相应单重损失率的适当假定来建立模型，这种模型的合适与否须经以后获得的数据检验。

一旦选定了称心的相应单重损失表，就可用 §7.5 的结果完成建立多重损失表的过程。从关于 $j=1,2,\cdots,m$ 及所有 x 的一组 $p'^{(j)}_x$ 出发，用 (7.5.2) 可算出 $p_x^{(\tau)}$，进而 $q_x^{(\tau)}=1-p_x^{(\tau)}$ 。

例 7.6.1: 对于以下给出的绝对损失率，用 (7.5.2) 及 (7.5.9) 得出相应的多重损失表。大致上，精算师已考查了有关成员组的特征，并认定以下单重损失表所列损失率是合适的，还假定损因 3 乃退休，可发生于 65 到 70 之间，而在 70 岁时是强制的。

x	$q'^{(1)}_x$	$q'^{(2)}_x$	$q'^{(3)}_x$
65	0.020	0.02	0.04
66	0.025	0.02	0.06
67	0.030	0.02	0.08
68	0.035	0.02	0.10
69	0.040	0.02	0.12

解: 公式 (7.5.2) 可改写成:

$$q_x^{(\tau)} = 1 - \prod_{j=1}^{3}(1 - q'^{(j)}_x).$$

同时根据 (7.5.9) 以及强制退休条件可算出以下表中的概率。至于多重损失表部分的建立, 与例 7.3.1 相同。

x	$q_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
65	0.07802	0.01940	0.01940	0.03912
66	0.10183	0.02401	0.01916	0.05867
67	0.12545	0.02851	0.01891	0.07803
68	0.14887	0.03290	0.01866	0.09731
69	0.17210	0.03720	0.01841	0.11649
70	1.00000	0.00000	0.00000	1.00000

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
65	1000.00	19.40	19. 40	39.21
66	921.99	22.14	17. 67	54.09
67	828.09	23.61	15. 66	64.62
68	724.20	23.83	13. 51	70.47
69	616.39	22.93	11. 35	71.80
70	510.31	0.00	0.00	510.31

当 $p'^{(j)}_x = 0$ 或 $p_x^{(\tau)} = 0$ 时, (7.5.9) 与 (7.5.10) 无法使用, 此时需要用其它办法来代替。其中一种特别调整方法基于相应单重损失表的损失分布假设, 而不考虑多重损失概率。首先考察在每一个单重损失表中的 (每一年龄) 损失均匀分布假设。以下只限于 3 种损失的场合, 有关方法与公式很容易推广到 $m > 3$ 的情形。在所述均匀分布假设下,

$$ {}_tp'^{(j)}_x = 1 - tq'^{(j)}_x \qquad j = 1,2,3; 0 \le t \le 1, \quad (7.6.1)$$

$$ {}_tp'^{(j)}_x \mu^{(j)}_{x+t} = \frac{d}{dt}(-{}_tp'^{(j)}_x) = q'^{(j)}_x. \quad (7.6.2)$$

据此，

$$
\begin{aligned}
q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\mu_{x+t}^{(1)}dt \\
&= \int_0^1 {}_tp'^{(1)}_x\mu_{x+t}^{(1)}{}_tp'^{(2)}_x{}_tp'^{(3)}_x dt \\
&= q'^{(1)}_x\int_0^1(1-tq'^{(2)}_x)(1-tq'^{(3)}_x)dt \\
&= q'^{(1)}_x[1-\frac{1}{2}(q'^{(2)}_x+q'^{(3)}_x)+\frac{1}{3}q'^{(2)}_xq'^{(3)}_x]. \quad (7.6.3)
\end{aligned}
$$

对 $q_x^{(2)}$ 与 $q_x^{(3)}$ 也成立类似公式。可以验证

$$
\begin{aligned}
&q_x^{(1)}+q_x^{(2)}+q_x^{(3)}=q'^{(1)}_x+q'^{(2)}_x+q'^{(3)}_x \\
&-[q'^{(1)}_xq'^{(2)}_x+q'^{(1)}_xq'^{(3)}_x+q'^{(2)}_xq'^{(3)}_x]+q'^{(1)}_xq'^{(2)}_xq'^{(3)}_x \\
&=1-[1-q'^{(1)}_x][1-q'^{(2)}_x][1-q'^{(3)}_x]=q_x^{(\tau)}. \quad (7.6.4)
\end{aligned}
$$

例 7.6.2: 假定相应单重损失表在每一年的损失均匀分布，根据例 7.6. 1 数据得出 65–69 岁的损失概率。

解: 运用 (7.6.3) 计算结果列表如下：

x	$q'^{(1)}_x$	$q'^{(2)}_x$	$q'^{(3)}_x$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
65	0.020	0.02	0.04	0.01941	0.01941	0.03921
66	0.025	0.02	0.06	0.02401	0.01916	0.05866
67	0.030	0.02	0.08	0.02852	0.01892	0.07802
68	0.035	0.02	0.10	0.03292	0.01867	0.09727
69	0.040	0.02	0.12	0.03723	0.01843	0.11643

这些概率与例 7.6.1 根据 (7.5.9) 得出的概率很接近。

在某些场合，根据已知事实需要用到特殊的分布，以下例子说明了对一种损失使用特殊分布的过程。

例 7.6.3: 考虑 3 种损因的场合，它们是死亡、残疾与退出。设死亡与残疾的绝对率分别为 $q'^{(1)}_x$, $q'^{(2)}_x$, 在相应损失表的每一年龄中均匀分布。设退出的绝对率为 $q'^{(3)}_x$, 并且只发生在年末。

(1) 给出 3 种损因在 x 岁到 $x+1$ 岁的 1 年内损失概率公式。

(2) 如果假定相应单重损失模型中的退出只发生在年中或年末，且两者比例相等，都为 $\frac{1}{2}q'^{(3)}$，亦给出损失概率公式。

注：到现在为止，除了可能确认的强制退休年龄外，多重损失模型都是完全连续型的，而且有关理论从多重损失模型开始，在定义效力 $\mu_{x+t}^{(j)}, j=1,2,\cdots,m$ 之后，过渡到单重损失表。在这个例子中，有一个单重损失表的损失离散地发生在所述时间段之末。本书并不打算对这种离散情形定义损失效力，只是直接由单重损失表建立多重损失模型，其中包含前已建立的 (7.2.17) 与 (7.5.2)。

解：(1) 图 7.6.1A 显示了给定单重损失表与一个多重损失表的生存因子，其中

$$ {}_tp_x^{(\tau)} = {}_tp_x'^{(1)} {}_tp_x'^{(2)} {}_tp_x'^{(3)} \qquad t \geq 0. $$

在 $t=1$ 时，${}_tp_x'^{(3)}$ 与 ${}_tp_x'^{(\tau)}$ 有间断，

$$ \begin{aligned} &\lim_{t\to 1} {}_tp_x^{(\tau)} = p_x'^{(1)} p_x'^{(2)} 1, \\ &p_x^{(\tau)} = p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). \end{aligned} $$

在多重损失表中，

$$ q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = 1 - p_x^{(\tau)} = 1 - p_x'^{(1)} p_x'^{(2)} [1 - q_x'^{(3)}]. $$

按均匀分布假设，

$$ \begin{aligned} q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)} \mu_{x+t}^{(1)} dt = \int_0^1 {}_tp_x'^{(1)} {}_tp_x'^{(2)} \mu_{x+t}^{(1)} dt \\ &= q_x'^{(1)} \int_0^1 [1 - tq_x'^{(2)}] dt = q_x'^{(1)} [1 - \frac{1}{2} q_x'^{(2)}]. \end{aligned} $$

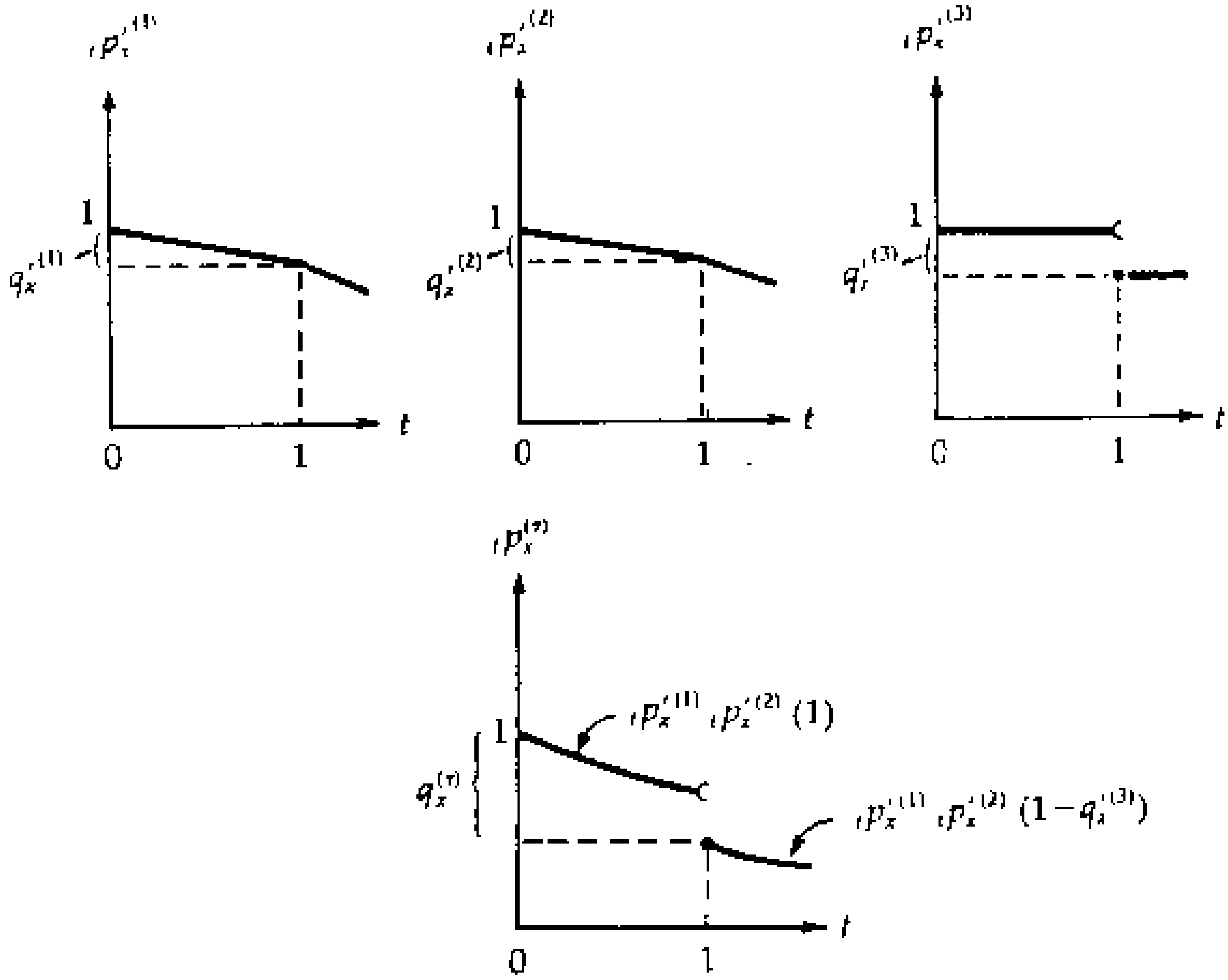


图 7.6.1A　生存因子 ${}_tp_x'^{(1)}, {}_tp_x'^{(2)}, {}_tp_x'^{(3)}$ 与 ${}_tp_x^{(\tau)}$

同理，

$$q_x^{(2)} = q_x'^{(2)}[1 - \frac{1}{2}q_x'^{(1)}].$$

于是

$$\begin{aligned} q_x^{(3)} &= q_x^{(\tau)} - q_x^{(1)} - q_x^{(2)} \\ &= 1 - p_x'^{(1)}p_x'^{(2)}[1 - q_x'^{(3)}] - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)} \\ &= p_x'^{(1)}p_x'^{(2)}q_x'^{(3)}. \end{aligned}$$

注意到

$$\lim_{t\to1-} {}_tp_x^{(\tau)} - \lim_{t\to1+} {}_tp_x^{(\tau)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)} = q_x^{(3)},$$

在 $t=1$ 处 ${}_tp_x^{(\tau)}$ 的间断等于 $q_x^{(3)}$ 。

(2) 此时，${}_tp'^{(1)}_x$ 及 ${}_tp'^{(2)}_x$ 仍与图 7.6.1A 所示相同，而 ${}_tp'^{(3)}_x$ 及 ${}_tp_x^{(\tau)}$ 则在 $t=1/2$ 与 $t=1$ 处有间断，如图 7.6.1B 所示。

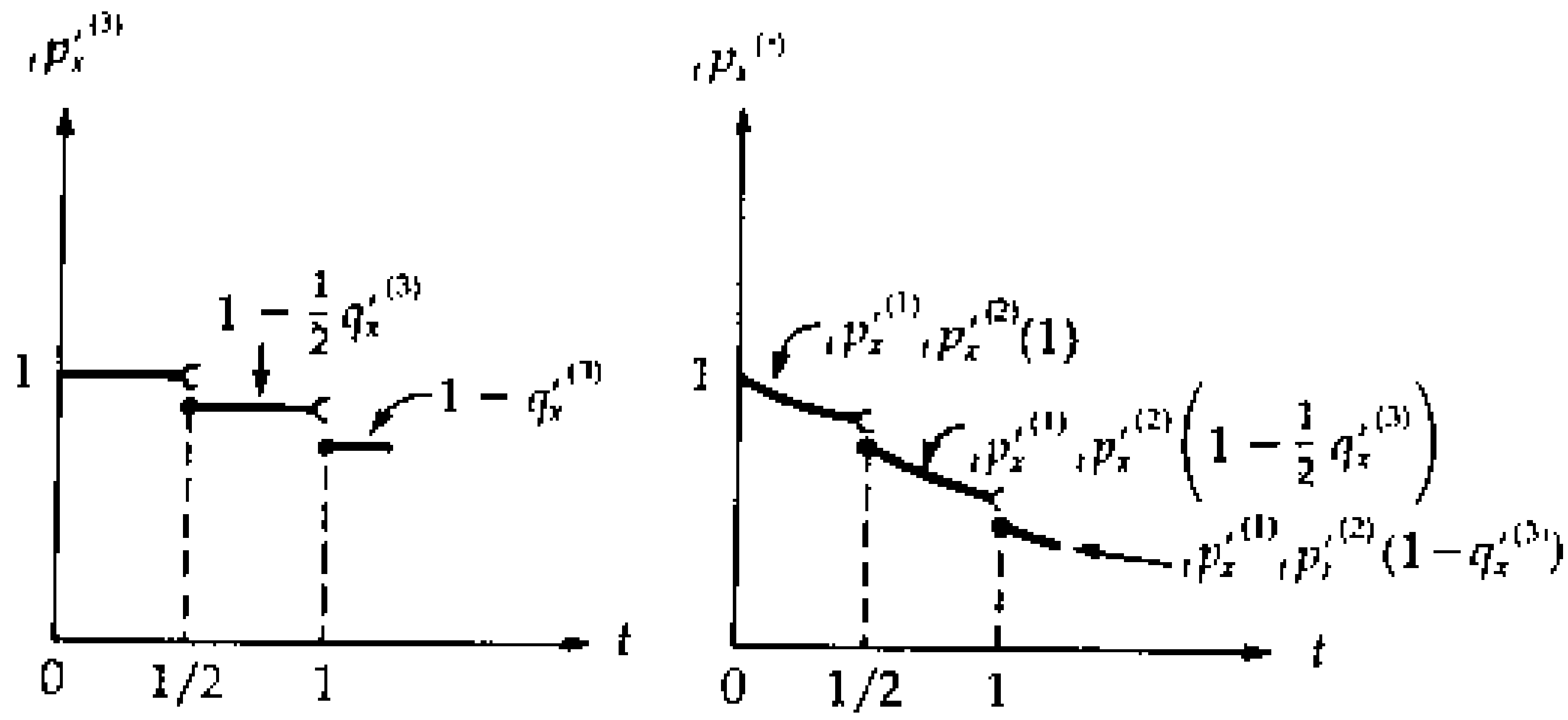


图 7.6.1B　生存因子 ${}_tp'^{(3)}_x$ 与 ${}_tp_x^{(\tau)}$

与前面类似，

$$
\begin{aligned}
q_x^{(1)} &= q'^{(1)}_x\int_0^{1/2}[1-tq'^{(2)}_x]dt+q'^{(1)}_x[1-\frac{1}{2}q'^{(3)}_x]\int_{1/2}^{1}[1-tq'^{(2)}_x]dt \\
&= q'^{(1)}_x[1-\frac{1}{2}q'^{(2)}_x-\frac{1}{4}q'^{(3)}_x+\frac{3}{16}q'^{(2)}_xq'^{(3)}_x].
\end{aligned}
$$

同理

$$q_x^{(2)}=q'^{(2)}_x[1-\frac{1}{2}q'^{(1)}_x-\frac{1}{4}q'^{(3)}_x+\frac{3}{16}q'^{(1)}_xq'^{(3)}_x].$$

于是

$$
\begin{aligned}
q_x^{(3)} &= 1-p_x^{(\tau)}-q_x^{(1)}-q_x^{(2)} \\
&= 1-p'^{(1)}_xp'^{(2)}_x[1-q'^{(3)}_x]-q_x^{(1)}-q_x^{(2)} \\
&= q'^{(3)}_x[1-\frac{3}{4}q'^{(1)}_x-\frac{3}{4}q'^{(2)}_x+\frac{5}{8}q'^{(1)}_xq'^{(2)}_x].
\end{aligned}
$$

§7.7 净趸缴保费及其数值计算

当受益赔付金额依赖于被保险人死亡方式时，就需要应用多重损失模型。在 $x+t$ 岁时原因为 j 的损失受益金额记为 $B_{x+t}^{(j)}$，则净趸缴保费为

$$\overline{A}=\sum_{j=1}^{m}\int_{0}^{\infty}B_{x+t}^{(j)}v^{t}{}_{t}p_{x}^{(\tau)}\mu_{x+t}^{(j)}dt. \tag{7.7.1}$$

在 $m=1$ 且 $B_{x+t}^{(j)}=1$ 时，上述 $\overline{A}$ 成为死亡即刻赔付终身人寿保险的净趸缴保费。与本章主旨较切合的例子是双倍补偿条款(double indemnity provision)，当死亡系意外事故所致时，提供双倍死亡受益。设 $J=1$ 代表死亡系意外事故引起，$J=2$ 代表其它情况的死亡，于是含有双倍补偿条款的 n 年期保险的净趸缴保费由下式给出：

$$\overline{A}=2\int_{0}^{n}v^{t}{}_{t}p_{x}^{(\tau)}\mu_{x+t}^{(1)}dt+\int_{0}^{n}v^{t}{}_{t}p_{x}^{(\tau)}\mu_{x+t}^{(2)}dt. \tag{7.7.2}$$

至此，以积分形式表示净趸缴保费并未完成数值计算的任务，为此，第一步将积分表达式分解成只涉及 1 年的积分，如对式 (7.7.2) 中第 1 个积分，有

$$\int_{0}^{n}v^{t}{}_{t}p_{x}^{(\tau)}\mu_{x+t}^{(1)}dt=\sum_{k=0}^{n-1}v^{k}{}_{k}p_{x}^{(\tau)}\int_{0}^{1}v^{s}{}_{s}p_{x+k}\mu_{x+k+s}^{(1)}ds.$$

现在假定，多重损失模型中的损失在每一年均匀分布，于是

$$\begin{aligned}\int_{0}^{n}v^{t}{}_{t}p_{x}^{(\tau)}\mu_{x+t}^{(1)}dt &= \sum_{k=0}^{n-1}v^{k+1}{}_{k}p_{x}^{(\tau)}q_{x+k}^{(1)}\int_{0}^{1}(1+i)^{1-s}ds\\ &= \frac{i}{\delta}\sum_{k=0}^{n-1}v^{k+1}{}_{k}p_{x}^{(\tau)}q_{x+k}^{(1)}.\end{aligned}$$

对式 (7.7.2) 中第二个积分运用类似的手段，并与以上所获结果相结合，得到

$$\begin{aligned}\overline{A} &= \frac{i}{\delta}[\sum_{k=0}^{n-1} v^{k+1}{}_kp_x^{(\tau)}(2q_{x+k}^{(1)} + q_{x+k}^{(2)})] \\ &= \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}{}_kp_x^{(\tau)}q_{x+k}^{(1)} + \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}{}_kp_x^{(\tau)}q_{x+k}^{(\tau)} \quad (7.7.3) \\ &= \overline{A}_{x:\overline{n}|}^{1(1)} + \overline{A}_{x:\overline{n}|}^{1},\end{aligned}$$

其中 $\overline{A}_{x:\overline{n}|}^{1(1)}$ 是对意外事故死亡保险的 1 单位定期保险的净趸缴保费，$\overline{A}_{x:\overline{n}|}^{1}$ 是对不管何种原因死亡均保险的 1 单位定期保险的净趸缴保费。这里的 ${}_kp_x^{(\tau)}$ 可作为生存组函数从死亡表获得，在 $q_{x+k}^{(1)}$ 能够获得的情况下，为计算 (7.7.3) 就毋须建立完全的二重损失表。

以上例子较为简单，其中受益额并不依赖于损失年龄，因而在同一年中就更不必说了。为研究更复杂的情形，考察 2 个损因的多重损失模型，为简单起见，取 $B_{x+t}^{(1)} = t$, $B_{x+t}^{(2)} = 0, t > 0$ 。此时，净趸缴保费为

$$\begin{aligned}\overline{A} &= \int_0^\infty tv^t{}_tp_x^{(\tau)}\mu_{x+t}^{(1)}dt \\ &= \sum_{t=0}^{\infty} v^k{}_kp_x^{(\tau)}\int_0^1 (k+s)v^s{}_sp_{x+k}^{(\tau)}\mu_{x+k+s}^{(1)}ds.\end{aligned}$$

在每 1 年的 (多重损失模型) 损失均匀分布假设下，

$$\begin{aligned}\overline{A} &= \sum_{k=0}^{\infty} v^{k+1}{}_kp_x^{(\tau)}q_{x+k}^{(1)}\int_0^1 (k+s)(1+i)^{1-s}ds \\ &= \sum_{k=0}^{\infty} v^{k+1}{}_kp_x^{(\tau)}q_{x+k}^{(1)}\frac{i}{\delta}(k+\frac{1}{\delta}-\frac{1}{i}). \qquad (7.7.4)\end{aligned}$$

量

$$k+\frac{1}{\delta}-\frac{1}{i}\cong k+\frac{1}{2}$$

可看作年度 $k+1$ 的等效受益金额，而熟知的项 i/δ 则是提供即刻赔付的校正系数。(7.7.4) 中的保费值可用

$$\sum_{k=0}^{\infty} v^{k+1/2}{}_kp_x^{(\tau)}q_{x+k}^{(1)}(k+\frac{1}{2}) \tag{7.7.5}$$

作很好的近似，它也可从按中点规则计算积分

$$\int_0^1 (k+s)(1+i)^{1-s}ds$$

的近似值得出。

在实践中，当 $B_{x+t}^{(j)}$ 较为复杂时，象 (7.7.5) 这样的近似公式被广泛使用。譬如，对 (7.7.1) 中的积分应用均匀分布假设，有

$$\sum_{k=0}^{\infty} v^{k+1}{}_kp_x^{(\tau)}q_{x+k}^{(j)}\int_0^1 B_{x+k+s}^{(j)}(1+i)^{1-s}ds.$$

再运用中点规则近似积分，就得出一个有用的净趸缴保费表达式

$$\sum_{k=0}^{\infty} v^{k+1/2}{}_kp_x^{(\tau)}q_{x+k}^{(j)}B_{x+k+1/2}^{(j)}. \tag{7.7.6}$$

在 §7.6 中，曾讨论过损失均匀分布假设不适用的情形，在那种场合，净趸缴保费须作调整。重新考察例 7.6.3，其中相应于退出原因的单重损失模型中，年中与年末损失比率各半，均为 $\frac{1}{2}q_x'^{(3)}$ 。在 $x+t,t>0$ 岁退出受益为 $B_{x+t}^{(3)}$ 的净趸缴保费为

$$\begin{aligned}\overline{A} \;=\; & \sum_{k=0}^{\infty} v^k{}_kp_x^{(\tau)}\{\frac{1}{2}q'^{(3)}_{x+k}v^{1/2}B_{x+k+1/2}^{(3)}[1-\frac{1}{2}q'^{(1)}_{x+k}][1-\frac{1}{2}q'^{(2)}_{x+k}] \\ & +\frac{1}{2}q'^{(3)}_{x+k}vB_{x+k+1}^{(3)}[1-q'^{(1)}_{x+k}][1-q'^{(2)}_{x+k}]\}.\end{aligned}$$

这里，所涉及的损失分布相应于单重损失表，而不是多重损失模型中的损失分布。一种可行的近似乃取利息因子的平均 $v^{3/4}$ 以及退出受益的平均

$$\hat{B}_{x+k}=\frac{1}{2}[B_{x+k+1/2}^{(3)}+B_{x+k+1}^{(3)}].$$

如果利息与受益项分别代之以几何平均与算术平均，那么利用例 7.6.3 中得出的 $q_{x+k}^{(3)}$ 表达式，可得

$$\begin{aligned}A &\cong \sum_{k=0}^{\infty} v^{k+3/4}{}_kp_x^{(\tau)}q_{x+3}^{(3)}\hat{B}_{x+k} \\ &= (1+i)^{1/4}\sum_{k=0}^{\infty} v^{k+1}{}_kp_x^{(\tau)}q_{x+k}^{(3)}\hat{B}_{x+k}.\end{aligned}$$

上式最后一个表达式可解释如下：在退出年末提供受益 $\hat{B}_{x+k}$ 并由 $(1+i)^{1/4}$ 作调整，以反映退出受益的支付平均而言要早 1/4 年。

为简明起见，这一章并未按第四章的方式叙述保费决定问题。其实，保费问题完全可以通过引入亏损函数并运用平衡原理解决。譬如，考虑当 (x) 在 r 岁之前因意外事故死亡时赔付 $2B$，其它原因死亡时赔付 B，而在 r 岁之后不论何种原因死亡均赔付 B 的保险，其保险费记为 $\overline{A}$ 。这里有两种损因，设 $J=1$ 对应意外事故原因，$J=2$ 对应非意外事故原因，保险人的亏损函数(参见第四章) 为

$$L=\begin{cases} 2Bv^T-\overline{A} & J=1 \\ Bv^T-\overline{A} & J=2 \end{cases}\Bigg\} 0<T\le r-x \\ \quad\; Bv^T-\overline{A} \qquad J=1,2 \qquad T>r-x.$$

根据平衡原理 $E[L]=0$,

$$\overline{A}=B\int_0^{r-x} v^t{}_tp_x^{(\tau)}\mu_{x+t}^{(1)}dt+B\int_0^{\infty} v^t{}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}dt.$$

容易验证，

$$\mathrm{Var}[L]=B^2[3\int_0^{r-x}v^{2t}{}_tp_x^{(\tau)}\mu_{x+y}^{(1)}dt+\int_0^{\infty}v^{2t}{}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}dt]-(\overline{A})^2.$$

对于净趸缴保费由 (7.7.1) 给出的一般情形，有

$$\begin{aligned}\mathrm{Var}[L] &= E[L^2]=\sum_{j=1}^{m}\int_0^{\infty}[B_{x+t}^{(j)}v^t-\overline{A}]^2{}_tp_x^{(\tau)}\mu_{x+t}^{(j)}dt\\ &= \sum_{j=1}^{m}\int_0^{\infty}[B_{x+t}^{(j)}v^t]^2{}_tp_x^{(\tau)}\mu_{x+t}^{(j)}dt-(\overline{A})^2.\end{aligned} \tag{7.7.7}$$

习　题

§7.2

1. 设 $\mu_{x+t}^{(j)}=\mu_x^{(j)},j=1,2,\cdots,m,t\geq 0$ 。得出

(1) $f(t,j)$,　　(2) $h(j)$,　　(3) $g(t)$

的表达式，并验证 T 与 J 独立。

2. 设 2 个损因的多重损失模型的损失效力为

$$\mu_{x+t}^{(1)}=\frac{1}{100-(x+t)}\qquad t<100-x,$$

$$\mu_{x+t}^{(2)}=\frac{2}{100-(x+t)}\qquad t<100-x.$$

对 $x=50$ 得出以下概率函数或概率密度函数的表达式。

(1) $f(t,j)$.　　(2) $g(t)$.　　(3) $h(j)$.　　(4) $h(j|t)$.

§7.3

3. 用例 7.3.1 中给出的多重损失概率计算：

(1) ${}_3p_{65}^{(\tau)}$.　　(2) ${}_{3|}q_{65}^{(1)}$.　　(3) ${}_3q_{65}^{(2)}$.

4. 下表所列多重损失概率运用于刚进入四年制大学的学生，每届有 1000 人。试问

整值学习时间	概率		
(学年初)	学业失败 ($j=1$)	其它原因辍学 ($j=2$)	完成一学年学业
0	0.15	0.25	0.60
1	0.10	0.20	0.70
2	0.05	0.15	0.80
3	0.00	0.10	0.90

(1) 毕业生人数的期望值是多少？方差又是多少？

(2) 4 年学习期间学业失败人数的期望值是多少？方差又是多少？

5. 根据习题 4 的数据构造多重损失表，并用来求

(1) 随机变量 J(离校方式) 的边际分布，这里 J 按学业失败、其它原因辍学、毕业分别取不同值。

(2) 给定第 3 年终止学业的情况下，终止方式的条件分布。

§7.4

6. 设 $\mu_x^{(1)} = 1/(a-x), 0 \le x < a$, 以及 $\mu_x^{(2)} = 1$, $l_0^{(\tau)} = a$, 导出以下量的表达式：

(1) $l_x^{(\tau)}$.　　(2) $d_x^{(1)}$.　　(3) $d_x^{(2)}$.

7. 设 $\mu_x^{(1)} = 2x/(a-x^2), 0 \le x < \sqrt{a}$, 以及 $\mu_x^{(2)} = c, c > 0$, 并假定 $l_0^{(\tau)} = 1000$, 导出 $l_x^{(\tau)}$ 的表达式。

8. 得出以下导数的表达式：

(1) $\frac{d}{dx}{}_tq_x^{(\tau)}$.　　(2) $\frac{d}{dx}{}_tq_x^{(j)}$.　　(3) $\frac{d}{dx}{}_tq_x^{(j)}$.

§7.5

9. 在多重损失模型中损失均匀分布假设下，用习题 4 的数据计算数值表 $q'^{(j)}_k, j=1,2,3$(其中 k 是整值持续时间)。

10. 设 $\mu_{x+t}^{(1)}$ 为常数 c, $0 \le t < 1$, 导出用 c 与 ${}_tp_x^{(\tau)}$ 表示的公式。

(1) $q'^{(1)}_x$.　　(2) $m_x^{(1)}$.　　(3) $q_x^{(1)}$.

11. 在适当的损失均匀分布假设下，证明

(1) $m_x^{(\tau)} = \frac{q_x^{(\tau)}}{1-(1/2)q_x^{(\tau)}}$.　(2) $m_x^{(j)} = \frac{q_x^{(j)}}{1-(1/2)q_x^{(\tau)}}$.

(3) $m'^{(j)}_x = \frac{q'^{(j)}_x}{1-(1/2)q'^{(j)}_x}$.　(4) $q_x^{(\tau)} = \frac{m_x^{(\tau)}}{1+(1/2)m_x^{(\tau)}}$.

(5) $q_x^{(j)} = \frac{m_x^{(j)}}{1+(1/2)m_x^{(\tau)}}$.　(6) $q'^{(j)}_x = \frac{m'^{(j)}_x}{1+(1/2)m'^{(j)}_x}$.

12. 将 $q'^{(j)}_x$, $q_x^{(j)}$, $m'^{(j)}_x$ 按大小排序，说明理由。

13. 在一个多重损失表中给出 $q'^{(1)}_{40} = 0.02$, $q'^{(2)}_{40} = 0.04$, 计算 $q_{40}^{(\tau)}$, 精确到第 4 位小数。

14. 对于一个双重损失表，给出 $m_{40}^{(\tau)} = 0.2$ 以及 $q'^{(2)}_{40} = 0.1$, 分别按以下假设计算 $q'^{(2)}_{40}$, 精确到第 4 位小数。

(1) 多重损失模型中损失均匀分布。

(2) 相应单重损失表中损失均匀分布。

15. 在多重损失模型中损失均匀分布的假设下，用习题 4 的数据构造数值表 $m_k^{(j)}, j = 1,2, k = 0,1,2,3$(其中 k 是整值持续时间), 精确到第 4 位小数。

16. 设损失可能归因于死亡 $(J=1)$ 、残疾 $(J=2)$ 或退休 $(J=3)$, 根据下表给出的绝对率，用 (7.5.9) 构造多重损失表。

年龄 x	$q'^{(1)}_x$	$q'^{(2)}_x$	$q'^{(3)}_x$
62	0.020	0.030	0.200
63	0.022	0.034	0,100
64	0.028	0.040	0.120

17. 根据习题 16 中的绝对损失率，用 中位率过渡 重新计算多重损失表。

提示：首先用公式

$$m'^{(j)}_x \cong \frac{q'^{(j)}_x}{1-(1/2)q'^{(j)}_x} \qquad j = 1,2,3$$

计算 $m'^{(j)}_x$, 这个公式在相应单重损失表的均匀分布假设下成立。接着假定 $m_x^{(j)} \cong m'^{(j)}_x, j = 1,2,3$, 并由

$$q_x^{(j)} = \frac{d_x^{(j)}}{l_x^{(\tau)}} = \frac{d_x^{(j)}}{l_x^{(\tau)} - (1/2)d_x^{(\tau)} + (1/2)d_x^{(\tau)}} = \frac{m_x^{(j)}}{1+(1/2)m_x^{(\tau)}}$$

得出 $q_x^{(j)}$，这个关系式在多重损失表的总损失均匀分布假设下成立，但是，在所述条件下

$$\begin{aligned}{}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)} &\neq {}_tp'^{(1)}_x\,{}_tp'^{(2)}_x\,{}_tp'^{(3)}_x \\ &= [1 - {}_tq'^{(1)}_x][1 - {}_tq'^{(2)}_x][1 - {}_tq'^{(3)}_x],\end{aligned}$$

即以上条件之间存在不相容性。不过就计算而言，结果还是相当精确的。

18. 指出以下关系式的成立依据：

(1) $m'^{(j)}_x \cong m_x^{(j)}$.　　　　(2) $\frac{q'^{(j)}_x}{1-(1/2)q'^{(j)}_x} \cong \frac{q_x^{(j)}}{1-(1/2)q_x^{(\tau)}}$.

验证以上关系式可导出

(3) $q_x^{(j)} \cong \frac{q'^{(j)}_x[1-(1/2)q_x^{(\tau)}]}{1-(1/2)q'^{(j)}_x}$.　　　　(4) $q'^{(j)}_x \cong \frac{q_x^{(j)}}{1-(1/2)(q_x^{(\tau)}-q_x^{(j)})}$.

并将 (3), (4) 与 (7.5.9),(7.5.10) 对照。

19. 在适当的损失均匀公布假设下,用例 7.5.1 的数值 $q_x^{(j)}, q'^{(j)}_x$ 计算 $m_x^{(j)}, m'^{(j)}_x, j = 1, 2, x = 65, \cdots, 69$(参见习题 11)。

20. 以下哪些陈述是可以接受的？需要时予以修正。

(1) $q_x^{(j)} \cong \frac{m_x^{(j)}}{1+(1/2)m_x^{(j)}}$.

(2) $\int_0^1 l_{x+t}^{(\tau)}dt \cong \frac{l_x^{(\tau)}}{1+(1/2)m_x^{(\tau)}}$.

(3) $q_x^{(1)} = q'^{(1)}_x[1-(1/2)q'^{(2)}_x]$，这里，在二重损失表中，相应单重损失在年龄 x 与 $x+1$ 之间均匀分布。

21. (1) 对于某个年龄 x，某种特殊损因 j 以及常数 K_j，证明以下条件等价。

① ${}_tq_x^{(j)} = K_j\,{}_tq_x^{(\tau)}$　　$0 \le t \le 1$.

② $\mu_{x+t}^{(j)} = K_j\mu_{x+t}^{(\tau)}$　　$0 \le t \le 1$.

③ $1 -_t q'^{(j)}_x = [1 -_t q_x^{(\tau)}]^{K_j}$　　$0 \le t \le 1$.

[提示：① ⇒ ② ⇒ ③ ⇒ ② ⇒ ①]

(2) 在多重损失表中，如果每个损因的常数效力假定

$$\mu_{x+t}^{(j)} = \mu_x^{(j)} \qquad 0 \le t \le 1, j = 1, 2, \cdots, m$$

成立，或者如果每个损因的均匀分布假定

$$_tq_x^{\prime(j)} = tq_x^{\prime(j)} \qquad 0 \le t \le 1, j = 1, 2, \cdots, m.$$

成立，那么请验证

$$_tq_x^{(j)} = k_j\,{}_tq_x^{(\tau)} \qquad 0 \le t \le 1, j = 1, 2, \cdots, m.$$

§7.6

22. 用习题 18 中 $q_x^{\prime(j)}$ 的公式重做习题 9 。

23. 在多重损失模型损失于每一年均匀分布的假设下，证明 $\mu_{x+1/2}^{(j)} = m_x^{(j)}$.

24. 如果给出

(1) $q_x^{\prime(1)}, q_x^{\prime(2)}, q_x^{(3)}$ 或者

(2) $q_x^{\prime(1)}, q_x^{(2)}, q_x^{(3)}$ 你将如何建立多重损失表？

25. 在例 7.6.2 中，设损失 3 在 69 岁并非均匀分布，而是服从型式

$$_tp_{69}^{\prime(3)} = \begin{cases} 1 - 0.12t & 0 < t < 1 \\ 0 & t = 1. \end{cases}$$

换言之，1 年中的绝对损失率为 0.12, 而到 70 岁，所有残存者都将因原因 3 而终止。这与 $q_{69}^{\prime(3)} = 1$ 假定一致。此时值 $q_{69}^{(3)}$ 是多少？

26. 在一个双重损失表中，原因 1 是死亡，原因 2 是退出，并假定：死亡在年龄 h 与 $h+1$ 之间均匀分布；年龄 h 与 $h+1$ 之间退出在达到 h 岁后立即发生。在这个表中，$l_{50}^{(\tau)} = 1000$, $q_{50}^{(2)} = 0.2$, $d_{50}^{(1)} = 0.06d_{50}^{(2)}$, 求 $q_{50}^{\prime(1)}$ 。

§7.7

27. 雇员在 30 岁参加一项受益计划，如果工作到强制退休年龄 70 岁，那么可得每年退休金 300 乘以服务年限；如果在强制退休前的工作期间死亡，那么立即支付受益金 20000; 如果在强制退休年龄之前除死亡以外其它原因离职，那么将得到一笔延期

生存年金，即从 70 岁开始岁入为 300 乘以服务年限。用积分以及连续年金给出 30 岁的雇员这项受益的精算现值表达式。

综合题

28. 基于三重损失表，(20) 不在 65 岁之前由于原因 2 终止的概率是什么？

29.

(1) 给定 $q'^{(1)}_x, q'^{(2)}_x, m^{(3)}_x, m^{(4)}_x$，如何据此建立受制于损失 1: 死亡， 2: 辞职， 3: 残疾， 4: 退休的在职服务群体的多重损失表？

(2) 基于第 (1) 小题的多重损失表，给出现在 y 岁在职者将来不因退休而是由于其它原因终止服务的概率表达式。

30. 证明并解释关系式

$$q^{(j)}_x = q'^{(j)}_x - \sum_{k \neq j} \int_0^1 {}_tp^{(\tau)}_x \mu^{(k)}_{x+t} {}_{1-t}q'^{(j)}_{x+t} dt.$$

31. 置

$$\omega^{(\tau)}(t) = \frac{{}_tp^{(t)}_x}{\int_0^1 {}_tp^{(\tau)}_x dt},$$

$$\omega^{(j)}(t) = \frac{{}_tp'^{(j)}_x}{\int_0^1 {}_tp'^{(j)}_x dt},$$

并设原因 j 以及至少另一个原因在区间 $0 \leq t \leq 1$ 上具有正的损失效力。

(1) 证明

① $\omega^{(\tau)}(0) > \omega^{(j)}(0)$.

② $\omega^{(\tau)}(1) < \omega^{(j)}(1)$. ③存在唯一满足 $0 < r < 1$ 的 r 使得 $\omega^{(\tau)}(r) = \omega^{(j)}(r)$.

(2) 置

$$-I = \int_0^r [\omega^{(j)}(t) - \omega^{(\tau)}(t)] dt,$$

证明

$$I = \int_r^1 [\omega^{(j)}(t) - \omega^{(\tau)}(t)]dt.$$

(3) 设 $\mu_{x+t}^{(j)}$ 在区间 $0 \leq t \leq 1$ 上为递增函数，用积分中值定理建立下列不等式：

$$\begin{aligned} m'^{(j)}_x - m^{(j)}_x &= \int_0^1 [\omega^{(j)}(t) - \omega^{(\tau)}(t)]\mu_{x+t}^{(j)}dt \\ &= \int_0^r [\omega^{(j)}(t) - \omega^{(\tau)}(t)]\mu_{x+t}^{(j)}dt \\ &\quad + \int_r^1 [\omega^{(j)}(t) - \omega^{(\tau)}(t)]\mu_{x+t}^{(j)}dt \\ &= -\mu_{x+t_0}^{(j)} I + \mu_{x+t_1}^{(j)} I \qquad 0 < t_0 < r < t_1 < 1. \end{aligned}$$

第八章 退休金计划估价理论

§8.1 引 言

在精算工作中，多重损失模型主要应用于退休金计划。这一章将讨论退休金计划参加者的分担额与受益额精算现值计算中用到的基本方法。计划的参加者可以是单个雇主的雇员，也可以是从事相似经营的一群雇主的雇员。这种计划在退休时通常为年老退休或伤残退休提供养老金。在辞职场合，可能退还所缴额的积累值或提供延期养老金。在其它缘由发生前死亡的话，可能向指定受益人支付一笔现金或提供收入支付。与保险中称为保费不同，这里为满足受益成本而须分担的费用称为醵出金，通常由计划参加者与发起者按不同比例分摊。

退休金计划可看作是购置延期生存年金及某些附属受益的一种体制，其购买方式为醵出金的某种定期年金。受益金与醵出金的精算现值平衡可以按个人计算，但更为经常的是按所有参加者集体的某种综合为基础进行计算，与此有关的方法是退休基金累积理论的内容。这一章只考虑个别估价(valuation)退休金计划参加者的受益金与醵出金精算现值，综合值可以通过对所有参加者的相加得到。估价退休金计划的受益及醵出的基本工具将在这一章里讨论，而它们应用于计划的基金累积方法将延至第十四章。

§8.2 基本函数

出发点是一个多重损失(服务)表，其中给出参加者的残存组在各年龄的状况，并可得出离职、在职死亡、残疾退休、适龄退休的概率。在 x 岁到 $x+1$ 岁年度的这些概率分别为 $q_x^{(w)}$，$q_x^{(d)}$，

$q_x^{(i)}, q_x^{(r)}$、与第七章的符号一致。以下也要使用第七章的残存组函数 $l_x^{(\tau)}$ 。设 a 是初始年龄，$l_a^{(\tau)}$ 是任意设定一个值，于是有

$$l_{x+1}^{(\tau)} = l_x^{(\tau)}[1-(q_x^{(w)}+q_x^{(d)}+q_x^{(i)}+q_x^{(r)})] = l_x^{(\tau)}p_x^{(\tau)}.$$

这个函数可用来计算 ${}_kp_x^{(\tau)}$，其表达式为

$${}_kp_x^{(\tau)} = \frac{l_{x+k}^{(\tau)}}{l_x^{(\tau)}}.$$

也可用直接递归，

$${}_kp_x^{(\tau)} = {}_{k-1}\,p_x^{(\tau)}p_{x+k-1}^{(\tau)}.$$

与服务表相联系的损失效力在绝大多数年龄是连续型的，它们分别记作 $\mu_x^{(w)}, \mu_x^{(d)}, \mu_x^{(i)}, \mu_x^{(r)}$ 。在某些年龄可出现间断，主要是合乎退休条件的最低年龄 α，以及强制退休的终极年龄 ω 。通常假定 $l_\omega^{(\tau)}=0$，但有时也可假定 $l_\omega^{(\tau)}\neq 0$ 而达到 ω 岁的所有退休正好发生在该年龄。

在早年服务期，离职率倾向较高，离职受益只能是参加者醵出金按利息的积累。经过一段时间，譬如说 5 年，离职者可获得延期退休金。如果这些情况成立，那么就可能需要在适当年数中使用选择离职率，残疾退休情况也提示需要选择基础，对选择的数学修正相对比较容易，如使用选择函数，理论的适应性就更广。在这一章中，只指出进入退休金计划的年龄，并不说明是否采用综合表、选择表或者选择与终极表。附录 2B 的示例服务表的进入年龄为 30，最低退休年龄为 $\alpha=60$，服务的终极年龄为 $\omega=71$，即 $l_{71}^{(\tau)}=0$.

在退休金计划中的主要受益是有资格获得的受益年金，要估价这种年金受益，就需要采用适当的死亡表，以区别残疾退休与适龄退休。相应的年金值将用右上标区分，在残疾退休场合为 $\bar{a}_{x+t}^{i}$，

适龄退休场合为 $\bar{a}^r_{x+t}$ 。退休金通常按月支付，用连续年金值近似是比较方便的办法。

某些退休金计划，尤其对于计时工资情形，受益金根据服务期每年的平均收入确定。另一些计划则按最终的平均薪水的一定百分比确定。在这些场合，为估计受益金，必须预计未来的薪水。计划发起人的分摊通常是薪水的一定百分比，这里，预计未来薪水也是重要的。为此，定义薪水函数：

$(AS)_{x+h}$ 是一个 x 岁进入、现在 $x+h$ 岁参予者的实际年薪(率)；

$(ES)_{x+h+t}$ 是预计 $x+h+t$ 岁时的年薪(率)。

另外，我们假定有一个薪水尺度函数 S_y，使得

$$(ES)_{x+h-t}=(AS)_{x+h}\frac{S_{x+h+t}}{S_{x+h}}. \tag{8.2.1}$$

这里，薪水尺度函数 S_y 不仅反映绩效与资历增长。譬如，在示例服务表（附录 2B）中： $S_y=s_y(1.06)^{y-30}$，其中因子 s_y 表示个人绩效与经验增加导致的薪水提升，而 6% 的累积因子则考虑了长期通胀及退休金计划所有成员的生产力提高而产生的影响。与函数 $l_x^{(\tau)}$ 类似， S_y 的某个值可任意选取，如附录 2B 示例服务表中， S_{30} 取为 1 。通常设 S_y 是一个阶梯函数，在每一年中为常数。

多重损失模型、薪水尺度、投资回报假设、残疾与适龄退休的适当年金值是决定退休金计划受益精算现值的基本因素，也是决定维持这些受益的醵出金的基础。以下各节将讨论退休金计划醵出金与各种类型受益金估价的基本公式。

§8.3 醵出金

醵出金有两种简单型式：每个参加者固定的年率与年薪的固定百分比。对于已达到 $x+h$ 岁的退休金计划参加者，每年(年

率)c 连续支付的未来醵出金精算现值可写成

$$c\int_0^{\omega-x-h} v^t\,{}_tp_{x+h}^{(\tau)}dt = c\sum_{k=0}^{\omega-x-h-1} v^k\,{}_kp_{x+h}\int_0^1 v^s\,{}_sp_{x+h+k}^{(\tau)}ds. \tag{8.3.1}$$

对上式右端项的积分用中点公式近似，得近似值

$$c\sum_{k=0}^{\omega-x-h-1} v^k\,{}_kp_{x+h}^{(\tau)}v^{1/2}\,{}_{1/2}p_{x+h+k}^{(\tau)} = c\sum_{k=0}^{\omega-x-h-1} v^{k+1/2}\,{}_{k+1/2}p_{x+h}^{(\tau)}, \tag{8.3.2}$$

它可以直接从支付发生在年中的假设得出。

如果醵出金是年薪的百分比 c, 那么当前年薪为 $(AS)_{x+h}$ 的参加者未来醵出金的精算现值可表示成

$$\begin{aligned} &c(AS)_{x+h}\int_0^{\omega-x-h} v^t\,{}_tp_{x+h}^{(\tau)}\frac{S_{x+h+t}}{S_{x+h}}dt \\ &= \frac{c(AS)_{x+h}}{S_{x+h}}\sum_{k=0}^{\omega-x-h-1} v^k\,{}_kp_{x+h}^{(\tau)}\int_0^1 v^s\,{}_sp_{x+h+k}^{(\tau)}S_{x+h+k}ds. \end{aligned} \tag{8.3.3}$$

若函数 S_y 在每一年中是常数，则可从积分中提出。进一步用中点公式近似积分，可得精算现值近似值

$$\frac{c(AS)_{x+h}}{S_{x+h}}\sum_{k=0}^{\omega-x-h-1} v^{k+1/2}\,{}_{k+1/2}p_{x+h}^{(\tau)}S_{x+h+k}. \tag{8.3.4}$$

例 8.3.1: 根据附录 2B 示例服务表以及年利率 6%, 给出现龄 50 岁参加者未来醵出金精算现值的公式：

(1) 每年定额 1200 。

(2) 初始定额 1200 且每年递增 100 。

(3) 初始定额 1200 且每年按复合增长率 4% 递增。

解: 在示例服务表中， $\omega = 71$, 这决定了求和范围。

(1) $1200\sum\limits_{k=0}^{20} v^{k+1/2}{}_{k+1/2}p_{50}^{(\tau)}$.

(2) $100\sum\limits_{k=0}^{20}(12+k)v^{k+1/2}{}_{k+1/2}p_{50}^{(\tau)}$.

(3) $1200\sum\limits_{k=0}^{20}(1.04)^k v^{k+1/2}{}_{k+1/2}p_{50}^{(\tau)}$.

最后一个表达式可整理成

$$1200\times(1.04)^{-1/2}\sum_{k=0}^{20}(v')^{k+1/2}{}_{k+1/2}p_{50}^{(\tau)},$$

其中 $v'=1.04/1.06$, 相当于年利率为

$$i'=\frac{1.06}{1.04}-1=0.019.$$

例 8.3.2: 在 超额型计划(excess−type plan) 中，受益与酬出的支付与薪水超过一系列收益水平 $H_0, H_1, \cdots, H_k, \cdots$ 相关, 其中 H_k 应用于未来年度 $k+1$ 。对现龄 50 岁年薪为 30000 的雇员, 设 $30000>H_0$ 且未来年薪保持高于收益水平 $H_k, k=1,2,\cdots$, 据示例服务表给出其数额为超额薪水 5% 的酬出金精算现值公式。

解: 所求精算现值为

$$0.05\sum_{k=0}^{20}\left[30000\frac{S_{50+k}}{S_{50}}-H_k\right]v^{k+1/2}\ {}_{k+1/2}p_{50}^{(\tau)}$$

§8.4 适龄退休受益

在退休金计划中，适龄退休的主要受益通常是延期年金。在规定酬出计划(defined contribution plans) 中，精算现值就是参加者的酬出按利息的积累，而受益则是由该积累购买的年金。在这种计划中，精算现值之确定通过累积过程完成。在其它计划中，退休受益由公式规定，正是这种 规定受益计划(defined benefit plans) 的精算现值，才是我们寻求的表达式。以下先考虑一般情形，然后考虑规定受益的更实用型式。

为此，引入函数 $R(x,h,t)$, 表示 x 岁进入、现龄 $x+h$ 岁、并将在 $x+h+t$ 岁定量的即期或延期年金的年受益收入 (率)。设收入保持为定额，在退休时的精算现值为 $R(x,h,t)\overline{a}^r_{x+h+t}$。值得注意，§7.7 中的受益是总额 B_{x+h+t}; 相应的量在这里是年金值 $R(x,h,t)\overline{a}^r_{x+h+t}$, 其计算是估价过程的预备步骤，随之，现龄 $x+h<\alpha$ 雇员的适龄退休受益之精算现值可用积分表示为

$$APV=\int_{\alpha-x-h}^{\omega-x-h} v^t {}_tp^{(\tau)}_{x+h}\mu^{(\tau)}_{x+h+t}R(x,h,t)\overline{a}^r_{x+h-t}dt. \qquad (8.4.1)$$

与 §7.7 一样，在实际计算精算现值时对积分作近似，假定每一年的退休均匀分布，则有

$$\begin{aligned}APV &= \\ &\sum_{k=\alpha-x-h}^{\omega-x-h-1} v^k {}_kp^{(\tau)}_{x+h}\int_0^1 v^s {}_sp^{(\tau)}_{x+k+h}\mu^{(r)}_{x+h+k+s}R(x,h,k+s) \\ &\qquad \overline{a}^r_{x+h+k+s}ds \\ &=\sum_{k=\alpha-x-h}^{\omega-x-h-1} v^k {}_kp^{(\tau)}_{x+h}q^{(r)}_{x+h+k}\int_0^1 v^s R(x,h,k+s)\overline{a}^r_{x+h+k+s}ds.\end{aligned}$$

用中点公式近似积分，得

$$APV\cong\sum_{k=\alpha-x-h}^{\omega-x-h-1} v^{k+1/2} {}_kp^{(\tau)}_{x+h}q^{(r)}_{x+h+k}R(x,h,k+\frac{1}{2})\overline{a}^r_{x+h+k+1/2}. \qquad (8.4.2)$$

这个公式可作为计算适龄退休受益之精算现值的一般途径。

一. $R(x,h,t)$ 与薪水无关

考虑受益收入为 b 乘以服务年限的情形，这时， $R(x,h,t)=b(h+t)$。如果只计算完整的服务年数，那么 $R(x,h,t)=b(h+k)$,

这里 $k=[t]$. 另一种变例是在超过一定年限后运用较低的比率，譬如 30 年，此时受益收入 (率) 为

$$R(x,h,t)=\begin{cases} b_1(h+t) & h+t\le 30 \\ 30b_1+b_2(h+t-30) & h+t>30. \end{cases}$$

例 8.4.1: 某退休金计划提供的基本受益年收入是每个服务年按每月 15 核算的，到 65 岁为止提供附加受益年收入是每个服务年按每月 10 核算的。根据示例服务表，给出 30 岁进入、现龄 40 岁参加者的这些受益之精算现值公式。

解: 对于基本受益， $R(30,10,t)=15\times 12\times(10+t)$，由 (8.4.2), 其精算现值为

$$180\sum_{k=20}^{30} v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}(10+k+\frac{1}{2})\overline{a}_{40+k+1/2}^{r}.$$

对于附加受益， $R(30,10,t)=10\times 12\times(10+t), 10+t\le 35$, 其精算现值为

$$120\sum_{k=20}^{24} v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}(10+k+\frac{1}{2})\overline{a}_{40+k+1/2:\overline{25-k-1/2|}}^{r}.$$

例 8.4.2: 在例 8.4.1 中，如果服务期超过 35 年的部分不纳入受益核算，那么所获公式需作何种修正？

解: 对于基本受益，现在的受益收入 (率) 为

$$R(30,10,k+\frac{1}{2})=\begin{cases} 180\times(10+k+1/2) & k<25 \\ 180\times 35 & 25\le k\le 30, \end{cases}$$

而附加受益的精算现值公式不变。

二. $R(x,h,t)$ **依赖于最终薪水**

先考虑受益收入是最终薪水的固定比例 g 的情形. 此时

$$R(x,h,t)=g(ES)_{x+h+t}=g(AS)_{x+h}\frac{S_{x+h+t}}{S_{x+h}}.$$

更为常见的情形是基于最后 m 年的平均薪金，按比例 g 得出受益收入，在这种情形下，当 $t \geq m$ 时，

$$\begin{aligned} R(x.h,t) &= g\frac{1}{m}\int_{t-m}^{t}(ES)_{x+h+s}ds \\ &= g\frac{(AS)_{x+h}}{m}\int_{t-m}^{t}\frac{S_{x+h+s}}{S_{x+h}}ds; \end{aligned} \tag{8.4.3}$$

而当 $t < m$ 时，

$$R(x,h,t) = g\frac{1}{m}\left[\int_{t-m}^{0}(AS)_{x+h+s}ds + \int_{0}^{t}(AS)_{x+h}\frac{S_{x+h+s}}{S_{x+h}}ds\right].$$

为数值计算 (8.4.3), 对 $k \leq t < k+1$, 用年中值

$$R(x,h,k+\frac{1}{2}) = g\frac{(AS)_{x+h}}{S_{x+h}}\frac{1}{m}\int_{k+1/2-m}^{k+1/2}S_{x+h+s}ds.$$

在通常 S_y 是每一年为常数的阶梯函数的假设下，有

$$\begin{aligned} R(x,h,k+\frac{1}{2}) &= g\frac{(AS)_{x+h}}{S_{x+h}}\frac{1}{m}\Big(\frac{1}{2}S_{x+h+k-m} + S_{x+h+k-m+1} \\ &\quad + \cdots + S_{x+h+k-1} + \frac{1}{2}S_{x+h+k}\Big). \end{aligned}$$

引入符号

$${}_mZ_y = \frac{1}{m}(\frac{1}{2}S_{y-m} + S_{y-m+1} + \cdots + S_{y-1} + \frac{1}{2}S_y). \tag{8.4.4}$$

以上表达式可写成

$$R(x,h,k+\frac{1}{2}) = g(AS)_{x+h}\frac{{}_mZ_{x+h+k}}{S_{x+h}}. \tag{8.4.5}$$

一种更普遍的最终薪水形式受益是按最后平均薪水与服务年限的乘积来核算，其代表公式为

$$R(x,h,t) = f(h+t)(AS)_{x+h}\left[\frac{1}{m}\int_{t-m}^{t}\frac{S_{x+h+s}}{S_{x+h}}ds\right],$$

其中 f 是设定的比例因子，譬如 $f=0.02$。对数值近似，可像 (8.4.5) 一样写出

$$R(x,h,k+\frac{1}{2})=f(h+k+\frac{1}{2})(AS)_{x+h}\frac{{}_mZ_{x+h+k}}{S_{x+h}}. \qquad (8.4.6)$$

在某些场合，只核算完整的服务年限. 此时

$$R(x,h,k+\frac{1}{2})=f(h+k)(AS)_{x+h}\frac{{}_mZ_{x+h+k}}{S_{x+h}}. \qquad (8.4.7)$$

例 8.4.3: 在某种分段计划(step−rate plan) 中，年度 $k+1$ 的退休受益收入 = 服务年限 × $[1.25\%H_k+1.75\%$(最后 3 年平均薪水超过 H_k 的部分)]。对于现龄 30 岁年薪 20000 的进入计划者，给出在 63 与 64 岁之间退休情形的年中受益收入率公式，这里假定最后 3 年平均薪水超过 H_{33}.

解:

$$\begin{aligned} R(30,0,33\frac{1}{2}) &= 33.5\left[0.0125H_{33}+0.0175(20000\frac{{}_3Z_{65}}{S_{30}}-H_{33})\right] \\ &= 33.5\times\left(350\frac{{}_3Z_{63}}{S_{30}}-0.005H_{33}\right). \end{aligned}$$

例 8.4.4: 在某种抵消计划(offset plan) 中，年受益收入 (率) 先按最后 3 年平均年薪的 2% 乘以服务年限核算，随后减去一项基于参加者社会保险受益的抵消额。抵消额等于从社会保险获得的初始退休受益的 50%, 对于 30 岁进入、现龄 40 岁、年薪为 30000 并估计 65 岁退休时的社会保险受益收入 (年率) 为 P 的计划参加者，给出正好在 65 岁退休的受益收入率公式。

解:

$$\begin{aligned} R(30,10,25) &= 35\left(0.02\times 30000\times\frac{{}_3\tilde{Z}_{65}}{S_{40}}\right)-0.5P \\ &= 21000\frac{{}_3\tilde{Z}_{65}}{S_{40}}-0.5P, \end{aligned}$$

其中

$$ {}_3\tilde{Z}_{65}=\frac{S_{62}+S_{63}+S_{64}}{3}. $$

例 8.4.5: 某种附加计划(add−on plan) 对每一年的服务提供最后 5 年平均报酬的 1.5% 基本受益收入，当 65 岁前退休时，还附加支付到 65 岁为止的每一年服务提供最后 5 年平均报酬的 0.5% 的补充受益。设最低退休年龄为 60 岁，强制退休年龄为 70 岁，且有些参加者一直工作到 70 岁。给出 30 岁开始服务、现龄 40 岁且年薪 30000 的参加者以上受益之精算现值公式。

解: 这里， $q_{40+k}^{(\tau)}=0,k=0,1,2,\cdots,19$ 和 ${}_{30}p_{40}^{(\tau)}\neq 0$ 。精算现值为

$$ \sum_{k=20}^{29} v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}(10+k+\frac{1}{2})\frac{{}_5Z_{40+k}}{S_{40}}450\overline{a}_{40+k+1/2}^{r} $$

$$ +v^{30}{}_{30}p_{40}^{(\tau)}40\frac{{}_5\tilde{Z}_{70}}{S_{40}}18000\overline{a}_{70}^{r} $$

$$ +\sum_{k=20}^{24} v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}(10+k+\frac{1}{2})\frac{{}_5Z_{40+k}}{S_{40}} $$

$$ 150\overline{a}_{40+k+1/2:\overline{25-k-1/2|}}^{r}, $$

其中

$$ {}_5\tilde{Z}_{70}=\frac{S_{65}+S_{66}+S_{67}+S_{68}+S_{69}}{5}. $$

例 8.4.6: 在例 8.4.5 中，如果核算受益收入的服务时间以 30 年为限，那么相应的受益精算现值如何改变？

解: 此时，对所考虑的参加者，到 60 岁时服务年限已达到 30 年，相应的精算现值公式简化为

$$ 13500\Big\{\sum_{k=20}^{29} v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}\frac{{}_5Z_{40+k}}{S_{40}}\overline{a}_{40+k+1/2}^{r}+v^{30}{}_{30}p_{40}^{(\tau)}\frac{{}_5\tilde{Z}_{70}}{S_{40}}\overline{a}_{70}^{r} $$

$$ +\frac{1}{3}\sum_{k=20}^{24} v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}\frac{{}_5Z_{40+k}}{S_{40}}\overline{a}_{40+k+1/2:\overline{25-k-1/2|}}^{r}\Big\}. $$

例 8.4.7: 给出例 8.4.6 中参加者的相应于 30 岁到 40 岁之间服务的退休受益之精算现值公式。

解: 30 岁到 40 岁之间的服务期为 10 年，是 30 的 1/3, 因此，相应受益的精算现值是按 30 年服务年限计算的例 8.4.6 中精算现值的 1/3, 只需将例 8.4.6 所得公式中的 13500 改为 4500 即可。

三. $R(x,h,t)$ 决定于整个服务期间的平均薪水

另一种类型的受益收入 (年率) 是整个服务期间收入的一定比例 f, 它也可看作为服务年限与整个服务期间平均薪水乘积的 f 倍。因此这种受益称为*服务期间平均受益*(career average benefits)。

服务期间平均退休受益的精算现值计算自然地分解成两部分：一部分针对薪水已知的过去服务期，另一部分针对薪水需估计的未来服务期。这里，与 (8.4.3) 相应的场合不同，过去薪水不仅仅用来核算接近退休的参加者受益，通常对所有参加者都要用到过去的实际薪水。设处于状态 $x+h$ 的参加者过去薪水总额为 $(TPS)_{x+h}$, 则相应的受益 (年率) 为 $f(TPS)_{x+h}$, 过去服务受益的精算现值为

$$f(TPS)_{x+h}\sum_{k=\alpha-x-h}^{\omega-x-h-1} v^{k+1/2}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(r)}\overline{a}_{x+h+k+1/2}^{r}. \quad (8.4.8)$$

而基于未来服务的受益收入率则是

$$f\int_0^t (ES)_{x+h+s}ds = f\frac{(AS)_{x+h}}{S_{x+h}}\int_0^t S_{x+h+s}ds.$$

对数值计算，当 S_{x+h+s} 是按年阶梯函数时，以上受益成为

$$f\frac{(AS)_{x+h}}{S_{x+h}}\left(\sum_{j=0}^{k-1} S_{x+h+j}+\frac{1}{2}S_{x+h+k}\right), \quad (8.4.9)$$

其中 $k=[t]$ 。于是未来服务受益的精算现值为

$$f\frac{(AS)_{x+h}}{S_{x+h}}\Big[\sum_{k=\alpha-x-h}^{\omega-x-h-1} v^{k+1/2}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(r)}\overline{a}_{x+h+k+1/2}^{r}$$

$$(\sum_{j=0}^{k-1}S_{x+h+j}+\frac{1}{2}S_{x+h+k})\Big]. \tag{8.4.10}$$

因为当 $k<\alpha-x-h$ 时 $q_{x+h+k}^{(r)}=0$, (8.4.10) 可改写成

$$f\frac{(AS)_{x+h}}{S_{x+h}}\Big[\sum_{k=0}^{\omega-x-h-1} v^{k+1/2}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(r)}\overline{a}_{x+h+k+1/2}^{r}$$

$$(\sum_{j=0}^{k-1}S_{x+h+j}+\frac{1}{2}S_{x+h+k})\Big]. \tag{8.4.11}$$

图 8.4.1 显示了以上二重求和的 j,k 值，其中 "o" 表示有乘积系数 1/2 的项。改变求和次序， (8.4.11) 成为

$$f\frac{(AS)_{x+h}}{S_{x+h}}\Big[\sum_{j=0}^{\omega-x-h-1} S_{x+h+j}(\frac{1}{2}v^{j+1/2}{}_jp_{x+h}^{(\tau)}q_{x+h+j}^{(r)}\overline{a}_{x+h+j+1/2}^{r}$$

$$+\sum_{k=j+1}^{\omega-x-h-1} v^{k+1/2}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(r)}\overline{a}_{x+h+k+1/2}^{r})\Big], \tag{8.4.12}$$

其中，内层和式当 $j=\omega-x-h-1$ 时为 0 。

表达式 (8.4.12) 可解释成年度 $j+1$ 的服务为年度 $j+1$ 以后的退休提供一个完全受益单位

$$f\frac{(AS)_{x+h}}{S_{x+h}}S_{x+h+j},$$

它为年度 $j+1$ 内的退休提供平均的 1/2 单位受益。

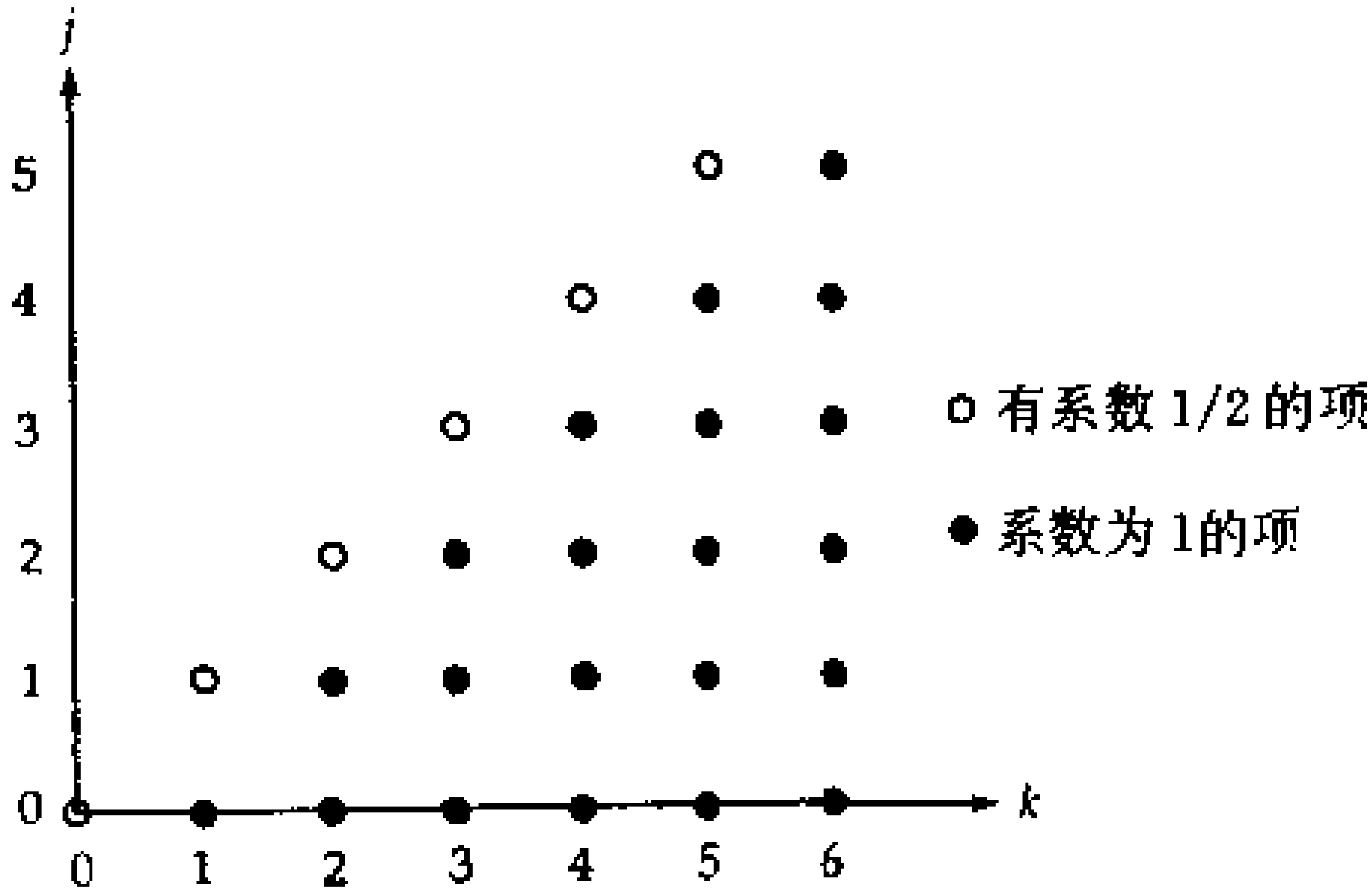


图 8.4.1　公式 (8.4.11) 与 (8.4.12) 的求和点

代替未来服务总受益的估价，有时会对决定相应于 $x+h$ 到 $x+h+1$ 岁之间服务收益之精算现值感兴趣，它就是 (8.4.12) 中第 1 个被加项 (即 $j=0$ 项)，消去 S_{x+h} 后成为

$$f(AS)_{x+h}\left[\frac{1}{2}v^{1/2}q_{x+h}^{(\tau)}\overline{a}_{x+h+1/2}^{r}\right.$$
$$\left.+\sum_{k=1}^{\omega-x-h-1}v^{k+1/2}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(\tau)}\overline{a}_{x+h+k+1/2}^{r}\right]. \quad (8.4.13)$$

这个精算现值可解释成在服务期平均计划的 $x+h$ 岁参加者相应于当前服务的受益成本。

例 8.4.8: (1) 某种服务期平均计划提供的退休收入为参加者服务期间薪水总额的 2% . 对 30 岁进入的、过去薪水总和为 200000 、现龄 40 岁且年薪为 25000 的参加者，写出其在 67 与 68 岁之间退休情形下总受益收入 (年率) 的表达式。

(2) 用附录示例服务表给出上述参加者未来服务的受益精算现值。

(3) 仍用示例服务表给出相应于从 40 到 41 岁服务的受益精算现值。

解: (1) 总的年中受益收入 (年率) 为

$$\begin{aligned} R(30,10,27\frac{1}{2}) &= 0.02\left[200000+\frac{25000}{S_{40}}\left(\sum_{j=0}^{26}S_{40+j}+\frac{1}{2}S_{67}\right)\right] \\ &= 4000+\frac{500}{S_{40}}\left(\sum_{j=0}^{26}S_{40+j}+\frac{1}{2}S_{67}\right). \end{aligned}$$

(2) 根据 (8.4.10), 所求精算现值为

$$\frac{500}{S_{40}}\sum_{k=20}^{30}v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}\left(\sum_{j=0}^{k-1}S_{40+j}+\frac{1}{2}S_{40+k}\right)\overline{a}^r_{40+k+1/2}.$$

或者根据 (8.4.12), 表达式为

$$\begin{aligned} &\frac{500}{S_{40}}\Big[\sum_{j=0}^{30}S_{40+j}\Big(\frac{1}{2}v^{j+1/2}{}_jp_{40}^{(\tau)}q_{40+j}^{(r)}\overline{a}^r_{40+j+1/2} \\ &+\sum_{k=j+1}^{30}v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}\overline{a}^r_{40+k+1/2}\Big)\Big]. \end{aligned}$$

由于当 $k<20$ 时; $q_{40+k}^{(r)}=0$, 和式中不少被加项都是 0.

(3) 将值为 0 的项去掉, 从 (8.4.13) 可得所求结果

$$500\sum_{k=20}^{30}v^{k+1/2}{}_kp_{40}^{(\tau)}q_{40+k}^{(r)}\overline{a}^{(r)}_{40+k+1/2}.$$

§8.5 残疾受益

前一节的类似过程也可用来估价残疾退休金，这种退休金通常依据致残时的薪水确定，可能有一个最低受益金，还有可能在某个年龄 (譬如 65 岁) 转为适龄退休金。我们以残疾退休金等于比例 f 乘以致残时年薪再乘以服务年限为例进行说明。假定参加者必须至少服务 5 年并且年龄不到 65 岁，在有资格获得残疾受益的这段时期内，受益收入 (年率) 的下限为 $10f$ 乘以致残时的年薪。对于 x 岁新进入的参加者，受益收入 (率) 函数为

$$R(x,0,t)=\begin{cases} 0 & 0\le t<5\text{或}t\ge 65-x \\ 10f(ES)_{x+t}=10f(AS)_x\frac{S_{x+t}}{S_x} & 5\le t<10 \\ tf(ES)_{x+t}=tf(AS)_x\frac{S_{x+t}}{S_x} & 10\le t<65-x. \end{cases} \tag{8.5.1}$$

以上残疾受益的精算现值由下式给出

$$\int_5^{65-x} v^t\,{}_tp_x^{(\tau)}\mu_{x+t}^{(i)}R(x,0,t)\bar{a}_{x+t}^i dt, \tag{8.5.2}$$

其近似值为

$$\sum_{k=5}^{64-x} v^{k+1/2}\,{}_kp_x^{(\tau)}q_{x+k}^{(i)}R(x,0,k+\frac{1}{2})\bar{a}_{x+k+1/2}^i. \tag{8.5.3}$$

这些表达式与 (8.4.1) 及 (8.4.2) 的差异在于，这里使用残疾损失效力及残疾损失概率，积分及求和限不同，并使用残疾年金值。

例 8.5.1: 某种退休金计划提供的残疾受益收入为最终薪水的 50%，但不超过最终薪水的 70 % 减去社会保险的残疾受益最初额。但参加者必须已服务了至少 3 年并在 65 岁之前因残疾退休。对于 30 岁进入的年薪为 15000 的参加者，设年龄在 y 与 $y+1(30\le y<65)$ 之间残疾退休的社会保险受益初始额估计为 $I_y,(30\le y<65)$，求该退休金计划的残疾受益收入 (率) 。

解: 如 $k=0,1,2$, 则 $R(30,0,k+1/2)=0$; 如 $3\le k\le 34$, 则

$$R(30,0,k+\frac{1}{2}) \;=\; \Big\{7500\frac{S_{30+k}}{S_{30}},\; \Big[10500\frac{S_{30+k}}{S_{30}}-I_{30+k}\Big]\Big\}.$$

的较小值。

§8.6 离职受益

一般有两种类型离职受益。经过若干年后, 譬如说 10 年后, 离职者有资格获得延期年金。例如, 考虑离职受益为始于 60 岁的延期年金, 其年收入 (率) 为 f 乘以离职时的服务年限再乘以离职时的年薪。在此情形下有

$$R(x,h,t)=\begin{cases}0 & h+t<10\\ f(h+t)(ES)_{x+h+t} & 10\le h+t<60-x,\end{cases}$$

其受益的精算现值近似表达式为

$$\sum_{k=l}^{59-x-h} v^{k+1/2}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(\omega)}R(x,h,k+\frac{1}{2})_{60-x-h-k-1/2|}\overline{a}^r_{x+h+k+1/2}, \tag{8.6.1}$$

这里, $l=\{10-h,0\}$ 的较大值, 使用 $\overline{a}^r$ 是基于假定: 退休人口的死亡表适用于离职者的延期退休年金计算。

另一种类型的离职受益应用于包含参加者醵出的计划, 这些计划通常涉及政府雇员。当参加者在有资格获得退休金受益之前离职时, 返还参加者的按利息累积的醵出金总额。与此类似的受益通常当参加者在职死亡时支付, 但此时对生存者也提供收入受益。这里, 我们只限于考虑离职时的退款受益, 并且只考虑有关过去及当年醵出退款。我们也可以接着考虑对未来醵出的退款进

行估价的服务期间平均公式，但这种公式较复杂并且在实践中可以避免。

为估价过去醵出退款受益，用 $(ATPC)_{x+h}$ 来记现龄 $x+h$ 岁参加者按利息累积的过去醵出金总额。设参加者的醵出金在未来按年利率 j 累积，那么 $x+h+t$ 岁离职时受益总额为

$$B(x,h,t)=(ATPC)_{x+h}(1+j)^{t}.$$

相应于过去醵出的退款受益精算现值近似为

$$(ATPC)_{x+h}\sum_{k=0}^{\beta-x-h-1} v^{k+1/2}{}_{k}p_{x+h}^{(\tau)}q_{x+h+k}^{(\omega)}(1+j)^{k+1/2},\quad(8.6.2)$$

其中，β 是有资格获得 (延期或即期) 退休金的年龄，$x+h<\beta$。这里假定，β 岁之后不再退款。

有关参加者薪水 $c\%$ 的当前醵出退款在当年离职时可用 $(1/2)(0.01c)\ (AS)_{x+h}$ 近似。相应于当前醵出的退款受益之精算现值为

$$0.01c(AS)_{x+h}\left\{\frac{1}{2}v^{1/2}q_{x+h}^{(\omega)}+\sum_{k=1}^{\beta-x-h-1} v^{k+1/2}{}_{k}p_{x+h}^{(\tau)}q_{x+h+k}^{(\omega)}(1+j)^{k}\right\},\quad(8.6.3)$$

其中 $x+h<\beta$。

例 8.6.1: 对 $j=i$，简化 (8.6.2) 与 (8.6.3)。

解: 当 $j=i$ 时，(8.6.2) 成为

$$\begin{aligned}&(ATPC)_{x+h}\sum_{k=0}^{\beta-x-h-1}{}_{k}p_{x+h}^{(\tau)}q_{x+h+k}^{(\omega)}\\&=(ATPC)_{x+h}Pr[(x+h)\text{将在有资格获得退休金之前离职}]\\&=(ATPC)_{x+h}\frac{l_{x+h}^{(w)}-l_{\beta}^{(w)}}{l_{x+h}^{(\tau)}},\end{aligned}$$

其中 $l_y^{(w)}$ 是 $l_y^{(\tau)}$ 个 y 岁残存在职者中将会辞职的期望人数。

公式 (8.6.3) 现在成为

$$
\begin{aligned}
&0.01(AS)_{x+h}v^{1/2}[\frac{1}{2}q_{x+h}^{(w)}+\sum_{k=1}^{\beta-x-h-1}{}_kp_{x+h}^{(\tau)}q_{x+h+k}^{(w)}] \\
&=0.01(AS)_{x+h}v^{1/2}\frac{(1/2)(l_{x+h}^{(w)}-l_{x+h+1}^{(w)})+l_{x+h+1}^{(w)}-l_{\beta}^{(w)}}{l_{x+h}^{(\tau)}}.
\end{aligned}
$$

§8.7 计算基数

在退休金计划的精算现值计算中，一些特殊的计算基数提供了一种传统的方法，这些计算基数在使用同一组精算假设进行大量的估价时会有用。退休金计算中用到的计算基数并未包含在国际精算符号规则里，但若干形式已在实践中广泛使用。

首先定义

$$D_x^{(\tau)}=v^x l_x^{(\tau)}. \tag{8.7.1}$$

它与第二、三章运用的函数 D_x 相似，上标 (τ) 表示它是通过多重损失表函数 $l_x^{(\tau)}$ 构造的。数值计算公式中经常出现象 $v^{k+1/2}{}_{k+1/2}p_x^{(\tau)}$ 这样的因子，现在可以写成

$$v^{k+1/2}{}_{k+1/2}p_x^{(\tau)}=\frac{\overline{D}_{x+k}^{(\tau)}}{D_x^{(\tau)}},$$

其中

$$\overline{D}_y^{(\tau)}=D_{y+1/2}^{(\tau)}. \tag{8.7.2}$$

另一些用来估价醵出的函数是

$$
\begin{aligned}
{}^S\overline{D}_y^{(\tau)} &= S_y\overline{D}_y^{(\tau)}, \qquad (8.7.3)\\
\overline{N}_x^{(\tau)} &= \sum_{y=x}^{\omega-1}\overline{D}_y^{(\tau)},
\end{aligned}
$$

$$ {}^{S}\overline{N}_x^{(\tau)} = \sum_{y=x}^{\omega-1} {}^{S}\overline{D}_y^{(\tau)}. \tag{8.7.4} $$

对于估价受益，基本的函数是

$$ \overline{C}_y^h = D_y^{(\tau)} v^{1/2} q_y^{(h)}. \tag{8.7.5} $$

与第二章类似，它含有一个 D 函数与损失概率 $q_x^{(h)}$ 。在其中的右上标，h 为 r 或 i 或 w 等，指明相应的损失。C 上面的横线则表示损失即时支付。另一些用来估价受益的函数是

$$ {}^{a}\overline{C}_y^h = \overline{C}_y^h \overline{a}_{y+1/2}^h, \tag{8.7.6} $$

$$ {}^{Sa}\overline{C}_y^h = S_y {}^{a}\overline{C}_y^h, \tag{8.7.7} $$

$$ {}^{Za}\overline{C}_y^h = {}_mZ_y {}^{a}\overline{C}_y^h. \tag{8.7.8} $$

相应的 $\overline{M}$ 与 $\overline{R}$ 函数也会被使用，例如

$$ {}^{Za}\overline{M}_x^h = \sum_{y=x}^{\omega-1} {}^{Za}\overline{C}_y^h, \tag{8.7.9} $$

$$ {}^{Za}R^h = \sum_{y=x}^{\omega-1} {}^{Za}\overline{M}_y^h. \tag{8.7.10} $$

如果在这些函数中，年金值并非终身生存年金，那么在上标 a 上加一撇。注意，当 $y < \alpha$ 时，${}^{Za}\overline{C}_y^r = 0, {}^{Za}\overline{M}_y^r = {}^{Za}\overline{M}_\alpha^r$, 其中 α 是最低退休年龄。

以下用退休金的计算基数表示相应于前面几节的精算现值公式。对于按年率 c 连续支付的未来醵出金精算现值，从 (8.3.2) 出发整理成

$$ c\sum_{k=0}^{\omega-x-h-1} \frac{v^{x+h+k+1/2} l_{x+h+k+1/2}^{(\tau)}}{v^{x+h} l_{x+h}^{(\tau)}} = c\frac{\overline{N}_{x+h}^{(\tau)}}{D_{x+h}^{(\tau)}}. \tag{8.7.11} $$

如果酬出金表示成薪水的一部分，那么从 (8.3.4) 出发整理成

$$
\begin{aligned}
& c(AS)_{x+h}\sum_{k=0}^{\omega-x-h-1}\frac{v^{x+h+k+1/2}l^{(\tau)}_{x+h+k+1/2}S_{x+h+k}}{v^{x+h}l^{(\tau)}_{x+h}S_{x+h}} \\
&= c(AS)_{x+h}\sum_{k=0}^{\omega-x-h-1}\frac{{}^S\overline{D}^{(\tau)}_{x+h+k}}{{}^S\overline{D}^{(\tau)}_{x+h}} = c\frac{(AS)_{x+h}{}^S\overline{N}^{(\tau)}_{x+h}}{{}^SD^{(\tau)}_{x+h}}.
\end{aligned}
\tag{8.7.12}
$$

由 (8.7.5) 及 (8.7.6)，表达式 $v^{k+1/2}{}_kp^{(\tau)}_{x+h}q^{(r)}_{x+h+k}\overline{a}^r_{x+h+k-1/2}$ 可写成 ${}^a\overline{C}^r_{x+h+k}/D^{(\tau)}_{x+h}$，于是适龄退休精算现值的一般公式 (8.4.2) 用计算基数表示为

$$
\sum_{k=\alpha_{T}x-h}^{\omega-x-h-1}\frac{R(x,h,k+1/2){}^a\overline{C}^r_{x+h+k}}{D^{(\tau)}_{x+h}}, \tag{8.7.13}
$$

其中 $x+h\leq\alpha$ 。特别是当 $R(x,h,k+1/2)$ 由 (8.4.5) 给出时，有

$$
\begin{aligned}
& g(AS)_{x+h}\sum_{k=\alpha-x-h}^{\omega-x-h-1}\frac{{}_mZ_{x+h+k}}{S_{x+h}}\frac{{}^a\overline{C}^r_{x+h+k}}{D^{(\tau)}_{x+h}} \\
&= g(AS)_{x+h}\sum_{k=\alpha-x-h}^{\omega-x-h-1}\frac{{}^{Za}\overline{C}^r_{x+h+k}}{{}^SD^{(\tau)}_{x+h}} \\
&= g(AS)_{x+h}\frac{{}^{Za}\overline{M}^r_{\alpha}}{{}^SD^{(\tau)}_{x+h}}.
\end{aligned}
\tag{8.7.14}
$$

由于当 $y<\alpha$ 时， $0=q^{(r)}_y=\overline{C}^r_y={}^{Za}\overline{C}^r_y$，以上精算现值也可表示成

$$
g(AS)_{x+h}\sum_{k=0}^{\omega-x-h-1}\frac{{}^{Za}\overline{C}^r_{x+h+k}}{{}^SD^{(\tau)}_{x+h}} = g(AS)_{x+h}\frac{{}^{Za}\overline{M}^r_{x+h}}{{}^SD^{(\tau)}_{x+h}}, \tag{8.7.15}
$$

它对 $x+h\le\alpha$ 与 $x+h>\alpha$ 都有效。这一重整公式的优越性还可在以下 $R(x,h,k+1/2)$ 依赖于服务年限的情形看到。

例如对于 (8.4.6), 其中

$$R(x,h,k+1/2)=f(h+k+\frac{1}{2})(AS)_{x+h}\frac{{}^{m}Z_{x+h+k}}{S_{x+h}}.$$

退休受益的精算现值从 $k=0$ 开始求和，为

$$\begin{aligned} & f(AS)_{x+h}\sum_{k=0}^{\omega-x-h-1}\frac{(h+k+1/2)^{Za}\overline{C}^{r}_{x+h+k}}{{}^{S}D^{(\tau)}_{x+h}} \\ & =f(AS)_{x+h}\frac{(h+1/2)^{Za}\overline{M}^{r}_{x+h}+{}^{Za}\,\overline{R}^{r}_{x-h+1}}{{}^{S}D^{(\tau)}_{x+h}} \end{aligned} \tag{8.7.16}$$

它对任何 $x+h$ 都成立，而当求和从 $k=\alpha-x-h$ 开始的话，相应公式为

$$f(AS)_{x+h}\frac{(\alpha-x+1/2)^{Za}\overline{M}_{\alpha}+{}^{Za}\,\overline{R}^{r}_{\alpha+1}}{{}^{S}D^{(\tau)}_{x+h}}\qquad x+h\le\alpha,$$

而 $x+h>\alpha$ 时还需用 (8.7.16) 补充。

如果象 (8.4.7) 那样只核算完整服务年数，那么精算现值为

$$f(AS)_{x+h}\frac{h^{Za}\overline{M}^{r}_{x+h}+{}^{Za}\,\overline{R}^{r}_{x+h+1}}{{}^{S}D^{(\tau)}_{x+h}}, \tag{8.7.17}$$

它可通过去掉 (8.7.16) 中不完整服务年的平均分数 1/2 得到。

在服务期间平均薪水情形的过去服务受益精算现值 (8.4.8) 可表示成

$$f(TPS)_{x+h}\frac{{}^{a}\overline{M}^{r}_{x+h}}{D^{(\tau)}_{x+h}}. \tag{8.7.18}$$

直接与未来服务受益精算现值 (8.4.10) 相当的公式为

$$f(AS)_{x+h}\sum_{k=a-x-h}^{\omega-x-h-1}\frac{\left(\sum\limits_{j=0}^{k-1}S_{x+h+j}+\frac{1}{2}S_{x+h+k}\right){}^{a}\overline{C}^{r}_{x+h+k}}{{}^{S}D^{(\tau)}_{x+h}}. \tag{8.7.19}$$

为用退休金计算基数得出相应于另一个公式 (8.4.12) 的表达式，将它写成

$$f(AS)_{x+h}\frac{\frac{1}{2}{}^{Sa}\overline{M}^{r}_{x+h}+\sum_{j=0}^{\omega-x-h-1}S_{x-h+j}{}^{a}\overline{M}^{r}_{x+h+j+1}}{{}^{S}D^{(\tau)}_{x+h}}.$$

定义

$${}^{S'a}\overline{M}^{r}_{y}=S_{y-1}{}^{a}\overline{M}^{r}_{y}, \tag{8.7.20}$$

这样以上精算现值可表示成

$$f(AS)_{x+h}\frac{(1/2)^{Sa}\overline{M}^{r}_{x+h}+{}^{S'a}\overline{R}^{r}_{x+h+1}}{{}^{S}D^{(\tau)}_{x+h}}. \tag{8.7.21}$$

以上涉及退休受益的步骤也适用于残疾或离职年金受益。于是，残疾受益的精算现值 (8.5.3) 可用计算基数写成

$$\sum_{k=5}^{64-x}\frac{R(x,0,k+1/2){}^{a}\overline{C}^{i}_{x+k}}{D^{(\tau)}_{x}}, \tag{8.7.22}$$

其中 ${}^{a}\overline{C}^{i}_{y}=v^{1/2}D^{(\tau)}_{y}q^{(i)}_{y}\overline{a}^{i}_{y+1/2}$。类似地，(8.6.1) 可用计算基数写成

$$f(AS)_{x+h}\sum_{k=l}^{59-x-h}\frac{(h+k+1/2)^{Sa'}\overline{C}^{\omega}_{x+h+k}}{{}^{S}D^{(\tau)}_{x+h}}, \tag{8.7.23}$$

其中 ${}^{Sa'}\overline{C}^{\omega}_{y}=S_{y}\overline{C}^{\omega}_{y\,60-y-1/2|}\overline{a}^{r}_{y+1/2}$。如果需要，也可用 ${}^{Sa'}\overline{M}^{\omega}_{y}$ 与 ${}^{Sa'}\overline{R}^{\omega}_{y}$ 表示这个精算现值。

引入符号 ${}^j\overline{C}_y^\omega=(1+j)^y\overline{C}_y^\omega$, (8.6 .3) 可改写成

$$0.01c(AS)_{x+h}\frac{\frac{1}{2}{}^j\overline{C}_{x+h}^\omega+\sum\limits_{k=1}^{\beta-x-h-1}{}^j\overline{C}_{x+h+k}^\omega}{(1+j)^{x+h}D_{x+h}^{(\tau)}}$$

$$=0.01c(AS)_{x+h}\frac{\frac{1}{2}{}^j\overline{C}_{x+h}^\omega+{}^j\overline{M}_{x+h+1}^\omega-{}^j\overline{M}_\beta^\omega}{(1+j)^{x+h}D_{x+h}^{(\tau)}}.\quad(8.7.24)$$

有关计算基数方法的更进一步说明，可在习题 13–19 中找到。

习　题

§8.2 及 §8.3

1. 假定对于 30 岁刚进入的新参加者，考虑到通货膨胀与生产力因素，薪水的年增长率将是 5%, 而且假定在 40 、 50 、 60 岁时因提升薪水增加 10% 。

(1) 用薪水尺度函数 S_{30+k} 来表示这些假定。

(2) 设 30 岁时年薪为 12000, 酿出金为薪水的 10%, 写出未来酿出的精算现值表达式。

2. 计划发起人每年贡献出参加者薪水中超过某个数额部分的 10%, 那个数额今年为 10000 并按年率 5% 递增。对于现在进入的年薪为 25000 的 35 岁参加者，给出发起人酿出的精算现值表达式。

§8.4

3. 在例 8.4.1 中，假定所有退休发生在 63 岁 (与示例服务表不同), 相应的精算现值简化成什么？

4. 某 25 岁参加者的当前年薪为 12000, 一种分级计划在年度 $k+1$ 退休时提供的收入 = 完整服务年数 ×(最后 3 年平均薪水不超过 15000×1.04^k 部分的 0 .01 倍 + 最后 3 年平均薪水超过 15000×1.04^k 部分的 0.015 倍), 写出受益收入 (率) 函数。

5. 某抵消计划的抵消额为服务年限乘以社会保险收入的 2%，但不超过社会保险收入的 50%。抵消之前该计划的受益收入 (率) 等于服务年限乘以最后 3 年平均薪水的 2%，对于年薪 30000 的 40 岁新进入者，分别给出以下情形的抵消后受益收入 (率) 公式：

(1) 正好在 65 岁退休，届时社会保险收入估计为 I_{65}。

(2) 在 68 到 69 岁退休，社会保险收入估计为 $I_{68\frac{1}{2}}$。

6. 某种计划到 65 岁为止，为每一年服务提供的受益收入 (年率) 是最后 3 年平均薪水的 2%，之后为每一年服务提供的受益收入是最后 3 年平均薪水的 $1\frac{1}{3}\%$。对于 30 岁进入、现龄 50 岁且年薪为 36000 的参加者，给出最早退休年龄为 55 岁、到 68 岁全部退休情况下受益的精算现值表达式。

7. 在习题 6 中，假定核算受益的服务年限最多为 35 年，那么受益的精算现值公式如何？

8. 写出与习题 6 中年龄在 30–50 岁服务期间有关的受益收入的精算现值。

9. 某种服务期间平均计划提供的退休收入为参加者服务期间薪水总额的 2%。最早退休年龄是 58 岁，到 68 岁时必须全部退休。对于 30 岁进入、现龄 50 岁、过去薪水总计为 400000 且当前年薪为 36000 的参加者，写出以下表达式：

(1) 正好在 65 岁退休情形的总受益收入 (年率)。

(2) 在 65 与 66 岁之间退休的年中总受益收入 (年率)。

(3) 相应于过去服务的退休受益精算现值。

(4) 相应于未来服务的退休受益精算现值。

§8.5

10. 对于例 8.5.1 的残疾受益，如果现龄 50 岁、具有 20 年服务期、年薪为 25000 的参加者在当年年中致残并且 $I_{50} = 8000$，那么其残疾受益的精算现值是多少？

§8.6

11. 某 35 岁参加者的过去醵出金累积总额为 5000，在达到

40 岁后有资格获得延期年金。假定酿出金按年利率 6% 累积，写出该参加者在 40 岁之前离职情况下过去酿出累积总额退款的精算现值表式。

12.

(1) 在 (8.6.3) 中，当前年离职项

$$0.01c(AS)_{x+h}\frac{1}{2}v^{1/2}q_{x+h}^{(\omega)}$$

改为用二重积分表示，其中一个变量为取得薪水增量的时间，另一个变量为离职时间。

(2) 在多重损失模型的离职平均分布假设下，求 (1) 中的积分。

(3) 按 $i=0.06$ 与 $j=0.04$ 求 (8.6.3) 中给出的相应项以及 (2) 中的积分，并比较两种结果。

§8.7

13. 用计算基数表示以下所列习题中的精算现值公式：

(1) 习题 1(2)。	(2) 习题 2。	(3) 习题 3。
(4) 习题 7。	(5) 习题 9(3)。	(6) 习题 9(4)。

14. 对于在 x 岁 $(x<\alpha)$ 进入退休金计划、年薪为 12000 的参加者，用退休金计算基数表示以下精算现值：

(1) 对每个完整服务年按最后 3 年平均薪水的 1% 核算的退休年金。

(2) 对每一服务年 (包括零数) 按最后 3 年平均薪水的 1% 核算的退休年金。

15. 在习题 14(2) 中，如果附加有至少服务 10 年的条件，那么精算现值公式变成什么样？

16. 设未来薪水的定额百分比 c 的酿出与习题 14(b) 中的退休受益等价，导出用退休金计算基数表示的 c 的公式。

17. 用退休金计算基数表示例 8.3.1 中给出的各公式。

18. 用退休金计算基数表示例 8.3.2 中给出的公式。

19. 用退休金计算基数表示以下所列例子中给出的公式：

(1) 例 8.4.5 (2) 例 8.4.7.

综合题

20. 对于例 8.4.8 中年薪为 30000 的 62 岁参加者，求相应于从年龄 62 至 63 这一年的受益的精算现值。

21. 设薪水尺度递增，对于年薪为 20000 的 25 岁新参加者：下列两个精算现值中哪个更大？

(A) $400\dfrac{(1/2)^{Za}\overline{M}_{25}^{r}+{}^{Za}\overline{R}_{26}^{r}}{{}^{S}D_{25}^{(\tau)}}$，这里 ${}_{m}Z_{y}$ 中的 $m=5$ 。

(B) $400\dfrac{(1/2)^{Sa}\overline{M}_{25}^{r}+{}^{S'a}\overline{R}_{26}^{r}}{{}^{S}D_{25}^{(\tau)}}$.

第九章 包括费用的保险模型

§9.1 引 言

第四章引入的平衡原理是用来决定保险费的，根据这个原理，受益与净保费的精算现值在保单生效时是相等的。在第五章中，该原理被用于在保单生效以后的时期决定责任准备金，责任准备金等于未来受益支付与未来保费收入的精算现值之差。

然而，以前各章所建立的模型并未容纳保险实践与经济现实中的一些因素。譬如，保险公司的支出不仅仅限于理赔支付，还包括税金、许可费以及保单销售服务等开销，这些费用必须由保费及投资收益弥补。这一章讨论的保险费与责任准备金模型就考虑了各种费用。

§9.2 一般费用

首先通过示例来说明考虑费用的主要想法。表 9.2.1A 与 9.2.1B 详细说明了涉及到的主要方面，其中项目的选择并不完全与实际对应，只是为计算与说明方便而设。

表 9.2.1A 保险描述

1. 保险计划：	向 (x) 发行年缴一次保费的 3 年期两全保险
2. 支付基础：	完全离散
3. 死亡率：	$q_x = 0.1, q_{x+1} = 0.1111, q_{x+2} = 0.5$
4. 利率：	年利率 $i = 0.15$
5. 保险金额：	1000
6. 费用：	
(1) 发生时间：	每个保单年度初支付
(2) 金额：	(由表 9.2.1B 给出)

表 9.2.1B　费用明细表

费用类别	时间			
	第一年		续年	
	保费的百分比	固定量	保费的百分比	固定量
推销佣金	10%	–	2%	–
营业费用	4%	3	–	1
税金、许可证等费用	2%	–	2%	–
保单维持	2%	1	2%	1
发行与等级分类	2%	4	–	–
总计	20%	8	6%	2

一、保费与责任准备金

表 9.2.1A 中描述性说明项 1 至 5 可用来根据平衡原理决定该保险的净年缴保费：$1000P_{x:\overline{3}|} = 288.41$。表 9.2.2 提供了净保费责任准备金的计算细节。

表 9.2.2　净保费责任准备金计算

(1) 结果	(2) 亏损	(3) 概率	(4) (2) × (3)
		在发行时 ($_0L$)	
$k=0$	581.16	0.1	58.12
$k=1$	216.94	0.1	21.69
$k\geq 2$	-99.76	0.8	-79.81
			$1000_0V_{x:\overline{3}|} = E[_0L] = 0.00$
			$\sigma(_0L) = 215.51$
		发行后 1 年 ($_1L$)	
$k=0$	581.16	0.1111	64.57
$k\geq 1$	216.94	0.8889	192.84
			$1000_1V_{x:\overline{3}|} = E[_1L] = 257.41$
			$\sigma(_1L) = 114.46$
		发行后 2 年 ($_2L$)	
$k\geq 0$	581.16	1	581.16
			$1000_2V_{x:\overline{3}|} = E[_2L] = 581.16$
			$\sigma(_2L) = 0$

最后可验证 $_3V_{x:\overline{3}|} = 1$：

$$1000(_2V_{x:\overline{3}|} + P_{x:\overline{3}|})(1+i) = 1000$$
$$(581.16 + 288.41) \times 1.15 = 1000$$

表 9.2.1B 记载的费用将被纳入修正的亏损变量，其中受益现值还须加上费用的现值，这一新的现值总额应与包括费用的保险费现值相抵消。表 9.2.3 根据表 9.2.1B 提供的信息建立，其中附加费用的年保费记为 G 。

表 9.2.3　增列费用的亏损变量 $({}_0L_e)$

结果	受益 + 费用	- 保费	概率
$k=0$	$1000v+(0.20G+8)$	$-G\ddot{a}_{\overline{1}\rvert}$	0.1
$k=1$	$1000v^2+(0.20G+8)+(0.06G+2)a_{\overline{1}\rvert}$	$-G\ddot{a}_{\overline{2}\rvert}$	0.1
$k\ge 2$	$1000v^3+(0.20G+8)+(0.06G+2)a_{\overline{2}\rvert}$	$-G\ddot{a}_{\overline{3}\rvert}$	0.8

附加费用的保险费也根据平衡原理决定，即由增列费用的亏损变量的期望值为 0, 得出附加费用的年保费

$$G=1000P_{x:\overline{3}\rvert}+\text{附加保费}(e)=288.41+43.94=332.35.$$

表 9.2.4 展示了有关亏损变量的期望值与标准差的计算过程，这些值在保单发行时以及之后 1 年、2 年时分别进行计算。总的责任准备金可分解成受益与费用两部分，在每一年，期望均衡净保费收入并不与期望受益支付相匹配，这导致非负的受益责任准备金。类似地，每一年期望均衡附加保费收入也并不与期望费用开支相匹配，这导致非正的费用责任准备金。

以上说明中涉及的一些关键点归结如下：

1. 原先引入的亏损变量在各不同时间衡量受益现值与净保费现值之差，这些变量可增加为包含费用与附加保费的亏损变量。

2. 平衡原理可用于决定附加费用的保险费及相应的总责任准备金 (受益责任准备金加上费用责任准备金) 。

3. 在早期保单年度，费用责任准备金通常是负值，其原因在于，费用开支序列递减，而附加保费则是均衡序列。

4. 对费用的分析测算在决定附加费用的保险费之前进行。

5. 增列费用的亏损变量之标准差可用于决定意外损失应急基金，该基金主要用来应付保费及投资收入与受益及费用支出可

表 9.2.4 增列费用的亏损变量期望值

结果	(受益现值 $-1000P_{x:\overline{3}\mid}\ddot{a}_{\overline{k+1}\mid}$)+(费用现值 $-e\ddot{a}_{\overline{k+1}\mid}$)	概率
	在发行时 $({}_0L_e)$	
$k=0$	$(869.57-288.41)+(74.47-43.94)$	0.1
$k=1$	$(756.14-539.20)+(93.55-82.15)$	0.1
$k\geq 2$	$(657.52-757.28)+(110.14-115.37)$	0.8
期望值：	受益准备金 + 费用责任准备金 = 责任准备金总额	
	0+0=0	
	$\sigma({}_0L_e)=226.82$	
	发行后 1 年 $({}_1L_e)$	
$k=0$	$(869.57-288.41)+(21.94-43.94)$	0.1111
$k\geq 1$	$(756.14-539.20)+(41.02-82.15)$	0.8888
期望值	受益责任准备金 + 费用责任准备金 = 责任准备金总额	
	$257.41-39.00=218.41$	
	$\sigma({}_1L_e)=120.47$	
	发行后 2 年 $({}_2L_e)$	
$k\geq 0$	$(869.57-288.41)+(21.94-43.94)$	1.0
期望值：	受益责任准备金 + 费用责任准备金 = 责任准备金总额	
	$581.16-22.00=559.16$	
	$\sigma({}_2L_e)=0$	

作为验证，最终的责任准备金总额 (3 年末) 为：

$$[E({}_2L_e)+G-(0.06G+2)](1+i)=1000$$

$$[559.16+332.35-21.94]\times 1.15=1000$$

能出现的不相配情况，这种情况发生的可能性源于受益赔付时间的随机性。

二. 会计事项

对于生产企业，产品通常在出售之前制作。对于大多数提供服务的商号，服务通常在收费之前履行。在这方面，保险经营不同寻常，承担风险的服务在收取保险费之后才履行。

会计报表的部分目标是将提供产品或服务的成本与出售所得的收入相配合，并据此衡量赢亏。人寿保险与年金经营机构的财务会计与很多企业不同，收入一般发生在成本支出之前。前面说明的责任准备金制、净均衡保费及附加费用的保费可用来改善保

费收入与相应支出的配合。

表 9.2.5 与 9.2.6 对前面的示例继续进行说明，其中假定，每份保单的年缴保费是附加费用的保险费 332.35 再加上为利润及应急所收费 10，合计 342.35 。这些会计报表按初始 10 个被保险人的决定性生存组导出，每一项会计栏目除以 10 后可解释成初始每个被保险人的期望数额。费用开支与投资收益完全按表 9.2.1A 与 9.2.1B 开列，并假设初始基金为 1000 。表中列 (1) 只以受益 (即净均衡保费) 责任准备金作为负债，列 (2) 则将受益与费用责任准备金合起来作为负债。

表 9.2.5　损益表 (初始 10 个被保险人)

(1) 净均衡保费责任准备金作为负债		(2) 受益与费用责任准备金作为负债
	第一个年度	
	收入	
3423.50	保费 (10 份)	3423.50
548.82	投资收益 (15%)	548.82
3972.32		3972.32
	支出	
	费用	
648.70	百分比 (20%)	648.70
80.00	固定量 (每份 8)	80.00
1000.00	理赔 (1 人)	1000.00
2316.69	责任准备金增加额	1965.69
4081.39		3730.39
−109.07	净收益	241.93
	第二个年度	
	收入	
3081.15	保费 (9 份)	3081.15
912.88	投资收益 (15%)	912.88
3974.03		3974.03
	支出	
	费用	
184.87	百分比 (6%)	184.87
18.00	固定费 (每份 2)	18.00
1000.00	理赔 (1 人)	10000.00

表 9.2.5(续)　损益表 (初始 10 个被保险人)

(1) 净均衡保费责任准备金作为负债		(2) 受益与费用责任准备金作为负债
2332.59	责任准备金增加额	2507.59
3535.46		3710.46
458.57	净收益	283.57
	第三个年度	
	收入	
2738.80	保费 (8 份)	2738.80
1283.59	投资收益 (15%)	1283.59
4022.39		4022.39
	支出	
	费用	
164.33	百分比 6%	164.33
16.00	固定费 (每份 2)	16.00
8000.00	理赔及到期 (8 人)	8000.00
−4649.28	责任准备增加额	−4473.28
3531.05		3707.05
491.34	净收益	315.34

注 1. 投资收益 =(上一年度末资产额 + 保费收入 − 费用支出)×0.15 。

注 2. 净收益总额 $= -109.07+458.57+491.34=241.93+283.57+315.34=840.84$ 。

注 3. 另一种计算方法：净收益总计 = 初始基金的利息收益 + 净利润附加保费的积累值 $=1000\times(1.15^3-1)+100\times0.80\times1.15^3+90\times0.94\times1.15^2+80\times0.94\times1.15=840.91$ 两者差异系舍入误差引起。

以上会计说明中涉及的一些关键点归结如下 (接着前面):

6. 当受益与费用责任准备金作为负债报告时，损益表中确认为净收益的数额比只用净保费责任准备金作为负债报告时的变动要小些。另外，净收益可与盈余的利息以及净利润附加费按利息的积累值相联系。

7. 3 年期的净收益总额并不受所选择的确认负债的方法影响。

表 9.2.6　资产负债表（初始 10 个被保险人）

(1) 净均衡保费责任准备金作为负债		(2) 受益与费用责任准备金作为负债
	第一年度	
3207.62	资产	3207.62
2316.69	负债（责任准备金）	1965.69
890.93	盈余	1241.93
3207.62		3207.62
	第二年末	
5998.78	资产	5998.78
4649.28	负债（责任准备金）	4473.28
1349.50	盈余	1525.50
5998.78		5998.78
	第三年末	
1840.84	资产	1840.84
0	负债（责任准备金）	0
1840.84	盈余	1840.84
1840.84		1840.84

注 1: 盈余增加额 = 净收益总额：$1840.84 - 1000 = 840.84$。

注 2: 盈余额 = 上一年度末盈余 + 净收益。

注 3: 资产 = 上一年度末资产 +（净收益 + 责任准备金增加额）

= 上一年度末资产 +（保费 + 投资收益 − 理赔 − 费用）。

8. 在实际运用时，期望结果并不象上例那样确切地实现。

§9.3　费用类型

保险企业的会计制度目的在于记录、分类与概括财务变动，同时还提供经营活动的数据：如销售量与销售额、理赔数、保费单据数等。在收集了这些信息后，可对经营活动的各主要费用项目进行分析，分析结果将有助于对未来出售的保险单决定附加保费。如果运用平衡原理，那么费用附加保费的精算现值等于保单所对应的费用支出的精算现值。

保险机构的费用归类与分配是一项令人头昏的工作。表 9.2.1 给出一个例子。

表 9.3.1　保险机构费用开支的一种分类方案

费用分类	成份
投资	(1) 分析
	(2) 购买、销售及服务成本
保险	
1. 新契约费	(1) 销售费用，包括代理人佣金及广告费
	(2) 风险分类，包括体检费用
	(3) 准备新保单及记录
2. 维持费	(1) 保费收取及会计
	(2) 受益变更及支付选择权准备
	(3) 与保单持有者联络
3. 营业费用	(1) 研究
	(2) 精算与一般法律服务
	(3) 普通会计
	(4) 税金、许可证等费用
4. 支付费用	(1) 理赔调查及辩护费
	(2) 受益支付费用

在决定附加费用的保险费时，一般只考虑保险费用，而将投资费用冲销投资收入，体现在保费中则适当降低设定的利率。

在某些例子中，费用项与经营活动有固定的关系，譬如，通常销售代理人的佣金按保费的百分比提取。§9.2 的示例中第 1 年佣金为保费的 10%，第 2 、第 3 年为 2% 。保险机构的税金一般也按保费的百分比交纳。

其它费用项目的分配就不那么显而易见了。这里，统计分析与直觉判断的混合常常被运用。在附加保费公式中，通常将新契约费分配于第一个保单年度。某些新契约费与保费数额相关，如佣金；另一些则与保险金额相关，如风险分类费用；而有些费用与保单或保费的额度无关，如档案记录的建立。

对费用的分类与分配是控制保险体系营运的重要管理手段。然而，在决定保费时，对费用的核算具有前瞻性质，目标是使未

来费用开支与未来附加保费相匹配，因此，按通货膨胀或收缩预期的费用开支趋势应考虑进去。

表 9.3.2 提供了按表 9.3.1 分类的保险费用与相应的附加因子的一个示例。

表 9.3.2　未来保险费用的分配示例

分类	第 1 年		
	每份保单	每 1000 保额	保费的百分比
1. 新契约费			
(1) 销售费用			
佣金	–	–	60%
销售事务	–	–	25%
其它	12.50	4.00	–
(2) 分类	18.00	0.50	–
(3) 发行与记录	4.00	–	–
2. 维持费	2.00	0.25	–
3. 营业费用			
(1), (2), (3)	4.00	0.25	–
(4) 税金	–	–	2%
小计 (1,2,3)	40.50	5.00	87%
4. 给付费用	每份保单 18.00 加上每 1000 保额 0.10		

分类	续年				
	每份保单	每 1000 保额	保费的百分比 (按保单年度)		
			2–9	10–15	16 以上
1. 新契约费					
(1) 销售费用					
佣金	–	–	7.0%	5.0%	3%
销售事务	–	–	2.5%	1.5%	1%
其它	–	–	–	–	–
(2) 分类	–	–	–	–	–
(3) 发行与记录	–	–	–	–	–
2. 维持费	2.00	0.25	–	–	–
3. 营业费用					
(1), (2), (3)	4.00	0.25	–	–	–
(4) 税金	–	–	2.0%	2.0%	2%
小计 (1,2,3)	6.00	0.50	11.5%	8.5%	6%
4. 给付费用	每份保单 18.00 加上每 1000 保额 0.10				

例 9.3.1: 对于按半连续基础向 x 发行的保险金额为 20000 的终身寿险保单，按表 9 .3.2 所列费用，根据平衡原理建立附加费用的年缴保费公式。

解: 设 G 是附加费用的保费，则由附加费用的保费精算现值 = 理赔、理赔费用及其它费用的精算现值，

$$\begin{aligned} G\ddot{a}_{[x]} &= 20020\overline{A}_{[x]} + [(140.50 + 0.87G) + 16a_{[x]} \\ &\quad +(0.115a_{[x]:\overline{8}|} + 0.085_{9|6}\ddot{a}_{[x]} + 0.06_{15|}\ddot{a}_{[x]})G], \end{aligned}$$

解得

$$\begin{aligned} G &= (20020\overline{A}_{[x]} + 140.50 + 16a_{[x]})/(\ddot{a}_{[x]} - 0.87 - 0.115 \\ &\quad (\ddot{a}_{[x]:\overline{9}|} - 1) - 0.085(\ddot{a}_{[x]:\overline{15}|} - \ddot{a}_{[x]:\overline{9}|}) - 0.06(\ddot{a}_{[x]} - \ddot{a}_{[x]:\overline{15}|})) \\ &= \frac{20020\overline{A}_{[x]} + 140.50 + 16a_{[x]}}{0.94\ddot{a}_{[x]} - 0.755 - 0.03\ddot{a}_{[x]:\overline{9}|} - 0.025\ddot{a}_{[x]:\overline{15}|}} \end{aligned}$$

在上例中，得出的是保险金额为 20000 保单的附加费用的保费。在实践中，保费通常以单位保险的费率表示。对于人寿保险，费率一般是 1000 元初始受益的保费；对于生存年金，费率一般是按每月一元收入计算的。当保费以费率形式表示时，产生了对不同金额保单按单位保单分配的费用问题，这个问题将在下一节讨论。

§9.4 单位保单的费用

设 $G(b)$ 是金额为 b 保单的附加费用的保费，并且

$$G(b)(1-f) = ab + c, \tag{9.4.1}$$

其中 a, c, f 是非负常数， $f < 1$ 。常数 a 表达了保险成本中直接与保险金额大小相关的部分，其中单位保险的净保费是最重要

的。常数 c 可解释成每份保单的费用，而常数 f 则代表与保费数额相关的费用在保费中的百分比。

公式 (9.4.1) 可写成

$$
\begin{aligned}
G(b) &= b\frac{a+c/b}{1-f} \\
&= bR(b),
\end{aligned}
\tag{9.4.2}
$$

其中

$$R(b)=\frac{a+c/b}{1-f}.$$

函数 $R(b)$ 是金额为 b 的保单的保险费费率，其表达式可简化为

$$R(b)=a'+\frac{c'}{b},$$

其中 $a'=a/(1-f), c'=c/(1-f)$ 。不过使用含 3 个常数的前一表达式通常较方便。

典型的费率函数 $R(b)$ 如图 9.4.1 所示，其中 m 是最低保单金额，$\bar{b}$ 是平均保单金额。图中给出了 3 种确定单位保单费用的 3 种方法：

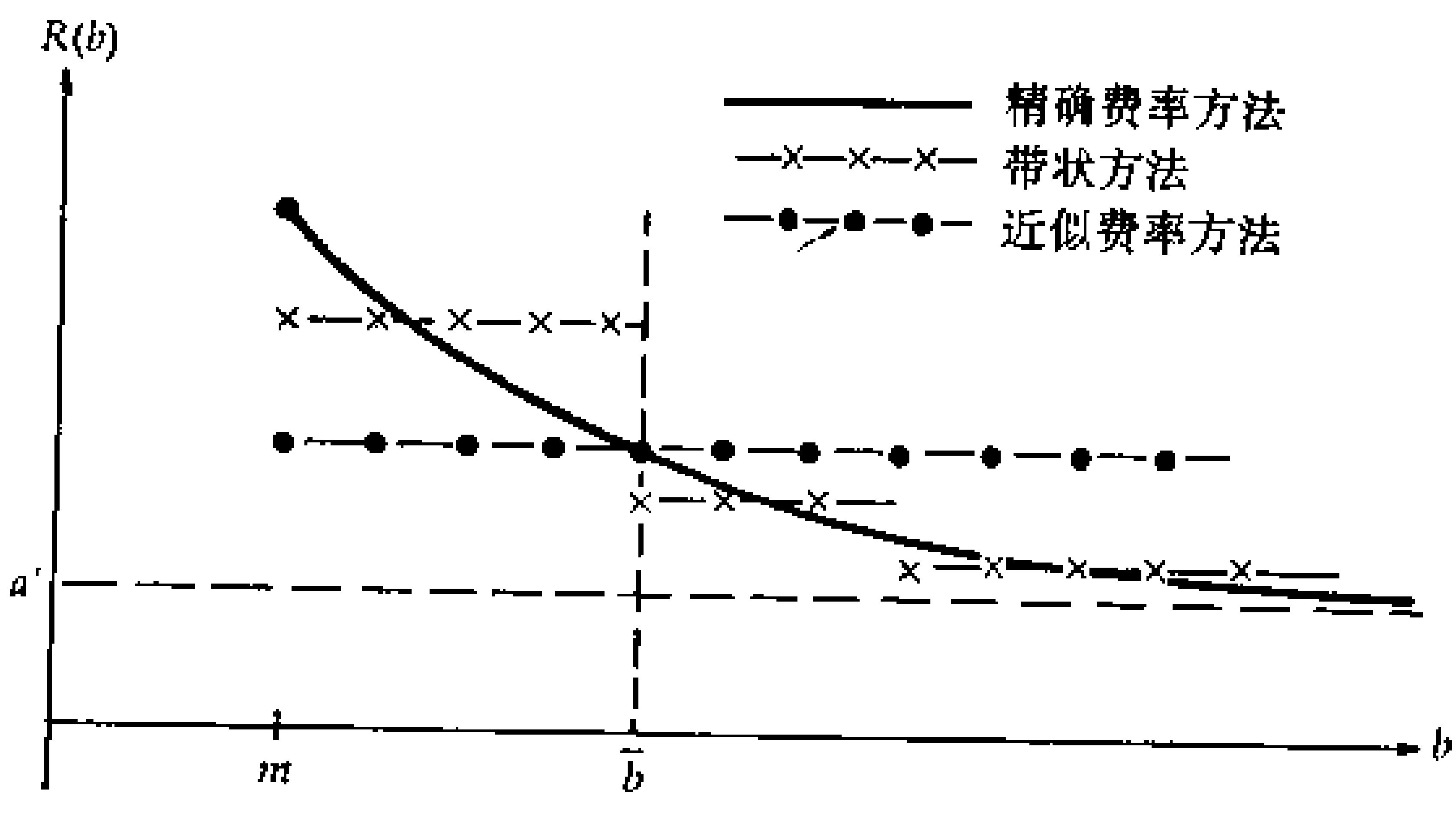


图 9.4.1　费率作为保险金额的函数

1. 发单费(policy fee)$c/(1-f)=c'$ 加上量 $b[a/(1-f)]=ba'$，得出附加费用的保费 (9.4 .2) 。按这个方法，实际费率就是 $R(b)$ 。

2. 用分段常数函数近似 $R(b)$，这个方法称为带状方法(band method) 。

3. 用常数 $R(\bar{b})$ 近似费率函数 $R(b)$ 。

第 2 种方法带状方法亦称为高额保险折扣优待方法(quantity discount approach)，因增加保险金额可明显降低费率而得名。按方法 3，当保单金额 $b<\bar{b}$ 时，附加保费的精算现值小于费用的精算现值，如果将保险金额看作随机变量，记为 B，那么保单的期望保费为

$$E[BR(\bar{b})]=\bar{b}R(\bar{b})=\frac{a\bar{b}+c}{1-f}.$$

这就是说，附加保费与费用的精算现值处于平衡。

例 9.4.1: 某种趸缴保费的寿险保单的保费构成如下：

净保费	$A_x=0.20$
费用	
推销佣金	保费的 7.5%
税金、许可证等费用	保费的 3.0%
单位保单费用：	
第一年	23.00
续年	2.50
每份保单理赔费用	12.00

计算 $R(b)$，这里保险金额以 10000 为单位， $i=0.06$ 。

解: 将有关数值代入 (9.4.1) 得

$$(1-0.105)G(b)=(1000b+12)\times 0.20+23+2.50a_x,$$

利用死亡均匀分布假设，

$$a_x=\frac{1-A_x}{d}-1=\frac{1-(\delta/i)\overline{A}_x}{d}-1=13.2353,$$

从而

$$0.895G(b) = 200b + 2.4 + 23 + 2.50 \times 13.2353,$$

$$G(b) = 223.46b + 65.35,$$

$$R(b) = 223.46 + \frac{65.35}{b}.$$

例 9.4.2: 某种按半连续基础发行的终身保单，费用分配如下：

	保费的百分比	每 1000 元保险金	单位保单
第一年	30%	3.00	10.00
续年	5%	0.50	2.50

(1) 假定单位保单费用按第 1 年与续年分别与保费匹配，写出附加费用的第 1 年保费与续年保费表达式。

(2) 写出每年须缴付的单一保费。

解: 设 $k(b)$ 是排除单位保单费用的保费，其中 b 的单位是千元。于是

$$\begin{aligned} k(b)\ddot{a}_x &= b[1000\overline{A}_x + 3 + 0.50a_x] + 0.30k(b) + 0.05k(b)a_x, \\ k(b) &= b\frac{1000\overline{A}_x + 0.5\ddot{a}_x + 2.50}{0.95\ddot{a}_x - 0.25} = ba'. \end{aligned}$$

当发单费用加进保费中去时，按保费百分比核算的费用也适用于单位保单费用，这样，第 1 年应缴保费为

$$\begin{aligned} k(b) &+ \frac{10}{0.70} = k(b) + 14.29 \\ &= ba' + 14.29 = b(a' + \frac{14.29}{b}), \end{aligned}$$

续年保费为

$$\begin{aligned} k(b) &+ \frac{2.50}{0.95} = k(b) + 2.63 \\ &= b(a' + \frac{2.63}{b}). \end{aligned}$$

(2) 设 g 是对应于单位保单费用的附加费，则

$$\begin{aligned} g\ddot{a}_x &= 10+2.5a_x+0.3g+0.05ga_x, \\ g &= \frac{2.5\ddot{a}_x+7.5}{0.95\ddot{a}_x-0.25}, \end{aligned}$$

所求保费为

$$k(b)+g=b(a'+\frac{g}{b}).$$

§9.5　会计计算基础

这一节要将 §9.2 中的许多论点细致化，以下将不断引用表 9.2.5 与 9.2.6 。

财务会计的一个目的，就是在会计周期内决定资产负债方程

$$A(h)=L(h)+U(h). \tag{9.5.1}$$

的变元。在 (9.5.1) 中， $A(h)$ 是资产额， $L(h)$ 是负债额， $U(h)$ 是所有人权益额 (在保险会计中称为盈余), h 是会计周期末的时间。盈余的改变可表示成

$$\begin{aligned} \triangle U(h) &= \triangle A(h)-\triangle L(h) \\ &= \text{在周期 h+1 的净收益}. \end{aligned} \tag{9.5.2}$$

下面按表 9.5.1 所述的条件说明这个基本模型。

表 9.5.1

1. 保险计划：	终身，单位金额
2. 支付基础：	完全离散
3. 年龄与发行时间：	在第一个会计年度初向 (x) 发行
4. 费用：	无费用及附加保费
5. 经验：	投资经验与以前设定的相同
	会计项目将以每个初始被保险人在保单发行时的期望值表示。

我们从责任准备金递归公式 (5.7.2) 开始，在那个公式中置 $b_h=1$, $\pi_{h-1}=P_x$, ${}_hV={}_hV_x$, 并乘以 ${}_{h-1}p_x(1+i)$, 有

$$
{}_{h-1}p_x({}_{h-1}V_x+P_x)(1+i)-{}_{h-1}p_x q_{x+h-1}={}_h p_x {}_hV_x \qquad h=1,2,\cdots. \tag{9.5.3}
$$

相应地在第 1 个会计年度末，按每个初始被保险人，期望资产负债方程为

$$
A(1)=L(1)
$$

或

$$
P_x(1+i)-q_x=p_x {}_1V_x. \tag{9.5.4}
$$

这是因为，在第一个会计年度里，期望资产改变如下：

增加量	保费收入 $=P_x$
	利息收入 $=P_x i$
减少量	死亡赔付 $=q_x$

$$
\begin{aligned}
A(1) &= A(0)+A(1)-A(0)\\
&= 0+P_x(1+i)-q_x\\
&= p_x {}_1V_x=L(1).
\end{aligned}
$$

等式 (9.5.4) 是 (9.5.3) 中 $h=1$ 的情形。由此可见，$A(1)-L(1)=U(1)=0$ 。

公式 (9.5.3) 也可用来得出以递归形式表示的相继年度的会计陈述。设会计年度 h 末有

$$
A(h)=L(h)={}_h p_x {}_hV_x.
$$

根据

$$
\begin{aligned}
\triangle A(h) &= \text{保费收入}+\text{利息收入}-\text{死亡赔付}\\
&= {}_hp_x P_x+{}_h p_x({}_hV_x+P_x)i-{}_h p_x q_{x+h},
\end{aligned}
$$

可得

$$
\begin{aligned}
A(h+1) &= A(h)+\triangle A(h)\\
&= {}_hp_{x\,h}V_x+\{{}_hp_x[P_x+({}_hV_x+P_x)i]-{}_h\,p_xq_{x+h}\}\\
&= {}_hp_x({}_hV_x+P_x)(1+i)-{}_h\,p_xq_{x+h}\\
&= {}_{h+1}p_{x\,h+1}V_x=L(h+1).
\end{aligned}
$$

在以上说明中，初始基金、利润或风险附加费都假定为 0, 按期望结果追踪得出 $A(h)-L(h)=U(h)=0,\qquad h=0,1,2,\cdots$ 。

现在将表 9.5.1 中的假定作一些修改，设净保费基础之上还加上正的附加费 c, 并设会计年度 h 之初 (时间为 $h-1$) 支付的费用每份残存保单为 e_{h-1} 。附加费常数中可含有利润成份，这样，附加保费 c 的精算现值可能大于系列费用 $e_{h-1},h=1,2,\cdots$ 的精算现值。

式 (9.5.3) 容纳附加保费与费用的增广形式为

$$
\begin{aligned}
&{}_{h-1}p_x\{[{}_{h-1}V_x+u(h-1)]+(P_x+c)-e_{h-1}\}(1+i)\\
-\ &{}_{h-1}p_xq_{x+h-1}={}_h\,p_x[{}_hV_x+u(h)]\qquad h=1,2,3,\cdots
\end{aligned}
\tag{9.5.5}
$$

其中， $u(h)$ 表示会计年度 h 之末每份残存保单的目标盈余。

从 (9.5.5) 中减去不含附加费的 (9.5.3), 得

$$
{}_{h-1}p_x[u(h-1)+(c-e_{h-1})](1+i)={}_h\,p_xu(h)\qquad h=1,2,\cdots. \tag{9.5.6}
$$

两端乘 v^h, 可整理成差分形式

$$
\triangle[v^{h-1}{}_{h-1}p_xu(h-1)]=v^{h-1}{}_{h-1}p_x(c-e_{h-1}). \tag{9.5.7}
$$

加上初始条件 $u(0)=0$ 可得出

$$
\sum_{j=1}^{h}\triangle[v^{j-1}{}_{j-1}p_xu(j-1)]=\sum_{j=1}^{h}v^{j-1}{}_{j-1}p_x(c-e_{j-1}),
$$

$$ {}_hp_xu(h)=\sum_{j=1}^{h}(1+i)^{h-j+1}{}_{j-1}p_x(c-e_{j-1}). \tag{9.5.8} $$

这表明，在会计年度 h 末的期望盈余是以前各年度对盈余贡献的积累值。这个结果可与表 9.2.5 附注 3 比较。

如果净保费责任准备金 (受益责任准备金) 作为负债，那么在以上理想化的保险体系中，会计年度 h 末以期望值作为各项目的会计报表如下：

<u>资产负债表</u>

(会计年度 h 末)

$$
\begin{aligned}
A(h) &= L(h)-U(h)\\
&={}_h p_x{}_hV_x+{}_h p_xu(h)\\
&={}_h p_x{}_hV_x+\sum_1^h(1+i)^{h-j+1}{}_{j-1}p_x(c-e_{j-1})
\end{aligned}
$$

<u>损益表</u>

(第 h 个会计年)

收入：

保费收入 ${}_{h-1}p_x(P_x+c)$

投资收益 ${}_{h-1}p_x[{}_{h-1}V_x+u(h-1)+P_x+c-e_{h-1}]i$

小计 ${}_{h-1}p_x[(P_x+c)(1+i)+({}_{h-1}V_x+u(h-1)-e_{h-1})i]$

死亡赔付 ${}_{h-1}p_xq_{x+h-1}$

费用 ${}_{h-1}p_xe_{h-1}$

责任准备金变化 ${}_hp_x{}_hV_x-{}_{h-1}p_x{}_{h-1}V_x$

小计 ${}_hp_x{}_hV_x-{}_{h-1}p_x({}_{h-1}V_x-e_{h-1})+{}_{h-1}p_xq_{x+h-1}$

净收益 (盈余变化)

$$ {}_{h-1}p_x[u(h-1)i+(c-e_{h-1})(1+i)] \tag{9.5.9} $$

表 9.2.5 与 9.2.6 的左边栏目提供了以上报表的数值说明，那里的报表是按决定性生存组编制的，而不是这里每个初始被保险人的期望值。

由 (9.5.9), 会计年度 h 末的期望盈余为

$$
\begin{aligned}
{}_hp_xu(h) \quad &= \quad {}_{h-1}p_xu(h-1)+{}_{h-1}p_x[u(h-1)i\\
&\qquad +(c-e_{h-1})(1+i)],
\end{aligned} \tag{9.5.10}
$$

与 (9.5.6) 一致，不过现在是从会计观点得出的。

这一章前面曾指出，在实践中，随着持续时间延续，费用倾向于递减。于是，期望盈余

$$ {}_hp_x u(h)=\sum_{j=1}^{h}(1+i)^{h-j+1}{}_{j-1}p_x(c-e_{j-1}). $$

通常对较小的 h 值为负，而对较大的 h 值则为正。藉以得出这一点的会计模型中，净保费责任准备金作为负债，附加保费是均衡的，而费用开支从保单发行开始随时间递减。

为避免起初出现资产少于负债的状况，可采取以下措施：
(1) 保险机构获得额外资本作为初始盈余 $u(0)$，以保持每一年度的 (期望) 盈余

$$ u0)(1+i)^h+\sum_{j=1}^{h}(1+i)^{h-j+1}{}_{j-1}p_x(c-e_{j-1}) $$

为正， $h=0,1,2,\cdots$ 。

(2) 各保单年度的附加保费不同，使得 $c_{h-1}-e_{h-1}\geq 0$，$h=1,2,3,\cdots$ 。

(3) 保险机构的负债可按某种修正责任准备金原则核算，以降低早期保单年度确认的负债。表 9.2.5 与 9.2.6 中列 (2) 使用的受益加上费用责任准备金方法即为一例。

§9.6 修正责任准备金方法

修正责任准备金方法在确定责任准备金时，从未来受益的精算现值里减去的项不直接使用一组均衡净保费的精算现值，而是采用*阶梯保费*(step premium) 制。尽管理论上可允许使用多个阶梯，但通常包含的不同保费水平不超过三个。这三个保费水平分别记为 α,β,P，其中 α 是第 1 年的净保费， β 是接下去 $j-1$ 年

的净保费，P 是起初 j 个保单年度之后的净均衡保费。这组保费需满足约束：

$$\begin{aligned}&\alpha+\beta a_{x:\overline{j-1}|}+P_{j|h-j}\ddot{a}_{x:\overline{h-j}|}=P\ddot{a}_{x:\overline{h}|},\\&\alpha+\beta a_{x:\overline{j-1}|}=P\ddot{a}_{x:\overline{j}|},\end{aligned}\tag{9.6.1}$$

其中 h 是缴费期年限。于是，这组保费的精算现值与净均衡保费(年缴额 P) 的精算现值相当。

按上一节中的符号，附加保费 c 可抵消第一个保单年度的部分费用。如果 $\alpha < P$, 那么按照修正责任准备金会计方法，$P+c-\alpha>c$ 可做得与第一年度的费用完全匹配。当 $\alpha<P$ 时，必有 $\beta>P$, 这一点可从 (9.6.1) 看出。事实上，由 (9.6.1) 的第二式可得

$$\begin{gathered}\alpha+\beta a_{x:\overline{j-1}|}=P(a_{x:\overline{j-1}|}+1),\\\beta=P+\tfrac{P-\alpha}{a_{x:\overline{j-1}|}}.\end{gathered}\tag{9.6.2}$$

另一个可从 (9.6.1) 获得的表达式为

$$\begin{gathered}\beta(\ddot{a}_{x:\overline{j}|}-1)=P\ddot{a}_{x:\overline{j}|}-\alpha,\\\beta=P+\tfrac{\beta-\alpha}{\ddot{a}_{x:\overline{j}|}}.\end{gathered}\tag{9.6.3}$$

这样，修正责任准备金方法可通过确定修正期年限 j 以及期间第 1 年保费 α 或续年保费 β 或差额 $\beta-\alpha$ 三者之一而得到体现。图 9.6.1 提供了这些关系的一个图解，其中阴影部分 A 与 B 所代表保费的精算现值相等，与之对应的关系式可从 (9.6.1) 看出：

$$P-\alpha=(\beta-P)a_{x:\overline{j-1}|}.$$

以上使用符号 α 与 β 分别记 j 年修正责任准备金方法中的第一年保费与续年保费，用一般符号 P 记净均衡保费。与此相应，以下将用符号 V^{Mod} 来记根据修正方法计算出的期末责任准备金。

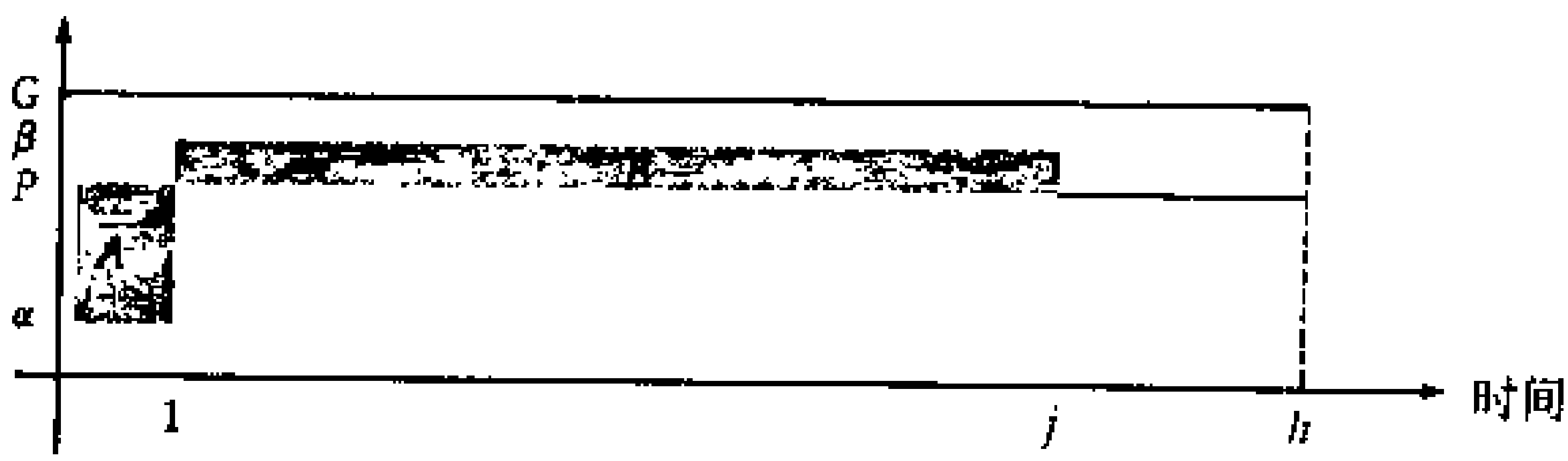


图 9.6.1　修正责任准备金方法中的保费

对于 h 年缴费、n 年期两全保险，采用 j 年修正期的修正责任准备金方法，下列公式可用于计算期末责任准备金。在修正期内，$k<j$，

$$\begin{aligned}
{}_k^hV_{x:\overline{n}|}^{Mod} &= A_{x+k:\overline{n-k}} - \beta\ddot{a}_{x+k:\overline{j-k}|} - {}_hP_{x:\overline{n}|}\,{}_{j-k|h-j}\ddot{a}_{x+k} \\
&= A_{x+k:\overline{n-k}|} - {}_hP_{x:\overline{n}|}\ddot{a}_{x+k:\overline{h-k}|} - (\beta - {}_hP_{x:\overline{n}|})\ddot{a}_{x+k:\overline{j-k}|} \\
&= {}_k^hV_{x:\overline{n}|} - (\beta - {}_hP_{x:\overline{n}|})\ddot{a}_{x+k:\overline{j-k}|}.
\end{aligned}$$

在修正期之后，在修正方法下的责任准备金与净均衡保费方法下的责任准备金相同，即 ${}_k^hV_{x:\overline{n}|}^{Mod} = {}_k^hV_{x:\overline{n}|}, k \ge j$ 。

例 9.6.1: 考虑完全连续基础的终身人寿保险，采用的修正责任准备方法的修正期为整个保单有效期，第一年的年保费（率）为 $\overline{\alpha}_x$，续年保费（年率）为 $\overline{\beta}_x$，其中，$\overline{\alpha}_x < \overline{P}(\overline{A}_x)$ 。定义未来亏损变量并写出可用来求责任准备金的方程。

解: 所求损失变量为

$${}_tL^{Mod} = \begin{cases} \left.\begin{array}{ll} v^U - \overline{\alpha}_x\overline{a}_{\overline{U}|} & 0 \le U < 1-t, \\ v^U - \overline{\alpha}_x\overline{a}_{\overline{1-t}|} - \overline{\beta}_{x\,1-t|}\overline{a}_{\overline{U-(1-t)}|} & U \ge 1-t, \end{array}\right\} & 0 \le t < 1 \\ v^U - \overline{\beta}_x\overline{a}_{\overline{U}|} & t \ge 1. \end{cases}$$

类似于 (5.2.2), 责任准备金

$$ {}_t\overline{V}(\overline{A}_x)^{Mod} = \begin{cases} \overline{A}_{x+t} - \overline{\alpha}_x \overline{a}_{x+t:\overline{1-t}|} - \overline{\beta}_{x\,1-t|}\overline{a}_{x+t} & 0 \le t < 1 \\ \overline{A}_{x+t} - \overline{\beta}_x \overline{a}_{x+t} & t \ge 1. \end{cases} $$

另外,

$$ {}_t\overline{V}(\overline{A}_x) - {}_t\overline{V}(\overline{A}_x)^{Mod} = [\overline{\beta}_x - \overline{P}(\overline{A}_x)]\overline{a}_{x+t} \qquad t \ge 1. $$

由于要求

$$ \overline{\alpha}_x \overline{a}_{x:\overline{1}|} + \overline{\beta}_{x\,1|}\overline{a}_x = \overline{P}(\overline{A}_x)(\overline{a}_{x:\overline{1}|} + {}_{1|}\overline{a}_x), \tag{9.6.4} $$

类似于 (9.6.2) 有

$$ \overline{\beta}_x = \overline{P}(\overline{A}_x) + \frac{[\overline{P}(\overline{A}_x) - \overline{\alpha}_x]\overline{a}_{x:\overline{1}|}}{{}_{1|}\overline{a}_x}. $$

并根据条件 $\overline{\alpha}_x < \overline{P}(\overline{A}_x)$ 可知

$$ \overline{\beta}_x > \overline{P}(\overline{A}_x). $$

于是

$$ {}_t\overline{V}(\overline{A}_x) - {}_t\overline{V}(\overline{A}_x)^{Mod} \ge 0 \qquad t \ge 1. $$

例 9.6.2: 对于例 9.6.1 中的情形, 导出 ${}_t\overline{V}(\overline{A}_x)^{Mod}$ 的后顾公式。

解: 在 $0 \le t < 1$ 时, 由

$$ \begin{aligned} \overline{A}_x &= \overline{\alpha}_x \overline{a}_{x:\overline{1}|} + \overline{\beta}_{x\,1|}\overline{a}_x \\ &= \overline{\alpha}_x(\overline{a}_{x:\overline{t}|} + \overline{a}_{x+t:\overline{1-t}|}\,{}_tE_x) + \overline{\beta}_x({}_{1-t|}\overline{a}_{x+t}\,{}_tE_x). \end{aligned} $$

用 §5.3 的符号可得

$$ \begin{aligned} {}_t\overline{V}(\overline{A}_x)^{Mod} &= \overline{A}_{x+t} - \overline{\alpha}_x \overline{a}_{x+t:\overline{1-t}|} - \overline{\beta}_{x\,1-t|}\overline{a}_{x+t} \\ &= \overline{A}_{x+t} - \frac{\overline{A}_x - \overline{\alpha}_x \overline{a}_{x:\overline{t}|}}{{}_tE_x} \qquad 0 \le t < 1. \\ &= \overline{\alpha}_x \overline{s}_{x:\overline{t}|} - {}_t\overline{k}_x. \end{aligned} $$

另外还有

$$ {}_t\overline{V}(\overline{A}_x) - {}_t\,\overline{V}(\overline{A}_x)^{Mod} = [\overline{P}(\overline{A}_x) - \overline{\alpha}_x]\overline{s}_{x:\overline{t}|} \qquad 0 \le t < 1. $$

在 $t \ge 1$ 时，由

$$ \overline{A}_x = \overline{\alpha}_x\overline{a}_{x:\overline{1}|} + \overline{\beta}_x({}_{1|t-1}\overline{a}_x + \overline{a}_{x+t\,t}E_x) $$

可得

$$ \begin{aligned} {}_t\overline{V}(\overline{A}_x)^{Mod} &= \overline{A}_{x+t} - \overline{\beta}_x\overline{a}_{x+t} \\ &= \overline{A}_{x+t} - \frac{\overline{A}_x - \overline{\alpha}_x\overline{a}_{x:\overline{1}|} - \overline{\beta}_{x\,1|t-1}\overline{a}_x}{{}_tE_x} \\ &= \frac{\overline{\alpha}_x\overline{a}_{x:\overline{1}|}}{{}_tE_x} + \overline{\beta}_x\overline{s}_{x+1:\overline{t-1}|} - {}_t\overline{k}_x. \end{aligned} $$

§9.7 完全初年定期制

为了与较高的第一年费用匹配而增加第一个保单年度有效附加保费 $G-\alpha$ 的责任准备金修正方法中，α 通常小于 P 。然而，在实践中，为了符合某些法规条例，α 的取值一般有一个下界。

负值的责任准备金负债在会计上实际上是资产，但未来的保费收集是不确定的，保险业监管部门一般不允许在资产负债表中出现负值的责任准备金。因此，实用的责任准备金修正制应避免第一个保单年度末出现负责任准备金。这意味着，对定额受益保单，α 的最小可能值按完全离散基础是 $A^{1}_{x:\overline{1}|}$. 这个结论可从以下推论看出：

$$ \begin{aligned} &{}_1V \ge 0, \\ &\alpha\ddot{s}_{x:\overline{1}|} - {}_1\,k_x \ge 0, \\ &\alpha \ge A^{1}_{x:\overline{1}|}. \end{aligned} \tag{9.7.1} $$

当 α 取为最低水平并且修正期为整个保费缴付期时，相应的方法称为 完全初年定期(full preliminary term)(FPT) 修正方法。在 FPT 修正制下，第一个保单年度末的责任准备金为 0 。

按此完全离散基础，续年保费 β 可利用类似于 (9.6.1) 的等式获得。用 A 表示净趸缴保费，$A(1)$ 表示始于 $x+1$ 岁的剩余受益保险的净趸缴保费，则

$$
\begin{aligned}
A^{1}_{x:\overline{1}|}+\beta_{1|h-1}\ddot{a}_{x} &= P\ddot{a}_{x:\overline{h}|} \\
&= A \\
&= A^{1}_{x:\overline{1}|}+{}_{1}E_{x}A(1)
\end{aligned}
\tag{9.7.2}
$$

或者

$$\beta=\frac{{}_{1}E_{x}A(1)}{{}_{1|h-1}\ddot{a}_{x}}=\frac{A(1)}{\ddot{a}_{x+1:\overline{h-1}|}}.$$

换言之，β 是岁数大 1 岁的类似保险的净年缴保费，在限定保费缴付期的场合，(β 的) 缴费期少 1 年，满期年龄与原保险相同。

按完全连续基础，第一年保费 (率)$\overline{\alpha}$ 的最小可能值是 $\overline{A}^{1}_{x:\overline{1}|}/\overline{a}_{x:\overline{1}|}$。此结果仍基于第一年末责任准备金非负：

$$
\begin{aligned}
&{}_{1}\overline{V}(\overline{A}_{x})\geq 0,\\
&\frac{\overline{\alpha}\overline{a}_{x:\overline{1}|}}{{}_{1}E_{x}}-{}_{1}\overline{k}_{x}\geq 0,\\
&\overline{\alpha}\geq\frac{\overline{A}^{1}_{x:\overline{1}|}}{\overline{a}_{x:\overline{1}|}}.
\end{aligned}
\tag{9.7.3}
$$

对于续年保费 (年率)$\overline{\beta}$，类似于 (9.7.2) 有

$$
\begin{aligned}
\overline{A}_{x:\overline{1}|}+\overline{\beta}_{1|h-1}\overline{a}_{x} &= \overline{P}(\overline{A})\overline{a}_{x:\overline{h}|} \\
&= \overline{A} \\
&= \overline{A}^{1}_{x:\overline{1}|}+{}_{1}E_{x}\overline{A}(1)
\end{aligned}
\tag{9.7.4}
$$

或者

$$\overline{\beta}=\frac{{}_{1}E_{x}\overline{A}(1)}{{}_{1|h-1}\overline{a}_{x}}.$$

完全初年定期 (FPT) 修正方法对会计报表的影响可从修正 (9.5.5) 得出。在 FPT 制下，附加费用的保费由

$$P_x + c = A^1_{x:\overline{1}|} + c_0 = \beta_x + c_1$$

给出，其中 c_0 是第一年的附加保费， c_1 是续年附加保费。仍用 $u(k)$ 记会计年度 k 之末每份残存保单的目标盈余，与 (9.5.5) 类似， $u(0)=0$,

$$\begin{aligned}&[(A^1_{x:\overline{1}|}+c_0)-e_0](1+i)-q_x = p_x u(1),\\ &(c_0-e_0)(1+i) = p_x u(1) \qquad k=0,\end{aligned} \tag{9.7.5A}$$

$$\begin{aligned}&{}_kp_x\{{}_kV_x^{FPT}+u(k)+(\beta_x+c_1)-e_k\}(1+i)-{}_kp_xq_{x+k}\\ &\quad={}_{k+1}p_x[{}_{k+1}V_x^{FPT}+u(k+1)] \qquad k=1,2,\cdots.\end{aligned} \tag{9.7.5B}$$

于是，当

$$c_0-e_0=(P_x+c-A^1_{x:\overline{1}|}-e_0)>0$$

时，以上理想化会计示例的第一年盈余是正的。在实际场合中，

$$c_0=P_x+c-A^1_{x:\overline{1}|}>c,$$

$p_xu(1)$ 将比以净均衡保费责任准备金作为负债时更大。

与 (5.7.2) 类似的递归公式是

$$A^1_{x:\overline{1}|}(1+i)-q_x=0, \tag{9.7.6A}$$

,

$${}_kp_x({}_kV_x^{FPT}+\beta_x)(1+i)-{}_kp_xq_{x+k}={}_{k+1}p_x{}_{k+1}V_x^{FPT}. \tag{9.7.6B}$$

从 (9.7.5B) 中减去 (9.7.6B) 得

$${}_kp_x[u(k)+c_1-e_k](1+i)={}_{k+1}p_xu(k+1) \qquad k=1,2,3,\cdots. \tag{9.7.7}$$

用 v^{k+1} 乘 (9.7.5A) 与 (9.7.7), 当 $k=0$ 时令 $c'_k=c_0$; 当 $k=1,2,\cdots$ 时令 $c'_k=c_1$, 于是有

$$\triangle[v^k{}_kp_xu(k)]=v^k{}_kp_x(c'_k-e_k). \tag{9.7.8}$$

由 $u(0)=0$ 及差分方程 (9.7.8) 可得,

$$\sum_{j=0}^{k-1}\triangle[v^j{}_jp_xu(j)]=\sum_{j=0}^{k-1}v^j{}_jp_x(c'_j-e_j), \tag{9.7.9}$$

$$_kp_xu(k)=\sum_{j=0}^{k-1}(1+i)^{k-j}{}_jp_x(c'_j-e_j). \tag{9.7.10}$$

与 (9.5.8) 一样, 在这个理想化模型中, 每份初始保单的期望盈余是附加保费超过费用部分的积累值。用 (9.5.7) 与 (9.7.8), 可分别得出完全初年定期 (FPT) 制责任准备金与净均衡保费 (NLP) 制责任准备金情形每份残存保单对盈余的年期望贡献, 并可比较如下:

FPT	NLP	
$c_0-e_0>c-e_0$		
$c_1-e_k<c-e_k$		$k=1,2,\cdots$

(9.7.11)

以上 (9.7.11) 所示对盈余的期望贡献不等式当 $\alpha<P$ 及 $\beta>P$ 时成立。

§9.8 美国保险监督官标准

如果采用保险企业资产负债表中责任准备金负债不能为负值的原则, 那么完全初年定期 (FPT) 制责任准备金方法确认的第一年期末责任准备金与净保费都是最低的。根据 (9.7.11), 按 FPT 与 NLP(净均衡保费) 制的第一年盈余贡献为

$$P+c-A^{\ 1}_{x:\overline{1}|}-e_0\overset{FPT}{=}c_0-e_0\overset{NLP}{>}c-e_0\,.$$

但困难的是，$P-A^{1}_{x:\overline{1}|}$ 的大小依赖于险种。譬如 $P_{x:\overline{n}|}$ 通常远大于 $P^{1}_{x:\overline{n}|}$，两全保险可用来冲销第一年费用的余地也就远大于定期保险。有学者认为，如果 $P-A^{1}_{x:\overline{1}|}$ 对低费率的保单提供的费用(冲销) 额度是合适的话，那么对高费率的保单这种费用额度就过多了。按照这个意见，对低费率的保单可令人满意地使用 FPT 方法，而对高费率保单则应使用一种第一年期末责任准备金为正的修正责任准备金方法。

一种更改的初年定期制需要给出一个决定规则，据此将保单分为低费率与高费率类别，对低费率保单允许使用 FPT 方法，而对高费率保单，$\beta,\beta-\alpha$ 或 $\alpha>A^{1}_{x:\overline{1}|}$ 三者之一以及修正期年限需另外确定。

政府制定法规的一个目的是维护被保险人的合法权益，减少保险公司违约的危险。为此，保险法规一般对计价方法及假设的选择有所限制。某些计价法确定了各种修正责任准备金标准，但与美国的精算师直接有关的是标准计价法规定的保险监督官计价标准，其要点如下：

(1) 高费率保单是那些满足 $\beta^{FPT}>{}_{19}P_{x+1}$ 的保单，其中 ${}_{19}P_{x+1}$ 是 20 年缴费期终身寿险的 FPT 制续年净保费。

(2) 对低费率保单采用 FPT 方法。

(3) 对高费率保单采用一个特别的保险监督官责任准备金计价方法 (Com)，其中，修正期等于缴费期，且

$$\beta^{Com}-\alpha^{Com}={}_{19}P_{x+1}-A^{1}_{x:\overline{1}|}.$$

应用 (9.5.3) 可得

$$\beta^{Com}=P+\frac{{}_{19}P_{x+1}-A^{1}_{x:\overline{1}|}}{\ddot{a}_{x:\overline{h}|}}, \tag{9.8.1}$$

这里 h 是保费缴费期限。

应用更改的初年定期计价制的一个问题是如何推广到非均衡保费与非定额受益的保单。在 §5.4 讨论了一种完全离散基础的一般保险，现用那种保险来说明问题。该保险当死亡发生在保单年度 $j+1$ 时，年末赔付死亡受益 b_{j+1}，在活着的情况下，年保费在缴费期内的保单年度初缴付，保单年度 $j+1$ 之初 (也就是时间 j) 缴付的毛保费为 G_j 。

以下阐述应用保险监督官制责任准备金标准于这个一般保险的若干规则 (按 Menge1946 年解释)，公式只对定期保险给出，对两全保险只作说明，并在例 9.8.1 中给出计算。

首先决定使用 FPT 方法的准则。第一步计算一个等价定额续保金额 $(ELRA)$，对上述一般保险，

$$ELRA = \frac{\sum_{j=0}^{n-2} b_{j+2} v^{j+1} {}_jp_{x+1} q_{x+1+j}}{A^{1}_{x+1:\overline{n-1}|}}. \tag{9.8.2}$$

对两全保险， $ELRA$ 只根据死亡受益计算，亦由 (9.8.2) 给出。然后，决定续年净保费与毛保费之平均比率

$$r_F = \frac{\sum_{j=0}^{n-2} b_{j+2} v^{j+1} {}_jp_{x+1} q_{x+1+j}}{\sum_{j=0}^{h-2} G_{j+1} v^j {}_jp_{x+1}}. \tag{9.8.3}$$

对两全保险，纯生存受益包括在 r_F 的分子中。

按 Menge 的解释，当

$$r_F G_0 \le ELRA_{19}P_{x+1} \tag{9.8.4}$$

时，可使用 FPT 方法，据此，净保费为 $\pi_0 = vb_1q_x$ 及 $\pi_j = r_F G_j, j = 1, 2, \cdots, h-1$，这里 h 是保费缴付期限。 $k \ge 1$ 时的

修正责任准备金为

$$ {}_kV^{Mod} = \sum_{j=0}^{n-k-1} b_{k+j+1} v^{j+1} {}_jp_{x+k} q_{x+k+j} - r_F \sum_{j=0}^{h-k-1} G_{k+j} v^j {}_jp_{x+k}. \tag{9.8.5} $$

当 (9.8.4) 不满足时，相当于 $\beta-\alpha$ 的第一年费用超额补贴为

$$ ELRA_{19}P_{x+1} - b_1 A^1_{x:\overline{1}|}. $$

净保费与毛保费之修正的平均比率 r_C 由下式决定：

$$ r_C = \frac{\sum\limits_{j=0}^{n-1} b_{j+1} v^{j+1} {}_jp_x q_{x+j} + (ELRA_{19}P_{x+1} - b_1 A_{x:\overline{1}|})}{\sum\limits_{j=0}^{h-1} G_j v^j {}_jp_x}. \tag{9.8.6} $$

此时，这个高费率情形的修正责任准备金为

$$ {}_kV^{Mod} = \sum_{j=0}^{n-k-1} b_{k+j+1} v^{j+1} {}_jp_{x+k} q_{x+k+j} - r_C \sum_{j=0}^{h-k-1} G_{k+j} v^j {}_jp_{x+k}. \tag{9.8.7} $$

对两全保险，r_C 的分子及 (9.8.5) 与 (9.8.7) 的右端需作适当调整以包括纯生存受益。

例 9.8.1: 根据保险监督官修正制计算投保年龄 35 岁的特殊 30 年期两全保单的净年缴保费。该保单的起初 20 年受益金为 150000, 以后为 100000, 期满受益亦为 100000 。毛保费前 10 年为 2500, 以后为 1250 。计算时采用附录的示例生命表以及 $i=0.06$ 。

解: $ELRA$ 根据死亡受益计算，

$$ \begin{aligned} ELRA &= 50000\frac{3M_{36}-M_{55}-2M_{65}}{M_{36}-M_{65}} \\ &= 130153.30. \end{aligned} $$

因子 r_F 由下式给出：

$$\begin{aligned} r_F &= \frac{50000(3M_{36} - M_{55} - 2M_{65} + 2D_{65})}{1250(2N_{36} - N_{45} - N_{65})} \\ &= 0.91014604. \end{aligned}$$

此外，

$$_{19}P_{36} = 0.0116543.$$

应用 FPT 必须满足 (9.8.4), 然而，

$$r_F G_0 = 0.91014604 \times 2500 = 2275.37 > 1516.85 = ELRA_{19}P_{36},$$

故不容许使用 FPT 方法。相应于 $\beta - \alpha$ 的第一年费用超额补贴为

$$ELRA_{19}P_{x+1} - b_1 A^{1}_{x:\overline{1}|} = 1516.8456 - 284.9395 = 1231.9061,$$

$$\begin{aligned} r_C &= \frac{50000(3M_{35} - M_{55} - 2M_{65} + 2D_{65}) + 1231.9061}{1250(2N_{35} - N_{45} - N_{65})} \\ &= 0.88223578. \end{aligned}$$

于是净续年保费为

$$\begin{aligned} 2500r_C &= 2205.59 \qquad \text{年度}2, 3, \cdots, 10, \\ 1250r_C &= 1102.79 \qquad \text{年度}11, 12, \cdots, 30. \end{aligned}$$

第一年净保费为 $2500r_C - 1231.9061 = 973.68$ 。

§9.9 加拿大修正制

加拿大保险法规定的修正责任准备金方法 (CAN) 允许精算师在选择假设时有较广的自由。以下描述的加拿大修正制按数学上等价的方式给出。

首先，令 E^{Can} 为第一年费用按均衡保费衡量的额外补贴，即

$$\alpha^{Can} = P - E^{Can}. \tag{9.9.1}$$

当修正期等于缴费期时，利用 (9.6.2) 可得

$$\beta^{Can} = P + \frac{E^{Can}}{a_{x:\overline{h-1}|}}, \tag{9.9.2}$$

其中

$$E^{Can} = min(a, b, c),$$

而 $a = 150\%$ 净均衡保费，$b=$ 新契约费，$c=$ 仍然提供管理费用及保单持有人分红时在第二及以后年中可收回费用的精算现值。

在 (9.9.2) 两端乘 $a_{x:\overline{h-1}|}$，加到 (9.9.1) 上去，得

$$\alpha^{Can} + \beta^{Can} a_{x:\overline{h-1}|} = A.$$

加拿大修正制责任准备金如下：

$${}_0V^{Can} + \alpha^{Can} = P - E^{Can}, \tag{9.9.3A}$$

$$\begin{aligned} {}_kV^{Can} &= A(k) - \beta^{Can}\ddot{a}_{x+k:\overline{h-k}|} && (9.9.3B) \\ &= {}_k^hV - \frac{E^{Can}\ddot{a}_{x+k:\overline{h-k}|}}{a_{x:\overline{h-1}|}}, && (9.9.3C) \end{aligned}$$

其中 $A(k)$ 是时间 k 时的未来受益精算现值。

另一种等价的描述为：在时间 0 的期末责任准备金定为 $-E^{Can}$，第一年净保费为 P，β^{Can} 按 (9.9.2) 确定。无论按何种方式，在时间 0 的期初责任准备金都是 $P - E^{Can}$，其后的期末责任准备金都由 (9.9.3B) 给出。

习　题

§9.2

1. (1) 某赌场在年度 A 的 7 月 1 日向 1000 位顾客每人收取 0.55 元，并立即投资于每半年利息率为 3% 的储蓄帐户基金。在年度 A+1 的 7 月 1 日，对应于每份顾客的共计 1000 枚分币将被掷出正反面。如正面向上，则可获 1 元奖金；如正面向下，奖金为 0 。对该赌场在年度 A 的 12 月 31 日的资产负债表与损益表填入应有的数字，其中负债使用精算现值。

资产负债表

资产		负债
储蓄帐户		准备金
		盈余

损益表

赌资收入	
利息收入	
准备金增加额	

(2) 在年度 A+1 的 7 月 1 日支付额随机变量 Y 服从二项分布，用正态近似估计

$$Pr[Y(1.03)^{-1} - a > 0],$$

其中 a 是在年度 A 的 12 月 31 日的资产。

(3) 如果该赌场只有一位顾客，那么第 (1) 小题中涉及的金额乘以 0.001, 证明此时第 (2) 小题中的概率等于 1/2.

2. 用于完全连续终身寿险保单的增列费用的亏损变量由下式给出：

$$L_e = L + X,$$

其中

$$L = v^T - \overline{P}(\overline{A}_x)\overline{a}_{\overline{T}|}$$

$$X = c_0 + (g - e)\overline{a}_{\overline{T}|}.$$

在这些表达式中，L 可解释成与保单的受益部分相联系的亏损变量，X 则与费用相联系，符号 c_0 表示非随机的初始费用，g 为连续维持费用 (率), e 是保费中的费用附加费。设平衡原理成立：$E[L] = E[X] = 0$, 证明

(1) $X = c_0 L$ 。

(2) $\mathrm{Var}[L_e] = (1 + c_0)^2 \mathrm{Var}[L]$ 。

§9.3

3. 考虑保额为 1000 到 65 岁为止的两全保险，保费按年缴付，向 40 岁被保险人发行。假定：

销售佣金为第 1 年附加费用的保费为 40%;

保单年度 2 至 10 的续保费佣金为附加费用的保费的 5%;

税金为每年附加费用的保费的 2%;

第 1 年的维持费用为每 1000 保额 12.50, 以后年度每 1000 保额 4.00;

净保费提供死亡即刻赔付受益，无死亡时保费调整；

使用 15 年选择与终极死亡表。

写出附加费用的保费表达式。

4. 某种一次性缴费的 n 年期两全保险的附加费用的保费按以下假设决定：

税金为附加费用的保费的 2.5%;

佣金为附加费用的保费的 4%;

每 1000 保额的其它费用为第 1 年 5, 以后每年 2.50;

受益在死亡即刻赔付，费用在每个保单年度初开支。

给出向 (x) 发行的受益金 1000 保单的附加费用的保费公式。

5. 对于金额为 1 完全连续支付基础的终身寿险保单，附加费用的保费根据以下费用开支清单计算：

初始费用 e_0;

每年 (包括第 1 年) 一笔费用 $e_1 + e_2 P_x$;

理赔支付成本 (与受益赔付同时开支)e_3 。

如 $G = aP_x + c$, 求 a 与 c 。

§9.4

6. 设

$$G(b) = \frac{b(a + c/b)}{1 - f},$$

$$R(b) = \frac{a}{1 - f} + \frac{c}{b - bf}.$$

在给定 $a/(1-f) = 25$, $c/(1-f) = 7.50$ 及最低保险金额 $m = 2$(单位：千元) 条件下，画出 $R(b)$ 的图形。

7. 根据习题 6 给定的条件，令 $t = 5$ 。验证

(1) $R(t) = 26.50$ 。

(2) $R(b) = R(t) + Z(b)$, 其中 $Z(b) = c/(1-f)b - c/(1-f)t$, 并画出第 (2) 小题中的函数 $Z(b)$ 图形。在这个问题中， t 称为支点, 当 $b < t$ 时 $Z(b) > 0$, 当 $b > t$ 时 $Z(b) < 0$ 。

8. 某险种每份保单金额的概率密度函数为

$$f(b) = kb^{-3} \qquad b > 10,$$

这里 b 的单位是千元。计算

(1) 规范常数 k 。

(2) 期望保单金额。

(3) 保险金分布的中位值。

(4) $R(b)$, 其中 $a = 25$, $f = 0.15$, $c = 12$ 。

§9.5

9. 与 (9.5.5) 类似的连续形式是微分方程

$$\begin{aligned}\frac{d}{dt}{}_tp_x[{}_t\overline{V}(\overline{A}_x) + \overline{u}(t)] &= {}_tp_x[\overline{P}(\overline{A}_x) + \delta_t\overline{V}(\overline{A}_x) \\ &\quad + \overline{c} - \overline{e}(t) + \delta\overline{u}(t) - \mu_{x+t}].\end{aligned}$$

用这个方程以及 (5.10.5) 给出的

$$\frac{d}{dt}[{}_tp_{x\,t}\overline{V}(\overline{A}_x)]$$

表达式，证明

$${}_tp_x\overline{u}(t) = \int_0^t e^{\delta(t-y)}{}_yp_x[\overline{c} - \overline{e}(y)]dy.$$

§9.6

10. 对于修正期等于缴费期的修正责任准备金方法，证明

$${}_kV_{x:\overline{n}|}^{Mod} = 1 - (\beta + d)\ddot{a}_{x+k:\overline{n-k}|}.$$

11. 某种完全连续终身寿险的修正责任准备金方法由下式确定：

$$\overline{\alpha}(t) = \frac{t}{m}\overline{\beta} \qquad 0 \le t < m,$$

其中 $\overline{\beta}$ 是 $t \ge m$ 时的均衡保费。

(1) 写出 $\overline{\beta}$ 的公式。

(2) 写出 ${}_t\overline{V}(\overline{A}_x)^{Mod}, t < m$ 的前瞻公式。

12. 对于修正期等于缴费期，${}_1V_x^{Mod} = K$ 的完全离散终身寿险，计算 α_x^{Mod} 与 β_x^{Mod} 。

13. 某种修正责任准备金方法将完全离散终身寿险保单的净年缴保费 P_x 代之以 (为责任准备金目的) 起初 n 年的年保费 α_x^{Mod} 及以后的年保费 β_x^{Mod} 。证明

$$\frac{\beta_x^{Mod} - P_x}{P_x - \alpha_x^{Mod}} = \frac{\ddot{a}_x}{{}_{n|}\ddot{a}_x} - 1.$$

14. 证明

$${}_kV - {}_kV^{Mod} = \left(\frac{\beta - \alpha}{\ddot{a}_{x:\overline{j}|}}\right)\ddot{a}_{x+k:\overline{j-k}|},$$

其中 j 是修正期限。注意，这个差额可解释为第 1 年费用超额补贴的未偿还部分。

§9.7

15. 证明

$$ {}_kV^{FPT}_{x:\overline{n}|} = 1 - \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x+1:\overline{n-1}|}} \qquad k = 1, 2, \cdots, n. $$

16. 初两年定期修正制责任准备金方法(2-year preliminary term reserve method) 有 3 个计价净保费：

第 1 年： $A^{1}_{x:\overline{1}|}$ 。

第 2 年： $A^{1}_{x+1:\overline{1}|}$ 。

以后： $x+2$ 岁的净均衡保费 (受益与缴费方式不变) 。

证明：按这个方法， (x) 的终身寿险保单的责任准备金为

$$ {}_1V^{Mod} = {}_2V^{Mod} = 0. $$

$$ {}_kV^{Mod} = {}_kV_x - (P_{x+2} - P_x)\ddot{a}_{x+k} \qquad k = 3, 4, 5, \cdots. $$

(这类责任准备金制在健康保险中较普遍。)

17. 某种责任准备金计价制提议第 1 年净保费 α 至少等于 $A^{1}_{x:\overline{1}|}$, 对有些保单及有些年龄可使用 FPT 方法，但续年净保费 β 与 α 之差不能超过 0.05 。设 $d = 0.03$, $\ddot{a}_x = 17$, $\ddot{a}_{x:\overline{12}|} = 9$, $A^{\ \,1}_{x:\overline{12}|} = 2/3$, $A^{1}_{x:\overline{1}|} = 0.01$ 。

(1) 计算 (x) 的终身寿险保单的 β 。

(2) 计算 ${}_{12}V^{Mod}_x$ 。

(3) 设 $\alpha = A^{1}_{x:\overline{1}|} = 0.01$, 对 (x) 的 12 年期两全保险计算 β 的试验值，并验证该 β 是不允许的 $(\beta - \alpha > 0.05)$ 。

(4) 用第 (3) 小题中的结果，计算 (x) 的 12 年两全保险的 β 。

(5) 计算 ${}_1V^{Mod}_{x:\overline{12}|}$ 。

§9.8

18. 对于 (x) 的完全离散缴费期为 15 年期两全保险，按保险监督官修正制写出第 1 年与净续年保费。

19. 某种更改的初年定期制如下：

保单分成两类，当 FPT 净续年保费大于 ${}_{19}P_{x+1}$ 时属类别 I，其余属类别 II；

对类别 I 的保单，第 1 年净保费与保险监督官责任准备金制相同，净续年保费使得在缴费期末或者当缴费期长于 15 年时在 15 年末达到净均衡保费责任准备金；

对类别 II 的保单，规定使用 FPT 方法。

对 (x) 的完全离散 20 年缴费 20 年期的两全保险，写出 α 与 β 的表达式。

20. 如 $\beta^{FPT} >_{19} P_{x+1}$, (x) 的完全离散定额寿险在 k 年的期末责任准备金按保险监督官制可写成

$$
{}_kV^{Com} = \frac{A^1_{x:\overline{1}|}}{{}_kE_x} +_{19} P_{x+1}\ddot{s}_{x+1:\overline{k-1}|} + T\ddot{s}_{x:\overline{k}|} - \frac{A^1_{x:\overline{k}|}}{{}_kE_x},
$$

导出 T 的表达式。

21. 设 $\beta^{FPT} >_{19} P_{x+1}$,

$$
b_{j+1} = 1 \qquad j = 0, 1, \cdots, n-1,
$$

$$
G_j = P(1+\lambda) \qquad j = 0, 1, \cdots, h-1.
$$

证明 (9.8.7) 成为保险监督官制责任准备金。

§9.9

22. 在采用加拿大修正制以前，加拿大法律规定的标准是，满足 $P > P_x$ 的保单定义为高费率的。对这种保单，修正期与缴费期相同，且 $P - \alpha = P_x - A^1_{x:\overline{1}|}$。对其它保单允许使用 FPT 方法。证明，对 n 年期两全保险，有

$$
\beta = P_{x:\overline{n}|} + \frac{P_x - A^1_{x:\overline{1}|}}{a_{x:\overline{n-1}|}}.
$$

综合题

23.

(1) 用以下方程计算表 9.2.5 中列 (2) 的净收益:

$$\text{净收益} = \text{盈余的利息收入} + \text{净利润附加费及其利息}.$$

(2) 比较第 (1) 小题中净收益表达式与相应的表达式 (9.5.9), 并对观点上的差异作出说明。

第十章　不没收受益与分红

§10.1　引　言

在第四与第五章讨论人寿保险净年保费及责任准备金时，建立了单重损失模型，在这个模型中，受益赔付的时间乃至金额由被保险人的死亡时间决定，保费则交付到死亡或根据保单规定的缴付期结束。在实践中，一般无法防止在死亡或缴费期结束前中断保费缴付，在这种情况下就产生了如何调和保单各方的利益问题。从多重损失理论导出的模型适合于检查这个问题，有关保单应考虑调和保险机构与被保险人这方面利益的讨论从很早起就已经展开。

决定保费及责任准备金必须采纳某种指导原则，类似地，决定不会因提前终止缴付保费而丧失的所谓*不没收受益*(nonforfeiture benefit) 也需要一个指导原则。这一节将采纳与美国保险法规很接近的一种简单操作原则，那就是，退出的被保险人所获值应使得用单重损失模型建立的受益、保费与责任准备结构在多重损失背景中仍保持适宜。

决定以上原则的动机是关于两类保单持有者对等权益的一种特定概念，这两类分别是那些在保险合同满期前终止的与不中断的保单持有人。很明显，构成对等权益概念的可能有各种不同涵义，从提前终止的保单持有人因未履约而不应享有不没收受益的观点，到应该偿还其所有保费的精算值 (当然要扣掉适当的成本费用) 这样的观点，应有尽有。美国所采用原则的基础是介于上述两个极端之间的权益概念，即退保的寿险保单持有者应享有不没收受益，但这些受益不应引起其他保单持有者的价格 − 受益结

构有所改变。

为说明以上所述原则，我们对完全连续支付基础的终身寿险保单，建立一个含死亡受益与退保受益的模型。将退保引入模型后不改变死亡效力，该效力在单重与双重损失模型中现在都记为 $\mu_{x+t}^{(1)}$，退保效力记为 $\mu_{x+t}^{(2)}$，另外

$$\mu_{x+t}^{(\tau)} = \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)}.$$

对多重损失模型，要求

$$\int_0^\infty \mu_{x+t}^{(\tau)} dt = \infty,$$

这样，

$$\lim_{t\to\infty} {}_tp_x^{(\tau)} = 0,$$

但 $\mu_{x+t}^{(2)}$ 不必具有这个性质。

由 (5.9.3),

$$\frac{d}{dt}{}_t\overline{V}(\overline{A}_x) = \overline{P}(\overline{A}_x) + \delta_t\overline{V}(\overline{A}_x) - \mu_{x+t}^{(1)}[1 -{}_t \overline{V}(\overline{A}_x)], \quad (10.1.1)$$

并回忆 §9.2 中

$$\frac{d}{dt}{}_tp_x^{(\tau)} = -{}_tp_x^{(\tau)}(\mu_{x+t}^{(1)} + \mu_{x+t}^{(2)}),$$

可得出以下的导数表达式，

$$\begin{aligned}
&\frac{d}{dt}[v^t{}_tp_x^{(\tau)}{}_t\overline{V}(\overline{A}_x)] \\
= \quad & v^t{}_tp_x^{(\tau)}[\overline{P}(\overline{A}_x) + \delta_t\overline{V}(\overline{A}_x) \\
& -\mu_{x+t}^{(1)}(1 -{}_t \overline{V}(\overline{A}_x))] - v^t{}_tp_x^{(\tau)}{}_t\overline{V}(\overline{A}_x)[\delta + \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)}] \\
= \quad & v^t{}_tp_x^{(\tau)}[\overline{P}(\overline{A}_x) - \mu_{x+t}^{(1)} - \mu_{x+t}^{(2)}{}_t\overline{V}(\overline{A}_x)]. \qquad (10.1.2)
\end{aligned}$$

对退保受益为 ${}_t\overline{V}(\overline{A}_x)$ 的终身寿险，其保费与责任准备金从一个双重损失模型导出，与 (10.1.1) 类似，有

$$\begin{aligned}\frac{d}{dt}({}_t\overline{V}(\overline{A}_x)^{\overset{2}{-}}) &= \overline{P}(\overline{A}_x)^{\overset{2}{-}} + \delta_t \overline{V}(\overline{A}_x)^{\overset{2}{-}} - \mu_{x+t}^{(1)}[1 -{}_t \overline{V}(\overline{A}_x)^{\overset{2}{-}}] \\ &\quad -\mu_{x+t}^{(2)}({}_t\overline{V}(\overline{A}_x) -{}_t \overline{V}(\overline{A}_x)^{\overset{2}{-}}), \qquad (10.1.3)\end{aligned}$$

其中的上标 $\overset{2}{-}$ 表示保费及责任准备金基于双重损失模型。当把责任准备金 ${}_t\overline{V}(\overline{A}_x)^{\overset{2}{-}}$ 看作抵消受益的储蓄基金时，式 (10.1.3) 中的最后一项是退保的净成本 [参见 (5.10.5)]，于是

$$\begin{aligned}\frac{d}{dt}[v^t{}_tp_x{}^{(\tau)}{}_t\overline{V}(\overline{A}_x)^{\overset{2}{-}}] &= v^t{}_tp_x^{(\tau)}\{\overline{P}(\overline{A}_x)^{\overset{2}{-}} + \delta_t\overline{V}(\overline{A}_x)^{\overset{2}{-}} \\ &\quad -\mu_{x+t}^{(1)}[1 -{}_t \overline{V}(\overline{A}_x)^{\overset{2}{-}}] - \mu_{x+t}^{(2)}[{}_t\overline{V}(\overline{A}_x) -{}_t \overline{V}(\overline{A}_x)^{\overset{2}{-}}]\} \\ &\quad -v^t{}_tp_x^{(\tau)}{}_t\overline{V}(\overline{A}_x)^{\overset{2}{-}}[\delta + \mu_{x+t}^{(1)} + \mu_{x+t}^{(2)}] \\ &= v^t{}_tp_x{}^{(\tau)}[\overline{P}(\overline{A}_x)^{\overset{2}{-}} - \mu_{x+t}^{(1)} - \mu_{x+t}^{(2)}{}_t\overline{V}(\overline{A}_x)]. \qquad (10.1.4)\end{aligned}$$

联合 (10.1.2) 与 (10.1.4) 可得

$$\frac{d}{dt}[v^t{}_tp_x^{(\tau)}({}_t\overline{V}(\overline{A}_x)^{\overset{2}{-}} -{}_t \overline{V}(\overline{A}_x))] = v^t{}_tp_x^{(\tau)}[\overline{P}(\overline{A}_x)^{\overset{2}{-}} - \overline{P}(\overline{A}_x)]. \qquad (10.1.5)$$

对上式从 $t=0$ 到 $t=\omega-x$ 积分，得

$$0 = \overline{a}_x^{(\tau)}[\overline{P}(\overline{A}_x)^{\overset{2}{-}} - \overline{P}(\overline{A}_x)], \qquad (10.1.6)$$

从而

$$\overline{P}(\overline{A}_x)^{\overset{2}{-}} = \overline{P}(\overline{A}_x).$$

于是 (10.1.5) 成为

$$\frac{d}{dt}\{v^t{}_tp_x{}^{(\tau)}[{}_t\overline{V}(\overline{A}_x)^{\overset{2}{-}} -{}_t \overline{V}(\overline{A}_x)]\} = 0,$$

并由此得出

$$ {}_s\overline{V}(\overline{A}_x)^{\overset{2}{-}} = {}_s\overline{V}(\overline{A}_x) \qquad s \geq 0. \tag{10.1.7} $$

因此，如果在完全连续终身寿险的双重损失模型中，退保受益是单重损失模型下的责任准备金，那么双重损失模型中的保费及责任准备金与单重损失模型下的相等。这个结果虽不准备直接应用于实际决定不没收受益问题，然而却为如何使退保或不没收受益对（单重损失模型中决定的）保费及责任准备金的影响最小提供了基本的思路。

§10.2 解约金

在上一节的说明中，未包括费用及相应的附加保费考虑，因此，那里的一般原则如被用来决定保费拖欠时的不没收受益值的话，就需要对这些因素进行补偿。考虑到对保险人在财务上不利时机退保的风险，因初始费用尚未从附加保费中收回而调整净均衡保费责任准备金的一种近似方法是，根据

$$ {}_kCV = {}_kV - {}_kSC \tag{10.2.1} $$

确定在保单发行后时间 k 的不没收受益现金值 ${}_kCV, k = 1, 2, 3, \cdots$，这里，${}_kCV$ 称为解约金(亦称现金价值，cash value)，${}_kV$ 是期末责任准备金，${}_kSC$ 是解约费用(surrender charge)。

量 ${}_kSC$ 的（英文）名称来自早期的保险法规与实践。美国某些州的法律多年来规定 ${}_kSC$ 的最大值为每 1 单位保险的 0.025. 在实践中，负值的 ${}_kCV$ 不可能向退保人收取，因此对每一单位保险，(10.2.1) 的值为正或 0.

在不没收受益现金价值的法则中，一个持久的主题是直接确认数额以及费用负担。与此相符的一种思路是确定单位保险的最低解约金为

$$ {}_kCV \quad = \quad A(k) - P^a\ddot{a}(k) $$

$$= {}_kV - (P^a - P)\ddot{a}(k), \tag{10.2.2}$$

其中 $A(k)$ 与 $\ddot{a}(k)$ 分别是在时间 k 适当的一般净趸缴保费与年金精算现值符号， $k = 1, 2, 3, \cdots$, ${}_kV$ 是同时的期末责任准备金，P 是净年缴保费， P^a 称为 调整保费(adjusted premium) 。(符号 $A(0)$ 与 $\ddot{a}(0)$ 将分别简写为 A 与 $\ddot{a}$.) 这样，问题就归结为确定调整保费。

北美精算学会研究不没收受益及相关事务的委员会在 1975 年报告中包含了确定调整保费的两种类型费用考虑，其一是每单位保险由整个缴费期内每年承担的均衡费用，记为 E, 其二是第 1 年额外费用补偿 E_1. 毛保费 G 假定是由调整保费与均衡年费用 E 所组成的，第 1 年的费用部分 E_1 假定是由调整保费提供的，于是

$$G = P^a + E, \tag{10.2.3A}$$

$$G\ddot{a} = (P^a + E)\ddot{a} = A + E_1 + E\ddot{a}. \tag{10.2.3B}$$

从 (10.2.3B) 可得

$$P^a = \frac{A + E_1}{\ddot{a}}, \tag{10.2.4}$$

利用 $\ddot{a} = a + 1$, 上式可改写成

$$P^a - E_1 + P^a a = A. \tag{10.2.5}$$

与 (9.6.1) 比较显示， $P^a - E_1$ 可看作修正责任准备金制中的第 1 年净保费 α, P^a 可看作其中的续年净保费 β. 一旦每单位保险的第一年费用补偿 E_1 确定，调整保费也就随之决定。

美国保险监督官协会 (NAIC) 的 1941 年人寿保险标准不没收法报告中规定了 E_1, 并用 (10.2.2) 确定每单位保险的最低解约金。在北美精算学会的 1975 年报告中，委员会推荐以 (10.2.2) 作为核算解约金的基本方法，但也推荐确定 E_1 时的某些变更与简

化。这些建议已反映在 1980 年的 NAIC 标准不没收法中，参见表 10.2.1 。

表 10.2.1　每单位保险的第一年费用补偿 E_1 之确定

1941	$0.4\min(P^a, 0.04) + 0.25\min(P^a, P_x^a, 0.04) + 0.02$
1980	$1.25\min(P, 0.04) + 0.01$

注意，以上表中 P^a 表示有关保单的调整保费， P 表示该保单的净保费率，而 P_x^a 则是 (x) 的终身寿险保单的调整保费率。

为实施 1941 年 NAIC 规则，必须先计算 P_x^a. 根据

$$P_x^a = \frac{A_x + E_1}{\ddot{a}_x},$$

以及

$$E_1 = \begin{cases} 0.4P_x^a + 0.25P_x^a + 0.02 = 0.65P_x^a + 0.02 & P_x^a < 0.04, \\ 0.016 + 0.01 + 0.02 = 0.046 & P_x^a \geq 0.04, \end{cases}$$

可得出

$$P_x^a = \begin{cases} \dfrac{A_x + 0.02}{\ddot{a}_x - 0.65} & P_x^a < 0.04 \\ \dfrac{A_x + 0.046}{\ddot{a}_x} & P_x^a \geq 0.04. \end{cases}$$

1980 年不没收法通过去掉 1941 年规则中的循环定义而达到简化的目的，其中直接利用净保费而不是调整保费的百分比来确定第 1 年费用补偿。

表 10.2.2 摘自 1941 年报告，其中给出了调整保费的可能范围，它们可从表 10.2.1 的第 1 行得出。这些调整保费用来确定单位保险的法定最低解约金，更高的解约金是容许的。以下表 10.2.3 可从表 10.2.1 的第 2 行得出。

例 10.2.1: 对表 10.2.2 及 10.2.3 中各行，计算相应的第 1 年费用补偿 E_1.

表 10.2.2　美国 1941 年报告中的调整保费

险种	调整保费范围 终身	 该险种	每单位保险的调整保费公式
终身	< 0.04	< 0.04	$(A_x + 0.02)/(\ddot{a}_x - 0.65)$
	≥ 0.04	≥ 0.04	$(A_x + 0.046)/\ddot{a}_x$
其它	< 0.04	< 0.04 且 $\leq P_x^a$	$(A + 0.02)/(\ddot{a} - 0.65)$
		< 0.04 但 $> P_x^a$	$(A + 0.02 + 0.25P_x^a)/(\ddot{a} - 0.4)$
		< 0.04	$A + 0.02/\ddot{a} - 0.65$
	< 0.04	≥ 0.04	$(A + 0.036 + 0.25P_x^a)/\ddot{a}$
	≥ 0.04	≥ 0.04	$(A + 0.046)/\ddot{a}$

表 10.2.3　美国 1980 年法规中的调整保费

险种	净保费范围	调整保费公式
任何险种	< 0.04	$(A + 1.25P + 0.01)/\ddot{a}$
	≥ 0.04	$(A + 0.06)/\ddot{a}$

解: 直接从表 10.2.1 及 P_x^a 的表达式，或由 (10.2.4) 的变形 $E_1 = P^a\ddot{a} - A = (P^a - P)\ddot{a}$, 都可算出表 10.2.2 中的 E_1 为

$$[\frac{0.65P_x + 0.02}{\ddot{a}_x - 0.65}]\ddot{a}_x, 0.046, [\frac{0.65P + 0.02}{\ddot{a} - 0.65}]\ddot{a},$$

$$[\frac{0.4P + 0.25P_x^a + 0.02}{\ddot{a} - 0.4}]\ddot{a}, 0.25P_x^a + 0.036, [\frac{0.65P + 0.02}{\ddot{a} - 0.65}]\ddot{a},$$

0.046.

表 10.2.3 中的 E_1 分别为 $1.25P + 0.01$, 0.06.

例 10.2.2: 对于 (20) 的缴费期为 10 年的 15 年期两全保险，用 1941 年报告以及 1980 年法规，并在实际的假设 $P_{20}^a < 0.04$ 与 ${}_{10}P_{20:\overline{15|}} \geq 0.04$ 的假设下，建立调整保费的公式。

解: 根据表 10.2.2 可得

$${}_{10}P^a_{20:\overline{15|}} = \frac{A_{20:\overline{15|}} + 0.036 + 0.25P_{20}^a}{\ddot{a}_{20:\overline{10|}}},$$

而用表 10.2.3 则得

$$
{}_{10}P^{a}_{20:\overline{15}|} = \frac{A_{20:\overline{15}|} + 0.06}{\ddot{a}_{20:\overline{10}|}}.
$$

在 §9.8 中，曾讨论对非均衡保费及非定额受益保单应用修正责任准备金制的某些问题，这种保单的不没收受益也出现类似情况，但有所不同。美国 1980 年不没收法引用了平均保险金额(average amount of insurance AAI) 这一概念，它表示开始 10 个保单年度的平均受益金额，按 §5.4 的一般保险符号，当 $n \geq 10$ 时

$$
AAI = \frac{\sum_{j=0}^{9} b_{j+1}}{10}.
$$

净均衡保费是

$$
P = \frac{\sum_{j=0}^{n-1} b_{j+1} v^{j+1} {}_{j}p_{x} q_{x+j}}{\ddot{a}_{x:\overline{h}|}}, \tag{10.2.6}
$$

其中 h 是缴费年限，当包括纯生存受益时，(10.2.6) 的分子还需增加一项。第 1 年费用补偿的公式依赖于 P 与 AAI 。

$$
E_1 = \begin{cases} 1.25P + 0.01AAI & P < 0.04AAI \\ 0.06AAI & P \geq 0.04AAI, \end{cases} \tag{10.2.7}
$$

而任何保单年度的调整保费为 r_N 乘以该年度的毛保费，

$$
r_N = \frac{E_1 + \sum_{j=0}^{n-1} b_{j+1} v^{j+1} {}_{j}p_{x} q_{x+j}}{\sum_{j=0}^{h-1} G_j v^j {}_{j}p_{x}}. \tag{10.2.8}
$$

最低解约金由下式给出：

$$
{}_{k}CV = \sum_{j=0}^{n-k-1} b_{k+j+1} v^{j+1} {}_{j}p_{x+k} q_{x+k+j} - r_N \sum_{j=0}^{h-k-1} G_{k+j} v^j {}_{j}p_{x+k}. \tag{10.2.9}
$$

如果包括纯生存受益，那么 (10.2.8) 及 (10.2.9) 需作相应修改。

例 10.2.3: 对于例 9.8.1 的特殊 30 年期两全保险单, 根据附录示例表以及利率 6%, 用 (10.2.7) 与 (10.2.8) 计算在时间 1,2,10,20 的最低解约金。

解: 开始 20 年的死亡受益为 150000, 以后为 100000, 但由于 AAI 依据开始的 10 个保单年度受益平均额， $AAI = 150000$. 该保单在投保年龄 35 岁时的净保费为

$$P = \frac{50000(3M_{35} - M_{55} - 2M_{65} + 2D_{65})}{N_{35} - N_{65}} = 1622.9358.$$

因为

$$P = 1622.94 \leq 6000 = 0.04AAI,$$

所以根据 (10.2.7),

$$E_1 = 1.25P + 0.01AAI = 3528.6698.$$

毛保费开始 10 年为每年 2500, 以后 20 年为每年 1250, 调整保费乘子

$$\begin{aligned} r_N &= \frac{E_1 D_{35} + 50000(3M_{35} - M_{55} - 2M_{65} + 2D_{65})}{1250(2N_{35} - N_{45} - N_{65})} \\ &= 0.9667466. \end{aligned}$$

用公式 (10.2.9) 可得下表所例解约金：

k	${}_kCV$
1	0.000
2	669.73
10	22519.81
20	48776.92

对于 ${}_1CV$, 公式 (10.2.9) 算出的数值为 -1483.53, 然而，负值的解约金不可能从退保人那里取得，故实际解约金为 0.

在这一节讨论的确定解约金的框架中，所使用的利率及生命表是重要的组成部分。在不没收价值法规的历史上，总的框架并不经常变动，而更新解约金的利率与死亡率基础的法定变化则较为频繁。1980 年法规提供的修订最高利率的公式部分地依赖于保单发行前一段时期的一个平均市场利率指标。

不没收价值既可以现金方式提供，也可以精算现值相当的保险受益方式提供，以下 §10.3 中将讨论这种选择权。另外，解约金还构成保单的另一重要条款 —— 保单低押条款的基础，保险人以保单的解约金为安全抵押，可提供保单持有人申请的额度不超过解约金的贷款。这种贷款的利率通常在保单上载明，不过从 70 年代开始，鉴于市场利率反复无常，保单贷款利率也有与市场利率相联系的倾向。

§10.3 保险选择权

以下给出不没收受益按精算现值相等的保险受益形式提供的三种主要形式。

一. 缴清保险

用平衡原理决定的按原保单提供但金额减少的保险作为解约金，这种形式的保险称为缴清保险。在时间 k 的缴清保险的受益金额 b_k 满足

$$\begin{gathered}{}_kCV = b_k A(k),\\ b_k = \frac{{}_kCV}{A(k)},\end{gathered} \tag{10.3.1}$$

其中 ${}_kCV$ 是在时间 k 的解约金，$A(k)$ 是单位保险的以后受益在时间 k 的净趸缴保费。

对于单位保险，在 ${}_kCV ={}_k V$(净均衡费责任准备金) 的特殊情形，符号 $W_k = b_k ={}_k V/A(k)$ 用来记缴清保险的金额。表 10.3.1 列出了 ${}_kW$ 与其它精算量之间的关系。

可用普通推理导出表 10.3.1 中的结果。1 单位终身寿险在

$x+k$ 岁时的净年缴保费 (完全离散基础) 为 P_{x+k}, 于是从 $x+k$ 岁开始的年缴费 P_x 只够提供 P_x/P_{x+k} 个单位的保险。由于 P_x 是实际每年缴付的净保费, 差 $1-P_x/P_{x+k}$ 就是在时间 k 的责任准备金所提供的未来保险的保额。这一推理也适用于其它保险。

表 10.3.1　减少后的缴清保险金额

特殊情形 $b_k = {}_kW = {}_kV/A(k)$	
完全连续基础	完全离散基础
终身寿险	
${}_k\overline{W}(\overline{A}_x) = \dfrac{\overline{A}_{x+k} - \overline{P}(\overline{A}_x)\overline{a}_{x+k}}{\overline{A}_{x+k}} = 1 - \dfrac{\overline{P}(\overline{A}_x)}{\overline{P}(\overline{A}_{x+k})}$	${}_kW_x = \dfrac{A_{x+k} - P_x\ddot{a}_{x+k}}{A_{x+k}} = 1 - \dfrac{P_x}{P_{x+k}}$
n 年缴费期终身寿险 $(k<n)$	
${}_k^n\overline{W}(\overline{A}_x) = \dfrac{\overline{A}_{x+k} - {}_n\overline{P}(\overline{A}_x)\overline{a}_{x-k:\overline{n-k}\rvert}}{\overline{A}_{x+k}} = 1 - \dfrac{{}_n\overline{P}(\overline{A}_x)}{{}_{n-k}\overline{P}(\overline{A}_{x+k})}$	${}_k^nW_x = \dfrac{A_{x+k} - {}_nP_x\ddot{a}_{x+k:\overline{n-k}\rvert}}{A_{x+k}} = 1 - \dfrac{{}_nP_x}{{}_{n-k}P_{x+k}}$
n 年期两全保险 $(k<n)$	
${}_k\overline{W}(\overline{A}_{x:\overline{n}\rvert}) = \dfrac{\overline{A}_{x+k:\overline{n-k}\rvert} - \overline{P}(\overline{A}_{x:\overline{n}\rvert})\overline{a}_{x+k:\overline{n-k}\rvert}}{\overline{A}_{x+k:\overline{n-k}\rvert}} = 1 - \dfrac{\overline{P}(\overline{A}_{x:\overline{n}\rvert})}{\overline{P}(\overline{A}_{x+k:\overline{n-k}\rvert})}$	${}_kW_{x:\overline{n}\rvert} = \dfrac{A_{x+k:\overline{n-k}\rvert} - P_{x:\overline{n}\rvert}\ddot{a}_{x+k:\overline{n-k}\rvert}}{A_{x+k:\overline{n-k}\rvert}} = 1 - \dfrac{P_{x:\overline{n}\rvert}}{P_{x+k:\overline{n-k}\rvert}}$

二. 展期定期保险

对于解约金方式为按原保单金额提供的缴清定期保险的情形, 用平衡原理决定缴清定期保险的期限 s. 与单位保险相联系的求解 s 的方程为

$$ {}_kCV = \overline{A}^{\,1}_{x+k:\overline{s}\rvert}, \tag{10.3.2}$$

在实践中, s 用线性插值决定。

在两全保险场合, 可能会出现缴清定期保险的期限 s 大于原保险剩余到期时间 $n-k$ 的情况, 此时缴清定期保险的期限为 $n-k$(到期时间与原两全保险相同) 。而解约金中未用于购买缴清

定期保险的多余部分则用于购买金额为

$$\frac{{}_kCV - \overline{A}^{\,1}_{x+k:\overline{n-k}|}}{A_{x+k:\overline{n-k}|}^{\quad\ 1}} \tag{10.3.3}$$

的纯生存保险。

如果金额为 b 的保单在解约时还欠有额度为 L 的保单贷款，那么展期定期保险通常提供的受益金额为 $b-L$. 若无此条款，则借款 L 的保单持有人就可通过解约使原本的死亡受益 $b-L$ 增加到 b. 在未偿还保单贷款的情形，(10.3.2) 修改为

$$b\,{}_kCV - L = (b-L)\overline{A}^{\,1}_{x+k:\overline{s}|}.$$

三、自动垫缴保费

有人并不把某些寿险保单中的自动垫缴保费贷款条款归作不没收受益。在保费拖欠发生时，该条款将以垫缴保费方式使保单继续有效，一直到解约金正好只够抵付垫缴保费所形成的贷款余额时为止，其中保费贷款余额因利息以及垫缴保费加入而在不断增加。设保费拖欠发生的时间为 k, 对于单位金额的完全连续保单，保费贷款期的最长时间 t 由以下方程决定：

$$G\bar{s}_{\overline{t}|i} = {}_{k+t}\,CV, \tag{10.3.4}$$

其中， G 是单位保险的 (年) 毛保费， ${}_{k+t}CV$ 是单位保险在时间 $k-t$ 时的解约金， i 是保单贷款利率。

在实际中， t 有时取为整数，满足

$$G\ddot{s}_{\overline{t}|i} \le {}_{k+t}CV$$

$$G\ddot{s}_{\overline{t+1}|i} > {}_{k-t+1}CV.$$

剩余的解约金 ${}_{k+t}CV - G\ddot{s}_{\overline{t}|i}$ 用于购买展期定期保险。

例 10.3.1: (x) 的单位金额完全连续终身人寿保险在 k 年末转为不没收受益。

(1) 设 ${}_kCV = {}_k\overline{V}(\overline{A}_x)$, 分别按①缴清保险与②展期定期保险, 给出刚改变后的保险的未来亏损方差与原保险在时间 k 的未来亏损方差之比。

(2) 如 $x=35, k=10$, 根据示例生命表以及利率 6%, 计算第 (1) 小题中所求①、②两种情形的比值。

解: (1) ①由 (5.2.4), 原保险在时间 k 时改变之前的方差为

$$(1+\frac{\overline{P}(\overline{A}_x)}{\delta})^2[{}^2\overline{A}_{x+k}-(\overline{A}_{x+k})^2]=\frac{{}^2\overline{A}_{x+k}-(\overline{A}_{x+k})^2}{(1-\overline{A}_x)^2}.$$

对于缴清保险, 回顾 §5.2 中 u 的定义, 亏损

$${}_k\overline{W}(\overline{A}_x)v^u-{}_k\overline{V}(\overline{A}_x)$$

的方差为

$$[{}_k\overline{W}(\overline{A}_x)]^2[{}^2\overline{A}_{x+k}-(\overline{A}_{x+k})^2],$$

所求的方差之比为

$$[{}_k\overline{W}(\overline{A}_x)]^2(1-\overline{A}_x)^2,$$

小于 1.

②在展期保险情形, 从 (10.3.2) 及 (2.2.5) 可得, 改变后的方差为

$${}^2\overline{A}^{\,1}_{x+k:\overline{s}|}-(\overline{A}^{\,1}_{x+k:\overline{s}|})^2.$$

于是所求的方差之比为

$$\frac{[{}^2\overline{A}^{\,1}_{x+k:\overline{s}|}-(\overline{A}^{\,1}_{x+k:\overline{s}|})^2](1-\overline{A}_x)^2}{{}^2\overline{A}_{x+k}-(\overline{A}_{x+k})^2}.$$

(2) ①按所给条件计算，

$$
\begin{aligned}
{}_{10}\overline{W}(\overline{A}_{35}) &= {}_{10}\overline{V}(\overline{A}_{35})/\overline{A}_{45} = 0.08604/0.20718 = 0.41529 \\
\overline{A}_{35} &= 0.13254,
\end{aligned}
$$

方差的比值为

$$[0.41529 \times (1 - 0.13254)]^2 = 0.13.$$

这就是说，缴清保险的亏损方差是原保险在时间 10 的亏损方差的 13%.

② 由

$$ {}_{10}\overline{V}(\overline{A}_{35}) = 0.08604 = \frac{(i/\delta)(M_{45} - M_{45+s})}{D_{45}}, $$

可知 s 在 19 与 20 之间。取 $s = 19$, 所求方差的比值为

$$\frac{{}^2\overline{A}^{1}_{45:\overline{19|}} - (\overline{A}^{1}_{45:\overline{19|}})^2}{{}^2\overline{A}_{45} - (\overline{A}_{45})^2}(1 - \overline{A}_{35})^2 = \frac{0.04308}{0.02922} \times 0.867646^2 = 1.11.$$

此时，方差上升为原保险的 111%.

§10.4 资产份额

人寿保险单是一个长期的契约，保险人的收入来自保费及投资收益，支出则由死亡与退保受益及费用开支组成。对一单位保险实际收取的毛保费会受到竞争的影响，不没收价值也会受竞争以及法规的影响。在精算现值意义上，需要对价格受益结构各因素平衡进行计算，这一节讨论的资产份额计算就用来满足这一需要。资产份额计算并不是过去结果的历史概括，而是一种前瞻性的计算，目的在于把握影响一组保单的期望财务状况的各主要因素。

对 (9.5.5) 推敲后，可得单位保险的以下等式：

$$
\begin{aligned}
&{}_{k+1}ASp_{x+k}^{(\tau)} = [{}_kAS + G(1-c_k) - e_k](1+i) \\
&-q_{x+k}^{(1)} - q_{x+k}^{(2)}{}_{k+1}CV \qquad k=0,1,2,3,\cdots \quad (10.4.1)
\end{aligned}
$$

其中：

${}_kAS$ 表示保单签发后 k 年、保单年度 $k+1$ 开始在即的期望资产份额；

G 表示毛保费；

c_k 表示时间 k 用于支付费用的毛保费比例；

e_k 表示时间 k 用于支付每份保单费用的金额；

$q_{x+k}^{(1)}$ 表示死亡引起的现龄 $x+k$ 岁被保险人在达到 $x+k+1$ 岁之前损失的概率；

$q_{x+k}^{(2)}$ 表示退保引起的现龄 $x+k$ 岁被保险人在达到 $x+k+1$ 岁之前损失的概率。

公式 (10.4.1) 基于完全离散支付基础，在死亡年末支付 1 单位受益，在退保年末支付解约金 ${}_{k+1}CV$.

公式 (10.4.1) 也可看作联系相继期末责任准备金的递归关系的一种推广，按各种不同方式改写后使人回想起责任准备金方程的类似处置。对 (10.4.1) 乘 $v^{k+1}l_{x+k}^{(\tau)}$，得

$$
\begin{aligned}
\triangle[l_{x+k}^{(\tau)}v^k{}_kAS] &= [G(1-c_k)-e_k]l_{x+k}^{(\tau)}v^k \\
&\quad -[d_{x+k}^{(1)} + d_{x+k}^{(2)}{}_{k+1}CV]v^{k+1}. \quad (10.4.2)
\end{aligned}
$$

于是

$$
\begin{aligned}
l_{x+k}^{(\tau)}v^k{}_kAS|_0^n &= \sum_{k=0}^{n-1}\{[G(1-c_k)-e_k]l_{x+k}^{(\tau)}v^k \\
&\quad -[d_{x+k}^{(1)} + d_{x+k}^{(2)}{}_{k+1}CV]v^{k-1}\}. \quad (10.4.3)
\end{aligned}
$$

如果 ${}_0AS=0$, 那么 ${}_nAS$ 等于

$$\frac{1}{l_{x+n}^{(\tau)}}\sum_{k=1}^{n-1}\{[G(1-c_k)-e_x]l_{x+k}^{(\tau)}(1+i)^{n-k}-$$
$$[d_{x+k}^{(1)}+d_{x+k}^{(2)}{}_{k+1}CV](1+i)^{n-k-1}\}. \tag{10.4.4}$$

在 (10.4.3) 中令 $n=\omega-x$ 并重新整理，有

$$G\ddot{a}_x^{(\tau)}=A_x^{(1)}+\sum_{k=0}^{\omega-x-1}[Gc_k+e_k]v^k{}_kp_x^{(\tau)}+\sum_{k=0}^{\omega-x-1}{}_kp_x^{(\tau)}q_{x+k}^{(2)}v^{k+1}{}_{k+1}CV. \tag{10.4.5}$$

上式可解释为应用平衡原理确定附加费用保费的一般公式，适当修改后可将这公式从终身寿险改为两全保险或定期保险情形。

将等式

$$p_{x+k}^{(\tau)}=1-q_{x+k}^{(1)}-q_{x+k}^{(2)}.$$

代入 (10.4.1), 可得

$$\begin{aligned}{}_{k+1}AS &= [{}_kAS+G(1-c_k)-e_k](1+i)-q_{x+k}^{(1)}(1-{}_{k+1}AS)\\ &\quad -q_{x+k}^{(2)}({}_{k+1}CV-{}_{k+1}AS).\end{aligned} \tag{10.4.6}$$

这一形式强调了差额 ${}_{k+1}CV-{}_{k+1}AS$ 在资产份额递进过程中的重要性。

资产份额计算可以视作追踪一组相似保单中每份残存保单期望的资产递进，对固定的毛保费、费用承担额及解约金的计算可用来核查价格－受益结构各种成份的平衡。计算的目的在于确定是否除最初的保单年度外， ${}_kAS\geq{}_kV$.

资产份额计算的另一个应用是决定 G. 为此，可设置一个资产份额目标，例如 $K>{}_{20}V$. 如果承担的费用及解约金已固定，那么 G 可利用 (10.4.4) 决定。设 H 是任意选取的一个毛保费尝试值， ${}_{20}AS_1$ 是用这个 H 以及当 $n=20$ 时由 (10.4.4) 得出的

结果，而设 K 是用所求保费 G 以及 $n=20$ 时由 (10.4.4) 得出的结果。于是

$$K -_{20} AS_1 = \frac{1}{l_{x+20}^{(\tau)}} \sum_{k=0}^{19} (G-H)(1-c_k) l_{x+k}^{(\tau)} (1+i)^{20-k}.$$

所求毛保费

$$G = H + \frac{(K -_{20} AS_1)_{20} p_x^{(\tau)} v^{20}}{\sum\limits_{k=0}^{19} (1-c_k)_k p_x^{(\tau)} v^k}. \tag{10.4.7}$$

式 (10.4.7) 右端第二项的影响在于校正尝试值 H, 以达到资产份额的目标值 K.

资产份额计算可能比这一节进行的更为精巧，例如，对受益在死亡即刻支付的情况，(10.4.1) 中项 $q_{x+k}^{(1)}$ 可乘以 i/δ. 对保费不是每年缴一次的情况，也可作相应调整。

例 10.4.1: 对 §9.1 中的示例，设 ${}_kCV = 1000{}_kV_{x:\overline{3}|} - 10$, $k=1,2,$, ${}_3CV = 1000$, 按以下双重损失表计算资产份额。

k	$p_{x+k}^{(\tau)}$	$q_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$
0	0.54	0.46	0.08	0.38
1	0.62	0.38	0.09	0.29
2	0.50	0.50	0.50	0.00

解: 用 (10.4.1) 作为模型，计算过程与结果如下表所示：

k	$\{[{}_kAS + G(1-c_k) - e_k](1+i) - 1000q_{x+k}^{(1)} -_{k+1} CV q_{x+k}^{(2)}\}$ $/p_{x+k}^{(\tau)} = {}_{k+1}AS$
0	$\{[0.00 + 332.35 \times 0.80 - 8] \times 1.15 - 80 - 247.41 \times 0.38\}$ $/0.54 = 226.94$
1	$\{[226.94 + 332.35 \times 0.94 - 2] \times 1.15 - 90 - 571.16 \times 0.29\}$ $/0.62 = 584.38$
2	$\{[584.38 + 332.35 \times 0.94 - 2] \times 1.15 - 500 - 1000.00 \times 0.00\}$ $/0.50 = 1058.01$

在这个例子中，最终的资产份额为 1058.01，大于最终的期末责任准备金 1,000 。

§10.5 经验调整

在一组保单发行之前用 (10.4.1) 计算的一系列资产份额，几乎可以肯定，不会正好等于每份残存保单的经验资产额。不过，§10.4 中的公式有助于洞察按期望结果衡量的财务赢亏的源泉。考虑 ${}_kAS$ 到 ${}_{k+1}\hat{AS}$ 的演变，后者带“帽子”(“∧”) 的 $k+1$ 年末资产份额系由 k 年末期望资产按经验成本要素计算得到，这里基于经验成本的要素符号带有“帽子”以示区别。设 $\hat{i}_{k+1}$ 是第 $k+1$ 个保单年度取得的经验利率，则经验资产份额由下式给出：

$$\begin{aligned}{}_{k+1}\hat{AS} \;=\;& [{}_kAS + G(1-\hat{c}_k) - \hat{e}_k](1+\hat{i}_{k+1}) - \hat{q}^{(1)}_{x+k}(1 - {}_{k+1}\hat{AS}) \\ & -\hat{q}^{(2)}_{x+k}({}_{k+1}CV - {}_{k+1}\hat{AS}). \qquad (10.5.1)\end{aligned}$$

从上式中减去追踪期望资产份额递进的式 (10.4.6)，得

$$\begin{aligned}{}_{k+1}\hat{AS} - {}_{k+1}AS =\;& ({}_kAS + G)(\hat{i}_{k+1} - i) & ① \\ & +[(Gc_k + e_k)(1+i) - G(\hat{c}_k + \hat{e}_k)(1+\hat{i}_{k+1})] & ② \\ & +[q^{(1)}_{x+k}(1 - {}_{k+1}AS) - \hat{q}^{(1)}_{x+k}(1 - {}_{k+1}AS)] & ③ \\ & +[q^{(2)}_{x+k}({}_{k+1}CV - {}_{k+1}AS) - \hat{q}^{(2)}_{x+k}({}_{k+1}\hat{CV} - {}_{k+1}\hat{AS})]. & ④\end{aligned}$$

$$(10.5.2)$$

在 (10.5.2) 中，经验资产份额与期望资产份额之差可分解成 4 个部分：第①部分涉及经验利率与设定利率之偏差；第②部分涉及经验费用与期望费用之差，并按利率调整；第③部分系设定的死亡成本与经验死亡成本之差；第④部分则是设定的退保成本与经验退保成本之差。

人寿保险的组织基于以下原则：对于发行的一批保单所承诺的受益支付及担负的费用开销，保费及其投资收入不足抵付的概

率应该非常小，并据此决定保费水平。按照这个原理（代替平衡原理)，亏损变量的期望值应为负值，以便产生一个安全保障额度，应付可能发生的不利偏差。由于不确定性随着时间的推移而消除，原价格 - 受益结构中为应付不利偏差而设置的保障额可随之释放并返还保单持有人，后者通过缴付高额保费承受了风险。这些不再需要的超出应付未来风险部分的余额称为*红利*(dividends)，式 (10.5.2) 的一个简化形式常被用于决定红利的分析。

我们从 (10.4.6) 的一种修正开始，用符号 ${}_kF$ 表示基金份额，代替资产份额 ${}_kAS$，基金份额 ${}_kF$ 不再是期望运行结果，而是事先设定的数额，以使所考虑的一组保单的未来保费与投资收入能以较高概率保证满足其受益责任与费用开支。于是

$$
\begin{aligned}
{}_{k+1}F = [{}_kF + G(1-c_k) - e_k](1+i) - q_{x+k}^{(1)}(1 - {}_{k+1}F) \\
- q_{x+k}^{(2)}({}_{k+1}CV - {}_{k+1}F),
\end{aligned}
\tag{10.5.3}
$$

这里，$c_k, e_k, q_{x+k}^{(1)}$ 与 $q_{x+k}^{(2)}$ 通常设置得比预期值高，i 的水平设置得比预期值低，以便提供填补不利偏差的保障额，这种安全保障使得动用外部基金的概率较低。

公式 (10.5.4) 描述了一单位保险的基金份额的递进关系，其中带有"帽子"的量系经验成本要素：

$$
\begin{aligned}
{}_{k+1}F + {}_{k+1}D \;=\; & [{}_kF + G(1-\hat{c}_k) - \hat{e}_k](1+\hat{i}_{k+1}) & (10.5.4)\\
& - \hat{q}_{x+k}^{(1)}(1 - {}_{k+1}F - {}_{k+1}D) & \\
& - \hat{q}_{x+k}^{(2)}({}_{k+1}CV - {}_{k+1}F - {}_{k+1}D). & (10.5.5)
\end{aligned}
$$

在 (10.5.4) 中，红利 ${}_{k+1}D$ 是经验基金份额与预先设定的基金份额目标 ${}_{k+1}F$ 之差。从 (10.5.4) 中减去 (10.5.3)，得

$$
\begin{aligned}
& {}_{k+1}D = ({}_kF + G)(\hat{i}_{k+1} - i) & ① \\
& + [(Gc_k + e_k)(1+i) - (G\hat{c}_k + \hat{e}_k)(1+\hat{i}_{k+1})] & ② \\
& + (1 - {}_{k+1}F)(q_{x+k}^{(1)} - \hat{q}_{x+k}^{(2)}) & ③
\end{aligned}
$$

$$+({}_{k+1}CV - {}_{k+1}F)(q^{(2)}_{x+k} - \hat{q}^{(2)}_{x+k}) \quad ④$$
$$+{}_{k+1}D(\hat{q}^{(1)}_{x+k} + \hat{q}^{(2)}_{x+k}). \quad ⑤$$

(10.5.5)

式 (10.5.5) 中各部分与以下经验要素相关：①涉及利息，②涉及费用，③涉及死亡情况，④涉及退保，⑤涉及 x 红利只支付给残存者。如果 ${}_{k+1}CV = {}_{k+1}F$，且红利也支付给死亡或退保的被保险人，并且将 $Gc_k + e_k$ 记为 E_k、$G\hat{c}_k + \hat{e}_k$ 记为 $\hat{E}_k$，那么 (10.5.5) 右端可写成三项形式：

$$\begin{aligned} {}_{k+1}D \;=\; & ({}_kF + G)(\hat{i}_{k+1} - i) \\ & +[E_k(1+i) - \hat{E}_k(1+\hat{i}_{k+1})] \\ & +(1 - {}_{k+1}F)(q^{(1)}_{x+k} - \hat{q}^{(1)}_{x+k}). \end{aligned} \tag{10.5.6}$$

§10.6 不同假设下的责任准备金

在前面章节中，作为模型中包含随机变量的后果，我们讨论了实际结果与期望结果的偏差。在上一节中，对资产份额的实际经验与期望经验之间的第二种类型差异及其偏差来源进行了分析以确定偏差的各来源成份。这个过程在从 (10.5.2), (10.5.5) 以及 (10.5.6) 中的稍许不同的模型里得到说明。这一节，将用联系相继责任准备金的递归方程，对不同精算假设下期望结果的第三种类型差异进行类似的分析，从而了解精算假设对责任准备金数额的影响。

以下理论的展开以 n 年期保险为模型：保额为 1，在死亡年末赔付；均衡保费 P 在 (x) 活着时每年年初缴付；无论按何种基础，责任准备金在 m 年末等于 W，$m \le n$(多数情形 m 可等于 n，但有时 $m = n$ 必须除外）。

设 P 是净均衡保费，$_hV$ 是在 h 年末的期末责任准备金，其计算根据的一组假设是：利率为 i 及死亡率为 q_h(系 q_{x+h} 简写)。另外，设 P' 与 $_hV'$ 分别是根据第二组假设算得的净均衡保费与责任准备金，依据是：利率为 i' 及死亡率为 q_h'. 由 (5.7.5), 对于保单年度 $h+1$, 有

$$({}_hV+P)(1+i)={}_{h+1}V+q_h(1-{}_{h+1}V), \tag{10.6.1}$$

$$({}_hV'+P')(1+i')={}_{h+1}V'+q'_h(1-{}_{h+1}V'). \tag{10.6.2}$$

置 $_hV'={}_hV+R_h$, 并将 (10.6.2) 整理成

$$\begin{aligned}({}_hV+P+R_h+P'-P)(1+i')={}&{}_{h+1}V+R_{h+1}\\&+q'_h(1-{}_{h+1}V-R_{h+1}),\end{aligned} \tag{10.6.3}$$

然后减去 (10.6.1), 将涉及函数 R_h 的项移至右端，可得

$$\begin{aligned}&({}_hV+P)(i'-i)+(P'-P)(1+i')\\&-(q'_h-q_h)(1-{}_{h+1}V)=p'_hR_{h+1}-R_h(1+i').\end{aligned} \tag{10.6.4}$$

将方程 (10.6.4) 的左端记为 S_h, 表达式 $[({}_hV+P)(i'-i)-(q'_h-q_h)(1-{}_{h+1}V)]$ 记为 c_h, 即 $S_h=(P'-P)(1+i')+c_h$, 其中 $(P'-P)(1+i')$ 与 h 无关，而 c_h 可能随 h 变化，称为临界函数(critical function)。如果 $i'>i, q'_h>q_h$, 那么临界函数等于期初责任准备金 $_hV+P$ 的超额利息减去风险净额 $1-{}_{h+1}V$ 的超额死亡成本。按这里引入的符号，(10.6.4) 成为

$$S_h=p'_hR_{h+1}-R_h(1+i'), \tag{10.6.5}$$

这是责任准备金差额函数 $R_h={}_hV'-{}_hV$ 的线性差分方程，且 $R_0={}_0V'-{}_0V=0$, $R_m=W-W=0$. 在 (10.6.5) 两端乘 D'_h(基于 i' 与 q'_h 的 D_{x+h}), 可得

$$D'_hS_h=(1+i')[D'_{h+1}R_{h+1}-D'_hR_h]. \tag{10.6.6}$$

于是

$$\sum_{h=0}^{k-1} D'_h S_h = (1+i')D'_k R_k,$$

$$R_k = \frac{v'}{D'_k}\sum_{h=0}^{k-1} D'_h S_h \tag{10.6.7}$$

$$R_m = 0 = \frac{v'}{D'_m}\sum_{h=0}^{m-1} D'_h S_h. \tag{10.6.8}$$

如果 S_h 是常数 S, 那么由 (10.6.8) 得 $S = 0$, 从而 $R_k = 0$, ${}_kV' = {}_kV, k = 0,1,\cdots,m$. 此时

$$\begin{aligned} S_h = 0 = &[({}_hV + P)(i' - i) - (q'_h - q_h)(1 - {}_{h+1}V)] \\ &+(P' - P)(1+i') = c_h + (P' - P)(1+i') \end{aligned} \tag{10.6.9}$$

称为两种假设下的责任准备金平衡方程(equation of equilibrium), 临界函数 c_h 为常数 $(P - P')(1+i')$, 即保费差按利率 i' 的一年积累值。换言之, 每年的超额利息减去超额死亡成本后与保费差平衡, 这种情况下责任准备金保持相同。

无论怎样, 方程 (10.6.7) 表明, 责任准备金差额 R_k 可表示成各年度 $S_h(h = 0,1,\cdots,k-1)$ 的精算积累值乘以 v'. 如果能获得 S_h 的有关信息 (如通过考察临界函数 c_h 获得), 那么就可能得出责任准备金差额的相关结论。

若 S_h 随 h 增加而递减 (记为 $S_h\downarrow$), 则 S_h 与 R_k 的图形 (将离散点画成连续线) 如图 10.6.1 所示。根据 (10.6.8), S_h 必定由正递减为负值, 当 S_h 为正时, R_k 递增; 而当 S_h 变负时, R_k 将递减 (如图 10.6.1) 。由于 $R_m = 0$, R_k 不可能为负值, 于是对 $k = 1,2,\cdots,m-1, R_k > 0$, 即 ${}_kV'$ 超过 ${}_kV$.

若 S_h 随 h 增加而递减 (记为 $S_h\uparrow$), 则 S_h 与 R_k 的图形如图 10 .6.2 所示。此时, S_h 必定由负递增为正值, 对 $k =$

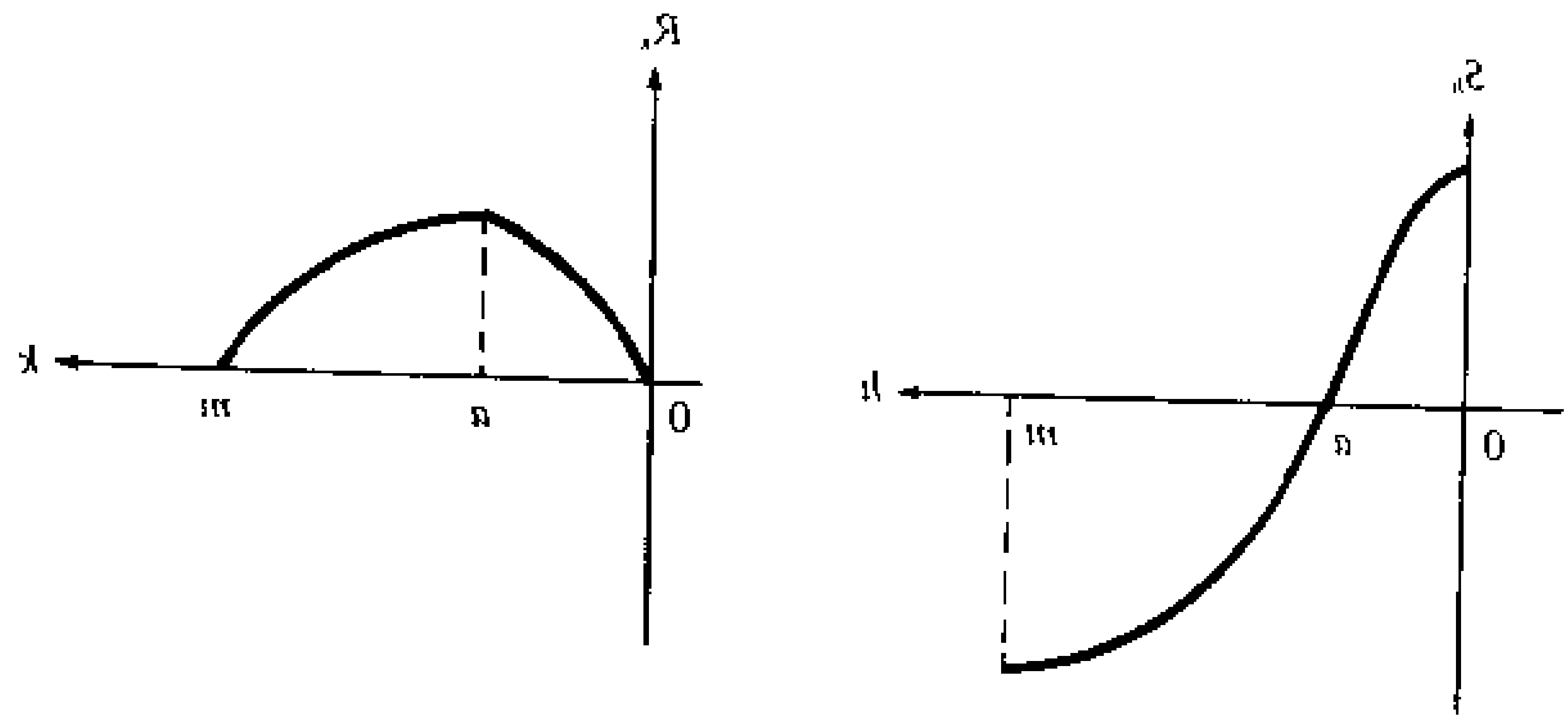


图 10.6.1　$S_h\downarrow$ 与 R_k 图形

$1, 2, \cdots, m-1, R_k<0$, 即 ${}_kV'$ 低于 ${}_kV$.

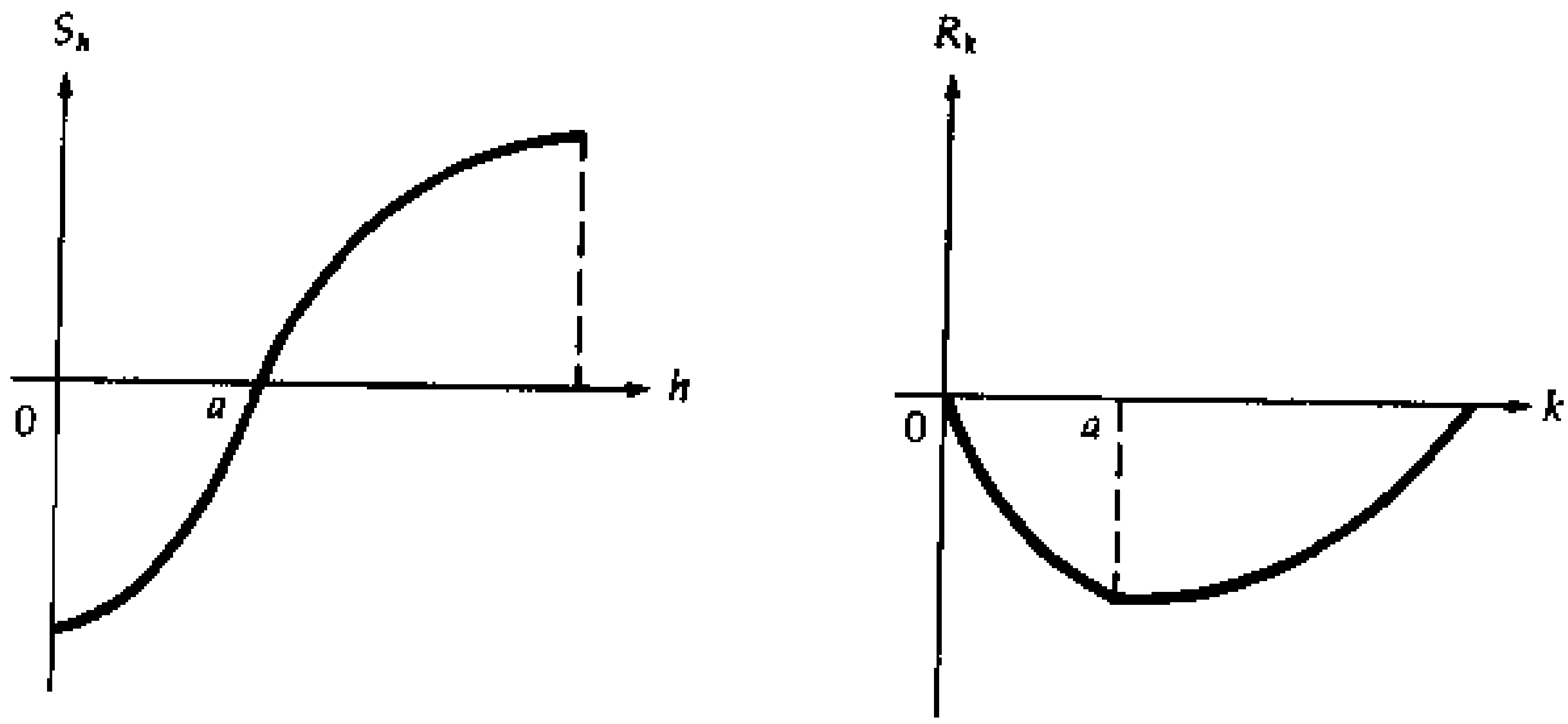


图 10.6.2　$S_h\uparrow$ 与 R_k 图形

为方便应用，条件 $S_h\downarrow$ 与 $S_h\uparrow$ 可分别改为等价的条件 $c_h\downarrow$ 与 $c_h\uparrow$. 前述结论概括如下：

定理 10.6.1: 对于保额为 1 死亡年末赔付的 n 年保险，设均衡年保费的缴费期为 n 年，m 年末的责任准备金为 $W(m \le n)$. P 与 ${}_hV$ 分别是以利率 i 及死亡率 q_h 为基础的净均衡保费与 h 年期末责任准备金，P' 与 ${}_hV'$ 则以利率 i' 及死亡率 q'_h 为基础。置临界函数

$$c_h = ({}_hV + P)(i' - i) - (q'_h - q_h)(1 - {}_{h+1}V) \qquad 0 \le h < m.$$

那么，当 c_h 随 h 增加而递减 (递增) 时，责任准备金 ${}_kV'$ 超过 (少于) 责任准备金 ${}_kV, 0 < k < m$. 当 c_h 是常数时，${}_kV'$ 等于 ${}_kV, 0 \le K \le m$.

如果两种假设中只是利率不同，例如 $i' = i + e, e > 0$, 则由于此时

$$c_h = ({}_hV + P)e,$$

当 ${}_hV \uparrow$ 时，$c_h \uparrow$. 于是有以下结果：

推论 10.6.1: 对于定理 10.6.1 中的保险 $(m = n)$, 如果责任准备金随时间递增，那么利率增加一个常数将导致时间 k 的责任准备金下降，$0 < k < n$.

如果两种假设中利率相同，但 $q'_h = q_h + e, e > 0$, 则

$$c_h = -e(1 - {}_{h+1}V).$$

当 ${}_hV \uparrow$ 时，$c_h \uparrow$. 于是应用定理 10.6.1 得

推论 10.6.2: 对于定理 10.6.1 中的保险 $(m = n)$, 如果责任准备金随时间递增，那么各死亡率增加一个常数将导致时间 k 的责任准备金下降，$0 < k < n$.

接下去考察，在 $i' = i$ 时，使得 ${}_kV' = {}_kV, 0 \le k \le m$ 成立的 q'_h 与 q_h 之间关系，此即

$$S_h = (P' - P)(1 + i) - (q'_h - q_h)(1 - {}_{h+1}V) = 0 \qquad 0 \le h < m. \tag{10.6.10}$$

在两全保险场合，由 ${}_{n-1}V' ={}_{n-1} V$，

$$ {}_{h-1}V' + P' = v ={}_{n-1} V + P, $$

从而 $P' = P$. 为避免这个平凡情形，可设 $m \le n-2$, 于是有

推论 10.6.3: 设 $i' = i$, 对于定理 10.6.1 中的保险 $(m \le n-2)$, ${}_kV' ={}_k V(0 \le k \le m)$ 的必要条件为

$$ q_h' = q_h + \frac{P' - P}{v(1 -{}_{h+1} V)} \qquad 0 \le h \le m. \tag{10.6.11} $$

反之，有

推论 10.6.4: 设 $i' = i$, 如果对于常数 $\triangle$,

$$ q_h' = q_h + \frac{\triangle}{v(1 - {}_{h+1}V)} \qquad 0 \le h < m, m \le n-2. \tag{10.6.12} $$

那么 $P + \triangle$ 是基于死亡率 q_h' 并使得开始 m 个保单年度的责任准备金等于 ${}_kV(0 \le k \le m)$ 的净年缴保费。

证: 将 (10.6.12) 写成以下形式

$$ q_h' = q_h + \frac{P + \Delta - P}{v(1 - {}_{n+1}v)} $$

整理后由 (5.7.6) 得

$$ P + \Delta - vq_h'(1 - {}_{n+1}V) = P - vq_h(1 - {}_{h+1}V) = v_{h+1}V - {}_hV $$

于是

$$ P + \Delta = vq_h'(1 - {}_{h+1}v) + v_{h+1}V - {}_hV \tag{10.6.13} $$

这就是说， $P + \Delta$ 是根据死亡率 q_h' 基础提供一年风险净额保险 (责任准备金保持为 ${}_kV$) 以及按储蓄基金建立责任准备金 ${}_kV$ 所需要的年存入额的均衡年支付额。在年度 $h+1$, 风险净额

$1-{}_{h+1}V$ 加上责任准备金 ${}_{h+1}V$ 正好提供完全的一单位保额，这样，$P+\Delta$ 提供定理 10.6.1 中保险的受益并保持责任准备金等于 ${}_kV, D\le k\le m$, 从而证明了推论 10.6.4 。当 $m\le\omega-x-2$ 时，推论 10.6.3 与 10.6.4 可应用于终身人寿保险。

在某些情况下，以上理论可用于比较两组假设下的净均衡保费，这可从以下例子中看到。

例 10.6.1: 设 $q'_h=q_h$, 但 $i'<i$. 对 n 年期两全保险，说明

$$P+d-d'>P'>P\frac{v'}{v}.$$

解: 此时

$$S_h=({}_hV+P)(i'-i)+(P'-P)(1+i'),$$

因 ${}_hV\uparrow, i'-i<0$, 故 $S_h\downarrow$, 从而 $S_0>0$. 由此得出

$$P(i'-i)+(P'-P)(1+i')>0,$$

$$P'(1+i')>P(1+i),$$

即

$$P'>P\frac{v'}{v}.$$

又从

$$P'=\frac{1}{\ddot{a}'_{x:\overline{n}|}}-d', P=\frac{1}{\ddot{a}_{x:\overline{n}|}}-d$$

以及 $\ddot{a}'_{x:\overline{n}|}>\ddot{a}_{x:\overline{n}|}$, 可直接得出

$$P+d-d'>P'.$$

习 题

§10.2

1. 根据条件：$A_x = 0.3208, \ddot{a}_x = 12, A_{x:\overline{n}|} = 0.5472, \ddot{a}_{x:\overline{n}|} = 8$，计算调整保费 $P^a_{x:\overline{n}|}$(按 1941 年报告)。

2. 设 $P_x > 0.04, i = 0.06$，按 1941 年报告与 1981 年法规，分别用 P_x 表示 P^a_x。

§10.3

3. 某公司打算采用新的生命表，指出如何决定投保年龄与从保单签发时算起的时间，使得终身寿险保单的减额缴清保险不没收受益增加以及减少。根据的只是以新老死亡表为基础的终身寿险保单的净年缴保费，并且假定，解约金始终是净保费责任准备金的同一百分比。

4. 考虑 (x) 投保的缴费期为 n 的 n 年期两全保险，保险金为一单位，支付基础为完全离散的。在拖欠保费的情况下，被保险人可选择：

(1) 减额缴清终身寿险，或

(2) 期限不超过原两全保险的展期定期保险以及 $x+n$ 岁时支付的减额生存保险。在时间 t 的解约金为 ${}_tV_{x:\overline{n}|}$，它可用来购买金额为 b 的缴清终身寿险，或用于购买金额为 1 的展期定期保险以及 $x+n$ 岁时的生存支付 f. 设 $A_{x+t:\overline{n-t}|} = 2A_{x+t}$，用 b, $A^{1}_{x+t:\overline{n-t}|}$ 及 ${}_{n-t}E_{x+t}$ 表示 f.

5. 向 (30) 发行的 1 单位完全连续 20 年期两全保险，在 10 年末中止，并且那时还有一笔以 ${}_{10}CV$ 为抵押的贷款额 L 尚未清偿。用净趸缴保费表达：

(1) 在保额为 $1-L$ 的展期定期保险可展延到原期满时的情况下，期满时的生存支付金额 E。

(2) 转为第 (1) 小题中展期定期保险与生存保险后 5 年时的

责任准备金。

6. 有人曾提出，减额缴清保险的金额应与已缴保费的年数成比例。用附录的示例生命表与 6% 利率比较 ${}^{20}_{10}W_{40}$ 和 ${}_{10}W_{40:\overline{20}|}$ 与按以上建议确定的缴清保险金额。

7. 证明

$$\frac{d}{dt}[{}_t\overline{W}(\overline{A}_{x:\overline{n}|})] = \frac{\overline{P}(\overline{A}_{x:\overline{n}|}) - \mu_{x+t}[1 -{}_t \overline{W}(\overline{A}_{x:\overline{n}|})]}{\overline{A}_{x+t:\overline{n-t}|}},$$

并解释这个方程。[提示:用 (5.10.3) 写出 ${}_t\overline{V}(\overline{A}_{x:\overline{n}|})$ 与 $\overline{A}_{x+t}:\overline{n-t|}$ 的导数。]

8. 在人寿保险的早期，一家保险公司的解约金定为

$${}_kCV = h(G_{x+h} - G_x)\ddot{a}(k) \qquad k = 1,2,\cdots,$$

其中 G 表示相应年龄的毛保费， $\ddot{a}(k)$ 表示始于 $x+k$ 岁并到缴费期结束为止的期初生存年金值， h 在实践中取为 2/3. 如果终身寿险保单的毛保费按 1980 年法规取为调整保费，并且 P_x 与 P_{x+k} 都小于 0.04, $h = 0.9$, 验证以上给出的解约金

$${}_kCV = (0.909 + 1.125P_x)_kV_x + 1.125(P_{x+k} - P_x).$$

9. 设 ${}_k\hat{W} ={}_k CV/A(k)$, 其中 ${}_kCV = A(k) - P^a\ddot{a}(k)$, P^a 是调整保费 [参见 (10.2.2)]. 对表 10.3.1 中的三种保单，按完全离散模型建立 ${}_k\hat{W}$ 与调整保费及净均衡保费的联系。

10. 设 ${}_kW^{Mod} ={}_k V^{Mod}/A(k)$, 其中 ${}_kV^{Mod}$ 是在第 k 个保单年度末的保险监督官标准责任准备金。对表 10.3.1 中的 3 种保单，按完全离散模型建立 ${}_kW^{Mod}$ 与续年保费及净均衡保费的联系。这里假定，在限定缴费期的计划中，缴费期少于 20 年。

11. 设 ${}_{k+1}CV ={}_{k+t} \overline{V}(\overline{A}_x)$.

(1) 证明：决定自动垫缴保费贷款期长度的方程 (10.3.4) 可写成 $H(t) = 0$, 其中 $H(t) = \overline{a}_xG\overline{s}_{\overline{t}|i} + \overline{a}_{x+k+t} - \overline{a}_x$ 。

(2) 验证：对于死亡效力递增的生存函数， $H(0)<0$, 且当 $t\to\infty$ 时， $H(t)$ 变成正的而又无界。

(3) 计算 $H'(t)$.

12. 与 (10.4.6) 相联系，设 ${}_{10}AS_1$ 是基于 G_1 的 10 年末资产份额，而 ${}_{10}AS_2$ 则基于 G_2. 写出 ${}_{10}AS_2-{}_{10}AS_1$ 的公式。

§10.5

13. 设 $\hat{G}$ 是根据实际死亡与费用假定确定的经验保费：

$$\hat{G}=v_{k+1}F-{}_kF+\hat{g}+v\hat{q}_{x+k}^{(1)}(1-{}_{k+1}F),$$

其中 $\hat{g}=G\hat{c}_k+\hat{e}_k, k=0,1,2,\cdots$. 另外，设 ${}_{k+1}CV={}_{k+1}F, k=0,1,2,\cdots$. 在这些假设下证明：

(1) ${}_{k+1}F=({}_kF+\hat{G}-\hat{g})(1+i)-\hat{q}_{x+k}^{(1)}(1-{}_{k+1}F)$.

(2) 如果红利也向死亡或退保的被保险人支付，那么 (10.5.4) 可写成

$${}_{k+1}F+{}_{k+1}D=[{}_kF-G-\hat{g}](1+\hat{i}_{k+1})-\hat{q}_{x+k}^{(1)}(1-{}_{k+1}F),$$

$${}_{k+1}D=(G-\hat{G})(1+\hat{i}_{k+1})+({}_kF+\hat{G}-\hat{g})(\hat{i}_{k+1}-i).$$

这个习题给出了红利计算的*经验保费方法*(experience premium method) 梗概。

14. 相继的生存年金递归关系为

$$(\ddot{a}_{x+h}-1)(1+i)=p_{x+h}\ddot{a}_{x+h+1}\qquad h=0,1,2,\cdots.$$

(1) 如实际的经验利率是 $\hat{i}_{h+1}$, 经验生存概率是 $\hat{p}_{x+h}$, 则基金的递归关系为

$$(\ddot{a}_{x+h}-1)(1+\hat{i}_{h+1})=\hat{p}_{x+h}(\ddot{a}_{x+h+1}+\triangle_{h+1}),$$

其中 $\triangle_{h+1}$ 是生存者份额的变化。证明

$$\triangle_{h+1}=\frac{(\hat{i}_{h+1}-i)(\ddot{a}_{x+h}-1)+(p_{x+h}-\hat{p}_{x+h})\ddot{a}_{x+h+1}}{\hat{p}_{x+h}},$$

并解释这个结果。

(2) 如年末的年金收入调整为年初的 r_{h+1} 倍，其中

$$(\ddot{a}_{x+h}-1)(1+\hat{i}_{h+1})=\hat{p}_{x+h}r_{h+1}\ddot{a}_{x+h+1},$$

用 i, $\hat{i}$, p_{x+h} 及 $\hat{p}_{x+h}$ 表示 r_{h+1}.

§10.6

15. 决定定理 10.6.1($m=n$) 中所述保险对责任准备金的影响，这里假定：责任准备金随着时间增加，p_x 随年龄递减，每个 p_x 都乘以 $(1+k), k>0$.

16. 对 §5.3 中的一般保险 (保单年度 $h+1$ 的年末死亡受益为 b_{h+1}, 年初净个缴保费为 $\pi_h, h=0,1,2,\cdots$), 证明 (10.6.4) 中的函数 S_h 成为

$$(\pi_h'-\pi_h)(1+i')+({}_hV+\pi_h)(i'-i)-(q_h'-q_h)(b_{h+1}-{}_{h+1}V).$$

17. (1) 对死亡年末赔付单位受益的终身人寿保险，证明 (10.6.12) 成为

$$q_y'=q_y+\frac{c}{v\ddot{a}_{y+1}}\qquad x\le y<x+m,$$

其中 $y=x+h, c=\ddot{a}_x\triangle$.

(2) 根据第 (1) 题中的死亡率 q_y' 以及利率 i, 求 x 岁到 $x+m$ 岁之间缴付的净年保费，使得与基于死亡率 q_y 及利率 i 且净年缴保费为 P_x 的终身寿险具有相同死亡受益，并且保持相同的责任准备金 ${}_kV_x, 0\le k\le m$.

第十一章　特殊年金与保险

§11.1　引　言

这一章考察各种提供特殊年金与保险受益的保单，决定相应的精算现值、净保费及毛保费、净保费责任准备金。

在 §11.2 中，我们将考察支付期可能长于年金领取者剩余寿命或者含有死亡受益的若干年金契约，这些合约产生于寿险保单的赔付选择权，也可能产生于退休金计划或个人年金保单。§11.3 包括了有密切联系的家庭收入保单。§11.4 考虑了一类在到期之前提供死亡受益的保单，其受益额为面额与责任准备金两者之中高的一项。受益水平与责任准备金依赖于投资结果的变额保险产品是 §16.5 的主题，当物价膨胀吞噬了用确定的货币单位表示的受益时，这些产品显出其重要性。在 §16.6 中，我们考察了在变更受益额及保费水平方面具有广泛可变性的新型保单。最后，§16.7 描述了各种形式的伤残保险，并讨论了在残疾保险中计算净保费及其责任准备金时常用的单重损失模型近似。

§11.2　特殊形式年金受益

这一节我们将注意力集中于计算特殊形式年金受益的精算现值。两种支付方式取决于已收毛保费，我们将确定这些情形的毛保费。以下将着眼于连续支付年金，并通过类比得出年付 m 次年金的相应结果。

以连续支付的年金为例，先分析所谓的 n 年确定和生命年金(n-year certain and life annuity)，这种年金保证支付 n 年。随后，如年金领取者仍活着，则继续支付直至死亡。其现值随机变

量 (年支付额 1) 为

$$Z = \begin{cases} \overline{a}_{\overline{n}|} & T \le n \\ \overline{a}_{\overline{T}|} & T > n, \end{cases} \tag{11.2.1}$$

精算现值为

$$E[Z] = \int_0^n a_{n\,t}p_x\mu_{x+t}dt + \int_n^\infty \overline{a}_{\overline{t}|\,t}p_x\mu_{x+t}dt. \tag{11.2.2}$$

用当期支付技巧可写出精算现值

$$\overline{a}_{\overline{n}|} \quad + \quad \int_n^\infty v^t{}_tp_xdt = \overline{a}_{\overline{n}|} + {}_{n|}\overline{a}_x \tag{11.2.3}$$

$$= \quad \overline{a}_{\overline{n}|} + \overline{a}_x - \overline{a}_{x:\overline{n}|}. \tag{11.2.4}$$

按第六章的观点，上述年金可看作最后生存年金 $\overline{a}_{\overline{x:\overline{n}|}}$. 由 (11.2.3) 或 (11.2.4) 可得

$$\overline{a}_{\overline{x:\overline{n}|}} = \overline{a}_{\overline{n}|} + {}_nE_x\overline{a}_{x+n}. \tag{11.2.5}$$

对于年付 m 次的离散年金，亦有

$$a^{(m)}_{\overline{x:\overline{n}|}} \quad = \quad a^{(m)}_{\overline{n}|} + a^{(m)}_x - a^{(m)}_{x:\overline{n}|} \tag{11.2.6}$$

$$= \quad a^{(m)}_{\overline{n}|} + {}_nE_xa^{(m)}_{x+n}. \tag{11.2.7}$$

上述年金的一种特殊形式为 分期退款年金(installment refund annuity), 它至少保证年金领取者 (或受益人) 可领回已缴的毛保费 (不计息) 。设趸缴毛保费为 G , 附加费为毛保费的 r 倍, 年金的年支付额 (率) 为 1, 则 G 应满足

$$G(1-r) = \overline{a}_{\overline{G}|} + {}_GE_x\overline{a}_{x+G}. \tag{11.2.8}$$

用整数值 G 尝试并进行插值可解得毛保费 G 的近似值，相应的年金是 G 年确定和生命年金 (G 可能是非整数) 。

一种包含保险受益的相关年金称为现金退款年金(cash refund annuity)，当年金领取者死亡时已领取的年金 (不计息) 若低于毛保费，则该年金立即将差额支付给受益人。设年金的年支付额为 1(连续支付)，G 是毛保费 (趸缴)，T 是死亡时间，那么现值随机变量为

$$Z=\begin{cases}\bar{a}_{\overline{T}|}+(G-T)v^T & T\leq G\\ \bar{a}_{\overline{T}|} & T>G,\end{cases} \tag{11.2.9}$$

精算现值为

$$\begin{aligned}E[Z] &= \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu_{x+t}dt+\int_0^G (G-t)v^t\,{}_tp_x\mu_{x+t}dt\\ &= \bar{a}_x+G\bar{A}^{\,1}_{x:\overline{G}|}-(\bar{I}\bar{A})^{\,1}_{x:\overline{G}|}.\end{aligned} \tag{11.2.10}$$

对分期退款年金来说，在附加费是毛保费 G 的 r 倍时，毛保费 G 由下式决定：

$$G(1-r)=\bar{a}_x+G\bar{A}^{\,1}_{x:\overline{G}|}-(\bar{I}\bar{A})^{\,1}_{x:\overline{G}|}. \tag{11.2.11}$$

当 (11.2.11) 的左端与右端之差对整数值 G 计算后，可用线性插值的方法逼近 G 。

例 11.2.1: 对于在年金领取者 (x) 死亡后继续支付 n 年的连续年金，计算精算现值。

解：设 T 是 (x) 的死亡时间，受益现值为

$$Z=\bar{a}_{\overline{T+n}|},$$

所求年金的精算现值为

$$\int_0^\infty \bar{a}_{\overline{t+n}|}\,{}_tp_x\mu_{x+t}dt.$$

分部积分后得

$$\bar{a}_{\overline{n}|}+\int_0^\infty v^{t+n}\,{}_tp_xdt=\bar{a}_{\overline{n}|}+v^n\bar{a}_x.$$

作变量代换 $t+n=s$, 上述精算现值可写成

$$\overline{a}_{\overline{n}|}+\int_n^\infty v^s\,{}_{s-n}p_x ds.$$

这个当期支付形式确认了在时间 n 之前支付是确定的，在时间 n 之后只有当 (x) 在 n 年前活着时才有支付。

§11.3 家庭收入保险

n 年家庭收入保险(n-year family income insurance) 当被保险人在 n 年内死亡时开始提供年金收入支付，直至 (从签单开始计算的) 第 n 年。这种收入受益的现值为

$$Z=\begin{cases} v^T\overline{a}_{\overline{n-T}|} & T\le n \\ 0 & T>n. \end{cases} \tag{11.3.1}$$

在抵押保障保单(mortgage protection policy) 中，年金值 $\overline{a}_{\overline{n-T}|}$ 代表未清偿抵押贷款余额，其中使用的抵押贷款利率 j 可不同于计算保险值的因子 v^T 中使用的利率 i. 抵押保障保单的现值随机变量为

$$Z=\begin{cases} v^T\overline{a}_{\overline{n-T}|j} & T\le n \\ 0 & T>n. \end{cases}$$

家庭收入受益的精算现值为

$$E[Z] = \int_0^n v^t\overline{a}_{\overline{n-t}|}\,{}_tp_x\mu_{x+t}dt \tag{11.3.2}$$

$$= \overline{a}_{\overline{n}|}-\int_0^n v^t\,{}_tp_x dt=\overline{a}_{\overline{n}|}-\overline{a}_{x:\overline{n}|}$$

$$= \int_0^n v^t(1-{}_tp_x)dt. \tag{11.3.3}$$

其解释如下：在满足 $t<n$ 的时刻 t, 只有当 (x) 已死亡时才有支付，其概率为 $1-{}_tp_x$ 。

年付 m 次的离散家庭收入保单有两种形式。其一，年金支付从被保险人死亡所在的 $\frac{1}{m}$ 年末开始，这里支付时间的衡量都是从保单签发日开始的。这种保单的精算现值为

$$a_{\overline{n}|}^{(m)} - a_{x:\overline{n}|}^{(m)}. \tag{11.3.4}$$

另一种受益支付从被保险人死亡时立即开始，每隔 $\frac{1}{m}$ 年付一次，在时间 n(从保单签发日开始计算) 前的最后一次为零数调整支付。设死亡时间为 t, $n-t=k-j/m+s$, $0 \le s < 1/m$, 在时间 $n-s$ 的最后零数支付额为

$$\frac{\delta}{d^{(m)}}\overline{a}_{\overline{s}|} = \frac{1-v^s}{d^{(m)}},$$

于是年金支付在时间 t 的现值为

$$\ddot{a}_{\overline{k+j/m}|}^{(m)} + v^{k+j/m}\frac{1-v^s}{d^{(m)}} = \frac{1-v^{n-t}}{d^{(m)}} = \ddot{a}_{\overline{n-t}|}^{(m)}.$$

据此可导出精算现值为

$$\begin{aligned}\int_0^n v^t \ddot{a}_{\overline{n-t}|}^{(m)} {}_tp_x \mu_{x+t} dt &= \frac{\delta}{d^{(m)}}\int_0^n v^t \overline{a}_{\overline{n-t}|} {}_tp_x \mu_{x+t} dt \\ &= \frac{\delta}{d^{(m)}}(\overline{a}_{\overline{n}|} - \overline{a}_{x:\overline{n}|}). \qquad (11.3.5)\end{aligned}$$

譬如当 $n=30$, $m=12$, $t=18.8$ 时， $k=11$, $j=2$,

$$s = n-t-k-\frac{j}{m} = \frac{1}{30},$$

$$\frac{1-v^s}{d^{(12)}} = 0.033382,$$

其中设利率 $i=0.06$ ，以上两值相差约 0.00005 。

例 11.3.1: 某种向 40 岁人签发的保单提供年支付为 1 的连续年金如下：当被保险人在 65 岁之前死亡时，提供家庭收入受

益至 65 岁 (假若未死本该达到 65 岁之时)，且至少支付 10 年；当被保险人在 65 岁还活着的话，提供至少确定支付 10 年的生存年金 (即 10 年确定和生命年金)。求精算现值。

解：用当期支付技巧，考虑在时间 t 的支付条件及相应的概率，列表如下：

时间	支付条件	概率
$0 < t \le 25$	(40) 已死	$1 - {}_tp_{40}$
$25 < t \le 35$	(40) 在 $t-10$ 时活着	${}_{t-10}p_{40}$
$t > 35$	(40) 还活着	${}_tp_{40}$

精算现值为

$$\int_0^{25} v^t(1 - {}_tp_{40})dt + \int_{25}^{35} v^t\,{}_{t-10}p_{40}dt + \int_{35}^{\infty} v^t\,{}_tp_{40}dt$$
$$= \overline{a}_{\overline{25}|} - \overline{a}_{40:\overline{25}|} + v^{10}(\overline{a}_{40:\overline{25}|} - \overline{a}_{40:\overline{15}|}) + \overline{a}_{40} - \overline{a}_{40:\overline{35}|}.$$

§11.4 退休收入保单

退休收入保单(retirement income policy) 是一种两全保单，其特征是到期额高于面额，到期额用于提供定额年金收入。因为责任准备金趋于到期额，从某一时间起，责任准备金将超过面额。当责任准备金超过面额时，死亡受益将等于责任准备金。

以下用完全连续模型进行分析。设 $1+k$ 是在时间 n 单位面额保单的到期额，$\overline{P}$ 是年保费 (年率)。如 a 是责任准备金等于 1 的时间，则受益额 b_t 为

$$b_t = \begin{cases} 1 & t \le a \\ {}_t\overline{V} & a < t \le n. \end{cases}$$

用后顾公式表达在时间 a 的责任准备金，可得

$$1 = {}_a\overline{V} = \overline{P}\overline{s}_{x:\overline{a}|} - {}_a\overline{k}_x = \frac{\overline{P}\overline{a}_{x:\overline{a}|} - \overline{A}^{1}_{x:\overline{a}|}}{{}_aE_x}. \tag{11.4.1}$$

另外，(5.10.3) 给出的责任准备金微分方程为

$$\frac{d}{dt}{}_t\overline{V} = \overline{P} + \delta_t\overline{V} - \mu_{x+t}(b_t - {}_t\overline{V}).$$

当 $t \geq a$ 时，$b_t = {}_t\overline{V}$, 涉及 μ_{x+t} 的项消失，解此微分方程 (或根据复利理论) 可导出

$$v^{t-a}{}_t\overline{V} = {}_a\overline{V} + \overline{P}\overline{a}_{\overline{t-a}|},$$

按 ${}_a\overline{V} = 1$、${}_n\overline{V} = 1 + k$ 得出

$$1 = (1+k)v^{n-a} - \overline{P}\overline{a}_{\overline{n-a}|}. \tag{11.4.2}$$

结合 (11.4.1) 与 (11.4.2), 有

$$\overline{P} = \frac{\overline{A}^{1}_{x:\overline{a}|} + {}_aE_x v^{n-a}(1+k)}{\overline{a}_{x:\overline{a}|} + {}_aE_x\overline{a}_{\overline{n-a}|}}. \tag{11.4.3}$$

从方程 (11.4.1) 可解出

$$\overline{P} = \frac{\overline{A}^{1}_{x:\overline{a}|} + {}_aE_x}{\overline{a}_{x:\overline{a}|}} = \overline{P}(\overline{A}_{x:\overline{a}|}),$$

于是

$$\overline{P} = \frac{1}{\overline{a}_{x:\overline{a}|}} - \delta. \tag{11.4.4}$$

类似地，从 (11.4.2) 可解出

$$\begin{aligned}\overline{P} &= \frac{(1+k)v^{n-a}}{\overline{a}_{\overline{n-a}|}} - \frac{1}{\overline{a}_{\overline{n-a}|}} = \frac{k}{\overline{s}_{\overline{n-a}|}} - \left(\frac{1}{\overline{a}_{\overline{n-a}|}} - \frac{1}{\overline{s}_{\overline{n-a}|}}\right)\\ &= \frac{k}{\overline{s}_{\overline{n-a}|}} - \delta.\end{aligned}$$

与 (11.4.4) 比较可得出 a 必须满足的条件

$$\frac{\bar{s}_{\overline{n-a}|}}{\bar{a}_{x:\overline{a}|}} = k. \tag{11.4.5}$$

据此可计算出 a, 随后根据 (11.4.4) 得出 $\overline{P}$.

至于责任准备金公式，当 $t<a$ 时用后顾方法，而 $t \geq a$ 时用前瞻方法最为方便：

$$ {}_t\overline{V} = \begin{cases} \overline{P}\bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x & t<a \\ (1+k)v^{n-t} - \overline{P}\bar{a}_{\overline{n-t}|} & a \leq t \leq n. \end{cases} \tag{11.4.6}$$

在 $t \geq a$ 时，也可用公式

$$ {}_t\overline{V} = (1+i)^{t-a} + \overline{P}\bar{s}_{\overline{t-a}|}.$$

完全离散的模型与此相似，只不过 a 是使得 ${}_aV \leq 1$ 且 ${}_{a+1}V > 1$ 的整数罢了。

§11.5 变额保险产品

这一节考察受益额与责任准备金依赖于保费投资结果的若干种保单，投资对象 (主要是股票) 一般在保单中载明。这些产品的最初目的在于通过将保费投资于股票获取较高的期望收益率，从而提供某种免受通货膨胀打击的途径。通常，这些保单在死亡率及附加费用方面按事先确立的基础计算，保单持有人既不会因这两方面的不利经验而被要求追加付费，也不会因这两方面的有利经验而额外得益，受益金额的变动只来源于保费投资收益率 (即利率) 的变化。

一、变额年金

首先考虑变额年金(variable annuity)。在缴费积累期内，一项基金由一次性或周期性存入方式进行累积，其利率决定于基金

的投资表现。在积累期内的死亡受益通常等于基金份额，退保受益 (解约价值) 通常是死亡受益额减去一笔解约费用。如不考虑退保，在缴费积累期内基金份额的增长由下式给出：

$$[F_k + \pi_k(1 - c_k) - e_k](1 + i'_{k+1}) = F_{k+1} + q_{x+k}(b_{k+1} - F_{k+1}) \tag{11.5.1}$$

这里 F_k 是在时间 k 的基金份额， π_k 是在时间 k 的毛保费， c_k 是保费 π_k 中用于费用开支的百分比， e_k 是不与保费成比例的费用， b_{k+1} 是从时间 k 至时间 $k+1$ 死亡的在时间 $k+1$ 的受益额， i'_{k+1} 是从时间 k 至 $k+1$ 这一年的实际净投资收益率 (扣除了投资费用) 。在积累期内，式 (11.5.1) 右端的第二项等于 0, 由此可见，基金恰好按利息 (即投资收益率) 累积。

在退休时，既有的基金份额用于购买缴清年金，后者的计算按预定的死亡率基础及*假设的投资收益率*(assumed inverstment return, AIR) 作出。如果假设的投资收益率 AIR 较低，那么初始的年金支付相对基金份额来说也比较低，但我们将会看到，年金的支付额具有某种增加的型式，以抵消部分通货膨胀影响。将 AIR 记为 i, 在时间 k 之后一年的实际投资收益率记为 i'_{k+1}, 如在时间 k 向活着的年金领取者此时及此后支付的年受益额设为 b_k, 则在时间 k 的责任准备金就是 $b_k\ddot{a}_{x+k}$, 这里 x 是退休时年龄， k 是从退休开始计算的时间。责任准备金演变的方程为

$$(b_k\ddot{a}_{x+k} - b_k)(1 + i'_{k+1}) = b_{k+1}p_{x+k}\ddot{a}_{x+k+1}. \tag{11.5.2}$$

但由 (3.8.4) 可知

$$(\ddot{a}_{x+k} - 1)(1 + i) = p_{x+k}\ddot{a}_{x+k+1}. \tag{11.5.3}$$

两式相除得出

$$b_{k+1} = b_k\frac{1 + i'_{k+1}}{1 + i}. \tag{11.5.4}$$

于是当 $i'_{k+1} > i$ 时，受益水平将提高，而较高的 AIR 可能导致受益额常常降低。

以上由 (11.5.4) 给出的结果对其它支付选择权也成立，对 n 年确定与生命年金可参见习题 15 。对年付 m 次场合也以稍微修改的形式出现。首先考虑按月调整支付额的情形，联系一年中第 1 与第 2 个月的年金值公式为

$$(\ddot{a}^{(12)}_{x+k} - \frac{1}{12})(1 + \frac{i^{(12)}}{12}) = {}_{\frac{1}{12}}\, p_{x+k}\ddot{a}^{(12)}_{x+k+\frac{1}{12}},$$

而基金份额的演变可表示成

$$(b_k\ddot{a}^{(12)}_{x+k} - \frac{b_k}{12})(1 + \frac{i'^{(12)}_{k+1}}{12}) = {}_{\frac{1}{12}}\, p_{x+k}b_{k+\frac{1}{12}}\ddot{a}^{(12)}_{x+k+\frac{1}{12}}.$$

相除得出

$$b_{k+\frac{1}{12}} = \frac{b_k(1 + i'^{(12)}_{k+1}/12)}{1 + i^{(12)}/12}. \qquad (11.5.5)$$

对年付 m 次但受益水平按年调整的情形，也成立同样结果。譬如在按月支付场合，由

$$\begin{aligned}
&(\ddot{a}^{(12)}_{x+k} - \ddot{a}^{(12)}_{x+k:\overline{1}|})(1 + i) = p_{x+k}\ddot{a}^{(12)}_{x+k+1}\\
&(b_k\ddot{a}^{(12)}_{x+k} - b_k\ddot{a}^{(12)}_{x+k:\overline{1}|})(1 + i'_{k+1}) = p_{x+k}b_{k+1}\ddot{a}^{(12)}_{x+k+1}
\end{aligned}$$

可得出与 (11.5.4) 相同的结果：

$$b_{k+1} = b_k\frac{1 + i'_{k+1}}{1 + i}.$$

变额寿险(variable life insurance) 的各种可能设计非常众多，以下讨论 3 种不同的设计，并以半连续终身寿险为例，受益额在每年初改变。

二. 完全变额人寿保险

第一种设计称为完全变额寿险(fully variable life insurance)，其受益金额根据投资结果变化且保费也同比变化。设 b_k 是在时间 k 之后一年内的受益额，则在时间 k 的应缴保费为 $b_k P(\overline{A}_x)$，一年的保险成本为 $b_k \overline{A}^{\,1}_{x+k:\overline{1|}}$。于是

$$[b_{k\,k}V(\overline{A}_x)+b_k P(\overline{A}_x)-b_k\overline{A}^{\,1}_{x+k:\overline{1|}}](1+i'_{k+1}) = p_{x+k}b_{k+1\,k+1}V(\overline{A}_x). \tag{11.5.6}$$

又根据

$$[{}_kV(\overline{A}_x) + P(\overline{A}_x) - \overline{A}^{\,1}_{x+k:\overline{1|}}](1+i) = p_{x+k\,k+1}V(\overline{A}_x), \tag{11.5.7}$$

可得

$$b_{k+1} = b_k\frac{1+i'_{k+1}}{1+i}, \tag{11.5.8}$$

它与变额年金的 (11.5.4) 相似。

三. 固定保费的变额人寿保险

第二种设计为固定保费的变额寿险(fixed premium variable life insurance)，它与完全变额设计的不同点在于净保费保持固定。考虑净保费为 $P(\overline{A}_x)$(即初始受益额为 1) 的情形，有

$$[b_{k\,k}V(\overline{A}_x)+P(\overline{A}_x)-b_k\overline{A}^{\,1}_{x-k:\overline{1|}}](1+i'_{k+1}) = p_{x+k}b_{x+k\,k+1}V(\overline{A}_x), \tag{11.5.9}$$

与 (11.5.7) 结合可导出

$$b_{k+1} = b_k\left[\frac{{}_kV(\overline{A}_x)+P(\overline{A}_x)/b_k-\overline{A}^{\,1}_{x+k:\overline{1|}}}{{}_kV(\overline{A}_x)+P(\overline{A}_x)-\overline{A}^{\,1}_{x+k:\overline{1|}}}\right]\frac{1+i'_{k+1}}{1+i}. \tag{11.5.10}$$

式 (11.5.9) 左端第一个因子可写成

$$(b_k-1)_kV(\overline{A}_x)+{}_kV(\overline{A}_x)+P(\overline{A}_x)-b_k\overline{A}^{\,1}_{x+k:\overline{1|}}.$$

这表明，固定净保费同时支持面额 1 与产生于实际投资回报的增加受益额 b_k-1。

四、缴清保险增额

最后考虑第三种设计，保费仍固定，但受益的改变以缴清保险的方式出现，这样，即使以后的实际投资收益率回落到 AIR，已增加的受益额将保持不会降低，而保费只用于支持原受益水平。责任准备金方程为

$$
\begin{aligned}
&[(b_k - 1)\overline{A}_{x+k} + {}_kV(\overline{A}_x) + P(\overline{A}_x) - b_k \overline{A}^{\,1}_{x+k:\overline{1}|}](1 + i'_{k+1}) \\
&= p_{x+k}[(b_{k+1} - 1)\overline{A}_{x+k+1} + {}_{k+1}V(\overline{A}_x)],
\end{aligned}
\qquad (11.5.11)
$$

其中当 $i'_{k+1} = i$ 时， $b_{k+1} = b_k$，于是

$$
\frac{1 + i'_{k+1}}{1+i} = \frac{(b_{k+1} - 1)\overline{A}_{x+k+1} + {}_{k+1}V(\overline{A}_x)}{(b_k - 1)\overline{A}_{x+k+1} + {}_{k+1}V(\overline{A}_x)}.
$$

将责任准备金的缴清保险公式

$$
{}_{k+1}V(\overline{A}_x) = \left[1 - \frac{P(\overline{A}_x)}{P(\overline{A}_{x+k+1})}\right]\overline{A}_{x+k+1}
$$

代入上式可获受益额的递推关系：

$$
b_{k+1} - \frac{P(\overline{A}_x)}{P(\overline{A}_{x+k+1})} = \left[b_k - \frac{P(\overline{A}_x)}{P(\overline{A}_{x+k+1})}\right]\frac{1 + i'_{k+1}}{1 - i}. \qquad (11.5.12)
$$

§11.6 可变计划产品

在七十年代初，保险公司开始提供若干种在保额、保费及保险计划等方面的改变有广泛选择权的保单。保险金的少量增加一般毋需新的可保性证明，但大额增加则需要这种证明。可供选择的计划包括各类均衡保费定额受益的定期、终身、限期缴费终身寿险及两全计划。一种特殊的红利选择权设计允许红利以直接相联系的净比率加入到现金价值中去，增大的现金价值则用于展延

定期计划的期限或增加终身计划的受益额。这些产品称为可变计划(flexible plan)，下面以特别简单的形式为例予以说明，并通过对第二种设计的描述来结束这一节，后一设计不如前一种那样注重保险计划，且与上一节的变额寿险计划具有某些共性。

一. 可变计划实例

我们将使用毛保费 G 与净保费 P 之间确定的关系：

$$0.8G = P. \tag{11.6.1}$$

另外，还使用完全初年定期制责任准备金与不没收价值。(值得注意，计价法及不没收法可能要求更高的值。) 定义 ${}_0V = -E$, 这里 E 是第一年费用超额补贴，根据完全初年定期制责任准备金修正方法，${}_1V = 0$, 于是按完全离散基础可得

$${}_0V + P = vq_xb,$$

$$E = -{}_0V = P - vq_xb, \tag{11.6.2}$$

其中 b 是初始死亡受益额，P 是初始净保费。设 h 是缴费年限，则

$${}_0V + P\ddot{a}_{x:\overline{h}|} = bA^{1}_{x:\overline{j}|}, \tag{11.6.3}$$

其中，在定期计划下 $j = h$, 在限期缴费终身寿险计划下 $j = \omega - x$. 责任准备金用后顾公式可写成

$${}_kV = \frac{{}_0V + P\ddot{a}_{x:\overline{k}|} - bA^{1}_{x:\overline{k}|}}{{}_kE_x}. \tag{11.6.4}$$

例 11.6.1: 考虑 35 岁签发的初始毛保费与保险金分别为 1000 与 120000 的保单，用示例生命表及 6% 利率决定 E, ${}_5V$, 以及该保险计划。

解：由 (11.6.1) 得 $P=800$, 于是

$$\begin{aligned} E &= -{}_0V = P - 120000vq_{35} = 572.05, \\ {}_5V &= \frac{-572.05 + 800\ddot{a}_{35:\overline{5}|} - 120000A^{\,1}_{35:\overline{5}|}}{{}_5E_{35}} = 2491.24. \end{aligned}$$

终身寿险的完全初年制的续期保费为 $120000P_{36}=1057.37$ 。既然净保费只有 800, 保险计划必为定期计划。用公式 (11.6.4) 可算出

$${}_{39}V = 3375.72, \qquad {}_{40}V = -1313.14,$$

可见保险计划的期限到 74 岁为止 (39 年) 。在时间 39 剩余的责任准备可用于提供接着的不到一年的定期保险，天数为

$$\frac{{}_{39}V}{120000A^{\,1}_{74:\overline{1}|}}365 = 230.$$

在变更保额或保费之时，可设新的责任准备金 ${}_kV'$ 等于在变更之时的原先完全初年定期制责任准备金 ${}_kV$ 。新的净保费 P', 保额 b' 与责任准备金的关系与 (11.6.3) 相似，为

$${}_kV' + P'\ddot{a}_{x+k:\overline{h}|} = b'A^{\,1}_{x+k:\overline{j}|}, \tag{11.6.5}$$

这里 h 是新计划的缴费期限， $j=h$ 或者 $\omega-x-k$. 与 (11.6.4) 类似，对 $g=1,2,3,\cdots$, 有

$${}_{k+g}V' = \frac{{}_kV' + P'\ddot{a}_{x+k:\overline{g}|} - b'A^{\,1}_{x+k:\overline{g}|}}{{}_gE_{x+k}}. \tag{11.6.6}$$

例 11.6.2: 上例中保单持有人希望在 5 年后将毛保费变更为 2000, 保险金改为 150000 。决定原保单签发 10 年后的责任准备金以及新的保险计划。

解：由 ${}_5V' = {}_5V = 2491.24, P' = 1600$, 代入 (11.6.6) 得

$$ {}_{10}V' = \frac{2491.24 + 1600\ddot{a}_{40:\overline{5}|} - 150000A^{1}_{40:\overline{5}|}}{{}_5E_{40}} = 10319.89. $$

鉴于 $2491.24 + 1600\ddot{a}_{40}$ 超过 $150000A_{40}$, 新计划是限期缴费的终身寿险。69 岁是责任准备金超过保额为 150000 的同龄终身寿险精算现值的第一个年龄，用 (11.6.6) 可算出

$$ {}_{34}V' = 75597.32, \qquad 150000A_{69} = 74954.44. $$

当保单持有者达到 69 岁时，保单可转成面额为

$$ 75597.32/A_{69} = 151287 $$

的缴清终身寿险保单。

例 11.6.3: 在例 11.6.1 中，如保单持有人希望在 5 年后将毛保费改为 2000, 计划改为最多缴至 60 岁为止的终身寿险，决定新的受益额。

解：此时 $P' = 1600$、由

$$ 2491.24 + 1600\ddot{a}_{40:\overline{20}|} = b'A_{40} $$

解得 $b' = 132090$ 。

例 11.6.4: 在例 11.6.1 中，如保单持有人希望将保额改为 150000, 计划改为到 65 岁为止的定期寿险，决定新的毛保费。

解：由

$$ 2491.24 + 0.8G'\ddot{a}_{40:\overline{25}|} = 150000A^{1}_{40:\overline{25}|}, $$

可解出 $G' = 895.00$ 。

二. 另一种设计

第二种设计将变额寿险与前面可变计划保单相结合，它对于保险计划的注重不象第一种设计那样强。此外，相对于受益额而言，更着重于以前称为风险净额的风险额 (risk amount)。在保单年度 $k+1$ 之初的风险量决定下含这个因子 r_k 的基金份额增长方程。以下分析以年度模型为例，不过在实践中，月度或更频繁的计算较为常见。与 (10.5.3) 相似但不允许退保的基金份额增长方程为

$$({}_kF + G - E - r_k \overline{A}^{\ 1}_{x+k:\overline{1|}})(1 + i'_{k+1}) = {}_{k+1}F. \tag{11.6.7}$$

注意这里是只按利息累积的，在发生死亡的情形，保单持有者既收到年初的基金份额 ${}_kF + G - E - r_k \overline{A}^{\ 1}_{x+k:\overline{1|}}$，也在死亡发生时获得按利息调整的风险额。对风险额的选择可使总受益额维持在某个大致的水平上，保单持有人对毛保费 G 以及风险额 r_k 的选取具有很大的灵活性。通常，i'_{k+1} 是至少等于某个最低利率 i 的投资收益率，风险费一般不超过 $r_k \overline{A}^{\ 1}_{x+k:\overline{1|}}$，其中 1 年定期保险的净趸缴保费以利率 i 及法定责任准备金计算中使用的死亡表为基础进行计算。

式 (11.6.7) 中的费用 E 常具有以下几种形式：(1) 按所有毛保费百分比收费；(2) 作为解约费用，按第一年保费的较高然而随时间下降的比例或按经办费 (如每次退保 25 元) 收取；(3) 每份保单统一收费，只在第 1 年收取或每个保单年度收取较低的数额；(4) 按每千元受益额的一个数额作为第 1 年收费。

最有可能受到法规制约的是第一年超额费用及风险收费。除了已叙述过的收费公式外，保险人也通过若干技术处理来补偿费用。其中的一些办法是：(1) 降低计息帐面额，对于保单现金价值一个起始范围 (譬如现金价值中起始的 1000)，只限于承诺的利率计息；(2) 净投资收益率与应用于现金价值的利息有一个 1% 到 $1\frac{1}{2}\%$ 的利差；(3) 犹如常规定期保险的保费中实际包含某些费用成份一样，风险收费中的一部分也这样予以确认。

如上所述，这种设计的重点不在保险计划。在任何时间，类似于例 11.6.1 及例 11 .6.2 所使用的计算可用于决定隐含于任何特殊型式的保费与受益、当前风险额、费用收费、利率及责任准备金。

§11.7　个人寿险中的残疾受益

普通人寿保险契约中一般都附有残疾受益，在全残而丧失工作能力后，可获得每千元面额 5 或 10 元的月收入补偿，或者得到免缴以后保费的受益。保单持有人在全残后通常须经过 3 、6 或 12 个月的等待期后方可获得残疾受益，但有时可追溯至等待期。残疾受益可能在寿险保单到期之前结束，典型的结束年龄为 60 或 65 岁。然而对于年金形式的受益，无论是残疾收入还是保费免缴，通常在一个更高的年龄终止，典型的是寿险保单到期日或缴清日。

一．残疾收入受益

考察 x 岁人到 y 岁为止在致残后 m 个月开始可获每月 1000 收入至 u 岁的残疾收入受益，按第七、第八章的记号，其精算现值为

$$12000 \sum_{k=0}^{y-x-1} v^k {}_k p_x^{(\tau)} v^{1/2} q_{x+k}^{(i)} v^{m/12} {}_{m/12}p^{i}_{[x+k+1/2]}$$

$$\ddot{a}^{(12)i}_{[x+k+1/2]+m/12:\overline{u-x-k-1/2-m/12}|}, \qquad (11.7.1)$$

这里，$[x+k+1/2]$ 表示致残年龄 (选择年龄)，$x+k+(1/2)+m/12$ 表示有资格受益的年龄。

这个公式在实践中通常作一些简化。首先，可设损失 i(即残疾) 只有当致残后生存到等待期 m 个月末时才认为发生，当致残后在等待期内死亡，作为损因 d(即死亡) 对待。这样，残疾者的

生存因子 ${}_{m/12}p^{i}_{[x+k+1/2]}$ 由于已计入 $q^{(i)}_{x+k}$ 而不必在 (11.7.1) 中出现。

其次，用连续年金

$$\overline{a}^{i}_{[x+k+1/2]+m/12:\overline{u-x-k-1/2-m/12|}}+\frac{1}{24} \tag{11.7.2}$$

近似 (11.7.1) 中的按月支付年金。在实践中，刚致残生命的总损失率由开始较高然后下降的死亡率与先取得峰值随后下降的实质恢复率组成。在这些场合，(11.7.2) 可从以下几个方面得到确认：与按月支付年金相比，连续年金失去每月支付额 $\frac{1}{12}$ 的大约 1/2 个月的利息。在因死亡或恢复而终止的情形，还失去部分支付，这部分支付在生存到年龄 u 的情形并不失去。因此连续年金用月支付的 1/2 作调整是方便甚至可能是保守的。

第三个近似涉及用标准的单重损失生命表代替死亡与残疾两重损失表，以及近似

$$\begin{aligned} q^{(i)}_{x+k} &= \int_0^1 {}_tp'^{(d)}_{x+k}\,{}_tp'^{(i)}_{x+k}\mu^{(i)}_{x+k+t}dt \\ &\approx {}_{1/2}p'^{(d)}_{x+k}\int_0^1 {}_tp'^{(i)}_{x+k}\mu^{(i)}_{x+k+t}dt = {}_{1/2}p_{x+k}q'^{(i)}_{x+k}. \end{aligned} \tag{11.7.3}$$

这些简化以 Phillips 近似(Phillips approximation) 著称，导致净保费及其责任准备金只是稍微有点偏离。

将以上简化施行于 (11.7.1)，可得精算现值的近似：

$$12000\sum_{k=0}^{y-x-1} v^{k+1/2}{}_kp_x\,{}_{1/2}p_{x+k}q'^{(i)}_{x+k}v^{m/12}$$

$$(\overline{a}^{i}_{[x+k+1/2]+m/12:\overline{u-x-k-1/2-m/12|}}+\frac{1}{24}). \tag{11.7.4}$$

可定义与第八章类似的计算基数，$D_x=v^xl_x$，

$$\overline{C}^{i}_{x}=v^{1/2}{}_{1/2}p_xq'^{(i)}_{x}D_x. \tag{11.7.5}$$

$$ {}^{u}\overline{C}^{i}_{x} = \overline{C}^{i}_{x} v^{m/12} \overline{a}^{i}_{[x+1/2]+m/12:\overline{u-x-1/2-m/12}|}. \tag{11.7.6} $$

$$ {}_{y}\overline{M}^{i}_{x} = \sum_{z=x}^{y-1} \overline{C}^{i}_{z} \tag{11.7.7} $$

$$ {}^{u}_{y}\overline{M}^{i}_{x} = \sum_{z=x}^{y-1} {}^{u}\overline{C}^{i}_{z}. \tag{11.7.8} $$

用这些记号，残疾收入受益的精算现值可写成

$$ \frac{12000\,{}^{u}_{y}\overline{M}^{i}_{x} + 500 v^{m/12}\,{}_{y}\overline{M}^{i}_{x}}{D_x}. \tag{11.7.9} $$

二. 保费免缴受益

设免缴的保费 P 每年分 g 次终身支付，并且受益追溯至等待期，即等待期内缴付的保费在等待期末退还。与残疾收入受益的主要区别在于保费免缴受益是始于等待期结束后第一个缴费日的年付 g 次年金。如假定在一年中伤残均匀发生，从等待期末到下次缴费日平均期为 $\frac{1}{2g}$，则等待期之后的免缴保费受益的年金值为

$$ {}_{1/(2g)|}\ddot{a}^{(g)i}_{[x+k+1/2]+m/12} \cong \overline{a}^{i}_{[x+k+1/2]+m/12}. \tag{11.7.10} $$

追溯的免缴保费受益平均约为 $(m/12)P$ 。当保费连续缴付或 m 是 g 的整数倍时，这很清楚。为了看一下其它情形可能发生的情况，考虑半年缴费且等待期为 4 个月的情况。如果等待期结束于下次缴费日前的 2 个月内，那么在等待期内未缴保费；否则在等待期内缴了保费 $(1/2)P$. 在一年中伤残均匀分布的假设下，平均的追溯受益额为

$$ [\frac{2}{6}\cdot 0 + \frac{4}{6}\cdot\frac{1}{2}]P = \frac{4}{12}P. $$

运用这里以及前面讨论残疾收入受益时的简化，保费免缴受益的精算现值可表示成

$$P\sum_{k=0}^{y-x-1} v^{k+1/2}{}_kp_{x}{}_{1/2}p_{x+k}q'^{(i)}_{x+k}v^{m/12}\left(\bar{a}^i_{[x+k+1/2]+m/12}+\frac{m}{12}\right). \tag{11.7.11}$$

用计算基数表示则成

$$P({}^{\omega}_{y}\overline{M}^i_x+\frac{m}{12}v^{m/12}{}_y\overline{M}^i_x)/D_x. \tag{11.7.12}$$

这些受益的净年保费可通过平衡原理获得。譬如，对于保费及其免缴受益持续至 65 岁而寿险缴清到 75 岁的保单，免缴受益的净年缴保费 ${}_{65-x}\pi_x$ 由下式决定：

$${}_{65-x}\pi_x\ddot{a}_{x:\overline{65-x}|}=\frac{P({}^{75}_{65}\overline{M}^i_x+\frac{m}{12}v^{m/12}{}_{65}\overline{M}^i_x)}{D_x},$$

这里 P 是全残后免缴的年保费 (寿险) 。上式可改写成

$${}_{65-x}\pi_x=\frac{P({}^{75}_{65}\overline{M}^i_x+\frac{m}{12}v^{m/12}{}_{65}\overline{M}^i_x)}{N_x-N_{65}}. \tag{11.7.13}$$

相应的责任准备金可用保费差公式表示：

$${}_kV=({}_{65-x-k}\pi_{x+k}-{}_{65-x}\pi_x)\ddot{a}_{x+k:\overline{65-x-k}|}. \tag{11.7.14}$$

残疾生命的期末责任准备金是在被保险人已具有伤残资格的假定下计算的未来残疾受益的精算现值。免缴保费额或者残疾收入率应乘以适合残疾者的生存年金精算现值。这个值考虑了致残年龄、伤残延续时间及受益终止年龄。

习　题

§11.2

1. 假设每年死亡服从均匀分布,试用第三章的结果将 (11.2.6) 中精算现值表示为

$$\frac{i}{i^{(m)}}\left[a_{\overline{n}|}+v^{n}{}_{n}p_{x}a_{x+n}+\left(\frac{1}{d}-\frac{1}{d^{(m)}}\right)v^{n}{}_{n}p_{x}A_{x+n}\right].$$

[提示：参考习题 3.17]

2. 证明 (11.2.1) 中定义的 Z 的方差和每年支付 1 的 n 年递延连续生命年金有相同的方差，因此， $\mathrm{Var}[Z]$ 可由 (3.3.20) 或习题 3.40 中相等的表达式给出。

3. 某部分现金立即偿还年金，其现值随机变量为：

$$Z^{*}=\begin{cases}\bar{a}_{\overline{T}|}+(\rho G-T)v^{T} & T<\rho G, 0<\rho<1\\ \bar{a}_{\overline{T}|} & T\geq\rho G.\end{cases}$$

a. 证明对一个部分现金立即偿还年金 (11.2.11) 可重写成：

$$G(1-r)=\bar{a}_{x}+\rho G\bar{A}^{1}_{x:\overline{\rho G}|}-(\overline{I}\bar{A})^{1}_{x:\overline{\rho G}|}.$$

b. 设 $H(G)=G(1-r)-\bar{a}_{x}-\rho G\bar{A}^{1}_{x:\overline{\rho G}|}+(\overline{I}\bar{A})^{1}_{x:\overline{\rho G}|}$, 求 G 使 $H(G)=0$ 。

(i) 求 $H'(G)$ 和 $H''(G)$

(ii) 在一个根的邻域内讨论 $H'(G)$ 和 $H''(G)$ 的符号。

4. 证明：对例 11.2.1 中的 Z 有：

$$\mathrm{Var}[Z]=\frac{v^{2n}[{}^{2}\bar{A}_{x}-\bar{A}_{x}^{2}]}{\delta^{2}}.$$

§11.3

5. 假设每年死亡服从均匀分布，试用第三章的结果将 (11.3.4) 中的精算现值表示为

$$\frac{i}{i^{(m)}}\left[a_{\overline{n}|}-a_{x:\overline{n}|}-\left(\frac{1}{d}-\frac{1}{d^{(m)}}\right)A^{1}_{x:\overline{n}|}\right].$$

[提示：参考习题 3.17]

6. 假设 Z 按 (11.3.1) 定义，证明

$$\mathrm{Var}[Z]=\frac{{}^{2}\overline{A}_{x:\overline{n}|}-\left(\overline{A}_{x:\overline{n}|}\right)^{2}}{\delta^{2}}.$$

7. 证明：对一个年金值按利息效力 δ' 计算的 n 年连续家庭收入保险。其精算现金值为：

$$\frac{\overline{A}^{1}_{x:\overline{n}|}-e^{-\delta' n}\,{}''\overline{A}^{1}_{x:\overline{n}|}}{\delta'}$$

其中 ${}''\overline{A}^{1}_{x:\overline{n}|}$ 按利息效力 $\delta''=\delta-\delta'$ 计算。

8. 一份保单从 (x) 死亡的日期开始提供每年为 1 的连续确定年金，若死亡发生在保单签发后的 15 年内，则年金付至保单签发后的 20 年年底。若死亡发生在保单签发后 15 至 20 年内，则年金支付 5 年。保单签发 20 年后停止生效，写出净趸缴保费 m 的表达式。

9. 一份合同规定，若被保险人在 20 年末还活着时可得 1000；若其在保单签发后的 20 年内死亡，则每月可得 10 的收入，该收入的第一笔支付在死亡的月末，但保单签发 20 年后则无支付。写出 x 岁时的净趸缴保费公式。

10. 证明

$$\overline{a}_{\overline{n}|}-\overline{a}_{x:\overline{n}|}=\frac{\overline{A}^{1}_{x:\overline{n}|}-v^{n}{}_{n}q_{x}}{\delta}.$$

11.

a. 联系例 11.3.1，将收入的现值表示为死亡时间的函数。

b. 运用综合支付技巧表达收入的精算现值。

c. 证明对 (b) 的结果运用分部积分得例 11.3.1 中的表达式。

§11.4

12. 对一个完全离散退休收入保单，定义 a 是满足 ${}_aV \le 1$ 及 ${}_{a-1}V > 1$ 的唯一整数，对该完全离散收入保单进行下列分析：

a. 前溯确定 ${}_aV$, 用净趸缴保费 P 表示，

b. 运用复利理论，后溯确定 ${}_aV$, 用 P 表示。

c. 令 ${}_aV$ 的两个表达式相等，求解 P 。

d. 从前文和不等式 ${}_aV \le 1, P \le P_{x:\overline{a}|}$, 及 ${}_{a+1}V > 1$ 和 $P > P_{x:\overline{a+1}|}$, 证明 a 是满足 $\frac{\bar{s}_{\overline{n-a}|}}{\ddot{a}_{x:\overline{a}|}} \ge k$ 的最大整数。

13. 将 Hattendorf 定理扩充到一般的完全连续保险，有

$$\begin{aligned}&\int_0^n (v^t b_t - \overline{P}\bar{a}_{\overline{t}|})^2 {}_tp_x \mu_{x+t} dt + (v^n b_n - \overline{P}\bar{a}_{\overline{n}|})^2 {}_np_x \\ = &\int_0^n v^{2t}(b_t - {}_t\overline{V})^2 {}_tp_x \mu_{x+t} dt,\end{aligned}$$

其中 n 是保费支付期，也是保险期限， b_t 是 t 时死亡受益、 b_n 是满期收益 (参见练习 5.45) 。运用上述方法，证明一个完全连续基础的退休收入保险，其损失变量的方差和一个 a 年 1 单位两全保险的损失变量的方差相等，为：

$$\frac{{}^2\overline{A}_{x:\overline{a}|} - \overline{A}_{x:\overline{a}|}^2}{(\delta \bar{a}_{x:\overline{a}|})^2}.$$

14. 证明对一个退休收入保单，若 $a = h+r$, 其中 $h = [a], 0 < r < 1$, 则在 $h+1$ 保单年死亡均匀分布的假设下成立

$$\bar{a}_{x:\overline{a}|} = \bar{a}_{x:\overline{h}|} + {}_hE_x \frac{(\delta - q_{x+h})\bar{a}_{\overline{r}|} + v^r r q_{x+h}}{\delta}.$$

§11.5

15. a. 证明

$$(\ddot{a}_{x:\overline{n}|}-1)(1+i)=p_x\ddot{a}_{x+1:\overline{n-1}|}+q_x\ddot{a}_{\overline{n-1}|}.$$

b. 证明对一个在 n 年确定和生命基础上支付的变额年金，(11.5.4) 成立。

16.

a. 将 (11.5.10) 重写成下列等价形式

$$b_{k+1}=\left[b_k-\frac{(b_k-1)P(\overline{A}_x)}{{}_1E_{x+k\,k+1}V(\overline{A}_x)}\right]\frac{1+i'_{k+1}}{1+i}.$$

b. 若 a 的公式中，$i'_{k+1}=i,k=0,1,2,\cdots$ 及 $b_0=1$ 。证明 $b_k+1=1,k=0,1,2,\cdots$

c. 若 a 的公式中，$i'_{k+h}=i$, 其中某个 $k>0,h=1,2,\cdots$, 证明 b_{k+h} 将收敛到 1 。

17. 假设 $p'_{x+h}=p_{x+h}$, 重解习题 11.14(b), 证明 $r_{h+1}=b_{h+1}/b_h$ 与 (11.5.4) 中结论一致。

18. 一个离散模型 F 的定额终身寿险，其中 $k+1$ 年的死亡受益 b_{k+1} 等于 $F_{k+1}+(1-{}_{k+1}V_x)=1+(F_{k+1}-{}_{k+1}V_x)$, 其中基金份额 F_x 满足递推方程

$$(F_k+P_x)(1+i'_{k+1})=q_{x+k}b_{k+1}+p_{x+k}F_{k+1},$$

其中 i'_{k+1} 为 $k+1$ 年相应投资的利率，保费 P_x 和准备金 ${}_kV_x$ 以利率 i 为基础。

a. 证明该递推方程能写成

$$(F_k+P_x)(1+i'_{k+1})-q_{x+k}(1-{}_{k+1}V_x)=F_{k+1},$$

并解释这个方程。

b. 若 $i'_{k+1}=i,k=0,1,2,\cdots$, 证明 $F_{k+1}={}_{k+1}V_x$, 且 b_{k+1} 在 1 时恒为常数。

c. 证明 $b_{k+1}=b_k+(F_k+P_x)i'_{k+1}-({}_kV_x+P_x)i$.

[注意：在这个方案中， $k+1$ 年的死亡受益是 b_{k+1} 而不是如 (11.5.9) 中的 b_k 。而且 $k+1$ 年初的 1 年期保险费用是 $b_{k+1}q_{x+k}/(1+i'_{k+1})$, 而不是 (11.5.9) 中的 $b_x q_{x+k}/(1+i)$ 。]

19. 例 11.6.1 中的保单持有者在 5 年后希望将保单转为年保费为 5000 至 65 岁的两全保险。试确定该变化产生的受益水平。

20. a. 习题 11.19 中的保单持有者决定选择到 65 岁的 160000 的两全保险，但仍将支付 5000 的毛年缴保费直至付清最终一笔分数保费，这笔分数保费在最后一笔完全保费的后一年支付。试问这笔分数保费将在什么年龄支付。

b. 对 (a) 中的保单，试问在转换成两全保险后的 10 年末责任准备金为多少？

§11.7

21.

a. 有一个签发给 (35) 的残疾收入保险，若 (35) 在 60 岁前致残且存活了 6 个月的等待期，则每月可得 1000 直至 65 岁。试用转换函数表示付至 60 岁的净趸缴保费。

b. 对 a 中的保险，写出其 10 年末的期末责任准备金公式。

22. 一附在 (x) 终身寿险保单后的残疾附加条款规定：若 (x) 在年龄 y 前变成全残且在 6 个月的等待期中仍为全残，可享受以下受益：

(1) 从等待期末开始，且在连续伤残的生命中都享受的月收入。

(2) 从寿险保费到期开始或在伤残的最初日子之后的整个连续伤残生命中都免缴寿险保费。

若该伤残附加条款的净趸缴保费为

$$\frac{1720\,{}_y^{\omega}\overline{M}_x^i+200v^{1/2}{}_y\overline{M}_x^i}{N_x-N_y},$$

试确定月收入的数量及免缴的年保费数量 (假设支付连续) 。

第十二章　多重生命续论

§12.1　引　言

在第六章里，我们定义了连生状况与最后生存者状况，并将这些状况的消亡时间 (剩余寿命) 随机变量用单个生命的剩余寿命来表示。为此，对第一至第三章的概念加以推广以得出只涉及两个生命的状况的精算函数。在两个生命的剩余寿命相互独立的假设下，多重生命的精算函数可用单个生命的函数来表达，从而使得利用现成的涉及单重生命的生命表成为可能。有关的概率、年金与保险、与两个生命的死亡次序相关的顺位函数已在第六章里作了讨论。

这一章要将这些内容推广到多于两个生命的情形，事实上在多于两个生命的场合，生存状况概念可以一般化 (见 §12.2 与 §12.3) 。以这些函数的数值计算作为最终目标，定理 12.2.1 可用来只根据连生状况的生存函数来表示这些一般状况的生存函数。利用剩余寿命的独立性假设，这些连生生存函数可作为个体生存概率的乘积求得。定理 12.2.1 是在所谓的包含 - 排斥方法中使用的概率论一般定理的一种形式。顺位概率与函数也被推广到多于两个生命的情形，并且在 §12.6 中还讨论了在退休计划中作为退休后死亡受益提供的继承年金。第四与第五章里的年保费模型也在 §12.7 中对多重生命状况的讨论中有所发展。

§12.2　更一般状况

对于 m 个生命： $(x_1),(x_2),\cdots,(x_m)$，至少 k 个生存者状况

(k-survivor status) 当其中至少有 k 个全活着时状况存在，而当第 $m-k+1$ 个死亡时状况消亡。至少 k 个生存者状况记为

$$\left(\overline{x_1x_2\cdots x_m}^{\,k}\right),$$

该状况至少存在 t 年的概率为 ${}_tp_{\overline{x_1x_2\cdots x_m}}^{\ \ k}$, 等等。$k=1$ 时就是第六章讨论的最后生存者状况，在记号中可省略；$k=m$ 时则成为连生状况；$k=0$ 时成为永久状况 (是确定的状况)。至少 k 个生存者状况的剩余寿命是 m 个寿命 $T(x_1),\cdots,T(x_m)$ 的第 k 个最大者。与第一及第六章的剩余寿命 (消亡时间) 相似，至少 k 个生存者状况的剩余寿命是从一个确定的起始时刻到一个随机的终止时刻状况存在所经历的时间。关于至少 k 个生存者状况的生存年金及保险的精算现值或净趸缴保费为

$$\bar{a}_{\overline{x_1x_2\cdots x_m}}^{\ \ k}=\int_0^\infty v^t\,{}_tp_{\overline{x_1x_2\cdots x_m}}^{\ \ k}dt \tag{12.2.1}$$

$$\bar{A}_{\overline{x_1x_2\cdots x_m}}^{\ \ k}=\int_0^\infty v^t\,{}_tp_{\overline{x_1x_2\cdots x_m}}^{\ \ k}\mu_{\overline{x_1x_2\cdots x_m}}^{\ \ k}(t)dt. \tag{12.2.2}$$

为了分析上的需要，可引入新的一类状况：当 m 个生命 $(x_1),(x_2),\cdots,(x_m)$ 中正好有 k 个活着时存在，而在其它情形不存在或消亡，这种状况称为恰好 k 个生存者状况([k]-deferred survivor status)，记为

$$\left(\overline{x_1x_2\cdots x_m}^{\,[k]}\right).$$

这类状况与以往的有所不同，起初状况并不存在 $(k<m)$，当第 $m-k$ 个生命死亡时开始存在，而当第 $m-k+1$ 个生命死亡时状况消亡。这是一种 (随机) 递延状况。$k=m$ 时与至少 m 个生存者状况一致；$k=0$ 时则成为从第 m 个生命死亡时起永久存在的状况。

譬如对于 $k<m$, ${}_tp_{\overline{x_1x_2\cdots x_m}}^{\ \ [k]}$ 是 m 个生命中恰有 k 个在时间 t 活着的概率，在 $t=0$ 时不等于 1, 这就不符合第一章里对生存

函数的要求。对 $k=0$, 当 $t\to\infty$ 时 ${}_tp_{\overline{x_1x_2\cdots xm}}^{[0]}$ 趋于 1, 也与第一章里的要求相抵触。此外，恰好 k 个生存者状况的存在时期并不等同于从初始时刻至消亡的时期。这意味着对这种新状况的年金受益必须小心定义。精算现值为 $\overline{a}_{\overline{x_1x_2\cdots x_m}}^{[k]}$ 的年金定义为在恰好 k 个生存者状况的未来存续期内的连续支付，因而它是一个具有随机递延期的延期年金。由于恰好 k 个生存者的消亡时间等于至少 k 个生存者状况的消亡时间，基于前者的保险受益本质上乃应用至少 k 个生存者状况。

例 12.2.1: 某种在 (w), (x), (y) 与 (z) 中任何一个活着时连续支付的年金，在每个死亡发生时年支付率减少 50% 。假定初始年受益率为 1, 用 $\overline{a}_{\overline{wxyz}}^{[k]}, k=1,2,3,4$ 表示这个年金的精算现值。

解：精算现值为

$$\overline{a}_{\overline{wxyz}}^{[4]}+\frac{1}{2}\overline{a}_{\overline{wxyz}}^{[3]}+\frac{1}{4}\overline{a}_{\overline{wxyz}}^{2}+\frac{1}{8}\overline{a}_{\overline{wxyz}}^{[1]}.$$

这个年金将在例 12.2.2 中继续讨论 (定理 12.2.1 之后) 。

在第六章里，最后生存者状况的概率及有关精算现值可用单重及连生状况的相应函数来表达。以下定理有助于对一般至少 k 个生存者状况得出同样结果。

定理 12.2.1: 设 ${}_tB_j=\sum {}_tp_{x_{k_1}x_{k_2}\cdots x_{k_j}}$, 和式中 $k_1,k_2,\cdots,k_j$ 取遍 $1,2,\cdots,m$ 的所有 j 个组合。那么对任意实数 $c_0,c_1,\cdots,c_m$, 成立

$$\sum_{j=0}^{m}c_j\,{}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]}=c_0+\sum_{j=1}^{m}\Delta^j c_0\,{}_tB_j, \tag{12.2.3}$$

其中 $\Delta c_k=c_{k+1}-c_k$, $\Delta^2c_k=\Delta c_{k+1}-\Delta c_k,\cdots$.

证：设 $A_j=\{T(x_j)>t\}, j=1,2,\cdots,m$, 则 ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]}$ 是这 m 个事件恰好有 j 个发生的概率。用 X_j 表示事件 A_j 的指示变量，即：当样本点在 A_j 中时 $X_j=1$, 不在 A_j 中时 $X_j=0$ 。

显然

$$
\begin{aligned}
{}_tB_j &= \sum_{i_1,i_2,\cdots,i_j} {}_tp_{x_{i_1}x_{i_2}\cdots x_{i_j}} = \sum_{i_1,i_2,\cdots,i_j} Pr[A_{i_1}A_{i_2}\cdots A_{i_j}] \\
&= \sum_{i_1,i_2,\cdots,i_j} E[X_{i_1}X_{i_2}\cdots X_{i_j}].
\end{aligned}
$$

作为移位算子 $E=1+\Delta$ 的函数，定义算子

$$
\begin{aligned}
\phi(E) &= (1+X_1\Delta)(1+X_2\Delta)\cdots(1+X_m\Delta) \\
&= (X_1E+1-X_1)(X_2E+1-X_2)\cdots(X_mE+1-X_m).
\end{aligned}
$$

上式右端的第 j 个因子当 $X_j=1$ 时为 E, 当 $X_j=0$ 时为 1, 于是

$$\phi(E)=Y_0+Y_1E+Y_2E^2+\cdots+Y_mE^m,$$

其中，指示变量 Y_j 当样本点恰好在 m 个事件 $A_1,A_2,\cdots,A_m$ 里的 j 个之中时为 1, 在其它样本点上为 0, 显然 $E[Y_j]={}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]}$.

移位算子 E 的 j 次幂 E^j 作用于 c_0 的结果是 c_j, 因此有

$$\phi(E)c_0=c_0Y_0+c_1Y_1+\cdots+c_mY_m,$$

期望值为

$$c_0{}_tp_{\overline{x_1x_2\cdots x_m}}^{[0]}+c_1{}_tp_{\overline{x_1x_2\cdots x_m}}^{[1]}+\cdots+c_m{}_tp_{\overline{x_1x_2\cdots x_m}}^{[m]}.$$

另一方面，显然有

$$
\begin{aligned}
\phi(E)c_0 &= (1+X_1\Delta)(1+X_2\Delta)\cdots(1+X_m\Delta)c_0 \\
&= c_0+\sum_{j=1}^{m}\Big(\sum_{i_1,i_2,\cdots,i_j} X_{i_1}X_{i_2}\cdots X_{i_j}\Big)\Delta j c_0,
\end{aligned}
$$

期望值为

$$c_0+\sum_{j=1}^{m}{}_tB_j\Delta^jc_0.$$

所以

$$\sum_{j=0}^{m} c_j {}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]} = c_0 + \sum_{j=1}^{m} {}_tB_j \Delta^j c_0.$$

定理 12.2.1 也适用于剩余寿命随机变量相关的生命，但在应用中我们将假定各生命的剩余寿命相互独立，从而 ${}_tB_j$ 中的项是个体生存概率的乘积。

例 12.2.2: 将例 12.2.1 中年金精算值用单重与连生状况的年金精算现值来表达。

解：精算现值为

$$\int_0^\infty v^t \left(\sum_{j=1}^{4} \left(\frac{1}{2}\right)^{4-j} {}_tp_{\overline{wxyz}}^{[j]}\right) dt.$$

其中系数及其差分如下：

j	c_j	Δc_j	$\Delta^2 c_j$	$\Delta^3 c_j$	$\Delta^4 c_j$
0	0	1/8	0	1/8	0
1	1/8	1/8	1/8	1/8	—
2	1/4	1/4	1/4	—	—
3	1/2	1/2	—	—	—
4	1	—	—	—	—

于是积分等于

$$\int_0^\infty v^t \left(\frac{1}{8} {}_tB_1 + \frac{1}{8} {}_tB_3\right) dt = \frac{1}{8}(\overline{a}_w + \overline{a}_x + \overline{a}_y + \overline{a}_z)$$
$$+ \frac{1}{8}(\overline{a}_{wxy} + \overline{a}_{wxz} + \overline{a}_{wyz} + \overline{a}_{xyz}).$$

这种表达式可通过将最后形式解释为年金组合而检验其合理性，在任何可能的情形，支付率的总和应等于原年金的支付率。例如，在以上例子中，原年金开始时支付率为 1，而最后形式相当于支付率均为 1/8 的 4 个单重生命与 4 个连生年金的组合；在第 1 个与第 2 个死亡之间，原年金支付率为 1/2，而 3 个单重生命

年金与 1 个连生年金合起来也提供相当的支付率；在其它情形的支付率可按类似的方式进行比较。

推论 12.2.1:

$$
{}_tp_{\overline{x_1x_2\cdots x_m}}^{[k]} = \sum_{j=k}^{m}(-1)^{j-k}\begin{pmatrix} j \\ k \end{pmatrix}{}_tB_j. \tag{12.2.4}
$$

证: 在定理 12.2.1 中，置 $c_k = 1, c_j = 0, j \neq k$ 。对这些 c_j,

$$
\Delta^j c_0 = (E-1)^j c_0 = (-1)^{j-k}\begin{pmatrix} j \\ k \end{pmatrix}, \qquad j = k, k+1, \cdots, m.
$$

例 12.2.3: 对于当 5 个生命中恰好有 3 个活着时每年连续支付 1 的年金，用连生年金的精算现值表达该年金的精算现值。

解: 用第三章的公式及 (12.2.4), 可得

$$
\begin{aligned}
\overline{a}_{\overline{x_1x_2x_3x_4x_5}}^{[3]} &= \int_0^\infty v^t {}_tp_{\overline{x_1x_2x_3x_4x_5}}^{[3]} dt \\
&= \int_0^\infty v^t({}_tB_3 - 4{}_tB_4 + 10{}_tB_5)dt \\
&= \overline{a}_{x_1x_2x_3} + \overline{a}_{x_1x_2x_4} + \text{另外 8 个三重生命连生年金值} \\
&\quad -4(\overline{a}_{x_1x_2x_3x_4} + \overline{a}_{x_1x_2x_3x_5} + \text{另 3 个四重连生年金值}) \\
&\quad +10\overline{a}_{x_1x_2x_3x_4x_5}.
\end{aligned}
$$

由关系式

$$
{}_tp_{\overline{x_1x_2\cdots x_m}}^{k} = \sum_{j=k}^{m} {}_tp_{\overline{x_1x_2\cdots x_m}}^{[j]}, \tag{12.2.5}
$$

可得出以下推论。

推论 12.2.2: 对任意实数 $d_0, d_1, \cdots, d_m$, 成立

$$
\sum_{j=0}^{m} d_j {}_tp_{\overline{x_1x_2\cdots x_m}}^{j} = d_0 + \sum_{j=1}^{m}\Delta^{j-1}d_1 {}_tB_j. \tag{12.2.6}
$$

证：根据 (12.2.5)，我们从

$$\sum_{h=0}^{m} d_{h\,t}p_{\overline{x_1x_2\cdots x_m}}^{\;\;h} = \sum_{h=0}^{m}\sum_{j=h}^{m} d_{h\,t}p_{\overline{x_1x_2\cdots x_m}}^{\;\;[j]}$$

开始，交换求和次序，可写出

$$\sum_{j=0}^{m} d_{j\,t}p_{\overline{x_1x_2\cdots x_m}}^{\;\;j} = \sum_{j=0}^{m}(\sum_{h=0}^{j} d_h)_t p_{\overline{x_1x_2\cdots x_m}}^{\;\;[j]},$$

其中，如定义 $c_j = \sum\limits_{h=0}^{j} d_h$, $j = 0, 1, \cdots, m$, 则符合 (12.2.3) 的形式。对这些 c, $c_0 = d_0$, $\Delta c_j = d_{j+1}$, $j = 0, 1, \cdots, m-1$ 。于是 $\Delta^j c_0 = \Delta^{j-1}(\Delta c_0) = \Delta^{j-1}d_1, j = 1, 2, \cdots, m$. 由 (12.2.3)，可得

$$\sum_{j=0}^{m} d_{j\,t}p_{\overline{x_1x_2\cdots x_m}}^{\;\;j} = d_0 + \sum_{j=1}^{m}\Delta^{j-1}d_{1\,t}B_j.$$

推论 12.2.2 可用于以连生与单重生命的生存函数表达至少 k 个生存者状况的生存函数。

推论 12.2.3:

$${}_tp_{\overline{x_1x_2\cdots x_m}}^{\;\;k} = \sum_{j=k}^{m}(-1)^{j-k}\begin{pmatrix} j-1 \\ k-1 \end{pmatrix}{}_tB_j. \tag{12.2.7}$$

证：在推论 12.2.2 中，置 $d_k = 1, d_j = 0, j \neq k$ 。对这些 d, $\Delta^{j-1}d_1 = (E-1)^{j-1}d_1 = (-1)^{j-k}\begin{pmatrix} j-1 \\ k-1 \end{pmatrix}$, $j = k, k+1, \cdots, m$ 。

由 (12.2.7) 中生存函数的表达式，可通过微分得到至少 k 个生存者状况剩余寿命 T 的概率密度函数的一个平行表达式：

$$\begin{aligned} f_T(t) &= \frac{d}{dt}(1 - {}_tp_{\overline{x_1x_2\cdots x_m}}^{\;\;k}) \\ &= \sum_{j=k}^{m}(-1)^{j-k}\begin{pmatrix} j-1 \\ k-1 \end{pmatrix}(-{}_tB_j'). \end{aligned} \tag{12.2.8}$$

依赖于 T 的一组支付现值的精算现值及其它概率分布特征可用 (12.2.7) 或 (12.2 .8) 决定，在这中间将使用 $-{}_tB_j'$ 是 m 个生命的 $\binom{m}{j}$ 个 $j-$ 连生状况的剩余寿命概率密度函数之和的事实。

例 12.2.4: 用连生及单重生命函数表示 3 个生命的最后生存者状况的 (1) 生存函数， (2) $E[v^T]$, (3) $E[\bar{a}_{\overline{T}|}]$,
其中 T 是 3 个生命的最后生存者状况的剩余寿命。

解： (1) 根据 (12.2.5), 可得

$$
\begin{aligned}
{}_tp_{\overline{x_1x_2x_3}} &= \sum_{j=1}^{3}(-1)^{j-1}\begin{pmatrix} j-1 \\ 0 \end{pmatrix}{}_tB_j = {}_tB_1 - {}_tB_2 + {}_tB_3 \\
&= \left({}_tp_{x_1} + {}_tp_{x_2} + {}_tp_{x_3}\right) - \left({}_tp_{x_1x_2} + {}_tp_{x_1x_3} + {}_tp_{x_2x_3}\right) + {}_tp_{x_1x_2x_3}.
\end{aligned}
$$

(2) 类似地利用 (12.2.8) 可得

$$
\begin{aligned}
\overline{A}_{\overline{x_1x_2x_3}} &= E[v^T] = \int_0^\infty v^t(-1)({}_tB'_1 - {}_tB'_2 + {}_tB'_3)dt \\
&= \overline{A}_{x_1} + \overline{A}_{x_2} + \overline{A}_{x_3} - (\overline{A}_{x_1x_2} + \overline{A}_{x_1x_3} + \overline{A}_{x_2x_3}) \\
&\quad + \overline{A}_{x_1x_2x_3}.
\end{aligned}
$$

(3) 用 $\bar{a}_{\overline{T}|}$ 代替 v^T, 可得

$$
\bar{a}_{\overline{x_1x_2x_3}} = E[\bar{a}_{\overline{T}|}] = \bar{a}_{x_1}+\bar{a}_{x_2}+\bar{a}_{x_3}-(\bar{a}_{x_1x_2}+\bar{a}_{x_1x_3}+\bar{a}_{x_2x_3})+\bar{a}_{x_1x_2x_3}.
$$

对一般状况，也成立 $v^T+\delta\bar{a}_{\overline{T}|}=1$, 我们可利用等式 $\overline{A}_{\overline{x_1x_2x_3}}+\delta\bar{a}_{\overline{x_1x_2x_3}}=1$ 由一个期望值计算另一个期望值。

通过对 (12.2.6) 两端求导，可推广有关的概率密度函数关系式，如用于在 m 个生命每个死亡时支付受益的保险。

例 12.2.5: 考虑有关 (x),(y) 与 (z) 的保险，在第 1 个死亡时支付 1, 在第 2 个死亡时支付 2, 在第 3 个死亡时支付 3 。用单重生命及连生状况单位保额保险的净趸缴保费表示该保险的净趸缴保费。

解：设 $f_j(t)$ 是至少 j 个生存者状况剩余寿命的概率密度函数，所求净趸缴保费为

$$\int_0^\infty v^t[1f_3(t)+2f_2(t)+3f_1(t)]dt.$$

按 (12.2.6) 的记号有：

j	d_j	Δd_j	$\Delta^2 d_j$	$\Delta^3 d_j$
0	0	3	-4	4
1	3	-1	0	—
2	2	-1	—	—
3	1	—	—	—

于是净趸缴保费成为

$$\int_0^\infty v^t(-1)(3{}_tB'_1-{}_tB'_2)dt$$
$$=3(\overline{A}_x+\overline{A}_y+\overline{A}_z)-(\overline{A}_{xy}+\overline{A}_{xz}+\overline{A}_{yz}).$$

12.3　复合状况

在前一节中，我们借助一般的至少 k 个生存者状况确定若干生命的各种状况。另一些状况可通过复合来确定。复合状况(compound status) 是由许多状况组合起来的状况，其中至少有一个状况涉及多于一个的生命。下面在例 12.3.1 中考察某些可能的情况。

例 12.3.1: 描述与以下精算现值与净趸缴保费相对应的年金与保险的支付条件。

(1) $\overline{a}_{\overline{wx}:\overline{yz}}$.　(2) $\overline{a}_{\overline{\overline{wx}:(yz)}}$.　(3) $\overline{a}_{\overline{(x:\overline{n|}):(yz:\overline{m|})}}$.

(4) $\overline{A}_{\overline{wx}:yz}$.　(5) $\overline{A}_{\overline{(wx):(yz)}}$.　(6) $\overline{A}_{\overline{(wx):y:z}}$.

解：(1) 年金在 (w) 与 (x) 至少有 1 个活着，并且 (y) 与 (z) 至少有 1 个活着的情况下以年率 1 提供连续支付。因此，当其中

有 3 个或 4 个活着时，或者当两个活着并且其中 1 个来自 (w) 与 (x) 而另 1 个来自 (y) 与 (z) 时，年金提供支付。

(2) 当 4 个人中至少有 2 个活着时，或者当只有 1 个活着并且是 (w) 与 (x) 中的 1 个时，年金按支付率 1 提供连续支付。

(3) 当 (x) 活着并且尚在 n 年内时，或者当 (y) 与 (z) 都活着并且尚在 m 年内时，年金按年率 1 提供连续支付。

(4) 1 单位保额在 (y) 第 1 个死亡或 (z) 第 1 个死时即刻支付，在其它情况下 (第 1 个死的是 (w) 或 (x)), 在第 2 个死亡之时即刻支付。

(5) 在前 2 个死亡者 1 个出自 (w) 与 (x) 这一对而另 1 个出自 (y) 与 (z) 那一对时，在第 2 个死亡之时即刻支付 1 单位保额，否则的话 (前 2 个死者出自同一对) , 在第 3 个死亡之时支付 1 单位。

(6) 一单位保额只有在 (y), (z) 以及 (w) 与 (x) 中的 1 个都死后才支付，换句话说，如果最后活着的是 (w) 或 (x), 那么在第 3 个人死亡时即刻支付，否则的话在第 4 个人 (最后 1 人) 死亡时即刻支付。

在应用中，这些精算现值或净趸缴保费的数值计算大多通过用单重生命及连生状况的有关函数表示而获得。第六章里有关 $T(xy)$, $T(\overline{xy})$, $T(x)$, $T(y)$ 之间的关系式以及 $K(xy)$, $K(\overline{xy})$, $K(x)$, $K(y)$ 之间的关系式对一般状况也成立。譬如，

$$v^{T(uv)} + v^{T(\overline{uv})} = v^{T(u)} + v^{T(v)}. \tag{12.3.1}$$

利用例 12.3.1 中的部分内容，我们来说明使用 (12.3.1) 及类似等式的过程。首先考虑 (5),

$$\bar{A}_{\overline{(wx):(yz)}} = \bar{A}_{wx} + \bar{A}_{yz} - \bar{A}_{wxyz},$$

这里 $(u) = (wx)$, $(v) = (yz)$ 。 $\bar{A}_{(wx):(yz)}$ 之所以能写成 $\bar{A}_{wxyz}$

的根据是

$$\begin{aligned}&\min\{\min[T(w),T(x)],\min[T(y),T(z)]\}\\&=\min[T(w),T(x),T(y),T(z)].\end{aligned}\tag{12.3.2}$$

关于例 12.3.1 中 (3) 有

$$\bar{a}_{\overline{(x:\overline{n}|):(yz:\overline{m}|)}}=\bar{a}_{x:\overline{n}|}+\bar{a}_{yz:\overline{m}|}-\bar{a}_{xyz:\overline{n}|},$$

这里设 $n\le m$ ，其中最后一项的根据是

$$\begin{aligned}&\min[T(x),T(y),T(z),T(\overline{n}|),T(\overline{m}|)]\\&=\min[T(x),T(y),T(z),T(\overline{n}|)].\end{aligned}$$

对例 12.3.1 中的其它情形，需要使用其它关系式。对于 (1),

$$\bar{a}_{\overline{wx}:\overline{yz}}=E[\bar{a}_{\overline{T}|}],\tag{12.3.3}$$

其中

$$\begin{aligned}T&=\min[T(\overline{wx}),T(\overline{yz})]\\&=\min\{\max[T(w),T(x),\max[T(y),T(z)]\}.\end{aligned}$$

对这个随机变量，象 (12.3.2) 那样的简单答案并不存在。为此，先假定 $T(\overline{wx})$ 与 $T(\overline{yz})$ 独立并考察 T 的生存函数 $s(t)$ 。于是，在独立情况下有

$$\begin{aligned}s(t)&=Pr[T>t]=Pr\{\min[T(\overline{wx}),T(\overline{yz})]>t\}\\&=Pr[T(\overline{wx})>t,T(\overline{yz})>t]\\&=Pr[T(\overline{wx})>t]Pr[T(\overline{yz})>t]\\&={}_tp_{\overline{wx}}\,{}_tp_{\overline{yz}}\qquad\qquad(12.3.4)\\&=({}_tp_w+{}_tp_x-{}_tp_{wx})({}_tp_y+{}_tp_z-{}_tp_{yz})\\&={}_tp_{wy}+{}_tp_{wz}+{}_tp_{xy}+{}_tp_{xz}-{}_tp_{wyz}-{}_tp_{xyz}\\&\quad-{}_tp_{wxy}-{}_tp_{wxz}+{}_tp_{wxyz};\end{aligned}$$

由此可得

$$\begin{aligned}\overline{a}_{\overline{wx}:\overline{yz}} &= \int_0^\infty v^t s(t)dt \\ &= \overline{a}_{wy} + \overline{a}_{wz} + \overline{a}_{xy} + \overline{a}_{xz} - \overline{a}_{wyz} - \overline{a}_{xyz} - \overline{a}_{wxy} \\ &\quad -\overline{a}_{wxz} + \overline{a}_{wxyz}. \end{aligned} \tag{12.3.5}$$

我们回到 (12.3.4)，并说明随机变量的一个平行的关系式在没有独立性的假设下成立。我们从以下断言开始：对所有的可能情况都成立

$$\begin{aligned}T(\overline{wx}:\overline{yz}) &= T(wy) + T(wz) + T(xy) + T(xz) \\ &\quad -T(wyz) - T(xyz) - T(wxy) \\ &\quad -T(wxz) + T(wxyz). \end{aligned} \tag{12.3.6}$$

各种可能的情况可按照 $T(w), T(x), T(y), T(z)$ 的次序归结起来，总共有 24 个互不相容的事件。由于以上断言关于 w 与 x 以及 y 与 z 是对称的，这样只有 6 种不同的可能情况需要验证。譬如当 $T(w) < T(x) < T(y) < T(z)$ 时，(12.3.6) 左端为 $T(\overline{wx}:\overline{yz}) = T(x)$，右端逐项相加为

$$\begin{aligned}&T(w) + T(w) + T(x) + T(x) - T(w) - T(x) - T(w) \\ &\quad -T(w) + T(w) = T(x),\end{aligned}$$

两者相等。其它情况可按同样方式得到验证。

与 (12.3.6) 平行的年金等式可通过类似的推理建立，于是

$$\begin{aligned}&\overline{a}_{\overline{T(\overline{wx}:\overline{yz})}|} = \overline{a}_{\overline{T(wy)}|} + \overline{a}_{\overline{T(wz)}|} + \overline{a}_{\overline{T(xy)}|} + \overline{a}_{\overline{T(xz)}|} \\ &-\overline{a}_{\overline{T(wyz)}|} - \overline{a}_{\overline{T(xyz)}|} - \overline{a}_{\overline{T(wxy)}|} - \overline{a}_{\overline{T(wxz)}|} + \overline{a}_{\overline{T(wxyz)}|}. \end{aligned} \tag{12.3.7}$$

在这个表达式两边取期望值就得出 (12.3.5) 。

这里强调一下独立性假设的两个方面。在建立 (12.3.7) 以及由 (12.3.7) 取期望值导出 (12.3.5) 时并未使用独立性假设。然而，为了从单位生命的生命表获得连生状况的函数，确实为方便而假定个体的剩余寿命相互独立。

§12.4　顺位概率与保险

这一节将第六章的顺位函数概念推广到 2 个生命以上的场合。我们从所求概率或净趸缴保费的积分表达式出发，然后改写成用基于第 1 个死亡 (先死) 的概率或净趸缴保费来表示的形式。这样就有可能利用第六章最后一节的技巧完成计算。在任何场合，都可使用数值积分方法。

为得出概率的积分表达式，我们将使用

$$Pr(A) = \int_{-\infty}^{\infty} Pr(A|T = t) f_T(t) dt, \tag{12.4.1}$$

其中 T 通常是某个生命的死亡时间。

例 12.4.1 用先死的顺位函数表示 ${}_nq_{wx\overset{2}{y}z}$.

解：用 A 表示相应的 (y) 在四人中第 2 个死并且死于 n 年内这个事件，那么

$${}_nq_{wx\overset{2}{y}z} = \int_0^n Pr[A|T(y) = t]\,{}_tp_y\mu_{y+t}dt.$$

如假定 $T(y)$ 与 $T(w), T(x), T(z)$ 独立，则

$$Pr[A|T(y) = t] = {}_tp_{\overline{wxz}}^{[2]} \qquad t < n,$$

于是根据推论 12.2.1 得

$$\begin{aligned} {}_nq_{wx\overset{2}{y}z} &= \int_0^n {}_tp_{\overline{wxz}}^{[2]}\,{}_tp_y\mu_{y+t}dt \\ &= \int_0^n ({}_tB_2 - 3{}_tB_3)\,{}_tp_y\mu_{y+t}dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^n ({}_tp_{wx} + {}_tp_{wz} + {}_tp_{xz} - 3{}_tp_{wxz}){}_tp_y\mu_{y+t}dt \\
&= {}_nq_{wxy}^{\;\;1} + {}_nq_{wyz}^{\;\;1} + {}_nq_{xyz}^{\;\;1} - 3{}_nq_{wxyz}^{\;\;\;1}.
\end{aligned}
$$

例 12.4.1 最后与以前结果相似的表达式没有用到独立性假设，这表明独立性假设并非必要。在习题 18 及 40 中的另一些推导也表明确实如此。

类似地可得出顺位保险的净趸缴保费：

$$E[Z] = \int_{-\infty}^{\infty} E[Z|T = t]f_T(t)dt. \tag{12.4.2}$$

例 12.4.2: 仅仅用先死顺位保险的净趸缴保费表示 $\overline{A}_{wxy}^{\;\;2}$。

解：设 Z 是保险受益在投保时的现值随机变量。鉴于保险在 (y) 死亡之时支付受益，选择 $T(y)$ 在 (12.4.2) 的条件期望中起 T 的作用。当 (y) 死于时间 t 时，只有在那时 $(w),(x)$ 中恰好有一个活着时才支付受益 1，于是

$$E[Z|T(y) = t] = v^t{}_tp_{\overline{wx}}^{[1]},$$

$$
\begin{aligned}
\overline{A}_{wxy}^{\;\;2} &= E[Z] = \int_0^{\infty} E[Z|T(y) = t]{}_tp_y\mu_{y+t}dt \\
&= \int_0^{\infty} v^t{}_tp_{\overline{wx}}^{[1]}{}_tp_y\mu_{y+t}dt \\
&= \int_0^{\infty} v^t({}_tB_1 - 2{}_tB_2){}_tp_y\mu_{y+t}dt \\
&= \overline{A}_{wy}^{\;\;1} + \overline{A}_{xy}^{\;\;1} - 2\overline{A}_{wxy}^{\;\;\;1}.
\end{aligned}
$$

§12.5 复合顺位函数

这一节中函数与上一节的有所不同，死亡受益也决定于支付受益时发生的死亡之前的死亡次序，这些死亡次序用放在有关生

命的符号之下的数字来表示。我们先考察以下几个符号，并注意记号中的可能差别。

如 ${}_nq_{\underset{1}{x}\overset{2}{y}z}$，表示 (y) 第 2 个死于 n 年内而且第一个死者为 (x) 的概率，相应的事件为 $T(x) < T(y) < \min(T(z), n)$ 。另一个符号 ${}_nq_{\underset{1}{x}y\overset{3}{z}}$ 涉及到的 3 个死亡顺序相同，所不同的是第 3 个死亡于 n 年内，相应事件为 $T(x) < T(y) < T(z) < n$ 。不过记号 ${}_nq_{x\underset{2}{y}\overset{3}{z}}$ 与后者含义相同，都表示 (z) 第 3 个死并死于 n 年内而且第 1 个死者为 (x)[即第 2 个死者为 (y)] 的概率。这类函数称为 **复合顺位函数**(compound contingent function).

例 12.5.1: 对于 $(w),(x),(y),(z)$ 按这个次序死亡，并且 (w) 与 (z) 的死亡间隔短于 10 年， (x) 与 (y) 的死亡间隔短于 5 年这一事件，导出概率表达式。

解：首先运用 (12.4.1) 的一个多元形式确定事件

$$A = \left\{ \begin{array}{c} T(w) < T(x) < T(y) < T(z) \\ T(z) - T(w) < 10 \\ T(y) - T(x) < 5 \end{array} \right\}. \tag{12.5.1}$$

选择 $T(w),T(x)$ 作为条件，这是因为它们包含在 $T(y)$ 与 $T(z)$ 的上下界中。于是

$$Pr(A) = \int_0^\infty \int_0^\infty Pr[A|(T(w) = r) \cap (T(x) = s)]g(r, s)dsdr, \tag{12.5.2}$$

其中 $g(r, s)$ 是 $T(w)$ 与 $T(x)$ 的联合概率密度函数。$Pr[A|(T(w) = r) \cap (T(x) = s)]$ 等于 $Pr[A^*]$, 其中

$$A^* = \left\{ \begin{array}{c} r < s < T(y) < T(z) < r + 10 \\ T(y) < s + 5 \end{array} \right\}.$$

概率用 $T(x) = s$ 且 $T(w) = r$ 条件下 $T(y)$ 与 $T(z)$ 的条件分布

计算。这样 $Pr(A^*)$ 可在随机变量 $T(y)$ 与 $T(z)$ 的样本空间中表示。两种情况在图 12.5.1 中给出。

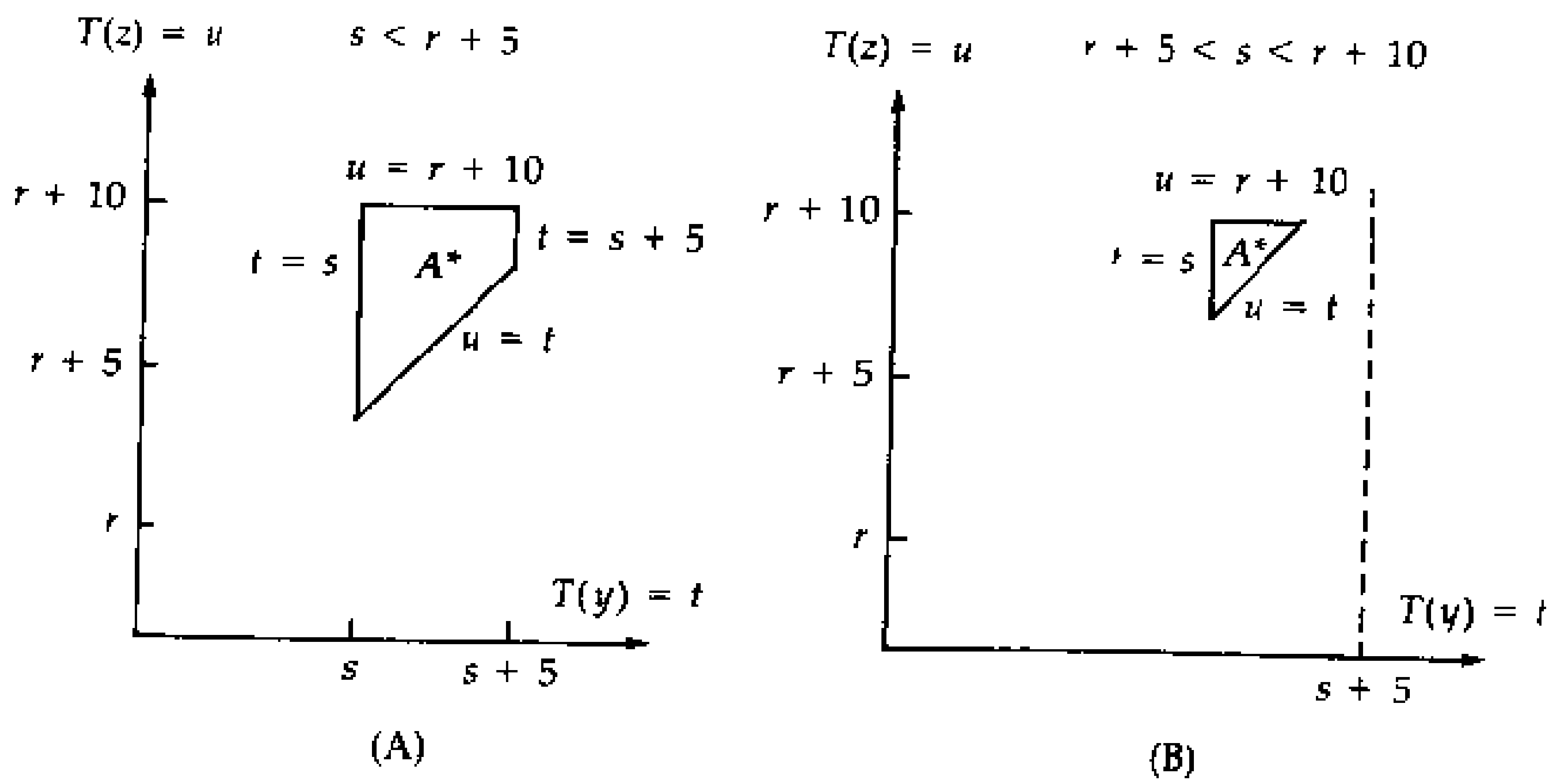


图 12.5.1　情形 (A):$s < r+5$; 情形 (B):$r+5 < s < r+10$

将 $T(x)=s$ 且 $T(w)=1$ 条件下 $T(y)$ 与 $T(z)$ 的条件概率密度函数记为 $h(t,u)$, 则有

$$Pr(A^*) = \begin{cases} \int_s^{s+5}\int_t^{r+10} h(t,u)dudt & r < s < r+5 \\ \int_s^{r+10}\int_t^{r+10} h(t,u)dudt & r+5 < s < r+10 \\ 0 & s > r+10\text{或}s < r. \end{cases}$$

代入 (12.5.2) 得

$$\begin{aligned} Pr(A) \;=\; & \int_0^\infty \int_r^{r+5} \int_s^{s+5} \int_t^{r+10} h(t,u)g(r,s)dudtdsdr \\ & + \int_0^\infty \int_{r+5}^{r+10} \int_s^{r+10} \int_t^{r+10} h(t,u)g(r,s)dudtdsdr. \end{aligned}$$

在剩余寿命相互独立的条件下，被积函数可用

$$_rp_w\mu_{w+r}\,{}_sp_x\mu_{x+s}\,{}_tp_y\mu_{y+t}\,{}_up_z\mu_{z+u}$$

取代。

下面考虑可用单重积分表示的若干复合顺位概率。首先应用 (12.4.1) 得出一个概率的各种等价形式。

例 12.5.2: 写出 ${}_nq_{\underset{1}{x}\overset{3}{y}z}$ 的三种不同的积分公式，并将其中的一个化成只依赖于第一个死亡的概率函数。

解：这里 $A=\{T(x)<T(g)<T(j)<n\}$ 。我们按每个剩余寿命作为条件写出 3 种积分公式：

$$_nq_{\underset{1}{x}y\overset{3}{z}}=\int_0^{\infty}Pr[A|T(x)=t]\,_tp_x\mu_{x+t}dt,$$

由

$$Pr[A|T(x)=t]=\begin{cases}0 & t>n\\ _tp_{yz}\ {}_{n-t}q_{y+t:\overset{2}{z+t}} & t\le n\end{cases}$$

得

$$_nq_{\underset{1}{x}y\overset{3}{z}}=\int_0^{n}{}_tp_{yz}\,{}_{n-t}q_{y+t:\overset{2}{z+t}}\,{}_tp_x\mu_{x+t}dt.$$

类似地有

$$\begin{aligned}_nq_{\underset{1}{x}y\overset{3}{z}} &= \int_0^{\infty}Pr[A|T(y)=t]\,_tp_y\mu_{y+t}dt\\ &= \int_0^{n}{}_tq_x\,{}_tp_z\,{}_{n-t}q_{z+t}\,{}_tp_y\mu_{y+t}dt.\end{aligned}$$

及

$$\begin{aligned}_nq_{\underset{1}{x}y\overset{3}{z}} &= \int_0^{\infty}Pr[A|T(z)=t]\,_tp_z\mu_{z+t}dt\\ &= \int_0^{n}{}_tq_{\overset{2}{xy}}\,{}_tp_z\mu_{z+t}dt.\end{aligned}$$

第二个积分公式可按以下途径用先死概率来表示：

$$\begin{aligned}
{}_nq_{\underset{1}{x}yz}^{\;\;3} &= \int_0^n (1-{}_tp_x)({}_tp_z-{}_np_z)\,{}_tp_y\mu_{y+t}dt \\
&= {}_nq_{yz}^{1} - {}_nq_{xyz}^{\;1} - {}_np_z({}_nq_y - {}_nq_{xy}^{\;1}).
\end{aligned}$$

例 12.5.3: 写出 ${}_nq_{\underset{12}{wx}\,yz}^{\;\;\;3}$ 的表达式。

解：相应的事件为

$$A=\{T(w)<T(x)<T(y)<T(z)\ 且 T(y)<n\},$$

于是

$$\begin{aligned}
{}_nq_{\underset{12}{wx}\,yz}^{\;\;\;3} &= \int_0^\infty Pr[A|T(y)=t]\,{}_tp_y\mu_{y+t}dt \\
&= \int_0^\infty {}_tq_{wx}^{\;2}\,{}_tp_z\,{}_tp_y\mu_{y+t}dt \\
&= \int_0^n ({}_tq_x - {}_tq_{wx}^{\;1})\,{}_tp_z\,{}_tp_y\mu_{y+t}dt \\
&= {}_nq_{yz}^{1} - {}_nq_{xyz}^{\;1} - \int_0^n {}_tq_{wx}^{\;1}\,{}_tp_z\,{}_tp_y\mu_{y+t}dt.
\end{aligned}$$

在这一节的例子中应用 (12.4.1) 时，我们使用了剩余寿命独立的假设来写出被积项中的因子。现在考虑在对每个涉及到的生命使用单重 Gompertz 死亡律时这些复合顺位概率的数值求解。

例 12.5.4: 在 Gompertz 死亡律之下，证明

$${}_\infty q_{\underset{12}{wx}\,yz}^{\;\;\;3} = {}_\infty q_{wxyz}^{\;1}\,{}_\infty q_{xyz}^{\;1} - {}_\infty q_{yz}^{1}.$$

解：在 (12.5.3) 中令 $n\to\infty$ 得

$${}_\infty q_{\underset{12}{wx}\,yz}^{\;\;\;3} = \int_0^\infty {}_tq_w\,{}_tp_{yz}\,{}_\infty q_{y+t:z+t}^{1}\,{}_tp_x\mu_{x+t}dt. \qquad (12.5.4)$$

在第六章的例子中曾经证明，在 Gompertz 死亡律之下成立

$$
{}_nq^{\ 1}_{xy} = \frac{c^x}{c^w}{}_nq_w, \tag{12.5.5}
$$

其中 $c^w = c^x + c^y$ 。将这个公式用于 (12.5.4) 被积函数中的 ${}_\infty q^{\quad 1}_{y+t:z+t}$，可得

$$
\begin{aligned}
{}_\infty q_{\underset{12}{wx}\overset{3}{y}z} &= \int_0^\infty \frac{c^{y+t}}{c^{y+t}+c^{z+t}}{}_tq_{w\,t}p_{yz\,t}p_x\mu_{x+t}dt \\
&= \frac{c^y}{c^y+c^z}({}_\infty q^{\ 1}_{xyz} - {}_\infty q^{\ \ 1}_{wxyz}).
\end{aligned}
$$

公式 (12.5.5) 可推广到多于两个生命的情形，用于以上表达式可得

$$
\begin{aligned}
{}_\infty q_{\underset{12}{wx}\overset{3}{y}z} &= \frac{c^y}{c^y+c^z}\left(\frac{c^x}{c^x+c^y+c^z} - \frac{c^x}{c^w+c^x+c^y+c^z}\right) \\
&= \left(\frac{c^w}{c^w+c^x+c^y+c^z}\right)\left(\frac{c^x}{c^x+c^y+c^z}\right)\left(\frac{c^y}{c^y+c^z}\right) \\
&= {}_\infty q^{\ 1}_{wxyz}\,{}_\infty q^{\ 1}_{xyz}\,{}_\infty q^{\ 1}_{yz}.
\end{aligned}
$$

§12.6 继承年金

继承年金(reversionary annuity) 为在一种状况 (v) 消亡后，而另一种状况 (u) 存在的情况下提供年金支付。连续支付的继承年金精算现值记为 $\bar{a}_{v|u}$ 。如果 (v) 是确定的期限，那么继承年金就成为 (u) 的延期生存年金；如果 (u) 是确定的期限，那么继承年金成为家庭收入保险。

考察在 (x) 死亡之后向 (y) 按年率 1 连续支付的年金，在时间 0 的现值为

$$
Z = \begin{cases} {}_{T(x)|}\bar{a}_{\overline{T(y)-T(x)}|} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases} \tag{12.6.1}
$$

于是对独立剩余寿命，年金的精算现值为

$$\begin{aligned}\overline{a}_{x|y} = E[Z] &= \int_0^\infty \int_t^\infty {}_{t|}\overline{a}_{\overline{s-t}|}\,{}_sp_y\mu_{y+s}\,{}_tp_x\mu_{x+t}dsdt \\ &= \int_0^\infty v^t\,{}_tp_y\Big[\int_t^\infty \overline{a}_{\overline{s-t}|}\,{}_{s-t}p_{y+t}\mu_{y+s}ds\Big]{}_tp_x\mu_{x+t}dt \\ &= \int_0^\infty v^t\,{}_tp_y\overline{a}_{y+t}\,{}_tp_x\mu_{x+t}dt.\end{aligned} \tag{12.6.2}$$

最后一个表达式具有 (12.4.2) 以 $T(x)$ 为条件的形式。

式 (12.6.1) 可改写成

$$Z = \begin{cases} \overline{a}_{\overline{T(y)}|} - \overline{a}_{\overline{T(x)}|} & T(x) \le T(y) \\ 0 & T(x) > T(y) \end{cases}$$

或者

$$Z = \begin{cases} \overline{a}_{\overline{T(y)}|} - \overline{a}_{\overline{T(x)}|} & T(x) \le T(y) \\ \overline{a}_{\overline{T(y)}|} - \overline{a}_{\overline{T(y)}|} & T(x) > T(y), \end{cases}$$

这表明

$$Z = \overline{a}_{\overline{T(y)}|} - \overline{a}_{\overline{T(xy)}|}. \tag{12.6.3}$$

由 (12.6.3) 可得

$$\overline{a}_{x|y} = E[Z] = E[\overline{a}_{\overline{T(y)}|}] - E[\overline{a}_{\overline{T(xy)}|}] = \overline{a}_y - \overline{a}_{xy}. \tag{12.6.4}$$

用当期支付技巧可得

$$\overline{a}_{x|y} = \int_0^\infty v^t\,{}_tp_y(1 - {}_tp_x)dt = \overline{a}_y - \overline{a}_{xy}.$$

这两个公式对一般状况 (u) 与 (v) 也成立，譬如

$$\overline{a}_{x:\overline{n}|\,|y} = \overline{a}_y - \overline{a}_{xy:\overline{n}|},$$

$$\overline{a}_{x|y:\overline{n}|} = \overline{a}_{y:\overline{n}|} - \overline{a}_{xy:\overline{n}|};$$

$$\overline{a}_{\overline{xy}|z} = \overline{a}_z - \overline{a}_{\overline{xy}:z} = \overline{a}_z - \overline{a}_{xz} - \overline{a}_{yz} + \overline{a}_{xyz}.$$

在以上例子中，确定的期限从时间 0 开始计算。对于在 (x) 死亡之后向 (y) 支付 n 年定期年金的继承年金，确定的期限是(随机)递延的状况，因此我们从最基本的原理出发。在年金签单时的现值为

$$Z = \begin{cases} 0 & T(y) \le T(x) \\ v^{T(x)}\overline{a}_{\overline{T(y)-T(x)}|} & T(x) < T(y) \le T(x)+n \\ v^{T(x)}\overline{a}_{\overline{n}|} & T(y) > T(x)+n. \end{cases}$$

按条件 $T(x)=t$ 运用 (12.4.2), 可写出精算现值

$$\begin{aligned} E[Z] &= \int_0^\infty E[Z|T(x)=t]{}_tp_x\mu_{x+t}dt \\ &= \int_0^\infty {}_tp_y v^t \overline{a}_{y+t:\overline{n}|}\,{}_tp_x\mu_{x+t}dt. \end{aligned} \tag{12.6.5}$$

将

$$\overline{a}_{y+t:\overline{n}|} = \int_t^{t+n} v^{s-t}{}_{s-t}p_{y+t}ds$$

代入 (12.6.5) 可得

$$E[Z] = \int_0^\infty \int_t^{t+n} v^s{}_sp_y\,{}_tp_x\mu_{x+t}dsdt.$$

接下去交换积分次序，

$$\begin{aligned} E[Z] &= \int_0^n \int_0^s v^s{}_sp_y\,{}_tp_x\mu_{x+t}dtds + \int_n^\infty \int_{s-n}^s v^s{}_sp_y\,{}_tp_x\mu_{x+t}dtds \\ &= \int_0^n v^s{}_sp_y(1-{}_sp_x)ds + \int_n^\infty v^s{}_sp_y({}_{s-n}p_x - {}_sp_x)ds \qquad (12.6.6) \\ &= \overline{a}_{y:\overline{n}|} - \overline{a}_{xy:\overline{n}|} + v^n{}_np_y\overline{a}_{x:y+n} - v^n{}_np_{xy}\overline{a}_{x+n:y+n}. \end{aligned}$$

式 (12.6.6) 是这个精算现值的当期支付形式。

在实践中年金是按离散基础支付的，这就需要区分支付期基于死亡日期的契约与支付期基于签约日期的契约。为表示前者，在年金符号上加一个帽子"^"。以下导出可用于离散年金的数值求解的表达式。对两个状况 (w) 与 (v), (12.6.3) 与 (12.6.4) 可修改后用在年付 m 次情形，

$$\ddot{a}_{v|u}^{(m)} = \ddot{a}_u^{(m)} - \ddot{a}_{uv}^{(m)}. \tag{12.6.7}$$

如果 (u) 及(uv) 的消亡时间在每一年都均匀分布，那么

$$\begin{aligned}\ddot{a}_{v|u}^{(m)} &= [\alpha(m)\ddot{a}_u - \beta(m)] - [\alpha(m)\ddot{a}_{uv} - \beta(m)] \\ &= \alpha(m)\ddot{a}_{v\,u}. \end{aligned} \tag{12.6.8}$$

例 12.6.1: 用 $\ddot{a}_{x|y}$ 与 $\ddot{a}_{x:\overline{n}|\,|y}$ 分别表示 (1) $\ddot{a}_{x|y}^{(12)}$　(2) $\ddot{a}_{x:\overline{n}|\,|y}^{(12)}$。

解：

(1) 设 $T(y)$ 在每一年龄中均匀分布，且 $T(xy)$ 在每一处中近似于均匀分布，根据 (12.6.8) 可得

$$\ddot{a}_{x|y}^{(12)} = \alpha(12)\ddot{a}_{x|y}.$$

(2) 由于 $(yx:\overline{n}|)$ 的消亡时间在第 n 年并不是均匀分布的，我们不能对第二个年金使用 (12.6.8)。为此重新考察 (12.6.8) 的论证，

$$\begin{aligned}\ddot{a}_{x:\overline{n}|\,|y}^{(12)} &= \ddot{a}_y^{(12)} - \ddot{a}_{yx:\overline{n}|}^{(12)} \\ &= [\alpha(12)\ddot{a}_y - \beta(12)] - [\alpha(12)\ddot{a}_{yx:\overline{n}|} - \beta(12)(1 - {}_nE_{xy})] \\ &= \alpha(12)\ddot{a}_{x:\overline{n}|\,|y} - \beta(12){}_nE_{xy}. \end{aligned}$$

对于支付期基于死亡日期的继承年金，只得诉诸最基本的原理。设 Z 是在 (x) 死亡日开始向 (y) 支付的年付 m 次期初年金

的现值随机变量，我们有

$$\begin{aligned}\mathring{a}_{y+t}^{(m)} &= E[Z] = \int_0^{\infty} E[Z|T(x)=t]_t p_x \mu_{x+t} dt \\ &= \int_0^{\infty} v^t{}_t p_y \ddot{a}_{y+t}^{(m)}{}_t p_x \mu_{x+t} dt.\end{aligned}$$

运用第三章中的有关等式可写出

$$\ddot{a}_{y+t}^{(m)} = \frac{\alpha(m)}{\alpha(\infty)}\overline{a}_{y+t} + \frac{\alpha(m)}{\alpha(\infty)}\beta(\infty) - \beta(m),$$

令

$$\varepsilon(m) = \frac{\alpha(m)}{\alpha(\infty)}\beta(\infty) - \beta(m) = \frac{i^{(m)} - \delta}{i^{(m)} d^{(m)}}, \qquad (12.6.9)$$

可得出

$$\begin{aligned}\mathring{a}_{x|y}^{(m)} &= \frac{\alpha(m)}{\alpha(\infty)}\int_0^{\infty} v^t{}_t p_y \overline{a}_{y+t}{}_t p_x \mu_{x+t} dt + \varepsilon(m)\int_0^{\infty} v^t{}_t p_y{}_t p_x \mu_{x+t} dt \\ &= \frac{\alpha(m)}{\alpha(\infty)}(\overline{a}_y - \overline{a}_{xy}) + \varepsilon(m)\overline{A}_{xy}^{1} \\ &= \alpha(m)\ddot{a}_{x|y} + \varepsilon(m)\overline{A}_{xy}^{1}. \qquad (12.6.10)\end{aligned}$$

§12.7　净保费与责任准备金

这一节考察本章保险的责任准备金。与第四章一样，净 (年) 保费根据平衡原理确定，净保费责任准备金则根据前瞻法作为在生存到所考虑时间的条件下未来亏损的条件期望值。

保险缴付期不应晚于理赔支付的时间，如在顺位保险情形，在已经可以断定不会有理赔支付的时候，缴费即应停止。当然缴费期始终可以更短。

在先死赔付的保险情形，保费只有当所有人都活着时才可能缴付。根据平衡原理，可得：

$$P_{xy}\ddot{a}_{xy} = A_{xy},$$

$$
{}_{10}P^{\{4\}}(\overline{A}^{\;1}_{\overline{xy}:\overline{20|}})\ddot{a}^{\{4\}}_{xy:\overline{10|}} = \overline{A}^{\;1}_{\overline{xy}:\overline{20|}},
$$

$$
P(\overline{A}^{1}_{xyz})\ddot{a}_{xyz} = \overline{A}^{1}_{xyz}.
$$

对于在第 2 个或以后死亡时赔付的保险有多于一种的自然缴费期。为了使对于某种保险受益的净保费最低，我们使用最长的缴费期。以下例子对若干情形说明了有关过程。

例 12.7.1: 用平衡原理写出以下净保费的方程。

(1) $P_{\overline{xy}}$. (2) $P(\overline{A}^{\;\;2}_{\overline{xyz}})$. (3) $P(\overline{A}_{\overline{wx}:yz})$.
(4) $P(\overline{A}^{\;\;2}_{xyz})$. (5) $P(\overline{A}^{\;\;2}_{\underset{1}{x}yz})$.

解：

(1) $P_{\overline{xy}}\ddot{a}_{\overline{xy}} = A_{xy}$.
(2) $P(\overline{A}^{\;\;2}_{\overline{xyz}})\ddot{a}^{\;\;2}_{\overline{xyz}} = \overline{A}^{\;\;2}_{\overline{xyz}}$.
(3) $P(\overline{A}_{\overline{wx}:yz})\ddot{a}_{\overline{wx}:yz} = \overline{A}_{\overline{wx}:yz}$.
(4) 只要 (y) 活着并且 (x) 与 (z) 中至少有一个还活着，受益支付仍有可能，于是，

$$
P(\overline{A}^{\;\;2}_{xyz})\ddot{a}_{y:\overline{xz}} = \overline{A}^{\;\;2}_{xyz}.
$$

(5) 当 (y) 与 (z) 都还活着时，受益支付仍有可能。因此合适的缴费期是 (yz) 的存在期，从而

$$
P(\overline{A}^{\;\;2}_{\underset{1}{x}yz})\ddot{a}_{yz} = \overline{A}^{\;\;2}_{\underset{1}{x}yz}.
$$

作为未来亏损的条件期望，净保费责任准备金依赖于计算中使用的状况的条件。对于先死赔付保险，责任准备金是唯一的，因为在保险终止前所有人都必须还活着。我们给出两种保险的责任准备金：

$$
{}_5V^{\;1}_{\overline{xy}:\overline{10|}} = A^{\;\;\;\;\;\;\;1}_{\overline{x+5:y+5}:\overline{5|}} - P^{1}_{\overline{xy}:\overline{10|}}\ddot{a}_{x+5:y+5:\overline{5|}},
$$

其中

$$P_{\overset{1}{xy}:\overline{10|}}\ddot{a}_{xy:\overline{10|}} = A_{\overset{1}{xy}:\overline{10|}},$$

$$_5V_{xyz}^{1} = A_{x+5:y+5:z-5}^{1} - P_{xyz}^{1}\ddot{a}_{x+5:y+5:z+5}.$$

对于在第 2 个或以后死亡时赔付的保险，净保费责任准备金既可以在哪些人还活着的条件下通过条件期望计算，也可以在保险尚未终止的条件下通过条件期望计算。考虑一个简单的情形，在 $(\overline{xy})$ 消亡之时赔付一单位的保险，$_tL$ 是在时间 t 的未来亏损，以哪些人还活着作为条件，我们有

$$E[_tL|T(x)>t,\ T(y)>t] = \overline{A}_{\overline{x+t:y+t}} - \overline{P}(\overline{A}_{\overline{xy}})\overline{a}_{\overline{x+t:y+t}}, \tag{12.7.1}$$

$$E[_tL|T(x)>t,\ T(y)\le t] = \overline{A}_{x+t} - \overline{P}(\overline{A}_{\overline{xy}})\overline{a}_{x+t}, \tag{12.7.2}$$

$$E[_tL|T(x)\le t,\ T(y)>t] = \overline{A}_{y+t} - \overline{P}(\overline{A}_{\overline{xy}})\overline{a}_{y+t}. \tag{12.7.3}$$

在保险尚未终止的条件下，我们需要 $E[_tL|T(\overline{xy})>t]$，它可通过以下和来计算：

$$\begin{aligned}
&E[_tL|T(x)>t,\ T(y)\le t]Pr[T(x)>t,\ T(y)\le t|T(\overline{xy})>t]\\
&+E[_tL|T(x)\le t,\ T(y)>t]Pr[T(x)\le t,\ T(y)>t|T(\overline{xy})>t]\\
&+E[_tL|T(x)>t,\ T(y)>t]Pr[T(x)>t,\ T(y)>t|T(\overline{xy})>t].
\end{aligned}$$

在这个表达式中，条件期望由 (12.7.1)-(12.7.3) 给出。在 $T(x)$ 与 $T(y)$ 独立的假设下，概率的形式如下，

$$\begin{aligned}
&Pr[T(x)>t, T(y)\le t|T(\overline{xy})>t]\\
&= \frac{_tp_x(1-{_tp_y})}{_tp_x(1-{_tp_y}) + {_tp_y}(1-{_tp_x}) + {_tp_x}{_tp_y}}
\end{aligned}$$

等等。将这些结合在一起，可得

$$_tV(\overline{A}_{\overline{xy}}) = \frac{1}{_tp_x(1-{_tp_y}) + {_tp_y}(1-{_tp_x}) + {_tp_x}{_tp_y}}$$

$$
\begin{aligned}
&\{{}_tp_x(1-{}_tp_y)[\overline{A}_{x+t}-\overline{P}(\overline{A}_{\overline{xy}})\overline{a}_{x+t}] \\
+\ & {}_tp_y(1-{}_tp_x)[\overline{A}_{y+t}-\overline{P}(\overline{A}_{\overline{xy}})\overline{a}_{y+t}] \\
+\ & {}_tp_x{}_tp_y[\overline{A}_{\overline{x+t:y+t}}-\overline{P}(\overline{A}_{\overline{xy}})\overline{a}_{\overline{x+t:y+t}}]\}.
\end{aligned}
$$

习　题

§12.2

1. 描述具有如下概率的事件，

a. ${}_tp_{wx}+{}_tp_{wy}+{}_tp_{wz}+{}_tp_{xy}+{}_tp_{xz}+{}_tp_{yz}$
$-3({}_tp_{wxy}+{}_tp_{wxz}+{}_tp_{wyz}+{}_tp_{xyz})+7{}_tp_{wxyz}$

b. ${}_tp_w+{}_tp_x+{}_tp_y+{}_tp_z-2({}_tp_{wx}+{}_tp_{wy}+{}_tp_{wz}+{}_tp_{xy}+{}_tp_{xz}$
$+{}_tp_{yz})+4({}_tp_{wxy}+{}_tp_{wxz}+{}_tp_{wyz}+{}_tp_{xyz})-8{}_tp_{wxyz}$.

2. 用 12.2 节中的推论证明 ${}_tp_{\overline{x_1x_2\cdots x_m}}^{[0]}=1-{}_tp_{\overline{x_1x_2\cdots x_m}}^{1}$.

3. 下表为利率是 $3\frac{1}{2}$ 的连生年金表的摘录：

连生状况	连生延付年金的精算现值
20:26:28	14.4
20:26:29	14.3
20:28:29	14.0
26:28:29	13.8
20:26:28:29	12.5

a. 计算在每年年末当 (20), (26), (28), (29) 中正好有 3 人存活时支付的年金的精算现值。

b. 对在 (20),(26), (28) 和 (29) 中第 2 个人死亡的年末支付 10000 的保险，计算其净趸缴保费。

4. 用 ${}_tB_j, j=1,2,3,4$ 表达 ${}_tP_{\overline{wxyz}}^{2}-{}_tP_{\overline{wxyz}}^{[2]}$ 。

5. 对在 (w) 活着，而 $(x),(y)$ 和 (z) 中最多只有 1 人活着时每年末支付 1 的年金，试用年金符号表示其精算现值。

6. 若 $\mu_{40+t}=0.002, 0\le t\le 10$ 及 $\delta=0.05$，计算 $\overline{A}_{40:40:40:40:40:\overline{10}|}$。

7. 有一项信托基金为 $(x),(y),(z)$ 提供收入。当 3 个人都活着时，该基金以年率 8 向每人提供连续的收入；当有 2 个人活着时，他们每人可得年率为 10 的收入；只有 1 个人活着时，他可得到年率为 15 的收入，计算下列支付的精算现值：

a. 所有的支付。

b. 所有给 (x) 的支付。

8. 有一份保单，当 4 个年龄为 x 中第 1 个人死亡时，它立即为其提供数额为 4 的死亡受益；第 2 个死亡时提供数额为 3 的受益，第 3 个死亡时提供数额为 2 的受益，最后一个人死亡时提供数额为 1 的受益。若 $\overline{A}_x=0.4$ 及 $\overline{A}_{xx}=0.5$，计算该保单的净趸缴保费。

§12.3

9. 用单个及连生年金符号表示在下列条件下每月支付 1000 的期末年金的精算现值：

a. 在未来 25 年中，(40) 和 (35) 正好只有 1 个人存活。

b. 在 65 岁之前，(40) 和 (35) 中至少有 1 人存活。

10. 用确定年金及单个和连生年金符号来表示下列式子

a. $\overline{a}_{\overline{x:y:\overline{n}|}}$。

b. $\overline{b}_{\overline{(25:\overline{40}):\overline{30}|}}$。

§12.4

11. 若在每一时刻，(x) 的死亡效力是 (y) 的 1/2，而 (z) 的死亡效力是 (y) 的 2 倍，试问在 3 人中，(x) 死于下列情况的概率是多少：

a. 第 1 个死亡。

b. 第 2 个死亡。

c. 第 3 个死亡。

12. 下列哪个 (些) 式子是正确的？改正其它式子。

Ⅰ. $\overline{A}_{\overline{wxy\overset{1}{z}}} = \overline{A}_{\overset{1}{w}xyz} + \overline{A}_{w\overset{1}{x}yz} + \overline{A}_{wx\overset{1}{y}z} + \overline{A}_{wxy\overset{1}{z}}$。

Ⅱ. $\overline{A}_{\overline{wxy\overset{3}{z}}} = \overline{A}_{\overset{2}{w}xyz} + \overline{A}_{w\overset{2}{x}yz} + \overline{A}_{wx\overset{2}{y}z} + \overline{A}_{wxy\overset{2}{z}}$。

Ⅲ. $\overline{A}_{wxy\overset{3}{z}} = \overline{A}_{w\overset{1}{z}} + \overline{A}_{x\overset{1}{z}} + \overline{A}_{y\overset{1}{z}} - (\overline{A}_{wx\overset{1}{z}} + \overline{A}_{wy\overset{1}{z}} + \overline{A}_{xy\overset{1}{z}}) + \overline{A}_{wxy\overset{1}{z}}$。

13. 对一个当 (x) 活过 (y) 时，在 (x) 死时支付的保险，用定积分写出其净趸缴保费。其中受益额等于从保单签发到 (y) 死亡的时间间隔。

14. 若 Gompertz 法则使用 $\mu_{40} = 0.003, \mu_{56} = 0.012$, 试计算，

a. ${}_{\infty}q_{40:\overset{2:3}{48}:56}$　　　　b. ${}_{\infty}q_{40:\overset{2}{48}:56}$。

[注意：在 a 中，记号 2:3 表示 (48) 是第 2 个或第 3 个死亡。]

15. 一个签发给 (x), (y) 和 (z) 的数额为 1 的保险，在 (x) 至少死了 10 年而 (y) 死了还不到 10 年的情况下，当 (z) 死时支付，试用死亡保险与纯生存保险的净趸缴保费来表示该保险的净趸缴保费。

16. 假设 (y) 或 (z) 活过 (x), 或 (y) 和 (z) 都活过 (x), 且两者都在 10 年期末之前死亡，对在 (x) 死后 10 年支付 1 的保险，求其净趸缴保费的不含积分的表达式。

17. 某特殊顺位保险，当 (30) 在 (60) 之前死亡或在 (60) 死后 5 年内死亡时，支付单位保额。另外，当 (30) 在 (60) 死后 5 年以外死亡而无受益时，退还不计利息的趸缴保费。设附加保费为净保费的 7.5%, 导出趸缴毛保费公式。

§12.5

18. 不使用独立假设，求关系式 ${}_nq_{wx\overset{1}{y}} = {}_nq_{wx\overset{1}{y}z} + {}_nq_{wx\overset{2}{y}\underset{1}{z}}$, 并用其去获得例 12.4.1 中的结果。

19. 不使用独立假设，建立下列关系式

$$\overline{A}_{x\overset{1}{y}} = \overline{A}_{x\overset{1}{y}z} + \overline{A}_{x\overset{2}{y}\underset{1}{z}},$$

$$\overline{A}_{yz}^{1} = \overline{A}_{xyz}^{\ 1} + \overline{A}_{\underset{1}{x}yz}^{\ 2},$$

并用其去获得例 12.4.2 中的结果。

20. 用以下方式表示 ${}_{\infty}q_{\underset{1}{w}xyz}^{\ 2}$:

a. 表示为一个定积分。

b. 用单次顺位概率符号表示。

21. 假设 Gompertz 规则适用，证明

a. ${}_tq_{xy}^{\ 2} = {}_tq_y - \frac{c^y}{c^x+c^y}{}_tq_{xy}$ 。

b. $\overline{A}_{\underset{1}{x}yz}^{\ 3} = \frac{c^x}{c^x+c^y}\overline{A}_z - \frac{c^z}{c^y+c^z}\overline{A}_{yz} + \frac{c^y}{c^x+c^y}\frac{c^z}{c^y+c^z}\overline{A}_{xyz}$ 。

22. 若对 $0 < x < 100$, $\mu_x = 1/(100-x)$ 适用于 (20),(40) 和 (60), 计算 a. ${}_{\infty}q_{20:40:60}^{\ \ \ 2}$ 。 b. ${}_{\infty}q_{20:40:60}^{\ \ \ 1}$ 。 c. ${}_{\infty}q_{20:40}^{\ \ \ 1}$ 。这说明在 Gompertz 规则基础上成立的 ${}_{\infty}q_{\underset{1}{x}yz}^{\ 2} = {}_{\infty}q_{xyz}^{1}\,{}_{\infty}q_{yz}^{1}$, 在一般情况下不成立。

23. 假设死亡表服从 Gompertz's 规则 ($c^8 = 2$), $\overline{A}_{54} = 0.3$, $\overline{A}_{62} = 0.4$, 且 $\overline{A}_{70} = 0.52$ 。试确定 $\overline{A}_{\underset{1}{54:54}:62}^{\ \ 2}$ 。

24. 已知 $\overline{A}_w = 0.6$, $\overline{A}_{wx}^{1} = 0.3$, $\overline{A}_{wxx}^{1} = 0.2$, $\overline{A}_{wxxx}^{1} = 0.1$, 计算

a. $\overline{A}_{wxxx}^{2}$ 。 b. $\overline{A}_{wxxx}^{4}$ 。 c. $\overline{A}_{w\underset{111}{xxx}}^{4}$ 。

25. 假设 (x), (y) 和 (z) 在未来 25 年内按以上顺序死亡，且每两人之间的死亡至少相差 10 年，试用积分形式表示其概率。

26. 假设 (10), (20) 和 (30) 都在达到 60 岁死去，且 (20) 是第 2 个死亡者，试用积分形式表达其概率。

27. 已知 ${}_{\infty}q_{xy}^{1} = 0.5537$, ${}_{\infty}q_{xz}^{1} = 0.6484$, ${}_{\infty}q_{xyz}^{1} = 0.5325$, ${}_{\infty}q_{xyz}^{\ 2} = {}_{\infty}q_{xyz}^{\ \ 3}$, 计算 ${}_{\infty}q_{x\underset{1}{y}z}^{\ 2}$.

28. 根据某死亡表已知，年龄分别为 70, 55 和 40 岁的 3 个人按以上顺序死亡且间隔时间不小于 15 年的概率为 0.048; 另外，2 个现龄为 70 岁的人中至少有 1 个在一个现龄为 55 岁的人死前

还能活 15 年的概率是 0.8 。试计算现龄为 40 岁的 2 个人都不能活到 70 岁的概率。

29. 下列哪个(些)式子正确？改正其它式子。

I. $\overline{A}_{\substack{wxyz\\12}}^{\;\;3} = \int_0^\infty v^t {}_tq_{wt}p_{xyz}\mu_{x+t}\overline{A}_{y+t}dt$ 。

II. $\int_0^{10}(1 - {}_{t+10}p_{50}){}_tp_{60}\mu_{60+t}dt$

$+ \int_{10}^\infty({}_{t-10}p_{50} - {}_{t+10}p_{50}){}_tp_{60}\mu_{60+t}dt$

$= \int_0^{10}(1 - {}_{t+10}p_{60}){}_tp_{50}\mu_{50+t}dt$

$+ \int_{10}^\infty({}_{t-10}p_{60} - {}_{t+10}p_{60}){}_tp_{50}\mu_{50+t}dt$ 。

III. ${}_{30}q^1_{40:50:60} + {}_{30}q^2_{40:50:60} = {}_{30}q^1_{40:\overline{50:60}}$ 。

§12.6

30. 证明

a. $\overline{A}^2_{xy} = \overline{A}_{xy}^{\;\,1} - \delta\overline{a}_{y|x}$ 。

b. $\frac{\partial}{\partial x}\overline{a}_{y|x} = \mu_x\overline{a}_{y|x} - \overline{A}^2_{xy}$ 。

31. 用 $\ddot{a}_{x|y:\overline{10}|}$ 表示 $\ddot{a}^{(12)}_{x|y:\overline{10}|}$ 。

32. 在下列情况下，不用积分形式分别写出支付年率为 1 的连续年金的精算现值：

a. 在 (y) 寿命期间及在 (y) 死后 10 年内，若 (x) 仍活着，则不支付年金。

b. 在 (y) 寿命期间及在 (y) 死后 10 年内，若 (x) 仍活着或 (y) 死在 (x) 之前，不支付年金。

33. 用年金和保险符号表示附加保费为毛保费的 8%，且能提供下列受益的毛趸缴保费：

一个关于 (x) 和 (y) 的年支付额为 1，递延 n 年并在第 1 个人死亡后减少 1/3 的最后生存者年金：若 (x) 死在 (y) 之前并在递延时期内，则减少后的年金从第 2 周年开始。若 (x) 死在 (y) 之后且在递延期间，趸缴保费在 (x) 死的那年末偿还。

34. 在 12.6 节中，若状况 (v) 消亡后状况 (u) 仍存活，则继承年金开始支付。该思想可推广到按规定次序的 2 个或多个死亡

发生时开始支付的年金。

a. 证明

$$\overline{a}_{xy|z}^{\ 2} = \overline{a}_{y|z} - \overline{a}_{xy|z}^{\ 1}.$$

b. 在 Gompertz 死亡表的基础上，证明

$$\overline{a}_{xy|z}^{\ 2} = \frac{c^x}{c^x + c^y}\overline{a}_z - \overline{a}_{yz} + \frac{c^y}{c^x + c^y}\overline{a}_{xyz}.$$

35. 一个顺位纯生存保险若 (x) 在 (y) 死后 n 年活着，则支付 1，求其净年缴保费的表达式。

36. 要获得相应于净趸缴保费 $A_{\underset{1}{w}xyz}^{\ \ 2}$ 的净年缴保费，需用什么年金的精算现值？

综合题

37. 一个保险，若 (x) 死在 $x+n$ 岁之前，(y) 死在 $y+m$ 岁之前，$m<n$，则在第 2 个死亡发生时那年末支付。

a. 证明其净趸缴保费可表示成

$$A_{\overline{xy}:\overline{m|}}^{1} + v^m {}_mp_x(1 - {}_mp_y)A_{x+m:\overline{n-m|}}^{1}.$$

b. 要获得其净年缴保费，用什么年金精算现值比较适合？

38. 一个 m 人集体同额分享一个年付为 1 连续支付的最后生存者年金收入。(x_1) 的份额的精算现值为

$$\sum_{j=0}^{m-1} \frac{1}{j+1}\overline{a}_{x_1:\overline{x_2x_3\cdots x_m}^{[j]}}.$$

证明该精算现值可表示成

$$\overline{a}_{x_1} - \tfrac{1}{2}(\overline{a}_{x_1x_2} + \cdots + \overline{a}_{x_1x_m}) + \tfrac{1}{3}(\overline{a}_{x_1x_2x_3} + \cdots + \overline{a}_{x_1x_{m-1}x_m}) - \cdots (-1)^{m-1}\tfrac{1}{m}\overline{a}_{x_1x_2\cdots x_m}.$$

[提示：对 $\sum_{j=0}^{m-1} \frac{1}{j+1}{}_tp_{\overline{x_2\cdots x_m}^{[j]}}$ 用定理 12.2.1]

39. 试用文字叙述 $\int_0^\infty v^t {}_tq_{xt}p_{yz}\mu_{y+t}\overline{A}_{\overset{1}{z+t}:\overline{10|}}dt$ 表示什么。

40. 在定理 12.2.1 的证明中，令

$$A_1 = \{T(y) < \min[n.T(x), T(z)]\},$$

$$A_2 = \{T(y) < \min[n, T(x), T(w)]\},$$

$$A_3 = \{T(y) < \min[n, T(w), T(z)]\}.$$

证明例 12.4.1 中的事件 A 和 A_1, A_2, A_3 中恰好有一个发生的事件一致。由此不用独立假设去求例 12.4.1 中的结果。[提示：讨论 $Pr(A_1) = {}_nq_{x\overset{1}{y}z}$, $Pr(A_1A_2) = Pr(A_1A_3) = Pr(A_2A_3) = Pr(A_1A_2A_3) = {}_nq_{wx\overset{1}{y}z}$ ，且 $Pr[A_1(非A_2)(非A_3)] = {}_nq_{\underset{1}{w}\,x\overset{2}{y}z}$.]

第十三章 人口理论

§13.1 引 言

第一章里的不少概念也构成人口数学理论的基本材料。例如，用于确定死亡时间随机变量分布并追踪生存组演变的生存函数，在建立人口模型时也起作用。

这一章建立的模型可应用于很多场合，如政治单位群体，职工群体甚至野生生物群体。

以下主要关心的是对于团体寿险的精算应用。在 §13.5 中，人口模型将用于研究向某个群体提供寿险受益的体系的演变。在第十四章研究向某个群体提供退休收入受益的体系的演变时，人口模型将用作有关模型的组成部分。

§13.2 Lexis 图

这一节将引入图示人口演变的一种很方便的方法。例如，劳动人口的历史可用被称为 Lexis 图(Lexis diagram) 的二维图中平行线表示 (见图 13.2.1) 。图中，个体加入劳动大军的进入点 (以进入时间与进入年龄为坐标) 是与该个体相联系的直线的一个端点，这条直线沿对角线路径延伸，终止于表示脱离劳动大军的退出点 (以退出时间与退出年龄为坐标) 。

对于图 13.2.1 中描画出的劳动人口，在相对于现在 $(t=0)$ 的过去时间 $t=-25$, 有 3 个在职劳动人口。现在 $(t=0)$ 有 2 个在职劳动人口。也许我们对他们未来的工作寿命感兴趣，图 13.2.1 中的虚线段就表示现在的 2 个在职劳动者的预期工作寿命。

以下评注概括了 Lexis 图的特征：

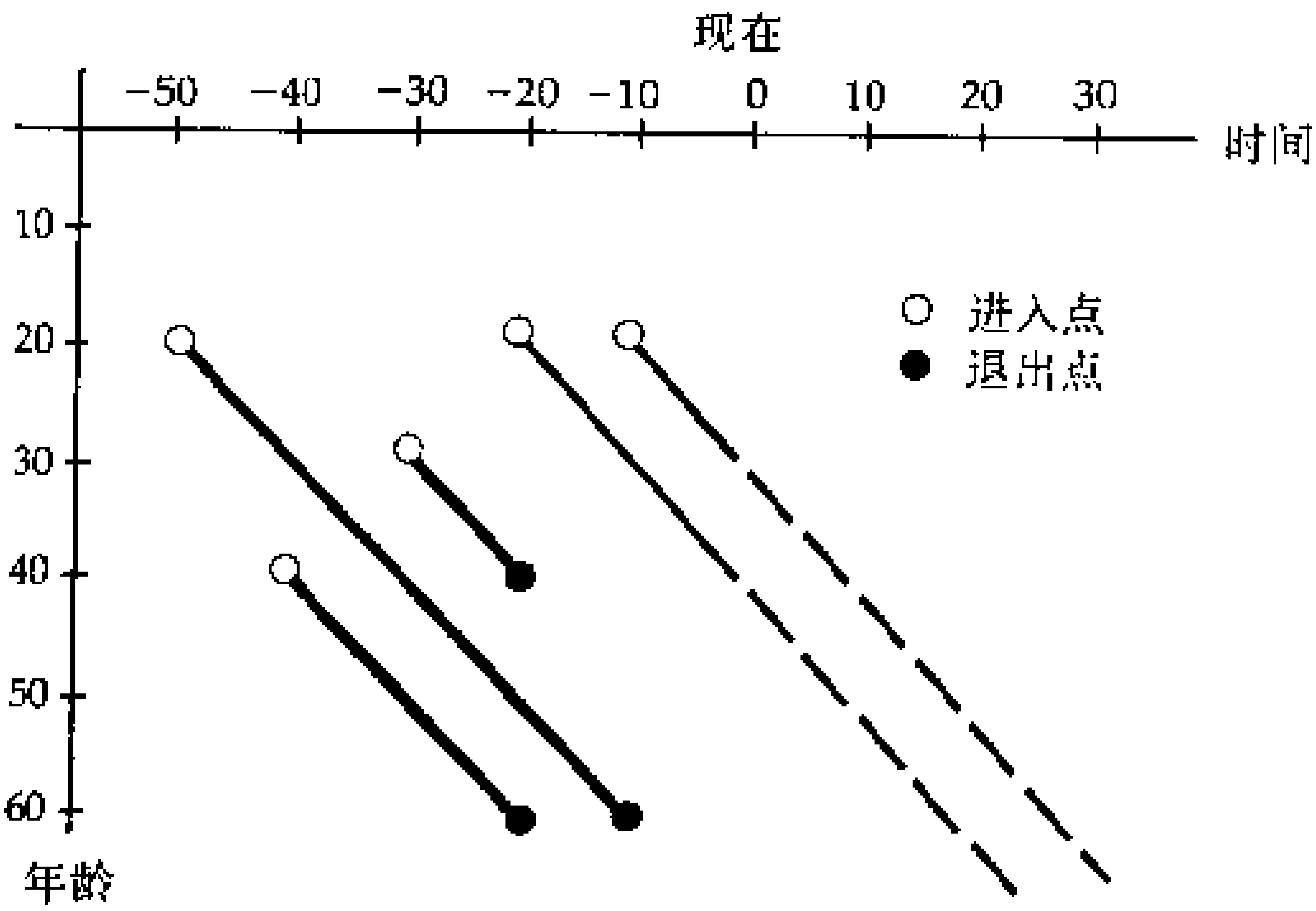


图 13.2.1　Lexis 图

1. 固定的时刻在图中表示为垂直线，在该时点的人口总数等于与该垂直线相交的平行直线段 (每条代表一个个体) 的条数。

2. 固定的年龄在图中表示为水平线，如果与某个个体相联系的直线段与处于 x_0 岁的水平线相交，那么该个体在 x_0 岁时是群体中的一个成员。

3. 在时间 t 达到年龄 x 的成员的出生时间为 $u=t-x$, 在 Lexis 图中用 x 与 t 作为坐标，而以后将经常使用变量 x 与 u 。

以上想法有很多推广。例如， Lexis 图可用于表示整个生存组的演变，而不仅仅是其中的各个个体，生存组是由一群具有公共出生期的个体组成的。在劳动者群体的模型里，可能有若干种

离开方式，进入也可能发生在不同年龄，这些可能性已在第七与第八章里讨论过。

以下两节建立的人口模型只涉及一种离开方式，可解释为死亡。类似地，只有出生将作为进入方式予以考虑。此外将采用决定性的方法。

§13.3 连续模型

这里将采用决定性的连续模型，并假定群体的增加源于出生，而减少则由于死亡。在时间 u 的 *出生密度函数*(density function of births) 记为 $b(u)$，即 $b(u)du$ 是时间 u 与 $u+du$ 之间的出生数。在时间 u 出生者的生存函数记为 $s(x,u)$，称为 *世代生存函数*(generation survival function)。定义

$$l(x,u)=b(u)s(x,u) \tag{13.3.1}$$

称为 *人口密度函数*(population density function)。对函数 $l(x,u)$ 的解释可借助于连续形式的 Lexis 图 (见图 13.3.1)，这个图以及本节以下的图 13.3.2– 图 13.3.6 都是二维的，它们被用来帮助解释一些微分项或表示积分区域。在每一种情形，都可以描绘出定义在这些时间 – 年龄平面上的函数的三维图形。

在 $l(0,u)du=b(u)du$ 个于时间 u 与 $u+du$ 之间出生的新生儿中，$l(x,u)du$ 个活到 x 岁。置 $t=x+u$，则 $dt=du$，前一表达式可解释为：

$$l(x,t-x)dt = \text{在时间 } t \text{ 与 } t+dt \text{ 之间达到 } x \text{ 岁的人数}. \tag{13.3.2}$$

于是在时间 t_0 与 t_1 之间达到 x 岁的人数为

$$\int_{t_0}^{t_1} l(x,t-x)dt. \tag{13.3.3}$$

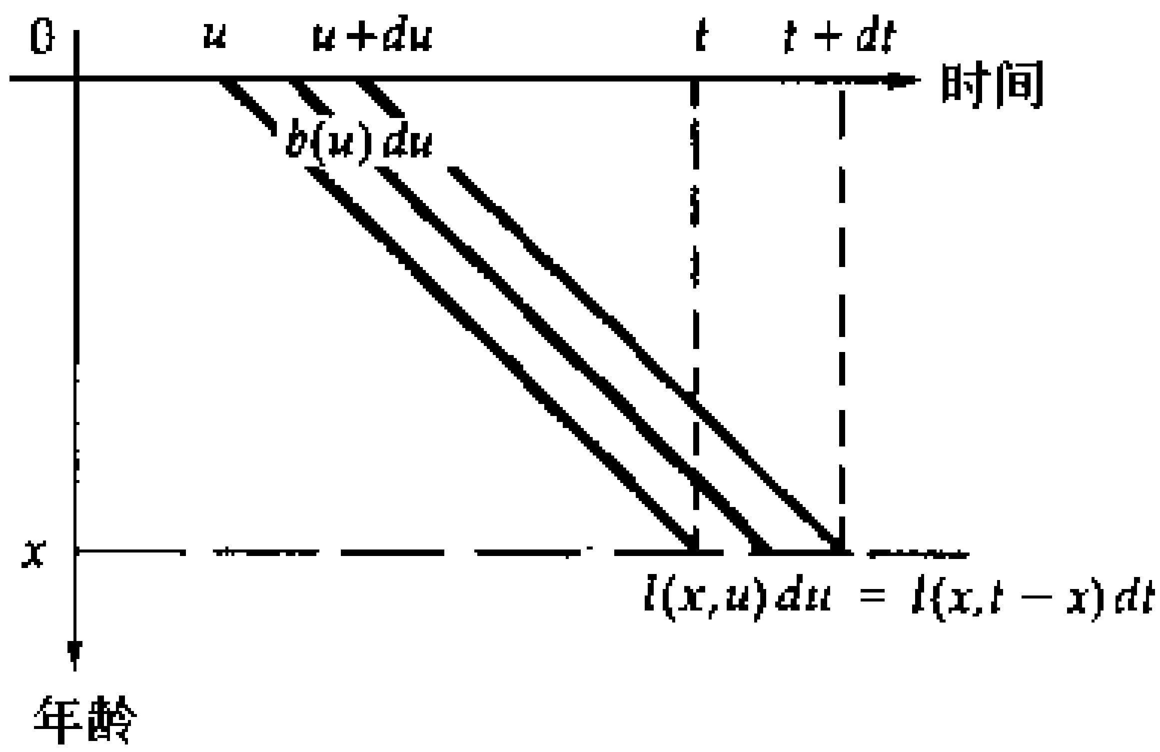


图 13.3.1　$l(x,u)$ 的解释

设 $x_0 < x_1$ 为两个年龄，t_0 是给定的时间。现在考虑另一个问题：在时间 t_0 有多少个年龄介于 x_0 与 x_1 岁之间的活着的生命。这些生命必定在时间 $t_0-(x_1-x_0)$ 与 t_0 之间已达到 x_0 岁，并且活到时间 t_0(参考图 13.3.2)。

其中，在时间 t 与 $t+dt$ 之间达到 x_0 岁的人数为 $l(x_0,t-x_0)dt$, 这些人中活到时间 t_0 的人数为

$$l(x_0,t-x_0)\frac{s(x_0+t_0-t,t-x_0)}{s(x_0,t-x_0)}dt=l(x_0+t_0-t,t-x_0)dt, \tag{13.3.4}$$

于是所求人数为

$$\int_{t_0-(x_1-x_0)}^{t_0} l(x_0+t_0-t,t-x_0)dt.$$

作变量代换 $x=x_0+t_0-t$, 积分可写成

$$-\int_{x_1}^{x_0} l(x,t_0-x)dx=\int_{x_0}^{x_1} l(x.t_0-x)dx, \tag{13.3.5}$$

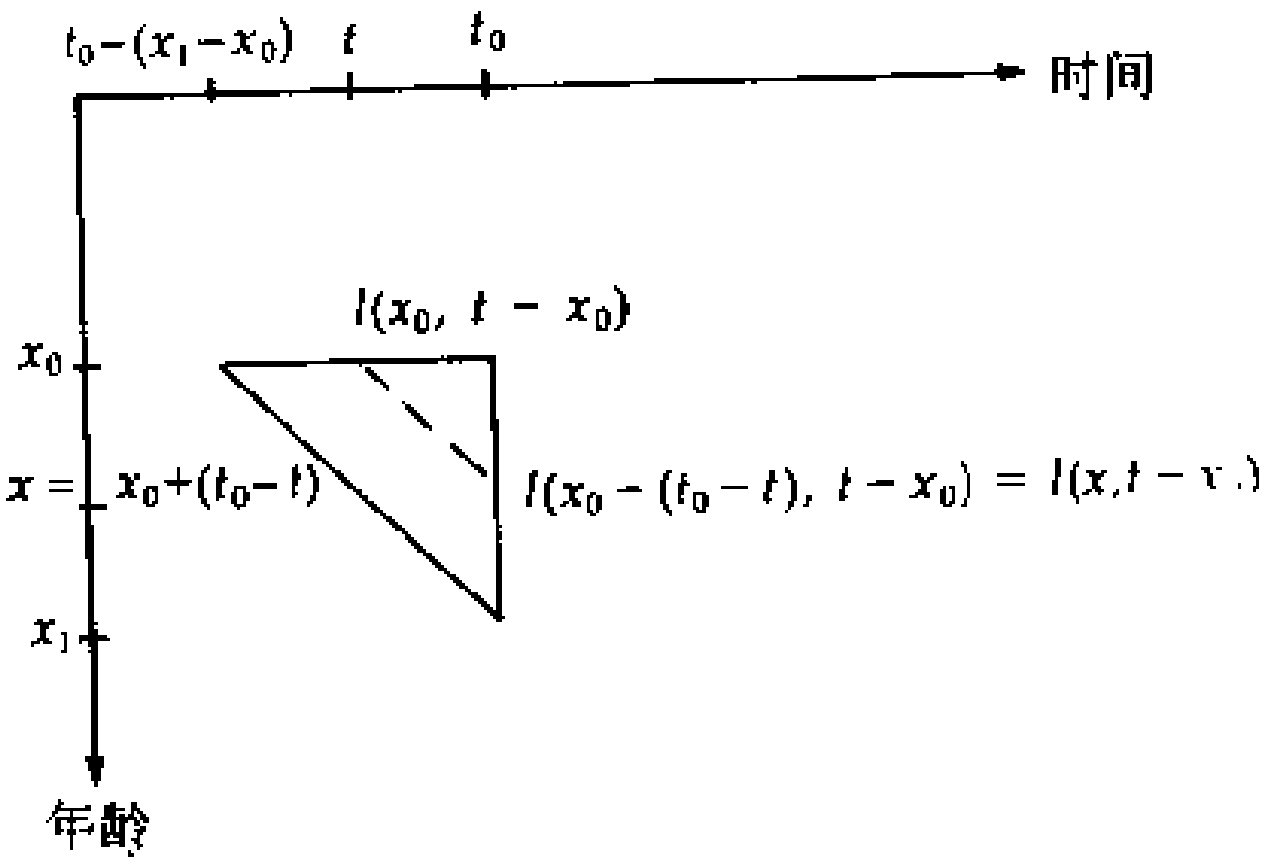


图 13.3.2　在时间 t_0 介于 x_0 与 x_1 岁的生命数

由此可得以下解释：

$$l(x,t_0-x)dx = \text{在时间}t_0\text{年龄介于}x\text{与 }x+dx\text{ 之间的人数.} \tag{13.3.6}$$

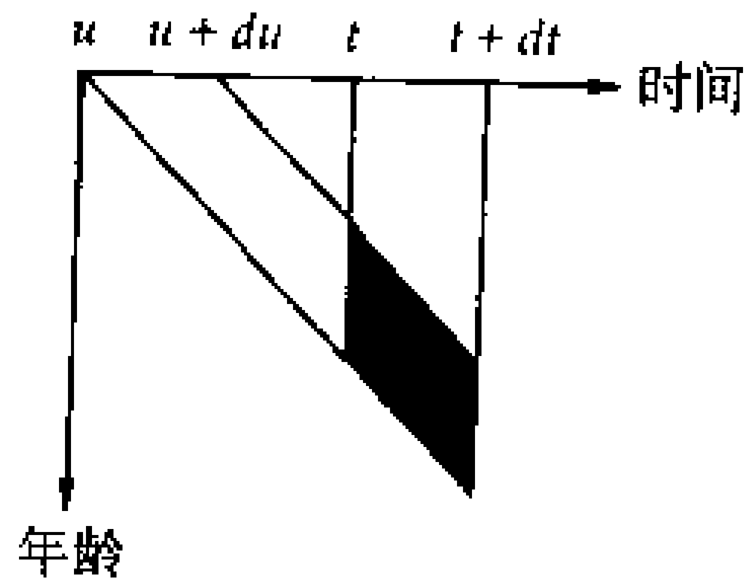
这样，人口密度函数有两种解释：其一为 (13.3.2) 与 (13.3.3)，其二为 (13.3.6) 与 (13.3.5)。两种解释中的第一种相当于对群体的 Lexis 图在 t 与 $t+dt$ 之间切片，第二种解释相当于对 Lexis 图在 x 与 $x+dx$ 之间切片。

在时间 u 出生的生命在 x 岁时死亡效力为

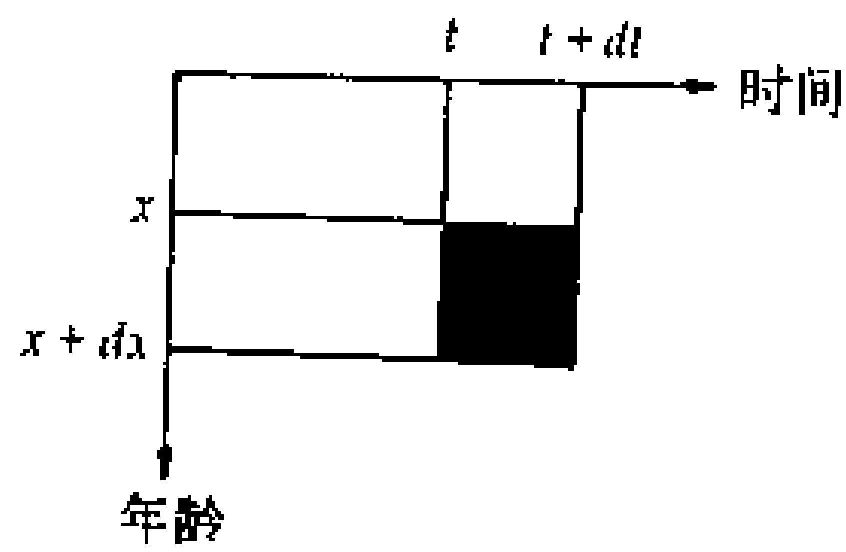
$$\mu(x,u) = -\frac{1}{s(x,u)}\frac{\partial}{\partial x}s(x,u) = -\frac{1}{l(x,u)}\frac{\partial}{\partial x}l(x,u), \tag{13.3.7}$$

称作*世代死亡效力*(generation force of mortality)。图 13.3.3 根据这一定义提供了三种解释，它们可按以下方式得到验证. 对二元人口密度函数乘以世代死亡效力，并实施相应的线性变换，并注意到变换的雅可比行列式为 1 。

A. $l(t-u,u)\mu(t-u,u)dudt$＝在u与$u-du$之间出生的那些人中死于时间t与$t+dt$之间的人数



B. 作代换$x=t-u$，我们有$l(x,t-x)\mu(x,t-x)dtdx$＝在时间t与$t+dt$之间死于年龄x与$x+dx$之间的人数



C. 作代换$x=t-u$，我们有$l(x,u)\mu(x,u)dudx$＝在u与$u+du$之间出生的那些人中死于年龄x与$x+dx$之间的人数

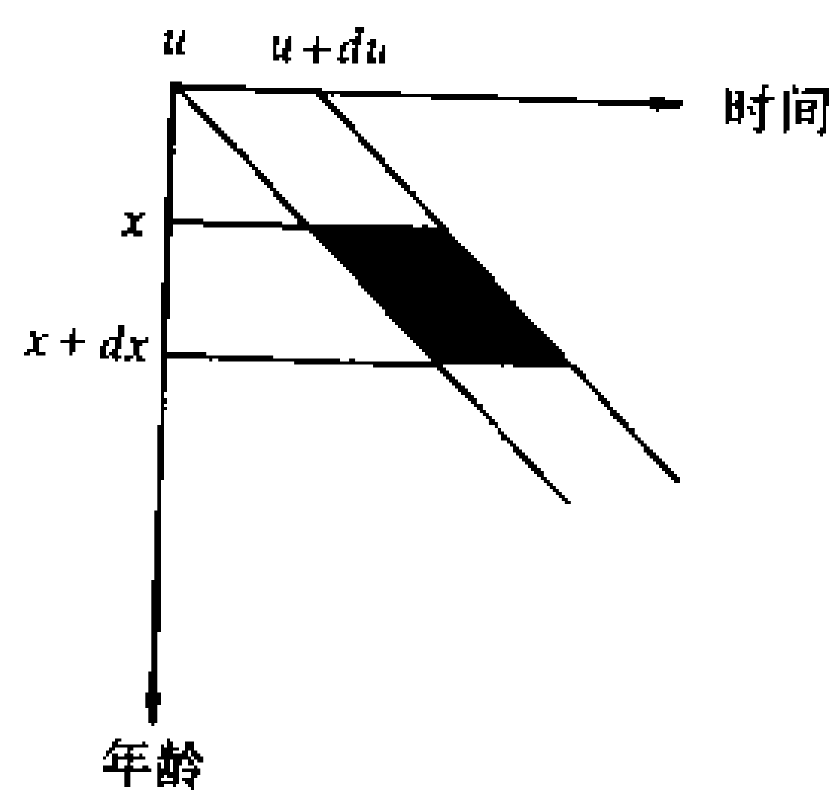


图 13.3.3 人口密度乘以世代死亡效力的解释

在由 Lexis 图描述的给定时间 – 年龄平面区域中的总死亡数，可通过对图 13.3.3 中的表达式在相应区域上积分得出，求解过程需要计算二重积分。

另一种方法称为 进入移出法(in-and-out method)，可为获得所需的死亡数提供一种较容易的途径。这个方法包括：决定进入以及移出有关区域的生命个数，这两个数字之差就是死亡数。在大多数场合，进入移出法只需要计算两个单重积分。

例 13.3.1: 在时间 t_0 与 t_0+1 之间达到 x_0 岁且在时间 t_0+3 之前死亡的人数有多少？

解：需要导出图 13.3.4 中所示梯形区域中的死亡数表达式。

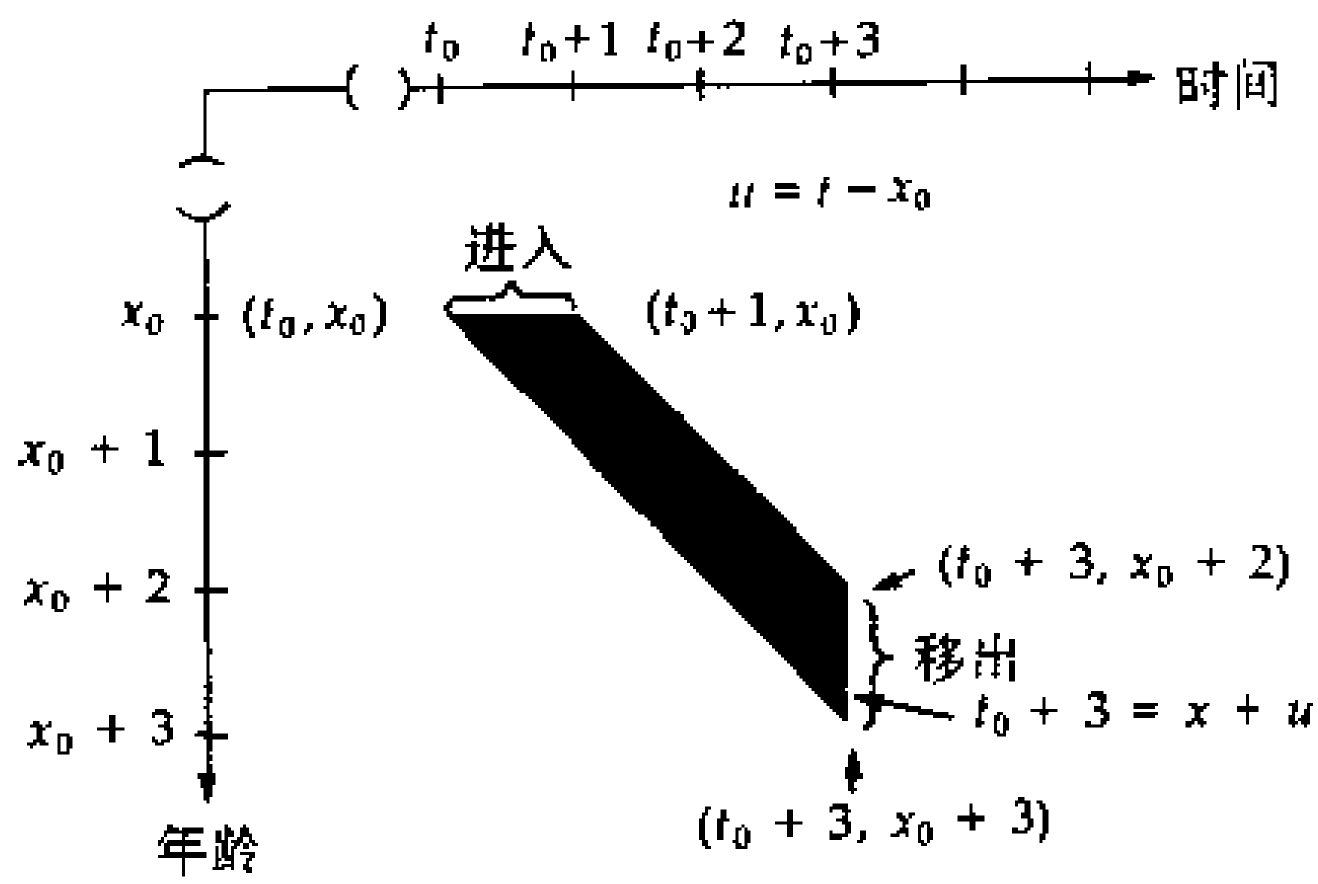


图 13.3.4 例 13.3.1 中计算死亡的区域

用二重积分可表示成

$$\int_{t_0-x_0}^{t_0+1-x_0}\int_{x_0}^{t_0+3-u} l(x,u)\mu(x,u)dxdu.$$

利用 (13.3.7) 可得死亡数为

$$\int_{t_0-x_0}^{t_0+1-x_0}\int_{x_0}^{t_0+3-u}[-\frac{\partial l(x,u)}{\partial x}]dxdu$$
$$=\int_{t_0-x_0}^{t_0+1-x_0}[-l(t_0+3-u,u)+l(x_0,u)]du$$
$$=\int_{t_0-x_0}^{t_0+1-x_0}l(x_0,u)du-\int_{t_0-x_0}^{t_0+1-x_0}l(t_0+3-u,u)du$$
$$=\int_{t_0}^{t_0+1}l(x_0,y-x_0)dy-\int_{x_0+2}^{x_0+3}l(w,t_0+3-w)dw.$$

用进入移出法，所求死亡人数是在时间 t_0 与 t_0+1 之间达到 x_0 岁的人数与在时间 t_0+3 时年龄介于 x_0+2 与 x_0+3 之间的人数之差。据此直接用单重积分表示所求死亡人数为

$$\int_{t_0}^{t_0+1}l(x_0,y-x_0)dy-\int_{x_0+2}^{x_0+3}l(w,t_0+3-w)dw,$$

与使用二重分积分得出的结果一致。

例 13.3.2: 决定在时间 t_0 时介于 20 与 40 岁并且将在 70 岁之前死亡的人数。

解：需要导出图 13.3.5 中所示梯形区域中的死亡数表达式。用二重积分方法可得所求人数为

$$\int_{t_0-40}^{t_0-20}\int_{t_0-u}^{70}l(x,u)\mu(x,u)dxdu.$$
$$=\int_{t_0-40}^{t_0-20}\int_{t_0-u}^{70}[-\frac{\partial l(x,u)}{\partial x}]dxdu$$
$$=\int_{t_0-40}^{t_0-20}[l(t_0-u,u)-l(70,u)]du$$
$$=\int_{t_0-40}^{t_0-20}l(t_0-u,u)du-\int_{t_0-40}^{t_0-20}l(70,u)du$$
$$=\int_{20}^{40}l(y,t_0-y)dy-\int_{t_0+30}^{t_0+50}l(70,w-70)dw.$$

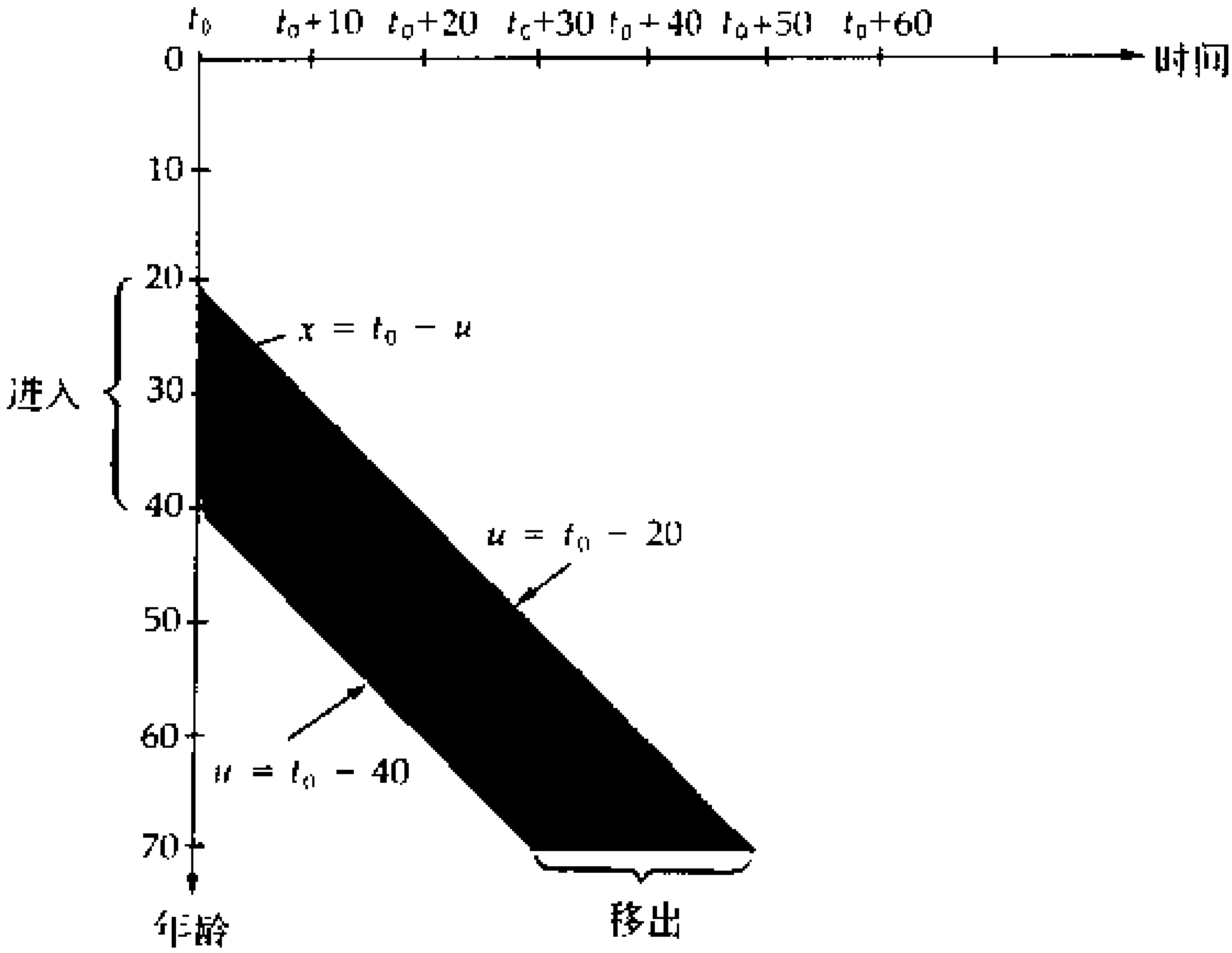


图 13.3.5　例 13.3.2 中计算死亡的区域

用进入移出法可得死亡人数为

$$\int_{20}^{40} l(x, t_0 - x)dx - \int_{t_0+30}^{t_0+50} l(70, t-70)dt,$$

它与二重积分结果是一致的。

§13.4　静止人口与稳定人口

这一节将考察 §13.3 中所描述模型的两种重要的特殊情形。

当 $l(x,u)$ 与 u 无关时，相应的模型称为静止人口(stationary population) 模型。此时

$$l(x,u)=bs(x), \tag{13.4.1}$$

其中 b 是常数出生密度，$s(x)$ 是不依赖于出生时间的生存函数。为了与 (1.3.1) 相一致，可将 (13.4.1) 改写为

$$l(x,u)=bs(x)=l_x, \tag{13.4.2}$$

这里 b 起着 l_0 的作用。

对于静止人口，(13.3.5) 成为

$$\int_{x_0}^{x_1} l(x,t_0-x)dx=\int_{x_0}^{x_1} l_x dx=T_{x_0}-T_{x_1},$$

即静止人口在时间 t_0 介于年龄 x_0 与 x_1 之间的人数可用 (1.5.10) 引入的函数 T_x 表示。而且图 13.3.3B 给出的 $l_x\mu_x$ 解释导致

$$\int_{x_0}^{x_1} l_x\mu_x dx=l_{x_0}-l_{x_1}$$

可作为在年龄 x_0 与 x_1 之间的死亡在任何时间 t 的密度。特别是，在年龄 x_0 及更高年龄的死亡的密度等于达到 x_0 岁的生命数在任何时间 t 的密度，后者的解释由 (13.3.2) 提供。这些事实表明静止人口这一名称是恰当的。

当人口密度函数具有以下形式

$$l(x,u)=e^{Ru}bs(x)=e^{Ru}l_x \tag{13.4.3}$$

时，相应的模型称为稳定人口(stable population) 模型。这里 $b>0$ 及 R 是常数，$s(x)$ 是不依赖于出生时间的生存函数。如 $R=0$，则稳定人口成为静止人口。在稳定人口中，在时间 u 的出生密度为 $e^{Ru}b=e^{Ru}l_0$ 。

稳定人口在时间 t 的总人口为

$$N(t) = \int_0^\infty l(x, t-x)dx = e^{Rt}\int_0^\infty e^{-Rx} l_x dx. \qquad (13.4.4)$$

由此可见，当 $R > 0$ 时人口指数式增长，而当 $R < 0$ 时人口指数式递减。

稳定人口中在时间 t 时介于年龄 x_0 与 x_1 之间的人数占总人口的比例为

$$\frac{\int_{x_0}^{x_1} l(x, t-x)dx}{\int_0^\infty l(x, t-x)dx} = \frac{\int_{x_0}^{x_1} e^{-Rx} l_x dx}{\int_0^\infty e^{-Rx} l_x dx}, \qquad (13.4.5)$$

与时间 t 无关。因此，尽管稳定人口的总规模可随时间改变，但相对的年龄分布保持不变。

稳定人口在时间 t 时介于 x_0 与 x_1 岁之间的人数可用第三章引入的计算基数 $\overline{N}_x$ 来表示：

$$\begin{aligned}\int_{x_0}^{x_1} l(x, t-x)dx &= e^{Rt}\int_{x_0}^{x_1} e^{-Rx} l_x dx \\ &= e^{Rt}(\overline{N}'_{x_0} - \overline{N}'_{x_1}). \qquad (13.4.6)\end{aligned}$$

这里及以下记号中的一撇表示计算时使用的利息效力等于稳定人口的 (瞬时) 变化率 R 。

根据 (13.3.7), 稳定人口的死亡效力为

$$\mu(x, u) = -\frac{1}{l(x, u)}\frac{\partial}{\partial x} l(x, u) = \mu_x,$$

它与出生时间无关。年龄介于 x_0 与 x_1 之间的人在时间 t 的死亡密度为

$$\begin{aligned}\int_{x_0}^{x_1} l(x, t-x)\mu(x, t-x)dx &= \int_{x_0}^{x_1} e^{R(t-x)} l_x \mu_x dx \\ &= e^{Rt}(\overline{M}'_{x_0} - \overline{M}'_{x_1}), \qquad (13.4.7)\end{aligned}$$

即年龄在 x_0 与 x_1 岁之间并在时间 t 与 $t+dt$ 之间死亡的人数为

$$e^{Rt}(\overline{M}'_{x_0} - \overline{M}'_{x_1})dt.$$

计算基数曾在第二章时引入，这里记号中的一撇表示按利息效力 R 计算。

以上关于稳定人口的事实与第二章里的一个等式结合，可证实稳定人口的一个性质：

$$x_0\text{ 岁以上人口在时间 } t \text{ 的变化率} = \frac{\text{在 } t \text{ 达到 } x_0 \text{ 岁的人口密度} - x_0 \text{ 岁以上人口在 } t \text{ 的死亡密度}}{\text{在时间 } t \text{ 的 } x_0 \text{ 岁以上人口数}}$$

$$= \frac{e^{Rt}(D'_{x_0} - \overline{M}'_{x_0})}{e^{Rt}\overline{N}'_{x_0}} = R$$

最后一步根据

$$\overline{A}'_{x_0} + R\overline{a}'_{x_0} = 1$$

从而得出

$$\overline{M}'_{x_0} + R\overline{N}'_{x_0} = D'_{x_0}.$$

例 13.4.1: 对于静止人口，0 岁人的 (完全) 期望剩余寿命等于时间 t 的总人口除以出生密度，即

$$\mathring{e}_0 = \int_0^\infty s(x)dx = \int_0^\infty \frac{l_x}{l_0}dx = T_0/b.$$

而对于稳定人口，类似的计算结果是什么？

解：稳定人口在时间 t 的总人口除以出生密度为

$$\frac{N(t)}{e^{Rt}b} = \frac{\int_0^\infty l(x, t-x)dx}{e^{Rt}l_0} = \frac{\int_0^\infty e^{-Rx}l_x dx}{l_0} = \overline{a}'_0.$$

当 $R > 0$ 时，$\overline{a}'_0 < \mathring{e}_0$; 当 $R < 0$ 时，$\overline{a}'_0 > \mathring{e}_0$。这表明，除了 $R = 0$ 的情况外，期望寿命无法从直接对稳定人口在某时刻的观察获得。

§13.5 精算应用

尽管由于生存函数或者出生密度的变化使得实际上稳定或静止人口的条件很难成立，但这些模型在人寿保险或退休收入体系的基金累积的替代计划研究中有用。这里所说的基金累积计划是指为提供寿险或年金受益所必须累积基金的预算计划。

在这一节与第十四章里，我们将脱离第二至十章建立的模型，这些模型的建立始于个别保单的运作考虑。这一节考察若干寿险的综合模型例子。在第十四章，将考察退休金体系的类似模型，那些模型对于向团体或群体提供死亡或退休受益的社会保险及团体保险体系尤其有关系。

例 13.5.1：设人口密度函数 $l(x,u)=b(u)s(x)$，其中生存函数 $s(x)$ 不依赖于出生时间 u 。假定这一人口中的每个人在达到 a 岁后就投保单位受益的完全连续终身寿险，证明

$$\overline{P}(\overline{A}_a)\int_a^\infty l(x,t-x)dx+\delta\int_a^\infty l(x,t-x)_{x-a}\overline{V}(\overline{A}_a)dx$$
$$=\int_a^\infty l(x,t-x)\mu_x dx+\frac{d}{dt}\int_a^\infty l(x,t-x)_{x-a}\overline{V}(\overline{A}_a)dx. \tag{13.5.1}$$

解：根据一般推理，在时间 t 的

保费收入 (年) 率 + 投资收入 (年) 率

= 受益支出 (年) 率 + 综合责任准备金的 (年) 变化率.

式 (13.5.1) 左端正是收入的来源：保费及利息，右端则是收入的分配：死亡受益及责任准备金的改变。

其解析证明如下：由 (5.10.5),

$$\frac{d}{dx}{}_{x-a}\overline{V}(\overline{A}_a)-\mu_{x\,x-a}\overline{V}(\overline{A}_x)+\mu_x=\overline{P}(\overline{A}_a)+\delta_{x-a}\overline{V}(\overline{A}_a), \tag{13.5.2}$$

乘 $l(x,t-x)=b(t-x)s(x)$, 并注意到

$$\frac{d}{dx}l(x,t-x)=-b'(t-x)s(x)-b(t-x)s(x)\mu_x,$$

可得

$$\frac{d}{dx}[l(x,t-x)_{x-a}\overline{V}(\overline{A}_a)]+b'(t-x)s(x)_{x-a}\overline{V}(\overline{A}_a)+l(x,t-x)\mu_x$$
$$=l(x,t-x)\overline{P}(\overline{A}_a)+\delta l(x,t-x)_{x-a}\overline{V}(\overline{A}_a). \qquad (13.5.3)$$

对 x 从 a 到 ∞ 积分，并注意到

$$\int_a^\infty b'(t-x)s(x)_{x-a}\overline{V}(\overline{A}_a)dx=\frac{d}{dt}\int_a^\infty b(t-x)s(x)_{x-a}\overline{V}(\overline{A}_a)dx, \qquad (13.5.4)$$

可得出 (13.5.1) 。

例 13.5.2: 上例中如果不按终身寿险保费积聚基金，而是按照同一时间收支相抵的摊派计划 (赋课计划) 累积的话，决定每个被保险成员摊定的缴付 (年) 率 π_t 。

解： π_t 由

$$\pi_t\int_a^\infty l(x,t-x)dx=\int_a^\infty l(x,t-x)\mu_x dx$$

决定，即

$$\pi_t=\frac{\int_a^\infty l(x,t-x)\mu_x dx}{\int_a^\infty l(x,t-x)dx}. \qquad (13.5.5)$$

例 13.5.3: 对于稳定人口，重新考察例 13.5.1 及 13.5.2 。

解： (1) 现在 $l(x,u)=e^{Ru}bs(x)$, 代入 (13.5.1) 并消去因子 e^{Rt}, 得

$$\overline{P}(\overline{A}_a)\int_a^\infty e^{-Rx}l_x dx+\delta\int_a^\infty e^{-Rx}l_{x}\,{}_{x-a}\overline{V}(\overline{A}_a)dx$$
$$=\int_a^\infty e^{-Rx}l_x\mu_x dx+R\int_a^\infty e^{-Rx}l_{x}\,{}_{x-a}\overline{V}(\overline{A}_a)dx \qquad (13.5.6)$$

根据例 13.5.1 的一般推理对 (13.5.6) 的解释，保费收入率与受益支出率之比为

$$\frac{\overline{P}(\overline{A}_a)\int_a^{\infty}e^{-Rx}l_x dx}{\int_a^{\infty}e^{-Rx}l_x\mu_x dx}=\frac{\overline{P}(\overline{A}_a)}{\overline{P}'(\overline{A}_a')}. \tag{13.5.7}$$

当 $R=0$ 时，静止人口的收入分配方程 (13.5.6) 成为

$$\overline{P}(\overline{A}_a)T_a+\delta\int_a^{\infty}l_{x\,x-a}\overline{V}(\overline{A}_a)dx=l_a. \tag{13.5.8}$$

而 (13.5.7) 给出的保费收入率与受益支出率之比则成为 $\overline{P}(\overline{A}_a)\,\overset{\circ}{e}_a$。

(2) 对于稳定人口，(13.5.4) 成为

$$\pi_t=\frac{\int_a^{\infty}e^{-Rx}l_x\mu_x dx}{\int_a^{\infty}e^{-Rx}l_x dx}=\overline{P}'(\overline{A}_a'). \tag{13.5.9}$$

对于静止人口，$R=0$, 此时 $\pi_t=1/\overset{\circ}{e}_a$.

注：对例 13.5.3 需要作一些特别说明。稳定人口中每个 a 岁以上成员按终身寿险与摊派累积方法所需的保费支付率分别为 $\overline{P}(\overline{A}_a)$ 与 $\overline{P}(\overline{A}_a')$。根据习题 21, 当死亡效力递增时，成立

$$\overline{P}(\overline{A}_a)>\overline{P}(\overline{A}_a')\qquad \delta<R,$$

$$\overline{P}(\overline{A}_a)=\overline{P}(\overline{A}_a')\qquad \delta=R,$$

$$\overline{P}(\overline{A}_a)<\overline{P}(\overline{A}_a')\qquad \delta>R.$$

这就是说，如果利息效力低于人口增长率，按摊派累积方法所需的保费支付 (率) 少于按终身寿险累积方法所需的；如果利息效力高于人口增长率，则终身寿险累积方法导致比摊派累积方法更小的保费 (支付率)。

例 13.5.4: 对于整理成以下形式的稳定人口收入分配方程 (13.5.8):

$$\int_a^{\infty}l_{x\,x-a}\overline{V}(\overline{A}_a)dx=\frac{l_a-\overline{P}(\overline{A}_a)T_a}{\delta},$$

提供一般推理解释。

解：从以上经整理的形式可以看出，综合责任准备金可看作以下两个永久年金的现值之差：

$$
\begin{aligned}
\frac{l_a}{\delta} &= \text{按年率 } l_a \text{ 支付死亡受益的连续永久年金的现值}\\
&\doteq \text{当前成员死亡受益的现值} - \text{未来成员死亡受益的现值},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\overline{P}(\overline{A}_a)T_a}{\delta} &= \text{按年率 } \overline{P}(\overline{A}_a)T_a \text{ 支付保费的连续永久年金的现值}\\
&= \text{当前成员保费的现值} - \text{未来成员保费的现值}.
\end{aligned}
$$

注意到未来成员的支付率为 $\overline{P}(\overline{A}_a)$ 的保费将从 a 岁开始支付，他们的保费现值等于他们的受益现值。这样，l_a/δ 与 $\overline{P}(\overline{A}_a)T_a/\delta$ 的解释中的第二个成份相互抵消，

$$
\begin{aligned}
\frac{l_a}{\delta} - \frac{\overline{P}(\overline{A}_a)T_a}{\delta} &= \text{当前成员的综合责任准备金}\\
&= \text{当前成员的受益现值} - \text{当前成员的保费现值}\\
&= \int_a^{\infty} l_{ax-a}\overline{V}(\overline{A}_a)dx.
\end{aligned}
$$

例 13.5.1, 例 13.5.3 及例 13.5.4 所讨论的寿险基金累积或预算方法中，基金已存在。这些例子假定群体的所有成员在达到进入年龄 a 岁时都从那个年龄 a 开始参加计划，并在此假设下考察了基金的特征。满足以上假设的体系称为已处于成熟状态(mature state)，在此之前，整个基金因净增加的进入者而处于增长状态。在这些例子中，需经过 $\omega-a$ 年基金才达到成熟状态。

§13.6 人口动力学

这一节回头考察出生密度函数 $b(t)$，目的是为 §13.3 的连续模型的发展奠定基础。除此之外，还将探讨 §13.4 中导致稳定或静止人口的条件。

在建立出生函数的数学模型过程中，需要引入**生育力函数**(force of birth function), 记为 $\beta(x,u)$ 。$\beta(x,t-x)dt$ 表示 x 岁妇女在时间 t 与 $t+dt$ 之间生育的女孩数，x 岁妇女自己是在时间 $t-x$ 出生的。生育力函数是各年龄及各代妇女生育的女婴瞬时出生率。

在时间 t 与 $t+dt$ 之间出生的女孩总数为

$$b_f(t)dt = [\int_0^{\infty} l_f(x,t-x)\beta(x,t-x)dx]dt. \tag{13.6.1}$$

在 (13.6.1) 中的下标 f 表示有关函数与女性生命相联系。总的出生数 (包括男孩) 可通过乘一个常数获得，该常数为女孩出生数与总出生数之比，在大多数人口中稍稍比 2 大些。

在 (13.6.1) 两端除以 dt, 并以 (13.3.1) 取代 $l_f(x,t-x)$, 我们可看出女性出生密度函数满足积分方程

$$b_f(t) = \int_0^{\infty} b_f(t-x)s_f(x,t-x)\beta(x,t-x)dx. \tag{13.6.2}$$

问题是在给定函数 $s_f(x,t-x)$ 与 $\beta(x,t-x)$ 时求解 $b_f(t)$ 。在 (13.6.2) 中，乘积函数 $s_f(x,t-x)\beta(x,t-x)$ 称为**净孕产函数**(net maternity function), 记为 $\phi(x,t-x)$ 。

这一节以下假定，净孕产函数不依赖于母亲的出生年代，即 $s_f(x,t-x)\beta(x,t-x)=\phi(x)$ 。在此假定下，积分方程 (13.6.2) 成为

$$b_f(t) = \int_0^{\infty} b_f(t-x)\phi(x)dx. \tag{13.6.3}$$

在这一节里，我们只限于考虑 (13.6.3) 的如下特殊形式解：

$$b_f(t) = be^{Rt}, \tag{13.6.4}$$

其中 b 是正常数，R 是方程

$$H(r) = 1 \tag{13.6.5}$$

的唯一实解，其中

$$H(r)=\int_0^\infty e^{-rx}\phi(x)dx.$$

事实上，将 (13.6.4) 直接代入 (13.6.3) 得

$$be^{Rt}=\int_0^\infty be^{R(t-x)}\phi(x)dx.$$

消去不依赖于 x 的因子，以上方程成为

$$1=\int_0^\infty e^{-Rt}\phi(x)dx=H(R).$$

方程 $H(r)=1$ 具有唯一实解这个结论的根据是以下几点：

1. $H'(r)=-\int_0^\infty xe^{-rx}\phi(x)dx<0$ 。
2. $H(0)=\int_0^\infty \phi(x)dx>0$ 。
3. $\lim_{r\to\infty}H(r)=0$ 。
4. $\lim_{r\to-\infty}H(r)=\infty$ 。这些事实连同

$$H''(r)=\int_0^\infty x^2e^{-rx}\phi(x)dx>0$$

的事实概括在图 13.6.1 中。从图 13.6.1 可以看出有唯一实解 R(图中画的是正的解，但 R 可能是负的) 。

在 $b_f(t)=b\epsilon^{Rt}$ 时， $l_f(x,t-x)=be^{R(t-x)}s_f(x)$, 女性人口是稳定人口。在 $R=0$ 的特殊情形，女性人口是静止人口。

为核实 R 究竟是正是负或是 0, 考虑数

$$\beta=H(0)=\int_0^\infty\phi(x)dx.$$

根据图 13.6.1 可得出以下结论：

1.$\beta>1$ 时， R 为正，人口是稳定且递增的。

2.$\beta=1$ 时， R 为 0, 人口是平稳的。

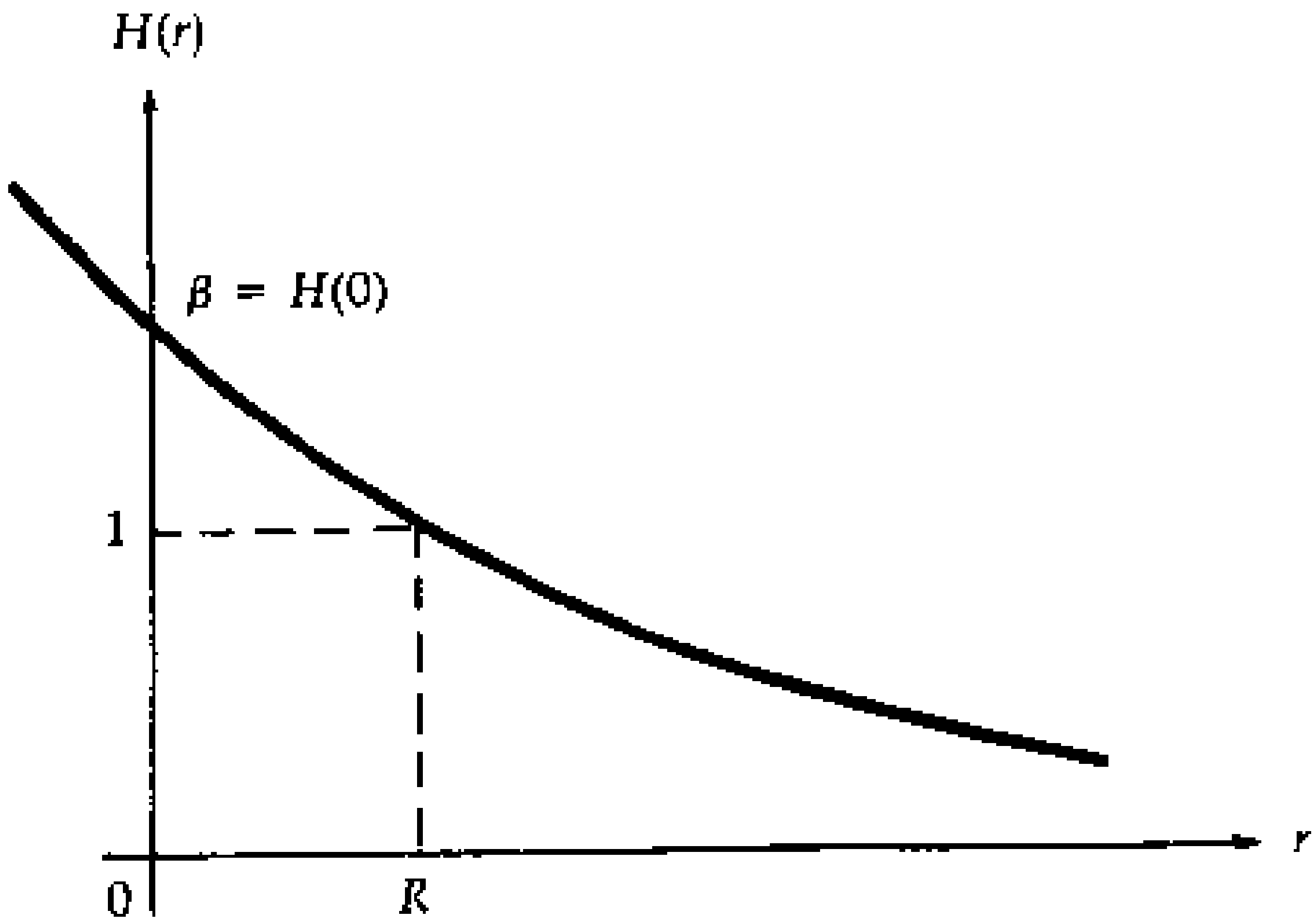


图 13.6.1　典型的函数 $H(r)$ 及公式 (13.6.5)

3.$\beta < 1$ 时，R 为负，人口是稳定且递减的。

由于 $H(0) = \beta$ 可解释为每个女性一生中生育的女孩数，它被称为 净再生育率(net reproduction rate)，参数 R 称为 人口的固有增长率(intrinsic rate of population growth)。

评注:

所有的人口都不会象这一节所推论的那样都是稳定的，模型的某些方面会与实际经验不一致。由 (13.6.2) 给出的基本模型建立在生存函数与生育力 (出生效力) 不随时间改变的假设之上，在 (13.6.3) 中还进一步局限于假设净孕产函数只依赖于母亲的年龄，而不依赖于母亲自己的出生年代。对公众健康的统计揭示了生存函数及生育力随着时间而有颇大变化。

此外，在解积分方程 (13.6.3) 时，我们只获得方程 $H(r) = 1$ 的实解。在复数域内，除了单个实解外还可确定无限多个解。方

程 $H(r)-1=0$ 的这些额外的 (复) 根使得 (13.6.3) 的一般解具有形式 $\sum_j c_j b_f^{(j)}(t)$, 其中的每个 $b_f^{(j)}$ 与 $H(r)-1=0$ 的一个根相联系。按共轭复数成对出现的复根，可使出生密度函数具有阻尼波动的结构。

人口理论是一些优雅的数学观念的集成，然而其中也有一些非常重要的统计问题，诸如根据已有的数据估计生存函数及生育力函数等这样一些关键的组成部分。作为人类社会的动态属性的反映，这些函数已被观察到随时间而变动。

正如所有描述自然现象的模型一样，人口数学模型只抓住了形成真实人口规模与年龄分布的动力的一小部分。即使稳定人口模型在某些时候是一个令人满意的近似，它也不可能长期适用。长期而论，在一个有限的星球上，常数 R 大于 0 是不可能的；同样，负的常数 R 持续太久的话，稳定人口将面临灭绝。

习　题

§13.2

1. 用图 13.2.1 的 Lexis 图计算

 a. 在时间 25 的雇员平均年龄。

 b. 已达到 50 岁的在职或曾在职的雇员人数。

 c. 在时间 25 的雇员中已达或将达 50 岁的在职雇员数。

§13.3

2. 令

$$\begin{aligned} b(u) &= 100[1+\cos(\pi u/200)] \qquad -\infty < u < \infty, \\ s(x,u) &= \cos(\pi x/200) \qquad 0 < x \le 100. \end{aligned}$$

计算在时间 25 到 100 间达到 50 岁的个体数。

3. 令

$$b(u) = 100(1-e^{-u/100}) \qquad u > 0,$$

$$s(x) = e^{-x/100} \qquad x > 0.$$

计算在时间 100, 年龄处在 25 岁到 50 岁间的个体数。

4. 用习题 3 确定的函数 $b(u)$ 和 $s(x,u)$ 计算在时间 50 和 51 之间将要达到 25 岁而又在时间 53 前死去的个体数。(这是例 13.3.1 的一个数值变形)

5. 假设 $s(x,u) = s(x)$, $b(u) = l_0$, 重解例 13.3.2 。

6. 用积分形式计算在时间 0 处在 20 到 50 岁间且在时间 50 之前活不到 80 岁的人数。

§13.4

7. a. 令 $N(t)$ 表示一个稳定人口在时间 t 时的人数，试证明 $dN(t)/dt = RN(t)$ 。

b. 定义时刻 t 出生率为 $b(t)/N(t)$, 对一个稳定人口证明其出生率 $i(t)$ 满足

$$i(t) = \left[\int_0^\infty e^{-Rx} s(x) dx\right]^{-1}.$$

8. 若 $\mu_x = ax, a > 0$ 且 $b(u) = be^{Ru}$, 试用 $N(0,1)$ 分布的分布函数 $\Phi(z)$ 来表示时间 t 时整个人口的规模。

9. 对一个稳定人口，试求时间 t 时在年龄 a 和 r 之间的人的平均年龄的表达式。假设 $R=0$, 重新求该表达式。

10. 若 $\overline{\mu}_x = \mu_x + 0.05/\mathring{e}_x$, 证明 $\overline{p}_x = p_x(T_{x+1}/T_x)^{0.05}$ 。

11. 已知 $s^*(x) = e^{-Rx}s(x), R \geq 0$ 是一个生存函数,

a. 写出 $s^*(x)$ 的概率密度函数和分布函数。

b. 证明生存函数为 $s^*(x)$, 现龄为 x_0 的期望剩余寿命的方差为 $2(\overline{I}\overline{a})'_{x_0} - \overline{a}'^2_{x_0}$ 。

12. 定义时刻 t 的粗死亡率为

$$\frac{\int_0^\infty l(x,t-x)\mu(x,t-x)dx}{\int_0^\infty l(x,t-x)dx}.$$

若该人口为稳定人口，证明粗死亡率等于 $i(t)-R$, 其中 $i(t)$ 是习题 7(b) 中定义的出生率。

§13.5

13. 假设一个稳定人口的生存函数为 $s(x)$, 且精算函数的计算也用同样的生存函数。证明并解释

$$l_r\overline{a}_r+\delta\int_r^{\infty}l_x\overline{a}_x dx=T_r.$$

14. 假设一个稳定人口的生存函数为 $s(x)$, 且精算函数的计算也用同样的生存函数。证明并解释下列等式：

a. $l(a,t-a)\overline{a}_{a:\overline{r-a|}}+\delta\int_a^r l(x,t-x)\overline{a}_{x:\overline{r-x|}}dx$
$=\int_a^r l(x,t-x)dx+R\int_a^r l(x,t-x)\overline{a}_{x:\overline{r-x|}}dx.$

[提示：用 (3.3.26)]

b. $l(a,t-a)\overline{A}_a+\delta\int_a^{\infty}l(x,t-x)\overline{A}_x dx$
$=\int_a^{\infty}l(x,t-x)\mu_x dx+R\int_a^{\infty}l(x,t-x)\overline{A}_x dx.$

15. 如果 $b(u)=100e^{0.01u}$, 年龄 $a=0$,$s(x)=e^{-x/50}$, 计算例 13.5.2 中终身寿险计划中用到的评价率 π_t

16. 一个人口在时间 t 时的老年依靠率为 $f(t)=\frac{\int_{65}^{\infty}l(x,t-x)dx}{\int_{20}^{65}l(x,t-x)dx}$。对一个稳定人口，证明

$$\frac{\partial}{\partial R}\log f(t)=\overline{x}_1-\overline{x}_2,$$

其中 $\overline{x}_1$ 是时间 t 时处于 20 岁到 65 岁间的人口的平均年龄， $\overline{x}_2$ 是时间 t 时在 65 岁上的人口的平均年龄。

§13.6

17. 已知纯孕产函数为

$$\phi(x)=x^{\alpha-1}e^{-\beta x}\qquad \alpha>0,\beta>0,$$

a. 计算 R 。

b. 若 $\alpha = 2$, $\beta = 1$ 该人口是静止人口还是稳定人口？

综合题

18. 已知一个人口在时间 t 时的数量满足微分方程

$$\frac{dN(t)}{dt} = \frac{c}{a}\{N(t)[a - N(t)]\} \qquad a > 0.$$

注意当 $N(t)$ 趋向于 a 时，人口规模的变化率趋向于 0.

a. 证明 logistic 函数 $N(t) = a[1 - be^{-ct}]^{-1}, b > 0$ 满足该微分方程。

b. 若 $c > 0$, 计算 $\lim_{t\to\infty} N(t)$, 并画出 $N(t)$ 的曲线。

c. 确定 $N(t)$ 的拐点的横坐标。

19. 若死力严格上增，证明

a. $s(x)s(y) \geq s(x+y)$, $x \geq 0, y \geq 0$.

b. $s(x)\int_0^\infty s(y)dy \geq \int_0^\infty s(x+y)dy$.

c. $s(x)\int_0^\infty s(y)dy \geq \int_x^\infty s(w)dw$.

d. $\int_0^\infty s(y)dy \geq \int_x^\infty \frac{s(w)}{s(x)}dw$.

20. 在习题 13.19 中，乘以 v^y, 证明 $\overline{a}_0 > \overline{a}_x$.

21.

a. 证明 $\overline{P}(\overline{A}_x)$ 可以写成死力 $\mu_{(x+t)}$ 的加权平均，其中权函数 $w(t,\delta) = \frac{v^t {}_tp_x}{\overline{a}_x}$ 。

b. 证明

(Ⅰ) $\int_0^\infty w(t,\delta)dt = 1$.

(Ⅱ) $\frac{\partial}{\partial t}w(t,\delta) \leq 0$.

(Ⅲ) $\frac{\partial}{\partial \delta}w(t,\delta) = \frac{v^t {}_tp_x[-t\overline{a}_x + (\overline{I}\overline{a})_x]}{(\overline{a}_x)^2}$.

c. 若死力严格上增，用结果 (b)(Ⅱ) 和 (b)(Ⅲ) 证明利息效力的增加会增加小死力的权数而减小大死力的权数。因此，若死力严格上增，利息效力的增加会减小 $\overline{P}(\overline{A}_x)$

第十四章　退休基金累积理论

§14.1　引　言

我们在第八章研究了退休金计划参加者的受益及醵出的精算现值，这些个别参加者的精算现值对于计划的精算现值总额决定过程是必不可少的输入值。这些受益精算现值总额连同当前资产一起与未来醵出的精算现值总额相平衡。必须与受益支付相平衡的综合醵出型式由*精算成本*(actuarial cost) 或由*基金累积方法*(funding method) 决定。这一章将定义在概括退休金计划累积状况中有用的函数，这些函数将用于描述精算成本方法，并探讨这些方法的性质。

为此，需用到第十三章的部分人口理论，并且这里的研究与例 13.5.1 — 13.5.4 中考察某人口群体寿险体系的替代基金累积或预算方法有相似之处。

为了将本章的观念与以前的结合在一起，读者应牢记这些观念的某些基本限制：

1. 为保障参加者的权益并限制收入中 (因向退休金计划醵出) 的延迟纳税数额，政府对精算成本方法有所规定，这些法规在实践中很重要，但这里不予讨论。

2. 退休金计划常常提供很多种类的受益。在退休受益以外，死亡受益与残疾受益是最常见的，在许多场合要求有法定的退保受益。这些受益中的某些精算现值已在第八章里决定。这一章的模型只提供退休收入受益，这一简化的目的在于将注意力集中于各种精算成本方法的性质。尽管大多数退休金计划提供的退休收入率或多或少地依赖于退休前的收入水平，但是在这一章里使用

的模型中初始退休金受益率只依赖于退休时的收入。这一简化同样是为了集中精力于精算成本方法。

3. 本书的一个持续的主题是，精算现值需要应用利息因子与未来可能有的支付的概率。这一章里的未来支付可能依赖于许许多多的不确定事件，然而，为了与研究精算成本方法的目标相一致，以下采取决定性的观点。

§14.2 模 型

设群体由 a 岁时加入计划的成员组成， r 是退休年龄。与此相适应的生存函数 $s(x), x \geq a$ 满足 $s(a)=1$ 。对于 $a \leq x < r$, 损失 (decrement) 可以是死亡或其它原因，但对于 $x \geq r$, 死亡是唯一的损因。在时间 u 的 a 岁加入者密度记为 $n(u)$, 在时间 t 达到 x 岁的人口 (参加者) 密度为

$$l(x,u)=n(u)s(x), \tag{14.2.1}$$

其中 $u=t-x+a$ 是进入计划的时间。公式 (14.2.1) 与 (13.3.1) 有联系，差别在于成为计划参加者相当于发生在 a 岁的出生。以下设生存函数不依赖于 u 。

设 x 岁成员在时间 0 的年薪 (率) 为 $w(x), a \leq x < r$. 用 $g(t)$ 表示反映通胀及生产力变化的时间因子，于是在时间 t 的 x 岁成员预期年薪 (率) 为

$$w(x)g(t) \qquad a \leq x < r. \tag{14.2.2}$$

这比 (8.2.1) 的简单模型更为实际。

根据 (13.3.6), 在时间 t 介于年龄 x 与 $x+dx$ 之间的 $l(x,t-x+a)dx$ 个成员的年薪 (率) 合计为

$$l(x,t-x+a)w(x)g(t)dx.$$

在时间 t 整个群体的年工资 (率) 总额为

$$\mathbf{W}(t)=\int_a^r l(x,t-x+a)w(x)g(t)dx. \tag{14.2.3}$$

公式 (14.2.3) 体现了本章以后部分使用的记号约定，黑体符号代表涉及所有参加者的量，于是 $\mathbf{W(t)}$ 表示在时间 t 的工资总额。

以下考虑的模型只在达到退休年龄 r 之后提供退休年金。设退休金的初始年支付率是最终年薪率的某个百分比，比例因子为 f，于是对于在时间 t 的退休者，预定的退休金年支付率为

$$fw(r)g(t). \tag{14.2.4}$$

而对于在时间 t 的 x 岁已退休者，预定的退休金支付年率为

$$fw(r)g(t-x+r)h(x) \qquad x\geq r, \tag{14.2.5}$$

其中 $h(x)$ 是一个应用于初始退休金的调整因子， $h(r)=1$ 。譬如 $h(x)$ 可具有指数形式 $\exp[\beta(x-r)]$，β 是常数 (瞬时) 增长率。

在我们讨论精算成本方法中，主要考虑的模型是*规定受益计划*(defined benefit plan)，这种计划规定了退休者获得的受益，我们将集中阐述产生一系列与受益支付相平衡的醵出与投资收入的精算成本方法。*规定醵出计划*(defined contribution plan) 的出发点有所不同，作为每个参加者的醵出是指定的，也许是常数或薪水的某个比例。精算问题在于计算精算现值与醵出的精算现值相等的受益水平，这可以在退休时用累积的醵出提供等价的退休年金来决定，也可用每年的醵出购买延期退休年金来逐年决定。

§14.3 期末基金累积

按*期末基金累积方法*(terminal funding method)，在职工退休时，一次性提出未来的退休金支付基金。在时间 t 整个群体按期

末基金累积方法所需的一次性醵出率或正规成本率(normal cost rate), 记为 $^{\mathbf{T}}\mathbf{P}(t)$, 它是在时间 t 所有 r 岁生存成员的未来退休金收入之精算现值关于时间的比率, 即 $^{\mathbf{T}}\mathbf{P}(t)dt$ 是在时间 t 与 $t+dt$ 之间的醵出金总额。从 r 岁开始且 x 岁时年收入率为 $h(x)$ 的连续支付生存年金的精算现值用 $\overline{a}_r^h$ 表示.

$$\overline{a}_r^h = \int_r^\infty e^{-\delta(x-r)} h(x) \frac{s(x)}{s(r)} dx. \tag{14.3.1}$$

根据 (13.3.2), 在时间 t 与 $t+dt$ 之间达到 r 岁的人数为 $l(r, t-r+a)dt$, 于是

$$^{\mathbf{T}}\mathbf{P}(t) = fw(r)g(t)l(r, t-r-a)\overline{a}_r^h. \tag{14.3.2}$$

为了说明这一理论,考虑具有以下特征的指数情形(exponential case):

$$n(u) = ne^{Ru}, \qquad g(t) = e^{\tau t}, \qquad h(x) = e^{\beta(x-r)}.$$

指数情形的局限性在于它不可能无限制地存在。当指数情形近似实现时, 三个主要经济上的瞬时变化率 —— 利率 δ, 工资增长率 τ 及退休金调整率 β 是相互关联的。

例如, 当 β 与通胀相联系时, 通常可假定利息效力 $\delta > \beta$, 尽管在某些非常的通胀期可能会出现相反的情况。(如果 $\beta > \tau$, 其结果是退休相对于在职工作而言将反而改善经济地位, 因此通常设 $\tau \geq \beta$).

例 14.3.1: 对于指数情形, 验证 $^{\mathbf{T}}\mathbf{P}(t+y) = e^{\rho y}\,{}^{\mathbf{T}}\mathbf{P}(t)$, 其中 $\rho = \tau + R$ 。

解: 由 (14.3.1) 可知

$$l(r, t+y-r+a) = ns(r)e^{R(t-y-r+a)},$$

又由 (14.3.1) 可知，

$$\overline{a}_r^h = \int_0^\infty e^{-(\delta-\beta)(x-r)}\frac{s(x)}{s(r)}dx = \overline{a}_r',$$

其中 $\overline{a}_r'$ 表示按利息效力 $\delta-\beta$ 计算的年金值。于是根据 (14.3.2) 得

$$\begin{aligned} {}^{\mathbf{T}}\mathbf{P}(t+y) &= fw(r)e^{\tau(t+y)}ns(r)e^{R(t+y-r+a)}\overline{a}_r' \\ &= e^{(\tau+R)y}fw(r)e^{\tau t}ns(r)e^{R(t-r+a)}\overline{a}_r' = e^{\rho y}\,{}^{\mathbf{T}}\mathbf{P}(t). \end{aligned}$$

其中的若干项可分别予以解释。变化率 $\rho=\tau+R$ 可解释为总的经济增长 (或下降) 率。项 $ns(r)$ 可解释为 l_r，它是受制于多重损因残存函数 $s(x)$,$a\le x\le r$ 的残存组在年龄 r 时的残存人数，该残存组在 a 岁有 n 个成员。

14.4 精算债务的积存

精算成本方法不同于以上的期末成本方法，未来待付退休金债务不是在达到退休年龄 r 时一次性确认，而是在计划参加者的工作期间就逐步予以确认，为表达始于 r 岁的退休金之精算债务的积存 (accrual of actuarial liability)，引入 *积存函数*(accrual function) $M(x)$，它是未来退休金之精算现值按精算成本法在 x 岁时应计的精算成本比例。 $M(x)$ 是非减右连续函数，且 $0\le M(x)\le 1$. 在 *初始基金累积方法*(initial funding method) 之下，所有的未来退休金负债在进入计划的 a 岁时一次性确认，此时

$$M(x)=\begin{cases} 0 & x<a \\ 1 & x\ge a. \end{cases}$$

对其它的精算成本方法，通常 $M(x)$ 连续且可表示成

$$M(x)=\int_a^x m(y)dy \qquad x\ge a, \qquad (14.4.1)$$

其中的非负函数 $m(x)$ 称为退休金积存密度函数(pension accrual density function), 通常当 $x>r$ 时 $m(x)=0$ 。这样，对于一般的精算成本方法 (不包括上述初始成本方法), $M(a)=0$ 成立，且当 $x>r$ 时 $M(x)=1$ 。在这种连续情形下，根据 (14 .4.1) 可得

$$m(x)=M'(x) \qquad a<x<r. \tag{14.4.2}$$

在 $M'(x)$ 的间断点，密度 $m(x)$ 无定义，我们可对它任意赋值，譬如，左极限值或右极限值。

引入积存函数的好处在于，可同时对各种各样精算成本方法建立退休金累积理论，而不必对每种方法分别建立。

例 14.4.1: 对于

$$M(x)=\bar{a}_{a:\overline{x-a}|}/\bar{a}_{a:\overline{r-a}|} \qquad a\le x\le r,$$

验证 (1) $M(x)$ 具有连续型积存函数的性质。

(2) $M(x)_{r-x|}\bar{a}_x$ 等于 a 岁时签发的从 r 岁开始 1 单位完全连续的延期终身生存年金 (缴费期为递延期) 在 x 岁时的责任准备金。

解：

(1) 对于 $a<x<r$,

$$M'(x)=m(x)=\frac{d}{dx}\frac{\int_a^x e^{-\delta(y-a)}s(y)dy}{\int_a^r e^{-\delta(y-a)}s(y)dy}=\frac{e^{-\delta(x-a)}s(x)}{\int_a^r e^{-\delta(y-a)}s(y)dy}>0. \tag{14.4.3}$$

又显然， $M(a)=0, M(r)=1$ 。

(2) 用责任准备金后顾公式可得 x 岁时 $(a<x<r)$ 的责任准备金为

$$\begin{aligned}\bar{P}({}_{r-a|}\bar{a}_a)\bar{s}_{a:\overline{x-a}|}&=\frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a}|}}\bar{s}_{a:\overline{x-a}|}\\&=\frac{{}_{x-a}E_a\,{}_{r-x|}\bar{a}_x}{\bar{a}_{a:\overline{r-a}|}}\frac{\bar{a}_{a:\overline{x-a}|}}{{}_{x-a}E_a}={}_{r-x|}\bar{a}_xM(x).\end{aligned}$$

§14.5 有关在职成员的基本函数

这一节将定义几个与模型计划中的退休金受益的基金累积有关系的基本函数，这些函数与在职群体相关，其符号前都冠以 a 作为前缀。

一. 在职群体未来退休金在时间 t 的精算现值 $(\mathbf{aA})(\mathbf{t})$

在时间 t, 在职群体的未来退休金的精算现值记为 $(\mathbf{aA})(t)$ 。由于在时间 t 的 $l(x, t-x+a)dx$ 个介于年龄 x 与 $x+dx$ 之间的成员在 $r-x$ 年后应提期末基金累积成本为 ${}^{\mathbf{T}}\mathbf{P}(t+r-x)dx$,

$$(\mathbf{aA})(t) = \int_a^r e^{-\delta(r-x)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)dx. \qquad (14.5.1)$$

例 14.5.1: 验证

$$\frac{d}{dt}(\mathbf{aA})(t) = e^{-\delta(r-a)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-a) - {}^{\mathbf{T}}\mathbf{P}(t) + \delta(\mathbf{aA})(t). \quad (14.5.2)$$

解：注意到

$$\frac{\partial}{\partial t}{}^{\mathbf{T}}\mathbf{P}(t+r-x) = -\frac{\partial}{\partial x}{}^{\mathbf{T}}\mathbf{P}(t+r-x), \qquad (14.5.3)$$

可导出

$$\begin{aligned}
\frac{d}{dt}(\mathbf{aA})(t) &= \int_a^r e^{-\delta(r-x)}\frac{\partial}{\partial t}{}^{\mathbf{T}}\mathbf{P}(t+r-x)dx \\
&= -\int_a^r e^{-\delta(r-x)}\frac{\partial}{\partial x}{}^{\mathbf{T}}\mathbf{P}(t+r-x)dx \\
&= -{}^{\mathbf{T}}\mathbf{P}(t) + e^{-\delta(r-a)}\,{}^{\mathbf{T}}\mathbf{P}(t-r-a) + \delta(\mathbf{aA})(t).
\end{aligned}$$

未来退休金的精算现值变化率等于有关新参加者 (他们在 $r-a$ 年之后将退休) 的未来期末基金累积率现值，减去有关现在退休的在职者的期末基金累积率，加上在时间 t 的精算现值利息。

二. 正规成本率 $\mathbf{P}(t)$

假定已选择了积存函数为 $M(t)$ 的一种精算成本方法，我们寻求模型计划的正规成本率的表示，即对于我们的连续模型显示将未来退休金受益精算现值分配于参加者在职期各不同估价时间的函数。

犹如 (14.5.1), 在时间 t 年龄介于 x 与 $x+dx$ 成员的未来期末基金累积成本为 ${}^{\mathbf{T}}\mathbf{P}(t+r-x)dx$, 而对于积存密度函数为 $m(x)$ 的精算成本方法，在时间 t 介于年龄 x 与 $x+dx$ 之间成员的应计 ($r-x$ 年末) 期末成本 (率) 为

$$
{}^{\mathbf{T}}\mathbf{P}(t+r-x)m(x)dx,
$$

因此相应的精算成本方法下 *正规成本率*(normal cost rate)

$$
\mathbf{P}(t)=\int_a^r e^{-\delta(r-x)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)m(x)dx, \tag{14.5.4}
$$

在时间 u 加入计划的 a 岁成员的期末成本率为 ${}^{\mathbf{T}}\mathbf{P}(u+r-a)$. 另一方面，根据 (14.5.4), 这些成员在时间 t 到 $t+dt$ 之间按精算成本方法应计醵出率为

$$
e^{-\delta(r-x)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)m(x)dt,
$$

其中 $x=a+t-u$, $u\le t\le u+r-a$, 这一醵出率在期末 ($r-x$ 年末 r 岁) 时为

$$
{}^{\mathbf{T}}\mathbf{P}(t+r-x)m(x)dt={}^{\mathbf{T}}\mathbf{P}(u+r-a)m(a+t-u)dt. \tag{14.5.5}
$$

因此从时间 u 到 $u+r-a$ 的累计期末醵出率为

$$
\int_u^{u+r-a}{}^{\mathbf{T}}\mathbf{P}(u+r-a)m(a+t-u)dt={}^{\mathbf{T}}\mathbf{P}(u+r-a),
$$

即期末成本率。

例 14.5.2: (1) 证明在指数情形成立

$$\mathbf{P}(t)=\exp\{-\delta[r-X(\theta)]\}^{\mathbf{T}}\mathbf{P}[t+r-X(\theta)], \tag{14.5.6}$$

其中

$$\theta=\delta-\rho=\delta-\tau-R,$$

且

$$e^{\theta X(\theta)}=\int_a^r m(x)e^{\theta x}dx. \tag{14.5.7}$$

(2) 解释 (14.5.6) 。

解： (1) 根据 (14.5.4) 以及例 14.3.1 的解,

$$\begin{aligned}
\mathbf{P}(t) &= \int_a^r e^{-\delta(r-x)\mathbf{T}}\mathbf{P}(t+r-x)m(x)dx \\
&= \int_a^r e^{-\delta(r-x)\mathbf{T}}\mathbf{P}\{t+[X(\theta)-x]+r-X(\theta)\}m(x)dx \\
&= \int_a^r e^{-\delta(r-x)}e^{\rho[X(\theta)-x]\mathbf{T}}\mathbf{P}[t+r-X(\theta)]m(x)dx \\
&= e^{[-\delta r+\rho X(\theta)]\mathbf{T}}\mathbf{P}[t+r-X(\theta)]\int_a^r e^{(\delta-\rho)x}m(x)dx.
\end{aligned}$$

用 $\delta-\theta$ 代替 ρ, 并利用 (14.5.7) 可得

$$\mathbf{P}(t)=e^{[-\delta r+(\delta-\theta)X(\theta)]\mathbf{T}}\mathbf{P}[t+r-X(\theta)]e^{\theta X(\theta)},$$

从而可得出 (14.5.6) 。

(2) 在时间 t 的年正规成本率及其利息足够提供 $r-X(\theta)$ 年之后的期末基金累积成本。数 $X(\theta)$ 的存在性由积分中值定理确保，并可解释为指数情形 ($\theta=\delta-\tau-R$) 与积存密度函数 $m(x)$ 相联系的正规成本支付的平均年龄。因此 $X(\theta)$ 依赖于利率、薪水以及人口变化率。

三. 精算积存负债 $(\mathbf{aV})(t)$

对于积存函数为 $M(x)$ 的精算成本方法，在职群体在时间 t 的**精算积存负债** (actuarial accrued liability) 为

$$(\mathbf{aV})(t) = \int_a^r e^{-\delta(r-x)}{}^{\mathbf{T}}\mathbf{P}(t+r-x)M(x)dx. \qquad (14.5.8)$$

用例 14.5.1 的方法可得

$$\begin{aligned}
\frac{d}{dt}(\mathbf{aV})(t) &= -\int_a^r e^{-\delta(r-x)}M(x)\frac{\partial}{\partial x}{}^{\mathbf{T}}\mathbf{P}(t+r-x)dx \\
&= -M(r)^{\mathbf{T}}\mathbf{P}(t) + e^{-\delta(r-a)}M(a)^{\mathbf{T}}\mathbf{P}(t+r-x) \\
&\quad + \int_a^r e^{-\delta(r-x)}[\delta M(x)+m(x)]^{\mathbf{T}}\mathbf{P}(t+r-x) \\
&= -{}^{\mathbf{T}}\mathbf{P}(t) + \delta(\mathbf{aV})(t) + \mathbf{P}(t), \qquad (14.5.9)
\end{aligned}$$

于是

$$\mathbf{P}(t) + \delta(\mathbf{aV})(t) = {}^{\mathbf{T}}\mathbf{P}(t) + \frac{d}{dt}(\mathbf{aV})(t). \qquad (14.5.10)$$

如将精算积存负债看作一项基金，正规成本按年支付率 $\mathbf{P}(t)$ 存入，在成员达到退休年龄时按期末成本率 ${}^{\mathbf{T}}\mathbf{P}(t)$ 转出，则 (14.5.10) 左端代表基金来源于正规成本和利息的收入率，而右端则是期末成本转出率和基金变化率。

例 14.5.3: 在指数情形成立

(1) $\mathbf{P}(t+y) = e^{\rho y}\mathbf{P}(t), \qquad \rho = \tau + R.$ (14.5.11)

(2) $(\mathbf{aV})(t+y) = e^{\rho y}(\mathbf{aV})(t)$ (14.5.12)

(3) $\mathbf{P}(t) + \theta(\mathbf{aV})(t) = {}^{\mathbf{T}}\mathbf{P}(t), \qquad \theta = \delta - \rho$ (14.5.13)

(4) $\theta > 0$, 则 $\mathbf{P}(t) < {}^{\mathbf{T}}\mathbf{P}(t)$. (14.5.14)

$\theta = 0$, 则 $\mathbf{P}(t) = {}^{\mathbf{T}}\mathbf{P}(t)$.

$\theta < 0$, 则 $\mathbf{P}(t) > {}^{\mathbf{T}}\mathbf{P}(t)$.

解：

(1) 将例 14.4.1 的结果代入 (14.5.4), 就可得出

$$\mathbf{P}(t+y) = e^{\rho y}\mathbf{P}(t).$$

(2) 与 (1) 类似，代入 (14.5.8) 可得

$$(\mathbf{aV})(t+y)=e^{\rho y}(\mathbf{aV})(t).$$

(3) 利用 (2) 的结果，有

$$\frac{d}{dt}(\mathbf{aV})(t)=\lim_{y\to 0}\frac{(\mathbf{aV})(t+y)-(\mathbf{aV})(t)}{y}=\rho(\mathbf{aV})(t), \tag{14.5.15}$$

代入 (14.5.10) 便得

$$\mathbf{P}(t)+(\delta-\rho)(\mathbf{aV})(t)={}^{\mathbf{T}}\mathbf{P}(t).$$

(4) 是 (3) 的推论。

四. 未来正规成本精算现值 (Pa)(t)

这一节开始时曾提到，在时间 t 介于年龄 x 与 $x+dx$ 之间的 $l(x,t-x+a)dx$ 个成员在 $r-x$ 年后退休时的期末基金累积成本为 ${}^{\mathbf{T}}\mathbf{P}(t+r-x)dx$ 。这些成员从年龄 y 到 $y+dy(x\le y<r)$ 的正规成本为

$$e^{-\delta(r-y)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)dxm(y)dy,$$

现值为

$$e^{-\delta(y-x)}e^{-\delta(r-y)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)dxm(y)dy. \tag{14.5.16}$$

因此，在职群体的*未来正规成本精算现值*(actuarial present value of future normal cost) 为

$$\begin{aligned}(\mathbf{Pa})(t) &= \int_a^r e^{-\delta(r-x)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)\int_x^r m(y)dydx\\ &= \int_a^r e^{-\delta(r-x)}\,{}^{\mathbf{T}}\mathbf{P}(t+r-x)[1-M(x)]dx.\end{aligned} \tag{14.5.17}$$

图 14.5.1 说明了产生 $(\mathbf{Pa})(\mathbf{t})$ 的想法。表达式 (14.5.16) 表示阴影区域成本元素在时间 t 的现值。在 (14.5.17) 中，内层积分表示沿对角线的元素和，而外层积分则是关于所有年龄的未来正规成本在时间 t 的现值。

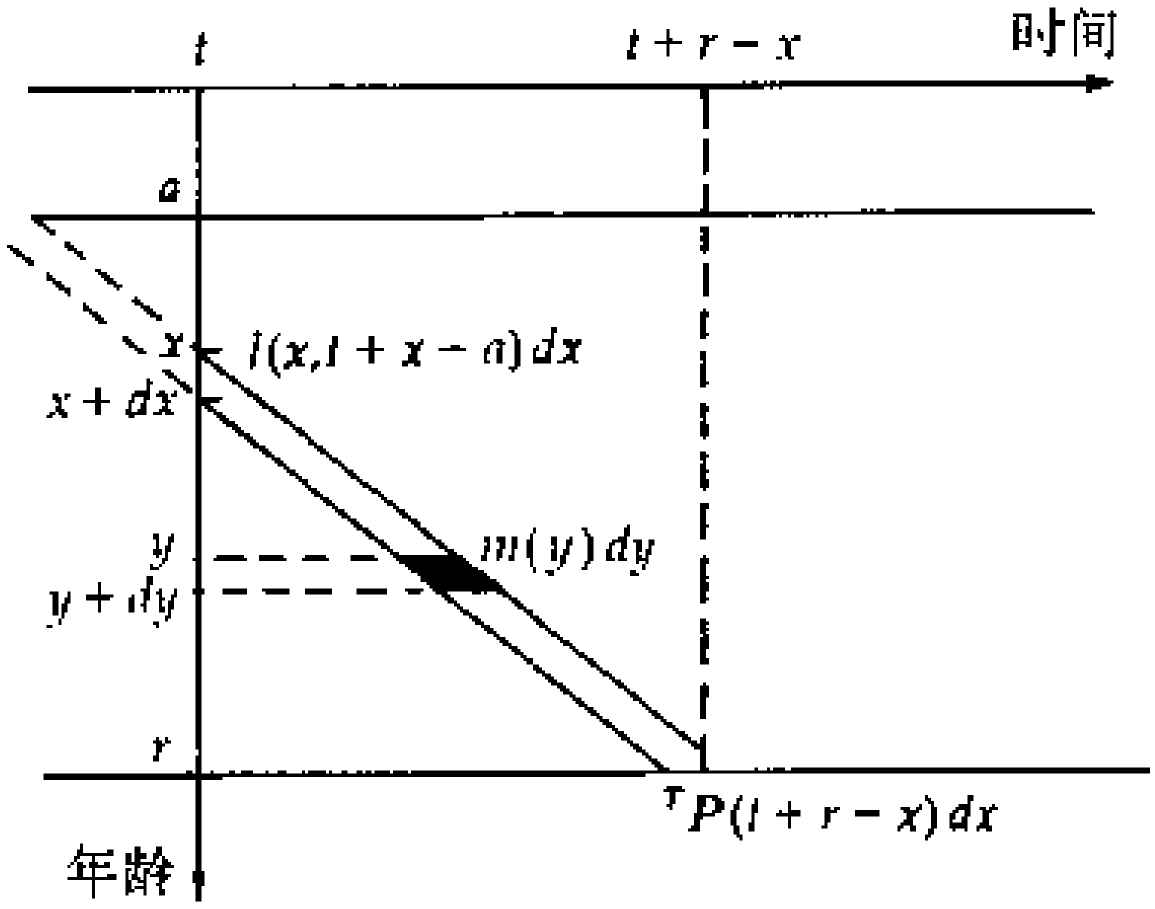


图 14.5.1 $(\mathbf{Pa})(\mathbf{t})$ 的形成

根据未来正规成本精算现值的含义或 (14.5.17), 可知

$$(\mathbf{Pa})(t) = (\mathbf{aA})(t) - (\mathbf{aV})(t), \tag{14.5.18}$$

或者

$$(\mathbf{aV})(t) = (\mathbf{aA})(t) - (\mathbf{Pa})(t). \tag{14.5.19}$$

式 (14.5.19) 与责任准备金的前瞻公式相仿，经常用于定义 $(\mathbf{aV})(t)$, 即

在职成员在时间 t 的精算负债

= 在职成员未来退休金的精算现值

− 未来正规成本的精算现值

按照与第五章相似的概念， $V = A - Pa$ 或 $A = V + Pa$，可得出与在职成员的未来退休金精算现值相平衡的是，有关在职者的精算积存负债和未来正规成本精算现值。

在职群体的未来退休金精算现值分解

$$(\mathbf{aA})(t) = (\mathbf{aV})(t) + (\mathbf{Pa})(t) \tag{14.5.20}$$

决定于积存函数 $M(x)$ 反映的精算成本方法。

例 14.5.4:

(1) 考虑两个积存函数 $M_{\mathrm{I}}(x)$, $M_{\mathrm{II}}(x)$ 。如果 $D(x) = M_{\mathrm{I}}(x) - M_{\mathrm{II}}(x)$ 满足 $D'(a) > 0$ 且 $D'(x) = 0$ 对 $a < x < r$ 只有唯一解，那么

$$(\mathbf{aV})_{\mathrm{I}}(t) > (\mathbf{aV})_{\mathrm{II}}(t).$$

(2) 设

$$M_{\mathrm{I}}(x) = \frac{\overline{a}_{a:\overline{x-a}|}}{\overline{a}_{a:\overline{r-a}|}}, \qquad M_{\mathrm{II}}(x) = \frac{x-a}{r-a},$$

验证它们满足 (1) 中的条件。

解： (1) 由于 $D(a) = D(r) = 0$, 根据所给条件可知

$$D(x) > 0 \qquad a < x < r,$$

于是

$$(\mathbf{aV})_{\mathrm{I}}(t) - (\mathbf{aV})_{\mathrm{II}}(t) = \int_a^r e^{-\delta(r-x)\mathbf{T}}\mathbf{P}(t+r-x)D(x)dx > 0.$$

(2) 参考例 14.4.1,

$$D'(x) = \frac{e^{-\delta(x-a)}s(x)}{\int_a^r e^{-\delta(y-a)}s(y)dy} - \frac{1}{r-a}.$$

由

$$\int_a^r e^{-\delta(y-a)}s(y)dy < \int_a^r dy = r-a,$$

可知

$$D'(a) = \frac{1}{\int_a^r e^{-\delta(y-a)}s(y)dy} - \frac{1}{r-a} > 0.$$

又根据 $D''(x) = -(\delta+\mu_x)e^{-\delta(x-a)}s(x)/\int_a^r e^{-\delta(y-a)}s(y)dy < 0$, $a < x < r$ 可知 $D'(x) = 0$ 在 $a < x < r$ 内至多只有一个解。于是对这两个具体的积存函数，成立

$$(\mathbf{aV})_{\mathrm{I}}(t) > (\mathbf{aV})_{\mathrm{II}}(t).$$

由 (14.5.20) 还可得

$$(\mathbf{Pa})_{\mathrm{II}}(t) > (\mathbf{Pa})_{\mathrm{I}}(t).$$

§14.6 个体精算成本方法

用积存函数 $M(x)$ 或它的导数确定的精算成本法是适用于每个参加者的个体成本方法。整个在职者群体的正规成本率或精算积存负债可通过每个计划参加者的正规成本率或精算积存负债相加得出。

对于现龄 x 岁 $(a \le x < r)$ 将在 r 岁退休并获得初始时为 1 单位退休金收入的个体在职者，受益的精算现值为

$$(aA)(x) = e^{-\delta(r-x)}\frac{s(r)}{s(x)}\bar{a}_x^h, \tag{14.6.1}$$

正规成本率为

$$P(x) = (aA)(x)m(x), \tag{14.6.2}$$

积存精算负债为

$$(aV)(x) = (aA)(x)M(x). \tag{14.6.3}$$

未来正规成本的精算现值定义为

$$(Pa)(x) = (aA)(x) - (aV)(x) = (aA)(x)[1 - M(x)]. \quad (14.6.4)$$

在时间 t, x 岁在职者 ($r-x$ 年后) 退休时的初始退休金年收入为 $fg(t+r-x)w(r)$ 。在时间 t 对所有在职参加者相加的综合未来退休年金精算现值为

$$\begin{aligned}
&\int_a^r fg(t+r-x)w(r)(aA)(x)l(x,t-x+a)dx \\
&= \int_a^r fg(t+r-x)w(r)e^{-\delta(r-x)}\overline{a}_r^h l(r,t-x+a)dx \\
&= \int_a^r e^{-\delta(r-x)\mathbf{T}}\mathbf{P}(t+r-x)dx = (\mathbf{aA})(t),
\end{aligned}$$

与 §14.5 的在职者群体退休金精算现值一致。

在 *应计受益成本方法*(accrued benefit cost method) 中，$M(x)$ 直接与个人参加者在 x 时应计积存受益相联系。如果在服务期内均匀地积存，那么

$$m(x) = \frac{1}{r-a}; \quad (14.6.5)$$

如果积存比例与工资总额挂勾，那么

$$m(x) = \frac{w(x)}{\int_a^r w(y)dy}; \quad (14.6.6)$$

如果积存比例与工资总额挂勾并且年薪具有指数式时间趋势，那么

$$m(x) = \frac{w(x)e^{\tau x}}{\int_a^r w(y)e^{\tau y}dy}. \quad (14.6.7)$$

对于 *参加年龄精算成本方法*(entry-age actuarial cost method), 预定受益按固定年率或年薪的固定比率酬出累积，此时，

$$m(x) = \frac{e^{-\delta x}s(x)}{\int_a^r e^{-\delta y}s(y)dy} \quad (14.6.8)$$

或

$$m(x)=\frac{e^{-\delta x}s(x)w(x)}{\int_a^r e^{-\delta y}s(y)w(y)dy}. \tag{14.6.9}$$

在酬出按年薪 (率) 的固定比率 π 进行累积的情形，

$$\pi=e^{-\delta(r-a)}\frac{s(r)}{s(a)}\overline{a}_r^h\Big/\int_a^r e^{-\delta(y-a)}\frac{s(y)}{s(a)}w(y)dy, \tag{14.6.10}$$

此时有

$$\begin{aligned}(\mathbf{aV})(x) &= e^{-\delta(r-x)}\frac{s(r)}{s(a)}\overline{a}_r^h-\pi\int_x^r e^{-\delta(y-x)}\frac{s(y)}{s(x)}w(y)dy\\ &= e^{-\delta(r-x)}\frac{s(r)}{s(x)}\overline{a}_r^h\left[1-\frac{\int_x^r e^{-\delta y}s(y)w(y)dy}{\int_a^r e^{-\delta y}s(y)w(y)dy}\right].\end{aligned}$$

由 (14.7.9) 得

$$(aV)(x)=(aA)(x)M(x).$$

这里，积存函数 $M(x)$ 的密度由 (14.6.9) 给出。如年薪具有指数式时间趋势，则

$$m(x)=\frac{e^{-\delta x}s(x)e^{\tau x}w(x)}{\int_a^r e^{-\delta y}s(y)e^{\tau y}w(y)dy}. \tag{14.6.11}$$

应计受益成本方法与参加年龄精算成本方法计算出的积存精算负债有所不同。前者一般系法规要求，并可用来与参加者沟通。两者的差异犹如普通保险中不没收受益与责任准备金的区别。

值得注意的是，计划发起者按个体精算成本方法支付的酬出率，通常不同于用该方法确定的总正规成本率，其一般理由有两个：首先，在计划开始或修订时，对于先前服务的精算积存负债可能改变；其次，精算假设一般不会准确无误地实现，由此导致基金盈余或亏损。如何调整酬出率来累积这种精算积存负债的改

变或者应付盈余或亏损，是涉及到法规的重要问题，所使用的特别调整并不受所选择的个体精算成本方法制约。

§14.7 总体成本方法

对于总体精算成本方法，需要以下几个函数：在时间 t 分配于在职群体的基金为 $(\mathbf{aF})(t)$，在职群体的年醵出率为 $(\mathbf{aC})(t)$，精算积存（应计）负债的未累积部分为 $(\mathbf{aU})(t)$. 显然，

$$(\mathbf{aU})(t) = (\mathbf{aV})(t) - (\mathbf{aF})(t), \tag{14.7.1}$$

$$\frac{d}{dt}(\mathbf{aF})(t) = (\mathbf{aC})(t) + \delta(\mathbf{aF})(t) - {}^{\mathbf{T}}\mathbf{P}(t). \tag{14.7.2}$$

式 (14.7.2) 右端的前两项为在职群体基金的两个来源：醵出与利息收入，第三项对应于期末基金累积成本转出为退休群体基金。

与精算成本方法相联系的醵出率的一个较为自然的形式为

$$(\mathbf{aC})(t) = \mathbf{P}(t) + \lambda(t)(\mathbf{aU})(t), \tag{14.7.3}$$

其中 $\mathbf{P}(t)$ 是正规成本率 [见 (14.5.4)], $\lambda(t)$ 确定了分摊 $(\mathbf{aU})(t)$ 的过程。

方程 (14.7.3) 指出了总体精算成本方法尚未提及的一个特征。这些方法规定的醵出率 $(\mathbf{aC})(t)$ 依赖于基金累积水平，即依赖 $(\mathbf{aU})(t)$ 的大小。这里因计划改变或盈亏而需作出的调整在精算成本方法中可自动完成，这是因为 $(\mathbf{aU})(t)$ 的值反映了这种改变及盈亏。

以下考察一种分摊过程，其中

$$\lambda(t) = \frac{1}{\mathbf{a}(t)}, \tag{14.7.4}$$

$$\mathbf{a}(t) = \frac{(\mathbf{Pa})(t)}{\mathbf{P}(t)}. \tag{14.7.5}$$

此时公式 (14.7.3) 可改写为

$$
\begin{aligned}
(\mathbf{aC})(t) &= \mathbf{P}(t)+\frac{(\mathbf{aV})(t)-(\mathbf{aF})(t)}{\mathbf{a}(t)} \\
&= \frac{(\mathbf{Pa})(t)+(\mathbf{aV})(t)-(\mathbf{aF})(t)}{\mathbf{a}(t)} \\
&= \frac{(\mathbf{aA})(t)-(\mathbf{aF})(t)}{\mathbf{a}(t)},
\end{aligned} \tag{14.7.6}
$$

即

$$
(\mathbf{aC})(t)\mathbf{a}(t)=(\mathbf{aA})(t)-(\mathbf{aF})(t). \tag{14.7.7}
$$

于是 $\mathbf{P}(t)\mathbf{a}(t)=(\mathbf{Pa})(t)$,$\mathbf{a}(t)$ 是一个单位定期年金的值，该定期年金以当前的正规成本率 $\mathbf{P}(t)$ 为定额收入率时，等价于当前在职成员的未来正规成本精算现值 $(\mathbf{Pa})(t)$ 。这里，记号的选取暗示了这种想法。

对于酿出由 (14.7.3) 给出的基金累积过程，(14.7.2) 成为

$$
\frac{d}{dt}(\mathbf{F})(t)=\mathbf{P}(t)+\lambda(t)(\mathbf{aU})(t)+\delta(\mathbf{aF})(t)-{}^{\mathbf{T}}\mathbf{P}(t). \tag{14.7.8}
$$

利用 (14.5.10) 可得

$$
\begin{aligned}
\frac{d}{dt}(\mathbf{aU})(t) &= \frac{d}{dt}(\mathbf{aV})(t)-\frac{d}{dt}(\mathbf{aF})(t) \\
&= -(\lambda(t)-\delta)(\mathbf{aU})(t).
\end{aligned} \tag{14.7.9}
$$

解这个微分方程可得

$$
(\mathbf{aU})(t)=(\mathbf{aU})(0)\exp[-\int_0^t(\lambda(s)-\delta)ds]. \tag{14.7.10}
$$

如果 $\lambda(t)$ 满足

$$
\lambda(t)-\delta\geq\varepsilon>0, \tag{14.7.11}
$$

那么当 $t\to\infty$ 时，$(\mathbf{aU})(t)\to 0$ 即 $(\mathbf{aF})(t)-(\mathbf{aV})(t)\to 0$.

满足 (14.7.11) 的总体成本方法渐近等价于用来估算 $(\mathbf{aV})(t)$ 及 $\mathbf{P}(t)$ 的积存函数所确定的个体成本方法。如 $\lambda(t)$ 由 (14.7.4) 给出，可能有不少积存函数使得 $(\mathbf{Pa})(u)/\mathbf{P}(t)$ 足够小以保证 (14.7.11) 成立，从而 $(\mathbf{aF})(t)-(\mathbf{aV})(t)\to 0$.

如果目标是在 n 年内分摊 $(\mathbf{aU})(0)$, 可取

$$\lambda(t)=\frac{1}{\bar{a}_{\overline{n-t|}}}\qquad 0<t<n. \tag{14.7.12}$$

此时由 (14.7.10) 得

$$\begin{aligned}(\mathbf{aU})(t) &= (\mathbf{aU})(0)\exp[-\int_0^t(\frac{1}{\bar{a}_{\overline{n-u|}}}-\delta)du] \\ &= (\mathbf{aU})(0)\exp[-\int_0^t\frac{1}{\bar{s}_{\overline{n-u|}}}du] \\ &= (\mathbf{aU})(0)\exp[+\int_0^t\frac{\delta du}{1-e^{(n-u)\delta}}] \\ &= (\mathbf{aU})(0)\exp[\ln\frac{e^{\delta n}-e^{\delta t}}{e^{\delta n}-1}] \\ &= (\mathbf{aU})(0)\frac{\bar{s}_{\overline{n|}}-\bar{s}_{\overline{t|}}}{\bar{s}_{\overline{n|}}}\qquad 0\le t\le n. \end{aligned} \tag{14.7.13}$$

于是在时间 n, 基金累积目标达到： $(\mathbf{aU})(n)=0$. 由 $\bar{s}_{\overline{n|}}-\bar{s}_{\overline{t|}}=\bar{a}_{\overline{n-t|}}(1+i)^n$ 还可知，

$$(\mathbf{aC})(t)=\mathbf{P}(t)+\frac{(\mathbf{aU})(0)}{\bar{a}_{\overline{n|}}}\qquad 0\le t\le n. \tag{14.7.14}$$

例 14.7.1: 考虑平稳计划， $n(a)=l_a$, $g(t)=1$, $h(x)=1$, 与定额参加年龄精算成本方法相联系的积存函数 $M(x)=\bar{a}_{a:\overline{x-a|}}/\bar{a}_{a:\overline{r-a|}}$ (参见例 14.4.1) 。

(1) 找出 λ 。

(2) 当 $(\mathbf{aF})(0)=0$ 时计算 $(\mathbf{aC})(0)$

解：公式 (14.3.2) 对平稳情形给出，对所有 t，

$$^{\mathbf{T}}\mathbf{P}(t) = fw(r)l_r\bar{a}_r,$$

于是由 (14.5.17)

$$(\mathbf{Pa})(t) = \int_a^r e^{-\delta(r-x)} fw(r)l_r\bar{a}_r\left(1 - \frac{\bar{a}_{a:\overline{x-a}|}}{\bar{a}_{a:\overline{r-a}|}}\right)dx.$$

根据 $M'(x) = m(x) = {}_{x-a}E_a/\bar{a}_{a:\overline{r-a}|}$ 及 (14.5.4) 得

$$\mathbf{P}(t) = \int_a^r e^{-\delta(r-x)} fw(r)l_r\bar{a}_r \frac{{}_{x-a}E_a}{\bar{a}_{a:\overline{r-a}|}}dx.$$

进而 (14.7.4) 给出

$$\begin{aligned}\mathbf{a}(t) &= \frac{\mathbf{Pa}(t)}{\mathbf{P}(t)} = \frac{\int_a^r e^{\delta x}{}_{x-a}E_a\bar{a}_{x:\overline{r-x}|}dx}{\int_a^r e^{\delta x}{}_{x-a}E_a dx} \\ &= \frac{\int_a^r l_x\bar{a}_{x:\overline{r-x}|}dx}{\int_a^r l_x dx}\end{aligned}$$

以及 $\lambda = 1/\mathbf{a}$.

(2) 将 (14.5.1) 代入 (14.7.5) 得

$$\begin{aligned}(\mathbf{aC})(0) &= \frac{\int_a^r e^{-\delta(r-x)} fw(r)l_x\bar{a}_r dx}{\mathbf{a}} \\ &= \frac{fw(r)l_r\bar{a}_r\bar{a}_{\overline{r-a}|}\int_a^r l_x dx}{\int_a^r l_x\bar{a}_{x:\overline{r-x}|}dx}.\end{aligned}$$

§14.8 有关退休成员的基本函数

这节讨论与退休群体相联系的基金累积理论中的几个函数，记号中将使用前缀 r 来表示涉及退休群体。

一. 未来受益的精算现值 $(\mathbf{rA})(t)$

在时间 t 介于年龄 x 与 $x+dx$ 之间的 $l(x,t-x+a)dx$ 个成员在 $x-r$ 年之前退休 $(x \geq r)$, 退休时的初始退休金年率为 $fw(r)g(t-x+r)$ 。对还活着的退休者的每单位初始退休金，剩余的精算现值为

$$\overline{a}_x^h = \int_x^\infty e^{-\delta(y-x)} h(y) \frac{s(y)}{s(x)} dy \qquad x \geq r, \tag{14.8.1}$$

其中 $s(y)$ 是仅仅基于死亡的单重损因生存函数。于是

$$(\mathbf{rA})(t) = \int_r^\infty l(x,t-x+a) fw(r) g(t-x+r) \overline{a}_x^h dx. \tag{14.8.2}$$

按 (14.2.1) 可知 $l(x,t-x+a) = n(t-x+a)s(x)$, 用 (14.8.1) 代入可将 $(\mathbf{rA})(t)$ 写成二重积分的形式：

$$(\mathbf{rA})(t) = \int_r^\infty n(t-x+a) fw(r) g(t-x+r) [\int_x^\infty e^{-\delta(y-x)} h(y) s(y) dy] dx. \tag{14.8.3}$$

与假设 $M(x)=1, x \geq r$ 相适应，在时间 t 对封闭的退休者群体而言没有未来正规成本。这样，相应于退休成员的精算积存负债等于他们的未来退休金精算现值，即

$$(\mathbf{rV})(t) = (\mathbf{rA})(t). \tag{14.8.4}$$

这与有关在职成员的式 (14.5.19) 形成对照。

二. 受益支付率 $\mathbf{B}(t)$

对退休成员有一个新的函数需要考虑，那是在时间 t 的受益支付率 $\mathbf{B}(t)$ 。在建立退休者未来受益精算现值的式 (14.8.2) 过程中，可以看到现龄介于 x 与 $x+dx$ 的退休者的退休金初始支付率为 $l(x,t-x-a)fw(r)g(t-x+r)dx$ 。在年龄 x, 这一支付率由因子 $h(x)$ 调整，因此

$$\mathbf{B}(t) = \int_r^\infty l(x,t-x+a) fw(r) g(t-x+r) h(x) dx. \tag{14.8.5}$$

用 (14.2.1) 可得出 $\mathbf{B}(t)$ 的另一形式

$$\mathbf{B}(t)=\int_r^{\infty} n(t-x+a)g(t-x+r)fw(r)s(x)h(x)dx. \quad (14.8.6)$$

对 (14.8.6) 给出的 $\mathbf{B}(t)$ 求导，可得

$$\begin{aligned}
\frac{d}{dt}\mathbf{B}(t) &= \int_r^{\infty} fw(r)s(x)h(x)\frac{\partial}{\partial t}[n(t-x+a)g(t-x+r)dx \\
&= -\int_r^{\infty} fw(r)s(x)h(x)\frac{\partial}{\partial x}[n(t-x+a)g(t-x+r)]dx \\
&= -fw(r)s(x)h(x)n(t-x+a)g(t-x+r)|_{x=r}^{x=\infty} \\
&\quad +\int_r^{\infty} fw(r)n(t-x+a)g(t-x+r)[s'(x)h(x) \\
&\quad +s(x)h'(x)]dx \\
&= [fw(r)l(r,t-r+a)g(t) \\
&\quad -\int_r^{\infty} fw(r)l(x,t-x+a)g(t-x+r)h(x)\mu_x dx] \\
&+ \int_r^{\infty} fw(r)l(x,t-x+a)g(t-x_r)h'(x)dx. \quad (14.8.7)
\end{aligned}$$

上式右端方括号内的项度量 替代影响(replacement effect), 第一项是新退休成员的初始退休金导致的受益支付 (率) 增长的变化率，第二项是在时间 t 死亡导致的受益支付 (率) 下降的变化率。方括号之外的项为 调整影响(adjustment effect), 它衡量在时间 t 受益支付率的调整额。

三. 分配方程

至此，我们可以对退休群体写出有关在职群体的 (14.5.10) 类似的公式：

$$^{\mathbf{T}}\mathbf{P}(t)+\delta(\mathbf{rV})(t)=\mathbf{B}(t)+\frac{d}{dt}(\mathbf{rV})(t). \qquad (14.8.8)$$

这个方程可根据复利理论来论证， $(\mathbf{rV})(t)$ 可看作是一项基金，其来源为期本基金累积成本与利息，其支出为退休金支付。总的收入率与支出率之差决定了该项基金额的变化率。

对 (14.8.8) 的验证可通过对 (14.8.3) 给出的 $(\mathbf{rA})(t) = (\mathbf{rV})(t)$ 求导得出：

$$
\begin{aligned}
\frac{d}{dt}(\mathbf{rV})(t) &= \int_r^\infty fw(r)\frac{\partial}{\partial t}[n(t-x+a)g(t-x+r)] \\
&\quad [\int_x^\infty e^{-\delta(y-x)}s(y)h(y)dy]dx \\
&= -fw(r)\int_r^\infty [\int_x^\infty e^{-\delta(y-x)}s(y)h(y)dy] \\
&\quad \frac{\partial}{\partial x}[n(t-x+a)g(t-x+r)]dx \\
&= -fw(r)\{n(t-x+a)g(t-x+r) \\
&\quad \int_x^\infty e^{-\delta(y-x)}s(y)h(y)dy|_{x=r}^{x=\infty} \\
&\quad - \int_r^\infty [\delta\int_x^\infty e^{-\delta(y-x)}s(y)h(y)dy - s(x)h(x)] \\
&\quad n(t-x+a)g(t-x+r)dx\} \\
&= {}^{\mathbf{T}}\mathbf{P}(t) + \delta(\mathbf{rV})(t) - \mathbf{B}(t),
\end{aligned}
$$

其中使用了 (14.8.6) 得出 $\mathbf{B}(t)$ 。

例 14.8.1: 对指数情形验证

(1) $\mathbf{B}(t+u) = e^{\rho u}\mathbf{B}(t), \qquad \rho = \tau + R$ 。 (14.8.9)

(2) $(\mathbf{rV})(t+u) = e^{\rho u}(\mathbf{rV})(t)$ 。 (14.8.10)

(3) ${}^{\mathbf{T}}\mathbf{P}(t) + \theta(\mathbf{rV})(t) = \mathbf{B}(t), \qquad \theta = \delta - \rho$ 。 (14.8.11)

(4) 若 $\theta > 0$, 则 ${}^{\mathbf{T}}\mathbf{P}(t) < \mathbf{B}(t)$ 。

若 $\theta = 0$, 则 ${}^{\mathbf{T}}\mathbf{P}(t) = \mathbf{B}(t)$ 。 (14.8.12)

若 $\theta < 0$, 则 ${}^{\mathbf{T}}\mathbf{P}(t) > \mathbf{B}(t)$.

解：

(1) 根据 (18.8.6),

$$
\mathbf{B}(t+u) = \int_r^\infty ne^{R(t+u-x+a)}e^{\tau(t+u-x+r)}fw(r)s(x)e^{\beta(x-r)}dx
$$

$$
\begin{aligned}
&= e^{(R+\tau)u}\int_r^{\infty} ne^{R(t-x+a)}e^{\tau(t-x+r)}fw(r)s(x)e^{\beta(x-r)}dx\\
&= e^{\rho u}\mathbf{B}(t).
\end{aligned}
$$

(2) 代入 (14.8.2) 并按 (1) 相同的方式可得所需结果。

(3) 将 (14.8.10) 改写成

$$
\frac{(\mathbf{rV})(t+u+-(\mathbf{rV})(t)}{u}=\frac{e^{\rho u}-1}{u}(\mathbf{rV})(t).
$$

令 $u\to 0$ 得

$$
\frac{d}{dt}(\mathbf{rV})(t)=\rho(\mathbf{rV})(t), \tag{14.8.13}
$$

代入 (14.8.8) 便得出 (3) 的结果。

(4) 这些不等式是 (14.8.11) 的直接推论，这里，指数情形中 $\theta=\delta-\tau-R$ 仍起着关键的作用。

例 14.8.2: 如果计划在固定薪水及定额退休金的平稳人口中运作，建立并解释公式

$$
(\mathbf{rV})(t)=fw(r)\frac{T_r-l_x\overline{a}_r}{\delta}. \tag{14.8.14}
$$

解：将 $h(x)=g(t)=1$, $\theta=\delta$, $\mathbf{B}(t)=fw(r)T_r$, ${}^{\mathbf{T}}\mathbf{P}(t)=fw(r)l_r\overline{a}_r$ 代入 (14.8.11) 可得 (14.8.14) 。为解释这个结果，注意到在平稳人口中，每年 $fw(r)$ 的退休金向所有年龄为 r 岁或更高的退休者连续支付。这包括了每年以年率 l_r 成为退休成员的未来新退休者的退休金，这些未来退休金支付形成现值为 $fw(r)l_r\overline{a}_r/\delta$ 的永久年金。以上两个永久年金的现值之差就是现龄为 r 岁或更高的参加者构成的封闭群体的未来退休金现值，即 $(\mathbf{rV})(t)$ 。

§14.9 有关在职成员与退休成员的基本函数

在 §14.5 与 §14.8 中，我们对在职成员与退休成员分别建立了基本函数。在很多情况下这是有用的区分。对这两个群体而言，

管理体制、精算估价问题乃至投资政策可能都是不同的。然而在另外一些场合，将在职与退休成员混合在一起的群体的有关基本函数是有用处的。

混合群体有关基本精算函数是 §14.5 中有关在职成员的基本函数与 §14.8 中有关退休成员的基本函数之和。这些函数概括在表 14.9.1 中。

表 14.9.1　退休金计划中有关在职成员、退休成员及混合成员群体的精算函数

函数	在职	退休	混合
在时间 t 的未来退休金精算现值	$(\mathbf{aA})(t)$	$(\mathbf{rA})(t)$	$\mathbf{A}(t)=(\mathbf{aA})(t)+(\mathbf{rA})(t)$
正规成本率	$\mathbf{P}(t)$	0	$\mathbf{P}(t)$
精算积存负债	$(\mathbf{aV})(t)$	$(\mathbf{rV})(t)$	$\mathbf{V}(t)=(\mathbf{aV})(t)+(\mathbf{rV})(t)$
未来正规成本的精算现值	$(\mathbf{Pa})(t)$	0	$(\mathbf{Pa})(t)$

我们可根据在职成员的收入分配方程 (14.5.10) 与退休成员的收入分配方程得出混合群体的收入分配方程

$$\frac{d}{dt}\mathbf{V}(t)=\mathbf{P}(t)+\delta\mathbf{V}(t)-\mathbf{B}(t) \tag{14.9.1}$$

或者

$$\mathbf{P}(t)+\delta\mathbf{V}(t)=\mathbf{B}(t)+\frac{d}{dt}\mathbf{V}(t).$$

在这个方程中，进入基金的正规成本与利息收入被分配于退休金受益支付与精算积存负债的变化。

为了对混合群体得出在综合基金累积下的有关公式，假定退休成员的退休金已得到完全累积，即 $(\mathbf{rV})(t)=(\mathbf{rF})(t)$ 。于是 (14.7.1) 可作为对所有成员的未累积精算负债而改写成

$$\begin{aligned}\mathbf{U}(t)&=\mathbf{V}(t)-\mathbf{F}(t)\\&=(\mathbf{aV})(t)+(\mathbf{rV})(t)-(\mathbf{aF})(t)-(\mathbf{rF})(t)\\&=(\mathbf{aU})(t).\end{aligned} \tag{14.9.2}$$

而且，鉴于对退休成员不存在醵出，所存成员的醵出率 $\mathbf{C}(t)$ 等于在职成员的醵出率 $(\mathbf{aC})(t)$ 。这样 (14.7.3) 可改写成

$$\mathbf{C}(t)=\mathbf{P}(t)+\lambda(t)\mathbf{U}(t). \tag{14.9.3}$$

当 $\lambda(t)=1/\mathbf{a}(t)$ 时，醵出率成为

$$\begin{aligned}\mathbf{C}(t) &= \frac{\mathbf{P}(t)\mathbf{a}(t)+\mathbf{V}(t)-\mathbf{F}(t)}{\mathbf{a}(t)}\\ &= \frac{\mathbf{A}(t)-\mathbf{F}(t)}{\mathbf{a}(t)}.\end{aligned} \tag{14.9.4}$$

因此，在 $(\mathbf{rF})(t)=(\mathbf{rV})$ 的情况下，(14.7.5) 给出的由在时间 t 的在职成员确定的综合成本方法结果，等价于 (14.9.4) 给出的由所有成员确定的结果。

习　题

§14.2

1. 一个平稳人口，其标准工资率为 w, 求它的 工资额函数$W(t)$ 。

§14.3

2. 在指数情形下，求工资额函数 $W(t)$ 。

3. 假设在时间 t 退休的人，其退休收入的初始年率为

$$\frac{f}{b}\int_{t-b}^{t} w(r-t+y)g(y)dy \qquad 0<b<r-a.$$

该模型计划的其它方面都不变，对这个建立在最终平均公式基础上的受益定义，

a. 证明时间 t 时的初始受益率为

$$\frac{f}{b}\int_0^b w(r-z)g(t-z)dz.$$

b. 写出时间 t 时的期末基金成本率的公式。

c. 重作例 14.3.1 。

4. 在时间 t 退休者的退休收入初始年率为 $c(r-a)we^{\tau t}$. 该模型计划的其它方面保持不变。对建立在服务年限和最终工资标准乘积基础上的该初始受益率,

a. 求时间 t 时的期末基金成本率公式。

b. 重解例 14.3.1 。

5. 若 $s(x)=e^{-u(x-a)}, a\leq x\leq r$, 在指数情形下求 ${}^{\mathbf{T}}\mathbf{P}(t)$ 。

§14.4

6. 在期末基金情形下求 $M(x)$ 。

§14.5

7. 用习题 5 中 $m(x)=1/(r-a)$ 时的假设求 $\mathbf{P}(t)$ 。

8. a. 证明

$$(aA)(t)=\int_t^{t+r-a} e^{-\delta(y-t)}{}^{\mathbf{T}}\mathbf{P}(y)dy.$$

b. 对 (a) 中式子求导, 以得到例 14.5.1 的另一解法。

9. 若 $e^{\theta X(\theta)}=\int_a^r e^{\theta x}m(x)dx$ 且 $\mu=\int_a^r xm(x)dx$, 证明

a. 若 $\theta>0$, 则 $X(\theta)>\mu$ 。

b. 若 $\theta<0$, 则 $X(\theta)<\mu$ 。

[提示: 用 Jensen 不等式]

10. 在指数情形下, 证明

a. $\mathbf{P}(t)={}^{\mathbf{T}}\mathbf{P}(t)\exp\{-\theta[r-X(\theta)]\}$ 。

b. $(\mathbf{aV})(t)={}^{\mathbf{T}}\mathbf{P}(t)\bar{a}_{\overline{r-X(\theta)}|\theta}=P(t)\bar{s}_{\overline{r-X(\theta)}|\theta}$ 。

11.

a. 若习题 10 中的模型计划是在 $g(t)=1$ 的平稳人口中实行, 则其公式会有什么变化?

b. 若 $\theta=\delta-\rho=0$, 习题 10 中的公式怎样变化?

12. a. 求适用于在时间 u 进入者的全部工资的正常成本率 $\pi(u)$ 。假设该模型计划的其它方面没有变化。

b. 用 (a) 中的结果，求 $m(x,u)$ 的相应函数。

13. a. 对模型计划证明

$$\mathbf{P}(t) = fw(r)s(r)\overline{a}_r^h \int_a^r e^{-\delta(r-x)} g(t+r-x)n(t-x+a)m(x)dx.$$

b. 若 $g(t)=1, n(t)=l_a$, 证明

$$\mathbf{P}(t) = fw(r)\int_a^r l_{x\,r-x}E_x \overline{a}_r^h m(x)dx,$$

其中 ${}_{r-x}E_x$ 对应的生存函数为 $s(x)$, 利息效力为 δ 。

14. 若 $g(t)=e^{\tau t}, n(t)=l_a$ 且

$$m(x) = \frac{w(x)e^{\tau x}}{\int_a^r w(y)e^{\tau y}dy},$$

证明

$$\mathbf{P}(t+u) = e^{\tau u}\mathbf{P}(t).$$

§14.6

15. 假设在时间 t 时年龄为 x 者的计划初始退休受益率为 $fw(r)g(t+r-x)$, 时间 t 时在年龄 x 到 $x+dx$ 之间的人数为 $n(t-x+a)s(x)dx$ 。

a. 证明 (14.5.1) 给出的 $(\mathbf{aA})(t)$ 等于

$$\int_a^r fw(r)g(t+r-x)n(t-x+a)s(x)(aA)(x)dx.$$

b. 证明 (14.5.4) 给出的 $\mathbf{P}(t)$ 等于

$$\int_a^r fw(r)g(t+r-x)n(t-x+a)s(x)P(x)dx.$$

c. 证明 (14.5.8) 给出的 $(\mathbf{aV})(t)$ 等于

$$\int_a^r fw(r)g(t+r-x)n(t-x+a)s(x)(aV)(x)dx.$$

d. 证明 (14.5.17) 给出的 $(\mathbf{Pa})(t)$ 等于

$$\int_a^r fw(r)g(t+r-x)n(t-x+a)s(x)(Pa)(x)dx.$$

16. 证明公式 (14.6.11) 。

§14.7

17. 证明

a. $-\int_0^t \frac{1}{\bar{s}_{\overline{n-y}|}}dy = \log\frac{\bar{s}_{\overline{n}|}-\bar{s}_{\overline{t}|}}{\bar{s}_{\overline{n}|}}$.

b. $-\int_0^t \frac{1}{\bar{a}_{\overline{n-y}|}}dy = \log\frac{\bar{a}_{\overline{n}|}-\bar{a}_{\overline{t}|}}{\bar{a}_{\overline{n}|}}$.

18. a. 在指数情形下导出 $\mathbf{a}(t)$ 的简化公式。

b. 在指数情形，当 $\theta=\delta-\rho=0$ 时， $\mathbf{a}(t)$ 变成什么？

§14.8

19. 在指数情形下，证明 $\mathbf{B}(t) = {}^{\mathbf{T}}\mathbf{P}(t)\bar{a}'^h_r/\bar{a}^h_r$, 其中

$$\bar{a}'^h_r = \int_r^\infty e^{-(\rho-\beta)(x-r)}\frac{s(x)}{s(r)}dx.$$

附录 1　正态分布表

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}u^2}du$$

x	0	1	2	3
0.0	0.5000	0.8413	0.9772	0.9987
0.1	0.5398	0.8643	0.9821	0.9990
0.2	0.5793	0.8849	0.9861	0.9993
0.3	0.6179	0.9032	0.9893	0.9995
0.4	0.6554	0.9192	0.9918	0.9997
0.5	0.6915	0.9332	0.9938	0.9998
0.6	0.7257	0.9452	0.9953	0.9998
0.7	0.7580	0.9554	0.9965	0.9999
0.8	0.7881	0.9641	0.9974	0.9999
0.9	0.8159	0.9713	0.9981	1.0000

某些标准正态分布函数值对应的点

$\Phi(x)$	0.800	0.850	0.900	0.950	0.975	0.990	0.995
x	0.842	1.036	1.282	1.645	1.960	2.326	2.576

附录 2　示例表

A. 示例生命表

基本函数表

年龄	l_x	d_x	$1000q_x$
0	100000.00	2042.1700	20.4217
1	97957.83	131.5672	1.3431
2	97826.26	119.7100	1.2237
3	97706.55	109.8124	1.1239
4	97596.74	101.7056	1.0421
5	97495.03	95.2526	0.9770
6	97399.78	90.2799	0.9269
7	97309.50	86.6444	0.8904
8	97222.86	84.1950	0.8660
9	97138.66	82.7816	0.8522
10	97055.88	82.2549	0.8475
11	96973.63	82.4664	0.8504
12	96891.16	83.2842	0.8596
13	96807.88	84.5180	0.8730
14	96723.36	86.0611	0.8898
15	96637.30	87.7559	0.9081
16	96549.54	89.6167	0.9282
17	96459.92	91.6592	0.9502
18	96368.26	93.9005	0.9744
19	96274.36	96.3596	1.0009
20	96178.00	99.0569	1.0299
21	96078.95	102.0149	1.0618
22	95976.93	105.2582	1.0967
23	95871.67	108.8135	1.1350
24	95762.86	112.7102	1.1770
25	95650.15	116.9802	1.2230
26	95533.17	121.6585	1.2735
27	95411.51	126.7830	1.3288
28	95284.73	132.3953	1.3895
29	95152.33	138.5406	1.4560
30	95013.79	145.2682	1.5289

年龄	l_x	d_x	$1000q_x$
31	94868.53	152.6317	1.6089
32	94715.89	160.6896	1.6965
33	94555.20	169.5052	1.7927
34	94385.70	179.1475	1.8980
35	94206.55	189.6914	2.0136
36	94016.86	201.2179	2.1402
37	93815.64	213.8149	2.2791
38	93601.83	227.5775	2.4313
39	93374.25	242.6085	2.5982
40	93131.64	259.0186	2.7812
41	92872.62	276.9271	2.9818
42	92595.70	296.4623	3.2017
43	92299.23	317.7619	3.4427
44	91981.47	340.9730	3.7070
45	91640.50	366.2529	3.9966
46	91274.25	393.7687	4.3141
47	90880.48	423.6978	4.6621
48	90456.78	456.2274	5.0436
49	90000.55	491.5543	5.4617
50	89509.00	529.8844	5.9199
51	88979.11	571.4316	6.4221
52	88407.68	616.4165	6.9724
53	87791.26	665.0646	7.5755
54	87126.20	717.6041	8.2364
55	86408.60	774.2626	8.9605
56	85634.33	835.2636	9.7538
57	84799.07	900.8215	10.6230
58	83898.25	971.1358	11.5752
59	82927.11	1046.3843	12.6181
60	81880.73	1126.7146	13.7604
61	80754.01	1212.2343	15.0114
62	79541.78	1302.9994	16.3813
63	78238.78	1399.0010	17.8812
64	76839.78	1500.1504	19.5231
65	75339.63	1606.2618	21.3203
66	73733.37	1717.0334	23.2871
67	72016.33	1832.0273	25.4391
68	70184.31	1950.6476	27.7932
69	68233.66	2072.1177	30.3680
70	66161.54	2195.4578	33.1833

年龄	l_x	d_x	$1000q_x$
71	63966.08	2319.4639	36.2608
72	61646.62	2442.6884	39.6240
73	59203.93	2563.4258	43.2982
74	56640.50	2679.7050	47.3108
75	53960.80	2789.2905	51.6911
76	51171.51	2889.6965	56.4708
77	48281.81	2978.2164	61.6840
78	45303.60	3051.9717	67.3671
79	42251.62	3107.9833	73.5589
80	39143.64	3143.2679	80.3009
81	36000.37	3154.9603	87.6369
82	32845.41	3140.4624	95.6134
83	29704.95	3097.6146	104.2794
84	26607.34	3024.8830	113.6861
85	23582.45	2921.5530	123.8867
86	20660.90	2787.9129	134.9367
87	17872.99	2625.4088	146.8926
88	15247.58	2436.7474	159.8121
89	12810.83	2225.9244	173.7533
90	10584.91	1998.1533	188.7738
91	8586.75	1759.6818	204.9298
92	6827.07	1517.4869	222.2749
93	5309.58	1278.8606	240.8589
94	4030.72	1050.9136	260.7258
95	2979.81	840.0452	281.9123
96	2139.77	651.4422	304.4456
97	1488.32	488.6776	328.3411
98	999.65	353.4741	353.5995
99	646.17	245.6772	380.2044
100	400.49	163.4494	408.1194
101	237.04	103.6560	437.2846
102	133.39	62.3746	467.6153
103	71.01	35.4358	498.9967
104	35.58	18.9023	531.2873
105	16.68	9.4105	564.3140
106	7.27	4.3438	597.8666
107	2.92	1.8458	631.7554
108	1.08	0.7163	665.7681
109	0.36	0.2517	699.9440
110	0.11	0.0793	734.9383

计算基数表一 (i=0.06)

年龄	D_x	N_x	S_x
0	100000.00	1680095.45	27526802.64
1	92413.05	1580095.45	25846707.19
2	87065.03	1487682.40	24266611.74
3	82036.31	1400617.38	22778929.33
4	77305.76	1318581.07	21378311.96
5	72853.96	1241275.31	20059730.89
6	68663.00	1168421.35	18818455.57
7	64716.38	1099758.35	17650034.22
8	60998.82	1035041.97	16550275.88
9	57496.23	974043.15	15515233.91
10	54195.50	916546.92	14541190.76
11	51084.50	862351.42	13624643.84
12	48151.94	811266.93	12762292.41
13	45387.31	763114.99	11951025.49
14	42780.83	717727.68	11187910.50
15	40323.37	674946.84	10470182.82
16	38006.37	634623.48	9795235.97
17	35821.78	596617.11	9160612.50
18	33762.02	560795.33	8563995.39
19	31819.93	527033.30	8003200.06
20	29988.76	495213.37	7476166.76
21	28262.14	465224.62	6980953.38
22	26634.09	436962.47	6515728.77
23	25098.94	410328.39	6078766.29
24	23651.37	385229.44	5668437.91
25	22286.35	361578.07	5283208.46
26	20999.15	339291.72	4921630.39
27	19785.29	318292.57	4582338.67
28	18640.57	298507.28	4264046.10
29	17561.00	279866.71	3965538.83
30	16542.86	262305.71	3685672.11
31	15582.61	245762.85	3423366.40
32	14676.93	230180.23	3177603.56
33	13822.67	215503.30	2947423.33
34	13016.88	201680.63	2731920.02
35	12256.76	188663.76	2530239.39
36	11539.70	176406.99	2341575.63
37	10863.21	164867.29	2165168.64

年龄	D_x	N_x	S_x
38	10224.96	154004.08	2000301.35
39	9622.73	143779.12	1846297.27
40	9054.46	134156.39	1702518.14
41	8518.19	125101.93	1568361.75
42	8012.06	116583.74	1443259.82
43	7534.35	108571.68	1326676.07
44	7083.41	101037.33	1218104.40
45	6657.69	93953.92	1117067.07
46	6255.74	87296.23	1023113.15
47	5876.18	81040.49	935816.92
48	5517.72	75164.31	854776.43
49	5179.14	69646.60	779612.12
50	4859.30	64467.45	709965.52
51	4557.10	59608.16	645498.07
52	4271.55	55051.05	585889.91
53	4001.66	50779.51	530838.86
54	3746.55	46777.85	480059.35
55	3505.37	43031.29	433281.51
56	3277.32	39525.92	390250.21
57	3061.66	36248.59	350724.30
58	2857.67	33186.93	314475.71
59	2664.71	30329.26	281288.77
60	2482.16	27664.55	250959.51
61	2309.44	25182.39	223294.97
62	2146.01	22872.95	198112.58
63	1991.37	20726.94	175239.63
64	1845.06	18735.57	154512.69
65	1706.64	16890.50	135777.13
66	1575.71	15183.86	118886.62
67	1451.90	13608.15	103702.76
68	1334.88	12156.25	90094.61
69	1224.32	10821.37	77938.36
70	1119.94	9597.05	67116.99
71	1021.49	8477.11	57519.94
72	928.72	7455.62	49042.83
73	841.44	6526.90	41587.20
74	759.44	5685.46	35060.30
75	682.56	4926.02	29374.84
76	610.64	4243.47	24448.82
77	543.54	3632.83	20205.36

年龄	D_x	N_x	S_x
78	481.14	3089.29	16572.53
79	423.33	2608.14	13483.24
80	369.99	2184.81	10875.10
81	321.02	1814.82	8690.28
82	276.31	1493.80	6875.46
83	235.74	1217.49	5381.66
84	199.21	981.75	4164.17
85	166.57	782.54	3182.42
86	137.67	615.97	2399.88
87	112.35	478.30	1783.90
88	90.42	365.95	1305.60
89	71.67	275.52	939.66
90	55.87	203.85	664.13
91	42.76	147.98	460.28
92	32.07	105.23	312.30
93	23.53	73.16	207.07
94	16.85	49.63	133.92
95	11.75	32.78	84.29
96	7.96	21.02	51.51
97	5.22	13.06	30.49
98	3.31	7.84	17.43
99	2.02	4.53	9.59
100	1.18	2.51	5.06
101	0.66	1.33	2.56
102	0.35	0.67	1.23
103	0.18	0.32	0.56
104	0.08	0.14	0.24
105	0.04	0.06	0.10
106	0.02	0.02	0.04
107	0.01	0.01	0.01
108	0.00	0.00	0.00
109	0.00	0.00	0.00
110	0.00	0.00	0.00

计算基数表二 (i=0.06)

年龄	C_x	M_x	R_x
0	1926.5755	4900.2575	121974.5461
1	117.0943	2973.6821	117074.2886
2	100.5108	2856.5877	114100.6065
3	86.9817	2756.0769	111244.0188
4	76.0003	2669.0952	108487.9418
5	67.1493	2593.0949	105818.8466
6	60.0413	2525.9455	103225.7518
7	54.3618	2465.9043	100699.8062
8	49.8349	2411.5425	98233.9020
9	46.2248	2361.7076	95822.3595
10	43.3309	2315.4828	93460.6519
11	40.9833	2272.1519	91145.1691
12	39.0469	2231.1687	88873.0172
13	37.3824	2192.1218	86641.8485
14	35.9103	2154.7394	84449.7268
15	34.5448	2118.8291	82294.9874
16	33.2805	2084.2843	80176.1583
17	32.1122	2051.0038	78091.8740
18	31.0353	2018.8916	76040.8702
19	30.0454	1987.8563	74021.9786
20	29.1381	1957.8109	72034.1223
21	28.3097	1928.6728	70076.3114
22	27.5563	1900.3631	68147.6386
23	26.8746	1872.8068	66247.2755
24	26.2613	1845.9322	64374.4687
25	25.7134	1819.6709	62528.5365
26	25.2281	1793.9575	60708.8656
27	24.8026	1768.7294	58914.9081
28	24.4344	1743.9268	57146.1787
29	24.1213	1719.4924	55402.2519
30	23.8610	1695.3711	53682.7595
31	23.6514	1671.5100	51987.3884
32	23.4906	1647.8586	50315.8784
33	23.3767	1624.3680	48668.0198
34	23.3080	1600.9913	47043.6517
35	23.2829	1577.6833	45442.6604
36	23.2997	1554.4004	43864.9771
37	23.3569	1531.1008	42310.5767

年龄	C_x	M_x	R_x
38	23.4531	1507.7439	40779.4760
39	23.5869	1484.2907	39271.7321
40	23.7569	1460.7038	37787.4414
41	23.9618	1436.9469	36326.7375
42	24.2001	1412.9851	34889.7907
43	24.4706	1388.7850	33476.8056
44	24.7717	1364.3144	32088.0206
45	25.1022	1339.5427	30723.7061
46	25.4604	1314.4406	29384.1634
47	25.8449	1288.9801	28069.7229
48	26.2539	1263.1352	26780.7427
49	26.6857	1236.8813	25517.6075
50	27.1383	1210.1957	24280.7262
51	27.6095	1183.0574	23070.5305
52	28.0972	1155.4479	21887.4731
53	28.5988	1127.3506	20732.0252
54	29.1113	1098.7519	19604.6746
55	29.6319	1069.6405	18505.9227
56	30.1571	1040.0086	17436.2822
57	30.6831	1009.8515	16396.2736
58	31.2057	979.1685	15386.4221
59	31.7204	947.9628	14407.2536
60	32.2223	916.2423	13459.2908
61	32.7057	884.0201	12543.0485
62	33.1646	851.3144	11659.0284
63	33.5925	818.1498	10807.7140
64	33.9824	784.5573	9989.5642
65	34.3265	750.5749	9205.0069
66	34.6167	716.2484	8454.4320
67	34.8444	681.6317	7738.1836
68	35.0005	646.7873	7056.5518
69	35.0755	611.7868	6409.7645
70	35.0597	576.7113	5797.9777
71	34.9434	541.6516	5221.2663
72	34.7168	506.7082	4679.6147
73	34.3706	471.9914	4172.9065
74	33.8959	437.6208	3700.9152
75	33.2850	403.7249	3263.2944
76	32.5312	370.4400	2859.5695
77	31.6300	337.9087	2489.1295

年龄	C_x	M_x	R_x
78	30.5786	306.2787	2151.2208
79	29.3771	275.7002	1844.9421
80	28.0289	246.3230	1569.2419
81	26.5407	218.2941	1322.9188
82	24.9234	191.7534	1104.6247
83	23.1918	166.8300	912.8713
84	21.3654	143.6382	746.0413
85	19.4675	122.2729	602.4031
86	17.5254	102.8054	480.1303
87	15.5697	85.2799	377.3249
88	13.6329	69.7102	292.0449
89	11.7485	56.0773	222.3347
90	9.9494	44.3288	166.2574
91	8.2660	34.3795	121.9286
92	6.7248	26.1135	87.5491
93	5.3465	19.3887	61.4356
94	4.1449	14.0421	42.0469
95	3.1256	9.8973	28.0048
96	2.2867	6.7716	18.1075
97	1.6183	4.4850	11.3359
98	1.1043	2.8667	6.8509
99	0.7241	1.7624	3.9842
100	0.4545	1.0384	2.2218
101	0.2719	0.5839	1.1834
102	0.1543	0.3120	0.5995
103	0.0827	0.1577	0.2875
104	0.0416	0.0749	0.1299
105	0.0196	0.0333	0.0549
106	0.0085	0.0138	0.0216
107	0.0034	0.0052	0.0079
108	0.0012	0.0018	0.0026
109	0.0004	0.0006	0.0008
110	0.0001	0.0002	0.0002

净趸缴保费表 (i=0.06)

年龄	$1000A_x$	$1000(^2A_x)$	$1000A_{xx}$
0	49.0026	25.9210	86.7274
1	32.1782	8.8846	53.6745
2	32.8098	8.6512	54.3565
3	33.5958	8.5072	55.3072
4	34.5265	8.4443	56.5060
5	35.5931	8.4548	57.9339
6	36.7876	8.5311	59.5733
7	38.1032	8.6667	61.4085
8	39.5342	8.8554	63.4258
9	41.0759	9.0918	65.6137
10	42.7246	9.3713	67.9626
11	44.4783	9.6903	70.4655
12	46.3360	10.0462	73.1176
13	48.2981	10.4373	75.9170
14	50.3669	10.8638	78.8643
15	52.5459	11.3268	81.9632
16	54.8404	11.8295	85.2203
17	57.2558	12.3749	88.6424
18	59.7977	12.9665	92.2366
19	62.4721	13.6080	96.0099
20	65.2848	14.3034	99.9697
21	68.2423	15.0569	104.1234
22	71.3508	15.8730	108.4786
23	74.6170	16.7566	113.0429
24	78.0476	17.7128	117.8241
25	81.6496	18.7472	122.8299
26	85.4300	19.8657	128.0682
27	89.3962	21.0744	133.5468
28	93.5555	22.3802	139.2737
29	97.9154	23.7900	145.2564
30	102.4835	25.3113	151.5028
31	107.2676	26.9520	158.0203
32	112.2754	28.7206	164.8162
33	117.5148	30.6259	171.8977
34	122.9935	32.6772	179.2716
35	128.7194	34.8843	186.9444
36	134.7002	37.2574	194.9221
37	140.9437	39.8074	203.2104

年龄	$1000A_x$	$1000(^2A_x)$	$1000A_{xx}$
38	147.4572	42.5455	211.8144
39	154.2484	45.4833	220.7386
40	161.3242	48.6332	229.9867
41	168.6916	52.0077	239.5619
42	176.3572	55.6199	249.4664
43	184.3271	59.4833	259.7015
44	192.6071	63.6117	270.2677
45	201.2024	68.0193	281.1642
46	210.1176	72.7205	292.3892
47	219.3569	77.7299	303.9398
48	228.9234	83.0624	315.8114
49	238.8198	88.7329	327.9986
50	249.0475	94.7561	340.4941
51	259.6073	101.1469	353.2895
52	270.4988	107.9196	366.3746
53	281.7206	115.0885	379.7377
54	293.2700	122.6672	393.3656
55	305.1431	130.6687	407.2435
56	317.3346	139.1053	421.3546
57	329.8381	147.9883	435.6810
58	342.6453	157.3280	450.2029
59	355.7466	167.1332	464.8990
60	369.1310	177.4113	479.7465
61	382.7858	188.1682	494.7213
62	396.6965	199.4077	509.7977
63	410.8471	211.1318	524.9491
64	425.2202	223.3401	540.1477
65	439.7965	236.0299	555.3647
66	454.5553	249.1958	570.5707
67	469.4742	262.8299	585.7356
68	484.5296	276.9212	600.8289
69	499.6963	291.4559	615.8203
70	514.9481	306.4172	630.6790
71	530.2574	321.7850	645.3750
72	545.5957	337.5361	659.8785
73	560.9339	353.6443	674.1606
74	576.2420	370.0803	688.1934
75	591.4895	386.8119	701.9503
76	606.6460	403.8038	715.4057
77	621.6808	421.0184	728.5362

年龄	$1000A_x$	$1000(^2A_x)$	$1000A_{xx}$
78	636.5634	438.4155	741.3197
79	651.2639	455.9528	753.7364
80	665.7528	473.5861	765.7683
81	680.0019	491.2699	777.3999
82	693.9837	508.9574	788.6175
83	707.6723	526.6012	799.4102
84	721.0431	544.1537	809.7690
85	734.0736	561.5675	819.6876
86	746.7428	578.7956	829.1617
87	759.0320	595.7923	838.1892
88	770.9244	612.5133	846.7701
89	782.4056	628.9163	854.9607
90	793.4636	644.9612	862.6027
91	804.0884	660.6105	869.8636
92	814.2727	675.8298	876.6969
93	824.0111	690.5878	883.1102
94	833.3008	704.8566	889.1137
95	842.1409	718.6116	894.7179
96	850.5327	731.8322	899.9341
97	858.4792	744.5012	904.7742
98	865.9855	756.6050	909.2506
99	873.0581	768.1335	913.3762
100	879.7048	779.0800	917.1638
101	885.9349	789.4411	920.6266
102	891.7587	799.2165	923.7777
103	897.1875	808.4086	926.6301
104	902.2338	817.0230	929.1969
105	906.9105	825.0679	931.4911
106	911.2323	832.5543	933.5261
107	915.2179	839.5011	935.3151
108	918.8880	845.9270	936.8720
109	922.2735	851.8502	938.2110
110	925.3803	857.1563	939.3470

B. 示例服务表

年龄	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	S_x
30	100000	100	19990	–	–	1.00
31	79910	80	14376	–	–	1.06
32	65454	72	9858	–	–	1.13
33	55524	61	5702	–	–	1.20
34	49761	60	3971	–	–	1.25
35	45730	64	2683	46	–	1.36
36	42927	64	1927	43	–	1.44
37	40893	65	1431	45	–	1.54
38	39352	71	1181	47	–	1.63
39	38053	72	959	49	–	1.74
40	36943	78	813	52	–	1.85
41	36000	83	720	54	–	1.96
42	35143	91	633	56	–	2.09
43	34363	96	550	58	–	2.22
44	33659	104	505	61	–	2.36
45	32989	112	462	66	–	2.51
46	32349	123	421	71	–	2.67
47	31734	133	413	79	–	2.84
48	31109	143	373	87	–	3.02
49	30506	156	336	95	–	3.21
50	29919	168	299	102	–	3.41
51	19350	182	293	112	–	3.63
52	28763	198	259	121	–	3.86
53	28185	209	251	132	–	4.10
54	27593	226	218	143	–	4.35
55	27006	240	213	157	–	4.62
56	26396	259	182	169	–	4.91
57	25786	276	178	183	–	5.21
58	25149	297	148	199	–	5.53
59	24505	316	120	213	–	5.86
60	23856	313	–	–	3552	6.21
61	19991	298	–	–	1587	6.56
62	18106	284	–	–	2692	6.93

年龄	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	S_x
63	15130	271	–	–	1350	7.31
64	13509	257	–	–	2006	7.70
65	11246	204	–	–	4448	8.08
66	6594	147	–	–	1302	8.48
67	5145	119	–	–	1522	8.91
68	3504	83	–	–	1381	9.35
69	2040	49	–	–	1004	9.82
70	987	17	–	–	970	10.31

附录3　符号索引

附录 4　精算函数符号的一般规定

精算函数由一个主干符号与一组辅助符号来表示，这些辅助符号包括字母、数字、双点、圆圈、帽子、水平线与垂直线等。主干符号表达函数的一般含义，而处在顶上与四周的辅助符号的选择与安置则给出确切含义。我们将概括选择与安置符号的规则，并在常见的应用范围内举例说明。

这个记号规则起源与 1898 年在伦敦召开的第二届国际精算师大会所采纳的国际精算记号 (IAN) 体系，该体系在国际精算协会的常设精算记号委员会指导下定期进行修正。IAN(International Actuarial Notation) 是一个基本的原则体系，它并不囊括精算应用的所有领域。本书遵从该体系的原则，并在需要时加以推广，用以构造相互一致的记号。

这个附录对于本书中出现的符号的基本型式为读者提供一个概览。尽管这是 IAN 的一个很好的导引，然而却并非包罗万象。作为权威出处可进一步参考以下文献：

Actuarial Society of America, "International Actuarial Notation," *Transactions*, XLVIII,1947:166–176.

Faculty of Actuaries, *Transactions*, XIX,1950:89.

Journal of the Institute of Actuaries, LXXV, 1949:121.

精算符号的表示可从下列图示看出端倪。 位置 I 代表主干符号，其它位置则安排上标或下标，其中的罗马数字对应于本附录的次序安排。

VI	IV	V
	I	
III		II

I　中心

利息理论：

i 年利率　　v 年末 1 之现值

δ 利息效力　　d (银行) 贴现率

生命表：

l 期望生存人数　　d 期望死亡人数

p 生存概率　　q 死亡概率

μ 死亡效力　　m 中位死亡率

L 生存组在一段时间内的期望生存总年数

T 生存组在未来总的期望生存总年数

寿险及生存险：

A 1 单位保险的精算现值 (净趸缴保费)

IA 初始 1 且年递增 1 保险的精算现值 (净趸缴保费)

DA 初始额等于期限且年递减 1 的定期保险精算现值 (净趸缴保费)

E 1 单位生存支付的精算现值

a 年支付 1 生存年金的精算现值

s 年支付 1 生存年金的精算现值

(Ia) 第一年支付为 1 且每年递增 1 生存年金的精算现值

(Da) 第一年支付额等于期限且每年递减 1 年金的精算积累值

P 净均衡年保费，例：P_x, $P(\overline{A}_x)$

V 责任准备金，例：${}_{10}V^{(4)}(\overline{A}_{x:\overline{n|}})$

W 用等于责任准备金的解约金购买的缴清保险的面额

计算基数：

D 向所有生存者支付 1 的精算现值

C 向所有在某年的死亡者赔付受益 1 的精算现值

N　从某年龄开始的 D 函数之和

M　从某年龄开始的 C 函数之和

S　从某年龄开始的 N 函数之和

R　从某年龄开始的 M 函数之和

退休金：

S　薪水尺度函数

Z　薪水尺度函数的平均值

II　右下

$x, 10$	开始年龄	例：$a_x, {}_5q_{10}, \overline{A}_x$
$\overline{n}\rvert, \overline{10}\rvert$	表示确定的期限	$A_{x:\overline{n}\rvert}, \ddot{a}_{\overline{10}\rvert}$
$[x], [35], [x]+t$	方括号中代表选择年龄	$l_{[x]+10}, A_{[35]}$
xyz或$x:y:z$	连生状况	$A_{xyz}, \ddot{a}_{25:\overline{10}\rvert}$
	强调连生状况	$A^{\ 1}_{xy:z}$
$\overset{2}{x}\underset{1}{y}z, {}^{1}_{x:\overline{10}\rvert}$	顺位	${}_\infty q^{\ \ 3}_{xyz}, \overline{A}^{1}_{x:\overline{n}\rvert}$
$\overline{xyz}, \overline{65:60}$	最后生存状况	$a_{\overline{xy}}, \overline{A}^{\ 12}_{\overline{65:60:\overline{10}\rvert}}$
$\frac{r}{xyz}, \frac{[r]}{x:y:10}$	右上方数字 r 不在方括号内时表示至少 r 个生存者状况	$\overline{A}^{\ \ 2}_{\overline{xyz}}$
	右上方数字 r 在方括号内时表示恰好 r 个生存者状况	$\overline{a}^{\ [2]}_{\overline{xyz}}$
$y\rvert x,\ 60\rvert 55$	表示始于垂线左边的状况消亡之时终结于垂线右边的状况消亡之时	$a_{x\rvert y}$

III　左下

$n, 15$	时间或期限 (有些场合省略的为 1) 对年保费 P 而言乃不同于期限的缴费期	${}_{20}V_{40}, {}_{n}p_{x}$, ${}_{25}P_{35:\overline{20}\vert}$
$m\vert n, m\vert$	延迟时间 (垂线左边) 及期限 (垂线右边)	${}_{m\vert n}q_{x}$
	一竖右边省略的有些场合为 1	${}_{k\vert}q_{x}$
	另一些场合为 ∞	${}_{n\vert}\bar{a}_{x}$

Ⅳ　中顶

$\ddot{}$	表示期初年金 (期末年金无冕)	$\ddot{a}_{x}, \ddot{s}_{\overline{40}\vert}$
$-$	连续给付或死亡即刻支付	$\bar{a}_{x}, \bar{A}_{30}$, ${}_{3}\bar{V}_{x}$
$\circ$	完全的 (即精确地计至死亡)	$\mathring{a}_{x}, \mathring{e}_{x}$

Ⅴ　右上

$(m), (12)$	表示年支付次数	$s_{\overline{10}\vert}^{(12)}, A_{x}^{(m)}$
	在多重损失模型中表示损因或所有原因	$q_{x}^{(2)}, {}_{t}p_{x}^{(\tau)}$
$\{m\}, \{1\}$	表示期初比例年金的年支付次数	$p_{30}^{\{1\}}, {}_{t}V^{\{2\}}(\bar{A}_{x})$
r, i	表示特殊的基础	$\ddot{a}_{65}^{r}, \ddot{a}_{[x]}^{i}$

Ⅵ　左上

$h, 2$	因左下方已被使用而表示缴费期	${}_{5}^{h}V_{30}$
	表示计算时利息效力为设定利息效力的某个倍数	${}^{2}\bar{A}_{x}, {}^{2}\ddot{a}_{20:\overline{10}\vert}$

习题答案

第一章

1.1.

$s(x)$	$F(x)$	$f(x)$	μ_x
$\cos x$	$1-\cos x$	$\sin x$	—
—	$1-e^{-x}$	e^{-x}	1
$\frac{1}{1+x}$	—	$\frac{1}{(1+x)^2}$	$\frac{1}{1+x}$

1.2. (1) $\exp\{-\frac{B}{\log c}(c^x-1)\}$　　(2) $\exp\{-ux^{n+1}\}$, $u=\frac{k}{n+1}$

(3) $(1+\frac{x}{b})^{-a}$

1.3. $\mu=\frac{x^2}{4}$, $f(x)=\frac{x^2}{4}e^{-x^3/12}$, $F(x)=1-e^{-x^3/12}$

1.4. (1) $\int_0^\infty \mu_x dx<\infty$

(2) 对某些 x (包括 $x=1,2$), $s'(x)>0$

(3) $\int_0^\infty f(x)dx\neq 1$

1.5. (1) $\frac{1}{100-x}$　　(2) $\frac{x}{100}$　　(3) $\frac{1}{100}$　　(4) $\frac{3}{10}$

1.7. 0.0020

1.8. $f(x)=C_{10}^x(0.77107)^x(0.22893)^{10-x}$,　　$x=0,1,\cdots,10$.

$E[L(65)]=7.7107$;　$Var[L(65)]=1.7652$.

1.9. (1) 都是 $\frac{9}{4}$　　(2) 都是 $\frac{27}{16}$　　(3) 都是 $-\frac{1}{3}$

1.10. (1) ${}_5q_0=0.01505$, 大于 ${}_5q_5=0.00150$ 的 10 倍.

(2) 0.15673

1.13. 1436.19

1.15. (1) $\frac{1}{c}$　　(2) $\frac{1}{c^2}$　　(3) $\frac{\log 2}{c}$

1.16. (1) $te^{-t^2/2}$　　(2) $\sqrt{\frac{\pi}{2}}$

1.17. (1) $\frac{100-x}{2}$　　(2) $\frac{(100-x)^2}{12}$　　(3) $\frac{100-x}{2}$

1.19. (1) $\frac{8}{9}$ (2) $\frac{1}{8}$ (3) $\frac{1}{8}$ (4) $\frac{1}{128}$ (5) $\frac{128}{3}$

1.23. 均匀分布： 0.98971

常数效力： 0.98966
Balducci: 0.98960

1.24. (1) 77.59 (2) 29.11

1.25. (1) 0.0440 (2) 0.0442

1.26.

	均匀分布	常数效力	Balducci
(1)	0.01270	0.01262	0.01254
(2)	0.01368	0.01377	0.01387
(3)	0.01377	0.01377	0.01377

1.30.(1) $\alpha/(\omega - x)$ $(\omega - x)/(\alpha + 1)$

1.31. (1) 0.00142 0.99867

1.32. 0.317， 0.140

1.33. 0.9792

1.34. $\log(1 - \frac{1}{2}q_x) - \log(1 - q_x)$

1.35. $q'_x < 2q_x$

1.37. (1) $(\frac{1+Bc^x}{1+B})^{-A/(B\log c)}$

1.38. (1) $\frac{5^7}{4^{10}} = 0.07451$ (2) 77.21

1.39. (2) $-\log(1 - q_x)$ (3) $\frac{-q_x^2}{(1-q_x)\log(1-q_x)}$ (4) $\frac{1}{45}$

第二章

2.5. (2) $\overline{A}^{\,1}_{x:\overline{n|}} = \frac{\mu_{x+n}}{\delta+\mu_{x+n}} A_{x:\overline{n|}}^{\;\;1}$

(3) $-\frac{\mu_{x+n}}{\delta+\mu_{x+n}}(A_{x:\overline{n|}}^{\;\;1})^2$, 其中 n 满足 (2)

(4) $n = \frac{\log 2}{\mu+\delta}$, $\min Cov[Z_1, Z_2] = -\frac{\mu}{4(\mu+\delta)}$

2.6. (1) 0.237832 (2) 0.416667

2.7. (1) 0.092099 (2) 0.055321

2.8. (1) $\frac{20}{3(100-x)}[1 - (\frac{20}{120-x})^3]$;

$\frac{20}{7(100-x)}[1 - (\frac{20}{120-x})^7] - \frac{20}{3(100-x)}[1 - (\frac{20}{120-x})^3]2.$

(2) $\frac{20}{3(100-x)}[10 - 10(\frac{20}{120-x})^2 - (100 - x)(\frac{20}{120-x})^3]$

2.10. (1) $\mu/(\mu + \delta)^2$ (2) $\mu[2/(\mu + 2\delta)^3 - \mu/(\mu + \delta)^4]$

2.11. (1) 0.407159 (2) 5.554541

2.13. (1) 0.5 (2) 0.05

2.14. (2) $(IA)_{x:\overline{m}|} = (IA)^{1}_{x:\overline{m}|} + mA_{x:\overline{m}|}^{\ \ 1}$

2.15. (1) $v^{k+(j+1)/m}$

$$A_x^{(m)} + \sum_{k=0}^{\infty} v^{k+1}{}_kp_x \sum_{j=0}^{m-1} {}_{j/m|1/m}q_{x+k}(1+i)^{1-(j+1)/m}$$

2.21. $\frac{1}{D_x}(2M_x - M_{65})$

2.22. $\frac{1000}{D_0}(\overline{R}_0 + \overline{R}_2 - 2\overline{R}_6 + 40\overline{M}_{21})$

2.23. (1) $\frac{1}{D_{30}}[\frac{i}{\delta}(R_{30} - R_{65} - 35M_{65}) + 35D_{65}]$

(2) $\frac{1}{D_{30}}[\frac{i}{\delta}(R_{30}-R_{65}-35M_{65})-\frac{i}{\delta}(\frac{1}{d}-\frac{1}{\delta})(M_{30}-M_{65})+35D_{65}]$

(3) $\frac{1}{D_{30}}[\frac{i}{\delta}(R_{30}-R_{40}-10M_{65})-\frac{i}{\delta}(\frac{1}{d}-\frac{1}{\delta})(M_{30}-M_{40})+10D_{65}]$

2.25. (1) $\pi = 9100/(14-k)$

(2) $1000000[{}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2] + (k\pi)^2[{}^2\overline{A}^{1}_{x:\overline{n}|} - (\overline{A}^{1}_{x:\overline{n}|})^2] - 2000k\pi\overline{A}^{1}_{x:\overline{n}|}A_{x:\overline{n}|}^{\ \ 1}$

2.26. (1) 0.307215 (2) 不会

2.27. (2) 0.001493 (3) 0.001493

第三章

3.1. 230.47 3.2. 4338.89

3.4. (1) 16.008, 12.761, 5.397

(2) 3.137, 10.230, 9.523

3.5. (1) 0.111, 0.251, 0.572

(2) 0.0251

3.7. $-Var[v^T] = -({}^2\overline{A}_x - (\overline{A}_x)^2)$

3.9. $\sum_{k=m}^{m+n-1} v^k{}_kp_x$; $\ddot{a}_{x:\overline{m+n}|} - \ddot{a}_{x:\overline{m}|}$; $\frac{A_{x:\overline{m}|} - A_{x:\overline{m+n}|}}{d}$;

${}_mE_x\ddot{a}_{x+m:\overline{n}|}$

3.13. $1 = ia_{x:\overline{n}|} + iA^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}$

3.16. $\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x)$, ${}_{n|}\ddot{a}_x - \frac{m-1}{2m}{}_nE_x$

3.22. (1) $\alpha(m)\ddot{s}_{25:\overline{40}|} - \beta(m)(\frac{1}{{}_{40}E_{25}} - 1)$

(2) ① 15.038 ② 196.380

3.23. (1) N_x/D_x (2) N_{x+1}/D_x (3) $(N_x - N_{x+n})/D_x$

(4) $(N_{x+1} - N_{x+n+1})/D_x$ (5) N_{x+n}/D_x

(6) N_{x+n+1}/D_x (7) $[N_x^{(m)} - N_{x+n}^{(m)}]/D_x$ (8) $N_{x+n}^{(m)}/D_x$

3.24. (1) $\tilde{N}_{60}^{(12)}/D_{60}$

(2) $(\tilde{N}_{40}^{(12)} - \tilde{N}_{65}^{(12)})/D_{40}$

(3) $\tilde{N}_{70}^{(12)}/D_{40}$

3.31. (2) $[{}^2\overline{A}_x - (\overline{A}_x)^2]/i^2$

3.34. (1) $\ddot{a}_x + 0.03(Ia)_x = (N_x + 0.03S_{x+1})/D_x$

(2) $\sum\limits_{k=0}^{\infty}(1.03)^k v^k {}_kp_x = \ddot{a}_x'$, 其中利率 $i' = \frac{i-0.03}{1.03}$

3.35. $\int_0^n (n-t)v^t {}_tp_x dt$

3.36. $1200[\tilde{N}_{30}^{(12)} + \tilde{N}_{40}^{(12)} + 3\tilde{N}_{50}^{(12)} + 5\tilde{N}_{60}^{(12)} - 10\tilde{N}_{70}^{(12)}]/D_{70}$

3.37. $\overline{a}_{35:\overline{25|}} - {}_{25}p_{35}\overline{a}_{\overline{25|}}$

3.38. $\ddot{a}_{x:\overline{n|}} - {}_np_x\ddot{a}_{\overline{n|}}$

3.39. $\frac{1}{12}\ddot{a}_{x:\overline{25|}} - \frac{25}{12}{}_{25}E_x$

3.41. $v^{2n}{}_np_x(1 - {}_np_x)\ddot{a}_{x+n}^2 + v^{2n}{}_np_x\frac{{}^2A_{x+n} - A_{x+n}^2}{d^2}$

3.44.

(1) $\alpha(m) = 1 + \frac{m^2-1}{12m^2}\delta^2 + \frac{2m^4-5m^2+3}{720m^4}\delta^4 + cdots$

$\beta(m) = \frac{m-1}{2m}[1 + \frac{m+1}{3m}\delta + \frac{m(m+1)}{12m^2}\delta^2$

$+\frac{(m+1)(6m^2-4)}{360m^3}\delta^3 + \cdots]$.

(2) $\alpha(\infty) = 1 + \frac{1}{12}\delta^2 + \frac{1}{360}\delta^4 + \cdots$

$\beta(\infty) = \frac{1}{2}[1 + \frac{1}{3}\delta + \frac{1}{12}\delta^2 + \frac{1}{60}\delta^3 + \cdots]$

3.46. $\frac{I}{\delta} + (J - \frac{I}{\delta})v^r; \frac{I}{\delta} + (J - \frac{I}{\delta})\overline{A}_x; (J - \frac{I}{\delta})^2({}^2\overline{A}_x - \overline{A}_x^2)$

3.47. (1) 14.353 (2) 13.350 (3). 1.002

3.51. (1) 14,624 (2) 15,422

3.54. (1) 488.23 (2) 700.48 (3) 531.77

第四章

4.1. 0, 0.1779

4.3. $\frac{\mu}{\mu+2\delta} = {}^2\overline{A}_x$

4.9. $A_x = (1-r)/(1+i-r)$, $\ddot{a}_x = (1+i)/(1+i-r)$,
$P_x = (1-r)/(1+i)$
$({}^2A_x - (A_x)^2)/(d\ddot{a}_x)^2 = (1-r)r/(1+2i+i^2-r)$
4.10. 0.01913987
4.12. 0.032868
4.13. 0.04128
4.14. 省略保费符号中共同的 $(\overline{A}_{40:\overline{25|}})$,

$$P \le P^{(2)} \le P^{\{4\}} \le P^{\{12\}} \le \overline{P}.$$

4.15. 100/99
4.16. 740.93
4.18.
(1) $(\overline{M}_x - \overline{M}_{x+30} + D_{x+30})/(N_x^{(12)} - N_{x+20}^{(12)})$
(2) $(\overline{M}_x - \overline{M}_{x+30} + D_{x+30})/(\overline{N}_x - \overline{N}_{x+20})$
(3) $(d^{(4)}/\delta)b$ (4) $N_{65}/(N_{25} - N_{45})$
4.19. $5000(d/\delta)(40\overline{M}_{30} - \overline{R}_{31} + \overline{R}_{71})/(\overline{N}_{30} - \overline{N}_{50})$
4.20. $P(A'^{1}_{45:\overline{20|}})$, 其中 $A'^{1}_{45:\overline{20|}}$ 是关于 (45) 在 $b_{k+1} = \ddot{s}_{\overline{k+1|}}$ 时的 20 年期净趸缴保费。
4.21. (1) 11.5451, 20.4106 (2) 6.3099, 25.6458
4.23. ${}_{25}P_{40}$
4.24. (2) $P^{(12)}(A_1^{(12)}) + d^{(12)}$
4.25. $100,000/[1.1\ddot{s}_{\overline{30|}} - 0.1\ddot{s}_{35:\overline{30|}}]$
4.26. 0.008
4.27. $[11,000M_x + 25(N_x - N_{x-20})]/[N_x - N_{x+20} - 1.1(R_x - R_{x+20})]$
4.28. $(M_{25} + M_{35})/(N_{25} + N_{35} - 2N_{65})$
4.29. $P^{(m)}(\overline{A}_x) = \overline{M}_x/N_x^{(m)}$; $P^{(m)}(\overline{A}^{1}_{x:\overline{n|}}) = [\overline{M}_x - \overline{M}_{x+n}]/[N_x^{(m)} - N_{x+n}^{(m)}]$;
$P^{(m)}(\overline{A}_{x:\overline{n|}}) = [\overline{M}_x - \overline{M}_{x-n} + D_{x+n}]/[N_x^{(m)} - N_{x+n}^{(m)}]$;

${}_hP^{(m)}(\overline{A}_x) = \overline{M}_x/[N_x^{(m)} - N_{x+h}^{(m)}]$;

${}_hP^{(m)}(\overline{A}_{x:\overline{n}|}) = [\overline{M}_x - \overline{M}_{x+n} + D_{x+n}]/[N_x^{(m)} - N_{x+h}^{(m)}]$

4.30. $L_1 = v^T - \overline{P}(\overline{A}_x)\overline{a}_{\overline{T}|} \equiv 1 - (1/\overline{a}_{\overline{T}|} = L_2$

4.31. (1) -0.08　　(2) 0.1296　　(3) 0.1587

第五章

5.1. ${}_tL = \begin{cases} v^U - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{\overline{U}|} & U < n-t \\ v^{n-t} - \overline{P}(\overline{A}_{x:\overline{n}|})\overline{a}_{n-t} & U \geq n-t \end{cases}$

5.2. $E[{}_tL] = \overline{a}_{x-t:\overline{n-t}|}$, $Var[{}_tL] = ({}^2\overline{A}_{x+t:\overline{n-t}|} - (\overline{A}_{x+t:\overline{n-t}|})^2)/\delta^2$

5.3. (1) $\overline{A}_{45:\overline{20}|} - {}_{20}\overline{P}(\overline{A}_{35:\overline{30}|})\overline{a}_{45:\overline{10}|}$

(2) $\overline{A}^1_{50:\overline{5}|}$

5.4. $\overline{A}_{50} - {}_{20}\overline{P}(\overline{A}_{40})\overline{a}_{50:nx10}$; $[{}_{10}\overline{P}(\overline{A}_{50}) - {}_{20}\overline{P}(\overline{A}_{40})]\overline{a}_{50:\overline{10}|}$;

$[1 - \frac{{}_{20}\overline{P}(\overline{A}_{40})}{{}_{10}\overline{P}(\overline{A}_{50})}]\overline{A}_{50}$; ${}_{20}\overline{P}(\overline{A}_{40})\overline{s}_{40:\overline{10}|} - {}_{10}\overline{k}_{40}$

5.6. $\overline{P}({}_{30|}\overline{a}_{35})\overline{s}_{35:\overline{20}|}$

5.8. (5.3.3)

5.10. $A_{50:\overline{10}|} - P_{40:\overline{20}|}\ddot{a}_{50:\overline{10}|}$, $(P_{50:\overline{10}|} - P_{40:\overline{20}|})\ddot{a}_{50:\overline{10}|}$,

$(1 - \frac{P_{40:\overline{20}|}}{P_{50:\overline{10}|}})A_{50:\overline{10}|}$, $P_{40:\overline{20}|}\ddot{s}_{40:\overline{10}|} - {}_{10}k_{40}$,

$1 - \frac{\ddot{a}_{50:\overline{10}|}}{\ddot{a}_{40:\overline{20}|}}$, $\frac{P_{50:\overline{10}|} - P_{40:\overline{20}|}}{P_{50:\overline{10}|} + d}$, $\frac{A_{50:\overline{10}|} - A_{40:\overline{20}|}}{1 - A_{40:\overline{20}|}}$

5.12. 1/5

5.13.

保险	完全连续	半连续	完全离散
30 年两全	0.17530	0.17504	0.17407
终身	0.08604	0.08566	0.08319
30 年定期	0.03379	0.03370	0.03273

5.14. (2) 及 (3)

5.16. 除 (4) 外

5.17. 全部

5.25. 责任准备金及保费省略 $(\overline{A}_{x:\overline{40}|})$。

(1) $\frac{1}{2}{}_{20}V + \frac{1}{2}{}_{20}V + \frac{1}{2}P$　　(2) $\frac{1}{2}{}_{20}\overline{V} + \frac{1}{2}{}_{20}\overline{V}$

(3) $\frac{1}{2}{}_{20}V^{(2)}+\frac{1}{2}{}_{21}V^{(2)}$ (4) $\frac{1}{3}{}_{20}V^{(2)}+\frac{2}{3}P^{(2)}$

(5) 与 (2) 相同 (6) $\frac{1}{3}{}_{20}\overline{V}+\frac{2}{3}{}_{21}\overline{V}+\frac{1}{3}P^{(2)}$

5.26. 0.05448

5.31. (1) 0.0067994 (2) 0.1858077 (3) 0.2012023

(4) 0.0275369 (5) 0.0255405

5.34. $-{}_tp_x[\delta\cdot{}_t\overline{V}(\overline{A}_x)+\overline{P}(\overline{A}_x)]$

5.35.

(1) $\{10,000[10\overline{M}_{30}-\overline{M}_{50}-\overline{M}_{55}-2\overline{M}_{60}-6\overline{M}_{65}]\div$ $\overline{(N_{30}-\overline{N}_{65})\}2\ddot{a}_{\overline{1}|/2}}$

(2) $60,000(\overline{M}_{60}-\overline{M}_{65})/D_{60}-P(\overline{N}_{60}-\overline{N}_{65})/D_{60}$,

其中 P 是 (a) 中大括号间的完全连续保费。

5.36.

(1) $100000D_{65}/(D_{35}-M_{35}+M_{65}$

(2) $[100000D_{65}+S(M_{35+k}-M_{65})]/D_{35+k}$

(3) $S(D_{35}-M_{35}+M_{35+k})/D_{35+k}$

5.37.

(1) $[(\overline{M}_{40}-\overline{M}_{65}+D_{65})-P(N_{40}^{(12)}-N_{50}^{(12)})]/D_{40}$; $[P(N_{30}^{(12)}-N_{40}^{(12)})-(\overline{M}_{30}-\overline{M}_{40})]/D_{40}$

(2) $(\overline{M}_{55}-\overline{M}_{65}+D_{65})/D_{55}$; $[P(N_{30}^{(12)}-N_{50}^{(12)})-(\overline{M}_{30}-\overline{M}_{55})]/D_{55}$

5.38. 0.008

5.39. 0.240

5.40.

(1) $P_{25}N_{35}/(N_{35}-N_{65})$

(2) $P_{25}(N_{25}-N_{35})/D_{35}-(M_{25}-M_{35})/D_{35}={}_{10}V_{25}$

(3) $1-P_{25}n_{65}/M_{35}$

(4) ${}_{20}V_{25}+(1-B)(M_{35}-M_{45})/D_{45}$

5.41.

(1) ${}_tp_x(\pi_t+\delta_t\overline{V}-b_t\mu_{x+t})$

(2) $v^t(\pi_t+\mu_{x+t}{}_t\overline{V}-b_t\mu_{x+t})$

(3) $v^t{}_tp_x[\pi_t-b_t\mu_{x+t}]$

5.43. (1) 0.0241821 (2) 0.0189660

5.47. (1) 和 (2) 1491.03 (3) 343.84 (4) 0

5.48.

(1) 1590915

(2) 6450962; 1495093, 是责任准备金的 1.00280 倍

(3) 5311375; 增补 3791, 是责任准备金的 0.002542 倍

(4) 对于 (2): 645096250; 149133281, 是责任准备金的 1.00028 倍

对于 (3): 531137500; 增补 37911, 是责任准备金的 0.00025 倍

5.49.

(1) 1104260 是这些保单的责任准备金

(2) 6450962; 1108438, 是责任准备金的 1.00378 倍

(3) 5311375; 增补 3791, 是责任准备金的 0.00343 倍

(4) 对于 (2): 645096250; 110467781, 是责任准备金的 1.00038 倍

对于 (3): 531137500; 增补 37911, 是责任准备金的 0.00034 倍

5.50.

(1) 用 ${}^h_k\overline{V}(\overline{A}_{x:\overline{n}|}),{}_h\overline{P}(\overline{A}_{x:\overline{n}|})$ 代替 ${}^h_kV(\overline{A}_{x:\overline{n}|}),{}_hP(\overline{A}_{x:\overline{n}|})$, 并在附号 N 上加横线。

(2) 用 ${}^h_kV_{x:\overline{n}|},{}_hP_{x:\overline{n}|}$ 代替 ${}^h_kV(\overline{A}_{x:\overline{n}|}),{}_hP(\overline{A}_{x:\overline{n}|})$, 并去掉附号 M 上的横线。

5.51. $5000[{}_{10}\overline{V}(\overline{A}_{30})+P^{(1)}(\overline{A}_{30})+{}_{11}\overline{A}_{30})]$

5.52. (1) 0.2 (2) 0.25 (3) 0.7584 (4) 0.27

第六章

6.1.

(1) ${}_np_{x}{}_np_y$

(2) ${}_np_x + {}_np_y - 2{}_np_{x}{}_np_y$

(3) ${}_np_x + {}_np_y - {}_np_{x}{}_np_y$

(4) $1 - {}_np_{x}{}_np_y$

(5) 与 (4) 一样

(6) $(1 - {}_np_x)(1 - {}_np_y) = 1 - {}_np_x - {}_np_y + {}_np_{x}{}_np_y$

6.3. ${}_nq_{xx}$

6.5. ${}_{n|}q_x + {}_{n|}q_y - {}_{n|}q_{x}{}_{n|}q_y$

否，因为对 ${}_{n|}q_{\overline{xy}}$ 而言，第二个死亡一定发生在年度 $n+1$，而对所求概率则不然。

6.6. 2/9

6.7. (1) 2/3 (2) 29/30 (3) 18.06 (4) 36.94 (5) 160.11 (6) 182.33 (7) 82.95 h. 0.49

6.8. $\mu_{xx}\ \mathring{e}_{xx}\ -1$

6.11. 531/2000

6.12. 每年末支付 1, 支付 n 年度且在此后 (xy) 存在的情况下支付的年金。

6.13. 在 (x) 死亡与 n 年末这两个区间的晚者支付 1 的保险。

6.15. $\overline{a}_{25:\overline{25}|} + \overline{a}_{30:\overline{20}|} - \overline{a}_{25:30:\overline{20}|}$

6.16. ${}_{20|}a_{30} + {}_{25|}a_{25} - {}_{25|}a_{25:30}$

6.17. $\frac{1}{6}\ddot{a}_{xy:\overline{n}|} + \frac{1}{2}\ddot{a}_{y:\overline{n}|} + \frac{1}{3}\ddot{a}_{x:\overline{n}|}$

6.18. $a_{x:\overline{n}|} + v^n{}_np_xa_{x+n:y:\overline{m-n}|}$

6.20. (1) $\ddot{a}_x^{(m)} + p(\ddot{a}_y^{(m)} - \ddot{a}_{xy}^{(m)})$

(2) $\ddot{a}_x^{(m)}/[\ddot{a}_x^{(m)} + p(\ddot{a}_y^{(m)} - \ddot{a}_{xy}^{(m)})]$

6.22. a. 7.0753 b. 7.0756

6.26. 1/3

6.30. ${}_{infty}q_{xy}^1 = {}_\infty q_{xy}^2$

6.33. $\overline{A}_{50} - \overline{A}^1_{50:20:\overline{20}|}$

6.34. $\overline{A}^1_{x:\overline{n}|} - \overline{A}^1_{xy} + {}_nE_x\overline{A}^{\ \ 1}_{x+n:y}$

6.35. 1/12

6.36. a. 0.2755 b. $\frac{1}{4}\overline{A}_{40:50}+0.0015\overline{a}_{40:50}$

6.37. 1/3, 52.68

6.38. (1) $\overline{a}_x+\overline{a}_{\overline{n}|}-\overline{a}_{x:\overline{n}|}$ (2) $v^n{}_nq_x$

6.40. $\mu_x\ \mathring{e}_{xy}-{}_\infty q_{xy}^1$

第七章

7.1. (1) $e^{-t\mu_x^{(\tau)}}\mu_x^{(j)}$ (2) $\mu_x^{(j)}/\mu_x^{(\tau)}$ (3) $e^{-t\mu_x^{(\tau)}}\mu_x^{(\tau)}$

7.2. a. $j(50-t)^2/50^3$ b. $3(50-t)^2/50^3$ c. $j/3$ d. $j/3$

7.3. ${}_3p_{65}^{(\tau)}=0.75321,\quad {}_3q_{65}^{(1)}=0.03766,\quad {}_3q_{65}^{(2)}=0.16504$

7.4. (1) 302.4 及 210.95

(2) 231.0 及 177.64

7.5.

(1) $h(1)=0.231;h(2)=0.4666;h(3)=0.3024$

(2) $h(1\ k=2)=0.25;h(2|k=2)=0.75;h(3|k=2)=0$

7.6. $l_x^{(\tau)}=(a-x)e^{-x};d_x^{(1)}=e^x(1-e^{-1});$

$d_x^{(2)}=(a-x-1)e^{-x}-(a-x-2)e^{-x-1}$

7.7. $1000[\frac{a-x^2}{a}]e^{-cx}$

7.8. (1) ${}_tp_x^{(\tau)}[\mu_{x+t}^{(\tau)}-\mu_x^{(\tau)}]$ (2) ${}_tp_x^{(\tau)}\mu_{x+t}^{(j)}+{}_tq_x^{(j)}\mu_{x^{(\tau)}}-\mu_x^{(j)}$ (3) ${}_tp_x^{(\tau)}\mu_{x+t}^{(j)}$

7.9.

k	$q'^{(1)}_k$	$q'^{(2)}_k$
0	0.17433	0.27332
1	0.11210	0.21163
2	0.05426	0.15410
3	0.00000	0.10000

7.10. (1) $1-e^{-c}$ (2) c (3) $c\int_0^1 {}_tp_x^{(\tau)}dt$

7.12. $m'^{(j)}_x\geq q'^{(j)}_x\geq q_x^{(j)}$

7.13. 0.0592

7.14. (1) 0.0909 (2) 0.0906

7.15.

k	$m_k^{(1)}$	$m_k^{(2)}$
0	0.18750	0.31250
1	0.11765	0.23529
2	0.05556	0.16667
3	0.00000	0.10526

7.16.

x	$p_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
62	0.76048	0.01767	0.02665	0.19520
63	0.85027	0.02054	0.03193	0.09726
64	0.82115	0.02578	0.03705	0.11603

7.19.

x	$m_x^{(1)}$	$m_x^{(2)}$	$m_x'^{(1)}$	$m_x'^{(2)}$
65	0.02073	0.05181	0.02073	0.05183
66	0.03141	0.06283	0.03144	0.06286
67	0.04233	0.07407	0.04237	0.07412
68	0.05348	0.08556	0.05355	0.08565
69	0.06486	0.09730	0.06499	0.09744

7.20. (1) 修正为 $m_x^{(j)}/(1+\frac{1}{2}m_x^{(\tau)})$

7.22.

k	$q_k'^{(1)}$	$q_k'^{(2)}$
0	0.17143	0.27027
1	0.11111	0.21053
2	0.05405	0.15385
3	0.00000	0.10000

7.24. (1) 由 $q_x^{(3)} = q_x'^{(3)}[1-\frac{1}{2}(q_x'^{(1)}+q_x'^{(2)})+\frac{1}{3}q_x'^{(1)}q_x'^{(2)}]$ 得出 $q_x'^{(3)}$, 然后用 (7.5.3)

(2) 由习题 18 得 $q_x^{(1)} \cong q_x'^{(1)}[1-\frac{1}{2}(q_x^{(2)}+q_x^{(3)})]$.

7.25. $q_{69}^{(3)} = 0.94434$

7.26. 0.015

7.27. 若 1 表示死亡, 2 表示因别的原因退出, 则精算现值为

$$20000\int_0^{40} v^t {}_tp_{30}^{(\tau)}\mu_{30+t}^{(1)}dt + 300\int_0^{40} v^t {}_tp_{30}^{(\tau)}\mu_{30+t}^{(2)} t_{40-t|}\overline{a}_{30+t}dt$$

$$12000v^{40}{}_{40}p_{30}^{(\tau)}\overline{a}_{70}.$$

7.28. $1-\sum_{k=0}^{44} d_{20+k}^{(2)}/l_{20}^{(\tau)} = 1-[l_{20}^{(2)}-l_{65}^{(2)}]/l_{20}^{(\tau)}$

7.29.

(1) 从 $q_x^{'(1)}, q_x^{'(2)}$ 近似得出 $m_x^{(1)}, m_x^{(2)}$, 或从 $m_x^{(3)}, m_x^{(4)}$ 近似得出 $q_x^{'(3)}, q_x^{'(4)}$。

(2). $1-\sum_{k=0}^{\infty} d_{y+k}^{(4)}/l_y^{(\tau)} = 1-l_y^{(4)}/l_y^{(\tau)}$

7.30. 当所有原因都起作用时由于原因 j 而损失的概率为

原因 j 的绝对损失率 − 其它原因 $k(k\neq j)$ 的损失发生并且此后在 (x) 达到 $x+1$ 岁前发生与 j 相联系的事件的概率

7.32.

(1) $f(t,j) = \frac{\theta\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1}e^{-\beta t} \qquad j=1, t\geq 0$

$= \frac{(1-\theta)\beta^\alpha t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)} \qquad j=2, t\geq 0,$

$$h(j) = \begin{cases} \theta & j=1 \\ 1-\theta & j=2, \end{cases}$$

$$g(t) = \frac{\beta^\alpha t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)}$$

(2) $E[T] = \frac{\alpha}{\beta}; Var[T] = \frac{\alpha}{\beta^2}$

(3) (Ⅰ) $L = \begin{cases} v^T - \overline{A} & J=1, \qquad T\geq 0 \\ -\overline{A} & J=2, T\geq 0 \end{cases}$

(Ⅱ) $\overline{A} = \theta(1+\delta/\beta)^{-\alpha}$

(Ⅲ) $Var[L] = \theta(1+2\delta/\beta)^{-\alpha} - \theta^2(1+\delta/\beta)^{-2\alpha}$

第八章

8.1. (1) 取 $S_{30}=1$, 则

$$S_{30+k} = \begin{cases} 1.05^k & 0\leq k<10 \\ 1.1\times 1.05^k & 10\leq k<20 \\ 1.1^2\times 1.05^k & 20\leq k<30 \\ 1.1^3\times 1.05^k & k\geq 30. \end{cases}$$

(2) $1200\sum_{k=0}^{\omega-31} v^{k+1/2}{}_{k+1/2}p_{30}^{(\tau)} S_{30+k}$

8.2. $0.1\sum_{k=0}^{\omega-31} v^{k+1/2}{}_{k+1/2}p_{35}^{(\tau)}[25000(S_{35+k}/S_{35}) - 10000(1.05)^k]$

8.3. $5940v^{23}{}_{23}p_{40}^{(\tau)}\overline{a}_{63}^{r}$; $3960v^{23}{}_{23}p_{40}^{(\tau)}\overline{a}_{63:\overline{2}|}^{r}$

8.4. 假设 $12000({}_3Z_{25+k}/S_{25}) < 15000(1.04)^k$, $k \le a$, 则

$$R(25, 0, k+1/2) = 120k({}_3Z_{25+k}/S_{25}) \qquad k \le a$$

$$R(25, 0, k+1/2) = k[180({}_3Z_{25+k}/S_{25}) - 75(1.04)^k] \qquad k > a.$$

8.5. (1) $R(40, 0, 25) = 15000({}_3\tilde{Z}_{65}/S_{40}) - 0.50I_{65}$, 其中 ${}_3\tilde{Z}_{65} = \frac{S_{62}+S_{63}+S_{64}}{3}$

(2) $R(40, 0, 28\frac{1}{2}) = 17100({}_3Z_{68}/S_{40}) - 0.50I_{68\frac{1}{2}}$

8.6. 这里 $\alpha = 55, \omega = 68$,

$\sum_{k=5}^{17} v^{k+1/2}{}_kp_{50}^{(\tau)}q_{50+k}^{(r)}(20+k+1/2)({}_3Z_{50+k}/S_{50})480\overline{a}_{50+k+1/2}^{r}$

$+\sum_{k=5}^{14} v^{k+1/2}{}_kp_{50}^{(\tau)}q_{50+k}^{(r)}(20+k+1/2)({}_3Z_{50+k}/S_{50})$

$240\overline{a}^{r}_{50+k+\frac{1}{2}:\overline{15-k-\frac{1}{2}}|}$

由于当 $k < 5$ 时, $q_{50+k}^{(r)} = 0$, 求和可从 $k = 0$ 开始。

8.7. 第一个和式中的最后 3 项改为

$$\sum_{k=15}^{17} v^{k+1/2}{}_kp_{50}^{(\tau)}q_{50+k}^{(r)}({}_3Z_{50+k}/S_{50})(16800)\overline{a}_{50+k+1/2}^{r}.$$

8.8. 在习题 8.6 的答案中, 用 20 取代 $(20+k+1/2)$ 。

8.9.

(1) $R(30, 20, 15) = 8000 + 720\sum_{j=0}^{14} S_{50+j}/S_{50}$

(2) $R(30, 20, 15\frac{1}{2}) = 8000 + 720[\sum_{j=0}^{14} S_{50+j} + (1/2)S_{65}]/S_{50}$

(3) $8000\sum_{k=8}^{17} v^{k+1/2}{}_kp_{50}^{(\tau)} q_{50+k}^{(\tau)} \overline{a}_{50+k+1/2}^{r}$

(4) $\sum_{k=0}^{17} v^{k+1/2}{}_kp_{50}^{(\tau)} q_{50+k}^{(r)} 720[(\sum_{j=0}^{k-1} S_{50+j} + (1/2)S_{50+k}) \div$

$S_{50}]\overline{a}_{50+k+1/2}^{r}$，它等于 $k=8$ 到 17 的和；或

$$\frac{720}{S_{50}}\Big[\sum_{j=0}^{17} S_{50+j}\big(\frac{1}{2}v^{j+1/2}p_{50}^{(\tau)} q_{50+j}^{(r)}\overline{a}_{50+j+1/2}^{r}$$

$$+\sum_{k=j+1}^{17} v^{k+1/2}{}_kp_{50}^{(\tau)} q_{50+k}^{(r)}\overline{a}_{50+k+1/2}^{r}\big)\Big]$$

8.10. $(0.7\times 25000-8000)\overline{a}_{50\frac{1}{2}}^{i}$

8.11. $5000\sum_{k=0}^{4} v^{k+1/2}{}_kp_{35}^{(\tau)} q_{35+k}^{(w)} (1.06)^{k+1/2}$

8.12.

(1) $0.01c(AS)_{x+h}\int_0^1 v^t{}_tp_{x+h}^{(\tau)}\mu_{x+h+1}^{(w)}\int_0^t(1+j)^{t-s}dsdt$

(2) $0.01c(AS)_{x+h}q_{x+h}^{(w)}\frac{1}{\log(1+j)}\big[\frac{1-((1+j)/(1+i))}{\log(1+i)-\log(1+j)}$
$-\frac{1-(1/(1+i))}{\log(1+i)}\big]$

(3) 0.4856429; 0.4873143; 两者都乘以 $0.01c(AS)_{x+h}q_{x+h}^{(w)}$

8.13.

(1) $1200[{}^S\overline{N}_{35}^{(\tau)}/{}^SD_{35}^{(\tau)}]$

(2) $2500[{}^S\overline{N}_{35}^{(\tau)}/{}^SD_{35}^{(\tau)}]-[1000(1.05)^{-1/2}\overline{N}_{35}^{'(\tau)}]/D'(\tau)_{35}$，
其中 $D_y^{'(\tau)}=v^y(1.05)^y l_y^{(\tau)}$

(3) $5940(D_{63}^{(\tau)}/D_{40}^{(\tau)})\overline{a}_{63}^{r}$; $3960(D_{63}^{(\tau)}/D_{40}^{(\tau)})\overline{a}_{63:\overline{2}|}^{r}$

(4) $\{480\sum_{k=5}^{14}(20+k+1/2)^{Za}\overline{C}_{50+k+1/2}^{r}$

$$+16800\sum_{k=15}^{17}{}^{Za}\overline{C}_{50+k}^{r}\}/{}^SD_{50}^{(\tau)},$$

其中 ${}^{za'}\overline{C}_{50+k+1/2}^{r}={}_3Z_{50+k}\overline{C}_{50+k}^{r}\overline{a}_{50+k+1/2:\overline{15-k-1/2}|}^{r}$.

8.14. (1) $\frac{120}{{}^SD_x^{(\tau)}}\sum_{\alpha-x}^{\omega-x-1}k^{Za}\overline{C}_{x+k}^{r}=120^{Za}\overline{R}_{x+1}^{r}/{}^SD_x^{(\tau)}$

(2) $120[\frac{1}{2}{}^{Za}\overline{M}_x^r + {}^{Za}\overline{R}_{x+1}^r]/{}^S D_x^{(\tau)}$

8.15. 若 $a-x\geq 10$, 则公式不变，若 $a-x<10$, 公式变为

$$120[10\frac{1}{2}{}^{Za}\overline{M}_{x+10}^r + {}^{Za}\overline{R}_{x+11}^r]^S D_x^{(\tau)}.$$

8.16. $[\frac{1}{2}{}^{Za}\overline{M}_x^r + {}^{Za}\overline{R}_{x+1}^r]/{}^S\overline{N}_x^{(\tau)}$ 8.17.

(1) $1200[\overline{N}_{50}^{(\tau)}/D_{50}^{(\tau)}]$

(2) $100[12\overline{N}_{50}^{(\tau)} + \overline{S}_{51}^{(\tau)}]/D_{50}^{(\tau)}$

(3) $12000(1.04)^{-1/2}[(\overline{N}_{50}^{'(\tau)}/D_{50}^{'(\tau)})]$, 其中 $D_{50}^{'(\tau)}$ 和 $\overline{N}_{50}^{'(\tau)}$ 按利率 $i'=0.02/1.04$ 计算。

8.18. $1500[{}^S\overline{N}_{50}^{(\tau)}/{}^S D_{50}^{(\tau)}] - 0.05[{}^H\overline{N}_{50}^{(\tau)}/D_{50}^{(\tau)}]$, 其中 ${}^H\overline{D}_{50+k}^{(\tau)} = H_k\overline{D}_{50+k}^{(\tau)}$.

8.19.

(1) $\{450[10\frac{1}{2}{}^{Za}\overline{M}_{40}^r + {}^{Za}\overline{R}_{41}^r] + 18000{}_5\tilde{Z}_{70}D_{70}^{(\tau)}\overline{a}_{70}^r$
$+150[\sum_{k=20}^{24}(10+k+\frac{1}{2})^{Za'}\overline{C}_{40+k}^r]\}/{}^S D_{40}^{(\tau)}$,

其中 ${}^{Za'}\overline{C}_y^r$ 结合了年金因子 $\overline{a}^r_{y+(1/2):\overline{65-y-(1/2)}|}$

(2) $4500\{{}^{Za}\overline{M}_{60}^r + {}_5\tilde{Z}_{70}D_{70}^{(\tau)}\overline{a}_{70}^r + \frac{1}{3}\sum_{k=20}^{24}{}^{Za'}\overline{C}_{40+k}^r\}/{}^S D_{40}^{(\tau)}$

8.20. $600[\frac{1}{2}v^{1/2}q_{62}^{(\tau)}\overline{a}_{62\frac{1}{2}}^r + \sum_{k=1}^{8}v^{k+1/2}{}_kp_{62}^{(\tau)}q_{62+k}^{(r)}\overline{a}_{62+k+1/2}^r]$

8.21. (A)

第九章

9.1. (1)

储蓄帐户	566.50	准备金	485.44
		盈余	81.06
	566.50		566.50
	赌资收入	550.00	
	利息收入	16.50	
	准备金增加额	566.50	
	净收入	81.06	

9.3. $\frac{1000\overline{A}_{[40]:\overline{25}|}+8.50+4\ddot{a}_{[40]:\overline{25}|}}{0.93\ddot{a}_{[40]:\overline{25}|}+0.05{}_{10}E_{[40]}\ddot{a}_{[40]+10:\overline{15}|}-0.35}$

9.4. $(1000\overline{A}_{x:\overline{n}|}+2.50+2.50\ddot{a}_{x:\overline{n}|})/0.935$

9.5. $a=(1+e_0+e_2+e_3), \qquad c=(e_1+e_0d)$

9.8. (1) $k=200$ (2) $b=20$

(3) $m=\sqrt{200}$ (4) $R(\bar{b})=30.12$

9.11. $\overline{\beta}=\frac{overline A_x}{(\overline{I}\overline{a})_{x:\overline{m}|}/m+{}_{m|}\overline{a}_x}$

${}_t\overline{V}(\overline{A}_x)^{Mod}=\overline{A}_{x+t}-\overline{\beta}[\overline{I}\overline{a})_{x+t:\overline{m-t}|}m+\frac{t}{m}\overline{a}_{x+t:\overline{m-t}|}+{}_{m-t|}\overline{a}_{x+t}]$

9.12. $\alpha_x^{Mod}=A^1_{x:\overline{1}|}+K_1E_x, \qquad \beta_x^{Mod}=P_{x+1}-K/\ddot{a}_{x-1}$

9.17. (1) $\beta=0.03, \quad \alpha=0.01, \quad \beta-\alpha<0.05$

(2) 0.28 (3) 0.0867 (4) 0.0278

9.18. $\beta^{Com}_{x:\overline{15}|}=P_{x:\overline{15}|}+\frac{{}_{19}P_{x+1}-A^1_{x:\overline{1}|}}{\ddot{a}_{x:\overline{15}|}}$

$\alpha^{Com}_{x:\overline{15}|}=\beta^{com}_{x:\overline{20}|}-({}_{19}P_{x+1}-A^1_{x:\overline{1}|})$

9.19. $\alpha=\beta^{Com}_{x:\overline{20}|}-({}_{19}P_{x+1}-A^1_{x:\overline{1}|}), \qquad \beta=(P_{x:\overline{20}|}\ddot{a}_{x:\overline{15}|}-\alpha)/a_{x:\overline{14}|}$

9.20. $T=\beta^{Com}-{}_{19}P_{x+1}$

第十章

10.1. 0.0738

10.2. $1.046P_x+0.0026, \qquad 1.06P_x+0.0034$

10.3. $\frac{P'_{x+t}}{P_{x+t}} \begin{matrix}>\\=\\<\end{matrix} \frac{P'_x}{P_x}$ 时，${}_tW'_x \begin{matrix}>\\=\\<\end{matrix} {}_tW_x$.

10.4. $\frac{b}{2}+(\frac{b}{2}-1)\frac{A^1_{x+t:\overline{n-t}|}}{{}_{n-t}E_{x+t}}$

10.5. (1) $[{}_{10}CV-L-(1-L)\overline{A}^1_{40:\overline{10}|}]/{}_{10}E_{40}$

(2) $(1-L)\overline{A}^1_{45:\overline{5}|}+E_5E_{45}$

10.6. ${}^{20}_{10}W_{40}=0.5829, {}_{10}W_{40:\overline{20}|}=0.6232$, 比例额 =0.5

10.9. 终身寿险： $1-P^a_x/P_{x+k}$

n 年缴费终身寿险： $1-{}_nP^a_x/{}_{n-k}P_{x+k}$

n 年两全保险： $1 - P^{a}_{x:\overline{n}|} / P_{x+k:\overline{n-k}|}$

10.10. 终身寿险： $1 - P_{x+1} / P_{x+k}$

n 年缴费终身寿险： $1 - \beta^{Com} / {}_{n-k}P_{x+k}$，
其中 $\beta^{Com} = {}_nP_x + ({}_{19}P_{x+1} - A^{1}_{x:\overline{1}|}) / \ddot{a}_{x:\overline{n}|}$

n 年两全保险： $1 - \beta^{Com} / P_{x+k:\overline{n-k}|}$，
其中 $\beta^{Com} = P_{x:\overline{n}|} + ({}_{19}P_{x+1} - A^{1}_{x:\overline{1}|}) / \ddot{a}_{x:\overline{n}|}$.

10.11. (3) $\overline{a}_x G e^{\delta t} + \overline{a}_{x+k+t}(\mu_{x+k+t} + \delta) - 1$

10.12. $(G_2 - G_1) \sum\limits_{k=0}^{9} (1 - c_k) l^{(\tau)}_{x+k} (1+i)^{10-k} / l^{(\tau)}_{x+10}$

10.14. (2) $\left(\frac{1+\hat{i}_{h+1}}{1+i}\right)\left(\frac{p_{x+h}}{p'_{x+h}}\right)$

10.15. ${}_hV' > {}_hV$

10.17. (2) $P_x + c/\ddot{a}_x$

第十一章

11.3. (2) ① $H'(G) = 1 - r - \rho \overline{A}^{1}_{x:\overline{\rho G}|}$;

$H''(G) = -\rho^2 {}_{\rho G}p_x \mu_{x+\rho G} v^{\rho G}$ (译注：原书无 $v^{\rho G}$)

11.8. $(\overline{A}^{1}_{x:\overline{15}|} - V^{20} {}_{15}q_x)/\delta + \overline{a}_{\overline{5}|}(\overline{A}^{1}_{x:\overline{20}|} - \overline{A}^{1}_{x:\overline{15}|})$，或

$\overline{a}_{\overline{20}|} - \overline{a}_{x:\overline{20}|} + v^{20} {}_{15}p_x \overline{a}_{x+15:\overline{5}|} - v^{20} {}_{20}p_x \overline{a}_{\overline{5}|}$

11.9. $[1000 A_{x:\overline{20}|}^{\ \ 1} + 120(a^{(12)}_{\overline{20}} - a^{(12)}_{x:\overline{20}|})] / \ddot{a}_{x:\overline{20}|}$

11.11.

$$(1)\ Z = \begin{cases} v^T \overline{a}_{\overline{25-T}|} & T \le 15 \\ v^T \overline{a}_{\overline{10}|} & 15 < T \le 25 \\ v^{25} \overline{a}_{\overline{10}|} & 25 < T \le 35 \\ v^{25} \overline{a}_{\overline{T-25}|} & T > 35 \end{cases}$$

(2) $\int_0^{15} v^t \overline{a}_{\overline{25-t}|} {}_tp_{40} \mu_{40+t} dt + \overline{a}_{\overline{10}|} \int_{15}^{25} v^t {}_tp_{40} \mu_{40+t} dt$
$+ v^{25} \overline{a}_{\overline{10}|} \int_{25}^{35} {}_tp_{40} \mu_{40+t} dt + v^{25} \int_{35}^{\infty} \overline{a}_{\overline{t-25}|} {}_tp_{40} \mu_{40+t} dt.$

11.19. 203421

11.20. (1) 55　　(2) 53759.04

11.21.

(1) $12000[{}^{65}_{60}\overline{M}^{i}_{35} + \frac{1}{24} v^{1/2} {}_{60}\overline{M}^{i}_{35}] / (N_{35} - N_{60})$

(2) $12000[{}^{20}_{15}\pi^i_{45} - {}^{30}_{25}\pi^i_{35}]\ddot{a}_{45:\overline{15}|}$，其中 $12000{}^{30}_{25}\pi^i_{35}$ 与 $12000{}^{20}_{15}\pi^i_{45}$ 分别是 (35) 与 (45) 的受益的净年保费。

11.22. 120, 280

第十二章

12.1.

(1) $(w),(x),(y),(z)$ 中偶数个至少活到时间 t 。

(2) $(w),(x),(y),(z)$ 中奇数个至少活到时间 t 。

12.3. (1) 6.5 (2) 3237

12.4. ${}_tB_3 - 3{}_tB_4$, 其中 ${}_tB_3 = {}_tp_{wxy} + {}_tp_{wxz} + {}_tp_{wyz} + {}_tp_{xyz}$, ${}_tB_t = {}_tp_{wxyz}$.

12.5. $a_w - (a_{wxy} + a_{wxz} + a_{wyz}) + 2a_{wxyz}$

12.6. 0.624

12.7.

(1) $15(\overline{a}_x + \overline{a}_y + \overline{a}_z) - 10(\overline{a}_{xy} + \overline{a}_{xz} + \overline{a}_{yz}) + 9\overline{a}_{xyz}$

(2) $15\overline{a}_x - 5(\overline{a}_{xy} + \overline{a}_{xz}) + 3\overline{a}_{xyz}$

12.8. 4.6

12.9.

(1) $12000[a^{(12)}_{40:\overline{25}|} + a^{(12)}_{35:\overline{25}|} - 2a^{(12)}_{40:35:\overline{25}|}]$

(2) $12000[a^{(12)}_{40:\overline{25}|} + a^{(12)}_{35:\overline{30}|} - a^{(12)}_{40:35:\overline{25}|}]$

12.10

(1) $\overline{a}_{\overline{n}|} + \overline{a}_x + \overline{a}_y - \overline{a}_{x:\overline{n}|} - \overline{a}_{y:\overline{n}|} - \overline{a}_{xy} + \overline{a}_{xy:\overline{n}|}$

(2) $\overline{a}_{25:\overline{40}|} + \overline{a}_{\overline{30}|} - \overline{a}_{25:\overline{30}|}$

12.11. (1) $\frac{1}{7}$ (2) $\frac{26}{105}$ (3) $\frac{64}{105}$

12.12. Ⅰ错误，右端应改为 $\overline{A}_{wxyz}^{\ 4} + \overline{A}_{wxyz}^{\ \ 4} + \overline{A}_{wxyz}^{\ \ \ 4} + \overline{A}_{wxyz}^{\ \ \ \ 4}$

Ⅱ正确，两端都在第二个死亡时提供一单位支付。

Ⅲ错误，正确的公式为 $\overline{A}_{wz}^{\ 1} + \overline{A}_{xz}^{\ 1} + \overline{A}_{yz}^{\ 1} - 2(\overline{A}_{wxz}^{\ \ 1} + \overline{A}_{wyz}^{\ \ 1} + \overline{A}_{xyz}^{\ \ 1}) + 3\overline{A}_{wxyz}^{\ \ \ 1}$.

12.13. $\int_0^\infty v^t {}_tp_{xy}\mu_{y+t} t\overline{A}_{x+t} dt$

12.14. (1)5/7 (2) 3/7

12.15. $A_{z:\overline{10}|}^{\ \ 1}[\overline{A}_{y:\overline{z+10}}^{\ \ 1} - \overline{A}_{xy:\overline{z+10}}^{\ \ \ 1}] - A_{yz:\overline{10}|}^{\ \ \ 1}[\overline{A}_{y+10:\overline{z+10}}^{\ \ \ \ \ 1}$
$-\overline{A}_{x:y+10:\overline{z+10}}^{\ \ \ \ \ \ 1}]$
12.16. $v^{10}[\overline{A}_{xy}^{1} + \overline{A}_{xz}^{1} - \overline{A}_{xyz}^{1}] - A_{y:\overline{10}|}^{\ \ 1}\overline{A}_{x:y+10}^{1} - A_{z:\overline{10}|}^{\ \ 1}\overline{A}_{x:z+10}^{1}$
$+A_{zy:\overline{10}|}^{\ \ \ 1}\overline{A}_{x:y+10:z+10}^{1}$
12.17. $(\overline{A}_{30:\overline{5}|}^{1} + A_{35:\overline{5}|}^{\ \ 1}\overline{A}_{35:60}^{1})/(\frac{1}{1.075} - A_{30:\overline{5}|}^{\ \ 1}\overline{A}_{35:60}^{\ \ \ 1})$
12.20.

(1) $\int_0^{\infty}(1 - {}_tp_w){}_tp_{xyz}\mu_{x+t}dt$
(2) ${}_{\infty}q_{xyz}^{1} - {}_{\infty}q_{wxyz}^{\ 1}$

12.22. (1) 25/72 (2) 19/36 (3) 5/8
12.23. 0.07
12.24. (1) 0.3 (2) 0.2 (3) 0.03
12.25. $\int_{10}^{15}(1 - {}_{t-10}p_x){}_tp_y\mu_{y+t}({}_{t+10}p_z - {}_{25}p_z)dt$
12.26. $\int_0^{30}(1 - {}_tp_{10}){}_tp_{20}\mu_{20+t}({}_tp_{30} - {}_{30}p_{30})dt$
$+ \int_0^{30}(1 - {}_tp_{30}){}_tp_{20}\mu_{20+t}({}_tp_{10} - {}_{50}p_{10})dt$
$+ \int_{30}^{40}(1 - {}_{30}p_{30}){}_tp_{20}\mu_{20+t}({}_tp_{10} - {}_{50}p_{10})dt$
12.27. 0.2145
12.28. 0.2704
12.29. I 错误，$\overline{A}_{\underset{12}{wxyz}}^{\ \ \ 3} = \int_0^{\infty} v^t{}_tq_{wt}p_{xyz}\mu_{x+t}\overline{A}_{\overline{y+t}:z+t}^{\ \ 1}dt$ 。

II 正确，两边都给出 (50) 与 (60) 相隔 10 年内死亡的概率。

III 正确，${}_{\infty}q_{40:\overline{50:60}}^{1}$ 是 (40) 在 (50) 与 (60) 中的生存者之前死亡的概率，也就是 (40) 第一个死或第二个死亡的概率。
12.31. $\alpha(12)\ddot{a}_{x|y:\overline{10}|} + \beta(12)v^{10}{}_{10}p_y{}_{10}q_x$
12.32.

(1) $\overline{a}_{\overline{10}|} - \overline{a}_{x:\overline{10}|} + v^{10}\overline{a}_y - v^{10}{}_{10}p_x\overline{a}_{x+10:y}$
(2) $\overline{a}_{\overline{10}|}\overline{A}_{xy}^{1} + v^{10}\overline{a}_{x|y}$

12.33. $G = [\frac{2}{3}(\ddot{a}_{x|y} + {}_{n|}\ddot{a}_x) + \frac{1}{3}{}_{n|}\ddot{a}_{xy}]/(0.92 - {}_{n|}A_{xy}^{2})$
12.35. $\overline{A}_{x+n:y}^{\ \ 1}/(\ddot{s}_{x:\overline{n}|} + \ddot{a}_{x+n:y})$

12.36. $\ddot{a}_{xyz}$

12.37. (2) $\ddot{a}_{\overline{xy}:\overline{m}|} + v^m{}_m p_x(1 - {}_m p_y)\ddot{a}_{x+m:\overline{n-m}|}$

12.39. 当 $(x),(y),(z)$ 按这个顺序死亡并且 (z) 在 (y) 死后 10 年内死亡时，在 (z) 死亡时支付 1 的保险的净趸缴保费。

第十三章

13.1. (1) 45　　(2) 2　　(3) 2

13.2. $2500\sqrt{2} + \frac{10000}{\pi}$

13.3. $10000[e^{-1/4} - e^{-1/2} - (e^{-1}/4)]$

13.4. $100^2[e^{-51/100} - e^{-50/100} + e^{-28/100} - e^{-27/100}]$
$+100[e^{-1/4} + e^{-53/100}]$

13.5. $T_{20} - T_{40} - 20l_{70}$

13.6. $\int_{20}^{50} l(x,-x)dx - \int_{70}^{80} l(x,50-x)dx - \int_{30}^{50} l(80,t-80)dt$

13.8. (b) $\frac{\sqrt{2\pi}}{\sqrt{a}}[1 - \Phi(R/\sqrt{a})]\exp[Rt + R^2/(2a)]$

13.9. $a + \frac{(\overline{I}\overline{a})'_{a:\overline{r-a}|}}{\overline{a}'_{a:\overline{r-a}|}}$; $\int_a^r x l_x dx/(T_a - T_r)$

13.11. (1) $e^{-Rx}s(x)(R + \mu_x)$; $1 - e^{-Rx}s(x)$

13.15. 0.02

13.17. (1) $[\Gamma(\alpha)]^{1/\alpha} - \beta$　　(2) 稳定 (静止)

13.18. (2) a　　(3) $(\log b)/c$

第十四章

14.1. $W(T_a - T_r)$

14.2. $ne^{R(t-a)+\tau t}\int_a^r e^{-Ry}s(y)w(y)dy$

14.3.

(2) $l(r,t-r+a)\overline{a}_r^h(f/b)\int_0^b w(r-y)g(t-y)dy$

(3) ${}^{\mathbf{T}}\mathbf{P}(t+u) = ne^{R(t+u-r+a)}s(r)\overline{a}'r(f/b)\int_0^b w(r-y)$
$e^{\tau(t+u-y)}dy = e^{\rho u}\,{}^{\mathbf{T}}\mathbf{P}(t)$.

14.4.

(1) $c(r-a)w(r)e^{\tau t}l(x,t-r+a)\overline{a}_h^r$

(2) ${}^{\mathbf{T}}\mathbf{P}(t+u) = c(r-a)w(r)e^{\tau(t+u)}ne^{R(t-r+a)}s(r)\overline{a}'r$

$= e^{\rho u}\,{}^{\mathbf{T}}\mathbf{P}(t)$

14.5. $e^{\rho t}e^{-(R+\mu)(r-a)}fw(r)\overline{a}'_r$

14.6. $M(x) = \begin{cases} 0 & x < r \\ 1 & x \geq r \end{cases}$

14.7. $fw(r)\overline{a}'_r e^{-(R+\mu)(r-a)}e^{\rho t}\frac{\overline{a}_{\overline{r-a}|\theta}}{r-a}$, 其中 $\theta = \delta - \rho$

14.11.

(1) $\mathbf{P}(t) = \exp(-\delta[r - X(\delta)])\,{}^{\mathbf{T}}\mathbf{P}(t)$;

$(\mathbf{aV})(t) = {}^{\mathbf{T}}\mathbf{P}(t)\overline{a}_{\overline{r-X(\delta)}|\delta} = \mathbf{P}(t)\overline{s}_{\overline{r-X(\delta)}|\delta}$

(2) $\mathbf{P}(t) = {}^{\mathbf{T}}\mathbf{P}(t)$; $(\mathbf{aV})(t) = {}^{\mathbf{T}}\mathbf{P}(t)(r-\mu)$,

其中 $\mu = \int_a^r xm(x)dx$ 。

14.12.

(1) $\frac{fw(r)g(u+r-a)e^{-\delta r}l(r,u)\overline{a}_r^h}{\int_a^r w(y)g(u+y-a)e^{-\delta y}l(y,u)dy}$

(2) $\frac{e^{-\delta x}l(x,u)w(x)g(u+x-a)}{\int_a^r e^{-\delta y}l(y,u)w(y)g(u+y-a)dy}$

14.18. (1) $\overline{a}_{\overline{X(\theta)-a}|\theta}$ (2) $\mu - a$, 其中 $\mu = \int_a^r xm(x)dx$ 。

参考文献

Actuarial Society of America. "International Actuarial Notation. "*Transactions* of the Actuarial Society of America, XLVIII, (1947): 166-176.

Allen, J. M. "On the Relation Between the Theories of Compound Interest and Life Contingencies. "*Journal of the Institute of Actuaries*, XLI, (1907): 305-337.

Allison, G. D. and winklevoss, H. E. "The Interrelationship Among Inflation Rates, Interest Rates, and Pension Costs," *Transactions* of the Society of Actuaries, XXVII, (1975): 197-210.

Arrow, K. J. "Uncertainty and the welfare of Medical Care." *The American Economic Review*, LIII, (1963): 941−973.

Baillie, D. C. "Actuarial Note: The Equation of Equilibrium. "*Transactions* of the Society of Actuaries, III, (1951): 74−81.

Bartlett, D. K. "Excess Ratio Distributions in Risk Theory." *Transactions* of the Society of Actuaries, XVII, (1965): 435−463.

Batten, R. W. 1978. *Mortality Table Construction*. Englewood Cliffs, New Jersey: prentice Hall, Inc.

Beard, R. E. Pentikainen, T., and Pesonen, E. 1977. *Risk Theory*; The Stochastic Basis of Inswrance. (2nded.). London: Methuen.

Becker, D., Bojrab, I., and Buchele, L. "Letters to the Editor. "*The Actuary*, Xi, (1977): 7.

Beekman, J. A. "A Ruin Function Approximation. "*Transactions* of the Society of Actuaries, XXI, (1969): 41−48, with discussion by N. L. Bowers, 275−277.

1974. *Two Stochastic Processes*. New York: Halsted Press.

Beekman, J. A., and Bowers, N. L. "An Approximation to the Finite Time Ruin Function." *Skandinavisk Aktuarietidskrift*, LV,(1972):41—56 and 128—137.

Bellhouse,D. R. and Panjer,H. H. "Stochastic Modelling of Interest Rates with Applications to Life Contingencies." *Journal of Risk and Insurance*,XLVII,(1980):91—110.

Bellman,R. E. ,Kalaba,R. E. ,and Lockett,J. 1966. *Numerical Inversion of the Laplace Transform:Applications to Biology, Economics,Engineering,and Physics.* New York:American Elsevier Publishing Company.

Bicknell,W. S. and Nesbitt,C. J. "Premiums and Reserves in Multiple Decrement Theory." *Transactions* of the Society of Actuaries,VIII,(1956):344—377.

Biggs,J. H. "Alternatives in Variable Annuity Benefit Design." *Transactions* of the Society of Actuaries,XXI,(1969):495—517.

Boermeester, J. M. "Frequency Distribution of Mortality Costs." *Transactions* of the Society of Actuaries,VIII,(1956):1—9.

Bohman,H. ,and Esscher F. "Studies in Risk Theory with Numerical Illustrations Concerning Distribution Functions and Stop—Loss Premiums." *Skandinavisk Aktuarietidskrift*,XLVI,(1963):173—225,and XLVII,(1964):1—40.

Borch,K. "An Attempt to Determine the Optimum Amount of Stop—Loss Reinsurance." *Transactions of the 16th International Congress of Actuaries*,I,(1960):597—610.

1974 *The Mathematical Theory of Insurance.* Lexington, Massachusetts:Lexington Books.

Bowers,N. L. "Expansions of Probability Density Functions as a Sum of Gamma Densities with Applications in Risk Theory." *Transactions* of the Society of Actuaries,XVIII,(1966):125—137.

"An Approximation to the Distribution of Annuity Costs." *Transactions* of the Society of Actuaries,XIX,(1967):295—309.

"An Upper Bound for the Net Stop—Loss Premium." *Transactions* of the Society of Actuaries,XXI,(1969):211—218.

Bowers, N. L., Jr., Hickman, J. C., and Nesbitt, C. J. "Introduction to the Dynamics of Pension Funding." *Transactions* of the Society of Actuaries, XXVIII, (1976): 177—203.

"The Dynamics of Pension Funding: Contribution Theory." *Transactions* of the Society of Actuaries, XXXI, (1979): 93—119.

Brillinger, D. R. "A Justification of Some Common Laws of Mortality." *Transactions* of the Society of Actuaries, XIII, (1961): 116—119.

Bühlmann, H. 1970. *Mathematical Methods in Risk Theory.* New York: Springer.

Chalke, S. A. and Davlin, M. F. "Universal Life Valuation and Nonforfeiture: A Generalized Model." *Transactions* of the Society of Actuaries, XXXV, (1983): 249—298.

Chamberlin, G. "The Proficient Instrument—a New Appraisal of the Commutation Function in the Context of Pension Fund Work." *Journal of the Institute of Actuaries Students' Society*, XXV, (1982): 1—46.

Chapin, W. L. "Toward Adjustable Individual Life Policies." *Transactions* of the Society of Actuaries, XXVIII, (1976): 237—269.

Chiang, C. L. 1968. *Introduction to Stochastic Processes in Biostatistics.* New York: John Wiley and Sons.

Cramér, H. 1930. *On the Mathematical Theory of Risk.* Stockholm: Centraltryckeriet.

Cueto, M. R. "Monetary Values for Ordinary Disability Benefits, Based on Period 2 of the 1952 Intercompany Study of the Society's Committee, with 2½% Interest." *Transactions* of the Society of Actuaries, VI, (1954): 108—177.

Cummins, J. D. 1973. *Development of Life Insurance Surrender Values in the United States.* Homewood, Illinois: Richard D. Trwin.

DeGroot, M. H. 1970. *Optimal Statistical Decisions.* New York: McGraw Hill.

1986. *Probability and Statistics.* Second Edition. Reading, Massachusetts: Addison—wesley.

DeVylder, F. "Martingales and Ruin in a Dynamic Risk Process." *Scandinavian Actuarial Journal*. (1978): 217—225.

Dropkin, L. B. "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records." *Proceedings of the Casualty Actuarial Society*, XLVI, (1959): 165—176.

Dubourdieu, J. 1952. *Théorie Mathématique Des Assurances*. Paris: Gauthier Villars.

Duncan, R. M. "A Retirement System Granting Unit Annuities and Investing in Equities." *Transactions* of the society of Actuaries, IV, (1952): 317—344.

Edelstein, Hermann. "How Accurate are Approximations?" *The Actuary*, XI, (1977): 2.

Elandt—Johnson, R. C. and Johnson, N. L. 1980. *Survival Models and Data Analysis*. New York: John Wiley and Sons.

Fassel, E. G. "Insurance for Face Amount or Reserve if Greater." *Record of the American Institute of Actuaries*, XIX, (1930): 233—246.

"Premium Rates Varying by Policy Size." *Transactions* of the Society of Actuaries, VIII, (1956): 390—419.

Feller, W. 1966. *An Introduction to Probability Theory and Its Applications, Vol. II*. New York: John Wiley and Sons.

1968. *An Introduction to Probability Theory and Its Applications. Vol. I* (3rd ed.). New York: John wiley and Sons.

Fraser, J. C. Miller, W. N., and Sternhell, C. M. "Analysis of Basic Actuarial Theory for Fixed Premium Variable Benefit Life Insurance." *Transactions* of the Society of Actuaries, XXI, (1969): 343—378, and discussions 379—457.

Frasier, W. M. "Second to Die Joint Life Cash Values and Reserves." *The Actuary*, XII, (1978): 3.

Fretwell, R. L. and Hickman, J. C. "Approximate Probability Statements about Life Annuity Costs." *Transactions of the Society of Actuaries*, XVI, (1964): 55—60.

Friedman, Milton and Savage. "The Utility Analysis of

Choices Involving Risk." *Journal of Political Economy*, LVI, (1948):279—304.

Gerber, H. U. "Martingales in Risk Theory." *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, LXXIII, (1973):205—216.

"The Dilemma between Dividends and Safety and a Generalization of the Lundberg—Cramér Formulas." *Scandinavian Actuarial Journal*, LXXIV, (1974):46—57.

"A Probabilistic Model for (Life) Contingencies and a Delta—free Approach to Contingency Reserves." *Transactions of the Society of Actuaries*, XXVIII, (1976):127—141.

1979. *An Introduction to Mathematical Risk Theory*. Huebner Foundation Monograph8, distributed by Richard D. Irwin (Homewood, III.).

"Principles of Premium Calculation and Reinsurance." *Transactions of the 21st International Congress of Actuaries*, I, (1980):137—142.

Gerber, H. U., and Jones, D. A. "Some Practical Considerations in Connection with the Calculation of Stop—Loss Premiums." *Transactions* of the Society of Actuaries, XXVIII, (1976): 215—232.

Gingery, S. W. "Special Investigation of Group Hospital Expense Insurance Experience." *Transactions* of the Society of Actuaries, IV, (1952):44—112.

Goovaerts, M. J., and Devylder, F. "Upper Bounds on Stop—Loss Premiums under Constraints on Claim Size Distributions as Derived from Representation Theorems for Distribution Functions." *Scandinavian Actuarial Journal*, LXXX, (1980): 141—148.

Greenwood, M., and Yule, G. U. "An Inquiry into the Nature of Frequency Distributions Representative of Multiple Happenings with Particular Reference to the Occurrence of Multiple Attacks of Disease or Repeated Accidents." *Journal of the Royal Statistical Society*, LXXXIII, (1920):255—279.

Greville, T. N. E. "Mortality Tables Analyzed by Cause of Death." *Record of the American Institute of Actuaries*, XXXVII, (1948): 283—294. (Discussion in XXXVIII, (1949): 77—79)

"Laws of Mortality which Satisfy a Uniform Seniority Principle." *Journal of the Institute of Actuaries*, LXXXII, (1956): 114—122.

Guertin, A. N. "Life Insurance Premiums." *Journal of Risk and Insurance*, 32, (1965): 23—50.

Halmstad, D. G. "Underwriting the Catastrophe Accident Hazard." *Transactions* of the Society of Actuaries, XXIV, (1972): D408—D418.

"Exact Numerical Procedures in Discrete Risk Theory." *Transactions of the 20th International Congress of Actuaries, III*, (1976): 557—562.

Hattendorf. *Rundschatt der Versicherungen*, XVIII, (1868).

Hickman, J. C. "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory." *Transactions* of the Society of Actuaries, XVI, (1964): 1—16.

Hogg, R. V. and Klugman, S. A. 1984. *Loss Distributions*. New York: John Wiley and Sons.

Hooker, P. F. and Longley—Cook, L. H. 1953. *Life and Other Contingencies, Vol. I*. Cambridge: Cambridge University Press.

1957. *Life and Other Contingencies, Vol. II*. Cambridge: Cambridge University Press.

Horn, R. G. "Life Insurance Earnings and the Release from Risk Policy Reserve." *Transactions* of the Society of Actuaries, XXIII, (1971): 391—399.

Hoskins, J. E. "A New Method of Computing Non—Participating Premiums." *Transactions* of the Actuarial Society of America, XXX, (1929): 140—166.

"Asset Shares and their Relation to Nonforfeiture Values." *Transactions* of the Actuarial Society of America, XL, (1939): 379—393.

Huffman, P. J. "Asset Share Mathematics." *Transactions* of the Society of Actuaries,XXX,(1978):277—296.

Hunter,A. and Phillips,J. T. 1932. *Disability Benefits in Life Insurance Policies*,The Actuarial Society of America,New York.

Institute of Actuaries,Faculty of Actuaries,"The A 1967—70 Tables for Assured Lives. "Institute of Actuaries,Staple Inn Hall,High Holborn,London WCIV7QJ,U. K. ;Faculty of Actuaries,23 St. Andrew Square,Edinburgh EH21AQ,U. K.

Jackson,R. T. "Some Observations on Ordinary Dividends." *Transactions* of the Society of Actuaries,XI,(1959):764—796.

Jenkins,W. A. "An Analysis of Self—selection,among Annuitants,Including Comparisons with Selection among Insured Lives. "*Transactions* of the Actuarial Society of America,XLIV,(1943):227—239.

Jordan,C. W. 1952 1st ed. , 1967 2nd ed. *Life Contingencies.* Chicago:Society of Actuaries.

Kabele,T. G. Discussions of "Expanded Structure for Ordinary Dividends"and "Extensions of Lidstone's Theorem. "*Transactions* of the Society of Actuaries,XXXVI,(1981):360,403.

Kahn,P. M. "Some Remarks on a Recent Paper by Borch." *ASTIN Bulletin*,I,(1961):265—272.

"An Introduction to Collective Risk Theory and Its Application to Stop—Loss Reinsurance. "Transactions of the Society of Actuaries,XIV,(1962):400—425.

Kellison,S. G. 1975. *Fundamentals of Numerical Analysis*, Homewood,Illinois:Richard D. Irwin.

Kendall,M. and Stuart,A. 1977. *The Advanced Theory of Statistics*,*Vol. I.* New York:MacMillan Publishing Co. ,Inc.

Keyfitz,N. 1968. *Introduction to the Mathematics of Population.* Reading,Massachusetts:Addison—Wesley.

1977. *Applied Mathematical Demography.* New York:John Wiley and Sons.

Keyfitz, N. and Beekman, J. 1984. *Demography Through Problems.* New York:Springer—Verlag.

King,G. 1887 1st ed. ,1902 2nd ed. ,*Institute of Actuaries'*

Textbook, Part II. London:Charles and Edwin Layton.

Kischuk, R. K. Discussion of "Fundamentals Pension Funding" by Bowers, Hickman and Nesbitt. *Transactions* of the Society of Actuaries, XXVIII, (1976): 205—211.

Lauer, J. A. 1967. "Apportionable Basis for Net Premiums and Reserves." *Transactions* of the Society of Actuaries, XIX, (1967): 13—23.

Lidstone, G. L. "Changes in Pure Premium Policy Values Consequent upon Variations in the Rate of Interest or Rate of Mortality." *Journal of the Institute of Actuaries*, 39, (1905): 209—252.

Linton, M. A. "Analysis of the Endowment Premium." *Transactions* of the Actuarial Society of America, XX, (1919): 430—439.

Lukacs, E. "On the Mathematical Theory of Risk." *Journal of the Institute of Actuaries Students' Society*, VIII, (1948): 20—37.

Lundberg, O. 1940. *On Random Processes and Their Application to Sickness and Accident Statistics*. Uppsala: Almqvist and Wiksells.

Macarchuk, J. "Some Observations on the Actuarial Aspects of the Insured Variable Annuity." *Transactions* of the Society of Actuaries, XXI, (1969): 529—538.

Maclean, J. B. and Marshall, E. W. 1937. *Distribution of Surplus*. Chicago: Society of Actuaries.

Makeham, W. M. "On the Application of the Theory of the Composition of Decremental Forces." *Journal of the Institute of Actuaries*, XVIII, (1874): 317—322.

Menge, W. O. "Forces of Decrement in a Multiple—Decrement Table" *Record of the American Institute of Actuaries*, XXI, (1932): 41—46.

"Commissioners Reserve Valuation Method." *Record of the American Institute of Actuaries*, XXXV, (1946): 258—300.

Mereu, J. A. "Some Observations on Actuarial Approximations. "*Transactions* of the Society of Actuaries, XIII, (1961): 87—102.

"Annuity Values Directly from Makeham Constants." *Transactions* of the Society of Actuaries. XIV, (1962): 269—286.

"An Algorithm for Computing Expected Stop—Loss Claims under a Group Life Contract. "*Transactions* of the Society of Actuaries, XXIV, (1972): 311—320.

"Letters to the Editor". *The Actuary*, XI, (1977): 8.

Miller. M. D. "Group Weekly Indemnity Continuation Table Study. "*Transactions* of the Society of Actuaries, III, (1951): 31—67.

Miller, W. N. "Variable Life Insurance Product Design." *Journal of the Risk and Insurance Association*, Vol. 38, (1971): 527—542.

Mood, A. M., Graybill, F. A. and Boes, D. C. (1974). *Introduction to the Theory of Statistics*, New York: McGraw Hill.

Myers. R. J. "Actuarial Analysis of Pension Plans under Inflationary Conditions. "*Transactions—16th International Congress of Actuaries*, 1, (1960): 301—315.

National Association of Insurance Commissioners. 1939. "Report of the Committee to Study the Need for a New Mortality Table and Related Topics. "Kansas City: National Association of Insurance Commissioners.

. 1941. "Report and Statements on Nonforfeiture Benefits and Related Matters. "Kansas City: National Association of Insurance Commissioners.

Neill, A. 1977. *Life Contingencies*. London: Heinemann.

Nesbitt, C. J. Discussion of "A Statistical Approach to Premiums and Reserves in Multiple Decrement Theory. "*Transactions* of the Society of Actuaries, XVI(1964): 149—153.

Nesbitt, C. J. and Van Eenam, M. L. "Rate Functions and Their Role in Actuarial Mathematics. "*The Record of the American Institute of Actuaries*, XXXVII, (1948): 202—222.

Noback, J. C. 1969. *Life Insurance Accounting: A Study of* *the Financial Statements of Life Insurance Companies in the United States.* Homewood, Illinois: Richard D. Irwin.

O'Grady, F. T. 1987. *Individual Health Insurance.* Itasca, Illinois: Society of Actuaries.

Panjer, H. H. "The Aggregate Claims Distribution and Stop — Loss Reinsurance." *Transactions* of the Society of Actuaries, XXXII, (1980): 523—535.

Pesonen, E. "On the Calculation of the Generalized Poisson Function." *ASTIN Bulletin*, IV, (1967): 120—128.

Pratt, J. W. "Risk Aversion in the Small and in the Large." *Econometrica*, XXXII, (1964): 122—136.

Preston, S. H., Keyfitz, N. and Schoen, R. "Cause — of — Death Life Tables: Application of a New Technique to Worldwide Data." *Transactions* of the Society of Actuaries, XXV, (1973): 83—109.

Promislow, S. D. "Extensions of Lidstone's Theorem." *Transactions* of the Society of Actuaries, XXXIII, (1981): 367—401.

Rasor, E. A. and Greville, T. N. E. "Complete Annuities." *Transactions* of the Society of Actuaries, IV, (1952): 574—582.

Rasor, E. A. and Myers. R. J. "Actuarial Note: Valuation of the Shares in a share-and-Share-Alike Last Survivor Annuity." *Transactions* of the Society of Actuaries, IV, (1952): 128—130.

Renyi, A. 1962. *Wahrscheinlichkeitsrechnung.* Berlin: Deutscher Verlagder Wissenschaften.

Richardson, C. F. B. "Expense Formulas for Minimum Nonforfeiture Values." *Transactions* of the Society of Actuaries, XXIX, (1977) 209—229.

Scher, E. "Relationships among the Fully Continuous, the Discounted Continuous, and the Semicontinuous Reserve Bases for Ordinary Life Insurance." *Transactions* of the Society of Actuaries, XXVI, (1974): 597—606.

Seal, H. L. 1969. *Stochastic Theory of a Risk Business*, New York: John Wiley and Sons.

"Studies in the History of Probability and Statistics. Multiple Decrements or Competing Risks. "*Biometrika*,LXIV,(1977): 429—439.

"From Aggregate Claims Distribution to Probability of Ruin. "*ASTIN Bulletin*,X,(1978):47—53.

1978. *Survival Probabilities—The Goal of Risk Theory*. New York:John Wiley and Sons.

Simon,L. J. "The Negative Binomial and the Poisson Distributions Compared. "*Proceedings of the Casualty Actuarial Society*,XLVII,(1960):20—24.

Smith,F. C. "The Use of Continuous Functions with Retirement Endowment Plan—Actuarial Note. "*Transactions* of the Society of Actuaries,XIII,(1961):364—367.

Society of Actuaries. 1962. "Monetary Tables of Disability Benefits based on 1952 Disability Study—Period 2 combined with the 1958 CSO Mortality Table,2½% Interest. "Chicago:Society of Actuaries.

"1965—70 Basic Tables. "*Transactions* of the Society of Actuaries,1973 Reports:199—223.

1976. "Report on Actuarial Principles and Practical Problems with regard to Nonforfeiture Requirements. "Chiago:Society of Actuaries.

Spurgeon,E. F. 1992 1st ed. ,1929 2nd ed. , 1932 3rd ed. *Life Contingencies*. Cambridge:Cambridge Univer sity Press.

Steffensen,J. F. "On Hattendorf's Theorem in the Theory of Risk. "*Skandinavisk Aktuarietidskrift*,XII,(1929):1—17.

Takács, L. 1967. *Combinatorial Methods in the Theory of Stochastic Processes*. New York:John Wiley & Sons.

Taylor,R. H. "The Probability Distribution of Life Annuity Reserves and Its Application to a Pension System. "*The Proceedings of the Conference of Actuaries in Public Practice*,II,(1952): 100—150.

Taylor,G. C. "Upper Bounds on Stop—Loss Premiums under Constraints on Claim Size Distributions. "*Scandinavian Actu-*

arial Journal, LXXVII, (1977): 94—105.

Tenenbein, A. and Vanderhoof, I. T. "New Mathematical Laws of Select and Ultimate Mortality." *Transactions* of the Society of Actuaries, XXXII, (1980): 119—158.

Thompson, J. S. "Select and Ultimate Mortality." *Transactions of the 10th International Congress of Actuaries*, II, (1934): 252—263.

Trowbridge, C. L. "Fundamentals of Pension – Funding." *Transactions*, Society of Actuaries, IV, (1952): 17—43.

"Funding of Group Life Insurance." *Transactions*, Society of Actuaries, VII, (1955): 270—284.

"The Unfunded Present Value Family of Pension Funding Methods." *Transactions* of the Society of Actuaries, XV, (1963): 151—169.

Trowbridge, J. R. "Assessmentism—An Alternative to Pensions Funding?" *Journal of the Institute of Actuaries*, 104, (1977): 173—204.

U. S. Department of Health and Welfare. Public Health Service. 1985. *United States Life Tables*: 1979—81. Washington D. C.: Government Printing Office.

White, R. P. and Greville, T. N. E. 1959. "On Computing the Probability that Exactly *k of n* Independent Events will Occur." *Transactions* of the Society of Actuaries, XI, (1959): 88—95.

Willett, A. H. 1951. The *Economic Theory of Risk and Insurance*. Philadelphia: University of Pennsylvania Press.

Williamson, W. R. "Selection." *Transactions* of the Actuarial Society of America, XLIII, (1942): 33—43.

Wooddy, J. 1973. *Study Notes for Risk Theory*. Chicago: Society of Actuaries.

Woolhouse, W. S. B. "On the construction of Tables of Mortality." *Journal of the Institute of Actuaries*, XIII, (1867): 75—102.

Ziock, R. W. "A Proof of Lidstone's Theorem." *Actuarial Research Clearing House*, 1978. 2, (1978): 273—274.

汉英名词对照

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译者的话

原书《 Actuarial Mathematics 》共十九章，其中的第一、二、十一、十二、十三章单独作为《风险理论》译出，现在的这本《精算数学》译自其余十四章。中译本前十章相当于原书的第三至十章及第十四、十五章，系北美精算师学会考试课程“精算数学”(编号为 150) 的指定内容。

原书作者在他们的序言里指出，这本书与先前英语版教材的最显著区别，在于更彻底地运用概率论方法论述寿险数理。为完整起见，书中对决定性观点仍作了介绍，依据决定性方法得出的结果通常都能从随机模型中作为期望值获得。本书的另一个特点是将寿险数学与风险理论有机地结合在一起，采用概率论观点正好为此提供了方便。

根据原书的序言，中译本的各章内容可按下列方式予以分类：

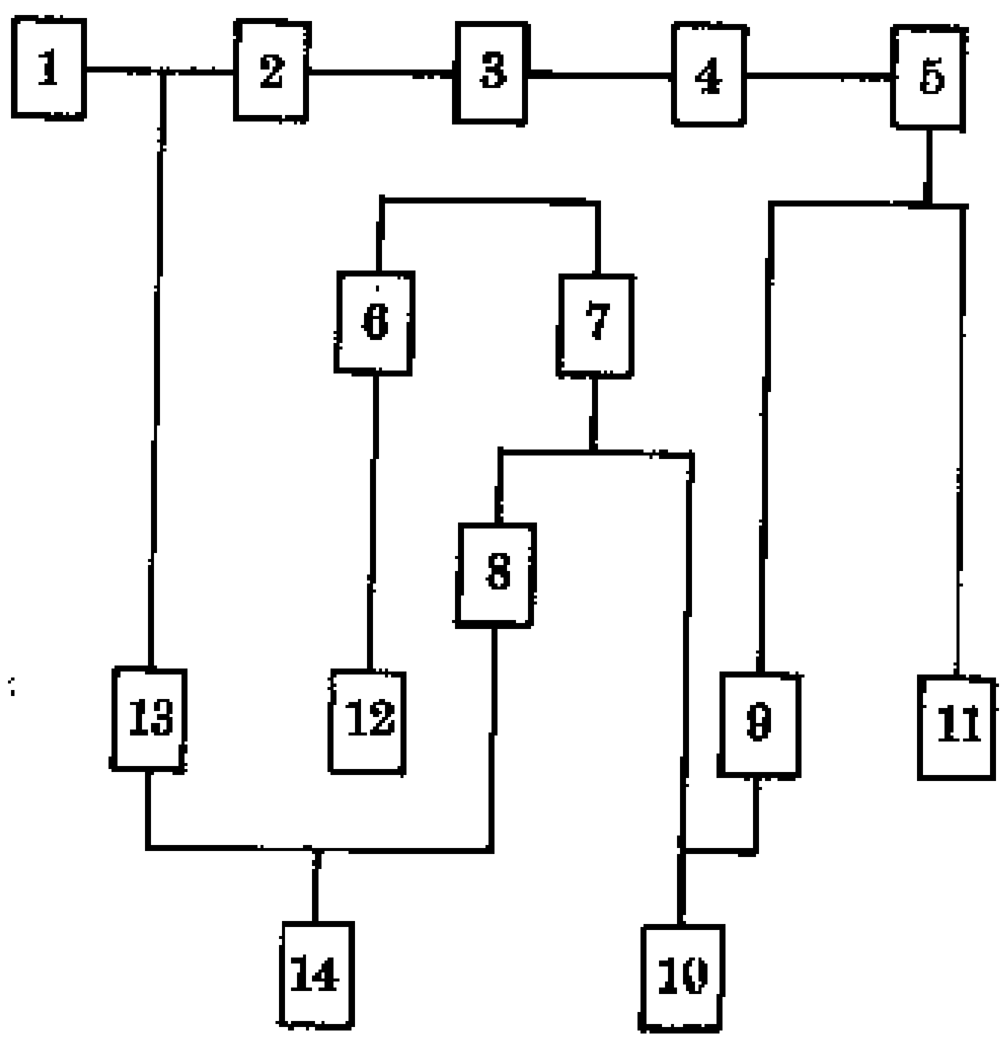
人寿保险	退休养老金保险
一，二，三，四，五，六，七	一，三，六，七，八
九，十，十一，十二	十三，十四

各章前后之间的依赖关系见附图所示。

原书每一章最后都有简短的一节，系有关材料来源与出处的注记，鉴于这些资料国内不易找到，中译本略去了这些内容，但附录 6 中的参考文献仍予保留。原书的定理、例题及图表是按章编号的，为便于查找，译文中都改成按节编号。此外，为方便阅读和理解，译文在个别地方作了适当修饰，譬如：第二章两个习题的部分内容编入了正文；第九章原书曾使用了与第三章不一致的年金记号“$_{m|}\ddot{a}_{x:\overline{n|}}$”，译文统一改成“$_{m|n}\ddot{a}_x$”；第十二章原书有一

个附录，系该章内定理的证明，中译本则直接编入正文；第十四章有几处的叙述顺序稍微作了一些改动等等，在此不一一列举。限于译者水平，译文中难免还会存在一些欠妥甚至违背原意的地方，恳请读者不吝指正。

本书最后四章的部分内容是由雷海斌先生翻译的；上海科技出版社汪沛霖先生在校阅编辑过程中提出了不少改进意见，使得译文纰谬大大减少；雷海斌与蔡志杰先生在打印与排版工作中也付出了艰辛的劳动。对此，译者谨向他们表示深深的敬意和由衷的感谢。



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