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Lauricella Function and the Conformal Mapping of Polygons

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Abstract—In this paper, some progress has been made in solving the problem of calculating the parameters of the Schwarz–Christoffel integral realizing a conformal mapping of a canonical domain onto a polygon. It is shown that an effective solution of this problem can be found by applying the formulas of analytic continuation of the Lauricella function $F_D^{(N)}$, which is a hypergeometric function of N complex variables. Several new formulas for such a continuation of the function $F_D^{(N)}$ are presented that are oriented to the calculation of the parameters of the Schwarz–Christoffel integral in the "crowding" situation. An example of solving the parameter problem for a complicated polygon is given.

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Dedicated to Viktor Pavlovich Maslov

1. INTRODUCTION

The Schwarz–Christoffel integral gives an analytic representation for a conformal mapping $w = \varphi(\zeta)$ of a canonical domain (half-plane, disk, or its exterior) onto a simply connected domain $\mathscr{B}$ bounded by a rectilinear polygonal contour [1]–[3]. The main difficulty in the practical application of this integral is in the need to find unknown parameters entering the integrand; see, about this, [4]–[14]. In this paper, the application of the formulas of the analytic continuation [15]–[17] of the hypergeometric Lauricella function $F_D^{(N)}$ to the efficient calculation of these parameters and to the construction of a conformal mapping of polygons is considered; concerning the function $F_D^{(N)}$, see [18]–[20].

Denote by $w_1, \dots, w_M$ the vertices of a polygon $\mathscr{B}$ and by $\pi\beta_1, \dots, \pi\beta_M$ the area-measured angles at these vertices. Suppose also that the point at infinity is not interior for the domain $\mathscr{B}$. Then the conformal mapping $\mu\colon \mathbb{H}^+ \xrightarrow{\text{conf}} \mathscr{B}$ of the upper half-plane $\mathbb{H}^+ = \{\operatorname{Im}\zeta > 0\}$ onto the domain $\mathscr{B}$ can be written according to [1]–[3] in the form of the following Schwarz–Christoffel integral:

$$w = \mu(\zeta) = \mathscr{K} \int_{\widetilde{\zeta}}^{\zeta} \prod_{j=1}^{M} (t - \zeta_j)^{\beta_j - 1}\, dt + \widetilde{w}, \tag{1.1}$$

where $\mathscr{K}$ and $\widetilde{w} := \mu(\widetilde{\zeta})$ are some constants and the real parameters $\zeta_j := \mu^{-1}(w_j)$ are preimages of the vertices of the M-gon $\mathscr{B}$. In accordance with the Riemann theorem [2], three of the quantities ζ_j can be arbitrarily given (when preserving the order of bypassing the domain) on the real axis $\mathbb{R} = \partial\mathbb{H}^+$. Here the other $M - 3$ parameters ζ_j and the factor $\mathscr{K}$ in the integrand remain still unknown, and their finding is a complicated and urgent problem.

Methods for calculating the unknown parameters of the Schwarz–Christoffel integral are proposed in [4]–[7], [21]–[24]. The problem of finding these parameters becomes especially difficult in the

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crowding situation, i.e., sharply nonuniform arrangement of the quantities ζ_j; see [7], [10], [25], [26]. The term "crowding" comes from the English word "crowd"; in the case under consideration (of a mapping of the half-plane), the crowding on $\mathbb{R}$ is understood in the spherical metric [3], [10]. It should be noted that, when using the Schwarz–Christoffel integral in applications, the situation of crowding occurs most often. In particular, the papers [7], [13], [21]–[23], [27], [28] were devoted to the solution of the problem of crowding, but it is still far from a complete solution.

A significant progress in solving the problem of crowding can be expected with the help of the results of the papers [15]–[17] using the analytic continuation of the Lauricella function $F_D^{(N)}$. Note that the relation of the function $F_D^{(N)}$ to the problem of parameters of the Schwarz–Christoffel integral was mentioned in [22]. The analytic continuation formulas constructed in [15]–[17] and also in Sec. 4 of this paper allow us to develop an algorithm, for calculating the unknown parameters entering (1.1), which is the more effective, the stronger is the crowding.

In the next section, Sec. 2, the necessary information concerning the Lauricella function $F_D^{(N)}$ is presented. In Sec. 3, the known system of equations for the desired parameters of the integral (1.1) is given; iterative methods of Newton type are traditionally used ([5], [10], [13]) to solve this system in the general case. Then the quantities entering this system are expressed using the Lauricella function $F_D^{(N)}$. In Sec. 4, analytic continuation formulas for $F_D^{(N)}$ are presented that enable us to calculate this function with high accuracy in the form of exponentially convergent series. Then, in Sec. 5, the efficiency of the proposed algorithm for calculating the parameters of the Schwarz–Christoffel integral is shown by an example of a domain of a complicated form.

2. LAURICELLA FUNCTION $F_D^{(N)}$

The hypergeometric Lauricella function $F_D^{(N)}(a_1,\dots,a_N;b,c;z_1,\dots,z_N)$ depends on N complex variables $z_1,z_2,\dots,z_N$ and complex parameters $a_1,a_2,\dots,a_N$, b, and c. This function, which is denoted for brevity by $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$, where $\mathbf{a}:=(a_1,\dots,a_N)$ and $\mathbf{z}:=(z_1,\dots,z_N)$ are vectors in $\mathbb{C}^N$, is defined in the form of the following N-fold series, see [18], [19]:

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}|}(a_1)_{k_1}\cdots(a_N)_{k_N}}{(c)_{|\mathbf{k}|}k_1!\cdots k_N!} z_1^{k_1}\cdots z_N^{k_N}; \tag{2.1}$$

here the summation is carried out over the multi-index $\mathbf{k}:=(k_1,\dots,k_N)$ for which $|\mathbf{k}|:=\sum_{j=1}^N k_j$; the Pochhammer symbol $(a)_m$, also denoted by (a,m), is defined by the equality ([19], [29])

$$(a)_m=(a,m):=\frac{\Gamma(a+m)}{\Gamma(a)}=\frac{(-1)^m\Gamma(1-a)}{\Gamma(1-a-m)}, \tag{2.2}$$

and, as is known, this symbol is a product of a finitely many factors of the form

$$(a)_0=1, \quad (a)_m=\begin{cases}(a)_m=a(a+1)\cdots(a+m-1), & m=1,2,\dots\,, \\ (-1)^m[(1-a)(2-a)\cdots((1-a)-m-1)]^{-1}, & m=-1,-2,\dots\,. \end{cases} \tag{2.3}$$

The parameter c in (2.1) does not take integer nonpositive values. The hypergeometric series (2.1) converges in the unit polydisk

$$\mathbb{U}^N:=\{\mathbf{z}\in\mathbb{C}^N : |z_j|<1,\ j=1,\dots,N\}. \tag{2.4}$$

By the *Lauricella function* $F_D^{(N)}$ one means the series (2.1) if the argument $\mathbf{z}$ belongs to the polydisk $\mathbb{U}^N$ or its analytic continuation if $\mathbf{z}\notin\mathbb{U}^N$.

The function $F_D^{(N)}$ satisfies the following system of N linear partial differential equations of the second order in the variables z_j, see [18] and, e.g., [19], [20]:

$$z_j(1-z_j)\frac{\partial^2 u}{\partial {z_j}^2}+(1-z_j)\sum_{\substack{k=1\\k\neq j}}^{N} z_k\,\frac{\partial^2 u}{\partial z_j\,\partial z_k}+[c-(1+a_j+b)z_j]\,\frac{\partial u}{\partial z_j}$$

$$- a_j \sum_{\substack{k=1\\k\neq j}}^{N} z_k \frac{\partial u}{\partial z_k} - a_j b u = 0, \qquad j = 1, \dots, N; \tag{2.5}$$

the parameters $\mathbf{a}$, b, and c are included in expressions for the coefficients of the equations. As is known [18], [19], the general solution of system (2.5) depends only on $(N+1)$ arbitrary complex constants; thus, it is overdetermined. The singular set $\mathscr{M}$ of this system is the union of hyperplanes

$$\mathscr{M}_j^{(\tau)} := \{\mathbf{z} \in \overline{\mathbb{C}}^N : z_j = \tau\},$$

where $\tau \in \mathscr{S} := \{0, 1, \infty\}$, and of the hyperplanes

$$\mathscr{M}_{j,l} := \{\mathbf{z} \in \overline{\mathbb{C}}^N : z_j = z_l\}; \tag{2.6}$$

here $j, l = 1, \dots, N$, $j \neq l$; the extended space $\overline{\mathbb{C}}^N$ is defined by $\overline{\mathbb{C}}^N = \underbrace{\overline{\mathbb{C}} \times \cdots \times \overline{\mathbb{C}}}_{N}$.

In [15]–[17], a system of formulas for the analytic continuation of the Lauricella function beyond the boundary of the unit polydisk $\mathbb{U}^N$ is constructed; these formulas have the form of finite sums

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{m=0}^{N} \lambda_m u_m(\mathbf{a}; b, c; \mathbf{z}), \tag{2.7}$$

where u_m are generalized hypergeometric series that are linear independent solutions of the system of partial differential equations (2.5) which is satisfied by the original series (2.1), and λ_m are some coefficients. Here the intersection of the domains of convergence of the representations (2.7) that were found in the above works [15]–[17] covers $\mathbb{C}^N \setminus \mathbb{U}^N$ in total (except for some hyperplanes).

Note that representations of the form (2.7) are a generalization of the classical formulas for the analytic continuation of the Gauss function of one variable in the case of an arbitrary number of variables, and the functions $u_m(\mathbf{z})$ occurring in (2.7) play the same role for system (2.5) as Kummer's canonical solutions [19], [29] for the hypergeometric equation, which is satisfied by the Gauss function. In Sec. 4, results of [15]–[17] are applied to derive some new formulas of the form (2.7) that are effective for the application under consideration to the parameter problem for the Schwarz–Christoffel integral.

Further, we present the integral representation for the Lauricella function $F_D^{(N)}$ which is indicated, e.g., in [19]. If the parameters b and c satisfy the conditions

$$\operatorname{Re} c > \operatorname{Re} b > 0, \tag{2.8}$$

then the following representation of Euler type holds for the series (2.1):

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{\prod_{j=1}^{N}(1-tz_j)^{a_j}}\, dt. \tag{2.9}$$

The restrictions (2.8) on the parameters of the Lauricella function $F_D^{(N)}$ can be removed if we use contour integrals of special form similar to (2.9), see [19]. Note that the integral on the right-hand side of formula (2.9), regarded as a function of the variables z_j, $j = 1, \dots, N$, is defined and single-valued in the domain

$$\mathbb{L}^N := \{\mathbf{z} \in \mathbb{C}^N : |\arg(1 - z_j)| < \pi, j = 1, \dots, N\}, \tag{2.10}$$

and can be represented in the polydisk $\mathbb{U}^N$ by the series (2.1).

Let us present the following relation used in what follows:

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{|\mathbf{k}_{\nu+1,N}|=0}^{\infty} \frac{(b)_{|\mathbf{k}_{\nu+1,N}|}}{(c)_{|\mathbf{k}_{\nu+1,N}|}} \frac{(a_{\nu+1})_{k_{\nu+1}} \cdots (a_N)_{k_N}}{k_{\nu+1}! \cdots k_N!} z_{\nu+1}^{k_{\nu+1}} \cdots z_N^{k_N}$$
$$\times F_D^{(\nu)}(a_1, \dots, a_\nu; b + |\mathbf{k}_{\nu+1,N}|, c + |\mathbf{k}_{\nu+1,N}|; z_1, \dots, z_\nu), \tag{2.11}$$

where we have introduced the notation $|\mathbf{k}_{\nu+1,N}| := \sum_{j=\nu+1}^{N} k_j$ for the partial sum of the components of the multi-index. Applying the continuation formulas found in [15]–[17] to functions $F_D^{(\nu)}$ used on the right-hand side of (2.11), one can obtain a number of new representations of the form (2.7) for the function $F_D^{(N)}$ (see Sec. 4) that are effective for its calculation beyond the boundary of the unit polydisk $\mathbb{U}^N$, where the series (2.1) converges.

Let us give another two relations useful to calculate $F_D^{(N)}$. Transforming the functions $F_D^{(\nu)}$ in formula (2.11) with the help of identities [19, Sec. 4.2, the formulas (4.2.4)] and changing the order of summation, we arrive at the following representation:

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = \prod_{j=1}^{\nu}(1-z_j)^{-a_j} \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}_{\nu+1,N}|}(c-b)_{|\mathbf{k}_{1,\nu}|}}{(c)_{|\mathbf{k}|}} \frac{(a_1)_{k_1}\cdots(a_N)_{k_N}}{k_1!\cdots k_N!} \times \left(\frac{z_1}{z_1-1}\right)^{k_1}\cdots\left(\frac{z_\nu}{z_\nu-1}\right)^{k_\nu} z_{\nu+1}^{k_{\nu+1}}\cdots z_N^{k_N}, \tag{2.12}$$

which holds in the domain

$$\big\{\mathbf{z}\in\mathbb{C}^N : |z_j| < |z_j-1|,\ |\arg(1-z_j)|<\pi,\ j=1,\dots,\nu,\ |z_\alpha|<1, \alpha=\nu+1,\dots,N\big\},$$

and also at the representation

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = (1-z_1)^{-b} \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b,|\mathbf{k}|)}{(c,|\mathbf{k}|)} \frac{(c-|\mathbf{a}_{1,\nu}|)_{k_1+|\mathbf{k}_{\nu+1,N}|}}{(c-|\mathbf{a}_{1,\nu}|)_{|\mathbf{k}_{\nu+1,N}|}k_1!} \frac{(a_2)_{k_2}\cdots(a_N)_{k_N}}{k_2!\cdots k_N!} \times \left(\frac{z_1}{z_1-1}\right)^{k_1}\left(\frac{z_1-z_2}{z_1-1}\right)^{k_2}\cdots\left(\frac{z_1-z_\nu}{z_1-1}\right)^{k_\nu}\left(\frac{z_{\nu+1}}{1-z_1}\right)^{k_{\nu+1}}\cdots\left(\frac{z_N}{1-z_1}\right)^{k_N}, \tag{2.13}$$

where

$$|\mathbf{a}_{1,\nu}| = \sum_{j=1}^{\nu} a_j, \qquad |\mathbf{k}_{\nu+1,N}| := \sum_{j=\nu+1}^{\nu} k_j, \qquad |\mathbf{k}| := \sum_{j=1}^{N} k_j,$$

which holds in the domain

$$\big\{\mathbf{z}\in\mathbb{C}^N : |z_1|+|z_\alpha| < |z_1-1|,\ \alpha=\nu+1,\dots,N, \quad |z_1-z_j|<|z_1-1|,\ j=2,\dots,\nu;\ |\arg(1-z_1)|<\pi\big\}.$$

Decomposing the integral on the right-hand side of (2.9) into two integrals I_1 and I_2 taken over the intervals $[0,1/2]$ and $[1/2,1]$, performing the substitutions $t=\tau_1/2$, $t=(1+\tau_2)/2$, $\tau_j\in(0,1)$, $j=1,2$, respectively, in these integrals, and applying the representation (2.9) to each of them, we arrive at the following relation:

$$\begin{aligned} F_D^{(N)}&(\mathbf{a};b,c;\mathbf{z}) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\Bigg[\frac{2^{-b}}{b} F_D^{(N+1)}\left(a_1,\dots,a_N,1+b-c;b,1+b;\frac{z_1}{2},\dots,\frac{z_N}{2},\frac{1}{2}\right) \\ &\quad + \frac{2^{b-c}}{c-b}\prod_{j=1}^{N}(1-z_j)^{-a_j} \\ &\quad\times F_D^{(N+1)}\left(a_1,\dots,a_N,1-b;c-b,1+c-b;\frac{z_1}{2(z_1-1)},\dots,\frac{z_N}{2(z_N-1)},\frac{1}{2}\right)\Bigg]. \end{aligned} \tag{2.14}$$

For the efficient calculation of the integrals (2.9) regarded as functions of the variable $\mathbf{z}=(z_1,\dots,z_N)$ in the unit polydisk (2.4), we can use the series (1.1). If $\mathbf{z}\in\mathbb{L}^N\setminus\mathbb{U}^N$, then, to calculate the integrals (2.9), the formulas of analytic continuation of the form (2.7) for the Lauricella function found in [15]–[17] and also in Sec. 4 of the present paper are very efficient.

3. SYSTEM OF EQUATIONS FOR THE PARAMETERS OF THE SCHWARZ–CHRISTOFFEL INTEGRAL

3.1. Derivation of Equations for ζ_j

Assume that the number of vertices w_j of the polygon $\mathscr{B}$ discussed in the introduction is $M = N + 3$. This choice of M is convenient to express below the elements of the system of equations for ζ_j in terms of the Lauricella function with the number of variables equal to N. Denote a mapping of the half-plane $\mathbb{H}^+$ onto the domain $\mathscr{B}$ by $w = \mu(\zeta)$ as above and the inverse function by $\zeta = \mu^{-1}(w)$. Let us enumerate the points w_j, the corresponding angles $\pi\beta_j$, and the preimages $\zeta_j = \mu^{-1}(w_j)$ in such a way that $j = 0, 1, \dots, N + 2$ (see Fig. 1); the quantities β_j are called the angle characteristics.

In Sec. 3.2 devoted to the representation of equations in terms of the Lauricella function, we consider the case in which all vertices w_j, $j = 0, \dots, N + 1$, are finite, i.e., the corresponding angle indicators satisfy the relations $1 \neq \beta_j \in (0, 2]$, $j = 0, \dots, N + 1$. For the sake of completeness, in Sec. 3.1 below, we present an information concerning the case of infinite vertices as well.

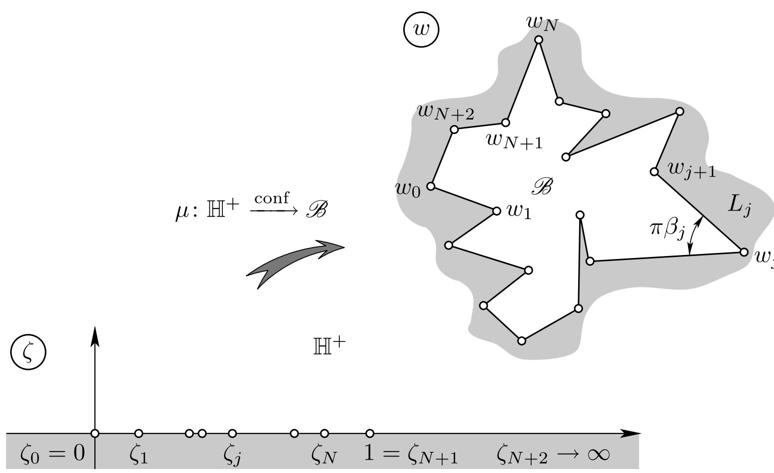

Fig. 1. Conformal mapping of a half-plane onto a polygon.

Let a mapping $w = \mu(\zeta)$ be such that the following conditions uniquely defining this mapping are satisfied:

$$\zeta_0 = \mu^{-1}(w_0) = 0, \qquad \zeta_{N+1} = \mu^{-1}(w_{N+1}) = 1, \qquad \zeta_{N+2} = \mu^{-1}(w_{N+2}) = \infty. \tag{3.1}$$

Taking into account these conditions, we write out the mapping $w = \mu(\zeta)$ in the form of the following Schwarz–Christoffel integral:

$$w = \mu(\zeta) = \mathscr{K}_0 \exp(i \arg(w_{N+2} - w_{N+1})) \int_{\widetilde{\zeta}}^{\zeta} t^{\beta_0 - 1} \left[\prod_{j=1}^{N} (t - \zeta_j)^{\beta_j - 1} \right] (t - 1)^{\beta_{N+1} - 1} \, dt + \widetilde{w}, \tag{3.2}$$

where $\mathscr{K}_0 > 0$ and $\mu(\widetilde{\zeta}) = \widetilde{w}$; recall that the integrand does not contain a factor corresponding to the preimage lying at the point at infinity, see [1], [3]. The unknown parameters in formula (3.2) are the preimages ζ_j, $j = 1, \dots, N$, and the factor $\mathscr{K}_0$ preceding the integral. To find these quantities, we formulate a system of $N + 1$ nonlinear equations, following the known method of [1] and [3].

The assumption that the vertices w_s and w_{s+1} are finite means the following relation:

$$\beta_s, \beta_{s+1} \in (0, 2]. \tag{3.3}$$

Calculating the modulus of the difference $|\mu(\zeta_{s+1}) - \mu(\zeta_s)|$ using formula (3.2) and equating the resulting value to the side length $L_s := |w_{s+1} - w_s|$, we arrive at the following equation:

$$\mathscr{K}_0 I_s(\mathbf{x}) = L_s, \tag{3.4}$$

where $\mathbf{x} = (\zeta_1, \ldots, \zeta_N)$ is a vector composed of the unknown preimages and $I_s(\mathbf{x})$ is the modulus of the integral in formula (3.2) taken between the points ζ_s and ζ_{s+1}:

$$I_s(\mathbf{x}) := \left| \int_{\zeta_s}^{\zeta_{s+1}} t^{\beta_0 - 1} \prod_{j=1}^{N} (t - \zeta_j)^{\beta_j - 1} (t-1)^{\beta_{N+1}-1}\, dt \right|. \tag{3.5}$$

If all vertices w_s are finite, possibly except for w_{N+2}, then equalities (3.4) for $s = 0, \ldots, N$ form the desired system of $N+1$ equations to determine the vector of preimages $\mathbf{x}$ and the coefficient $\mathscr{K}_0$, see [1], [3], [10].

For the case in which some vertex w_k is infinitely distant, i.e., the angle characteristic satisfies the inequality $\beta_k \le 0$, the integrals I_{k-1} and I_k defined by formula (3.5) diverge. Thus, instead of equations (3.4), (3.5) with the numbers $k-1$ and k, one should use other relations that would make it possible to connect the parameters of the mapping (3.2) with geometric parameters of the polygon $\mathscr{B}$. If there is a relation

$$\beta_k \in (-2, 0) \setminus \{-1\}, \tag{3.6}$$

then the pair of equations (3.4), (3.5) with numbers $k-1$ and k can be replaced by the following two equations:

$$\mathscr{K}_0 I_k^{\pm}(\mathbf{x}) = H_k^{\pm}; \tag{3.7}$$

here $I_k^{\pm}$ are integrals of the form

$$I_k^{\pm}(\mathbf{x}) := \left| \int_{\Gamma_k^{\pm}} t^{\beta_0 - 1} \prod_{j=1}^{N} (t - \zeta_j)^{\beta_j - 1} (t-1)^{\beta_{N+1}-1}\, dt \right|, \tag{3.8}$$

where $\Gamma_k^{\pm}$ are rectifiable Jordan contours; Γ_k^{-} starts at the point ζ_{k-1}, continues in the upper half-plane, bypasses the point ζ_k clockwise and returns to the point ζ_{k-1} by points of the lower half-plane; the points ζ_j, $j = 0, 1, \ldots, N+1$, $j \ne k-1, k$ lie outside the domain bounded by Γ_k^{-}. The contour Γ_k^{+} starts at ζ_{k+1}, continues in the upper half-plane, bypasses ζ_k counterclockwise, and returns to the point ζ_{k+1} by points of the upper half-plane; here the points ζ_j, $j = 0, 1, \ldots, N+1$, $j \ne k, k+1$, lie outside the domain bounded by Γ_k^{+}. On the right-hand side of (3.7), the value $H_k^{-} := |z_{k-1}^{*} - z_{k-1}|$ is the distance from the vertex z_{k-1} to its reflection z_{k-1}^{*} relative to the side (z_k, z_{k+1}), and $H_k^{+} := |z_{k+1}^{*} - z_{k+1}|$ is the distance from the vertex z_{k+1} to its reflection z_{k+1}^{*} relative to the side (z_{k-1}, z_k).

If $\beta_k = 0$ or $\beta_k = -1$, and $\beta_{k-1}, \beta_{k+1} \in (0, 2)$, then the pair of equations (3.7) with the numbers $k-1$ and k can be replaced by the following equation (written in the complex form):

$$\mathscr{K}_0 \widetilde{I}_k(\mathbf{x}) = w_{k+1} - w_{k-1}, \tag{3.9}$$

$$\widetilde{I}_k(\mathbf{x}) := \int_{\Gamma_k} t^{\beta_0 - 1} \prod_{k=1}^{N} (t - \zeta_k)^{\beta_k - 1} (t-1)^{\beta_{N+1}-1}\, dt, \tag{3.10}$$

where the integration contour Γ_k connecting ζ_{k-1} and ζ_{k+1} lies (with the exception of endpoints) in $\overline{\mathbb{H}}^{+} \setminus \{\zeta_0, \ldots, \zeta_{N+2}\}$.

3.2. Representation of Equations for Parameters Using the Lauricella Function $F_D^{(N)}$

For the subsequent considerations, it is convenient to introduce the vector $\mathbf{a}$ and numbers b and c associated with the exponents β_j of the angles of the polygon by the formulas

$$\begin{gathered} \mathbf{a} = (a_1, \ldots, a_N), \qquad a_j := 1 - \beta_j, \quad j = 1, \ldots, N, \\ b := -1 + \sum_{j=0}^{N+1} (1 - \beta_j), \qquad c := \sum_{j=0}^{N} (1 - \beta_j), \end{gathered} \tag{3.11}$$

and thus $1-\beta_{N+1}=1+b-c$ and $1-\beta_0=c-\sum_{j=1}^{N}a_j$, and also define the function

$$\mathscr{Y}(\mathbf{x})=\left(\zeta_1,\frac{\zeta_1}{\zeta_2},\dots,\frac{\zeta_1}{\zeta_N}\right), \tag{3.12}$$

depending on N complex variables $(\zeta_1,\dots,\zeta_N)=\mathbf{x}$. Moreover, we will use the notation

$$|\mathbf{a}_{s,l}|:=\sum_{j=s}^{l}a_j, \qquad |\mathbf{a}|:=|\mathbf{a}_{1,N}|=\sum_{j=1}^{N}a_j, \tag{3.13}$$

where the numbers a_j are related to the angle characteristics of the polygon $\mathscr{B}$ by formula (3.11). Further, let us write the quantities (3.3) occurring in the left-hand side of (3.4) using the Lauricella function.

As above, assume that conditions (3.3) are satisfied for all $s=0,1,\dots,N+1$. Transforming the points ζ_s and ζ_{s+1} by the change of variables $t(\xi)=\zeta_s+(\zeta_{s+1}-\zeta_s)\xi$ to the ends of the unit closed interval $\xi\in[0,1]$ and using the integral representation (2.9), we write out the integrals (3.5) in the following form:

$$I_k(\mathbf{x})=C_k\mathscr{I}_k(\mathbf{a};b,c;\mathbf{x}), \qquad k=0,\dots,N, \tag{3.14}$$

where the coefficients C_k are given by the equalities

$$\begin{gathered} C_0=\frac{\Gamma(1+|\mathbf{a}|-c)\Gamma(1-a_1)}{\Gamma(2+|\mathbf{a}_{2,N}|-c)}\,, \qquad C_N=\frac{\Gamma(1-a_N)\Gamma(c-b)}{\Gamma(1+c-b-a_N)}\,, \\ C_k=\frac{\Gamma(1-a_k)\Gamma(1-a_{k+1})}{\Gamma(2-a_k-a_{k+1})}\,, \qquad k=1,\dots,N-1, \end{gathered} \tag{3.15}$$

and the quantities $\mathscr{I}_k(\mathbf{x})=\mathscr{I}_k(\mathbf{a};b,c;\mathbf{x})$, $k=0,1,\dots,N$, are defined using the Lauricella function $F_D^{(N)}$ by the formulas

$$\mathscr{I}_0(\mathbf{a};b,c;\mathbf{x}):=\zeta_1^{1+|\mathbf{a}_{2,N}|-c}\left(\prod_{j=2}^{N}\zeta_j^{-a_j}\right)F_D^{(N)}(\mathbf{a}_0;b_0,c_0;\mathbf{x}_0), \tag{3.16}$$

$$\begin{aligned} \mathscr{I}_k(\mathbf{a};b,c;\mathbf{x}):=\ &\zeta_k^{|\mathbf{a}|-c}(\zeta_{k+1}-\zeta_k)^{1-a_k-a_{k+1}}(1-\zeta_k)^{c-b-1} \\ &\times\prod_{j=1}^{k-1}(\zeta_k-\zeta_j)^{-a_j}\prod_{j=k+2}^{N}(\zeta_j-\zeta_k)^{-a_j}F_D^{(N)}(\mathbf{a}_k;b_k,c_k;\mathscr{Y}(\mathbf{1}-\mathbf{x}_k)), \end{aligned} \tag{3.17}$$

$$\mathscr{I}_N(\mathbf{a};b,c;\mathbf{x}):=\zeta_N^{|\mathbf{a}|-c}(1-\zeta_N)^{c-b-a_N}\prod_{j=1}^{N-1}(\zeta_N-\zeta_j)^{-a_j}F_D^{(N)}(\mathbf{a}_N;b_N,c_N;\mathscr{Y}(\mathbf{1}-\mathbf{x}_N)); \tag{3.18}$$

here the parameters $\mathbf{a}_k$, b_k, and c_k are written out using the quantities (3.11):

$$\mathbf{a}_0:=(a_2,\dots,a_N,1+b-c), \qquad b_0:=1+|\mathbf{a}|-c, \qquad c_0:=2+|\mathbf{a}_{2,N}|-c, \tag{3.19}$$

$$\begin{gathered} \mathbf{a}_k:=(c-|\mathbf{a}|,a_1,\dots,a_{k-1},a_{k+2},\dots,a_N,1+b-c), \\ b_k:=1-a_k, \quad c_k:=2-a_k-a_{k+1}, \qquad k=1,\dots,N, \end{gathered} \tag{3.20}$$

$$\mathbf{a}_N:=\Big(c-|\mathbf{a}|,a_1,\dots,a_{N-1}\Big), \qquad b_N:=1-a_N, \qquad c_N:=1+c-b-a_N, \tag{3.21}$$

the vectors $\mathbf{x}_k$ are expressed using the preimages ζ_j, $j=1,\dots,N$, according to the following formulas:

$$\begin{gathered} \mathbf{x}_0:=\left(\frac{\zeta_1}{\zeta_2},\dots,\frac{\zeta_1}{\zeta_N},\zeta_1\right), \qquad \mathbf{x}_N:=\left(\frac{1}{\zeta_N},\frac{\zeta_1}{\zeta_N},\dots,\frac{\zeta_{N-1}}{\zeta_N}\right), \\ \mathbf{x}_k:=\left(\frac{\zeta_{k+1}}{\zeta_k},\frac{\zeta_1}{\zeta_k},\dots,\frac{\zeta_{k-1}}{\zeta_k},\frac{\zeta_{k+2}}{\zeta_k},\dots,\frac{\zeta_N}{\zeta_k},\frac{1}{\zeta_k}\right), \qquad k=1,\dots,N-1, \end{gathered} \tag{3.22}$$

and $\mathscr{Y}(\mathbf{x})$ is determined from (3.12).

The case in which one or more vertices w_j, $j = 0, \dots, N+2$, are points at infinity is not considered in the present paper. We only note that the left-hand sides of equations (3.7) and (3.9) can also be represented in terms of the Lauricella function. At the same time, to write out the integrals in formula (3.7) using this function, it is necessary to use the representations for this function that are specified, e.g., in [19] with the help of integrals over loop-shaped Pochhammer contours rather than the Euler-type representation (2.9). To represent the integral in formula (3.10), where $\beta_k = 0$ or $\beta_k = -1$ and $\beta_{k-1}, \beta_{k+1} \in (0,1)$, using the function $F_D^{(N)}$, one can use Jacobi-type relations for this functions; see [15].

3.3. On the Numerical Solution of the System of Equations for the Parameters and the Computing of the Mapping Function $\mu(\zeta)$

To solve the system of nonlinear transcendental equations specified in Sec. 3.1 for the parameters, numerical methods of the type of Newton's iterative procedure are applied (see, e.g., [5], [10], [13]). The main difficulties encountered in the implementation of these numerical algorithms are related, first, to high-precision calculation of the integrals arising in it and, second, to the choice of a good initial approximation for the unknown parameters.

In the previous subsection, Sec. 3.2, the elements of the system of equations were expressed in terms of integrals of the form (2.9), and thus the problem of calculating the integrals occurring in the system is reduced to the question of efficient calculation of the Lauricella function $F_D^{(N)}$. If the vector argument $\mathscr{Y}(\mathbf{1} - \mathbf{x}_k)$ of the function $F_D^{(N)}$ entering the corresponding expression (3.17), (3.18) lies in the unit polydisk, then the N-fold series (2.1) can be used to calculate the function. At the same time, as a rule, when implementing the Newton method for solving the system, considered in Subsecs. 3.1, 3.2, the opposite situation occurs in which $\mathscr{Y}(\mathbf{1} - \mathbf{x}_k) \notin \mathbb{U}^N$. In some cases, the transformation formulas (2.12), (2.13) are useful. However, a universal algorithm for computing the functions $F_D^{(N)}(\mathscr{Y}(\mathbf{1} - \mathbf{x}_k))$ is provided by the analytic continuation formulas of the form (2.7). The formulas of this continuation required for the development of an algorithm for solving the problem of parameters for the Schwarz–Christoffel integral are obtained in [15]–[17] and also in Sec. 4.

The problem of constructing the initial approximation is solved in the present paper by the method of continuation with respect to the parameter. Note that, in [21], [22], [30], to construct approximations, it was proposed to use asymptotics for the parameters which, in some cases, can be effectively constructed using the theory of variation of the mapping function when deforming the domain [31].

After the parameters of the conformal mapping (3.2) are defined, the next problem is to calculate the mapping function $w = \mu(\zeta)$ itself. Suppose that some vertex w_m is finite. Then, assuming that $\widetilde{\zeta} = \zeta_m$ and $\widetilde{w} = w_m$ in the integral (3.2) and changing the variable in this integral, $t = \zeta_m + (\zeta - \zeta_m)\xi$, $\xi \in (0,1)$, we arrive at the following representation, taking into account the Euler-type formula (2.9):

$$\begin{aligned}\mu(\zeta) &= w_m + \mathscr{Q}_m e^{i\pi\theta_m}(\zeta - \zeta_m)^{1-a_m} \\ &\quad\times F_D^{(N+1)}(c - |\mathbf{a}|, a_1, \dots, a_{m-1}, a_{m+1}, \dots, a_N, 1 + b - c; 1 - a_m, 2 - a_m; \mathscr{Y}(\mathbf{1} - \mathbf{X}_m)), \end{aligned} \tag{3.23}$$

where the parameters a_j, b, and c are defined in formula (3.11), the function $\mathscr{Y}$ is of the form (3.12), $|\mathbf{a}| = \sum_{j=1}^N a_j$, and the constants $\mathscr{Q}_m$ and θ_m and the vector $\mathbf{X}_m(\zeta)$ are given by the equalities

$$\mathscr{Q}_m = \mathscr{K}_0 (1 - a_m)^{-1} \zeta_m^{|\mathbf{a}| - c} \prod_{j=1}^{m-1} (\zeta_m - \zeta_j)^{-a_j} \Bigg(\prod_{j=m+1}^{N} (\zeta_j - \zeta_m)^{-a_j} \Bigg) (1 - \zeta_m)^{c-b-1},$$

$$\theta_m = \frac{\arg(w_{N+2} - w_{N+1})}{\pi} + c - b - \Bigg(\sum_{l=m+1}^{N} a_l \Bigg) - 1,$$

$$\mathbf{X}_m(\zeta) = \left(\frac{\zeta}{\zeta_m}, \frac{\zeta_1}{\zeta_m}, \dots, \frac{\zeta_{m-1}}{\zeta_m}, \frac{\zeta_{m+1}}{\zeta_m}, \dots, \frac{\zeta_N}{\zeta_m}, \frac{1}{\zeta_m} \right);$$

here $m = 1, \dots, N$, and the cases $m = 0$ and $m = N + 1$ require simple modifications of these equalities and of (3.23). To calculate the conformal mapping using formula (3.23) depending on the domain of variation of ζ, we use representations (2.1), (2.12), and (2.13) and formulas for the analytic continuation of the Lauricella function; see Sec. 4.

4. FORMULAS FOR THE ANALYTIC CONTINUATION OF THE LAURICELLA FUNCTION

In this section, formulas are constructed for the analytic continuation to the domain in which $|z_j| > 1$ or $|1 - z_j| < 1$, $j = 1, \dots, \nu$, and the relations $|z_j| < 1$, $j = \nu + 1, \dots, N$, hold for the other variables. Here we consider the case in which some of the variables with the numbers $j = 1, \dots, \nu$ form one or more sets in such a way that, inside every set, the distances between variables are small as compared with their moduli or the distances to one. In this connection, it will be convenient for us to modify the notation for the variables and parameters of the Lauricella function.

4.1. Notation in Use

Let us represent the argument $\mathbf{z} = (z_1, \dots, z_N)$ of the function $F_D^{(N)}$ in the form of the union of $q + 1$ families $\mathbf{z}^{(s)}$, $s = 1, \dots, q + 1$:

$$\mathbf{z} = (\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \dots, \mathbf{z}^{(q+1)}), \quad \mathbf{z}^{(s)} := (z_1^{(s)}, z_2^{(s)}, \dots, z_{p_s}^{(s)}), \qquad s = 1, \dots, q + 1, \tag{4.1}$$

where the family $\mathbf{z}^{(s)}$ with the number s consists of p_s elements, and thus

$$\sum_{s=1}^{q+1} p_s = N, \qquad \sum_{s=1}^{q} p_s = \nu, \qquad p_{q+1} = N - \nu.$$

It is easy to establish a connection between the notation (4.1) and the former one used in formula (2.1):

$$z_j^{(s)} = z_m, \qquad m = \sum_{l=1}^{s-1} p_s + j, \tag{4.2}$$

and, therefore, for example,

$$\mathbf{z}^{(1)} = (z_1, \dots, z_{p_1}), \qquad \mathbf{z}^{(2)} = (z_{p_1+1}, \dots, z_{p_1+p_2}), \qquad \mathbf{z}^{(q+1)} = (z_{\nu+1}, \dots, z_N).$$

Assume that the components of the vector $\mathbf{z}^{(s)}$, i.e., the variables $z_j^{(s)}$ with the same superscript, are "close enough" in the sense that will be refined below in Theorems 1, 2 concerning the analytic continuation. We redenote accordingly the components of the vector parameter $\mathbf{a}$ of the Lauricella function and the components of the multi-index $\mathbf{k}$ in formula (2.1):

$$\begin{aligned} \mathbf{a} &= (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(q+1)}), \qquad \mathbf{a}^{(s)} := (a_1^{(s)}, a_2^{(s)}, \dots, a_{p_s}^{(s)}), \\ \mathbf{k} &= (\mathbf{k}^{(1)}, \dots, \mathbf{k}^{(q+1)}), \qquad \mathbf{k}^{(s)} := (k_1^{(s)}, k_2^{(s)}, \dots, k_{p_s}^{(s)}), \end{aligned} \tag{4.3}$$

and thus

$$a_j^{(s)} = a_m, \quad k_j^{(s)} = k_m, \qquad m = \sum_{l=1}^{s-1} p_s + j. \tag{4.4}$$

In what follows, we use the following notation for the partial sums of the vectors $\mathbf{a} = (a_1, \dots, a_N)$ and $\mathbf{k} = (k_1, \dots, k_N)$:

$$|\mathbf{a}_{s,l}| := \sum_{k=s}^{l} a_k, \quad |\mathbf{a}| := |\mathbf{a}_{1,N}|, \qquad |\mathbf{k}_{s,l}| := \sum_{n=s}^{l} k_n, \quad |\mathbf{k}| := |\mathbf{k}_{1,N}|. \tag{4.5}$$

This and (4.3) imply, for example, the relations

$$|\mathbf{a}_{1,p_s}^{(s)}| = \sum_{l=1}^{p_s} a_l^{(s)}, \quad |\mathbf{k}_{1,p_s}^{(s)}| = \sum_{l=1}^{p_s} k_l^{(s)}, \qquad \sum_{s=1}^{q+1} |\mathbf{a}_{1,p_s}^{(s)}| = |\mathbf{a}|, \quad \sum_{s=1}^{q+1} |\mathbf{k}_{1,p_s}^{(s)}| = |\mathbf{k}|, \tag{4.6}$$

which are needed below.

In this section, formulas are constructed for the analytic continuation of the Lauricella function $F_D^{(N)}$ to a neighborhood of hyperplanes defined by formula (2.6), as well as to their intersections, more precisely, in the new notation (4.1), the formulas that are valid near the set $\bigcap_{s=1}^{q} \mathscr{M}_s$, where

$$\mathscr{M}_s := \{z_1^{(s)} = \dots = z_{p_s}^{(s)}\}, \qquad s = 1, \dots, q. \tag{4.7}$$

Here we separately consider neighborhoods of the points $(\infty, \dots, \infty)$ and $(1, \dots, 1)$; see Sec. 4.2 and Sec. 4.3, respectively.

4.2. Analytic Continuation in the Case of Variables with Large Modulus

Introduce the domain $\mathbb{O}^N = \mathbb{O}^N(p_1, \dots, p_{q+1})$ as the intersection :

$$\mathbb{O}^N = \bigcap_{j=0}^{q} \mathbb{O}_j^N, \tag{4.8}$$

where the domains $\mathbb{O}_j^N = \mathbb{O}_j^N(p_1, \dots, p_{q+1})$, $j = 0, \dots, q$, are defined by the following formulas:

$$\begin{aligned} \mathbb{O}_j^N := \big\{ \mathbf{z} \in \mathbb{C}^N : {} & |z_1^{(\alpha)}| > 1 + |z_1^{(\alpha)} - z_n^{(\alpha)}|,\ \alpha = 1, \dots, j,\ n = 2, \dots, p_\alpha, \\ & |z_1^{(j)}| > 1 + |z_n^{(\alpha)} - z_1^{(\alpha)}| + |z_1^{(j)} - z_m^{(j)}|,\ \alpha = j+1, \dots, q, \\ & n = 2, \dots, p_\alpha,\ m = 2, \dots, p_j,\ |z_l^{(q+1)}| < 1,\ l = 1, \dots, p_{q+1}; \\ & |\arg(-z_k)| < \pi,\ k = 1, \dots, q \big\}, \qquad j = 1, \dots, q. \end{aligned} \tag{4.9}$$

$$\begin{aligned} \mathbb{O}_0^N := \big\{ \mathbf{z} \in \mathbb{C}^N : {} & |z_1^{(\alpha)}| > 1 + |z_1^{(\alpha)} - z_n^{(\alpha)}|,\ \alpha = 1, \dots, q,\ n = 2, \dots, p_\alpha; \\ & |z_l^{(q+1)}| < 1,\ l = 1, \dots, p_{q+1},\ |\arg(-z_k)| < \pi,\ k = 1, \dots, q \big\}. \end{aligned} \tag{4.10}$$

The following assertion holds, which establishes formulas for the analytic continuation of the Lauricella function $F_D^{(N)}$ that are valid near the intersection $\bigcap_{s=1}^{q} \mathscr{M}_s$ of the hyperplanes (4.7) and correspond to the case in which all variables $z_l^{(s)}$, $l = 1, \dots, p_s$, $s = 1, \dots, q$, are large in modulus and the other variables $z_l^{(q+1)}$, $l = 1, \dots, p_{q+1}$, have the modulus less than one.

Theorem 1. *If none of the numbers* $(b - \sum_{s=1}^{j} |\mathbf{a}_{1,p_s}^{(s)}|)$, $j = 1, \dots, q$, *is an integer, then the analytic continuation of the series* (2.1) *to the domain* $\mathbb{O}^N$ *defined by the relations* (4.8)–(4.10) *is given by the formula*

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{j=0}^{q} \sigma_j \mathscr{U}_j^{(\infty)}(\mathbf{a}; b, c; \mathbf{z}), \tag{4.11}$$

where the functions $\mathscr{U}_0^{(\infty)}$ *and* $\mathscr{U}_j^{(\infty)}$ *are defined by the equalities*

$$\mathscr{U}_0^{(\infty)}(\mathbf{a}; b, c; \mathbf{z}) = \left(\prod_{s=1}^{q} (-z_1^{(s)})^{-|\mathbf{a}_{1,p_s}^{(s)}|} \right) \sum_{|\mathbf{k}|=0}^{\infty} \Lambda_0(\mathbf{a}; b, c; \mathbf{k}) \mathbf{Z}_0^{\mathbf{k}}, \tag{4.12}$$

$$\mathscr{U}_j^{(\infty)}(\mathbf{a}; b, c; \mathbf{z})$$

$$= (-z_1^{(j)})^{\sum_{s=1}^{j-1}|\mathbf{a}_{1,p_s}^{(s)}|-b}\biggl(\prod_{s=1}^{j-1}(-z_1^{(s)})^{-|\mathbf{a}_{1,p_s}^{(s)}|}\biggr)\sum_{|\mathbf{k}|=0}^{\infty}\Lambda_j(\mathbf{a};b,c;\mathbf{k})\mathbf{Z}_j^{\mathbf{k}}, \qquad j=1,\dots,q; \tag{4.13}$$

here the coefficients Λ_j *are determined by the following equalities*:

$$\begin{aligned}\Lambda_0(\mathbf{a};b,c;\mathbf{k}) &= \frac{(1+\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|-c, \sum_{s=1}^{q}k_1^{(s)}-|\mathbf{k}^{(q+1)}|)}{(1+\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|-b, \sum_{s=1}^{q}k_1^{(s)}-|\mathbf{k}^{(q+1)}|)}\\ &\quad\times\prod_{s=1}^{q}\biggl[\frac{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_s}^{(s)}|)}{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{2,p_s}^{(s)}|)k_1^{(s)}!}\prod_{l=2}^{p_s}\frac{(a_l^{(s)},k_l^{(s)})}{k_l^{(s)}!}\biggr]\prod_{l=1}^{p_{q+1}}\frac{(a_l^{(q+1)},k_l^{(q+1)})}{k_l^{(q+1)}!},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\Lambda_j(\mathbf{a};b,c;\mathbf{k}) &= \frac{(b-\sum_{s=1}^{j-1}|\mathbf{a}_{1,p_s}^{(s)}|, \sum_{s=j}^{q+1}|\mathbf{k}_{2,p_s}^{(s)}|+\sum_{s=j}^{q+1}k_1^{(s)}-\sum_{s=1}^{j-1}k_1^{(s)})}{(|\mathbf{a}_{1,p_j}^{(j)}|,|\mathbf{k}_{2,p_j}^{(j)}|)(1+b-\sum_{s=1}^{j}|\mathbf{a}_{1,p_s}^{(s)}|, \sum_{s=j+1}^{q+1}|\mathbf{k}_{2,p_s}^{(s)}|+\sum_{s=j}^{q+1}k_1^{(s)}-\sum_{s=1}^{j-1}k_1^{(s)})}\\ &\quad\times\biggl(\prod_{s=1}^{q}\prod_{l=2}^{p_s}\frac{(a_l^{(s)},k_l^{(s)})}{k_l^{(s)}!}\biggr)\biggl(\frac{(1-c+b,k_1^{(j)})}{k_1^{(j)}!}\prod_{s=1,s\neq j}^{q}\frac{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_s}^{(s)}|)}{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{2,p_s}^{(s)}|)k_1^{(s)}!}\biggr)\\ &\quad\times\prod_{l=1}^{p_{q+1}}\frac{(a_l^{(q+1)},k_l^{(q+1)})}{k_l^{(q+1)}!}.\end{aligned} \tag{4.15}$$

In formulas (4.12) *and* (4.13), *the following notation* $\mathbf{Z}_j^{\mathbf{k}}=\prod_{s=1}^{q+1}\prod_{l=1}^{p_s}(Z_j^{(s,l)})^{k_l^{(s)}}$ *is used, where the numbers* $Z_j^{(s,l)}$ *are the elements of the vectors* $Z_j^{(s)}=\{Z_j^{(s,1)},\dots,Z_j^{(s,p_s)}\}$, $j=0,\dots,q$, $s=1,\dots,q+1$, *defined for* $j=0$ *by the equalities*

$$\begin{aligned}Z_0^{(s)} &= \biggl(\frac{1}{z_1^{(s)}},\frac{z_1^{(s)}-z_2^{(s)}}{z_1^{(s)}},\dots,\frac{z_1^{(s)}-z_{p_s}^{(s)}}{z_1^{(s)}}\biggr), \qquad s=1,\dots,q,\\ Z_0^{(q+1)} &= \bigl(z_1^{(q+1)},\dots,z_{p_{q+1}}^{(q+1)}\bigr), \qquad j=q+1,\end{aligned} \tag{4.16}$$

and, for $j=1,\dots,q$, *by the equalities*

$$\begin{aligned}Z_j^{(s)} &= \biggl(\frac{z_1^{(j)}}{z_1^{(s)}},\frac{z_1^{(s)}-z_2^{(s)}}{z_1^{(s)}},\dots,\frac{z_1^{(s)}-z_{p_s}^{(s)}}{z_1^{(s)}}\biggr), \qquad s=1,\dots,j-1,\\ Z_j^{(j)} &= \biggl(\frac{1}{z_1^{(j)}},\frac{z_1^{(j)}-z_2^{(j)}}{z_1^{(j)}},\dots,\frac{z_1^{(j)}-z_{p_j}^{(j)}}{z_1^{(j)}}\biggr), \qquad s=j,\\ Z_j^{(s)} &= \biggl(\frac{z_1^{(s)}}{z_1^{(j)}},\frac{z_2^{(s)}-z_1^{(s)}}{z_1^{(j)}},\dots,\frac{z_{p_s}^{(s)}-z_1^{(s)}}{z_1^{(j)}}\biggr), \qquad s=j+1,\dots,q,\\ Z_j^{(q+1)} &= \biggl(\frac{z_1^{(q+1)}}{z_1^{(j)}},\dots,\frac{z_{p_{q+1}}^{(q+1)}}{z_1^{(j)}}\biggr), \qquad s=q+1;\end{aligned} \tag{4.17}$$

the coefficients σ_j *have the form*

$$\begin{aligned}\sigma_0 &= \frac{\Gamma(c)\Gamma(b-\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|)}{\Gamma(b)\Gamma(c-\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|)},\\ \sigma_j &= \frac{\Gamma(c)\Gamma(b-\sum_{s=1}^{j-1}|\mathbf{a}_{1,p_s}^{(s)}|)\Gamma(\sum_{s=1}^{j}|\mathbf{a}_{1,p_s}^{(s)}|-b)}{\Gamma(|\mathbf{a}_{1,p_j}^{(j)}|)\Gamma(b)\Gamma(c-b)}, \qquad j=1,\dots,q.\end{aligned} \tag{4.18}$$

The functions (4.12) *and* (4.13) *are linearly independent solutions of system* (2.5) *in the domain* $\mathbb{O}^N$.

To prove the theorem, we use the decomposition (2.11) and note that, taking into account the notation (4.1)–(4.5) of Sec. 4.1, we obtain the following relations for the parameters and variables of the Lauricella function:

$$\begin{aligned} \mathbf{a}^{(q+1)} &= (a_1^{(q+1)}, \dots, a_{p_{q+1}}^{(q+1)}) = (a_{\nu+1}, \dots, a_N), \\ \mathbf{z}^{(q+1)} &= (z_1^{(q+1)}, \dots, z_{p_{q+1}}^{(q+1)}) = (z_{\nu+1}, \dots, z_N), \\ \mathbf{k}^{(q+1)} &= (k_1^{(q+1)}, \dots, k_{p_{q+1}}^{(q+1)}) = (k_{\nu+1}, \dots, k_N), \\ |\mathbf{k}_{\nu+1,N}| &= \sum_{s=\nu+1}^{N} k_s = \sum_{s=1}^{p_{q+1}} k_s^{(q+1)} = |\mathbf{k}^{(q+1)}|, \end{aligned} \tag{4.19}$$

and the function $F_D^{(\nu)}$ can be represented in the form

$$\begin{aligned} &F_D^{(\nu)}(a_1, \dots, a_\nu; b + |\mathbf{k}_{\nu+1,N}|, c + |\mathbf{k}_{\nu+1,N}|; z_1, \dots, z_\nu) \\ &\qquad = F_D^{(\nu)}(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(q)}; b + |\mathbf{k}^{(q+1)}|, c + |\mathbf{k}^{(q+1)}|; \mathbf{z}^{(1)}, \dots, \mathbf{z}^{(q)}). \end{aligned} \tag{4.20}$$

Rewriting formula (2.11), performing the substitutions (4.19) and (4.20), and applying to the function $F_D^{(\nu)}$ in (2.11) and (4.20) the formulas of analytic continuation that are established in [16, Theorem 2.2], we arrive at the assertion of the theorem. The assertion claiming that the functions (4.12) and (4.13) are linearly independent solutions of the system of equations which is satisfied by the Lauricella function is verified by their direct substitution into (2.5). We do not present cumbersome manipulations with the help of which the above reasoning is made.

4.3. Analytic Extension for the Case of Variables Close to One

Introduce the domain $\mathbb{P}^N = \mathbb{P}^N(p_1, \dots, p_{q+1})$ as the intersection

$$\mathbb{P}^N = \bigcap_{j=0}^{q} \mathbb{P}_j^N, \tag{4.21}$$

where the domains $\mathbb{P}_j^N = \mathbb{P}_j^N(p_1, \dots, p_{q+1})$, $j = 0, \dots, q$, are defined by the following formulas:

$$\begin{aligned} \mathbb{P}_j^N := \big\{ \mathbf{z} \in \mathbb{C}^N : {}& |1 - z_1^{(j)}| + |z_1^{(\alpha)} - z_n^{(\alpha)}| < 1,\ \alpha = 1, \dots, j,\ n = 2, \dots, p_\alpha; \\ & |1 - z_1^{(j)}| + |z_1^{(j)} - z_m^{(j)}| + |z_1^{(\alpha)} - z_n^{(\alpha)}| < |1 - z_1^{(\alpha)}|,\ m = 2, \dots, p_j, \\ & \alpha = j+1, \dots, q,\ n = 2, \dots, p_\alpha;\ |z_l^{(q+1)}| < 1,\ l = 1, \dots, p_{q+1}, \\ & |\arg(1 - z_k)| < \pi,\ k = 1, \dots, q \big\}, \qquad j = 1, \dots, q, \end{aligned} \tag{4.22}$$

$$\begin{aligned} \mathbb{P}_0^N := \big\{ \mathbf{z} \in \mathbb{C}^N : {}& |1 - z_1^{(\alpha)}| + |z_1^{(\alpha)} - z_n^{(\alpha)}| < 1,\ \alpha = 1, \dots, q,\ n = 2, \dots, p_\alpha; \\ & |z_l^{(q+1)}| < 1,\ l = 1, \dots, p_{q+1} \big\}. \end{aligned} \tag{4.23}$$

The following assertion holds, which establishes formulas for the analytic continuation of the Lauricella function $F_D^{(N)}$ that are valid near the intersection $\bigcap_{s=1}^{q} \mathscr{M}_s$ of the hyperplanes (4.7) and correspond to the case in which all variables $z_l^{(s)}$, $l = 1, \dots, p_s$, $s = 1, \dots, q$, are close to one and the other variables $z_l^{(q+1)}$, $l = 1, \dots, p_{q+1}$, have the moduli less than one.

Theorem 2. *If none of the numbers* $(c - \sum_{s=1}^{j} |\mathbf{a}_{1,p_s}^{(s)}| - b)$, $j = 1, \dots, q$, *is an integer, then the analytic continuation of the series* (2.1) *to the domain* (4.21)–(4.23) *is given by the formula*

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{j=0}^{q} \varkappa_j \mathscr{U}_j^{(1)}(\mathbf{a}; b, c; \mathbf{z}), \tag{4.24}$$

where the functions $\mathscr{U}_j^{(1)}$, $j=0,\dots,q$, *are defined by the equalities*

$$\mathscr{U}_0^{(1)}(\mathbf{a};b,c;\mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \Lambda_0(\mathbf{a};b,c;\mathbf{k})\mathbf{Z}_0^{\mathbf{k}}, \tag{4.25}$$

$$\mathscr{U}_j^{(1)}(\mathbf{a};b,c;\mathbf{z}) = (1-z_1^{(j)})^{c-\sum_{s=1}^{j}|\mathbf{a}_{1,p_s}^{(s)}|-b}\biggl(\prod_{l=j+1}^{q}(1-z_1^{(l)})^{-|\mathbf{a}_{1,p_l}^{(l)}|}\biggr) \times \sum_{|\mathbf{k}|=0}^{\infty}\Lambda_j(\mathbf{a};b,c;\mathbf{k})\mathbf{Z}_j^{\mathbf{k}}, \quad j=1,\dots,q. \tag{4.26}$$

Here the coefficients Λ_j *have the form*

$$\Lambda_0(\mathbf{a};b,c;\mathbf{k}) = \frac{(b)_{|\mathbf{k}|}}{(c-\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_{q+1}}^{(q+1)}|)(1-c+\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|+b,\sum_{s=1}^{q}|\mathbf{k}_{1,p_s}^{(s)}|)} \times \prod_{s=1}^{q}\biggl[\frac{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_s}^{(s)}|)}{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{2,p_s}^{(s)}|)k_1^{(s)}!}\prod_{l=2}^{p_s}\frac{(a_l^{(s)},k_l^{(s)})}{k_l^{(s)}!}\biggr]\prod_{l=1}^{p_{q+1}}\frac{(a_l^{(q+1)},k_l^{(q+1)})}{k_l^{(q+1)}!}, \tag{4.27}$$

$$\begin{aligned}\Lambda_j&(\mathbf{a};b,c;\mathbf{k})\\ &= \frac{(c-\sum_{s=1}^{j-1}|\mathbf{a}_{1,p_s}^{(s)}|-b,\sum_{s=j}^{q}k_1^{(s)}-\sum_{s=1}^{j-1}k_1^{(s)}-\sum_{s=1}^{j-1}|\mathbf{k}_{2,p_s}^{(s)}|)}{(1+c-\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|-b,\sum_{s=j}^{q}k_1^{(s)}-\sum_{s=1}^{j-1}k_1^{(s)}-\sum_{s=1}^{j}|\mathbf{k}_{2,p_s}^{(s)}|)(|\mathbf{a}_{1,p_j}^{(j)}|,|\mathbf{k}_{2,p_j}^{(j)}|)}\\ &\quad\times\biggl(\prod_{s=1}^{q}\prod_{l=2}^{p_s}\frac{(a_l^{(s)},k_l^{(s)})}{k_l^{(s)}!}\biggr)\\ &\quad\times\biggl(\frac{(c-\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_{q+1}}^{(q+1)}|+k_1^{(j)})}{(c-\sum_{s=1}^{q}|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_{q+1}}^{(q+1)}|)k_1^{(j)}!}\prod_{s=1,s\neq j}^{q}\frac{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{1,p_s}^{(s)}|)}{(|\mathbf{a}_{1,p_s}^{(s)}|,|\mathbf{k}_{2,p_s}^{(s)}|)k_1^{(s)}!}\biggr)\\ &\quad\times\prod_{l=1}^{p_{q+1}}\frac{(a_l^{(q+1)},k_l^{(q+1)})}{k_l^{(q+1)}!}.\end{aligned} \tag{4.28}$$

In formulas (4.27) *and* (4.28) *we use the notation* $\mathbf{Z}_j^{\mathbf{k}} = \prod_{s=1}^{q+1}\prod_{l=1}^{p_s}(Z_{j,l}^{(s)})^{k_l^{(s)}}$, *where the numbers* $Z_j^{(s,l)}$ *are the entries of the vectors* $Z_j^{(s)} = \{Z_{j,1}^{(s)},\dots,Z_{j,p_s}^{(s)}\}$, $j=0,\dots,q$, $s=1,\dots,q+1$, *defined for* $j=0$ *by the equalities*

$$\begin{aligned}Z_0^{(s)} &= \Bigl(1-z_1^{(s)},z_1^{(s)}-z_2^{(s)},\dots,z_1^{(s)}-z_{p_s}^{(s)}\Bigr), \qquad s=1,\dots,q,\\ Z_0^{(q+1)} &= \Bigl(z_1^{q+1},\dots,z_{p_{q+1}}^{q+1}\Bigr), \qquad s=q+1,\end{aligned} \tag{4.29}$$

and, for $j = 1, \dots, q$, by the following equalities:

$$
\begin{aligned}
Z_j^{(s)} &= \left(\frac{1 - z_1^{(s)}}{1 - z_1^{(j)}}, \frac{z_1^{(s)} - z_2^{(s)}}{1 - z_1^{(j)}}, \dots, \frac{z_1^{(s)} - z_{p_s}^{(s)}}{1 - z_1^{(j)}}\right), && s = 1, \dots, j-1,\\
Z_j^{(j)} &= \left(1 - z_1^{(j)}, \frac{z_1^{(j)} - z_2^{(j)}}{1 - z_1^{(j)}}, \dots, \frac{z_1^{(j)} - z_{p_j}^{(j)}}{1 - z_1^{(j)}}\right), && s = j,\\
Z_j^{(s)} &= \left(\frac{1 - z_1^{(j)}}{1 - z_1^{(s)}}, \frac{z_2^{(s)} - z_1^{(s)}}{1 - z_1^{(s)}}, \dots, \frac{z_{p_s}^{(s)} - z_1^{(s)}}{1 - z_1^{(s)}}\right), && s = j+1, \dots, q,\\
Z_j^{(q+1)} &= \left(z_1^{q+1}, \dots, z_{p_{q+1}}^{q+1}\right), && s = q+1.
\end{aligned}
\tag{4.30}
$$

The coefficients $\varkappa_j$, $j = 0, \dots, q$, in formula (4.24) *have the form*

$$
\begin{aligned}
\varkappa_0 &= \frac{\Gamma(c)\Gamma(c - \sum_{s=1}^{q} |\mathbf{a}_{1,p_s}^{(s)}| - b)}{\Gamma(c - \sum_{s=1}^{q} |\mathbf{a}_{1,p_s}^{(s)}|)\Gamma(c-b)},\\
\varkappa_j &= \frac{\Gamma(c)\Gamma(c - \sum_{s=1}^{j-1} |\mathbf{a}_{1,p_s}^{(s)}| - b)\Gamma(\sum_{s=1}^{j} |\mathbf{a}_{1,p_s}^{(s)}| + b - c)}{\Gamma(|\mathbf{a}_{1,p_j}^{(j)}|)\Gamma(b)\Gamma(c-b)}, \qquad j = 1, \dots, q.
\end{aligned}
\tag{4.31}
$$

The functions (4.25), (4.26) *are linearly independent solutions of system* (2.5) *in the domain* (4.21)–(4.23).

Proof. To prove the theorem, we rewrite formula (2.11), performing the changes of variables (4.19) and (4.20), and apply the formulas of analytic continuation that are established in [17, Theorem 2.2] to the function $F_D^{(\nu)}$ in (2.11) and (4.20); as a result, we arrive at the assertion of the theorem. The assertion claiming that functions (4.25) and (4.26) are linearly independent solutions of the system of equations satisfied by the Lauricella function is verified by their direct substitution into (2.5). As above, we do not present cumbersome (but, in essence, not very complicated) calculations. □

5. AN EXAMPLE OF THE SOLUTION OF THE CROWDING PROBLEM

Let us show the results of the previous sections, Secs. 2–4, by the example of constructing a conformal mapping of a 10-angle domain $\mathscr{M}$ depicted in Fig. 2 whose form is borrowed from the paper [32] devoted to the calculation of the electric field in a technical device of a complicated form.

Denote the vertices of the polygon $\mathscr{M}$ by M_j and their coordinates by w_j, $j = 0, \dots, 9$. The corresponding angles of the polygon are equal to

$$
\pi\beta_j = \frac{\pi}{2}, \quad j = 0, 2, 3, 4, 8, 9; \qquad \pi\beta_1 = \pi\beta_5 = \frac{3\pi}{2}, \qquad \pi\beta_6 = \pi\alpha, \qquad \pi\beta_7 = \pi(2 - \alpha), \tag{5.1}
$$

where the value α is determined by the position of the vertices. We carry out calculations for the following values:

$$
\begin{gathered}
w_0 = i, \qquad w_1 = 1 + i, \qquad w_2 = 1, \qquad w_3 = 3.5, \qquad w_4 = 3.5 + 2i, \qquad w_5 = 1 + 2i,\\
w_6 = 1 + 3i, \qquad w_7 = 2 + 5i, \qquad w_8 = 2 + 9i, \qquad w_9 = 9i,
\end{gathered}
\tag{5.2}
$$

here $\alpha \approx 1.14758$.

The normalization of the mapping $\mu\colon \mathbb{H}^+ :\xrightarrow{\text{conf}} \mathscr{M}$ is as follows, see Figs. 2(a) and 2(b):

$$
\mu(0) = w_0, \qquad \mu(1) = w_8, \qquad \mu(\infty) = w_9. \tag{5.3}
$$

A function $\mu(\zeta)$, subjected to conditions (5.3) can be written in the form of the Schwarz–Christoffel integral

$$
z = \mu(\zeta) = w_0 - \mathscr{K}_0 \int_0^{\zeta} \left(\frac{(t - \zeta_1)(t - \zeta_5)}{t(t - \zeta_2)(t - \zeta_3)(t - \zeta_4)(t - 1)}\right)^{1/2} (t - \zeta_6)^{\alpha - 1}(t - \zeta_7)^{1 - \alpha}\, dt, \tag{5.4}
$$

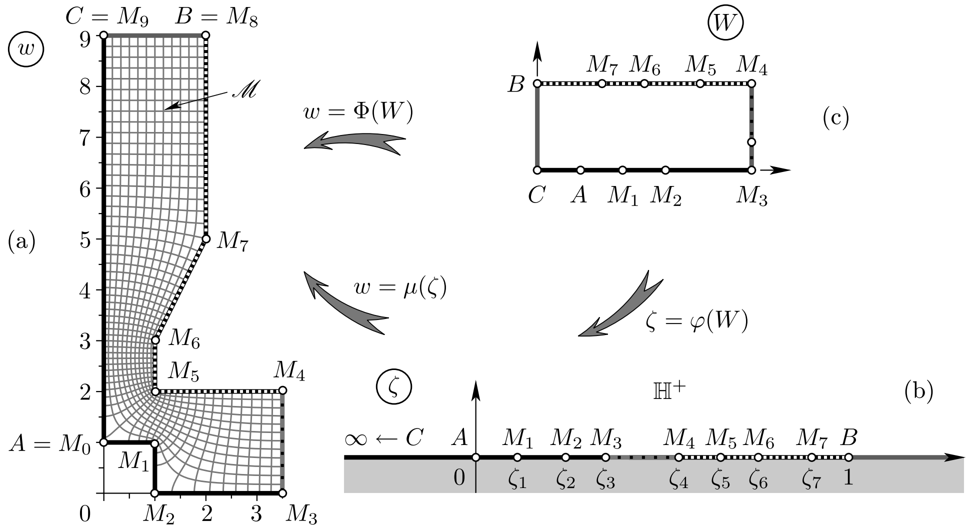

Fig. 2. Example of computing a conformal mapping.

where the vector of preimages $(\zeta_1, \dots, \zeta_7) =: \mathbf{x}$ and the pre-integral factor $\mathscr{K}_0 > 0$ are the unknown quantities. To find ζ_j, $k = 1, \dots, 7$, we have the following system of equations:

$$\frac{I_k(\mathbf{x})}{I_0(\mathbf{x})} = \frac{L_k}{L_0}, \qquad k = 1, \dots, 7, \tag{5.5}$$

where

$$I_k = \left| \int_{\zeta_k}^{\zeta_{k+1}} \left(\frac{(t-\zeta_1)(t-\zeta_5)}{t(t-\zeta_2)(t-\zeta_3)(t-\zeta_4)(t-1)} \right)^{1/2} (t-\zeta_6)^{\alpha-1}(t-\zeta_7)^{1-\alpha}\, dt \right|, \quad k = 0, \dots, 7.$$

After solving the system, the quantity $\mathscr{K}_0$ is calculated by the formula $\mathscr{K}_0 = L_0/I_0(\mathbf{x})$, where $\mathbf{x}$ is the vector of preimages found from system (5.5).

To express the elements of this system in terms of the function $F_D^{(7)}$, we first of all calculate the quantities (3.11), taking into account (5.1):

$$\mathbf{a} = (a_1, \dots, a_7) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 1-\alpha, \alpha-1 \right), \qquad b = \frac{1}{2}, \qquad c = 1, \tag{5.6}$$

and then, using (5.6), we find the vectors in (3.20) and the scalars in (3.20). After that, we calculate the coefficients C_k in (3.14) by formulas (3.15) and write out the vectors $\mathbf{x}_k$ defined by (3.22):

$$\begin{gathered} \mathbf{x}_0 := \left(\frac{\zeta_1}{\zeta_2}, \dots, \frac{\zeta_1}{\zeta_7}, \zeta_1 \right), \qquad \mathbf{x}_7 := \left(\frac{1}{\zeta_7}, \frac{\zeta_1}{\zeta_7}, \dots, \frac{\zeta_6}{\zeta_7} \right), \\ \mathbf{x}_k := \left(\frac{\zeta_{k+1}}{\zeta_k}, \frac{\zeta_1}{\zeta_k}, \dots, \frac{\zeta_{k-1}}{\zeta_k}, \frac{\zeta_{k+2}}{\zeta_k}, \dots, \frac{\zeta_7}{\zeta_k}, \frac{1}{\zeta_k} \right), \qquad k = 1, \dots, 6. \end{gathered} \tag{5.7}$$

Finally, we obtain the expressions for the integrals I_k in system (5.5) in the form

$$I_k(\mathbf{x}) = C_k \mathscr{I}_k(\mathbf{a}; b, c; \mathbf{x}), \qquad k = 0, \dots, 7,$$

where the $\mathscr{I}_k$ are found using formulas (3.16)–(3.18), in which one needs to put $N = 7$ and substitute the calculated parameters and variables $\mathbf{a}_k$, b_k, c_k and $\mathbf{x}_k$ corresponding to the polygon $\mathscr{M}$:

$$\begin{aligned}
I_0(\mathbf{x}) &= \frac{\pi}{2}\biggl(\zeta_1 \prod_{j=2}^{7} \zeta_j^{-a_j}\biggr) F_D^{(7)}(\mathbf{a}_0; 1/2, 2; \mathbf{x}_0),\\
I_7(\mathbf{x}) &= \frac{\sqrt{\pi}\,\Gamma(2-\alpha)}{\Gamma(5/2-\alpha)}\\
&\qquad \times \zeta_7^{-1/2}\biggl(\prod_{j=1}^{6}(\zeta_7-\zeta_j)^{-a_j}\biggr)(1-\zeta_7)^{3/2-\alpha} F_D^{(7)}(\mathbf{a}_7; 2-\alpha, 5/2-\alpha; \mathscr{Y}(\mathbf{1}-\mathbf{x}_7)), \qquad (5.8)\\
I_k(\mathbf{x}) &= \frac{\Gamma(1-a_k)\Gamma(1-a_{k+1})}{\Gamma(2-a_k-a_{k+1})}\,\frac{(\zeta_{k+1}-\zeta_k)^{1-a_k-a_{k+1}}}{\zeta_k^{1/2}(1-\zeta_k)^{1/2}}\\
&\qquad \times \prod_{j=1, j\neq k,k+1}^{7}(|\zeta_k-\zeta_j|)^{-a_j} F_D^{(7)}(\mathbf{a}_k; b_k, c_k; \mathscr{Y}(\mathbf{1}-\mathbf{x}_k)), \qquad k=1,\dots,6;
\end{aligned}$$

the function $\mathscr{Y}$ is defined in (3.12). System (5.5), (5.8) is solved numerically by Newton's method using the (known) continuation method by the parameter to find the initial approximation. The key point is the application of formulas of the analytic continuation of the Lauricella function $F_D^{(7)}$ occurring in (5.8), providing an efficient (practically, with computer precision) calculation of this function at every step of Newton's iterative algorithm.

The parameters of the Schwarz–Christoffel integral (5.10) were found by numerically solving system (5.5), (5.8) by the above method with the accuracy of 14 significant figures (in calculations, the standard mantissa was used with 16 significant figures). It should be noted that, as the calculations show, the points ζ_j, $j=1,\dots,7$, are located within the interval $(0,1)$ extremely uneven, i.e., there is a crowding effect. The distances between the majority of adjacent preimages are very small, for example,

$$\begin{aligned}
\zeta_1 &= 0.97128856311256\times 10^{-8}, & \zeta_2-\zeta_1 &= 1.6211586580969\times 10^{-7},\\
\zeta_3-\zeta_2 &= 0.5734139025052\times 10^{-9}, & \zeta_4-\zeta_3 &= 1.6717435578084\times 10^{-8},\\
\zeta_5-\zeta_4 &= 1.3733659034931\times 10^{-6}, & \zeta_6-\zeta_5 &= 0.86825507732556\times 10^{-6},\\
\zeta_7-\zeta_6 &= 0.96617657001692\times 10^{-2}.
\end{aligned}$$

At the same time, the distance $1-\zeta_7 = 0.99024882796294$ is close to the length of the entire unit interval on which all 7 preimages are placed. These results well illustrate the term "crowding": the points ζ_j, $j=1,\dots,7$, are "crowding" on the interval $(0,\zeta_7)$, whose length is two orders of magnitude less than the length of the interval $(0,1)$, and the distance between adjacent points among the indicated ζ_k, $j=1,\dots,5$, are several orders of magnitude smaller than the length of $(0,\zeta_7)$.

To illustrate the mapping $\mu\colon \mathbb{H}^+ \xrightarrow{\text{conf}} \mathscr{M}$, it is convenient to use an auxiliary conformal mapping $\varphi\colon \Pi \xrightarrow{\text{conf}} \mathbb{H}^+$ of the rectangle Π onto the half-plane $\mathbb{H}^+$, see Fig. 2 (b), (c), and first map the Cartesian grid natural for Π into $\mathbb{H}^+$, and then, using the function $z=\mu(\zeta)$, transfer the grid to the domain $\mathscr{M}$. Thus, we depict in the domain $\mathscr{M}$ the image of the rectangular Cartesian grid (originally constructed in the rectangle Π) under the mapping $\Phi\colon \Pi \xrightarrow{\text{conf}} \mathscr{M}$. Subjecting the mapping φ to the following conditions:

$$\varphi(ih)=1, \qquad \varphi(0)=\infty, \qquad \varphi(d)=\zeta_3, \qquad \varphi(d+ih)=\zeta_4 \tag{5.9}$$

(i.e., the points B, C, M_3, and M_4 of the boundary $\partial\Pi$ pass to the points of the same name on $\partial\mathbb{H}^+$), we find the mapping in the form

$$\varphi(W) = 1-\frac{1-\zeta_3}{\operatorname{sn}^2(k,W)}, \qquad k=\frac{1-\zeta_4}{1-\zeta_3}; \tag{5.10}$$

here $\operatorname{sn}(k,W)$ stands for the Jacobi elliptic function [33] with modulus k calculated via the parameters of the conformal mapping (5.4) according to the formula specified in (5.10). In this case, the length d

and the height h of the rectangle are equal to the elliptic integrals $K(k)$ and $K'(k)$, respectively. In formula (5.10), the quantities ζ_1 and ζ_2 are the parameters (computed above) of the Schwarz–Christoffel integral (5.4).

In Fig. 2(a), we show the image of the Cartesian grid (constructed in the rectangle Π) under the mapping $\Phi(W) = \mu \circ \varphi(W)$, where $z = \mu(\zeta)$ is the Schwarz–Christoffel integral (5.4) and $\zeta = \varphi(W)$ is the auxiliary mapping (5.10) of the rectangle onto the half-plane.

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