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Convergence Proofs and Error Estimates for an Iterative Method for Conformal Mapping

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Summary. The iterative method as introduced in [8] and [9] for the determination of the conformal mapping Φ of the unit disc onto a domain G is here described explicitly in terms of the operator K, which assigns to a periodic function u its periodic conjugate function Ku. It is shown that whenever the boundary curve Γ of G is parametrized by a function η with Lipschitz continuous derivative $\dot{\eta} \neq 0$, then the method converges locally in the Sobolev space W of 2π-periodic absolutely continuous functions with square integrable derivative. If η is in a Hölder class $C^{2+\mu}$, the order of convergence is at least $1+\mu$. If Γ is in $C^{l+1+\mu}$ with $l \geqq 1$, $0<\mu<1$, then the iteration converges in $C^{l+\mu}$. For analytic boundary curves the convergence takes place in a space of analytic functions.

For the numerical implementation of the method the operator K can be approximated by Wittich's method, which can be applied very effectively using fast Fourier transform. The Sobolev norm of the numerical error can be estimated in terms of the number N of grid points. It is $O(N^{1-l-\mu})$ if Γ is in $C^{l+1+\mu}$, and $O(\exp(-\tau N/2))$ if Γ is an analytic curve. The number τ in the latter formula is bounded by $\tau \leqq \log R$, where R is the radius of the largest circle into which Φ can be extended analytically such that $\Phi'(z) \neq 0$ for $|z|<R$. The results of some test calculations are reported.

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1. Introduction

Let Γ be a simple closed smooth curve in the complex plane represented by a 2π-periodic function $\eta(s)=x(s)+iy(s)$. Then Γ divides $\mathbb{C}$ into two regions G^+ and G^-. We assume that G^+ is the bounded component and $0 \in G^+$. Similarly the unit circle $\Delta := \{z: |z|=1\}$ divides the plane into the disc D^+ and its exterior D^-.

It is well known that there exists a unique conformal mapping $\Phi: D^+ \to G^+$ normalized by the conditions

$$\Phi(0)=0, \qquad \Phi'(0)>0. \tag{1.1}$$

Similarly there is a unique conformal mapping Ψ: $D^- \to G^-$ with a simple pole in ∞ and

$$\Psi'(\infty)>0. \tag{1.2}$$

We assume that η has a continuous derivative $\dot\eta$ with $\dot\eta \neq 0$, and that the curve is oriented in the counterclockwise direction. By the Osgood-Carathéodory theorem, the functions Φ and Ψ can be extended continuously to Δ. They map Δ onto Γ. Hence the values of Φ and Ψ on the boundary can be represented in the following way

$$\Phi(e^{it})=\eta(S(t)), \qquad S(t)-t \ \ 2\pi\text{-periodic}, \tag{1.3}$$

$$\Psi(e^{it})=\eta(T(t)), \qquad T(t)-t \ \ 2\pi\text{-periodic}. \tag{1.4}$$

In view of Cauchy's formula Φ and Ψ can easily be reconstructed from their boundary values. Therefore the conformal mapping is completely determined by the real functions S or T, resp.

The tangent angle $\theta(s)$ of the curve is defined by the formula

$$\dot\eta(s)=|\dot\eta(s)| \exp(i\theta(s)). \tag{1.5}$$

Since $\dot\eta$ is a continuous function with winding number 1, $\theta(s)$ can be chosen as a continuous function such that $\theta(s)-s$ is 2π-periodic. By forming the t-derivative in (1.3) we obtain the equation

$$i\Phi'(e^{it})e^{it}=\dot\eta(S(t))S'(t) \tag{1.6}$$

which implies

$$\log(\Phi'(e^{it}))=\log(|\dot\eta(S(t))|\, S'(t))+i(\theta(S(t))-t-\tfrac{\pi}{2}). \tag{1.7}$$

In view of conformality $\Phi'(z)\neq 0$ in D^+. Therefore the function $\log\Phi'$ is analytic in D^+. It follows from $\Phi'(0)>0$, that $\log\Phi'(0)$ is real up to multiples of $2\pi i$, hence

$$\alpha := \frac{1}{2\pi}\int_0^{2\pi}(\theta(S(t))-t)\,dt=\tfrac{\pi}{2} \ (\mathrm{mod}\, 2\pi) \tag{1.8}$$

and $\cot\alpha=0$.

2. Conjugate Functions

The conjugate function Ku of a function $u\in L^2(0,2\pi)$ is defined by

$$Ku(\tau):=\frac{1}{2\pi}\int_0^{2\pi} u(t)\cot\frac{\tau-t}{2}\,dt. \tag{2.1}$$

The right hand side exists for almost all τ as a Cauchy principal value integral. If u is represented by the Fourier series

$$u(t)\sim a_0+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu t+b_\nu\sin\nu t) \tag{2.2}$$

then the conjugate function is represented by the conjugate series

$$K u(\tau) \sim \sum_{\nu=1}^{\infty} (a_\nu \sin \nu\tau - b_\nu \cos \nu\tau). \tag{2.3}$$

The importance of the operator K in conformal mapping ([2], p. 62ff.) is mainly due to the following property:

Let $F(z)$ be analytic in D^+ and continuous in $\overline{D^+}$. If u and v are the real and imaginary parts of the boundary values of F, i.e. $F(e^{it}) = u(t) + i v(t)$, then

$$v(t) = \operatorname{Im} F(0) + K u(t). \tag{2.4}$$

Let $F(z)$ be analytic in $D^- \cup \{\infty\}$ and continuous in $\overline{D^-}$ with boundary values $F(e^{it}) = u(t) + i v(t)$, then

$$v(t) = \operatorname{Im} F(\infty) - K u(t). \tag{2.5}$$

Let W be the Sobolev space of 2π-periodic functions u which are primitives of functions $u' \in L^2(0, 2\pi)$. (W is the class $\mathfrak{K}$ of [2], p. 68). The norm on W is defined by

$$\|u\|_W := \max(\|u\|_0, \|u'\|_2) \tag{2.6}$$

in terms of the maximum norm $\|\cdot\|_0$ in $C[0, 2\pi]$ and the norm $\|\cdot\|_2$ in $L^2(0, 2\pi)$. We adopt on L^2 the norm

$$\|f\|_2 := \left(\int_0^{2\pi} |f(t)|^2 \, dt\right)^{1/2}$$

which differs by a factor of $\sqrt{2\pi}$ from the norm used in [2], p. 63. It follows immediately from (2.2) and (2.3) that Ku is in L^2 whenever $u \in L^2$ and

$$\|K u\|_2 \leqq \|u\|_2 \tag{2.7}$$

with equality for all u with $a_0 = 0$ in (2.2), or equivalently for all u with $\int u \, dt = 0$ (see [2], p. 63).

Lemma 1. *If $u \in W$ then $K u \in W$ and*

$$\|K u\|_0 \leqq \sqrt{\frac{\pi}{6}} \, \|u'\|_2, \tag{2.8}$$

$$\|K u\|_W \leqq \|u\|_W. \tag{2.9}$$

Proof. We conclude from [2], p. 64, that $K u \in W$, and furthermore

$$(K u)' = K(u'), \tag{2.10}$$

$$\|(K u)'\|_2 = \|u'\|_2, \tag{2.11}$$

$$\int K u \, dt = 0. \tag{2.12}$$

The inequality (2.8) could be derived immediately from (2.11) and (2.12) using the inequality D.28 of Levin and Stečkin [5]. We give here a simple direct proof. Because of (2.12), Ku has a zero t_1. We assume without loss of generality $t_1=0$, hence

$$Ku(\tau)=\int_0^{2\pi}(Ku)'(t)\,\chi_\tau(t)\,dt \tag{2.13}$$

with $\chi_\tau(t)=1$ for $0\leqq t\leqq\tau$ and $=0$ otherwise.

From $\int(Ku)'dt=0$ it follows that the function $\phi_1\equiv 1$ is orthogonal to $(Ku)'$. In view of (2.12) the same is true for $\phi_2(t)=1-t$. Hence we can replace in (2.13) the function χ_τ by $\chi_\tau+\lambda_1\phi_1+\lambda_2\phi_2$ for any real numbers λ_1,λ_2. We apply the Schwarz inequality and obtain

$$|Ku(\tau)|\leqq\|(Ku)'\|_2\,\|\chi_\tau+\lambda_1\phi_1+\lambda_2\phi_2\|_2. \tag{2.14}$$

A simple calculation shows that

$$\min_{\lambda_1,\lambda_2}\|\chi_\tau+\lambda_1\phi_1+\lambda_2\phi_2\|^2=\frac{1}{2\pi}\,\tau(2\pi-\tau)-\frac{3}{8\pi^3}\,(\tau(2\pi-\tau))^2$$

which is $\leqq\pi/6$ for all $\tau\in[0,2\pi]$. This proves the lemma in view of (2.14) and (2.11). □

The Hölder spaces $C^{l+\mu}$ are defined for $l=0,1,2,\ldots$ and $0<\mu\leqq 1$. They consist of all 2π-periodic functions ϕ, whose l-th derivative exists and satisfies a Hölder condition with exponent μ

$$|\phi^{(l)}(t_1)-\phi^{(l)}(t_2)|\leqq C|t_1-t_2|^\mu. \tag{2.15}$$

The infimum of all numbers C for which (2.15) is valid is denoted by $[\phi^{(l)}]_\mu$. Then the space $C^{l+\mu}$ provided with the norm

$$\|\phi\|_{l+\mu}:=\max(\|\phi\|_0,[\phi^{(l)}]_\mu)$$

is a Banach space. It is a consequence of the well-known Plemelj-Privalov theorem ([6], p. 46) that K is a bounded operator on $C^{l+\mu}$ for $0<\mu<1$.

In view of (2.4) the operator K can be applied to construct an analytic function with prescribed real part on the boundary. We shall show now that K can also be used to solve explicitly more general boundary value problems. This will be very important for the next section.

Let $A\neq 0$ be a complex function in W. Assume that A is represented in the form

$$A(t)=\rho(t)\exp(-i\beta(t))$$

with continuous real functions $\rho>0$ and β. The problem of finding a function F, which is analytic in the disc D^+ continuous in $\overline{D^+}$ and satisfies

$$\mathrm{Re}(A(t)F(e^{it}))=c(t) \tag{2.16}$$

for a given 2π-periodic real function $c\in W$ is called a *Riemann-Hilbert problem*. This kind of problem is investigated by Muskhelishvili in [6], p. 99ff. We pursue a different approach, leading to an explicit representation of the solution of (2.16) which involves only the operator K.

Furthermore, we consider a more general problem insofar as we search for solutions of (2.16) which have a zero of order $\geqq p$ at 0 for some prescribed integer number p. In case $p<0$ the function F must be analytic for $0<|z|<1$ with a pole of order $\leqq -p$ at 0.

Let m be the winding number of $\bar{A}$

$$m:=\frac{1}{2\pi}(\beta(2\pi)-\beta(0)). \tag{2.17}$$

Then $\beta(t)-mt$ is a 2π-periodic function. Therefore the following definition makes sense:

$$w(t):=K(\beta(t)-mt). \tag{2.18}$$

In view of Lemma 1 this function is in W. By hypothesis also c and ρ are in W. Therefore the function

$$q(t):=c(t)\,e^{w(t)}/\rho(t) \tag{2.19}$$

is also in W. So one can apply the operator K again and form the function

$$F(e^{it}):=(q(t)+iK\,q(t))\exp(i\beta(t)-w(t)). \tag{2.20}$$

Lemma 2. *For $m\geqq p$ the problem* (2.16) *has a solution for each function c. In case $m<p$ a solution exists* iff *the function q as defined in* (2.19) *satisfies the conditions*

$$\int q(t)\cos\nu t\,dt=\int q(t)\sin\nu t\,dt=0 \quad \textit{for} \ \ \nu=0,\ldots,p-m-1. \tag{2.21}$$

Whenever a solution exists F as defined in (2.20) *is a particular solution. For $m<p$ this solution is unique. In case $m\geqq p$ the leading term in the Laurent series of the particular solution* (2.20) *is*

$$F(z)=(-1)^m\hat{q}\exp(i\hat{\beta})\,z^m+\ldots \tag{2.22}$$

where $\hat{q}$ and $\hat{\beta}$ are the mean-values of the functions q and β over $[0,2\pi]$. *The general solution of the homogeneous problem* (2.16) *has the form*

$$F_0(e^{it})=\sum_{\nu=0}^{m-p} i(a_\nu\cos\nu t+b_\nu\sin\nu t)\exp(i\beta(t)-w(t)) \tag{2.23}$$

with arbitrary real parameters a_ν and b_ν.

Proof. It follows from (2.18) and (2.4) that

$$V(e^{it})=-w(t)+i(\beta(t)-mt) \tag{2.24}$$

are the boundary values of a function $V(z)$, which is analytic in D^+. We insert the Ansatz $F(z)=z^pQ(z)\exp(V(z))$ into (2.16), observe that the factors $\exp(\pm i\beta)$ cancel and obtain for the analytic function Q the boundary problem

$$\operatorname{Re}(e^{i(p-m)t}Q(e^{it}))=c(t)\,e^{w(t)}/\rho(t)=q(t). \tag{2.25}$$

We consider two cases:

a) If $m<p$, then $\tilde{Q}(z) := z^{(p-m)} Q(z)$ is an analytic function which can easily be reconstructed from (2.25) using (2.4)

$$\tilde{Q}(e^{it}) = q(t) + i(K q(t) + \operatorname{Im} \tilde{Q}(0)). \tag{2.26}$$

In order that $Q = z^{(m-p)} \tilde{Q}$ is also an analytic function, $\tilde{Q}$ must have a zero of order $p-m$ at 0. In view of (2.3) this is equivalent to $\operatorname{Im} \tilde{Q}(0) = 0$ and the conditions (2.21).

b) In case $m \geqq p$ we obtain immediately a particular solution

$$Q(z) = z^{(m-p)} \tilde{Q}(z) \tag{2.27}$$

where $\tilde{Q}$ is defined as in (2.26). The boundary values of the general solution of the homogeneous problem (2.25) are represented by

$$Q_0(e^{it}) = i e^{i(m-p)t} \sum_{\nu=0}^{m-p} (a_\nu \cos \nu t + b_\nu \sin \nu t) \tag{2.28}$$

with arbitrary real constants a_ν, b_ν. When Q is inserted into the Ansatz for F the factors $e^{\pm i(m-p)t}$ cancel and we obtain (2.20) and (2.23). From (2.27) it follows that

$$F(z) = \tilde{Q}(0) \exp(V(0)) z^m + \dots .$$

The values of $\tilde{Q}$ and V at 0 can be calculated by means of Cauchy's formula from the boundary values given in (2.26) and (2.24). This proves (2.22). $\square$

For comparison of Lemma 2 with the results of [6], p. 104, we note that $\kappa := 2m$ is the index of the Riemann-Hilbert problem in the sense of Muskhelischvili.

The exterior problem, where F is required to be analytic for $|z|>1$ (including the point ∞), can be reduced to (2.16) by the observation that the reflected function

$$F^*(z) := \overline{F(1/\bar{z})}, \qquad |z|<1$$

is analytic in the disc and satisfies $F^*(e^{it}) = \overline{F(e^{it})}$, which leads to the problem

$$\operatorname{Re}(\bar{A}(t) F^*(e^{it})) = c(t)$$

for the function F^*.

3. The Iterative Method

Recall that the conformal mapping $\Phi: D^+ \to G^+$ is characterized by three properties

a) Φ is an analytic function in the disc D^+ which satisfies $\Phi(0)=0$, $\operatorname{Im} \Phi'(0) = 0$,

b) Φ maps the circle $\Delta = \partial D^+$ onto the boundary curve $\Gamma = \partial G^+$,

c) $\Phi'(0) > 0$.

The idea of our iterative scheme is to construct two sequences of functions, namely a sequence of functions Φ_k, analytic in D^+ with $\Phi_k(0)=0$ and $\operatorname{Im}\Phi'_k(0)=0$, and a sequence of continuous mappings $f_k\colon \Delta\to\Gamma$. So, each of the functions has one of the properties a) or b). If both sequences converge and $\Phi_k - f_k \to 0$ on Δ, then the Φ_k converge to a function, which has both properties a) and b). Hence the limit is the conformal mapping function Φ if $\Phi'(0)$ is positive.

Each function of the sequence f_k can be represented in the form

$$f_k(e^{it}) = \eta(S_k(t)), \tag{3.1}$$

where S_k is a continuous real function with the property that $S_k(t)-t$ can be extended to a continuous 2π-periodic function for all t. The functions S_k are defined iteratively. We assume that $\dot\eta$ is Lipschitz continuous and start with a function S_0 which has the property that $S_0(t)-t$ is in the Sobolev space W. If the function S_k is determined for some $k \geqq 0$, and if $S_k(t)-t$ is in W, then we put

$$S_{k+1}(t) = S_k(t) + U_k(t) \tag{3.2}$$

and determine the shift U_k from the condition that there exists an analytic function Φ_{k+1} which agrees with f_{k+1} in the terms of first order in U_k. Hence Φ_{k+1} and U_k are connected by the equation

$$\Phi_{k+1}(e^{it}) = \eta(S_k(t)) + U_k(t)\,\dot\eta(S_k(t)) \tag{3.3}$$

and Φ_{k+1} is subject to the constraints

$$\Phi_{k+1}(0) = 0, \qquad \operatorname{Im}\Phi'_{k+1}(0) = 0. \tag{3.4}$$

Since U_k is real, (3.3) is equivalent to the Riemann-Hilbert problem

$$\operatorname{Im}(\Phi_{k+1}(e^{it})/\dot\eta(S_k(t))) = \operatorname{Im}(\eta(S_k(t))/\dot\eta(S_k(t))). \tag{3.5}$$

This is of the form (2.16) if we put $A(t) = 1/\dot\eta(S_k(t))$, $F = -i\Phi_{k+1}$. Then β can be expressed in terms of the tangent angle θ as defined in (1.5). We obtain $\beta(t) = \theta(S_k(t))$ and the number m defined in (2.17) is

$$m = \frac{1}{2\pi}(\theta(S_k(2\pi)) - \theta(S_k(0))) = 1.$$

It follows from the hypotheses, that the function A and the right hand side in (3.5) are in W. Therefore we can apply Lemma 2 with $p=1$ and see that there is always a solution of problem (3.5) which satisfies $\Phi_{k+1}(0)=0$. The general solution can be obtained by the following three operations

$$w_k(t) := K(\theta(S_k(t)) - t), \tag{3.6}$$

$$q_k(t) := \operatorname{Im}(\eta(S_k(t))\exp(w_k(t) - i\theta(S_k(t)))), \tag{3.7}$$

$$\Phi_{k+1}(e^{it}) = (i q_k(t) - \gamma_k - K q_k(t))\exp(i\theta(S_k(t)) - w_k(t)) \tag{3.8}$$

with an arbitrary real number γ_k. It follows from (2.22) that the leading term of Φ_{k+1} is

$$\Phi_{k+1}(z)=(i\hat{q}_k-\gamma_k)\exp(i\alpha_k)z+\dots$$
$$\alpha_k:=\frac{1}{2\pi}\int_0^{2\pi}(\theta(S_k(t))-t)\,dt. \tag{3.9}$$

Hence the condition $\operatorname{Im}\Phi'_{k+1}(0)=0$ is satisfied for

$$\gamma_k=\hat{q}_k\cot\alpha_k. \tag{3.10}$$

So, we have proven that there is a unique solution of (3.5) subject to the constraints (3.4) if α_k is not a multiple of π. The solution can be obtained by the operations (3.6)-(3.8) which involve only the operator K of conjugation.

From (3.3) it follows that

$$U_k=\operatorname{Re}((\Phi_{k+1}-f_k)/\dot{\eta}).$$

We insert the representation (3.8), observe that the factors $\exp(\pm i\theta(S_k(t)))$ in Φ_{k+1} and $\dot{\eta}$ cancel and obtain finally

$$U_k(t)=-\operatorname{Re}\left(\frac{\eta(S_k(t))}{\dot{\eta}(S_k(t))}\right)-\frac{\gamma_k+Kq_k}{|\dot{\eta}(S_k(t))|\cdot\exp(w_k(t))}. \tag{3.11}$$

We have shown, that there is a unique real function U_k such that there exists an analytic function Φ_{k+1} which satisfies (3.3) and (3.4). This function U_k can explicitly be calculated by means of the formulas (3.6), (3.7), (3.9), (3.10) and (3.11), which use only the operation K of conjugation twice. All information about the curve we need is the parametrization $\eta(s)$, its derivative $\dot{\eta}(s)$ and the tangent angle $\theta(s)$.

The function $\theta(S_k(t))$ is in W. Hence K can be applied in (3.6) and yields a function $w_k\in W$. It follows that q_k is in W. Therefore the right hand side in (3.11) is defined and gives a function $U_k\in W$. Therefore S_{k+1} has the same properties which were assumed for S_k. Hence the iteration can be continued as long as α_k is not a multiple of π. Comparison of (3.9) with (1.8) shows, that α_k is an approximation for $\alpha=\pi/2$. Hence the exceptional case does not arise as long as $\|S_k-S\|_0$ is small. We shall see in the next section, that these conditions which ensure that the iteration is well defined in W are also sufficient for convergence.

The exterior problem can be treated quite similarly. The sequence f_k of mappings of Δ into Γ is represented in the form

$$f_k(e^{it})=\eta(T_k(t)) \tag{3.1a}$$

with real continuous functions T_k such that $T_k(t)-t$ are continuous 2π-periodic functions on $\mathbb{R}$. If T_k is already constructed for some $k\geqq 0$, we put

$$T_{k+1}(t):=T_k(t)+U_k(t) \tag{3.2a}$$

and determine the shift U_k from the condition, that there exists a function Ψ_{k+1}

analytic for $|z|>1$ except for a simple pole in ∞ with real valued residue, such that

$$\Psi_{k+1}(e^{it})=\eta(T_k(t))+U_k(t)\dot{\eta}(T_k(t)). \tag{3.3a}$$

Since U_k is real, Ψ_{k+1} must solve the Riemann-Hilbert problem

$$\operatorname{Im}(\Psi_{k+1}(e^{it})/\dot{\eta}(T_k(t)))=\operatorname{Im}(\eta(T_k(t))/\dot{\eta}(T_k(t))). \tag{3.5a}$$

In view of Schwarz' reflection principle the function

$$\Psi^*_{k+1}(z):=\overline{\Psi_{k+1}(1/\bar{z})}, \qquad |z|\leqq 1$$

is analytic in the disc, except for a simple pole in 0 with real residue. For $|z|=1$.

$$\Psi^*_{k+1}(z)=\overline{\Psi_{k+1}(z)}$$

holds true. We insert this into (3.5a) and obtain for Ψ_{k+1} the equation

$$\operatorname{Im}(\Psi^*_{k+1}(e^{it})/\overline{\dot{\eta}(T_k(t))})=-\operatorname{Im}(\eta(T_k(t))/\dot{\eta}(T_k(t))) \tag{3.5b}$$

which resembles closely (3.5) with the only difference, that θ is replaced by $-\theta$, which implies $m=-1$. On the other hand a simple pole is admitted. So we can apply Lemma 2 with $m=-1$ and $p=-1$ and arrive at the following formulas:

$$w_k(t):=-K(\theta(T_k(t))-t), \tag{3.6a}$$

$$q_k(t)=-\operatorname{Im}(\eta(T_k(t))\exp(w_k(t)-i\theta(T_k(t))), \tag{3.7a}$$

$$\Psi^*_{k+1}(e^{it})=(iq_k(t)-\gamma_k-Kq_k(t))\exp(-i\theta(T_k(t))-w_k(t)), \tag{3.8a}$$

$$\Psi_{k+1}(e^{it})=-(iq_k(t)+\gamma_k+Kq_k(t))\exp(i\theta(T_k(t))-w_k(t)). \tag{3.8b}$$

According to (2.22) the residue of the function Ψ^*_{k+1} in 0 is equal to $(-\gamma_k+\hat{q}_k)\exp(-i\alpha_k)$ where α_k is defined as in (3.9) with S_k replaced by T_k. The residue is real-valued if we choose

$$\gamma_k=-\hat{q}_k\cot\alpha_k. \tag{3.10a}$$

We insert (3.8b) into (3.3a) and obtain

$$U_k(t)=-\operatorname{Re}\left(\frac{\eta(T_k(t))}{\dot{\eta}(T_k(t))}\right)-\frac{\gamma_k+Kq_k}{|\dot{\eta}(T_k(t))|\exp(w_k(t))}. \tag{3.11a}$$

The iterative step for the exterior problem is described by (3.6a), (3.7a), (3.9), (3.10a), (3.11a), (3.2a). Corresponding formulas for the interior and exterior problem differ only in the signs in some places. Therefore we restrict our attention in the following completely to the interior problem. It is obvious how the results can be carried over to the exterior problem.

4. Convergence in W

Theorem 1. *If η has a Lipschitz continuous derivative $\dot{\eta}$, then for each number $M>0$ there exists a number $\varepsilon_M>0$ with the following property: If S_0 is a* *function so that* $S_0(t)-t$ *is in* W *and* S_0 *satisfies* $\|S_0'\|_2 \leqq M$ *and*

$$\left|\alpha_0 - \frac{\pi}{2}\right| \leqq \pi/8, \tag{4.1}$$

if furthermore there exists a function Φ_0 *analytic in* D^+ *with*

$$\Phi_0(0)=0, \qquad \Phi_0'(0)>0, \tag{4.2}$$

if Φ_0 *is continuous in* $\overline{D^+}$ *such that the boundary function* $\Phi_0(e^{it})$ *is in* W *and satisfies*

$$\|\eta(S_0(t)) - \Phi_0(e^{it})\|_W \leqq \varepsilon_M, \tag{4.3}$$

then $\|S - S_k\|_W \to 0$. *For* U_k *the estimate*

$$\|U_k\|_W \leqq C\,\|U_{k-2}\|_W^{3/2} \tag{4.4}$$

holds. If η *has a second derivative* $\ddot{\eta}$ *which is Hölder continuous with exponent* μ, $0<\mu\leqq 1$, *then the increments* U_k *satisfy the inequalities*

$$\|U_k\|_W \leqq C\,\|U_{k-1}\|_W^{1+\mu} \tag{4.5}$$

$$\|U_k\|_W \leqq C\,\|U_{k-2}\|_W^{(1+\mu)(3+\mu)/2}. \tag{4.6}$$

Proof. Since $\dot{\eta}$ is Lipschitz continuous, it is absolutely continuous and $\ddot{\eta} \in L^\infty$. One can choose $\ddot{\eta}$ as a bounded function. From the hypothesis $\dot{\eta} \neq 0$ it follows that $|\dot{\eta}| \geqq c > 0$. Therefore the function $\log\dot{\eta}$ is also absolutely continuous with derivative $\ddot{\eta}/\dot{\eta}$. From the representation $\dot{\eta} = \rho\exp(i\theta)$ with $\rho(s) := |\dot{\eta}(s)|$ it follows that ρ and θ are absolutely continuous. The derivatives satisfy the equation

$$\ddot{\eta}/\dot{\eta} = \dot{\rho}/\rho + i\dot{\theta}.$$

Therefore $\dot{\rho}, \dot{\theta}$ are also in L^∞. First we prove the convergence assuming the additional hypothesis

$$\sum_{l=0}^{k-1} \|U_l\|_W \leqq \delta \leqq 1. \tag{4.7}$$

Then $\|S_k'\|_2 \leqq M + \delta$. It is easily calculated that (4.7) implies that there are constants $C_V \geqq 1$, C_P, C_T depending only on M such that

$$\|\theta(S_k(t)) - t\|_W \leqq C_V, \qquad \|1/\rho(S_k(t))\|_W \leqq C_P, \qquad \|\exp(\pm i\theta(S_k(t)))\|_W \leqq C_T.$$

If δ is so small that $8\delta\|\dot{\theta}\|_\infty \leqq \pi$, then $|\cot\alpha_k| \leqq 1$. From (2.9) we conclude $\|w_k\|_W \leqq C_V$ and $\|\exp(\pm w_k)\|_W \leqq C_V\exp(C_V)$. We subtract Φ_k on both sides of (3.3) and obtain an equation

$$\Phi_k^*(e^{it}) = f_k - \Phi_k + U_k\dot{\eta}(S_k(t)) \tag{4.8}$$

for the function $\Phi_k^* = \Phi_{k+1} - \Phi_k$ which is analytic in D^+ and satisfies $\Phi_k^*(0)=0$, $\operatorname{Im}\Phi_k^{*\prime}(0)=0$. One can determine U_k from (4.8) with the same set of formulas (3.6) to (3.11) if only f_k is replaced by $f_k^* := f_k - \Phi_k$. We distinguish the pertaining auxiliary quantities q_k, γ_k etc. by a star. Using these modified formulas we obtain

$$\|q_k^*\|_W \leqq 3\, C_V\, C_T \exp(C_V)\, \|f_k^*\|_W, \tag{4.9}$$

$$\|\gamma_k^* + K q_k^*\|_W \leqq (3\, C_V\, C_T + 1) \exp(C_V)\, \|f_k^*\|_W. \tag{4.10}$$

In view of (2.9) we finally find

$$\|U_k\|_W \leqq C_U\, \|f_k^*\|_W \tag{4.11}$$

with $C_U := 3\, C_P(\exp(2\, C_V)\, C_V(3\, C_V\, C_T + 1) + C_T)$. From the definition it follows that

$$\|q_k^*\|_0 \leqq \exp(C_V)\, \|f_k^*\|_0. \tag{4.12}$$

We use now Warschawski's inequality (see [2], p. 68), (2.7), (2.9), (4.9) and (4.12) and obtain

$$\begin{aligned} \|K q_k^*\|_0^2 &\leqq \|K q_k^*\|_2\, \|K q_k^*\|_W \leqq \|q_k^*\|_2\, \|q_k^*\|_W \\ &\leqq 3\sqrt{2\pi}\, C_V\, C_T \exp(2\, C_V)\, \|f_k^*\|_0\, \|f_k^*\|_W. \end{aligned}$$

From this estimate and (4.10) we now conclude that

$$\|U_k\|_0 \leqq C_0(\|f_k^*\|_0\, \|f_k^*\|_W)^{1/2}. \tag{4.13}$$

In order to estimate $f_k - \Phi_k$ we start from the representation

$$f_k(t) - \Phi_k(e^{it}) = \eta(S_{k-1} + U_{k-1}) - \eta(S_{k-1}) - \dot{\eta}(S_{k-1})\, U_{k-1}. \tag{4.14}$$

At first we note that the right hand side of (4.14) is equal to $U_{k-1} I_{k-1}$ with

$$I_{k-1} := \int_0^1 (\dot{\eta}(S_{k-1} + \tau\, U_{k-1}) - \dot{\eta}(S_{k-1}))\, d\tau.$$

We estimate the integrand using the Lipschitz condition and obtain

$$\|f_k - \Phi_k\|_0 \leqq [\dot{\eta}]_1\, \|U_{k-1}\|_0^2. \tag{4.15}$$

Since $\dot{\eta}$ is Lipschitz continuous, the function in (4.14) is absolutely continuous with derivative

$$\begin{aligned} (f_k - \Phi_k)' &= \dot{\eta}(S_{k-1} + U_{k-1})(S'_{k-1} + U'_{k-1}) - \dot{\eta}(S_{k-1})\, S'_{k-1} \\ &\quad - \ddot{\eta}(S_{k-1})\, S'_{k-1}\, U_{k-1} - \dot{\eta}(S_{k-1})\, U'_{k-1} \\ &= (\dot{\eta}(S_{k-1} + U_{k-1}) - \dot{\eta}(S_{k-1}))\, U'_{k-1} \\ &\quad - S'_{k-1}\, U_{k-1} \int (\ddot{\eta}(S_{k-1} + \tau\, U_{k-1}) - \ddot{\eta}(S_{k-1}))\, d\tau. \end{aligned} \tag{4.16}$$

From this representation it follows that

$$|(f_k - \Phi_k)'| \leqq [\dot{\eta}]_1\, \|U_{k-1}\|_0\, |U'_{k-1}| + 2\, \|\ddot{\eta}\|_\infty\, \|U_{k-1}\|_0\, |S'_{k-1}|.$$

If $\ddot{\eta}$ is Hölder continuous the last term on the right hand side of (4.16) can be estimated by

$$[\ddot{\eta}]_\mu\, \|U_{k-1}\|_0^{1+\mu}\, |S'_{k-1}|.$$

In any case it follows that $f_k-\Phi_k$ is in W and

$$\|(f_k-\Phi_k)'\|_2 \leqq C_3 \|U_{k-1}\|_0(\|U'_{k-1}\|_2+\|U_{k-1}\|_0^{\mu}). \tag{4.17}$$

The case of $\ddot{\eta}\in L^\infty$ can be included in this formula and the following as the limit $\mu=0$. In view of (4.15) the same type of estimate holds true for $\|f_k-\Phi_k\|_W$. We insert (4.15) and (4.17) into (4.13) and (4.11) and obtain

$$\|U_k\|_0 \leqq C_4\cdot \|U_{k-1}\|_0^{3/2}\,\|U_{k-1}\|_W^{\mu/2} \tag{4.18}$$

$$\|U_k\|_W \leqq C_5 \|U_{k-1}\|_0(\|U'_{k-1}\|_2+\|U_{k-1}\|_0^{\mu}) \tag{4.19}$$

$$\|U_k\|_W \leqq C_6 \|U_{k-1}\|_W^{1+\mu}. \tag{4.20}$$

The last line already proves (4.5). We assume that

$$C_4\cdot \|U_0\|_0^{1/2} \leqq q<1. \tag{4.21}$$

Then it follows from (4.18) that

$$\|U_k\|_0 \leqq q^k \|U_0\|_0 \qquad \text{for } k\geqq 0$$

and from (4.19) that

$$\|U_k\|_W \leqq C_5 q^{k-1} \|U_0\|_0 \qquad \text{for } k\geqq 1 \tag{4.22}$$

whence the convergence of $S_k-S_0=U_0+U_1+\ldots+U_{k-1}$ in W follows. In order to complete the proof we must show that (4.7) and (4.21) can be satisfied. We use (4.22), (4.11) and (4.3) and obtain

$$\sum_{l=0}^{k-1} \|U_l\|_W \leqq \|U_0\|_W \left(1+C_5\sum_{l=0}^{k-2} q^l\right) \leqq (1+C_5/(1-q))\, C_U \varepsilon_M.$$

Therefore by choosing ε_M sufficiently small both conditions (4.7) and (4.21) can be satisfied for any δ and q in $(0,1]$. Combining (4.18), (4.19) and (4.20) and evaluating for k and $k-1$ yields finally (4.6) and (4.4). □

With regard to applications an advantage of this theorem is that it does not require the theoretical mapping function Φ. One has only to assume that $\eta(S_0)$ is sufficiently near to the set of analytic functions. If we specify $\Phi_0=\Phi$ we obtain the following

Corollary 1. *If η is in $C^{2+\mu}$, then there exists a number $\varepsilon>0$ such that for each initial function S_0 with $\|S_0-S\|_W\leqq\varepsilon$ the iteration converges in W.*

From the proof of Theorem 1 one can also deduce an error estimate

Corollary 2. *If the hypotheses of Theorem 1 are satisfied, then there exists to each M a constant C_M which depends only on M and the function η, such that*

$$\|S_0-S\|_W \leqq C_M \|\eta(S_0(t))-\Phi_0(e^{it})\|_W. \tag{4.23}$$

The proof of Theorem 1 shows how such a constant C_M can be calculated explicitly from η. We shall see later that (4.23) is useful to obtain error estimates for numerically calculated approximations for S.

5. Convergence in Hölder Spaces

Theorem 2. *If η is in $C^{l+1+\mu}$, $l \geqq 1$, and ν is a number with $0<\nu<1$ and $\nu \leqq \mu$, then there is for each number $M>0$ a number $\delta_M>0$ with the property: If $S_0(t)-t \in C^{l+\nu}$, $\|S_0(t)-t\|_{l+\nu} \leqq M$ and $|\alpha_0-\pi/2| \leqq \pi/8$ and if furthermore there exist a function Φ_0 analytic in D^+, continuous in $\overline{D^+}$, with boundary function $\Phi_0(e^{it}) \in C^{l+\nu}$, such that Φ_0 satisfies* (4.2) *and*

$$\|\eta(S_0(t))-\Phi_0(e^{it})\|_{l+\nu} \leqq \delta_M \tag{5.1}$$

then $\|S-S_k\|_{l+\nu} \to 0$.

Proof. For $l \geqq 1$ the restriction of each $\psi \in C^{l+\nu}$ to $[0, 2\pi]$ is in W and there is a constant C_0 so that

$$\|\psi\|_W \leqq C_0 \|\psi\|_{l+\nu}.$$

Therefore $\|S_0(t)-t\|_{l+\nu} \leqq M$ entails $\|S_0'\|_2 \leqq M' := C_0(M+1)$. If we choose δ_M so small that $C_0 \delta_M \leqq \varepsilon_{M'}$ with the number $\varepsilon_{M'}$ of Theorem 1, then $\|S_k - S\|_W \to 0$. Now we assume that $\Phi_k(e^{it})$ is in $C^{l+\nu}$ and

$$\sum_{\kappa=0}^{k-1} \|U_\kappa\|_{l+\nu} \leqq D. \tag{5.2}$$

Then $\|S_k(t)-t\|_{l+\nu} \leqq M+D$ and the $(l+\nu)$-norms of $\theta(S_k(t))-t$, $\exp(\pm i\theta(S_k(t)))$, $1/\dot{\eta}(S_k(t))$, w_k and $\exp(\pm w_k)$ are uniformly bounded. It follows from the representations (3.8) and (3.11) that $\Phi_{k+1}(e^{it})$ and $U_k(t)$ are in $C^{l+\nu}$ and as in the proof of Theorem 1, that there is a constant C_U such that

$$\|U_k\|_{l+\nu} \leqq C_U \|f_k - \Phi_k\|_{l+\nu}. \tag{5.3}$$

From the representation (4.14) we conclude

$$\|f_k-\Phi_k\|_{l+\nu} \leqq C_2(\|U_{k-1}\|_0 \|I_{k-1}\|_{l+\nu} + \|U_{k-1}\|_{l+\nu} \|I_{k-1}\|_0).$$

In I_{k-1} the Hölder norm of the integrand can be estimated in view of

$$\|\dot{\eta}(S_{k-1}(t)+\tau U_{k-1}(t))\|_{l+\nu} \leqq c_1 + C_1 \cdot D.$$

Therefore $\|I_{k-1}\|_{l+\nu} \leqq C_3$. We estimate $\|I_{k-1}\|_0$ as in (4.15), use the hypothesis (5.2) and obtain

$$\|f_k - \Phi_k\|_{l+\nu} \leqq C_4 \cdot \|U_{k-1}\|_0$$

which in view of (5.3) implies

$$\|U_k\|_{l+\nu} \leqq C_U \cdot C_4 \|U_{k-1}\|_0. \tag{5.4}$$

Since the hypotheses of Theorem 1 are satisfied, the right hand side in

$$\sum_{\kappa=0}^{k-1} \|U_\kappa\|_{l+\nu} \leqq C_U \left(\delta_M + C_4 \sum_{\kappa=0}^{k-2} \|U_\kappa\|_0\right)$$

can be estimated by $C_5 \cdot \delta_M$. Therefore δ_M can be chosen so small, that the assumption (5.2) is valid for all k. Hence the estimate (5.4) proves the convergence in $C^{l+\nu}$. □

In the hypotheses of Theorem 2 are satisfied, then according to a theorem of Warschawski [7] the boundary mapping $\Phi(e^{it})$ as well as $S(t)-t$ are in $C^{l+1+\nu}$. Therefore one can choose $\Phi_0=\Phi$. We start from the representation

$$\begin{aligned}\eta(S_0(t))-\Phi(e^{it})&=\eta(S_0(t))-\eta(S(t))\\&=(S_0(t)-S(t))\int_0^1 \dot{\eta}(S(t)+\tau(S_0(t)-S(t)))\,d\tau\end{aligned}$$

and observe that the $(l+\nu)$-norm of the integrand is bounded as long as S_0-S is bounded in $C^{l+\nu}$. Therefore the left hand side of (5.1) can be estimated by $C\|S_0-S\|_{l+\nu}$. So we obtain the following

Corollary. *If η is in $C^{l+1+\mu}$, $l\geqq 1$, and ν is a number with $0<\nu<1$ and $\nu\leqq\mu$, then there exists a $\delta>0$ such that for each initial function S_0 with $\|S_0-S\|_{l+\nu}\leqq\delta$ the iteration converges in $C^{l+\nu}$.*

Though η is in $C^{l+1+\mu}$, the convergence of S_k takes place in $C^{l+\nu}$ and at first glance the limit element S seems to be also only in $C^{l+\nu}$. But it follows that $\theta(S(t))-t$ is in $C^{l+\nu}$. The operator K maps $C^{l+\nu}$ into $C^{l+\nu}$. So we can conclude that the right hand side of (1.7) is in $C^{l+\nu}$. Therefore $\Phi(e^{it})$ is in fact in $C^{l+1+\nu}$ in accordance with the theorem of Warschawski.

6. Analytic Curves

For any fixed positive number τ let $\mathfrak{S}_\tau$ be the strip

$$\mathfrak{S}_\tau:=\{z\colon |\operatorname{Im} z|<\tau\}$$

and A_τ the set of bounded analytic functions on $\mathfrak{S}_\tau$. With the norm

$$\|\psi\|_\tau:=\sup_{z\in\mathfrak{S}_\tau}|\psi(z)|.$$

A_τ is a Banach-space.

Theorem 3. *Assume that $\eta(s)$ is a bounded analytic function in a strip $\mathfrak{S}_\sigma$ for some $\sigma>0$, and $|\dot{\eta}(s)|\geqq c_1>0$ and $|\ddot{\eta}(s)|\leqq C_1$ in $\mathfrak{S}_\sigma$. Then there exists for each $M>0$, $0<\sigma_0<\sigma$ a number δ_{M,σ_0} with the property: If S_0 is analytic in $\mathfrak{S}_\tau$ such that $S_0(t)-t$ is real and 2π-periodic for real t, if S_0 satisfies $|\operatorname{Im} S_0(t)|\leqq\sigma_0$ and $|S_0(t)-t|\leqq M$ for $t\in\mathfrak{S}_\tau$ and $\left|\alpha_0-\frac{\pi}{2}\right|\leqq\frac{\pi}{8}$, and if furthermore there exists a function $\Phi_0(z)$ which is bounded and analytic for $|z|<e^\tau$ and satisfies $\Phi_0(0)=0$, $\Phi_0'(0)>0$, such that*

$$\|\eta(S_0(t))-\Phi_0(e^{it})\|_\tau\leqq\delta_{M,\sigma_0} \tag{6.1}$$

then the sequence $S_k(t)-t$ converges in A_τ. The convergence is quadratic, i.e. there is a constant C such that

$$\|U_k\|_\tau\leqq C\cdot\|U_{k-1}\|_\tau^2.$$

Proof. For each $\psi \in A_\tau$ the restriction to $[0, 2\pi]$ is in W and there is a constant C_0 such that

$$\|\psi\|_W \leqq C_0 \|\psi\|_\tau.$$

Therefore $\|S'_0\|_2 \leqq M' = 1 + C_0 M$ and if δ_M is chosen so that $C_0 \delta_M \leqq \varepsilon_{M'}$ holds true, with the number $\varepsilon_{M'}$ of Theorem 1, then $\|S_k - S\|_W \to 0$. We assume at first

$$\sum_{l=0}^{k-1} \|U_l\|_\tau \leqq \sigma - \sigma_0.$$

Then S_k maps $\mathfrak{S}_\tau$ into $\mathfrak{S}_\sigma$ and $\eta(S_k(t))$ and $\dot{\eta}(S_k(t))$ are in A_τ. The representation

$$U_k(t) = \frac{\Phi_{k+1}(e^{it}) - \eta(S_k(t))}{\dot{\eta}(S_k(t))} \tag{6.2}$$

holds true for real t. The function Φ_{k+1} is analytic in D^+. Hence $\Phi_{k+1}(e^{it})$ is analytic for $\operatorname{Im} t > 0$, which implies that $U_k(t)$ is analytic for $0 < \operatorname{Im} t < \tau$. Furthermore U_k is continuous for $0 \leqq \operatorname{Im} t < \tau$ and is real whenever t is real. Thus Schwarz's reflection

$$U_k(\bar{t}) = \overline{U_k(t)} \tag{6.3}$$

yields an analytic continuation to a function $U_k \in A_\tau$. Therefore the right hand side of (6.2) is also in A_τ, in particular Φ_{k+1} is analytic for $|z| < e^\tau$. Starting from the representation (4.8) we obtain

$$|\Phi_k^*(z)| \leqq C \|f_k - \Phi_k\|_W \qquad \text{for } |z| \leqq 1.$$

Since $|\dot{\eta}(s)| \geqq c_1 > 0$ in $\mathfrak{S}_\sigma$, the estimate

$$|U_k(t)| \leqq C_U \|\eta(S_k(t)) - \Phi_k(e^{it})\|_\tau$$

holds true for $0 \leqq \operatorname{Im} t < \tau$. In view of (6.3) this implies

$$\|U_k\|_\tau \leqq C_U \|\eta(S_k(t)) - \Phi_k(e^{it})\|_\tau. \tag{6.4}$$

As in (4.8) the difference on the right hand side can be represented by a Taylor remainder term

$$\eta(S_k(t)) - \Phi_k(e^{it}) = U_{k-1}^2 \int_0^1 (1-\lambda)\ddot{\eta}(S_{k-1} + \lambda U_{k-1})\, d\lambda$$

whence the estimate

$$\|\eta(S_k(t)) - \Phi_k(e^{it})\|_\tau \leqq \|\ddot{\eta}\|_\sigma \|U_{k-1}\|_\tau^2$$

follows. We combine this with (6.4) and obtain

$$\|U_k\|_\tau \leqq C \|U_{k-1}\|_\tau^2.$$

The rest of the proof is as in Theorem 1. □

Corollary. *Let η be analytic in a strip $\mathfrak{S}_\sigma$, and assume $|\dot{\eta}(s)| \geqq c_1 > 0$ and $|\ddot{\eta}(s)| \leqq C_1$ in $\mathfrak{S}_\sigma$.*

Let $\tau_0>0$ be so that S is analytic in the strip $\mathfrak{S}_{\tau_0}$ and S: $\mathfrak{S}_{\tau_0}\to\mathfrak{S}_\sigma$. Then for each τ in $0<\tau<\tau_0$ there exists $\delta_\tau>0$ such that for all functions S_0 which are analytic in $\mathfrak{S}_\tau$ and satisfy $\|S-S_0\|_\tau<\delta_\tau$ the iteration converges in A_τ.

With the definitions and the hypotheses of Theorem 3

$$\Phi(e^{it}) := \eta(S(t)) \tag{6.5}$$

defines an analytic continuation of Φ to the circle $|z|<e^\tau$. In view of conformality

$$i\Phi'(e^{it})e^{it} = \dot{\eta}(S(t))S'(t) \tag{6.6}$$

is nonzero for $0\leqq \operatorname{Im} t<\tau$. Therefore $S'(t)\neq 0$ in this domain. The function $S(t)$ is real for real t, hence $S(\bar{t})=\overline{S(t)}$. Therefore $S'(t)\neq 0$ in $\mathfrak{S}_\tau$. Since also $\dot{\eta}\neq 0$ by hypothesis it follows that $\Phi'(z)\neq 0$ for $|z|<e^\tau$. So we obtain the following estimate:

If R is the radius of the largest circle, so that Φ has an analytic continuation for $|z|<R$ such that $\Phi'(z)\neq 0$ for $|z|<R$, then for any analytic parametrization η the number τ in Theorem 3 satisfies

$$e^\tau \leqq R. \tag{6.7}$$

We show now that this bound is best possible. To this aim we use the parametrization $\eta(s)=\Phi(e^{is})$ of the curve by the conformal mapping. Then the parameter map $S(t)$ is the identity $S(t)=t$. Let τ be any number in the interval $0<\tau<\log R$. Then for any choice of σ with $\tau<\sigma<\log R$ the function η satisfies the hypotheses of the last corollary on $\mathfrak{S}_\sigma$ and one can choose $\tau_0=\sigma$. The corollary then yields local convergence in A_τ.

7. The Wittich Method

For the numerical implementation of the iterative method the main problem consists in the approximation of the operator K. We describe here the most important method, which is due to Wittich (see [2], p. 74). Let n be a natural number and $N=2n$. The interval $[0,2\pi]$ is divided into N intervals of equal length by the grid points

$$t_\nu = \frac{\nu-1}{N}2\pi, \qquad \nu=1,2,\ldots,N.$$

For each continuous function $u(t)$ there exists a trigonometric polynomial

$$\tilde{u}(t) := \sum_{\mu=0}^{n} (\tilde{a}_\mu \cos\mu t + \tilde{b}_\mu \sin\mu t) \tag{7.1}$$

with $\tilde{u}(t_\nu)=u(t_\nu)$ for $\nu=1,2,\ldots,N$. The interpolating function $T_N u := \tilde{u}$ is unique if the normalization $\tilde{b}_n=0$ is imposed.

The conjugate function Ku is then approximated by $K_N u := K(T_N u)$ i.e. in view of (2.3)

$$K_N u := \sum_{\mu=1}^{n} (\tilde{a}_\mu \sin \mu t - \tilde{b}_\mu \cos \mu t). \tag{7.2}$$

We give some estimates for K_N in W. At first we note that $K_N u - K u = K(T_N u - u)$. In view of (2.9) it follows that the error

$$\|K_N u - K u\|_W \leqq \|T_N u - u\|_W \tag{7.3}$$

is essentially the interpolation error. If $u \in W$ is expressed by the Fourier series (2.2), then

$$\tilde{a}_0 = a_0 + \sum_{k=1}^{\infty} a_{kN}, \qquad \tilde{a}_n = a_n + \sum_{k=1}^{\infty} a_{kN+n},$$

$$\tilde{a}_\mu = a_\mu + \sum_{k=1}^{\infty} (a_{kN+\mu} + a_{kN-\mu}), \qquad \mu = 1, \ldots, n-1,$$

$$\tilde{b}_\mu = b_\mu + \sum_{k=1}^{\infty} (b_{kN+\mu} - b_{kN-\mu}), \qquad \mu = 1, \ldots, n-1.$$

Lemma 3. *For $u \in W$ the trigonometric interpolation polynomial $\tilde{u} := T_N u$ satisfies the inequalities:*

$$\|(\tilde{u})'\|_2 \leqq \tfrac{\pi}{2} \|u'\|_2, \tag{7.4}$$

$$\|K_N u\|_W \leqq \tfrac{\pi}{2} \|u\|_W, \tag{7.5}$$

$$\|\tilde{u}\|_W \leqq 3\, \|u\|_W. \tag{7.6}$$

Proof. The L^2-norm of the derivative u' can be represented in terms of the quantities

$$\begin{aligned} A_l &:= l^2 a_l^2 + \sum_{k=1}^{\infty} [(kN+l)^2 a_{kN+l}^2 + (kN-l)^2 a_{kN-l}^2], \qquad l = 1, \ldots, n-1 \\ A_0 &:= \sum_{k=1}^{\infty} k^2 N^2 a_{kN}^2, \qquad A_n := \sum_{k=0}^{\infty} (kN+n)^2 a_{kN+n}^2 \end{aligned} \tag{7.7}$$

and the analogously defined B_l as follows

$$\|u'\|_2^2 = \pi \sum_{l=0}^{n} (A_l + B_l).$$

By maximizing the right hand side in

$$|\tilde{a}_l| \leqq |a_l| + \sum_{k=1}^{\infty} (|a_{kN+l}| + |a_{kN-l}|)$$

with the constraint (7.7) it follows that for $l = 1, \ldots, n-1$

$$l^2 \tilde{a}_l^2 \leqq A_l \cdot \sum_{k=-\infty}^{+\infty} (1 + kN/l)^{-2} \leqq \frac{\pi^2}{4} \cdot A_l.$$

This inequality remains true for $l=n$. For the coefficients $\tilde{b}_l$ an analogous estimate follows. So we obtain

$$\|(\tilde{u})'\|_2^2 \leqq \pi \sum_{l=1}^{n} l^2(\tilde{a}_l^2+\tilde{b}_l^2) \leqq \frac{\pi^2}{4} \cdot \|u'\|_2^2$$

whence (7.4) and (7.5) follow.

Similarly as above one can estimate

$$|\tilde{a}_0 - a_0| \leqq \frac{\pi}{N}\sqrt{\frac{A_0}{6}} \leqq \frac{\|u'\|_2}{N}\sqrt{\frac{\pi}{6}}.$$

The inequality (7.6) then follows from

$$\begin{aligned}\|\tilde{u}\|_W &\leqq |a_0| + |\tilde{a}_0 - a_0| + \|\tilde{u} - a_0\|_W \\ &\leqq \|u\|_W + |\tilde{a}_0 - a_0| + \sqrt{\frac{\pi}{6}}\,\|(\tilde{u})'\|_2 \leqq \left(1+\left(\frac{1}{N}+\frac{\pi}{2}\right)\sqrt{\frac{\pi}{6}}\right)\|u\|_W. \quad \square\end{aligned}$$

Let $E_n u$ be the truncated Fourier series of u, i.e.

$$E_n u(t) := \sum_{\nu=0}^{n} a_\nu \cos \nu t + \sum_{\nu=1}^{n-1} b_\nu \sin \nu t.$$

Then $T_N E_n u = E_n u$ and therefore the error of the interpolation can be represented by the error of the approximation by E_n.

$$T_N u - u = T_N(u - E_n u) - (u - E_n u).$$

Using the estimate (7.6) we obtain

$$\|T_N u - u\|_W \leqq 4\,\|E_n u - u\|_W. \tag{7.8}$$

We observe that $\|E_n u - u\|_W \leqq \|u\|_W$. So we derive from (7.3) and (7.8) the general estimate

$$\|K_N u - K u\|_W \leqq 4\,\|u\|_W \quad \text{for all } u \in W. \tag{7.9}$$

Now we are in position to estimate the error of Wittich's method for sufficiently smooth functions.

Lemma 4. a) *If u is analytic in a strip $\mathfrak{S}_\tau$ (see Sect. 6) with $\tau > 0$, then*

$$\|K_N u - K u\|_W \leqq c \cdot \|u\|_\tau N \exp(-\tau N/2). \tag{7.10}$$

b) *If u is in $C^{k+\mu}$ with $k+\mu > 1$, then*

$$\|K_N u - K u\|_W \leqq c \cdot \|u\|_{k+\mu} N^{1-(k+\mu)}. \tag{7.11}$$

Proof. a) We use (7.3) and observe that

$$\|T_N u - u\|_W \leqq C\,\|(T_N u - u)'\|_0. \tag{7.12}$$

The right hand side in (7.12) can be estimated by the inequality (1.4) of [3] and the second estimate on p. 148 of [3]. This yields (7.10).

b) In view of (7.3) and (7.8) it is sufficient to estimate $E_n u - u$. Since this is a Fourier series without constant term, the Sobolev norm is the L^2-norm of the derivative

$$\|(E_n u - u)'\|_2^2 = \pi \left(\sum_{v=n+1}^{\infty} v^2 a_v^2 + \sum_{v=n}^{\infty} v^2 b_v^2 \right). \tag{7.13}$$

We estimate the right hand side by a method which is known from the proof of a theorem of Bernstein [10], p. 240. The k-th derivative $u^{(k)}$ has Fourier coefficients $v^k a_v$ and $v^k b_v$. Therefore

$$u^{(k)}(t+h) - u^{(k)}(t-h) \sim 2 \sum v^k \sin vh (b_v \cos vt - a_v \sin vt),$$

$$\frac{1}{\pi} \int_0^{2\pi} (u^{(k)}(t+h) - u^{(k)}(t-h))^2 \, dt = 4 \sum_{v=1}^{\infty} v^{2k} (a_v^2 + b_v^2) \sin^2 vh.$$

The left hand side does not exceed $2(2h)^{2\mu}$ since u is in $C^{k+\mu}$. On setting $h = \pi/2^{l+1}$ we obtain

$$\sum v^{2k} (a_v^2 + b_v^2) \sin^2 (v\pi/2^{l+1}) \leqq \frac{1}{2} \left(\frac{\pi}{2^l}\right)^{2\mu}.$$

Since $\sin^2(v\pi/2^{l+1}) \geqq 1/2$ for $2^{l-1} < v \leqq 2^l$ it follows that

$$\sum_{2^{l-1} < v \leqq 2^l} v^{2k} (a_v^2 + b_v^2) \leqq \left(\frac{\pi}{2^l}\right)^{2\mu}.$$

This inequality entails

$$\sum_{2^{l-1} < v \leqq 2^l} v^2 (a_v^2 + b_v^2) \leqq \left(\frac{\pi}{2^l}\right)^{2\mu} \cdot 2^{-(l-1)2(k-1)} = C \cdot 2^{-2l(k+\mu-1)}.$$

Since $k + \mu > 1$ one can sum over l which yields

$$\sum_{v > 2^{l-1}} v^2 (a_v^2 + b_v^2) \leqq C \cdot 2^{-2l(k+\mu-1)}.$$

By choosing l so that $2^{l-1} < n \leqq 2^l$ we obtain for the right hand side of (7.13) the bound $Cn^{-2(k+\mu-1)}$ whence (7.11) follows.

8. Numerical Implementation of the Iterative Scheme

The iteration scheme can be performed numerically in the following way. Let n be a natural number and $N = 2n$. Choose equidistant points

$$t_v := (v-1) \cdot 2\pi/N, \qquad v = 1, 2, \ldots, N$$

in the interval $[0, 2\pi]$. We denote the numerically calculated analogues of the functions S_k, v_k etc. and the numbers α_k, $\hat{q}_k$ by $\tilde{S}_k$, $\tilde{v}_k$, $\tilde{\alpha}_k$, $\tilde{\beta}_k$. Assume that S_0

satisfies the hypotheses of Theorem 1 and put $\tilde{S}_0 := S_0$. If $\tilde{S}_k$ is determined, then calculate in close analogy with the formulas (3.6) to (3.11)

$$\tilde{v}_k(t) := \theta(\tilde{S}_k(t)) - t, \tag{8.1}$$

$$\tilde{w}_k := K_N \tilde{v}_k, \tag{8.2}$$

$$\tilde{q}_k := \operatorname{Im}(\exp(\tilde{w}_k - i\theta(\tilde{S}_k))\,\eta(\tilde{S}_k)), \tag{8.3}$$

$$\tilde{\alpha}_k := \frac{1}{N} \sum_{v=1}^{N} \tilde{v}_k(t_v), \qquad \tilde{\beta}_k := \frac{1}{N} \sum_{v=1}^{N} \tilde{q}_k(t_v), \tag{8.4}$$

$$\tilde{r}_k := \tilde{\beta}_k \cot \tilde{\alpha}_k + K_N \tilde{q}_k, \tag{8.5}$$

$$\tilde{U}_k := -\tilde{r}_k \exp(-\tilde{w}_k)/|\dot{\eta}(\tilde{S}_k)| - \operatorname{Re}(\eta(\tilde{S}_k)/\dot{\eta}(\tilde{S}_k)), \tag{8.6}$$

$$\tilde{S}_{k+1} := \tilde{S}_k + \tilde{U}_k. \tag{8.7}$$

Notice that all these functions must be evaluated only at the grid points t_v, and that the numbers $\tilde{S}_{k+1}(t_v)$, $v = 1, \ldots, N$ depend only on the values of $\tilde{S}_k$ at the points t_v. The result of the k-th iteration are the numbers $\tilde{S}_k(t_v)$, $v = 1, \ldots, N$.

All functions $\tilde{v}_k$, $\tilde{w}_k$ etc. are defined in terms of certain trigonometric polynomials on the whole interval $[0, 2\pi]$. Therefore also the functions $\tilde{S}_k$ are defined on $[0, 2\pi]$. But the calculation of the function $\tilde{S}_k$ needs all the auxilary quantities, which determine $\tilde{U}_0, \tilde{U}_1, \ldots, \tilde{U}_{k-1}$. Therefore the whole function $\tilde{S}_k$ cannot easily be reconstructed. Instead we use in the k-th iteration as an approximation for the function S the trigonometric interpolation polynomial

$$S_k^*(t) := t + T_N(\tilde{S}_k(t) - t). \tag{8.8}$$

This function can easily be determined from the actual available data.

The method requires in the computer storage of the order $O(N)$ and if the operator K_N is evaluated by fast Fourier transform (see [4]), computing time of the order $O(N \log N)$. Therefore this method is a very efficient algorithm for the numerical computation of conformal mappings.

For sufficiently smooth curves the numerical error $S_k - \tilde{S}_k$ can be estimated:

Theorem 4. a) *Assume that η is in $C^{l+1+\mu}$, $l \geqq 2$, $0 < \mu < 1$ and the hypotheses of Theorem 2 are satisfied. Then there exist constants C_k and N_k such that*

$$\|S_k - \tilde{S}_k\|_W \leqq C_k N^{1-l-\mu} \quad \textit{for } N \geqq N_k. \tag{8.9}$$

b) *If η is analytic, the hypotheses of Theorem 3 are satisfied and S_k converges in A_τ, then there exist constants C_k and N_k such that*

$$\|S_k - \tilde{S}_k\|_W \leqq C_k N e^{-\tau N/2} \quad \textit{for } N \geqq N_k. \tag{8.10}$$

Proof of statement a) by induction. We assume that (8.9) is true for some $k \geqq 0$ (it is trivial for $k = 0$) and prove it for $k+1$. We denote the error by $E_k := S_k - \tilde{S}_k$. Since

$$E_{k+1} = E_k + U_k - \tilde{U}_k \tag{8.11}$$

we have to estimate the difference

$$U_k - \tilde{U}_k = \frac{\tilde{r}_k \exp(-\tilde{w}_k)}{|\dot{\eta}(\tilde{S}_k)|} - \frac{r_k \exp(-w_k)}{|\dot{\eta}(S_k)|} + \operatorname{Re}\left(\frac{\eta(\tilde{S}_k)}{\dot{\eta}(\tilde{S}_k)} - \frac{\eta(S_k)}{\dot{\eta}(S_k)}\right). \tag{8.12}$$

The last term in (8.12) has the form $H(\tilde{S}_k) - H(S_k)$ with the function $H(s) := \operatorname{Re}(\eta(s)/\dot{\eta}(s))$. It follows from the hypotheses that $H \in C^{1+1}$. So we obtain the estimate

$$\|H(\tilde{S}_k) - H(S_k)\|_W \leqq C\,\|E_k\|_W. \tag{8.13}$$

We define $v_k(t) := \theta(S_k(t)) - t$. The function θ is also in C^{1+1}, whence the estimate

$$\|v_k - \tilde{v}_k\|_W \leqq C\,\|E_k\|_W \tag{8.14}$$

follows. By hypothesis θ is in $C^{l+\mu}$ and it follows from Theorem 2 that $\|v_k\|_{l+\mu} \leqq C$ uniformly for all k. Using (7.5), (7.11), (8.14) and the induction hypothesis (8.9) for E_k we obtain

$$\begin{aligned}\|w_k - \tilde{w}_k\|_W &\leqq \|(K - K_N)\, v_k\|_W + \|K_N(v_k - \tilde{v}_k)\|_W \\ &\leqq C N^{1-l-\mu}.\end{aligned} \tag{8.15}$$

The same type of estimate holds true for $\|\exp(\pm w_k) - \exp(\pm \tilde{w}_k)\|_W$ and consequently also for $\|q_k - \tilde{q}_k\|_W$. Let us consider the difference of the numbers

$$\begin{aligned}\alpha_k - \tilde{\alpha}_k &= \frac{1}{2\pi}\int v_k\, dt - \frac{1}{N}\sum \tilde{v}_k(t_\nu) \\ &= \frac{1}{2\pi}\int v_k\, dt - \frac{1}{N}\sum v_k(t_\nu) + \frac{1}{N}\sum (v_k(t_\nu) - \tilde{v}_k(t_\nu)).\end{aligned} \tag{8.16}$$

The last term on the right hand side can be estimated by $\|v_k - \tilde{v}_k\|_0$, hence by (8.14). The first two terms represent the error of a quadrature formula. We represent v_k by a Fourier series, observing that the quadrature formula is exact for all terms $\cos \nu t$ and $\sin \nu t$ except for $\nu = mN$ with $m = 1, 2, \ldots$. The estimate for the Fourier coefficients of v_k then yields the estimate $O(N^{-l-\mu})$ for the error of the quadrature formula and finally

$$|\alpha_k - \tilde{\alpha}_k| \leqq C N^{1-l-\mu}. \tag{8.17}$$

For N sufficiently large, the right hand side is less than $\pi/8$. In the proof of Theorem 1 we have shown that $|\cot \alpha_k| \leqq 1$. Hence also $\cot \tilde{\alpha}_k$ is bounded. For $|\hat{q}_k - \tilde{\beta}_k|$ we obtain the same estimate as in (8.17). Define $r_k := \gamma_k + K q_k$. Since the sequence q_k is bounded in W we can estimate $r_k - \tilde{r}_k$ as in (8.15). It follows that also $\tilde{w}_k$ and $\tilde{r}_k$ are bounded in W and the first difference on the right hand side of (8.12) can be estimated by the differences $w_k - \tilde{w}_k$ and $r_k - \tilde{r}_k$ and E_k. In view of (8.15), (8.13) and the inductive hypothesis (8.9) this implies

$$\|E_{k+1}\|_W \leqq C N^{1-l-\mu}$$

which completes the inductive step.

With the same argument one can also prove statement b). One has only to use (7.10) instead of (7.11). □

Corollary. *Theorem* 4 *remains valid, if $\tilde{S}_k$ is replaced by S_k^* as defined in* (8.8).

Proof. We start from the representation

$$S_k^* - S_k = (T_N - I)(S_k(t) - t) + T_N(\tilde{S}_k - S_k)$$

with the identity operator I. We observe that $S_k - t$ is either in $C^{l+\nu}$ or in A_τ and estimate the first term on the right hand side as in the proof of Lemma 4. A bound for the last term follows from (7.6) and Theorem 4. □

After each iterative step the analytic function

$$\tilde{\Phi}_{k+1}(e^{it}) := -(\tilde{r}_k - i\tilde{q}_k) \exp(it - \tilde{w}_k + i T_N \tilde{v}_k) \tag{8.18}$$

is defined in terms of trigonometric polynomials. Corollary 2 to Theorem 1 can be used to estimate the error of the function S_{k+1}^* as defined in (8.8).

Lemma 5. *For sufficiently large N the estimate*

$$\|S_{k+1}^* - S\|_W \leqq C_M \|\eta(S_{k+1}^*(t)) - \tilde{\Phi}_{k+1}(e^{it})\|_W$$

holds true with the same constant C_M as in (4.23), *provided the right hand side is less than $C_M \varepsilon_M/2$ with ε_M of Theorem* 1.

By definition for any function u

$$P(e^{it}) := T_N u + i K_N u \tag{8.19}$$

is the restriction to the circle of a polynomial

$$P(z) = \sum_{\nu=0}^{n} d_\nu z^\nu \tag{8.20}$$

with the properties

$$\operatorname{Re} P(e^{it_\nu}) = u(t_\nu), \qquad \nu = 1, \ldots, N \tag{8.21}$$

$$\operatorname{Im} d_0 = \operatorname{Im} d_n = 0. \tag{8.22}$$

There is exactly one polynomial of degree $\leqq n$ which satisfies (8.21) and (8.22). Its boundary values are represented by (8.19).

The polynomial $V_k(z)$ of degree $\leqq n$ with boundary values $-\tilde{w}_k + i T_N \tilde{v}_k$ is determined only by $\tilde{\eta}$. The function $\tilde{\Phi}_{k+1}$ with the boundary values (8.18) is of the form

$$\tilde{\Phi}_{k+1}(z) = z Q_k(z) \exp(V_k(z)) \tag{8.23}$$

with a polynomial Q_k of degree $\leqq n$. The coefficient of z^n in Q_k is real. It follows from the construction of Q_k and V_k and the property (8.21) that $\tilde{\Phi}_{k+1}$ solves the discrete analogon to (3.3).

$$\Phi_{k+1}(e^{it_\nu}) = \eta(\tilde{S}_k(t_\nu)) + \tilde{U}_k(t_\nu)\,\tilde{\eta}(\tilde{S}_k(t_\nu)), \qquad \nu = 1, \ldots, N, \tag{8.24}$$

and satisfies

$$\tilde{\Phi}_{k+1}(0)=0, \qquad \operatorname{Im}\tilde{\Phi}'_{k+1}(0)=0. \tag{8.25}$$

On the other hand, if V_k is already determined from $\tilde{\eta}$, the numbers $\tilde{U}_k(t_\nu)$, $\nu=1,\dots,N$ are uniquely defined by the demand that there exists a polynomial Q_k of degree $\leqq n$ with real highest order coefficient such that the analytic function (8.23) satisfies (8.24) and (8.25).

Now the question arises of whether one could not try to satisfy (8.24) and (8.25) with another $(N+1)$-parameter family of analytic functions $\tilde{\Phi}_{k+1}$. We note that the Ansatz (8.23) has two very important features:

1) The solution of (8.24) can be calculated very efficiently by two operations K_N, which means four real FFT over N points.

2) The numerical iteration scheme follows closely an iteration scheme which converges to the conformal mapping Φ. Hence the result of the numerical calculation is close to Φ.

These features of our method become especially apparent when compared with other similar methods. Fornberg [1] proposed to solve (8.24) with a polynomial $\Phi_{k+1}(z)$ of degree n. He observes that such a polynomial can rather effectively be calculated by a conjugate gradient method in combination with FFT.

The computational cost depends on the number of steps which are performed in the conjugate gradient method. Numerical experience reported in [1] shows that the determination of Φ_{k+1} to sufficient accuracy requires about 30 complex FFTs over $N/2$ points. This is clearly higher than the cost of our method.

A serious drawback is up to now that the theoretical foundation of Fornberg's method is rather poor. The arguments in [1], p. 390, are more motivating than rigorous. It is desirable to obtain more information concerning conditions for convergence, the interpretation of the results and error estimates.

9. Numerical Experiences

We have done calculations (on the Amdahl 470 of the IPP in Garching) for several curves Γ with different degrees of regularity. The shift $\|U_k\|_0 := \max(|U_k(t_\nu)|,\ \nu=1,\dots,N)$ decreases very rapidly in the first few iterations. This reflects the quadratic convergence of the method. But after some iterations $\|U_k\|_0$ decreases more slowly and may finally even increase (see Table 1 in [8]). This behaviour is according to Theorem 4 due to the numerical error. The numerical scheme need not converge. We found in our calculations, that it is best to continue the iteration as long as $\|U_k\|_0$ decreases.

We have done several test calculations for regions with known mapping functions. All calculations were started with $S_0(t)=t$. After the k-th iteration we calculated the numerical error

$$E = \max_{\nu=1,\dots,N} |(\eta(\tilde{S}_k(t_\nu)) - \Phi(e^{it_\nu})| \tag{9.1}$$

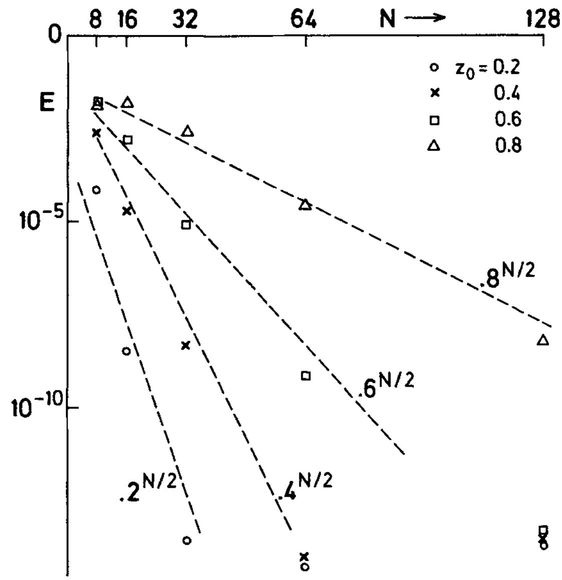

Fig. 1. Numerical errors for eccentric circles and approximation by $C|z_0|^{N/2}$

with the numerically calculated parameter function $\tilde{S}_k$ at the grid points. In the following we choose $k=7$. We compare the numerical error with the estimates of Theorem 4 as well as with the errors in calculations with other methods.

a) For eccentric circles represented by

$$\eta(s)=z_0+e^{is}, \qquad 0<z_0<1, \tag{9.2}$$

the mapping function Φ is the linear function

$$\Phi(z)=z_0+\frac{z-z_0}{1-z_0 z}. \tag{9.3}$$

It has a pole at $z_1=1/z_0$. In Fig. 1 we show E for several choices of z_0 in dependence on the number of grid points N. It turns out that the E in the experiments is rather well represented by the estimate $E=O(|z_0|^{N/2})$ which follows from Theorem 4.

b) We modify the first example by applying a power function

$$\eta(s)=(a+(z_0+e^{is}))^l-a^l \tag{9.4}$$

with a real number a and a natural number l. The theoretical mapping function Φ^* is easily obtained

$$\Phi^*(z)=(a+\Phi(z))^l-a^l \tag{9.5}$$

in terms of the function Φ of (9.3). It has a singularity at $z_1=1/z_0$. Furthermore, $\Phi^{*\prime}$ has a zero at $z_2=a/(z_0(z_0+a)-1)$. From Theorem 4 and the remark

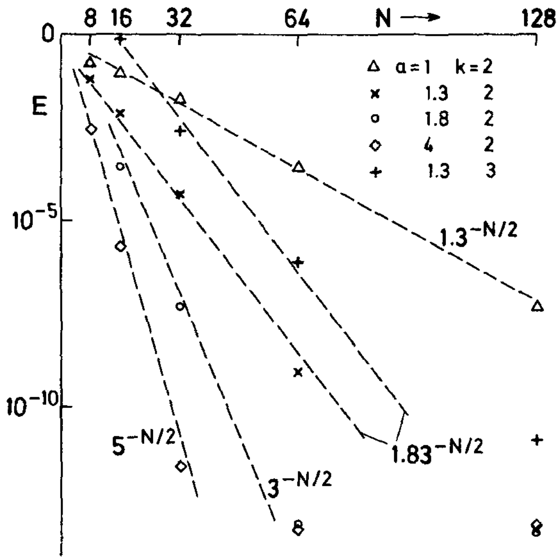

Fig. 2. Numerical errors for example b) with $z_0 = 0.2$ and approximation by the error bound (9.6)

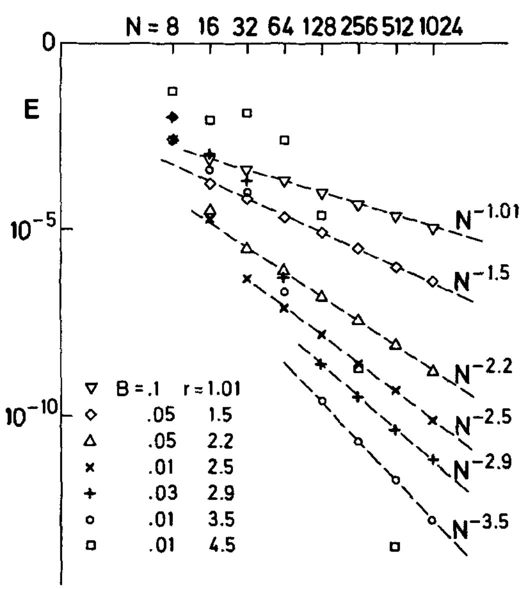

Fig. 3. Numerical errors for example c) with $A = 1$ and approximation by CN^{-r}

at the end of Sect. 5 one obtains an error estimate

$$E = O(\max(|z_1|^{-N/2}, |z_2|^{-N/2})). \tag{9.6}$$

The results shown in Fig. 2 confirm this estimate. It gives evidence that the attainable accuracy is not only limited by the singularities of the mapping function outside the disc but also by the zeroes of its derivative.

Table 1. Numerical errors E in the calculated values of the mapping function for the circle with centre at z_0. Calculations with N grid points. First line: Fornberg's method, second line: Theodorsen's method, lower line: our method

	z_0			
	0.2	0.4	0.6	0.8
$N=16$	0.98E-6	0.45E-3	0.15E-1	0.21E-0
	0.12E-6	0.65E-4	0.36E-1	
	0.32E-7	0.22E-4	0.15E-2	0.16E-1
$N=32$	0.25E-11	0.30E-6	0.25E-3	0.35E-1
	0.16E-12	0.23E-7	0.28E-4	
	0.27E-13	0.45E-8	0.82E-5	0.27E-2
$N=64$	0.24E-14	0.13E-12	0.70E-7	0.76E-3
	0.17E-14	0.81E-14	0.43E-8	
	0.58E-14	0.10E-13	0.67E-9	0.27E-4
$N=128$	0.13E-14	0.22E-14	0.77E-14	0.61E-6
	0.28E-14	0.92E-14	0.23E-13	
	0.22E-13	0.33E-13	0.48E-13	0.67E-8

c) Let r be a positive real number, $r=l+\mu$, then

$$\eta(s) := A \cdot e^{is} + B \left|2 \sin \frac{s}{2}\right|^r \exp\left(ir \frac{(s-\pi)}{2}\right) - C, \quad 0 \leqq s \leqq 2\pi, \tag{9.7}$$

is for $0<\mu<1$ a function in the Hölder class $C^{l+\mu}$. If $C=0.4A+0.6^r B$, and $|A|>|B| r 2^{r-1}$, then the function

$$\Phi(z) = A\Psi(z) + B(1-\Psi(z))^r - C, \tag{9.8}$$

which involves the linear function $\Psi(z) := (5z+2)/(5+2z)$, maps the disc conformally onto the region bounded by the curve parametrized by (9.7). Figure 3 shows that the numerical errors behave for large N like $CN^{-(l+\mu)}$. This is better than the estimate $O(N^{2-(l+\mu)})$, which follows from Theorem 4. Note that the method works well also for $r=1.01$ and $r=1.5$ where η is not Lipschitz continuous.

In order to get an impression how the accuracy of our method compares with that of other methods we have done calculations for some examples also with the methods of Fornberg [1] and Theodorsen.

With Fornberg's method we performed 7 "outer iterations" and stopped the "inner iterations" of the conjugate gradient method as soon as the changes became less than 10^{-12}. This criterion was on the average fulfilled after 3 steps. The iterations with Theodorsen's method were continued as long as the error decreased. It does not converge for $z_0=0.8$. The numerical error for the series of calculations for eccentric circles are listed in Table 1.

Our method clearly yields the most accurate results. The errors of the Theodorsen method are larger by a factor of 3 and the errors of the Fornberg method by a factor 10–100. The lower left part of this table suggests that other methods may be less sensitive to rounding errors.

For calculations for reflected ellipses ([2], p. 264) Theodorsen's method yielded the most accurate results, while the errors of Fornberg's as well as of our method were larger by a factor 7–20.

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