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Conformal Mapping by the Method of Alternating Projections

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Dedicated to the memory of Peter Henrici

Summary. The functional analytic principle of alternating projections is used to construct an iterative method for numerical conformal mapping of the unit disc onto regions with smooth boundaries. The result is a simple method which requires in each iterative step only two complex Fourier transforms. Local convergence can be proved using a theorem of Ostrowski. Convergence is linear. The asymptotic convergence factor is equal to the spectral radius of a certain operator. A version with overrelaxation as well as a discretized version are discussed along the same lines. For regions which are close to the unit disc convergence is fast. For some familiar regions the convergence factors can be calculated explicitly. Finally, the method is compared with Theodorsen's.

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1 Introduction

Let G be a bounded simply connected region in the complex plane $\mathbb{C}$ with $0 \in G$. We assume that the boundary curve $\Gamma := \partial G$ of G is parametrized by a differentiable 2π-periodic complex function $\eta(s)$, whose derivative $\dot{\eta}$ is continuous, does not vanish and has winding number 1 on $[0, 2\pi]$. This means, that $\eta(s)$ surrounds G once in the counterclockwise direction, when s passes from 0 to 2π.

By $\hat{\Phi}$ we denote the conformal mapping of the unit disc $D := \{z : |z| < 1\}$ onto the region G normalized by the conditions

$$\hat{\Phi}(0) = 0, \quad \hat{\Phi}'(0) > 0. \tag{1.1}$$

The smoothness of the boundary implies, that $\hat{\Phi}$ can be extended to a continuous function on the closed disc $\bar{D}$, and

$$\hat{\Phi}(e^{it}) \in \Gamma \quad \text{for all } t. \tag{1.2}$$

In view of this, the conformal mapping is characterized by two properties. First, it is a function analytic in the disc, continuous to the boundary and normalized by (1.1). Secondly, it maps D onto G or at least it maps the circle onto the curve Γ, i.e. satisfies (1.2). Several of the known iteration procedures for calculating the conformal mapping numerically construct essentially two sequences of functions, namely a sequence of normalized analytic functions in the disc and a sequence of functions which map the unit circle into the boundary curve Γ. The former sequence is constructed by means of the operator K of conjugation. Therefore, the alternation between these two types of functions is very natural for an iterative solution of the conformal mapping problem.

In this paper we use the functional analytic principle of alternating projections devised by J. von Neumann [6] for the construction of such an alternating sequence of functions. We benefit from the fact that all projectors needed can be constructed very easily in terms of Fourier series. The local convergence of the resulting method can be proved for sufficiently smooth boundary curves Γ by using a generalization of a theorem of Ostrowski [8, 4]. Convergence is linear with a factor which is the spectral radius of a certain operator. The convergence can be accelerated by overrelaxation.

For numerical purposes the method can be discretized on a grid of N equidistant points. We prove, that for sufficiently large N this discrete method also converges locally. The limit functions are interpolating polynomials whose existence and relevance for conformal mapping were recently investigated [12]. The method uses in each iterative step only two complex Fourier transforms (no conjugation). It is therefore one of the simplest methods available. We give for nearly circular regions estimates for the convergence factors in terms of a quantity which measures in a sense the distance from the circle. Finally, we compare the method of alternating projections with the method of Theodorsen, and report on a series of test calculations. The speed of convergence of both methods is comparable. A shortcoming of the Theodorsen method is, that it is applicable only for star-shaped regions and in addition requires even that a representation of the boundary curve in polar coordinates is available. The method of alternating projections is more flexible in this respect. It is applicable for any region (not necessarily star-shaped) and any parametrization of the boundary curve.

2 Functional Analytic Tools

In this paper all spaces, also the spaces of complex valued functions, are considered as linear spaces over the scalar field $\mathbb{R}$ of real numbers.

Let L^2 be the space of 2π-periodic complex functions f which are square integrable over $[0, 2\pi]$. With inner product

$$(f, g)_2 := \frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} f(t)\overline{g(t)}dt$$

this is a Hilbert space over $\mathbb{R}$.

Let W be the Sobolev space of 2π-periodic absolutely continuous functions f such that the derivative f' is in L^2. With the inner product

$$(f,g)_W := (f,g)_2 + (f',g')_2$$

this is a Hilbert space over $\mathbb{R}$. By L^2_r and W_r we denote the subspaces of real functions ϕ in L^2 and W, respectively.

Each function f in L^2 has a Fourier series

$$f(t) \sim \sum_{\nu=-\infty}^{+\infty} a_\nu e^{i\nu t}. \tag{2.1}$$

By splitting this series at the coefficient a_1 one obtains a representation $f = f^- + f^+$ with

$$f^-(t) \sim \sum_{\nu=-\infty}^{0} a_\nu e^{i\nu t} + i(\operatorname{Im} a_1)e^{it}, \qquad f^+(t) \sim (\operatorname{Re} a_1)e^{it} + \sum_{\nu=2}^{\infty} a_\nu e^{i\nu t}. \tag{2.2}$$

This effects a direct sum decomposition $L^2 = L^{2-} \oplus L^{2+}$. Functions in W can be split in the same way, which leads to the analogous direct sum decomposition $W = W^- \oplus W^+$. The summands W^- and W^+ are orthogonal.

If $\dot{\eta}$ is Hölder continuous, the function $\Phi(t) := \hat{\Phi}(e^{it})$ is Hölder continuously differentiable [9]. In particular, Φ is in W. The fact, that Φ is the boundary function of an analytic function which is restricted by (1.1) can now be expressed simply by saying $\Phi \in W^+$.

The boundary function maps the unit circle into the boundary of G. Therefore there is a real function $\hat{U}(t)$ such that

$$\Phi(t) = \eta(t + \hat{U}(t)) \tag{2.3}$$

holds for all t. It is a consequence of the implicit function theorem that $\hat{U}$ is continuously differentiable, hence $\hat{U} \in W_r$.

For the investigation of the right hand side of (2.3) we need the following facts which follow from Lemma 6 in [12].

Lemma 1. *If η is twice differentiable and $\ddot{\eta}$ is Hölder continuous, then*

$$E(U) := \eta(t + U(t))$$

defines a continuous mapping of W_r into W. It is Fréchet differentiable. The derivative $E'(U)$ of E at $U \in W_r$ is

$$E'(U)V = \dot{\eta}(t + U(t)) \cdot V(t),$$

i.e., it is the operator of multiplication with the function $\dot{\eta}(t + U(t))$.

In view of this lemma the set

$$M := \{\eta(t + U(t)) : U \in W_r\}$$

is a differentiable manifold in W parametrized by W_r. The tangent space T_U at a specified point $\eta(t + U(t))$ is given in parametric form by

$$T_U := \{\eta(t + U(t)) + \dot{\eta}(t + U(t)) \cdot V(t);\ V \in W_r\}.$$

Now we get a simple geometric interpretation of Eq. (2.3): The function Φ is in the intersection of the manifold M and the linear space W^+:

$$\Phi \in M \cap W^+ . \tag{2.4}$$

3 The Method of Alternating Projections

The method of alternating projections was first used by von Neumann [6, p. 55] to determine the intersection of two closed linear spaces. It can easily be adapted to calculate the intersection of a manifold with a linear space as required by (2.4).

The operator P which associates to a function f with Fourier series (2.1) the function f^+ defined in (2.2) is an orthogonal projector in L^2 as well as in W.

Assume that at the beginning of the k-th iteration there is a function $U_k \in W_r$. Then the function $\eta(t+U_k(t))$ is an element of the manifold M. By means of the projector P this function is mapped onto W^+. The result is an analytic function Φ_k. In the second step this function Φ_k is projected onto the tangent space T_k of the manifold M at the point $\eta(t+U_k(t))$. The image is an element of T_k, i.e., it is a function of the form $\eta(t+U_k(t))+\dot{\eta}(t+U_k(t))\cdot V_k(t)$. A projection onto T_k, which is orthogonal in L^2, is obtained by putting

$$V_k(t) := \operatorname{Re}\left(\frac{\Phi_k(t)-\eta(t+U_k(t))}{\dot{\eta}(t+U_k(t))}\right). \tag{3.1}$$

(In general this mapping from W^+ to T_k is not orthogonal in W!) The result of the k-th iteration is then $U_{k+1}(t) := U_k(t)+V_k(t)$.

These geometric considerations are the motivation for the following alternating projection iteration scheme (AP-method for short), which can be written in a more compact form:

AP-Method

1. Start with a function $U_0 \in W_r$.

2. Given U_k for $k \geq 0$ calculate for $\nu = 1, 0, -1, -2, \ldots$ the Fourier coefficients a_ν of the function $\eta(t+U_k(t))$, i.e. form

$$a_\nu := \frac{1}{2\pi} \int_0^{2\pi} \eta(t+U_k(t)) e^{-i\nu t} dt .$$

3. Calculate the new iterate

$$U_{k+1}(t) := U_k(t) - \operatorname{Re}\left(\frac{i(\operatorname{Im} a_1)e^{it} + \sum_{\nu=-\infty}^{0} a_\nu e^{i\nu t}}{\dot{\eta}(t+U_k(t))}\right).$$

This is a stationary iteration scheme of the form

$$U_{k+1} = HU_k \tag{3.2}$$

with the nonlinear operator H defined by

$$HU(t) := U(t) - \mathrm{Re}\left(\frac{(I-P)\eta(t+U(t))}{\dot{\eta}(t+U(t))}\right). \tag{3.3}$$

I denotes the identity operator. The projection P of M onto W^+ is orthogonal in W, but the mapping of W^+ to T_k defined by (3.1) is not. Therefore the method described above is the method of alternating projections in L^2. From the construction we may expect only convergence in L^2. We aim to prove convergence in W. To achieve this we have to assume more stringent smoothness properties of η.

The function $\hat{U}$ is a fixed point of the iteration (3.2), since $(I-P)\eta(t+\hat{U}(t))=0$. We want to apply a theorem of Ostrowski in order to prove convergence. To this aim we investigate the derivative of the mapping $H: W_r \to W_r$.

With the methods used in the proof of Lemma 5 in [12] one can show the following: If η is three times differentiable and $\dot{\eta}$ is Hölder continuous, then H is Fréchet differentiable. The derivative $H'(\hat{U})$ at $\hat{U}$ is

$$H'(\hat{U})\phi := \phi(t) - \mathrm{Re}\left(\frac{(I-P)(\dot{\eta}(t+\hat{U}(t))\cdot\phi(t))}{\dot{\eta}(t+\hat{U}(t))}\right). \tag{3.4}$$

After introducing functions r and θ by $\dot{\eta}(t+\hat{U}(t)) = r(t)\exp(i\theta(t))$ with $r>0$ one can write formula (3.4) in the form

$$H'(\hat{U})\phi = \frac{1}{r} A(r\phi) \tag{3.5}$$

with the operator

$$A\psi := \mathrm{Re}\,(e^{-i\theta}P(e^{i\theta}\psi)). \tag{3.6}$$

Similarly as in [11] one can express the operator P in terms of singular integrals. By inserting these into (3.6) one obtains the representation

$$A = \tfrac{1}{2}(I-R) - R_1, \tag{3.7}$$

where R and R_1 are integral operators with kernels

$$R(t,s) := \begin{cases} -\dfrac{1}{2\pi}\,\dfrac{\sin(\theta(t)-\theta(s)-(t-s)/2)}{\sin((t-s)/2)}, & \text{for } t \neq s, \\ -\dfrac{1}{2\pi}\,(2\theta'(t)-1), & \text{for } t=s, \end{cases} \tag{3.8}$$

$$R_1(t,s) := \frac{1}{2\pi}\sin(\theta(t)-t)\cdot\sin(\theta(s)-s). \tag{3.9}$$

Since η is twice continuously differentiable, the kernels $R(t,s)$ and $R_1(t,s)$ are continuous. The operator R_1 is of rank 1. Its range consists of multiples of the function $\sin(\theta(t)-t) \in W_r$. Therefore R_1 maps W_r into W_r. Since by the definition (3.6) the operator A also maps W_r into W_r, the same is true for the integral operator $R = I - 2(A+R_1)$.

On the other hand the integral operators R and R_1 are in a natural way defined on L^2_r. Consequently, A can be extended by (3.7) to an operator $A: L^2_r \to L^2_r$.

Therefore, we can consider the operators A, R, R_1 either as operators, which map W_r into W_r, or as operators, which map L_r^2 into L_r^2. We shall show in the next lemma, that the spectral properties of these operators are independent of the choice of the underlying space.

Lemma 2. *The operator R is compact in L_r^2 as well as in W_r. For each real number μ the spectra of $R+\mu R_1$ in W_r and in L_r^2 are the same. The spectrum consists in any case only of the point 0 and of isolated eigenvalues $\lambda \neq 0$ which are all real.*

Proof. 1) Following [11, Sect. 4] we associate with each function $V \in W$ an operator R_V defined by

$$R_V \psi := \mathrm{Re}\,(\bar{V} \cdot P_0(V \cdot \psi)),$$

in terms of the operator P_0, which assigns to each f with Fourier series (2.1) the function

$$P_0 f(t) \sim \sum_{\nu=-\infty}^{0} a_\nu e^{i\nu t} - \sum_{\nu=1}^{+\infty} a_\nu e^{i\nu t}.$$

It was shown in Lemma 3 of [11] that in the case of a finite trigonometric polynomial

$$V_n(t) = \sum_{\nu=-n}^{n} v_\nu e^{i\nu t}$$

the associated operator R_{V_n} is a finite dimensional operator with range contained in the linear span of $\{1, e^{\pm it}, \ldots, e^{\pm int}\}$. Each function $V \in W$ can be approximated in the W-norm by the finite sections V_n of its Fourier series. It follows from Lemma 2 in [11] that $\|R_V - R_{V_n}\| \to 0$ holds true in the L^2-norm as well as in the W-norm. Consequently, for each $V \in W$ the operator R_V can be approximated by finite-dimensional operators. Therefore, the operator R_V is compact in L_r^2 as well as in W_r. This applies in particular to the operator R which is of the form R_V with $V = e^{i\theta}$. Since the one-dimensional operator R_1 is trivially compact, $R+\mu R_1$ is compact. Therefore, the spectrum of $R+\mu R_1$ consists only of isolated eigenvalues $\lambda \neq 0$ and the point 0.

2) Each isolated eigenvalue $\lambda \neq 0$ is the limit point of eigenvalues λ_n of the approximating operators $R_{V_n} + \mu R_1$. On the other hand each limit point $\neq 0$ of λ_n is eigenvalue of $R+\mu R_1$ (see e.g. [3, p. 71]). This holds true in L_r^2 as well as in W_r. Since the range of the approximating finite-dimensional operators $R_{V_n} + \mu R_1$ is contained in W_r their spectra are independent of whether these operators are considered in W_r or L_r^2. Therefore, the spectrum of $R+\mu R_1$ is also independent.

3) Since the kernels (3.8) and (3.9) of the integral operators R and R_1 are symmetric, $R+\mu R_1$ is selfadjoint in L_r^2. Therefore, each eigenvalue λ is a real number. □

This lemma assures, that the spectrum of A is independent of whether the operator is considered in W_r or in L_r^2. Since $H'(\hat{U})$ is similar to A in view of (3.5), the same is true for $H'(\hat{U})$. Therefore we can omit in the next lemma the specification of the space.

Lemma 3. *The spectrum of the operator $H'(\hat{U})$ consists only of the point $\frac{1}{2}$ and of isolated eigenvalues $\lambda \neq \frac{1}{2}$ which are all real and contained in the interval $0 \leq \lambda < 1$.*

Proof. 1) In view of (3.5) $H'(\hat{U})$ is similar to A. Therefore it has the same spectrum as A. It follows from Lemma 2 and the representation (3.7), that the spectrum of A consists of the point $\frac{1}{2}$ and of eigenvalues $\lambda \neq \frac{1}{2}$ which are all isolated and real.

2) The representation (3.6) implies that

$$\begin{aligned} \|A\psi\|_2 &= \|\operatorname{Re}(e^{-i\theta}P(e^{i\theta}\psi))\|_2 \leqq \|e^{-i\theta}P(e^{i\theta}\psi)\|_2 \\ &= \|P(e^{i\theta}\psi)\|_2 \leqq \|e^{i\theta}\psi\|_2 = \|\psi\|_2 . \end{aligned} \tag{3.10}$$

Therefore, $|\lambda| \leqq 1$ for all eigenvalues λ of A. On the other hand, replacing P by the complementary projector $I-P$ shows that also $\|(I-A)\psi\|_2 \leqq \|\psi\|_2$, which implies $|1-\lambda| \leqq 1$. Both inequalities can be combined to $0 \leqq \lambda \leqq 1$.

3) If $\lambda = 1$ is an eigenvalue of A, then there exists a function $\psi_1 \in L_r^2$ such that $A\psi_1 = \psi_1$. This implies in particular

$$\|A\psi_1\|_2 = \|\psi_1\|_2 .$$

Therefore, in the chain of inequalities (3.10) equality must hold for ψ_1 at each place. We consider the first $\leqq$-sign and use the simple observation that $\|\operatorname{Re} f\|_2 \leqq \|f\|_2$ holds true with equality if and only if $\operatorname{Im} f = 0$. Applied to the first $\leqq$-sign in (3.10) this yields

$$\operatorname{Im}(e^{-i\theta}p(t)) = 0 \tag{3.11}$$

for the function $p := P(e^{i\theta}\psi_1) \in W^+$. We make use of the fact, that p is the boundary function of an analytic function $\hat{p}$ normalized by the conditions

$$\hat{p}(0) = 0, \qquad \operatorname{Im}\hat{p}'(0) = 0 . \tag{3.12}$$

Then (3.11) is a homogeneous Riemann-Hilbert problem for the analytic function $\hat{p}$. If we consider at first only the first of the conditions (3.12), then we obtain from the general theory (Lemma 2 in [10]), that the general solution of (3.11) which also satisfies $\hat{p}(0) = 0$, is represented by

$$\hat{p}(e^{it}) = a_0 e^{it} \exp(i(\theta(t) - t) - w(t))$$

with a real number a_0 and the conjugate function w of $\theta(t) - t$. From this we obtain $\hat{p}'(0) = a_0 e^{i\alpha}$ with

$$\alpha := \frac{1}{2\pi} \int_0^{2\pi} (\theta(t) - t)\, dt = \pi/2 \pmod{2\pi} . \tag{3.13}$$

The last equality stems from the fact, that θ is derived from the conformal mapping (see e.g. formula (1.8) in [10]). It follows that $\hat{p}'(0) = ia_0$ is purely imaginary. Therefore the second of the conditions (3.12) implies $a_0 = 0$, hence $p = 0$, and finally $\psi_1 = 0$ in view of the fact that (3.10) holds for ψ_1 with equality. Therefore $\lambda = 1$ cannot be an eigenvalue of A. □

We denote by $r_\sigma(B)$ the spectral radius of an operator B. We have seen, that $\hat{U}$ is a fixed point of the AP-method. The preceeding lemma shows that the Fréchet derivative of the iteration operator H has spectral radius less than 1. It follows from the theorem of Ostrowski generalized for operators in Banach spaces [4] that $\hat{U}$ is an attractive fixed point for the iteration method. We summarize these results in the following:

Theorem 1. *If η is three times differentiable and $\tilde{\eta}$ is Hölder continuous, then there is an $\varepsilon > 0$ such that for each start function $U_0 \in W_r$ with $\|U_0 - \hat{U}\|_W < \varepsilon$ the iterates U_k obtained by the alternating projection method converge to $\hat{U}$ in the W-norm. Convergence is linear with an asymptotic convergence factor*

$$q_{\text{con}} = \tfrac{1}{2} r_\sigma (I - R - 2R_1). \tag{3.14}$$

In view of the definition of the convergence factor [7, p. 288] the last statement of the theorem means that

$$\overline{\lim} (\|U_k - \hat{U}\|_W)^{\frac{1}{k}} \leqq q_{\text{con}}$$

as $k \to \infty$ for all start functions U_0 with $\|U_0 - \hat{U}\|_W < \varepsilon$. Roughly spoken, the error is reduced in each iteration by about the factor q_{con}.

4 Alternating Projections with Overrelaxation

Lemma 3 shows that $H'(\hat{U})$ is positive semi-definite. This expresses the fact, which is intuitively clear, that the iterative process (3.2) obtained by alternating projections approaches the stationary point from one side. Furthermore, the spectral radius of $H'(\hat{U})$ can never be less than $\frac{1}{2}$. This sets the lower bound $\frac{1}{2}$ to the convergence factor (3.14) of the iteration. As we shall see now, this situation can be improved by introducing a relaxation factor. We consider the following alternating projection method with overrelaxation (OAP-method for short):

OAP-Method

1. Start with a function $U_0 \in W_r$.
2. Given U_k for $k \geqq 0$ calculate for $\nu = 1, 0, -1, -2, \ldots$ the Fourier coefficients a_ν of the function $\eta(t + U_k(t))$, i.e. form

$$a_\nu := \frac{1}{2\pi} \int_0^{2\pi} \eta(t + U_k(t)) e^{-i\nu t} dt$$

3. Calculate the new iterate

$$U_{k+1}(t) := U_k(t) - \operatorname{Re} \left(\frac{i(\operatorname{Im} a_1) e^{it} + 2 \sum_{\nu=-\infty}^{0} a_\nu e^{i\nu t}}{\dot{\eta}(t + U_k(t))} \right). \tag{4.1}$$

Observe that in (4.1) the relaxation factor 2 is applied only to the terms with $\nu \leqq 0$ but not to the e^{it}-term. This iteration can be described in a quite similar way as above. One has just to use instead of the projector $I - P$ the operator Q_2, which assigns to each function f with Fourier series (2.1) the function

$$Q_2 f \sim 2 \sum_{\nu=-\infty}^{0} a_\nu e^{i\nu t} + i(\operatorname{Im} a_1) e^{it}. \tag{4.2}$$

Then the OAP-method is a stationary iteration scheme of the form

$$U_{k+1} = H_2 U_k \tag{4.3}$$

with the nonlinear operator

$$H_2 U(t) := U(t) - \mathrm{Re}\left(\frac{Q_2 \eta(t+U(t))}{\dot{\eta}(t+U(t))}\right). \tag{4.4}$$

Since $Q_2 \eta(t+\hat{U}(t)) = 0$ the function $\hat{U}$ is a fixed point of this iteration. Under the same hypotheses on η as before, H_2 is Fréchet differentiable. The derivative $H_2'(\hat{U})$ at $\hat{U}$ is

$$H_2'(\hat{U})\phi := \phi(t) - \mathrm{Re}\left(\frac{Q_2(\dot{\eta}(t+\hat{U}(t))\cdot\phi(t))}{\dot{\eta}(t+\hat{U}(t))}\right).$$

This can be written in the form

$$H_2'(\hat{U})\psi = \frac{1}{r} A_2(r\psi) \tag{4.5}$$

with the operator

$$A_2\psi := \psi - \mathrm{Re}(e^{-i\theta} Q_2(e^{i\theta}\psi)). \tag{4.6}$$

One can represent this operator in terms of the integral operators R and R_1 with the kernels defined in (3.8) and (3.9) in the following way:

$$A_2 = -R - R_1. \tag{4.7}$$

Lemma 4. *The spectrum of the operator $H_2'(\hat{U})$ consists only of the point 0 and isolated eigenvalues $\lambda \neq 0$, which are all real and contained in the interval $-1 < \lambda < 1$.*

Proof. 1) It follows from (4.7) and Lemma 2 that A_2 is a compact operator, and that the spectrum consists only of isolated real eigenvalues $\lambda \neq 0$ and the point 0.

2) From the representation

$$A_2\psi := \mathrm{Re}(e^{-i\theta}(I - Q_2)(e^{i\theta}\psi))$$

and the estimate $\|(I-Q_2)f\|_2 \leqq \|f\|_2$ we obtain $\|A_2\psi\|_2 \leqq \|\psi\|_2$, from which $-1 \leqq \lambda \leqq 1$ follows.

3) If $\lambda = 1$ is an eigenvalue of A_2, then there exists ψ_+ such that $A_2\psi_+ = \psi_+$, which in view of (4.6) implies

$$\mathrm{Re}(e^{-i\theta} Q_2(e^{i\theta}\psi_+)) = 0. \tag{4.8}$$

The function $q(t) := Q_2(e^{i\theta}\psi_+)$ is in W^-. Therefore, it is of the form $q(t) = \hat{q}(e^{it})$ with a function $\hat{q}$ analytic outside the disc D with a pole of first order in ∞ with imaginary residue. In view of (4.8) the function $\hat{q}$ solves the homogeneous Riemann-Hilbert problem:

$$\mathrm{Re}(e^{-i\theta}\hat{q}(e^{it})) = 0. \tag{4.9}$$

It follows from [10] that (4.9) has a non trivial solution with a simple pole in ∞. This solution is unique up to a constant real factor. Property (3.13) implies that the residue of the pole in ∞ is real. Therefore, the requirement, that $\hat{q}$ has imaginary residue, enforces $\hat{q} = 0$.

Hence the component of $e^{i\theta}\psi_+$ in W^- vanishes and $e^{i\theta}\psi_+ = p$ with a function $p \in W^+$. But this implies that $\mathrm{Im}(e^{-i\theta}p(t)) = 0$, which is impossible for nonzero

functions $p \in W^+$, as we have seen in part 3 of the proof of Lemma 3. Hence $\psi_+ = 0$, and 1 cannot be an eigenvalue of A_2.

4) If $\lambda = -1$ is an eigenvalue of A_2, then there exists ψ_- such that $A_2\psi_- = -\psi_-$, which in view of (4.6) implies

$$\mathrm{Re}(e^{-i\theta}Q_2(e^{i\theta}\psi_-)) = 2\psi_- .$$

Because of $\|Q_2 f\|_2 \leqq 2\|f\|_2$ it follows from (4.6) that $\|\mathrm{Re}(e^{-i\theta}Q_2(e^{i\theta}\psi))\|_2 \leqq 2\|\psi\|_2$ with equality only if

$$Q_2(e^{i\theta}\psi) = 2e^{i\theta}\psi . \tag{4.10}$$

The function $q_- := Q_2(e^{i\theta}\psi_-)$ is the boundary function $q_-(t) = \hat{q}_-(e^{it})$ of a function $\hat{q}_-$ analytic in the exterior of the disc. It follows from (4.10) that the e^{it}-term of $\hat{q}_-$ vanishes. Therefore, $\hat{q}_-$ is analytic also at the point ∞. Furthermore, this $\hat{q}_-$ solves the homogeneous Riemann-Hilbert problem $\mathrm{Im}(e^{-i\theta}\hat{q}_-(e^{it})) = 0$. But Lemma 2 of [10] shows that this problem has no nontrivial solution $\hat{q}$, which is analytic outside D including ∞. Therefore, $\psi_- = 0$, and -1 is not an eigenvalue of A_2.

5) The representation (4.5) shows that $H_2'(\hat{U})$ is similar to A_2. Therefore the spectra coincide. □

This lemma shows, that the Fréchet derivative of the iteration operator H_2 has spectral radius less than 1. Again we can invoke the generalized theorem of Ostrowski [4] in order to show that $\hat{U}$ is an attractive fixed point for this iteration method:

Theorem 2. *If η is three times differentiable and $\bar{\eta}$ is Hölder continuous, then there is an $\varepsilon > 0$ such that for each start function $U_0 \in W_r$ with $\|U_0 - \hat{U}\|_W < \varepsilon$ the iterates U_k obtained by the alternating projection method with overrelaxation converge to $\hat{U}$ in the W-norm. Convergence is linear with an asymptotic convergence factor*

$$q_{\mathrm{con}} = r_\sigma(R + R_1) . \tag{4.11}$$

5 The discretized AP-Method

Let $t_\mu := (\mu - 1)2\pi/N$, $\mu = 1, \ldots, N$, be N equidistant grid points in the interval $[0, 2\pi)$ with $N = 2n$ an even number. By $W_N^\pm$ we denote spaces of in a certain way normalized Fourier polynomials

$$W_N^+ := \{p(t) = \sum_{\nu=1}^{n+1} p_\nu e^{i\nu t}, \quad \mathrm{Im}\, p_1 = \mathrm{Im}\, p_{n+1} = 0\}, \tag{5.1}$$

$$W_N^- := \{q(t) = \sum_{\nu=-n+1}^{1} q_\nu e^{i\nu t}, \quad \mathrm{Re}\, q_{-n+1} = \mathrm{Re}\, q_1 = 0\}. \tag{5.2}$$

We denote vectors in $\mathbb{R}^N$ and $\mathbb{C}^N$ by boldface letters and use the euclidean norm $\|\cdot\|_2$. We assemble the grid points t_μ to a vector $\mathbf{t}$ with components t_μ. If f is a continuous function and $\mathbf{s} \in \mathbb{R}^N$ we mean by $f(\mathbf{s})$ the vector with components $f(s_\mu)$. The product $\mathbf{x} \cdot \mathbf{y}$ and quotient $\mathbf{x}/\mathbf{y}$ of vectors $\mathbf{x}$ and $\mathbf{y}$ are understood as the vectors with components $x_\mu \cdot y_\mu$ and x_μ / y_μ, respectively.

Now we can introduce and investigate the discrete alternating projection method. We study only the form with overrelaxation. The normal form is obtained by omitting the factor 2 in (5.4).

Discretized OAP-Method

1. Start with a vector $\mathbf{s}_0 \in \mathbb{R}^N$.

2. Given a vector $\mathbf{s}_k$ for $k \geqq 0$ with components $s_{k,\mu}$ calculate for $v = 1, 0, -1, \ldots, 1-n$ the Fourier coefficients

$$a_v := \frac{1}{N} \sum_{\mu=1}^{N} \eta(s_{k,\mu}) e^{-ivt_\mu}. \tag{5.3}$$

3. Form the new vector $\mathbf{s}_{k+1}$ with components

$$s_{k+1,\mu} := s_{k,\mu} - \mathrm{Re}\left(\frac{i(\mathrm{Im}\, a_1) e^{it_\mu} + i(\mathrm{Im}\, a_{1-n}) e^{i(1-n)t_\mu} + 2 \sum_{v=2-n}^{0} a_v e^{ivt_\mu}}{\dot{\eta}(s_{k,\mu})} \right). \tag{5.4}$$

Observe that in (5.4) the relaxation factor 2 is applied only to the terms with indices $2-n \leqq v \leqq 0$ but not to the terms e^{it} and $e^{i(1-n)t}$. In view of the periodicity $a_{v+N} = a_v$ of the Fourier coefficients (5.3) one can formulate the discrete OAP-method also in terms of Fourier coefficients with positive indices.

The iteration is computationally very simple, since it requires in each step only two complex Fourier transforms of length N, which can be performed efficiently by fast Fourier transform.

The iteration can be expressed as a stationary iteration scheme $\mathbf{s}_{k+1} = B\mathbf{s}_k$ with the nonlinear operator

$$B\mathbf{s} := \mathbf{s} - \mathrm{Re}\left(\frac{Q_d \eta(\mathbf{s})}{\dot{\eta}(\mathbf{s})} \right). \tag{5.5}$$

The operator Q_d is modeled in analogy to the Q_2 defined in (4.2). It picks the non-analytic part of the trigonometric interpolation polynomial in the following way: If the vector $\mathbf{x} = (x_1, \ldots, x_N)$ has the representation

$$x_\mu = \sum_{v=1-n}^{n} b_v e^{ivt_\mu}$$

then $Q_d \mathbf{x}$ is the vector $\mathbf{y}$ with components

$$y_\mu := i(\mathrm{Im}\, b_1) e^{it_\mu} + i(\mathrm{Im}\, b_{1-n}) e^{i(1-n)t_\mu} + 2 \sum_{v=2-n}^{0} b_v e^{ivt_\mu}. \tag{5.6}$$

By comparison of (5.6) with (5.2) we notice, that $\mathbf{y} = q(\mathbf{t})$ with a function $q \in W_N^-$. Therefore one can associate with Q_d an operator Q_d^- of $\mathbb{C}^N$ into W_N^- which assigns to each $\mathbf{x}$ the function $q \in W_N^-$ such that $Q_d \mathbf{x} = q(\mathbf{t})$.

Before we start to investigate the convergence of the method, we must identify the possible fixed points. To this aim we recall from [12] the following results:

Theorem 3. *If η is twice differentiable with Lipschitz continuous $\ddot{\eta}$, then there is a number N_0 such that for all $N > N_0$ the following holds true:*

1) *There exist $p_N^* \in W_N^+$ and a vector $\mathbf{s}_N^*$ in $\mathbb{R}^N$ such that*

$$p_N^*(\mathbf{t}) = \eta(\mathbf{s}_N^*).$$

2) *If $p \in W_N^+$ satisfies*

$$\operatorname{Im}(\overline{\dot{\eta}(\mathbf{s}_N^*)} \cdot p(\mathbf{t})) = 0,$$

then $p = 0$.

We recall also from [12] that there is $\varepsilon > 0$ such that for N large enough there is exactly one p_N^* and $\mathbf{s}_N^*$ with $\|p_N^* - \Phi\|_W < \varepsilon$ and the properties mentioned in the theorem. The sequence of p_N^* converges to $\Phi(t) := \hat{\Phi}(e^{it})$ in the W-norm as $N \to \infty$. This property connects the interpolating polynomials p_N^* with the conformal mapping $\hat{\Phi}$. Therefore each method for calculating these polynomials is a method for approximating $\hat{\Phi}$. We shall see now, that the iteration method described above converges to $\mathbf{s}_N^*$.

Let N be fixed, but large enough, such that the statements of Theorem 3 are valid. We omit the subscript N in the sequel. The vector $\mathbf{s}^*$ is a fixed point of the iteration, since $Q_d \eta(\mathbf{s}^*) = 0$. Since η is twice differentiable, the nonlinear operator B is differentiable. The derivative at $\mathbf{s}^*$ is

$$B'(\mathbf{s}^*)\mathbf{u} = \mathbf{u} - \operatorname{Re}\left(\frac{Q_d(\dot{\eta}(\mathbf{s}^*) \cdot \mathbf{u})}{\dot{\eta}(\mathbf{s}^*)}\right).$$

Theorem 4. *For each fixed $N > N_0$ the vector $\mathbf{s}^* := \mathbf{s}_N^*$ from Theorem 3 is a fixed point of the discretized OAP-iteration method. There is a neighbourhood of $\mathbf{s}^*$, such that when the iteration is started with an $\mathbf{s}_0$ in this neighbourhood, the sequence of iterates $\mathbf{s}_k$ converges to $\mathbf{s}^*$.*

Proof. 1) We introduce the vector $\mathbf{e}$ with components $e_\mu = \dot{\eta}(s_\mu^*)/|\dot{\eta}(s_\mu^*)|$. Then the operator $B'(\mathbf{s}^*)$ is similar to the operator A_d defined by

$$A_d \mathbf{u} := \operatorname{Re}(\bar{\mathbf{e}} \cdot (I - Q_d)(\mathbf{e} \cdot \mathbf{u})).$$

From $|e_\mu| = 1$ it follows that $\|\mathbf{e} \cdot \mathbf{x}\|_2 = \|\mathbf{x}\|_2$. The operator $I - Q_d$ multiplies all Fourier coefficients with either ± 1 or 0. Hence $\|I - Q_d\|_2 = 1$. Therefore, $\|A_d\|_2 \leqq 1$, which implies $|\lambda| \leqq 1$ for all eigenvalues λ of A_d.

2) As in the proof of Lemma 14 in [12] it can be shown that $I + A_d$ is represented by a symmetric positive semidefinite matrix. Hence A_d is also symmetric and the eigenvalues λ of A_d are all real and as shown in the first part of the proof all λ are in the interval $-1 \leqq \lambda \leqq 1$.

3) If $\lambda = +1$ is an eigenvalue of A_d, then there exists $\mathbf{u}_+ \in \mathbb{R}^N$ such that

$$\operatorname{Re}(\bar{\mathbf{e}} \cdot Q_d(\mathbf{e} \cdot \mathbf{u}_+)) = 0. \tag{5.7}$$

The function $q := Q_d^-(\mathbf{e} \cdot \mathbf{u}_+)$ is in W_N^-. The function $p(t) := ie^{int}q(t)$ is in W_N^+. In view of (5.7) it satisfies $\operatorname{Im}(\bar{\mathbf{e}} \cdot p(\mathbf{t})) = 0$, since $\exp(int_\mu) = \pm 1$ is real for all μ. It follows from property 2 of Theorem 3, that this implies $p = 0$, hence $q = 0$. This means that the component of $\mathbf{e} \cdot \mathbf{u}_+$ in W_N^- vanishes. Therefore, there must be a

$p_+ \in W_N^+$ such that $\mathbf{e} \cdot \mathbf{u}_+ = p_+(\mathbf{t})$. But since $\mathbf{u}_+$ is a real vector, this implies $\mathrm{Im}(\bar{\mathbf{e}} \cdot p_+(\mathbf{t})) = 0$, which again in view of property 2 of Theorem 3 enforces $p_+ = 0$ and as a consequence $\mathbf{u}_+ = 0$. Therefore, $\lambda = +1$ cannot be an eigenvalue of A_d.

4) If $\lambda = -1$ is an eigenvalue of A_d, then there exists $\mathbf{u}_- \in \mathbb{R}^N$ such that

$$\mathrm{Re}(\bar{\mathbf{e}} \cdot Q_d(\mathbf{e} \cdot \mathbf{u}_-)) = 2\mathbf{u}_- . \tag{5.8}$$

Since $\|Q_d \mathbf{x}\|_2 \leqq 2\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^N$ with equality only if $Q_d \mathbf{x} = 2\mathbf{x}$, the left hand side of (5.8) is $\leqq 2\|\mathbf{u}_-\|_2$ with equality only if

$$Q_d(\mathbf{e} \cdot \mathbf{u}_-) = 2(\mathbf{e} \cdot \mathbf{u}_-) . \tag{5.9}$$

This implies that the function $q_- := Q_d^-(\mathbf{e} \cdot \mathbf{u}_-)$ in W_N^- is of the special form

$$q_-(t) = \sum_{l=-n+2}^{0} c_l e^{ilt} . \tag{5.10}$$

In view of (5.9) this function satifies

$$\mathrm{Im}(\bar{\mathbf{e}} \cdot q_-(\mathbf{t})) = 0 \tag{5.11}$$

since $\mathbf{u}_-$ is a real vector. The coefficients c_1 and c_{-n+1} vanish in the representation (5.10) of q_-. Therefore, the function $p_-(t) := e^{int} q_-(t)$ is in W_N^+. By inserting this into (5.11) one obtains $\mathrm{Im}(\bar{\mathbf{e}} \cdot p_-(\mathbf{t})) = 0$, which implies $p_- = 0$ in view of the property 2 of Theorem 3. Hence $q_- = 0$ and finally $\mathbf{u}_- = 0$ in view of (5.9). Therefore, $\lambda = -1$ cannot be an eigenvalue of A_d.

5) We have shown, that the spectral radius of A_d is less than 1. The same is true for the matrix B', which is similar to A_d. It follows now from the theorem of Ostrowski [8, 7, p. 300] that the fixed points $\mathbf{s}^*$ is attractive. □

6 Nearly Circular Regions

Regions which are in a sense close to the unit disc D must be considered as "simple regions". The conformal mapping for these regions should be easy to calculate by almost every method. We shall show in this section, that for nearly circular regions the convergence factors for the OAP method are close to zero, hence convergence is very fast. For several standard regions, the factors can even be calculated explicitly.

For the unit disc $G = D$ with parametrization of the boundary $\eta(s) = e^{is}$, we have $\hat{\Phi}(z) = z$ and therefore $e^{i\theta} = ie^{it}$. We insert this into (4.6) and obtain $A_2 \psi = 0$ for each ψ, i.e., A_2 is the null operator. Hence for the mapping of the disc onto itself by the OAP method the convergence factor is 0, which means superlinear convergence.

In the definition of the OAP method there is some arbitrariness in the choice of the relaxation factors in the operator Q_2 in (4.2). In fact one could have chosen for each coefficient a_ν a different factor μ_ν in an interval $0 < \varepsilon \leqq \mu_\nu \leqq 2$ for $\nu \leqq 0$ and $0 < \mu_1 < 2$ to achieve the convergence result of Theorem 2. But the choice of the operator Q_2 in (4.2) is optimal in the sense that it is the only choice for which one achieves for the disc $G = D$ superlinear convergence.

We consider now regions whose boundary curve is parametrized by

$$\eta(s) = (1 + \xi(s)) e^{is} \tag{6.1}$$

with a real function ξ which is small in some sense. Then

$$\dot{\eta}(\hat{S}(t)) = (1+\xi(t))ie^{it} + \xi'(t)e^{it} + \ddot{\eta}(t)\cdot(\hat{S}(t)-t) + \dots \tag{6.2}$$

The first two terms on the right hand side are simply the derivative of (6.1) with respect to the parameter. The third term accounts for the fact, that we have to take this derivative not at the values t, but at the disturbed values $\hat{S}(t)$. For the circle $\eta(s) = e^{is}$ and therefore $\ddot{\eta}(s) = -e^{is}$. The shift $\hat{S}(t)-t$ in the parameter function is in first order equal to $K\xi$ in view of the variational formula of Lavrentiev [5, p. 442]. The dots in (6.2) remind us of terms of higher order in ξ which are omitted. The expression (6.2) can then be written in the following form:

$$\dot{\eta}(\hat{S}(t)) = (1+\xi(t))ie^{it} + \xi' e^{it} - e^{it}K\xi + \dots$$

From this we obtain that

$$e^{i\theta} = (i+\tau)e^{it} + \dots \tag{6.3}$$

depends only on the function

$$\tau := \xi' - K\xi\,. \tag{6.4}$$

We insert (6.3) into (4.6) and get

$$A_2\psi = \operatorname{Re}((i-\tau)(I-Q_3)((i+\tau)\cdot\psi)) + \dots \tag{6.5}$$

with the operator $Q_3 f := e^{-it}Q_2(e^{is}\cdot f)$. We have noted before that $A_2 = 0$ for $\tau = 0$. Therefore, we can reduce (6.5) to

$$A_2\psi = \operatorname{Re}(\tau(I-Q_3)(i\psi)) + \operatorname{Im}((I-Q_3)(\tau\cdot\psi)) + \dots \tag{6.6}$$

The operator Q_3 is closely related to the operator K of conjugation. For each f with Fourier series (2.1) $(I-Q_3)f = \operatorname{Re} a_0 + iKf$ holds true. We insert this into (6.6) and obtain

$$A_2\psi = K(\tau\cdot\psi) - \tau\cdot K\psi + \dots \tag{6.7}$$

This implies the general estimate $r_\sigma(A_2) \leqq \|A_2\|_2 \leqq 2\|\tau\|_\infty + \dots$ in terms of the maximum norm $\|\tau\|_\infty$ of τ and the estimate for the convergence factor

$$q_{\text{con}} \leqq 2\|\tau\|_\infty + \dots \tag{6.8}$$

In special cases, where $\tau(t) = \cos mt$ or $\sin mt$ with some integer $m \geqq 1$ the operator A_2 in (6.7) (with higher order terms omitted) reduces to a finite dimensional operator whose range is spanned by the trigonometric functions $\{\cos lt, \sin lt,\ l \leqq m\}$. Using these trigonometric functions as basis, the operator is represented by a matrix which after suitable rearrangement has zeros everywhere except in the antidiagonal (the diagonal from top right to bottom left). The eigenvalues of this operator are therefore easily determined. They are $\lambda = \pm 1/\sqrt{2}$ and for $m \geqq 2$ in addition $\lambda = \pm 1$.

We consider three examples.

1) The eccentric circles with parametric representation

$$\eta(s) = \sqrt{1+p^2+2p\cos s}\,e^{is} = (1+p\cos s)e^{is} + O(p^2)$$

fit into this framework with $\xi(s) = p\cos s$. We calculate $\tau(s) = -2p\sin s$ and a convergence factor of $\sqrt{2}p$.

2) The ellipse with eccentricity ε parametrized by

$$\eta(s)=(1+\varepsilon\cos s)^{-1}e^{is}=(1-\varepsilon\cos s)e^{is}+O(\varepsilon^2)$$

is quite analogous to the first example, only with ε instead of p. The convergence factor is $\sqrt{2}\varepsilon$.

3) The inverted ellipses with parametric representation

$$\eta(s)=\sqrt{1-(1-p^2)\cos^2 s}\,e^{is}=(1-(1-p)/2-\tfrac{1}{2}(1-p)\cos 2s)e^{is}+O((1-p)^2)$$

are also of the form (6.1) with $\xi(s)=-\frac{1}{2}(1-p)\cos 2s$ and $\tau(s)=\frac{3}{2}(1-p)\sin 2s$. The theory yields a convergence factor of $\frac{3}{2}(1-p)$ in this case.

7 Comparison with the Theodorsen Method

If the region G is starshaped with respect to 0, then the boundary curve can be parametrized in the form $\eta(s)=\varrho(s)e^{is}$. The boundary mapping function $\hat{S}(t)$ of the conformal mapping is then characterized by Theodorsen's equation (see e.g. [1, p. 64])

$$\hat{S}(t)=t+K\log\varrho(\hat{S}(t)). \tag{7.1}$$

In general this equation is solved by iteration

$$S_{k+1}(t)=t+K\log\varrho(S_k(t)). \tag{7.2}$$

This is a stationary iteration process $U_{k+1}=H_T U_k$ in W_r with the nonlinear operator $H_T U:=K\log\varrho(U(t)+t)$ and the fixed point $\hat{U}(t)=\hat{S}(t)-t$. The derivative of H_T has the simple form

$$H_T'\phi=K(\sigma(U(t)+t)\cdot\phi)$$

with the function $\sigma(s):=\dot{\varrho}(s)/\varrho(s)$. It follows, that $\|H_T'\|_2\leqq\|\sigma\|_\infty$. Therefore, Ostrowski's theorem guarantees convergence if $\|\sigma\|_\infty<1$.

One obtains basically two sequences of functions. The first

$$F_k(e^{it}):=\varrho(S_k(t))\exp(iS_k(t))$$

maps the unit circle into the boundary curve Γ. The second

$$\Psi_k(e^{it}):=\varrho(S_k(t))\exp(iS_{k+1}(t))$$

is analytic in view of the construction of S_{k+1} by (7.2).

The decomposition $f(t)=\operatorname{Re}f(t)+i\operatorname{Im}f(t)$ effects a direct sum decomposition $W=W_r\oplus W_r$. Replacing the second S_k in

$$\log\varrho(S_k(t))+i(S_k(t)-t) \tag{7.3}$$

by the S_{k+1} defined by (7.2) amounts to a projection parallel to the imaginary summand W_r of W in order to gain analyticy. Then replacing also the first S_k in (7.3) by S_{k+1} is a projection parallel to the real summand W_r of W to construct a function of the form $\log\varrho(S(t))$. Therefore, Theodorsen's method is also a method of "alternating projections", but in contrast to the method described before, in the

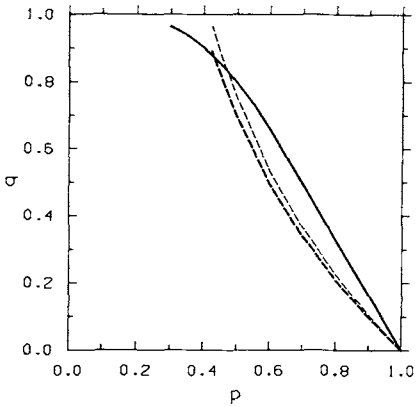

Fig. 1. Convergence factors q for inverted ellipses with parameter p for the OAP-method *(solid line)* and the Theodorsen method *(dashed line)*. The thin dashed line is $\|\sigma\|_\infty$

Theodorsen method the projectors are fixed, namely parallel to the imaginary or real summands of W. As a consequence it converges only for a restricted class of problems, namely for those, which satisfy $\|\sigma\|_\infty < 1$.

The AP-method on the other hand uses projectors, which are adapted to the individual problem. Therefore it can achieve local convergence for all boundary curves. To illustrate this behaviour we have performed a series of test calculations for inverted ellipses (see the third example in Sect. 6). We used a grid with $N=32$ points and started with $s_{0,\mu}=t_\mu+\zeta_\mu$, where the ζ_μ are independent random variables uniformly distributed in the interval $0\leqq\zeta\leqq 0.01$. The convergence factors are shown in Fig. 1 in dependence of the parameter p. That of the Theodorsen method follows closely the quantity $\|\sigma\|_\infty = 1-p+O((1-p)^2)$. The convergence factor of the OAP-method is about $1.5(1-p)$ for p close to 1 as predicted in Sect. 6, but then bends over as it approaches 1 and stays less than 1. As we have seen in Sect. 6 (and in other series of test calculations), the situation shown in Fig. 1 is typical.

Following a suggestion of Gaier [1, p. 105], overrelaxation has also been applied to speed up Theodorsen's method. A detailed discussion is in [2] where also nearly optimal underrelaxation parameters for certain classes of regions (including all the examples discussed in Sect. 6) are calculated.

There is an essential difference to the overrelaxation introduced in Sect. 4 insofar as the relaxation parameter for the Theodorsen method depends on the region while the relaxation parameter of 2 in (4.1) does not. In order to get a fair comparison we have drawn in Fig. 1 the performance of two methods which both do not depend explicitly on the shape of the region. The success of underrelaxation in the Theodorsen method draws our attention to the possibility that one can design more efficient methods by taking into account additional information about the region.

Finally, we emphasize that the OAP method does not require, that the boundary curve is represented in polar coordinates. Therefore, it can also be applied for regions G which are not starshaped. This makes the OAP-method a rather flexible, simple and convenient tool for numerical conformal mapping.

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