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ORIGINAL PAPER

The Approximate Conformal Mapping of a Disk onto Domain with an Acute Angle

P. N. Ivanshin[1] · **E. A. Shirokova**[1]

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Abstract

We apply boundary curve reparametrization to find the approximate conformal analytical map of the unit disk onto an arbitrary given finite domain with a boundary smooth at every point but fininte number of acute angle points. In order to do this we first solve the Fredholm equation and then construct spline-interpolation deformation of the auxiliary functions. The integral equation in its turn is reduced to approximate solution of a linear system over Fourier coefficients of the angle reparametrization. The resulting mapping is reconstructed as a Cauchy integral. The method presentation is supported by demonstration of some examples. This method is applicable to the case of finitely connected domains with finite number of angle points at their boundary.

Keywords Angle · Fredholm integral equation · Conformal mapping · Linear system

Introduction

It is known that analytical functions play an important role in solution of many mathematical and mechanical problems, e.g. in the reconstruction of plane potentials and solution of Laplace equation [1]. Conformal mappings of certain domains then present a particular interest. A number of authors applied conformal mappings of the unit disk onto a given domain with singularities at its boundary to plane boundary value problems for corresponding domains by symmetry methods, zipper technique [2]. Lately mathematicians introduced new numerical methods in construction of conformal mappings [3]. All these methods possess different time complexity. Recall, for instance, the Wegmann numerical conformal mapping construction based on solution of Riemann–Hilbert problem. This method involves collocation processes [4, 5].

Recall several types of the so-called canonical slit regions for conformal analytical mappings of multiply connected domains [6]. Two of these canonical regions are 1) a unit disk with concentric circular slits, 2) a unit disk with radial slits. So the construction method allows generalization to multiply-connected domains. Nasser introduced the construction

✉ P. N. Ivanshin
pivanshi@yandex.ru

1 Lobachevskiy Institute of Mathematics and Mechanics, Kazan Federal University, Kremlevskaya str., 18, Kazan, Russia 420008

of connected finitely-connected domains mapping onto the canonical regions. The mapping function then is the solution of a Riemann-Hilbert problem. He solved this problem as a boundary integral equation of the second kind with the generalized Neumann kernel. It is known that the free summand of this integral equation contains among other elements a singular integral with singularity in the form of the cotangent function approximated by Wittich's technique. Nasser discretized the integral equation by the Nyström method and the trapezoidal quadrature formula in order to obtain a finite linear system [7–13]. The method given in [6–8, 14–16] is based also on the solution of a certain integral equation through its finite dimensional approximation. Construction of the mapping is reduced there to solution of a finite linear system over the unknown Fourier coefficients. We apply Hilbert formula to the summands with the cotangent singularity.

Some of the problems of physics are connected with domains with boundary angle points. It is not difficult to find an appropriate mapping onto such domain by application of the integral equation method when the inner angles of the domain exceed the angle π. Note that in the case of the boundary angle point with the inner angle less than π the kernel of the corresponding Neumann equation has the singularity at this point [19, 20]. The numerical solution of the mapping problem for these domains is also difficult. Some work is done regarding the optimal mesh structure of n-gonal domains [17]. Certain authors considered asymptotic expansions with logarythmic terms [18]. Hough and Papamichael dealt with polygonal domains gluing splines and singular functions [19]. Construction of conformal mapping for domains with angle points via application of the additional conformal mappings and the continued fractions was presented in [20].

Here we conformally and analytically map the unit disk onto a 1-connected domain with acute angle points at its boundary applying both trigonometric polynomials and splines.

We first map the unit disk onto a similar domain with the smooth boundary. We recall the standard method based on solution of the integral equation over the imaginary part of the boundary value of an auxiliary regular function and the linear system truncation.Then the analytical function is restored via Cauchy integral. This construction is given in Sect. 2. The method is adequate for the case of a smooth boundary of the domain. Presence of an acute angle boundary point of the given domain brings to the linear system solution disturbance. Therefore the dependence of the polar angle of the unit circle on the boundary point of the given domain fails to be monotonic.

After we construct the mapping with nonmonotonic polar angle of the unit circle we consider the piecewise correction of this polar angle function with the help of monotonic splines in the neighbourhoods of the preimages of the angle points. Now the Cauchey integral is constructed with application of reparametrization of the boundary parameter with the inverse of spline function. Here we have no need in gluing together two sets of solutions and simply modify the already constructed one as in [19]. This is the content of Sect. 3.

Approximate Conformal Mapping of the Unit Disk onto a Domain Similar to the given One by Means of Integral Equation

Consider a simply or 1-connected domain D_z bounded by the Jordan simple curve L_0 given by the relation $z = z_0(t)$, $t \in [0, 2\pi]$.

Assume that the boundary curve L_0 is given by the exponential polynomial in the complex plane $z = x + iy$:

$$z_0(t) = \sum_{k=-m}^{n} d_k e^{ikt},\ t \in [0, 2\pi].$$

Let this parametrization trace the domain D_z counterclockwise along L_0.

We present construction of the mapping of the unit disk onto a domain similar to D_z due to [7].

We assume that $0 \in D_z$ without loss of generality.

First we constuct the function $\zeta(z)$ analytic in the domain D_z. This function maps the domain D_z conformally onto the unit disk D_ζ with the correspondence $\zeta(0) = 0$. There naturally appear function $\psi(z) = \log \frac{z}{\zeta(z)}$ analytic in D_z. Recall that $\psi(z_0(t)) = \log \frac{z_0(t)}{e^{i\theta_0(t)}}$ is the boundary value of the function analytic in D_z if and only if the following boundary relation holds true:

$$\log \frac{z_0(t)}{e^{i\theta_0(t)}} = \frac{1}{\pi i} \int_0^{2\pi} \log \frac{z_0(\tau)}{e^{i\theta_0(\tau)}} \left[\log(z_0(\tau) - z_0(t))\right]'_\tau d\tau. \tag{1}$$

holds for $t \in [0, 2\pi]$.

Consider the imaginary part of Eq. (1):

$$\begin{aligned} q_0(t) = \frac{1}{\pi} \int_0^{2\pi} q_0(\tau)[\arg\,(z_0(\tau) - z_0(t))]'_\tau d\tau \\ -\frac{1}{\pi} \int_0^{2\pi} \log|z_0(\tau)| \left[\log|z_0(\tau) - z_0(t)|\right]'_\tau d\tau, \end{aligned} \tag{2}$$

where $q_0(t) = \arg(z_0(t)) - \theta_0(t)$, $\theta_0(t)$ is the mapping of the parameter of L_0 onto the unit circle polar angle.

In order to find a better approximation of the mapping function we differentiate with respect to t equality (2) and integrate the result by parts. We then obtain the relation on the new unknown function $q_0'(t)$:

$$q_0'(t) = \frac{1}{\pi} \int_0^{2\pi} q_0'(\tau) K(\tau, t) d\tau + P(t), \tag{3}$$

where

$$K(\tau, t) = -[\arg(z_0(\tau) - z_0(t))]'_t, \quad L(\tau, t) = [\log|z_0(\tau) - z_0(t)|]'_t,$$

$$P(t) = \frac{1}{\pi} \int_0^{2\pi} [\log|z_0(\tau)|]'_\tau L(\tau, t) d\tau.$$

The kernel L possesses a singularity given by cot $\frac{\tau-t}{2}$ as follows:

$$(\log|z_0(\tau)-z_0(t)|)'_t = \mathrm{Re}\left(\log\sum_{k=1}^{n} d_k\left[e^{ik\tau}-e^{ikt}\right]\right)'_t = \mathrm{Re}\left(\log\sin\frac{\tau-t}{2}+\right.$$

$$\left.+\log\left[\sum_{k=1}^{n} d_k e^{ikt}\sum_{l=0}^{k-1} e^{il(\tau-t)}-\sum_{k=1}^{m_s} d_{(-k)}e^{-ik\tau}\sum_{l=0}^{k-1} e^{il(\tau-t)}\right]\right)'_t$$

$$=-\frac{1}{2}\cot\frac{\tau-t}{2}+\left(\log\left|\sum_{k=1}^{n} d_k e^{ikt}\sum_{l=0}^{k-1} e^{il(\tau-t)}-\sum_{k=1}^{m} d_{(-k)}e^{-ik\tau}\sum_{l=0}^{k-1} e^{il(\tau-t)}\right|\right)'_t.$$

Cauchy principal value integral

$$\frac{1}{\pi}\int_0^{2\pi}[\log|z_0(\tau)|]'\cot\frac{\tau-t}{2}d\tau$$

can be found via Hilbert formula [9] as in [8].

If we constuct the solution of Eq. (3) as the infinite convergent Fourier series

$$q_0'(t)=\sum_{l=1}^{\infty}\alpha_l\cos lt+\beta_l\sin lt,$$

given

$$P(t)=\sum_{l=1}^{\infty}\gamma_l\cos lt+\kappa_l\sin lt,\quad t\in[0,2\pi],$$

we can rewrite Eq. (3) as the linear integral operator equation system as follows:

$$\begin{pmatrix} I-K_{c,c} & -K_{c,s} \\ -K_{s,c} & I-K_{s,s}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\gamma\\ \kappa\end{pmatrix}. \tag{4}$$

Here

$$\alpha=(\alpha_1,\alpha_2,\ldots)^T,\ \beta=(\beta_1,\beta_2,\ldots)^T,\ \gamma=(\gamma_1,\gamma_2,\ldots)^T,\ \kappa=(\kappa_1,\kappa_2,\ldots)^T.$$

The last operator system is in fact the infinite system over the unknown Fourier coefficients α_l, β_l, $l\in\mathbb{N}$, of the function $q_0'(t)$. Here we calculate the coefficients of double Fourier expansions of the integral operator kernels and equate the coefficients with the same trigonometric functions [6].

The solution $q_0'(t)$ of system (4) allows to obtain the polar angle function: $\theta_0(t) = \arg(z_0(t)) - q_0(t)$. The function $\theta_0(t)$ should monotonically increase as t changes from 0 to 2π, $\theta_0(2\pi)-\theta_0(0)=2\pi$. Now the Cauchy integral gives us the analytic function $f(\zeta)$ that maps the unit disk onto the given domain D_z:

$$f(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{z_0(t)e^{i\theta_0(t)}\theta_0'(t)dt}{e^{i\theta_0(t)}-\zeta}. \tag{5}$$

We consider a solution of a truncated finite system as an approximate solution of the infinite one. So we simply take the unknown Fourier coefficients of the function $q_0'(t)$ from the solution of the finite system.

Then the approximate solution of system (4) is in fact a Fourier polynomial of a finite degree:

$$\tilde{q}_0'(t) = \sum_{l=1}^{M} \alpha_l \cos lt + \beta_l \sin lt, \tag{6}$$

given

$$\tilde{P}(t) = \sum_{l=1}^{M} \gamma_l \cos lt + \kappa_l \sin lt, \quad t \in [0, 2\pi].$$

So we reduce Fredholm integral equation of the second kind given by Eq. (3) to the finite linear system over the unknown Fourier coefficients α_j and β_j:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_M \\ \beta_1 \\ \vdots \\ \beta_M \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_M \\ \kappa_1 \\ \vdots \\ \kappa_M \end{pmatrix}. \tag{7}$$

Vectors $(\gamma_1, \dots, \gamma_M)^T$, $(\kappa_1, \dots, \kappa_M)^T$ of system (7) possess cooordinates

$$\gamma_j = \frac{1}{\pi} \int_0^{2\pi} P(t) \cos jt dt, \ \kappa_j = \frac{1}{\pi} \int_0^{2\pi} P(t) \sin jt dt, \quad j = 1, \dots, M.$$

Also system (7) contain the matrices $A, B, C, D \in Mat(M \times M)$ whose elements equal

$$A_{jk} = \delta_{jk} - \frac{1}{\pi^2} \int_0^{2\pi} \cos k\tau d\tau \int_0^{2\pi} K(\tau, t) \cos jt dt,$$

$$B_{jk} = -\frac{1}{\pi^2} \int_0^{2\pi} \sin k\tau d\tau \int_0^{2\pi} K(\tau, t) \cos jt dt,$$

$$C_{jk} = -\frac{1}{\pi^2} \int_0^{2\pi} \cos k\tau d\tau \int_0^{2\pi} K(\tau, t) \sin jt dt,$$

$$D_{jk} = \delta_{jk} - \frac{1}{\pi^2} \int_0^{2\pi} \sin k\tau d\tau \int_0^{2\pi} K(\tau, t) \sin jt dt,$$

for $j, k = 1, \dots, M$. Here δ_{jk} is the Kronecker δ-function.

We reconstruct the approximate value $\tilde{q}(t)$ of the function $q_0(t)$, via its derivative (6) by equating the constant summand to 0:

$$\tilde{q}(t) = \sum_{l=1}^{M} \frac{\alpha_l}{l} \sin lt - \frac{\beta_l}{l} \cos lt,$$

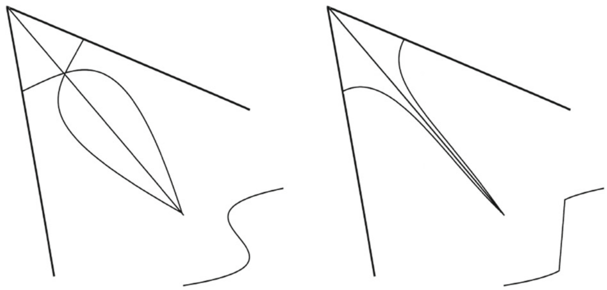

Fig. 1 Normals overlap in the neighbourhood of the angle point

This auxiliary function $\tilde{q}(t)$ approximating the function $q_0(t)$, $t \in [0, 2\pi]$, allows us to find the correspondence between the boundary parameter of L_0 and the polar angle of the circle boundary of D_ζ via the formula $\tilde{\theta}(t) = \arg z_0(t) - \tilde{q}(t)$.

Finally we reconstruct the approximate analytic function mapping D_ζ onto D_z as the Cauchy integral similarly to (5):

$$\tilde{f}(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z_0(t) e^{i\tilde{\theta}(t)} \tilde{\theta}'(t) dt}{e^{i\tilde{\theta}(t)} - \zeta}. \tag{8}$$

It is known that system (4) possesses the exact solution and the approximate solution of system (7) converges to the exact one as $M \to \infty$. The time complexity of the method is $O(M \ln M)$. The convergence rate equals $O\left(\frac{1}{M^2}\right)$ if L_0 is a smooth curve [7].

Since we reduce integral equation system (3) first to infinite linear system (4) and then to the finite one given by relation (7) we need to describe the asymptotic behaviour of the elements of the infinite and the truncated matrices of relations (4) and (7) in the case of an angular point at the boundary of D_z. The elements of the infinite matrices K_{cc}, K_{cs}, K_{sc} and K_{ss} in (4) are the double Fourier series coefficients of the kernel $K(\tau, t) = -[\arg(z_0(\tau) - z_0(t))]'_t$. This kernel is not continuous at the point t_0 if the boundary of D_z contains the angle point $z(t_0)$ with the inner angle $\lambda\pi$ when $0 < \lambda < 1$ but has the singularity of type $\frac{1}{|\tau - t|^{1-\lambda}}$, [19, 20]. This singularity disturbes the convergence of the finite system (7) to the infinite system (4) so that the corresponding approximation $\tilde{\theta}(t)$ of the polar angle $\theta_0(t)$ fails to be monotonic and therefore the function (8) can not provide an adequate mapping.

Reparametrization in the Neighbourhood of the Singular Point

We suppose that the boundary of D_z contains the acute angle point at $z_0(t_0)$ with the inner angle $\lambda\pi$, $\lambda < 1$. The first step in our construction of the approximate mapping function of the unit disk onto D_z is approximate solution of integral Eq. (3).

The approximation $\tilde{\theta}(t)$ to $\theta_0(t)$ fails to be monotonic in the neighbourhood of the parameter value t_0, so the normals to the boundary of D_z overlap at the neighbourhood of the angle point as it is demonstrated by Fig. 1a. In order to save the function monotonicity we cut the fold (see Fig. 1b) in the following way.

Note that the polar angle $\theta_0(t)$ vanishes at the point t_0. Really, since L_0 contains an angle point the mapping function in the neighbourhood of its preimage has the following

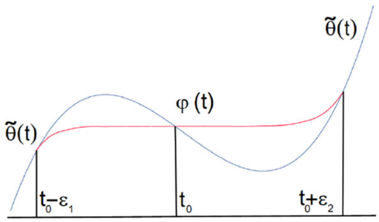

Fig. 2 Spline in the neighbourhood of the angle point

representation:

$$f(\zeta) = f\left(e^{i\theta_0(t_0)}\right) + \left(\zeta - e^{i\theta_0(t_0)}\right)^{\lambda} g(\zeta),$$

so

$$f'(\zeta) = \frac{\Psi(\zeta)}{\left(\zeta - e^{i\theta_0(t_0)}\right)^{1-\lambda}}.$$

Since

$$f'\left(e^{i\theta_0(t)}\right) = \frac{z_0'(t)}{ie^{i\theta_0(t)}\theta_0'(t)}$$

the singularity at the parameter value $t = t_0$ appears only if $\theta_0'(t_0) = 0$.

Therefore we replace the function $\tilde{\theta}(t)$ by a spline $\phi(t)$ on a segment $[t_0 - \epsilon_1, t_0 + \epsilon_2]$ so that $\phi'(t) > 0$, $t \in [t_0 - \epsilon_1, t_0) \cup (t_0, t_0 + \epsilon_2]$, $\phi'(t_0) = 0$ and the continuous function

$$\breve{\theta}(t) = \begin{cases} \tilde{\theta}(t), & t \in [0, 2\pi] \setminus [t_0 - \epsilon_1, t_0 + \epsilon_2], \\ \phi(t), & t \in [t_0 - \epsilon_1, t_0 + \epsilon_2], \end{cases}$$

monotone increases on $[0, 2\pi]$ (see Fig. 2).

Due to monotonic increase of the spline $\phi(t)$ we can construct the inverse also monotone increasing function $t = t(\phi)$, $\phi \in [\tilde{\theta}(t_0 - \epsilon_1), \tilde{\theta}(t_0 + \epsilon_2)]$. Now we rewrite formula (8) in the case of an acute angle point t_0:

$$\hat{f}(\zeta) = \frac{1}{2\pi} \int\limits_{t_0+\epsilon_2}^{2\pi+t_0-\epsilon_1} \frac{z_0(t)e^{i\tilde{\theta}(t)}\tilde{\theta}'(t)dt}{e^{i\tilde{\theta}(t)} - \zeta} + \frac{1}{2\pi} \int\limits_{\tilde{\theta}(t_0-\epsilon_1)}^{\tilde{\theta}(t_0+\epsilon_2)} \frac{z_0(t(\phi))e^{i\phi}d\phi}{e^{i\phi} - \zeta}. \tag{9}$$

Example 1 Unit disk with two orthogonal tangent lines. The left part of Fig. 3is the result of the first step, cubic spline. The second step also with the cubic spline is the right part of Fig. 3, red lines show the target domain boundary. The picture shows us that the approximation is better than in [20], the order of approximation being $O(0.001)$ in contrast with $O(0.01)$ of [20] with the same size of the system (6).

Example 2 A semidisk of radius 1. The left part of Fig. 4is the result of the first step, cubic spline. The second step with the linear spline is the right part of Fig. 3, red lines show the target domain boundary. The lower image is the angle reparametrization.

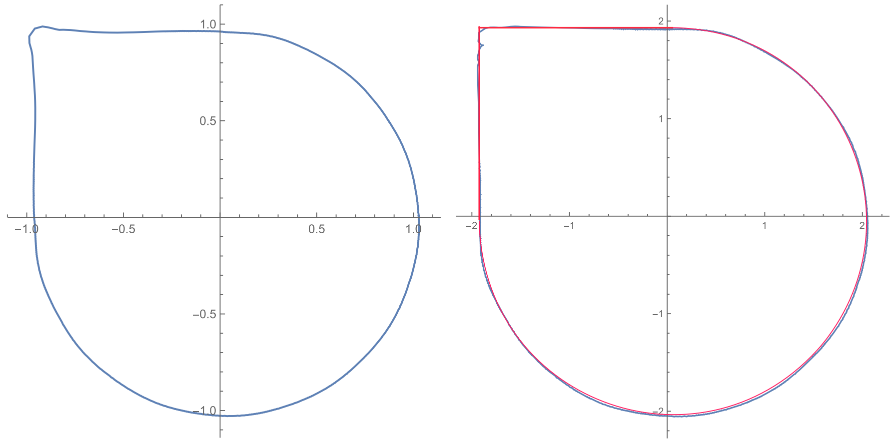

Fig. 3 Domain with one angle

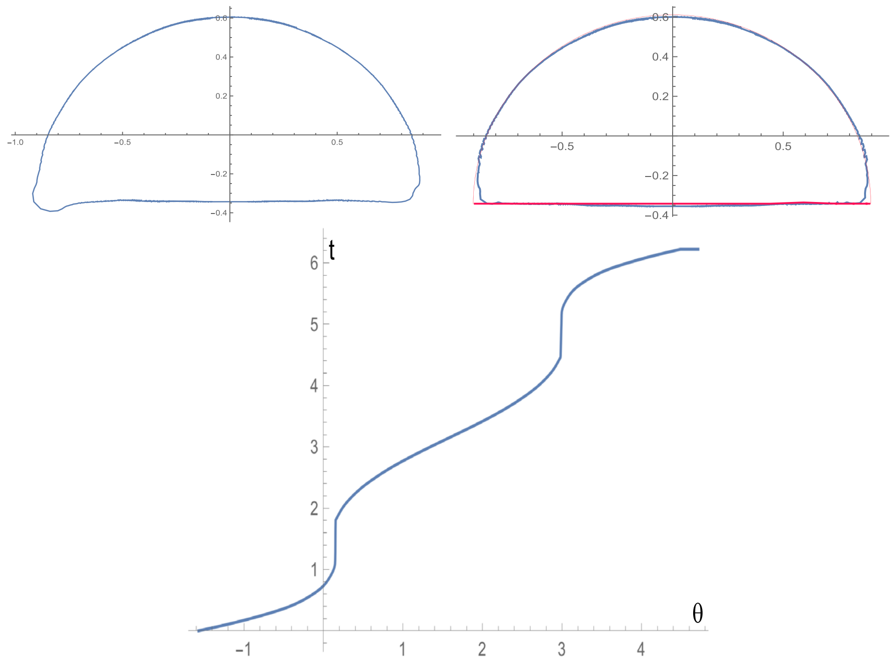

Fig. 4 Semidisk and the angle reparametrization

Example 3 A figure bounded by the curve $e^{it} + \frac{1}{4}e^{-i2t} + \frac{1}{8}ie^{-i3t}$, $0 \leq t \leq 2\pi$. It can be seen here that not only angles but also domains with relatively thin boundary elements can be approximated by the proposed approach. The first two parts of Fig. 5are the results of the first step, cubic spline and the level line of the second step, linear spline; red lines show the target domain boundary. The lower image is the angle reparametrization.

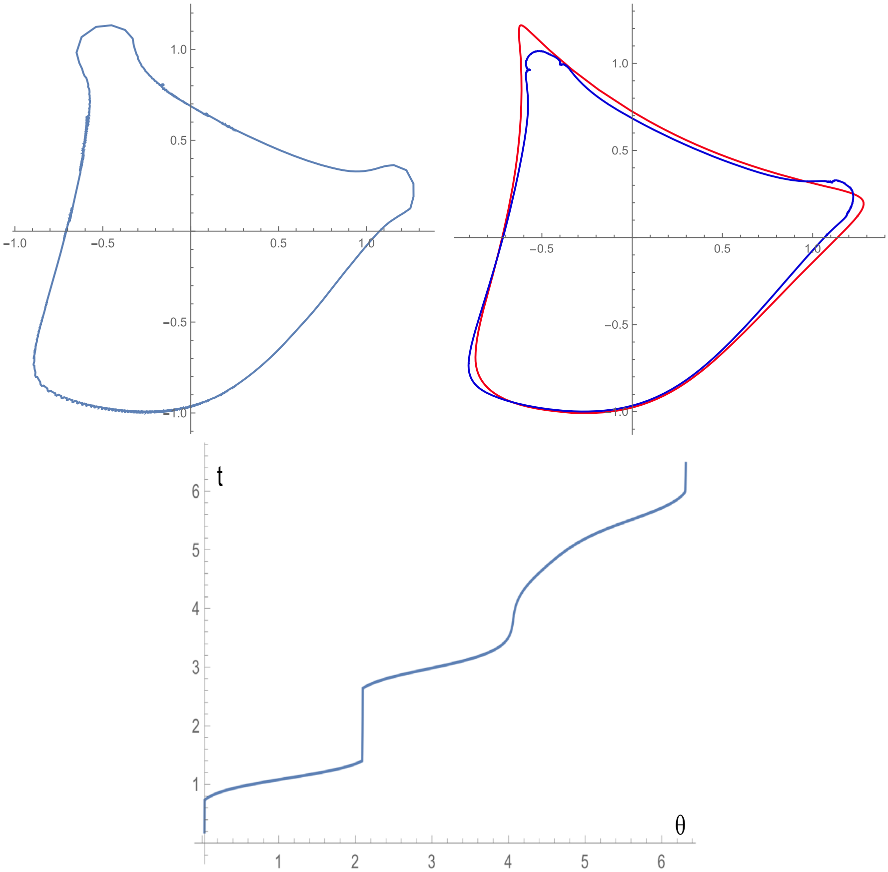

Fig. 5 Two approximations and the angle reparametrization

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Availability of Data and Material On demand.

Code Availability Fortran code and.nb available on demand.

Declarations

Conflict of interest We hereby state that the article "The approximate conformal mapping of a disk onto domain with an acute angle" does not involve any conflict of interest.

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