Conformal Mesh Mappings

Bachelor Thesis D-MATH ETHZ
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Abstract

When performing shape optimization one encounters the problem of conformal mapping of a relatively simple onto a potentially very complicated domain. The Riemann mapping theorem guarantees the existence of such a conformal map, but does not provide any explicit construction of it. In practice, one has to resort to numerical methods to construct the conformal map. This text is intended to give a general overview of currently known numerical algorithms for this very problem, and to provide a comparison of their suitability in terms of input and output format as well as their performance in terms of accuracy and speed. One of the algorithms is then implemented in the context of a 2-dimensional shape optimization problem, and its performance is evaluated.

Keywords: numerical conformal mapping, Riemann mapping theorem, Schwarz-Christoffel mapping, conjugate function method, Fourier series representation, curvilinear triangular mesh, mesh generation, finite element method, Laplace-Beltrami equations, spectral conformal parameterization, Ricci flow, charge simulation method, RT-algorithm, CRDT (cross-ratios Delaunay triangulation) https://observablehq.com/@jrus/scpie

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1 Introduction

1.1 Problem Setting

2 Theoretical Background

2.1 Conformal Mappings

Definition 1. *A conformal mapping, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation w=f(z) that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative.*

This type of mapping is useful as some mesh properties remain regular under such transformations. This ensures that cells do not become too stretched or overlap, which would cause numerical issues or even solver failure [Wec19]. In two dimensions, conformality can be achieved by enforcing that the mapping satisfies the Cauchy-Riemann equations. In higher dimensions, slmcoawrown

2.2 Riemann Mapping Theorem

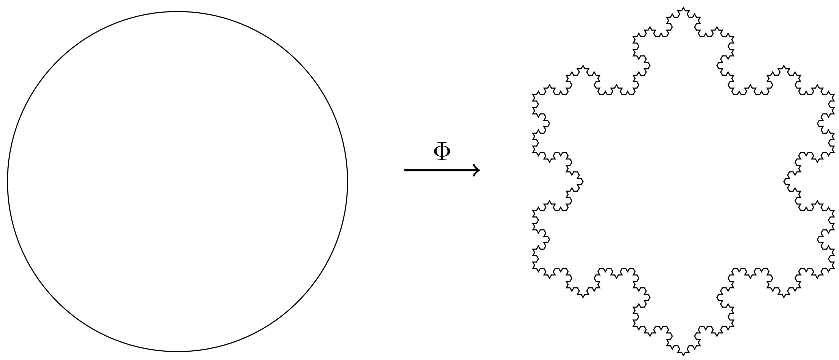
The reason why we use conformal mappings for shape optimisation in two dimensions is the following

Theorem 1 (Riemann Mapping Theorem)**.** *If U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from U onto the open unit disk*

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

This mapping is known as a Riemann mapping.

The beauty of the Riemann mapping theorem lies in its weight of implications, i.e. the fact that it guarantees the existence of a conformal map between any two simply connected domains in the complex plane, provided they are not the entire plane. The existence of this Riemann map is a priori not obvious: Even relatively simple Riemann mappings (for example a map from the interior of a circle to the interior of a square) have no explicit formula using only elementary functions. Simply connected open sets in the plane can be highly complicated, for instance, the boundary can be a nowhere-differentiable fractal curve of infinite length, even if the set itself is bounded. One such example is the Koch curve. The fact that such a set can be mapped in an angle-preserving manner from the nice and regular unit disc seems counter-intuitive.



proof.

2.3 Riemann-Hilbert Problem

2.4 Hilbert Transform

2.5 Mixed boundary conditions (Wec 4.3)(?)

2.6 Hilbert theory and different inner products

From complex analysis we know that a complex differentiable function with non-vanishing complex derivative is conformal, and complex differentiability in two dimensions can be caracterised by the Cauchy-Riemann equations (necessary and sufficient condition). Denoting

$$B := \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}$$

we want to find a deformation Φ such that $B\Phi = 0$.

The Cauchy-Riemann equations however do not guarantee a solution for arbitrary boundary data. A holomorphic map from the boundary of the unit disk onto some boundary of a convex set in $\mathbb{R}^2$ satisfies Cauchy-Riemann equations if the boundary of the target set can be described by non-negative Fourier frequencies [PROOF]. Thus, the choice of parametrization of the target region's boundary is another challenge posed when solving conformal mapping problems, along with finding a holomorphic deformation Φ.

In order to keep the approximation error as low as possible while maintaining good solver performance, we typically want uniform meshes, e.g. where the triangulation is close to equilateral. The quality of a mesh is measured in terms of its individual cells (?) where for a mesh $\mathcal{M} := K$ of triangles K such that $\bar{\Omega} = \mathbb{U}_{K \in \mathcal{M}} K$ we define $h(K)$ the diameter of the smallest K-circumscribing ball and $\rho(K)$ the diameter of the largest ball inscribed in K. Then a measure for the quality of K is the ratio of these diameters

$$\eta(K) := \frac{h(K)}{\rho(K)} \in [1, \infty)$$

[Wec19]. ”’ [Wec19]: Mesh quality measures comment p 54 ”’

2.7 Crowding

Wegmann [Weg05] proved the following result.

Theorem 2. *When the region G can be enclosed in a rectangle with sides a and b, $b \leq a$, such that G touches both small sides (see Figure 4) then the conformal mapping $\phi : D \rightarrow G$ satisfies*

$$\|\phi'\|_D \geq b\psi(b/a)$$

with a function $\psi(\tau)$ which behaves for small τ like

$$\psi(\tau) \approx \frac{1}{2\pi\sqrt{\epsilon}} \exp\left(\frac{\pi}{2\tau}\right).$$

Crowding is cumbersome for all methods which work with grid points. On the other hand, methods which approximate the mapping functions by polynomials also face severe problems when the target region is elongated. It follows that, for the mapping of the disk to a region of aspect ratio 1 910 by a polynomial, the degree must be of several millions. In any case, the number of grid points and the degree of the approximating polynomials increase both like exp(zr/2r) as the aspect ratio, r, tends to zero. DeLillo [37] has shown how crowding affects the accuracy of numerical computations. Crowding also limits the practical usefulness of conformal maps. This was demonstrated by DeLillo [36] for the Laplace equation. Crowding has also been observed for regions with elongated sections (”fingers”). For ”pinched” regions, such as the interior of an inverted ellipse, ill conditioning occurs of a less severe, algebraic nature (DeLillo [37]).

2.7.1 The operator R

In some conformal mapping methods, boundars value problems as follows occur,

$$\Psi(e^{it}) = B(t) + A(t)U(t)$$

where $A, B : \mathbb{C} \rightarrow \mathbb{C}$ and $U : \mathbb{C} \rightarrow \mathbb{R}$. Multiplication with $\bar{A}$ yields the RH problem

$$Im(A\bar{(t)}\Psi(e^{it})) = Im(A\bar{(t)}B(t))$$

This problem can be solved by the operator R_β, which is defined as follows:

This follows from the following theorem:

Theorem 3. *There exists a function Ψ analytic in D with $\Psi(0) = 0$ satisfying the boundary problem if and only if U is a solution of the Fredholm integral equation of the second kind*

$$(I + R_\beta)U = g$$

with the right-hand side

$$g := -Re(e^{-i\beta}(I - iK + J)/B).$$

Where K is the conjugation operator and J the averaging operator.

2.8 Mapping from the region to the disk

2.8.1 Extremum Principles

Both the Bergman and the Szegö norms are very useful for characterizing the conformal mapping from G to a disk by extremum principles:

Theorem 4. *Let F be the conformal mapping from G to a disk normalized by $f(0) = 0, f'(0) = 1$. Then*

1. *(Principle of minimum area) F' is the unique function which minimizes $||f||_B$ among all functions $f \in B(G)$ satisfying $f(O) = 1$.*
2. *(Principle of minimum length) $\sqrt{F'}$ is the unique function which minimizes $||f||_S$ among all functions $f \in S(G)$ satisfying $f(O) = 1$.*

2.8.2 Osculation Methods

The osculation method (Schmiegungsverfahren) of Koebe [140] approximates F by a com- position of elementary maps. It is universally applicable, since it requires no hypotheses at all concerning the boundary 0G of the region G.

2.9 Mapping from the disk to the region

3 Existing Methods

- Numerical stability of different conformal mapping methods - Boundary discretization requirements (how smooth does γ need to be?) - Mesh quality preservation - how does the mapping affect triangle quality? - Computational complexity - Input and output formats

4 Proposed Method

4.1 Choice/ Justification

criteria: - Accuracy for domains with sharp corners or high curvature - Speed for practical mesh sizes - Robustness - does it fail for certain domain shapes? - Implementation complexity given your timeline - Jacobian computation - analytical vs numerical differentiation

4.2 Implementation

- Separate modules for boundary parameterization, mapping computation, Jacobian eval, and mesh transformation - plot original vs. mapped grids (e.g., Matplotlib quiver for Jacobians) to spot issues early.

4.3 Numerical Experiments/ Testing

check angle preservation (e.g., via dot products on mapped vectors) and scale factors ($det(D\Phi) > 0$, $|\frac{\partial \Phi}{\partial z}|$ constant in theory). - Test suite: Use known exact mappings (e.g., disk to square via Schwarz-Christoffel) for error metrics (L2 norm on boundary points). - Metrics: Runtime for N points, mesh quality post-mapping (e.g., min/max angles in triangles, shape regularity ratio). - Real-world applicability: Apply to a sample FEM problem (e.g., Poisson equation on Ω) and compare accuracy/speed vs. uniform mesh. - Robustness: Vary boundary complexity (smooth vs. corners), noise in Fourier coeffs, mesh resolutions. - Debugging: Use assertions for bijectivity (e.g., check injectivity numerically) - Error handling - what happens with degenerate inputs?

4.4 Results

5 Conclusion

References

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