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Revisiting the hilbert transform of periodic functions

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Abstract In this expository paper, we present a self-contained derivation of the Hilbert transform for periodic functions and revisit established results concerning its global behavior, particularly its strong boundedness in L_p spaces. We follow a novel perspective that relies on basic real analysis concepts, introduced in [21].

Keywords Periodic functions · Weierstrass M-test · Sine product formula

1 Introduction

The Hilbert transform is a fundamental operator in harmonic analysis, with extensive applications in both theoretical and practical domains. It plays a pivotal role in areas such as Fourier series convergence, complex analysis, potential theory, and wavelet theory. Previous studies have comprehensively examined the properties and applications of the Hilbert transform for periodic functions [1, 2, 5, 12], providing valuable insights into its behavior and utility.

The boundedness of the Hilbert transform ensures that it preserves the integrability of functions, which is crucial for many applications in analysis and signal processing. This property allows it to be reliably used to analyze and process sig-

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nals across various domains, making it an indispensable tool in modern mathematical analysis and engineering disciplines.

This note presents a novel perspective on obtaining the Hilbert transform for periodic functions. Unlike traditional approaches, which primarily rely on the framework of distribution theory, ours builds upon the work of Salwinski [21], leveraging fundamental concepts from real analysis. This approach avoids the technical complexities associated with distribution theory, making the subject more accessible to a broader audience. Additionally, we discuss the strong boundedness of the Hilbert transform on the Lebesgue spaces $L_p(\mathbb{R})$.

We start by saying that the Hilbert transform of a sufficiently well-behaved function f is defined as

$$\begin{aligned} Hf(x) &= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} f(x-t)\frac{dt}{t} \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|t|>\epsilon} f(x-t)\frac{dt}{t} \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-t|>\epsilon} \frac{f(t)}{x-t}\,dt \\ &= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(t)}{x-t}\,dt. \end{aligned} \tag{1}$$

The Hilbert transform for a periodic function with period 2π is given by

$$Hf(x) = \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} f(x-t)\cot\left(\frac{t}{2}\right)dt. \tag{2}$$

Early references to this formula or equivalent variations include Kellogg [11], Hilbert [10, 11], and Hardy and Littlewood [8]. A more general definition of the Hilbert transform is possible when the period is taken as 2T. In this case, the generalization of equation (2) can be seen in [16, 25]:

$$H_T f(x) = \frac{1}{2T}\,\text{p.v.}\int_{-T}^{T} f(x-t)\cot\left(\frac{\pi t}{2T}\right)dt. \tag{3}$$

Equation (3) can be rearranged as follows:

$$H_T f(x) = \frac{1}{2T}\,\text{p.v.}\int_{0}^{T} [f(x-t)-f(x+t)]\cot\left(\frac{\pi t}{2T}\right)dt. \tag{4}$$

Several authors have studied the properties of the Hilbert transform of periodic functions, among which we can mention Tauber [22], Hilbert [9, 10], Titchmarsh [23], Cossar [2], Easthen [5], Zygmund [26], Botzer and Nessel [1], Pandey [15], Zhizhiashvili [25], and King [12], among others.

The need to determine the Hilbert transform for periodic functions arises in various applications. For instance, it is possible to express certain properties of conformal mappings, such as those involving frequency-dependent quantities like generalized susceptibilities and scattering amplitudes, using Fourier series. This approach leverages the periodic nature or the symmetry properties of the mappings when they are analyzed on the complex plane. Conformal mappings such as those mapping the upper half-plane to the unit disc can encode boundary behaviors that depend on complex variables. These mappings often appear in contexts like scattering theory, where the mappings preserve angles and may relate to analytic continuations of scattering amplitudes. When dealing with physical quantities like susceptibilities or amplitudes, their dependence on frequency can often exhibit periodic or quasi-periodic characteristics, making Fourier representation a natural choice.

In the context of dispersive-dissipative dynamics, the real part of a complex-valued function, often representing phenomena like wave dispersion or propagation, is called the dispersive component. The imaginary part of the function is the dissipative component and it is related to energy dissipation or absorption. From the series representation of the dispersive component, the dissipative component is determined from the conjugate series, which is the Hilbert transform of the original series. The converse of the latter statement also applies.

Notice that in the limit $T \to \infty$, equation (4) reverts back to the standard definition given in (1):

$$\begin{aligned} \lim_{T\to\infty} H_T f(x) &= \lim_{T\to\infty} \frac{1}{2T} \text{ p.v.} \int_{-T}^{T} f(x-t) \cot\left(\frac{\pi t}{2T}\right) dt \\ &= \frac{1}{\pi} \text{ p.v.} \int_{-\infty}^{\infty} \frac{f(x-t)}{x-t}\, dt. \end{aligned} \tag{5}$$

If one formally differentiates the well-known Fourier series

$$\log\left|\sin\left(\frac{x}{2}\right)\right| = -\log(2) - \sum_{n=1}^{\infty} \frac{\cos(nx)}{n},$$

one would get

$$\frac{1}{2}\cot\left(\frac{x}{2}\right) = \sum_{n=1}^{\infty} \sin(nx).$$

Now, as it happens, one cannot just differentiate any Fourier series as one wishes. Note that for general $x \in (0, 2\pi)$ the term $\sin(nx) \nrightarrow 0$ as $n \to \infty$, so the series on the right-hand side does not converge. One approach to address this issue would be to interpret everything in terms of distributions; however, we do not require this theory for our purposes.

2 Technical results

In this section, we establish several key technical results that will be crucial for our subsequent proof of the Euler sine product formula. This formula is fundamental in our derivation of the Hilbert transform.

Lemma 2.1 *For* $0 < a < 1$ *and a constant* B*, the inequality* $-\ln(1-a) < a + Ba^2$ *holds.*

Proof We consider the identity

$$-\ln(1-a) = a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \cdots = a + a^2\left(\frac{1}{2} + \frac{a}{3} + \frac{a^2}{4} + \cdots\right).$$

Define $B_n = \frac{1}{2} + \frac{a}{3} + \frac{a^2}{4} + \cdots$; since B_n is convergent, we have $\sup_n B_n = B$. Therefore, we obtain

$$-\ln(1-a) < a + Ba^2 \quad \text{for } 0 < a < 1,$$

which concludes the proof. □

Theorem 2.1 *The series* $\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 k^2}\right)$ *uniformly converges for* $-k\pi \leq x \leq k\pi$ *and any positive integer* k*.*

Proof Let $M_k = \frac{x^2}{\pi^2 k^2} + B\frac{x^4}{\pi^4 k^4}$ for $x \in [-k\pi, k\pi]$ and any positive integer k. Observe that

$$\sum_{k=1}^{\infty} M_k = \frac{x^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} + B\frac{x^4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

is a convergent series. Leveraging Lemma 2.1, we have $\left|\ln\left(1 - \frac{x^2}{\pi^2 k^2}\right)\right| \leq M_k$, and by the Weierstrass M-test, the series $\sum_{k=1}^{\infty}\ln\left(1 - \frac{x^2}{\pi^2 k^2}\right)$ converges uniformly. This implies that $\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 k^2}\right)$ uniformly converges, thus completing the proof. □

Theorem 2.2 *Let* $\varphi : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *be a function defined by*

$$\varphi(x) = \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{x - \pi k} + \frac{1}{x + \pi k}\right).$$

Then φ *can be differentiated termwise.*

Proof Define $\varphi_n : \Omega \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\varphi_n(x) = \frac{1}{x} + \sum_{k=1}^{n} \left(\frac{1}{x - \pi k} + \frac{1}{x + \pi k} \right), \quad n = 0, 1, 2, \ldots$$

Consider any $x \in \Omega$. For all $k > \frac{\sqrt{2}}{\pi}|x|$, the reverse triangle inequality yields

$$|x^2 - \pi^2 k^2| \geq \pi^2 k^2 - |x|^2 > \pi^2 k^2 - \frac{\pi^2 k^2}{2} = \frac{\pi^2 k^2}{2},$$

implying $\left| \frac{1}{x^2 - \pi^2 k^2} \right| \leq \frac{2}{\pi^2 k^2} := M_k$. By the Weierstrass M-test, the partial sum $\varphi_n(x) = \frac{1}{x} + 2x \sum_{k=1}^{n} \frac{1}{x^2 - \pi^2 k^2}$ converges absolutely. As a result, they converge to the limit function

$$\varphi(x) = \frac{1}{x} + 2x \sum_{k=1}^{\infty} \frac{1}{x^2 - \pi^2 k^2}.$$

We now prove that the convergence is uniform on compact subsets of Ω. Let K be such a subset, and let $\epsilon > 0$ be given. A uniform bound M exists on $|x|$ for all $x \in K$, i.e., $|x| \leq M$. Furthermore, there exists a starting index n_0 such that for any $n > n_0$,

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} < \frac{\epsilon \pi^2}{4M}.$$

For any $n > \max(n_0, \frac{\sqrt{2}}{\pi} M)$ and for $x \in K$, we have

$$\begin{aligned} |\varphi(x) - \varphi_n(x)| &= \left| 2x \sum_{k=n+1}^{\infty} \frac{1}{x^2 - \pi^2 k^2} \right| \\ &\leq 2M \sum_{k=n+1} \left| \frac{1}{x^2 - \pi^2 k^2} \right| \\ &\leq \frac{2M}{\pi^2} \sum_{k=n+1}^{\infty} \frac{2}{k^2} < \epsilon. \end{aligned}$$

This demonstrates that the convergence of the sequence $\{\varphi_n\}$ to φ on Ω is uniformly on compact subsets, allowing termwise differentiation for the limit function φ. □

3 The Euler Sine product formula

In 1734, Leonhard Euler conjectured that the sine function could be expressed as an infinite product, namely

$$\sin(x) = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right) \tag{6}$$

for all real values of x. While Euler's approach is considered less rigorous by today's standards, a comprehensive description of his intuitive method can be found in [4, pp. 214–216].

Despite the non-rigorous nature of his derivation, Euler provided compelling arguments supporting the validity of his sine product formula, some of which are discussed in [13]. He explored numerous consequences of this formula, most notably using it to solve the Basel problem, which established that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi}{6}$$

and resolved a longstanding question that had baffled his contemporaries for nearly a century. Euler's original derivation can be found in [6], with summaries in [3, p. 47], [4, pp. 216–134], and [18, p. 12].

Throughout the remainder of this note, we will utilize the following well-known results:

1.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\, dt &= \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos^{2(n-1)}(t)\, dt \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^0(t)\, dt \\
&= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \\
&= \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}, \qquad \text{for } n \geq 0.
\end{aligned}$$

2.

$$\sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{n+1}} \leq \int_0^{\frac{\pi}{2}} \cos^{2n+1}(t)\, dt \leq \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{1+\frac{1}{2}}}$$

for $n \geq 1$.

For further details on inequality (B), refer to [21].

The next two results are proven in [21]; however, for the sake of completeness and the benefit of the reader, we shall provide their proofs.

Lemma 3.1 *Suppose f is a Riemann integrable function such that* $|f(t) - f(0)| \leq Mt$ *for some positive constant M and* $0 \leq t \leq \frac{\pi}{2}$*. Then, we have*

$$\lim_{n\to\infty} \frac{\int_0^{\frac{\pi}{2}} f(t)\cos^n(t)\,dt}{\int_0^{\frac{\pi}{2}} \cos^n(t)\,dt} = f(0). \tag{7}$$

Proof Since $\tan(t) \geq t$ for all $t \in [0, \frac{\pi}{2}]$, and using the well-known identity $1 + \tan^2(t) = \sec^2(t)$, we have $\sec^2(t) \geq 1 + t^2$. This implies $\cos(t) \leq \frac{1}{\sqrt{1+t^2}}$ for all $t \in [0, \frac{\pi}{2}]$. Next, for $n \geq 3$, we have

$$\begin{aligned}
&\left| \int_0^{\frac{\pi}{2}} f(t)\cos^n(t)\,dt - f(0)\int_0^{\frac{\pi}{2}} \cos^n(t)\,dt \right| \\
&\quad \leq \int_0^{\frac{\pi}{2}} |f(t) - f(0)|\cos^n(t)\,dt \\
&\quad \leq M\int_0^{\frac{\pi}{2}} \frac{t}{(1+t^2)^{\frac{n}{2}}}\,dt \\
&\quad \leq \frac{M}{n-2}.
\end{aligned}$$

Therefore, combining the above estimate with inequality (B), we obtain

$$\left| \frac{\int_0^{\frac{\pi}{2}} f(t)\cos^n(t)\,dt}{\int_0^{\frac{\pi}{2}} \cos^n(t)\,dt} - f(0) \right| \leq \sqrt{\frac{2}{\pi}}\frac{\sqrt{n+1}}{n-2}M$$

for $n \geq 3$, from which the conclusion follows by taking the limit as $n \to \infty$. □

Theorem 3.1 (Euler's sine product formula) *For all real* $x \in [-k, k]$, $k = 1, \ldots$ *we have*

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right).$$

Proof Suppose $n \geq 2$ and $x \in [-k, k]$, $k = 1, \ldots$ except 0 or $\pm\frac{n}{2}$. Integrating by parts twice, we have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \cos^n(t)\cos(2xt)\,dt = \\
&= -\frac{\cos^n(t)\sin(2xt)}{2x}\Big|_0^{\frac{\pi}{2}} + \frac{n}{2x}\int_0^{\frac{\pi}{2}} \cos^{n-1}(t)\sin(t)\sin(2xt)\,dt \\
&= -\frac{\cos^n(t)\sin t\cos(2xt)}{2x}\Big|_0^{\frac{\pi}{2}} + \frac{1}{2x}\int_0^{\frac{\pi}{2}} [(n-1)\cos^{n-2}(t)(\cos^2(t)-1) \\
&\quad + \cos^n(t)]\cos(2xt)\,dt \\
&= \frac{n}{4x^2}\int_0^{\frac{\pi}{2}} \cos^n(t)\cos(2xt)\,dt - \frac{n(n-1)}{4x^2}\int_0^{\frac{\pi}{2}} \cos^{n-2}(t)\cos(2xt)\,dt.
\end{aligned}$$

This rearrangement gives the reduction formula:

$$\int_0^{\frac{\pi}{2}} \cos^n(t)\cos(2xt)\,dt = \frac{n-1}{n}\left(1 - \frac{4x^2}{n^2}\right)^{-1}\int_0^{\frac{\pi}{2}} \cos^{n-2}(t)\cos(2xt)\,dt,$$

valid for $n \geq 2$. For even exponents, where $n \geq 1$, we find:

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\cos(2xt)\,dt \\
&= \frac{2n-1}{2n}\left(1 - \frac{x^2}{n^2}\right)^{-1}\int_0^{\frac{\pi}{2}} \cos^{2(n-1)}(t)\cos(2xt)\,dt \\
&= \prod_{k=1}^{n}\frac{2k-1}{2k}\prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2}\right)^{-1}\int_0^{\frac{\pi}{2}} \cos^{2(0)}(t)\cos(2xt)\,dt \\
&= \frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}\prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2}\right)^{-1}\frac{\sin(\pi x)}{\pi x}.
\end{aligned}$$

By replacing the first product with the integral in result (A) and rearranging, we obtain the suggestive formula:

$$\sin(\pi x) = \pi x\prod_{k=1}^{n}\left(1 - \frac{x^2}{k^2}\right)\frac{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\cos(2xt)\,dt}{\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\,dt},$$

where $x \in [-k, k]$ and $n \geq 1$. By Lemma 3.1, we have

$$\begin{aligned}
\sin(\pi x) &= \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \lim_{n\to\infty} \frac{\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\cos(2xt)\,dt}{\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\,dt} \\
&= \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) \cos(0) \\
&= \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right),
\end{aligned}$$

which completes the proof of the Euler sine product formula. $\square$

Remark 3.1 From (6), we have

$$\sin(x) = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 k^2}\right). \tag{8}$$

4 The Hilbert transform for periodic functions

To derive the Hilbert transform for periodic functions, we start from the foundational identity (8). To begin, Let's consider the absolute values of the terms that might potentially be negative within this identity:

$$|\sin(x)| = |x| \prod_{k=1}^{\infty} \left|\left(1 - \frac{x^2}{\pi^2 k^2}\right)\right|. \tag{9}$$

Taking the logarithm at both sides of (9) we obtain,

$$\log|\sin x| = \log|x| + \sum_{k=1}^{\infty} \log\left|1 - \frac{x^2}{\pi^2 k^2}\right|. \tag{10}$$

By applying Theorem 2.1, we can differentiate both sides of (10) with respect to x term by term, resulting in

$$\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2 - \pi^2 k^2},$$

which is uniformly convergent by Theorem 2.2.

Now, let's consider a periodic function f with period $2T$. We have:

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{f(s)}{x-s}\,ds\\
&= \frac{1}{\pi}\Bigg\{\cdots+\text{p.v.}\int_{-5T}^{-3T}\frac{f(s)}{x-s}\,ds+\text{p.v.}\int_{-3T}^{-T}\frac{f(s)}{x-s}\,ds\\
&\quad+\text{p.v.}\int_{-T}^{T}\frac{f(s)}{x-s}\,ds+\text{p.v}\int_{T}^{3T}\frac{f(s)}{x-s}\,ds+\text{p.v.}\int_{3T}^{5T}\frac{f(s)}{x-s}\,ds\\
&\quad+\text{p.v.}\int_{5T}^{7T}\frac{f(s)}{x-s}\,ds+\cdots\Bigg\},
\end{aligned}
$$

where we've used the periodicity of $f(s)$, i.e., $f(s) = f(s+2Tk)$ for $k \in \mathbb{Z}$. Simplifying further, we express this as:

$$
Hf(x) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\text{p.v.}\int_{-T}^{T}\frac{f(s)}{x-s+2Tk}\,ds.
$$

In particular, for $f \in L_p(\mathbb{R})$ with $1 < p \leq q < \infty$, from Hölder's inequality

$$
\sum_{k=-\infty}^{\infty}\int_{-T}^{T}\frac{f(s)}{x-s+2Tk}\,ds \leq \sum_{k=-\infty}^{\infty}\left(\frac{1}{x-s+2Tk}\right)^{\frac{1}{q}}\Bigg|_{-T}^{T}\|f\|_p < \infty,
$$

so we can apply the Beppo Levi theorem to interchange the summation with the integral:

$$
\begin{aligned}
Hf(x) &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \text{p.v.} \int_{-T}^{T} \frac{f(s)}{x-s+2Tk}\, ds \\
&= \frac{1}{\pi}\, \text{p.v.} \sum_{k=-\infty}^{\infty} \int_{-T}^{T} \frac{f(s)}{x-s+2Tk}\, ds \\
&= \frac{1}{\pi}\, \text{p.v.} \int_{-T}^{T} f(s) \left(\sum_{k=-\infty}^{\infty} \frac{1}{x-s+2Tk} \right) ds \\
&= \frac{1}{\pi}\, \text{p.v.} \int_{-T}^{T} f(s) \left[\sum_{k=1}^{\infty} \frac{1}{x-s+2Tk} + \sum_{k=-\infty}^{0} \frac{1}{x-s+2Tk} \right] ds \\
&= \frac{1}{\pi}\, \text{p.v.} \int_{-T}^{T} f(s) \left[\frac{1}{x-s} + \sum_{k=1}^{\infty} \left(\frac{1}{x-s+2Tk} + \frac{1}{x-s-2Tk} \right) \right] ds \\
&= \frac{1}{\pi}\, \text{p.v.} \int_{-T}^{T} f(s) \left[\frac{1}{x-s} + \sum_{k=1}^{\infty} \left(\frac{1}{\frac{(x-s)}{2T}+k} + \frac{1}{\frac{(x-s)}{2T}-k} \right) \right] ds \\
&= \frac{1}{\pi} \frac{2T}{2T}\, \text{p.v.} \int_{-T}^{T} f(s) \left[\frac{1}{\frac{(x-s)}{2T}} + \sum_{k=1}^{\infty} \left(\frac{1}{\frac{(x-s)}{2T}+k} + \frac{1}{\frac{(x-s)}{2T}-k} \right) \right] ds \\
&= \frac{1}{2T}\, \text{p.v.} \int_{-T}^{T} f(s) \cot\left(\frac{\pi(x-s)}{2T} \right) ds.
\end{aligned}
$$

Therefore, for a periodic function f with period $2T$ belonging to $L_p(\mathbb{R})$, and even more for f belonging to $L_2([-\pi, \pi])$, we have:

$$
Hf(x) = \frac{1}{2T}\, \text{p.v.} \int_{-T}^{T} f(s) \cot\left(\frac{\pi(x-s)}{2T} \right) ds.
$$

As an example, let's compute the Hilbert transform of the periodic function $f(t) = \sin(t)$ (period π), which belongs to $L_2([-\pi, \pi])$.

Example 4.1 *We find its Hilbert transform of the periodic function* $f(t) = \sin(t)$ *(period* π*), which even more belongs to* $L_2([-\pi, \pi])$.

Proof (Solution)

$$\begin{aligned}
Hf(x) &= \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} \sin(t)\cot\left(\frac{x-t}{2}\right)dt\\
&= \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} \sin(x-t)\cot\left(\frac{t}{2}\right)dt\\
&= \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} (\sin(x)\cos(t)-\cos(x)\sin(t))\cot\left(\frac{t}{2}\right)dt\\
&= -\frac{\cos(x)}{2\pi}\int_{-\pi}^{\pi} \sin(t)\cot\left(\frac{t}{2}\right)dt\\
&= -\frac{\cos(x)}{2\pi}\int_{-\pi}^{\pi} 2\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\cot\left(\frac{t}{2}\right)dt\\
&= -\frac{\cos(x)}{2\pi}\int_{-\pi}^{\pi} 2\cos^2\left(\frac{t}{2}\right)dt\\
&= -\frac{\cos(x)}{\pi}\int_{0}^{\pi} (\cos(t)+1)\,dt\\
&= -\cos(x).
\end{aligned}$$

Thus, $H(\sin)(x) = -\cos(x)$. $\square$

For further exploration of the Hilbert transform of periodic functions, refer to [1] and [15]. To examine the Hilbert transform of periodic functions with arbitrary periods, consult [15] and [17].

The next notable example provided by Weeden and Zygmund (see [24]) illustrates that, in general, if a function f is integrable, the Hilbert transform *Hf* need not be integrable. This result emphasizes the intricacies of the Hilbert transform and its implications in harmonic analysis, particularly concerning the preservation of integrability under transformation.

Example 4.2 *Hf is not necessarily bounded on* $L_1(\mathbb{R})$.

Proof (Solution) On one hand, let f be any periodic integrable function in $(0, \frac{\pi}{2})$ and zero elsewhere in $(-\pi, \pi)$. Then for $x \in (0, \frac{\pi}{2})$ we have

$$Hf(x) = \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} f(t)\cot\left(\frac{x-t}{2}\right)\,dt = \frac{1}{2\pi}\,\text{p.v.}\int_{0}^{\frac{\pi}{2}} f(t)\cot\left(\frac{x-t}{2}\right)\,dt.$$

Select $x \in (-\frac{\pi}{2}, 0)$ splitting the preceding integral

$$\begin{aligned}
&\frac{1}{2\pi}\,\text{p.v.}\int_{0}^{\frac{\pi}{2}} f(t)\cot\left(\frac{x-t}{2}\right)\,dt \\
&= \frac{1}{2\pi}\,\text{p.v.}\int_{0}^{|x|} f(t)\cot\left(\frac{x-t}{2}\right)\,dt + \frac{1}{2\pi}\,\text{p.v.}\int_{|x|}^{\frac{\pi}{2}} f(t)\cot\left(\frac{x-t}{2}\right)\,dt,
\end{aligned}$$

and noting that the second integral on the right hand side must be negative, yields

$$\begin{aligned}
\frac{1}{2\pi}\,\text{p.v.}\int_{0}^{\frac{\pi}{2}} f(t)\cot\left(\frac{x-t}{2}\right)\,dt &\le \frac{1}{2\pi}\,\text{p.v.}\int_{0}^{|x|} f(t)\cot\left(\frac{x-t}{2}\right)\,dt \\
&= -\frac{1}{2\pi}\,\text{p.v.}\int_{0}^{|x|} f(t)\cot\left(\frac{x-t}{2}\right)\,dt,
\end{aligned}$$

and thus

$$|Hf(x)| \ge \frac{1}{2\pi}\,\text{p.v.}\int_{0}^{|x|} f(t)\cot\left(\frac{x-t}{2}\right)\,dt.$$

For $t \in (0, |x|)$. It follows that

$$\tan\left(\frac{t-x}{2}\right) = \tan\left(\frac{1+|x|}{2}\right) \le \tan(|x|),$$

hence

$$\cot\left(\frac{t-x}{2}\right) \ge \cot(|x|)$$

and so

$$|Hf(x)| \geq \frac{1}{2\pi} \text{ p.v.} \int_0^{|x|} f(t) \cot\left(\frac{t-x}{2}\right) dt \geq \frac{\cot(|x|)}{2\pi} \int_0^{|x|} f(t)\, dt. \tag{11}$$

Now choosing

$$f(t) = \begin{cases} \dfrac{1}{t \log^2(t)} & \text{for } t \in (0, \frac{1}{2}) \\ 0 & \text{for } x \in [\frac{1}{2}, \frac{\pi}{2}). \end{cases}$$

Then

$$\int_{-\pi}^{\pi} |f(t)|\, dt = \int_0^{\frac{1}{2}} \frac{dt}{t \log^2(t)} = \frac{1}{\log 2}.$$

So that $f \in L_1(-\pi, \pi)$.

On the other hand, from (11) it follows that

$$|Hf(x)| \geq \frac{\cot(|x|)}{2\pi \log^2(|x|)}.$$

which in the vicinity of $x = 0$ can be expressed as follows

$$|Hf(x)| \geq \frac{1}{2\pi |x| \log^2(x)} \geq \frac{1}{2\pi |x| \log(x)}.$$

Now, for small α, write

$$\int_{-\pi}^{\pi} |Hf(x)|\, dx = \int_{-\pi}^{-\alpha} |Hf(x)|\, dx + \int_{-\alpha}^{\alpha} |Hf(x)|\, dx + \int_{\alpha}^{\pi} |Hf(x)|\, dx,$$

hence

$$\int_{-\pi}^{\pi} |Hf(x)|\, dx \geq \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{dx}{|x| \log(x)} = \frac{1}{\pi} \int_0^{\alpha} \frac{dx}{x \log(x)} = \infty.$$

Therefore, $Hf \notin L_1(-\pi, \pi)$. □

By its definition the following is clear.

Proposition 4.1 *Let f and g be two periodic functions with period T such that f and g belongs to* $L_p(\mathbb{R})$ *with* $\alpha, \beta \in \mathbb{R}$. *Then*

$$H(\alpha f + \beta g)(x) = \alpha Hf(x) + \beta Hg(x).$$

The following result establishes an important relationship between the Hilbert transforms of periodic functions. It showcases the symmetry between *Hf* and *Hg* with respect to their corresponding functions *f* and *g*.

Theorem 4.1 *Let f and g two periodic functions with period T such that* $f \in L_p(\mathbb{R})$ *and* $g \in L_q(\mathbb{R})$ *with* $1 < p < q < \infty$. *Then*

$$\int_{-T}^{T} Hf(x)g(x)\,dx = -\int_{-T}^{T} f(x)Hg(x)\,dx.$$

Proof Let $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$ periodic with period *T*, invoking the Fubini Theorem we have

$$\begin{aligned}
\int_{-T}^{T} Hf(x)g(x)\,dx &= \int_{-T}^{T} \left[\frac{1}{2\pi} \text{p.v.} \int_{-T}^{T} f(x) \cot\left(\frac{x-t}{2}\right) dt \right] g(x)\,dx \\
&= \int_{-T}^{T} f(x) \left[\frac{1}{2\pi} \text{p.v.} \int_{-T}^{T} g(x) \cot\left(\frac{x-t}{2}\right) dx \right] dt \\
&= -\int_{-T}^{T} f(x) \left[\frac{1}{2\pi} \text{p.v.} \int_{-T}^{T} g(x) \cot\left(\frac{t-x}{2}\right) dx \right] dt \\
&= -\int_{-T}^{T} f(t)Hg(t)\,dt.
\end{aligned}$$

Hence

$$\int_{-T}^{T} Hf(x)g(x)\,dx = -\int_{-T}^{T} f(x)Hg(x)\,dx,$$

thus Theorem 4.1 is completely proved. □

5 L_p strong boundedness for the Hilbert transform

In this section, we establish the boundedness of the Hilbert transform on the Lebesgue spaces $L_p(\mathbb{R})$ for $1 < p < \infty$.

Theorem 5.1 *Let $f \in L_p(\mathbb{R})$ with $1 < p < 2$. Then*

$$\|Hf\|_{L_p} \leq C\|f\|_{L_p}.$$

Proof The first step so is by breaking up a function $f \in L_p(\mathbb{R})$ for $1 < p < 2$, as $f = g + b$ where g and b are in L_1 and L_2 respectively. One way of achieving this decomposition is by truncating |f| on its range, that is, by setting $g = f\chi_{\{|f|>\lambda\}}$ and $b = f\chi_{\{|f|\leq\lambda\}}$ for some $\lambda > 0$ (typically b is the tail part of f and g is the singular part of f). Since $1 - p < 0$ and $2 - p > 0$, we see that

$$\int_{\mathbb{R}} |g(x)|\,dx = \int_{|f|>\lambda} |f(x)|^p |f(x)|^{1-p}\,dx \leq \lambda^{1-p} \int_{|f|>\lambda} |f(x)|^p\,dx \leq \lambda^{1-p}\|f\|_{L_p}$$

and

$$\int_{\mathbb{R}} |b(x)|^2\,dx = \int_{|f|\leq\lambda} |f(x)|^p |f(x)|^{2-p}\,dx \leq \lambda^{2-p} \int_{|f|\leq\lambda} |f(x)|^p\,dx < \lambda^{2-p}\|f\|_{L_p}$$

so that g and b are indeed in the appropriate spaces. Since for $f \in L_p$, $f = g + b$. Then $Hf(x) = Hg(x) + Hb(x)$, and so

$$D_{Hf}(\lambda) \leq D_{Hg}(\lambda/2) + D_{Hb}(\lambda/2),$$

where $D_f(\lambda) = m(\{x \in \mathbb{R} : |f(x)| > \lambda\})$ and m stand for the Lebesgue measure in $\mathbb{R}$.

Using Cavalieri's principle we obtain

$$
\begin{aligned}
\|Hf\|_{L_p}^p &= p\int_0^\infty \lambda^{p-1} D_{Hf}(\lambda)\,d\lambda \\
&\le p\int_0^\infty \lambda^{p-1} D_{Hg}(\lambda/2)\,d\lambda + p\int_0^\infty \lambda^{p-1} D_{Hb}(\lambda/2)\,d\lambda \\
&\le B_1\int_0^\infty \lambda^{p-2}\left(\int_{|f|>\lambda} |f(x)|\,dx\right) d\lambda + B_2\int_0^\infty \lambda^{p-3}\left(\int_{|f|\le\lambda} |f(x)|^2\,dx\right) d\lambda \\
&= B_1\int_0^\infty \lambda^{p-2}\left(\int_0^{|f|} |f(x)|\,dx\right) d\lambda + B_2\int_0^\infty \lambda^{p-3}\left(\int_{|f|}^\infty |f(x)|^2\,dx\right) d\lambda \\
&= B_1\int_0^\infty \lambda^{p-2}\left(\int_0^{|f|} |f(x)|\,dx\right) d\lambda + B_2\int_0^\infty \lambda^{p-3}\left(\int_{|f|}^\infty |f(x)|^2\,dx\right) d\lambda \\
&= B_1\int_0^\infty |f(x)|\left(\int_0^{|f|} \lambda^{p-2}\,d\lambda\right) dx + B_2\int_0^\infty |f(x)|^2\left(\int_{|f|}^\infty \lambda^{p-3}\,d\lambda\right) dx \\
&= \left(\frac{B_1}{p-1}+\frac{B_2}{p-2}\right)\int_0^\infty |f(x)|^p\,dx.
\end{aligned}
$$

Finally

$$\|Hf\|_{L_p} \le C\|f\|_{L_p}$$

while $C = \dfrac{B_1}{p-1}+\dfrac{B_2}{p-2}$. The proof is complete. □

Theorem 5.2 *Let* $f \in L_p(\mathbb{R})$ *with* $2 < p < \infty$. *Then*

$$\|Hf\|_{L_p} \le C\|f\|_p.$$

Proof Let $f \in L_p$ with $\frac{1}{p}+\frac{1}{q} = 1$, since $2 < p < \infty$, then we get that $1 < q < 2$. Hence by Theorem 5.1

$$\|Hg\|_{L_q} \le C\|g\|_{L_q} \tag{12}$$

for $g \in L_q$. By Theorem 4.1 as well as the Hölder inequality and (12) we have

$$\begin{aligned}
\|Hf\|_{L_p} &= \sup_{\|g\|_{L_q}=1} \left| \int_{-T}^{T} Hf(x)g(x)\,dx \right| \\
&= \sup_{\|g\|_{L_q}=1} \left| -\int_{-T}^{T} f(x)Hg(x)\,dx \right| \\
&\leq \sup_{\|g\|_{L_q}=1} |f|_p |H(g)|_{L_q} \\
&\leq C\|f\|_{L_p},
\end{aligned}$$

which prove that H is bounded on $L_p(\mathbb{R})$ for $2 < p < \infty$. □

Gohberg and Krupnik [7] determined the norm of the Hilbert transform (2) for $p = 2$, namely, we have

$$\|Hf\|_{L_2} = 1.$$

We conclude this paper by demonstrating the boundedness of the Hilbert transform on $L_p(\mathbb{R})$ for $1 < p < \infty$, utilizing the Hilbert transform defined on the unit circle $\mathbb{T}$.

Lemma 5.1 (M. Riez, see [20]) *The Hilbert transform is a bounded linear operator on $L_p(\mathbb{T})$ with $1 < p < \infty$. That is for $f \in L_p(\mathbb{T})$*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |Hf(e^{i\theta})|^p\,d\theta \leq \frac{1}{2\pi} M_p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p\,d\theta,$$

where

$$Hf(e^{i\theta}) = \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} f(e^{i\theta})\cot\left(\frac{\theta - t}{2}\right)\,dt$$

and $\mathbb{T}$ is the unit circle.

The best constant M_p was found by Pichorides [19] and it is

$$M_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if} \quad 1 < p < 2 \\ \cot\left(\frac{\pi}{2p}\right) & \text{if} \quad 2 < p < \infty. \end{cases}$$

Theorem 5.3 *Let $f \in L_p(\mathbb{R})$ with $1 < p < \infty$. Then*

$$\|Hf\|_{L_p} \le M_p \|f\|_{L_p}.$$

Proof Let $f \in L_p(\mathbb{R})$ and define

$$H_n f(x) = \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(y) \cot\left(\frac{x-y}{2n}\right) dy.$$

By (5) we have

$$\lim_{n\to\infty} H_n f(x) = Hf(x).$$

Now, by Lemma 5.1 the Hilbert transform $H_n f$ is bounded on $L_p(\mathbb{T})$, so we have

$$\begin{aligned}
\int_{-n\pi}^{n\pi} |H_n f(x)|^p \, dx &= n \int_{-\pi}^{\pi} |H_n f(nx)|^p \, dx \\
&\le n M_p \int_{-\pi}^{\pi} |f(nx)|^p \, dx \\
&= M_p \int_{-n\pi}^{n\pi} |f(x)|^p \, dx \\
&\le M_p \int_{-\infty}^{\infty} |f(x)|^p \, dx.
\end{aligned}$$

Thus for any $n \ge m$,

$$\int_{-n\pi}^{n\pi} |H_n f(x)|^p \le M_p \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

By Fatou's Lemma

$$\begin{aligned}\int_{-n\pi}^{n\pi} |Hf(x)|^p\,dx &= \int_{-n\pi}^{n\pi} \liminf_{n\to\infty} |H_n f(x)|^p\,dx \\ &\leq \liminf_{n\to\infty} \int_{-n\pi}^{n\pi} |H_n f(x)|^p\,dx \\ &\leq M_p \int_{-\infty}^{\infty} |f(x)|^p\,dx.\end{aligned}$$

Now, let $m \to \infty$ to get

$$\int_{-\infty}^{\infty} |Hf(x)|^p\,dx \leq M_p \int_{-\infty}^{\infty} |f(x)|^p\,dx,$$

hence Theorem 5.3 is completely proved. □

We would like to acknowledge that the anonymous referee brought our attention to the paper [14], in which the authors studied the truncated Hilbert transform. Among other results, they established an isometric property for the aforementioned transform. Following the ideas in [14], we may obtain similar results to those in [14], which we leave for further studies.

6 Statements and declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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