Chapter 6
The Numerical Solution of Riemann–Hilbert Problems

Before we begin this chapter, we remark that other approaches to the numerical solution of Riemann–Hilbert problems have appeared in the literature (see, for example, [88, 40, 65]). These methods are typically based on discretizing the singular integral, or regularized singular integral, with a quadrature rule, e.g., the trapezoidal rule. Significant care must be employed in evaluating such discretized integrals near the contour of integration, and successful methodologies exist to deal with this issue [70]. Our method relies on a basis expansion and an explicit application of the Cauchy transform. Our evaluations and solutions are valid uniformly up to the contour of integration without the need for additional schemes.

With this in mind we consider the numerical solution of general RH problems, posed on complicated contours $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_L$, where Γ_j are Möbius transformations of the unit interval $M_j : \mathbb{I} \to \Gamma_j$. Our approach is based on solving the SIE (2.43):

$$\begin{aligned} \mathscr{C}[G;\Gamma]u &= G - I \quad \text{for} \\ \mathscr{C}[G;\Gamma]u &\triangleq \mathscr{C}_\Gamma^+ u - \mathscr{C}_\Gamma^- u G = u - \mathscr{C}_\Gamma^- u(G-I). \end{aligned}$$

Note that each row of the system is independent; therefore it is possible to reduce the problem to several vector-valued RH problems:

$$\mathscr{C}[G;\Gamma](\boldsymbol{e}_j^\top u) = \boldsymbol{e}_j^\top G - \boldsymbol{e}_j^\top.$$

In the presentation below, we primarily use the full matrix version, though in some cases we specialize to the 2×2 case where we solve for the rows separately.

To numerically approximate solutions to this SIE, we replace the infinite-dimensional operator with finite-dimensional matrices, reducing the problem to finite-dimensional linear algebra. In Section 6.1, we introduce the family of *projection methods* for solving general linear equations. Convergence for general projection methods is discussed in Section 6.1.1, subject to an assumption restricting the growth of the norm of the inverse of the underlying linear system.

To construct the required matrices, we exploit the fact that we can readily evaluate the Cauchy transform applied to a mapped Chebyshev basis *pointwise* to construct a *collocation method*, as described in Section 6.1.2. This gives us Chebyshev coefficients such that

$$u(s) \approx \sum_{k=0}^{n_j - 1} u_{kj}^n T_k(M_j^{-1}(s)) \qquad \text{for} \qquad s \in \Gamma_j,$$

where the unknown coefficients u_{kj}^n are determined by imposing that the SIE holds at a sequence of points. This leads to an approximation to the solution of the RH problem

$$\Phi(z) \approx \Phi_n(z) \triangleq I + \sum_{j=1}^{L} \sum_{k=0}^{n_j-1} u_{kj}^n \mathscr{C}_{\Gamma_j}[T_k \circ M_j^{-1}](z),$$

where the resulting Cauchy transforms can be evaluated via Chapter 5. Collocation methods are a type of projection method and hence fit directly into the general framework of Section 6.1. For an approximation resulting from a projection method converging to the true solution, it is sufficient to show that the approximation to u converges in $Z(\Gamma)$; see Corollary 5.35.

6.1 ▪ Projection methods

Suppose we are given an infinite-dimensional linear equation

$$\mathscr{A} u = f,$$

where $\mathscr{A} : X \to Y$ is an invertible linear map and $f \in Y$ for two vector spaces X and Y. The goal of projection methods is to use *projection operators* to convert the infinite-dimensional equation to a finite-dimensional linear system.

Assume we are given a projection operator $\mathscr{I}_n : Y \to Y_n$ where $Y_n \subset Y$ is finite-dimensional, and suppose X_n is a finite-dimensional subspace of X. The projection method considers the finite-dimensional operator $\mathscr{A}_n : X_n \to Y_n$ defined by

$$\mathscr{A}_n \triangleq \mathscr{I}_n \mathscr{A}|_{X_n}.$$

An approximation to $u \approx u^n \in X_n$ is obtained by solving the finite-dimensional equation

$$\mathscr{A}_n u^n = \mathscr{I}_n f.$$

Example 6.1. The canonical concrete example of a projection method is the *finite-section method*. In this case $X = Y = \ell^2$, $X_n = Y_n = \mathbb{C}^n$, and $\mathscr{I}_n$ is the *truncation operator*:

$$\mathscr{I}_n \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

If we write $\mathscr{A}$ in matrix form by its action on the canonical basis $\boldsymbol{e}_k$,

$$\mathscr{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

then the finite-section method consists of solving the finite-dimensional linear system

$$\mathscr{A}_n u^n = \boldsymbol{f}_n \quad \text{for} \quad \mathscr{A}_n = \mathscr{I}_n \mathscr{A}|_{\mathbb{C}^n} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}. \quad \blacksquare$$

6.1.1 ▪ Convergence of projection methods

We now turn our attention to bounding the error of projection methods. To facilitate this, it helps to have another projection operator $\mathscr{P}_n : X \to X_n$. Note that there can be many such projection operators $\mathscr{P}_n$, the choice of which can impact the bound.

Lemma 6.2. *Suppose $\mathscr{P}_n : X \to X_n$ and $\mathscr{I}_n : Y \to Y_n$ are projection operators, where $X_n \subset X$ and $Y_n \subset Y$. Then we have*

$$\|u - u^n\|_X \le \left(1 + \|\mathscr{A}_n^{-1}\|_{\mathscr{L}(Y_n, X_n)} \|\mathscr{I}_n\|_{\mathscr{L}(Y,Y_n)} \|\mathscr{A}\|_{\mathscr{L}(X,Y)}\right) \|u - \mathscr{P}_n u\|_X. \tag{6.1}$$

Proof. We have

$$u^n \triangleq \mathscr{A}_n^{-1} \mathscr{I}_n f = \mathscr{A}_n^{-1} \mathscr{I}_n \mathscr{A} u.$$

Thus we have

$$\begin{aligned} u - u^n &= u - \mathscr{P}_n u + \mathscr{P}_n u - \mathscr{A}_n^{-1} \mathscr{I}_n \mathscr{A} u \\ &= u - \mathscr{P}_n u + \mathscr{A}_n^{-1} \mathscr{A}_n \mathscr{P}_n u - \mathscr{A}_n^{-1} \mathscr{I}_n \mathscr{A} u \\ &= u - \mathscr{P}_n u + \mathscr{A}_n^{-1} (\mathscr{A}_n \mathscr{P}_n u - \mathscr{I}_n \mathscr{A} u) \\ &= u - \mathscr{P}_n u + \mathscr{A}_n^{-1} \mathscr{I}_n \mathscr{A} (\mathscr{P}_n u - u) \\ &= \left(I - \mathscr{A}_n^{-1} \mathscr{I}_n \mathscr{A}\right)(u - \mathscr{P}_n u), \end{aligned}$$

where I is taken to mean the identity operator on X. ☐

Provided the projection $\mathscr{P}_n u$ converges to u faster than the other operator norms grow, the method converges. The power of this lemma is that the rate of decay of $\|\mathscr{P}_n u - u\|_X$ can be extremely fast due to high regularity properties of the solution u, which implies fast convergence of u^n to u, even though $\mathscr{A}_n$ is constructed in low regularity spaces and no information about high regularity is used in the numerical scheme.

Remark 6.1.1. *This is in contrast to many other numerical methods — e.g., the finite-element method — where the convergence rate is limited by the regularity properties used in the construction of the numerical scheme.*

In the context of the SIEs associated with RH problems, we want to take $X = Z(\Gamma)$ as introduced in Definition 5.33 to ensure uniform convergence, due to Corollary 5.35. In this context, it is clear that the finite-section method is not appropriate: the truncation operator does not preserve the zero-sum condition encoded in $Z(\Gamma)$. We therefore must use an alternative.

6.1.2 ▪ Collocation methods

In practice, we use a *collocation method* for solving RH problems, which will prove convenient for enforcing the zero-sum condition. We first present collocation methods, before seeing that they are, in fact, a special case of projection methods.

Assume that $\mathscr{A} : X \to Y$, where $Y \subset C^0(\Gamma)$ (i.e., piecewise continuous functions; see (A.1)), so that evaluation is a well-defined operation. Then we can apply the operator $\mathscr{A}$ to a basis $\{\psi_k\} \subset X$ pointwise, i.e., we can evaluate $\mathscr{A}\psi_k(s)$ exactly for $s \in \Gamma$. Given n

collocation points $s^n = [s_1, \ldots, s_n]$, $s_j \in \Gamma$, we can construct an $n \times n$ linear system that consists of imposing that the linear equation holds pointwise:

$$\sum_{k=0}^{n-1} u_k^n \mathscr{A} \psi_k(s_j) = f(s_j) \quad \text{for} \quad j = 1, \ldots, L.$$

In other words, we calculate u_k^n by solving the linear system

$$A_n \begin{bmatrix} u_0^n \\ \vdots \\ u_{n-1}^n \end{bmatrix} = \begin{bmatrix} f(s_1) \\ \vdots \\ f(s_n) \end{bmatrix} \quad \text{for} \quad A_n = \begin{bmatrix} \mathscr{A}\psi_0(s_1) & \cdots & \mathscr{A}\psi_{n-1}(s_1) \\ \vdots & \ddots & \vdots \\ \mathscr{A}\psi_0(s_n) & \cdots & \mathscr{A}\psi_{n-1}(s_n) \end{bmatrix}.$$

We then arrive at the approximation

$$u^n = \sum_{k=0}^{n-1} u_k^n \psi_k. \tag{6.2}$$

Convergence bounds: The same argument as in projection methods allows us to bound the error.

Corollary 6.3. *Define the evaluation operator* $\mathscr{E}_n : C^0(\Gamma) \to \mathbb{C}^n$ *by*

$$\mathscr{E}_n f = [f(s_1), \ldots, f(s_n)]^\top,$$

the expansion operator $\mathscr{B}_n : \mathbb{C}^n \to X_n$ *by*

$$\mathscr{B}_n \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix} = \sum_{k=0}^{n-1} u_k \psi_k,$$

and the domain space by $X_n = \operatorname{span}\{\psi_0, \ldots, \psi_{n-1}\}$*. For any projection operator* $\mathscr{P}_n : X \to X_n$*, and any choice of norm on* $\mathbb{C}^n$*, we have the bound*

$$\|u - u^n\|_X \le \left(1 + \|\mathscr{B}_n\|_{\mathscr{L}(\mathbb{C}^n, X)} \|A_n^{-1}\|_{\mathscr{L}(\mathbb{C}^n)} \|\mathscr{E}_n\|_{\mathscr{L}(Y, \mathbb{C}^n)} \|\mathscr{A}\|_{\mathscr{L}(X,Y)}\right) \|u - \mathscr{P}_n u\|_X,$$

where u^n *is* (6.2).

Proof. Similar to the argument of Lemma 6.2, we have

$$u - u^n = (I - \mathscr{B}_n A_n^{-1} \mathscr{E}_n \mathscr{A})(u - \mathscr{P}_n u). \qquad \square$$

In the setting of RH problems, X has special structure that may not be satisfied by the basis: for example, the basis may be piecewise mapped Chebyshev polynomials but X imposes the zero-sum condition. Thus we consider the case where there is a larger space Z that contains both X and the basis: $\psi_k \in Z$ and $X \subset Z$ but $\psi_k \notin X$. This raises an issue: we can no longer define the collocation matrix by applying the operator to the basis, as the operator is only defined on X. Instead, we assume we are given a matrix A_n (in our case, built from the finite-part Cauchy transform) that is equivalent to $\mathscr{A}$ for all expansions in the basis that are also in X. The argument of Corollary 6.3 remains valid, under the condition that the collocation method maps to the correct space.

Corollary 6.4. *Define the domain space* $X_n = \operatorname{span}\{\psi_0, \ldots, \psi_{n-1}\} \cap X$. *Suppose we are given an invertible matrix* A_n *satisfying*

$$A_n \begin{bmatrix} u_0 \\ \vdots \\ u_{n-1} \end{bmatrix} = \mathscr{E}_n \mathscr{A} \sum_{k=0}^{n-1} u_k \psi_k$$

whenever $\sum_{k=0}^{n-1} u_k \psi_k \in X_n$, *which we use in the definition of* u^n *via* (6.2). *For any projection operator* $\mathscr{P}_n : X \to X_n$, *and any choice of norm on*

$$V_n \triangleq \mathscr{E}_n \mathscr{A} X_n \subset \mathbb{C}^n, \tag{6.3}$$

we have the bound

$$\|u - u^n\|_X \le \left(1 + \|\mathscr{B}_n A_n^{-1}\|_{\mathscr{L}(V_n,X)} \|\mathscr{E}_n\|_{\mathscr{L}(Y,V_n)} \|\mathscr{A}\|_{\mathscr{L}(X,Y)}\right) \|u - \mathscr{P}_n u\|_X \tag{6.4}$$

provided that

$$\mathscr{B}_n A_n^{-1} : V_n \to X.$$

Furthermore, for any choice of norm on $\mathbb{C}^n$

$$\|u - u^n\|_X \le \left(1 + \|\mathscr{B}_n\|_{\mathscr{L}(\mathbb{C}^n,Z)} \|A_n^{-1}\|_{\mathscr{L}(\mathbb{C}^n)} \|\mathscr{E}_n\|_{\mathscr{L}(Y,\mathbb{C}^n)} \|\mathscr{A}\|_{\mathscr{L}(X,Y)}\right) \|u - \mathscr{P}_n u\|_X.$$

Proof. We have the identity

$$u - u^n = (I - \mathscr{B}_n A_n^{-1} \mathscr{E}_n \mathscr{A})(u - \mathscr{P}_n u)$$

from before, where the right-hand side must be in X from the assumptions. Taking the X-norm of both sides gives (6.4). We also have

$$\|u - u^n\|_X \le \left(1 + \|\mathscr{B}_n\|_{\mathscr{L}(S_n,X)} \|A_n^{-1}\|_{\mathscr{L}(V_n,S_n)} \|\mathscr{E}_n\|_{\mathscr{L}(Y,V_n)} \|\mathscr{A}\|_{\mathscr{L}(X,Y)}\right) \|u - \mathscr{P}_n u\|_X,$$

where $S_n = A_n^{-1} V_n \subset \mathbb{C}^n$. We use $\{\|v\|_X = 1\}$ to denote the set of all vectors in X with unit norm. Then if V_n and S_n are equipped with the same norm as $\mathbb{C}^n$,

$$\|\mathscr{B}_n\|_{\mathscr{L}(S_n,X)} = \sup_{\|v\|_{S_n}=1} \|\mathscr{B}_n v\|_X = \sup_{\|v\|_{S_n}=1} \|\mathscr{B}_n v\|_Z \le \sup_{\|v\|_{\mathbb{C}^n}=1} \|\mathscr{B}_n v\|_Z = \|\mathscr{B}_n\|_{\mathscr{L}(\mathbb{C}^n,Z)},$$

$$\|A_n^{-1}\|_{\mathscr{L}(V_n,S_n)} = \sup_{\|v\|_{V_n}=1} \|A_n^{-1} v\|_{S_n} = \sup_{\|v\|_{V_n}=1} \|A_n^{-1} v\|_{\mathbb{C}^n} \le \sup_{\|v\|_{\mathbb{C}^n}=1} \|A_n^{-1} v\|_{\mathbb{C}^n} = \|A_n^{-1}\|_{\mathscr{L}(\mathbb{C}^n)},$$

$$\|\mathscr{E}_n\|_{\mathscr{L}(Y,V_n)} = \sup_{\|v\|_Y=1} \|\mathscr{E}_n v\|_{V_n} = \sup_{\|v\|_Y=1} \|\mathscr{E}_n v\|_{\mathbb{C}^n} = \|\mathscr{E}_n\|_{\mathscr{L}(Y,\mathbb{C}^n)},$$

and the corollary follows. ∎

Collocation methods as projection methods: We remark that collocation methods can also be viewed as a special family of projection methods, via an interpolation operator that takes V_n to Y.

Corollary 6.5. *Assume that there is an interpolation operator* $\mathscr{T}_n : V_n \to Y \subset C^0(\Gamma)$ *defined on* V_n *from* (6.3) *so that* $\mathscr{T}_n v(s_k) = e_k^\top v$ *for all* $v \in V_n$. *Then collocation methods are equivalent to a projection method with the projection operator* $\mathscr{I}_n = \mathscr{T}_n \mathscr{E}_n$, *the domain space* $X_n = \operatorname{span}\{\psi_0, \ldots, \psi_{n-1}\} \cap X$, *and* $\mathscr{A}_n = \mathscr{T}_n A_n \mathscr{B}_n^{-1}$.

This interpretation will prove useful in the next chapter.

Chebyshev collocation methods: We will use the Chebyshev bases introduced in Chapter 4 for the construction of collocation methods for RH problems. In the case where Γ has a single component and $M : \mathbb{I} \to \Gamma$ is a map from the unit interval, we use mapped Chebyshev polynomials as the basis $\psi_k(s) = T_k(M^{-1}(s))$ and mapped Chebyshev points as the collocation points $\boldsymbol{s}^n = M(\boldsymbol{x}^n)$, where $\boldsymbol{x}^n$ was defined in Definition 4.7. When $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_L$ and $M_j : \mathbb{I} \to \Gamma_j$, we take $n = n_1 + \cdots + n_L$ and use the basis

$$\begin{cases} T_k(M_j^{-1}(s)) & \text{if } s \in \Gamma_j, \\ 0 & \text{otherwise} \end{cases} \quad \text{for} \quad k = 0, \ldots, n_j - 1 \quad \text{and} \quad j = 1, \ldots, L. \tag{6.5}$$

(We omit the relationship between the basis defined above and the precise ordering of ψ_k, as it is immaterial.) We use the union of mapped Chebyshev points as our collocation points:

$$\boldsymbol{s}^n = M_1(\boldsymbol{x}^{n_1}) \cup \cdots \cup M_L(\boldsymbol{x}^{n_L}).$$

Note that this choice repeats nonsmooth points for every contour that contains them. We treat these as different points attached with the direction of approach:

$$M_j(-1) + 0\mathrm{e}^{\mathrm{i} \arg M_j'(-1)} \qquad \text{and} \qquad M_j(1) - 0\mathrm{e}^{\mathrm{i} \arg M_j'(1)},$$

recalling that $\arg M_j'(\pm 1)$ gives the angle that Γ_j leaves $M_j(-1)$ and approaches $M_j(1)$.

Finally, when the spaces are two-component row vectors, we take $n = 2(n_1 + \cdots + n_L)$ and use the bases

$$\begin{cases} \left[T_k(M_j^{-1}(s)),\, 0\right] & \text{if } s \in \Gamma_j, \\ [0, 0] & \text{otherwise} \end{cases} \quad \text{and} \quad \begin{cases} \left[0,\, T_k(M_j^{-1}(s))\right] & \text{if } s \in \Gamma_j, \\ [0, 0] & \text{otherwise} \end{cases}$$

for $k = 0, \ldots, n_j - 1$ and $j = 1, \ldots, L$.

Zero-sum spaces: For our setting, we want to work within the space $Z(\Gamma)$: in particular, we choose $X_n = Z_n(\Gamma) \triangleq Z(\Gamma) \cap \operatorname{span}\{\psi_0(s), \ldots, \psi_{n-1}(s)\}$, where the basis is constructed using mapped Chebyshev polynomials. Applying a truncation operator such as $\mathcal{I}_n$ to each component of Γ does not take $Z(\Gamma)$ to $Z_n(\Gamma)$; hence we cannot use it as $\mathcal{P}_n$. Instead, we use the projection consisting of evaluation and interpolation:

$$\mathcal{P}_n = \mathcal{B}_n \mathtt{T}_n \mathcal{E}_n,$$

which returns the piecewise-mapped Chebyshev polynomial that interpolates at Chebyshev points, including all self-intersection points. We have a guarantee that $\|\mathcal{P}_n u - u\|_{Z(\Gamma)}$ tends to zero by the theory of Chapter 4, with the speed of convergence dictated by the regularity of u.

6.2 ▪ Collocation method for RH problems

We now turn to the problem of constructing a collocation method to solve the SIE

$$\mathcal{C}[G;\Gamma]u = G - I.$$

In order to successfully construct a collocation method, our goal is to choose the basis and collocation points so that (1) we can evaluate the singular integral operator applied to the basis pointwise, (2) the collocation method converges, and (3) the collocation method converges *fast*. The last property is critical, not only because we aim to have high accuracy numerics, but also for the practical reason that the linear collocation system is dense, and dense linear algebra quickly breaks down as the dimension of the system increases.

6.2.1 ▪ Collocation method for a scalar RH problem on the unit interval

The singularities of the Cauchy transform at the endpoints of Γ present several difficulties for the construction of a collocation method. We first consider a simple example that avoids these issues: an RH problem for $\Gamma = \mathbb{I}$, where G is a smooth scalar function satisfying $G(\pm 1) = 1$.

We use the Chebyshev basis $\psi_k(x) = T_k(x)$ with the Chebyshev points. We thus need to evaluate

$$\mathscr{C}[G;\mathbb{I}]T_k(x) = T_k(x) + (1 - G(x))\mathscr{C}_{\mathbb{I}}^- T_k(x)$$

at the Chebyshev points. While $\mathscr{C}_{\mathbb{I}}^- T_k(x)$ has a logarithmic singularity at ± 1, this is canceled out by the decay of $1 - G(x)$, and hence we have $\mathscr{C}[G;\mathbb{I}]T_k(\pm 1) = T_k(\pm 1)$. We can therefore construct the *collocation matrix*

$$C_n[G;\mathbb{I}] \triangleq \begin{bmatrix} T_0(-1) & \cdots & T_{n-1}(-1) \\ \mathscr{C}[G;\mathbb{I}]T_0(x_2^n) & \cdots & \mathscr{C}[G;\mathbb{I}]T_{n-1}(x_2^n) \\ \vdots & \ddots & \vdots \\ \mathscr{C}[G;\mathbb{I}]T_0(x_{n-1}^n) & \cdots & \mathscr{C}[G;\mathbb{I}]T_{n-1}(x_{n-1}^n) \\ T_0(1) & \cdots & T_{n-1}(1) \end{bmatrix}.$$

Provided it is nonsingular, we solve the linear system

$$C_n[G;\mathbb{I}] \begin{bmatrix} u_0^n \\ \vdots \\ u_{n-1}^n \end{bmatrix} = \begin{bmatrix} G(x_1) - 1 \\ \vdots \\ G(x_n) - 1 \end{bmatrix},$$

giving the approximation

$$u(x) \approx u^n(x) \triangleq \sum_{k=0}^{n-1} u_k^n T_k(x).$$

In other words, we approximate the solution to the RH problem as

$$\Phi(z) \approx \Phi_n(z) \triangleq 1 + \mathscr{C}_{\mathbb{I}} u^n(z) = 1 + \sum_{k=0}^{n-1} u_k^n \mathscr{C}_{\mathbb{I}} T_k(z).$$

The terms $\mathscr{C}_{\mathbb{I}} T_k(z)$ can be calculated via the formulae in Section 5.6.

Convergence: The construction of a numerical approximation to the solution of an RH problem thus proceeds without difficulty. Convergence of the approximation is a more delicate question. If we assume a simple bound on the growth of the inverse of the collocation method, we can guarantee spectral convergence.

We recall Definition 2.45 and prove the following.

Theorem 6.6. *Assume that the solution* $u \in H^{\lambda+3}(\mathbb{I})$, $u(\pm 1) = 0$, *and*

$$\|C_n[G;\mathbb{I}]^{-1}\|_{\mathscr{L}(\ell^\infty)} = O(n^{\beta-3})$$

for some $\beta \geq 3$. *Then*

$$|\Phi(z) - \Phi_n(z)| = O(n^{\beta-\lambda}) \quad \textit{as} \quad n \to \infty$$

holds uniformly in z.

Proof. It is sufficient by Section 5.7.1 to prove that

$$\|u^n - u\|_{Z(\mathbb{I})} = O(n^{\beta-\lambda}),$$

i.e., we need $u^n \in Z(\mathbb{I})$ and the coefficients of u^n to converge to the coefficients of u at the desired rate in the $\ell^{2,1}$ norm. We will bound each operator in Corollary 6.4.

Note that, for some constant D_1,

$$\|f\|_{H_z^1(\mathbb{I})} \leq D_1 \|f\|_{Z(\mathbb{I})}$$

since every function in $Z(\mathbb{I})$ has a uniformly converging derivative and vanishes to first order at ± 1. Therefore, we have that

$$\|\mathscr{C}[G;\mathbb{I}]\|_{\mathscr{L}(Z(\mathbb{I}),H_z^1)} \leq D_1 \|\mathscr{C}[G;\mathbb{I}]\|_{\mathscr{L}(H_z^1)}$$

is bounded, by Theorem 2.50. The evaluation operator is also bounded $H^1(\mathbb{I}) \to \mathbb{C}^n$ by Sobolev embedding, using the supremum norm attached to $\mathbb{C}^n$: there exists a constant D_2 such that for every u

$$\|\mathscr{E}_n u\|_{\mathbb{C}^n} \leq \|u\|_{L^\infty(\mathbb{I})} \leq D_2 \|u\|_{H^1(\mathbb{I})}.$$

Finally, the expansion operator satisfies

$$\|\mathscr{B}_n\|_{\mathscr{L}(\ell^\infty,\ell^{2,1})} \leq n^3,$$

using the trivial estimate

$$\|\boldsymbol{u}\|_{\ell^{2,1}} = \sum_{k=0}^{n-1} |u_k|(k+1)^2 \leq n^3 \|\boldsymbol{u}\|_{\ell^\infty}.$$

We thus have one last criterion, that

$$\mathscr{B}_n C_n[G;\mathbb{I}]^{-1} : V_n \to X_n,$$

where $V_n = \operatorname{ran} \mathscr{E}_n \mathscr{C}[G;\mathbb{I}]Z(\mathbb{I}) \subset \left\{\boldsymbol{u} : \boldsymbol{e}_1^\top \boldsymbol{u} = \boldsymbol{e}_n^\top \boldsymbol{u} = 0\right\}$, i.e., the space of vectors whose first and last entries are zero. This follows since the first and last rows of $C_n[G;\mathbb{I}]$ correspond to

$$\sum_{k=0}^{n-1} u_k^n T_k(\pm 1);$$

hence if the right-hand side has zero first and last rows, the approximate solution vanishes at ± 1. As a consequence of Proposition B.9, we have $\|\mathscr{P}_n u - u\|_{Z(\mathbb{I})} = O(n^{-\lambda})$. Putting everything together, we have

$$\begin{aligned}
&\|u^n - u\|_{Z(\mathbb{I})} \\
&\quad \leq \left(1 + \|\mathscr{B}_n\|_{\mathscr{L}(\ell^\infty,Z(\mathbb{I}))} \|C_n[G;\mathbb{I}]^{-1}\|_{\mathscr{L}(\ell^\infty)} \|\mathscr{E}_n\|_{\mathscr{L}(H^1(\mathbb{I}),\ell^\infty)} \|\mathscr{A}\|_{H_z^1(\mathbb{I})}\right) \|\mathscr{P}_n u - u\|_{Z(\mathbb{I})} \\
&\quad = \left[1 + n^3 \mathscr{O}(n^{\beta-3})\right] \mathscr{O}(n^{-\lambda}) \|\mathscr{P}_n u - u\|_{Z(\mathbb{I})} = \mathscr{O}(n^{\beta-\lambda}) \|\mathscr{P}_n u - u\|_{Z(\mathbb{I})}. \qquad \square
\end{aligned}$$

6.2.2 ▪ Construction of a collocation method for matrix RH problems on multiple contours

We now consider the construction of a collocation method for our general family of RH problems. In this case, we use the piecewise Chebyshev basis of (6.5), with the mapped Chebyshev points with the argument information included in the self-intersection points. Then we will construct a collocation matrix $\mathrm{FP}\, C_n[G;\Gamma]$ using the finite-part Cauchy transform.

We now construct the collocation matrix in the scalar case. (The construction in the vector and matrix case follows in a straightforward manner.) Begin with a single contour Γ, where $M : \mathbb{I} \to \Gamma$ is a Möbius transformation, so that Γ is a line segment, ray, or circular arc. We obtain a discretization of the Cauchy operator on Γ alone as

$$\mathrm{FP}\, C^{\pm}_{n\Gamma} \triangleq \begin{bmatrix} \mathrm{FP}\mathscr{C}^{\pm}_{\Gamma}[T_0 \circ M^{-1}](M(x^n_1)) & \cdots & \mathrm{FP}\mathscr{C}^{\pm}_{\Gamma}[T_{n-1} \circ M^{-1}](M(x^n_1)) \\ \mathscr{C}^{\pm}_{\Gamma}[T_0 \circ M^{-1}](M(x^n_2)) & \cdots & \mathscr{C}^{\pm}_{\Gamma}[T_{n-1} \circ M^{-1}](M(x^n_2)) \\ \vdots & \ddots & \vdots \\ \mathscr{C}^{\pm}_{\Gamma}[T_0 \circ M^{-1}](M(x^n_{n-1})) & \cdots & \mathscr{C}^{\pm}_{\Gamma}[T_{n-1} \circ M^{-1}](M(x^n_{n-1})) \\ \mathrm{FP}\mathscr{C}^{\pm}_{\Gamma}[T_0 \circ M^{-1}](M(x^n_n)) & \cdots & \mathrm{FP}\mathscr{C}^{\pm}_{\Gamma}[T_{n-1} \circ M^{-1}](M(x^n_n)) \end{bmatrix},$$

recalling that $M(x^n_1)$ and $M(x^n_n)$ are endowed with the argument that Γ leaves the respective point. This gives us a discretization of $\mathscr{C}[G;\Gamma]$ as

$$\mathrm{FP}\, C_n[G;\Gamma] \triangleq I - \boldsymbol{G} \cdot \mathrm{FP}\, C^{-}_{n\Gamma},$$

where $\boldsymbol{G} = \operatorname{diag}\left\{G(M(x^n_1)) - 1, \ldots, G(M(x^n_n)) - 1\right\}$. (In the matrix and vector case, this becomes a matrix corresponding to pointwise multiplication on the right.)

We can similarly construct a discretization of the Cauchy operator on Γ evaluated on another contour Ω given by a Möbius transformation $N : \mathbb{I} \to \Omega$:

$$\mathrm{FP}\, C_{n\Gamma}|_{m\Omega} \triangleq \begin{bmatrix} \mathrm{FP}\mathscr{C}_{\Gamma}[T_0 \circ M^{-1}](N(x^m_1)) & \cdots & \mathrm{FP}\mathscr{C}_{\Gamma}[T_{n-1} \circ M^{-1}](N(x^m_1)) \\ \mathscr{C}_{\Gamma}[T_0 \circ M^{-1}](N(x^m_2)) & \cdots & \mathscr{C}_{\Gamma}[T_{n-1} \circ M^{-1}](N(x^m_2)) \\ \vdots & \ddots & \vdots \\ \mathscr{C}_{\Gamma}[T_0 \circ M^{-1}](N(x^m_{m-1})) & \cdots & \mathscr{C}_{\Gamma}[T_{n-1} \circ M^{-1}](N(x^m_{m-1})) \\ \mathrm{FP}\mathscr{C}_{\Gamma}[T_0 \circ M^{-1}](N(x^m_m)) & \cdots & \mathrm{FP}\mathscr{C}_{\Gamma}[T_{n-1} \circ M^{-1}](N(x^m_m)) \end{bmatrix},$$

noting that the finite-part Cauchy transform becomes a standard Cauchy transform when Γ and Ω are disjoint. We now decompose the full collocation matrix into its action on each subcomponent Γ:

$$\mathrm{FP}\, C_n[G;\Gamma] \triangleq \begin{pmatrix} \mathrm{FP}\, C_{n_1}[G_1;\Gamma_1] & -\boldsymbol{G}_1 \cdot \mathrm{FP}\, C_{n_2\Gamma_2}|_{n_1\Gamma_1} & \cdots & -\boldsymbol{G}_1 \cdot \mathrm{FP}\, C_{n_L\Gamma_L}|_{n_1\Gamma_1} \\ -\boldsymbol{G}_2 \cdot \mathrm{FP}\, C_{n_1\Gamma_1}|_{n_2\Gamma_2} & \mathrm{FP}\, C_{n_2}[G_2,\Gamma_2] & \cdots & -\boldsymbol{G}_2 \cdot \mathrm{FP}\, C_{n_L\Gamma_L}|_{n_2\Gamma_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{G}_L \cdot \mathrm{FP}\, C_{n_1\Gamma_1}|_{n_L\Gamma_L} & -\boldsymbol{G}_L \cdot \mathrm{FP}\, C_{n_2\Gamma_2}|_{n_L\Gamma_L} & \cdots & \mathrm{FP}\, C_{n_L}[G_L,\Gamma_L] \end{pmatrix},$$

where $\boldsymbol{G}_j = \operatorname{diag}\left\{G(M_j(x^{n_j}_1)), \ldots, G(M_j(x^{n_j}_{n_j}))\right\}$.

Remark 6.2.1. *More explicit formulae for the collocation matrices in every special case are given in* [96] *and are implemented in* [93].

Convergence: To establish convergence, we need to show that $C_n[G;\Gamma]^{-1}$ maps the correct spaces so that we can appeal to Corollary 6.4. Before continuing, we use the following proposition, which is in the same vein as Theorem 2.77 and compiled from the arguments that preceded it. This result is used repeatedly to simplify the arguments of the proofs by facilitating the reversal of orientations.

Proposition 6.7. *Suppose* $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_L$, *where* $M_j : \mathbb{I} \to \Gamma_j$. *Consider* Γ'', *where we reverse the orientation of one contour:* $\Gamma'' = M_1(-\mathbb{I}) \cup \Gamma_2 \cup \cdots \cup \Gamma_L$. *Then the RH problem* $[G,\Gamma]$ *has the same solution as* $[\tilde{G};\Gamma'']$, *where* $\tilde{G}(s) = G^{-1}(s)$ *for* $s \in \Gamma_1$ *and* $\tilde{G}(s) = G(s)$ *otherwise. Furthermore, for any* $u \in L^2(\Gamma)$, $\mathscr{C}[G;\Gamma]u|_{\Gamma_1} = -\mathscr{C}[\tilde{G};\Gamma'']u|_{\Gamma_1}$ *and* $\mathscr{C}[G;\Gamma]u|_{\Gamma_j} = \mathscr{C}[\tilde{G};\Gamma'']u|_{\Gamma_j}$ *for* $j = 2,\ldots,L$. *Specifically, if* u *solves* $\mathscr{C}[G;\Gamma]u = G - I$, *then* $\mathscr{C}[\tilde{G};\Gamma'']\tilde{u} = \tilde{G} - I$, *where* $\tilde{u}|_{\Gamma_1} = -u|_{\Gamma_1}$ *and* $\tilde{u}|_{\Gamma_j} = u|_{\Gamma_j}$ *for* $j = 2,\ldots,L$. *These properties carry over to the collocation discretization.*

We now describe precisely the space in which the right-hand side of the collocation lives.

Remark 6.2.2. *Unlike the analytic treatment (recall Definition* 2.54*) we do not use a decomposition of the jump matrix* $G(s) = X_-^{-1}(s)X_+(s)$ *in what follows. Part of the reasoning for this is that we only need to account for the first-order zero-sum/product condition in our discretization.*

Definition 6.8. *A function* f *defined on* Γ *satisfies* the cyclic junction condition with respect to G *if, for every* $a \in \gamma_0$, *we have*

$$\sigma_1 f_1 G_2^{\sigma_2} \cdots G_\ell^{\sigma_\ell} + \sigma_2 f_2 G_3^{\sigma_3} \cdots G_\ell^{\sigma_\ell} + \cdots + \sigma_{\ell-1} f_{\ell-1} G_\ell^{\sigma_\ell} + \sigma_\ell f_\ell = 0,$$

where $\Gamma_{j_1},\ldots,\Gamma_{j_\ell}$ *are the components of* Γ *that contain* a *as an endpoint,* $G_i = G|_{\Gamma_{j_i}}(a)$ *and* $f_i = f|_{\Gamma_{j_i}}(a)$, *and* $\sigma_i = +1$ *if* Γ_{j_i} *is oriented outwards and* -1 *if* Γ_{j_i} *is oriented inwards from* a.

Lemma 6.9. *If* G *satisfies the product condition and* $u \in Z(\Gamma)$, *then* $\mathscr{C}[G;\Gamma]u$ *satisfies the cyclic junction condition with respect to* G.

Proof. Define $f = \mathscr{C}[G;\Gamma]u$, let a be a nonsmooth point of Γ, and let $\Gamma_{i_1},\ldots,\Gamma_{i_\ell}$ be the contours with a as an endpoint, which we assume are oriented outwards (using Proposition 6.7). Because $u \in Z(\Gamma)$, we know $\mathscr{C}u(z)$ is continuous in each sector of the complex plane off Γ in a neighborhood of a, and therefore $\mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_{i+1}}) = \mathscr{C}_\Gamma^+ u(a + 0\mathrm{e}^{\mathrm{i}\varphi_i})$, where $\varphi_i = \arg M'_{j_i}(-1)$ is the angle that Γ_{j_i} leaves a. Also, from Plemelj's lemma, we have $\mathscr{C}_\Gamma^+ u(a + 0\mathrm{e}^{\mathrm{i}\varphi_i}) = u_i + \mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_i})$. Putting these together, we have for $i = 2,\ldots,\ell$

$$\begin{aligned}
f_i &= u_i + \mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_i})(I - G_i) = u_i + \mathscr{C}_\Gamma^+ u(a + 0\mathrm{e}^{\mathrm{i}\varphi_{i-1}})(I - G_i) \\
&= u_i + u_{i-1} - u_{i-1}G_i + \mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_{i-1}})(I - G_i) \\
&= u_i + u_{i-1} - u_{i-1}G_i + \mathscr{C}_\Gamma^+ u(a + 0\mathrm{e}^{\mathrm{i}\varphi_{i-2}})(I - G_i) \\
&= u_i + u_{i-1} + u_{i-2} - (u_{i-1} + u_{i-2})G_i + \mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_{i-2}})(I - G_i) \\
&\ \vdots \\
&= u_i + \cdots + u_1 - (u_{i-1} + \cdots + u_1)G_i + \mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_1})(I - G_i).
\end{aligned}$$

Now consider plugging this expression into

$$f_1 G_2 \cdots G_\ell + f_2 G_3 \cdots G_\ell + \cdots + f_{\ell-1} G_\ell + f_\ell,$$

to show that it vanishes. Due to cancellation between terms, we have

$$(I - G_1) G_2 \cdots G_\ell + (I - G_2) G_3 \cdots G_\ell + \cdots + (I - G_{\ell-1}) G_\ell + (I - G_\ell) = I - G_1 \cdots G_\ell = 0,$$

where the final equality follows from the product condition, which shows that the sum over the $\mathscr{C}_\Gamma^- u(a + 0\mathrm{e}^{\mathrm{i}\varphi_1})(I - G_i) G_{i+1} \cdots G_\ell$ terms are canceled. Similar cancellation implies that

$$\begin{aligned} u_1 G_2 \cdots G_\ell + [(u_2 + u_1) - u_1 G_2] G_3 \cdots G_\ell + \cdots + \left[(u_\ell + \cdots + u_1) - (u_{\ell-1} + \cdots + u_1) G_\ell\right] \\ = u_\ell + \cdots + u_1 = 0, \end{aligned}$$

where the final equality follows from the zero-sum condition. Putting these two cancellations together shows that f satisfies the required condition. ∎

We now show that, since the right-hand side satisfies the cyclic jump condition, the solution to the collocation system satisfies the zero-sum condition. There is a technical requirement on G for this to be true.

Definition 6.10. *The nonsingular junction condition is satisfied if, for all $a \in \gamma_0$,*

$$(\varphi_1 + 2\pi - \varphi_\ell) I + \sum_{j=2}^{L} (\varphi_j - \varphi_{j-1}) G_j \cdots G_\ell$$

is nonsingular (in other words, its determinant is nonzero), where we use the notation of Definition 6.8.

Lemma 6.11. *If $C_n[G;\Gamma]$ is nonsingular, G satisfies the product condition and nonsingular junction condition, and f satisfies the cyclic junction condition, then $\mathscr{B}_n C_n[G;\Gamma]^{-1} \mathscr{E}_n f \in Z_n(\Gamma)$.*

Proof. Let $u^n \triangleq \mathscr{B}_n C_n[G;\Gamma]^{-1} \mathscr{E}_n f$ be the solution to the collocation system, which we want to show is in $Z(\Gamma)$. Using the same notation as in the preceding proof, let $a \in \gamma_0$ and again assume the subcontours which contain a as an endpoint are oriented outwards. Define

$$C_i^\pm = \mathrm{FP}\, \mathscr{C}_\Gamma^\pm u^n(a + 0\mathrm{e}^{\mathrm{i}\varphi_i}).$$

By construction, the collocation system imposes that

$$C_i^+ - C_i^- G_i = f_i,$$

where f_i is $f|_{\Gamma_{j_i}}(a)$. From Corollary 5.37 we have

$$C_{i+1}^- = C_i^+ + (\varphi_{i+1} - \varphi_i) S,$$

where $S = -\frac{1}{2\pi}\sum_{i=1}^{\ell} u^n(a + 0\mathrm{e}^{\mathrm{i}\varphi_i})$ (which we want to show is zero). We get

$$\begin{aligned}
C_\ell^+ &= C_\ell^- G_\ell + f_i = C_{\ell-1}^+ G_\ell + S(\varphi_\ell - \varphi_{\ell-1})G_\ell + f_\ell \\
&= C_{\ell-1}^- G_{\ell-1} G_\ell + S(\varphi_\ell - \varphi_{\ell-1})G_\ell + f_{\ell-1}G_\ell + f_\ell \\
&= C_{\ell-2}^- G_{\ell-2} G_{\ell-1} G_\ell + S\left[(\varphi_{\ell-2} - \varphi_{\ell-1})G_{\ell-1}G_\ell + (\varphi_\ell - \varphi_{\ell-1})G_\ell\right] \\
&\qquad + f_{\ell-2}G_{\ell-1}G_{\ell-2} + f_{\ell-1}G_\ell + f_\ell \\
&\;\vdots \\
&= C_1^- G_1 \cdots G_\ell + S\sum_{i=2}^{\ell}(\varphi_i - \varphi_{i-1})G_i \cdots G_\ell + \sum_{i=1}^{\ell} f_i G_{i+1} \cdots G_\ell \\
&= C_\ell^+ + S\left[(\varphi_1 + 2\pi - \varphi_\ell)I + \sum_{i=2}^{\ell}(\varphi_i - \varphi_{i-1})G_i \cdots G_\ell\right],
\end{aligned}$$

where we used the cyclic junction condition to remove the sums involving f_i. The nonsingular junction condition means that the equality holds only if $S = 0$. □

With this property out of the way, the proof of convergence of the collocation method follows almost identically to Theorem 6.6.

Corollary 6.12. *Assume that* Γ *is bounded, the solution* $u \in H^{\lambda+3}(\Gamma)$, *and*

$$\|C_n[G;\Gamma]^{-1}\|_{\mathscr{L}(\ell^\infty)} = \mathscr{O}(n^{\beta-3}).$$

Then

$$|\Phi(z) - \Phi_n(z)| = \mathscr{O}(n^{\beta-\lambda}) \quad \text{as} \quad n \to \infty$$

holds uniformly in z.

Failure of the nonsingular junction condition: Now suppose the nonsingular junction condition is not satisfied at a single point $a \in \gamma_0$. Consider the modified collocation system with the condition

$$C_\ell^+ - C_\ell^- G_\ell = f_\ell$$

replaced with the condition that $S = \sum_{i=1}^{\ell} u^n(a + 0\mathrm{e}^{\mathrm{i}\varphi_i}) = 0$, ensuring the solution is in the correct space. Assuming that the resulting system is nonsingular, we have an approximation that satisfies the zero-sum condition, and now we will show that $C_\ell^+ - C_\ell^- G_\ell = f_\ell$ is still satisfied. We have continuity of the Cauchy transform, and hence

$$C_{j+1}^- = C_j^+ \quad \text{and} \quad C_1^- = C_\ell^+.$$

Thus, using the cyclic jump condition and the product condition,

$$\begin{aligned}
C_\ell^- G_\ell + f_\ell &= C_{\ell-1}^+ G_{\ell-1} G_\ell + f_\ell = \cdots = C_1^- G_1 \cdots G_{\ell-1} G_\ell + \sum_{j=1}^{\ell} f_j G_{j+1} \cdots G_\ell \\
&= C_\ell^+,
\end{aligned}$$

and the removed condition is still satisfied.

Remark 6.2.3. *The implementation in* [93] *does not take into account this possible failure or the remedy, as it does not arise generically.*

6.3 ▪ Case study: Airy equation

In this section we numerically solve the RH problem for the Airy equation introduced in Section 1.3. This demonstrates the deformation procedure that is required to obtain an RH problem that is sufficiently regular and has smooth solutions. Initially, one may want to apply the numerical method directly to the RH problem in Figure 1.5, but a simple computation shows that it does not satisfy the product condition (Definition 2.55) that is required for smooth solutions which is, in turn, required for convergence. The issue arises from singularities of $z^{-1/4}$ and $z^{3/2}$ at $z=0$. So, we demonstrate an important technique for moving contours away from such singularities.

Deformations: When deforming, it is beneficial to consider modifying $\boldsymbol{y}$ first because its jump matrices have no branch cuts. Define

$$J_0=\begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix},\quad J_1=\begin{bmatrix} 1 & 0 \\ -\mathrm{i} & 1 \end{bmatrix},\quad J_2=\begin{bmatrix} 1 & -\mathrm{i} \\ 0 & 1 \end{bmatrix},$$

and the jump conditions for $\boldsymbol{y}$ are shown in Figure 6.1. While $\boldsymbol{y}$ does not have nice asymptotic behavior at infinity, it is apparent that this RH problem satisfies the product condition of any order. We define $\boldsymbol{y}_1$ in terms of $\boldsymbol{y}$ within an open neighborhood, say $|z|<1$, of the origin in Figure 6.2. Note that $\boldsymbol{y}_1=\boldsymbol{y}$ for $|z|>1$. The jumps satisfied by $\boldsymbol{y}_1$ are shown in the right panel of Figure 6.2. The next task is to deal with the large z behavior of $\boldsymbol{y}_1$ so that the resulting function tends to $[1,\ 1]$ at infinity. While this was accomplished in some sense in Section 1.3, the jump matrix should tend to the identity matrix at infinity for the RH problem to be k-regular for any k. As an intermediate step define

$$\boldsymbol{\phi}_1(z)=\boldsymbol{y}_1(z)\begin{cases} 2\sqrt{\pi}z^{1/4}\begin{bmatrix} e^{\frac{2}{3}z^{3/2}} & 0 \\ 0 & e^{-\frac{2}{3}z^{3/2}} \end{bmatrix} & \text{if } |z|>1, \\ I & \text{if } |z|<1. \end{cases}$$

Then, the same computations as in Section 1.3 show that $\boldsymbol{\phi}_1^+(z)=\boldsymbol{\phi}_1^-(z)J_0$ for $z\in\mathbb{R}^-\setminus\{|z|\leq 1\}$ and $\boldsymbol{\phi}_1\sim[1,\ 1]$ at infinity. Next, we modify $\boldsymbol{\phi}_1$ so that the resulting function has jumps that all tend to the identity. We look to find a 2×2 matrix solution of

$$H^+(z)=H^-(z)J_0,$$

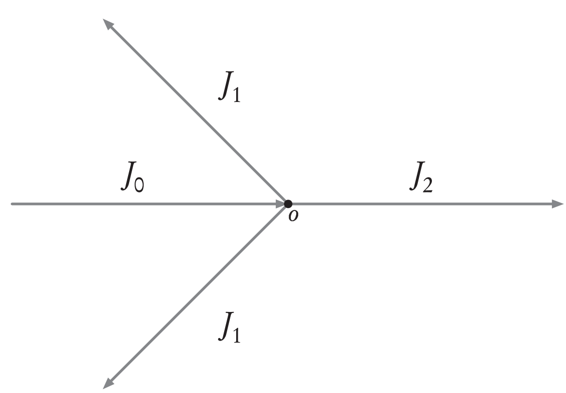
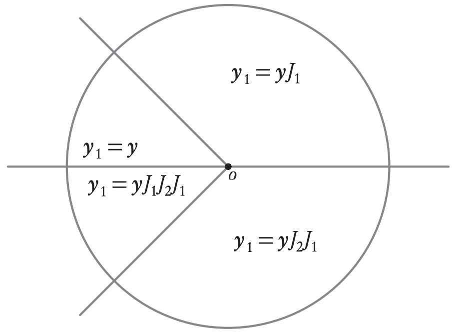


Figure 6.1. *The jumps satisfied by $\boldsymbol{y}$.*



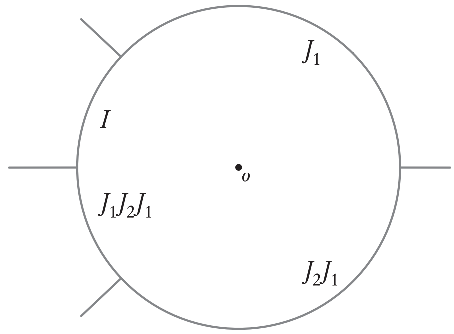


Figure 6.2. *The definition of* $\boldsymbol{y}_1$ *in* B *(left). The jumps satisfied by* $\boldsymbol{y}_1$ *(right). Circular contours have counterclockwise orientation.*

under the condition that $([1,\ 1] + \mathcal{O}(z^{-1}))H(z) = [1,\ 1] + \mathcal{O}(z^{-1})$. We multiply the function $\boldsymbol{\phi}_1$ by H^{-1} for $|z| > 1$ to effectively remove the jump of J_0 on $\{z < 0\}$. When we do this, we want to preserve the $[1,\ 1]$ condition at infinity. A suitable choice is

$$H(z) = \frac{1}{2}\begin{bmatrix} 1 & 1 - \dfrac{1}{\sqrt{z}} \\ 1 & 1 + \dfrac{1}{\sqrt{z}} \end{bmatrix}, \tag{6.6}$$

and a straightforward calculation verifies the required properties when $\sqrt{z}$ is given its canonical branch cut.

Finally, define

$$\boldsymbol{\phi}_2(z) = \boldsymbol{y}_1(z)\begin{cases} R(z) & \text{if } |z| > 1, \\ I & \text{if } |z| < 1, \end{cases} \quad R(z) = 2\sqrt{\pi}z^{1/4}\begin{bmatrix} e^{\frac{2}{3}z^{3/2}} & 0 \\ 0 & e^{-\frac{2}{3}z^{3/2}} \end{bmatrix} H(z).$$

So then $\boldsymbol{\phi}_2(z) = [1,1] + \mathcal{O}(z^{-1})$ and it satisfies the jumps in Figure 6.3. In practice, infinite contours can be truncated following Proposition 2.78 when, say, $|z| \geq 15$ so that $e^{-2/3|z|^{3/2}} \approx 10^{-17}$.

Numerical results: To examine accuracy for large z an asymptotic expansion $\tilde{\boldsymbol{\phi}}_m(z)$ of $\boldsymbol{\phi}(z)$ to order m is readily computed [91]. We compare our numerical approximation $\boldsymbol{\phi}^{6n}(z)$ of $\boldsymbol{\phi}$, with n collocation points per contour, to $\tilde{\boldsymbol{\phi}}_m(z)$ in Figure 6.4. Accuracy is maintained for large values z.

6.4 ▪ Case study: Monodromy of an ODE with three singular points

In this section we demonstrate the statements made in Section 1.4 numerically. This involves first solving the ODE

$$Y'(z) = \sum_{k=1}^{3} \frac{A_k}{z-k} Y(z), \tag{6.7}$$

to compute the monodromy matrices M_1 and M_2. With the monodromy matrices in hand, we must solve Problem 1.4.1.

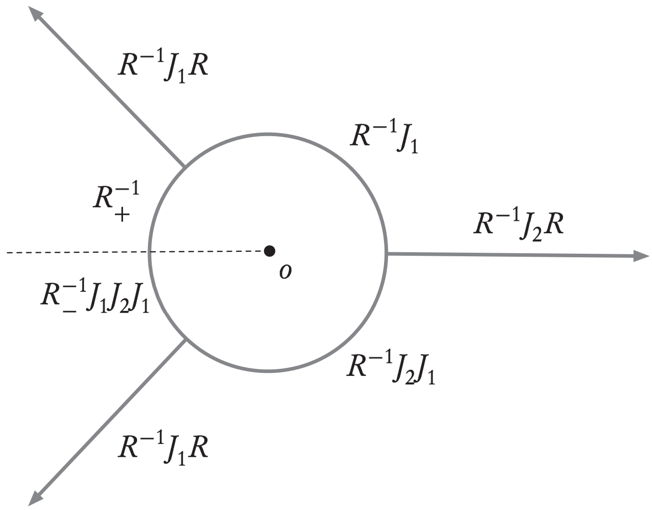


Figure 6.3. *The jump contours and jump conditions for ϕ_3. This RH problem can be solved numerically. Circular contours have counterclockwise orientation.*

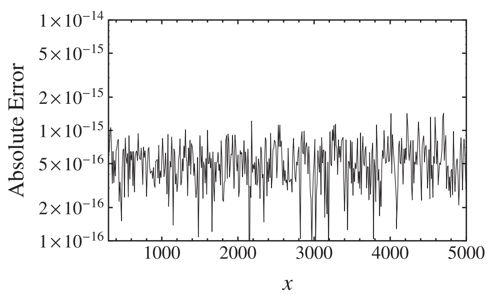


Figure 6.4. *A plot of $\|\tilde{\boldsymbol{\phi}}_6^+(x)-\hat{\boldsymbol{\phi}}_-^{360}(x)\|_2$ versus x. High accuracy is maintained as a function of x.*

Computing the monodromy matrices: We integrate (6.7) along paths in the complex plane. This is accomplished using complex "time-stepping." For example, the first-order Euler scheme becomes

$$Y(z+\mathrm{e}^{\mathrm{i}\theta}\Delta|z|)\approx Y(z)+\Delta|z|\mathrm{e}^{\mathrm{i}\theta}\sum_{k=1}^{3}\frac{A_k}{z-k}Y(z),\quad A_3=-A_1-A_2,$$

where $\Delta|z|$ is the time step. In this example, we use Runge–Kutta 4 instead of the Euler scheme; see [76]. We integrate this along the six contours in Figure 6.5. The contours that are close to one another can be taken to overlap and thus reduce the overall computation. Let $\tilde{Y}$ be the approximation of Y found using this method. We define approximations of the monodromy matrices

$$\tilde{M}_1=(\tilde{Y}^-(1.5))^{-1}\tilde{Y}^+(1.5),$$
$$\tilde{M}_2=(\tilde{Y}^-(2.5))^{-1}\tilde{Y}^+(2.5),$$
$$\tilde{M}_3=(\tilde{Y}^-(3.5))^{-1}\tilde{Y}^+(3.5).$$

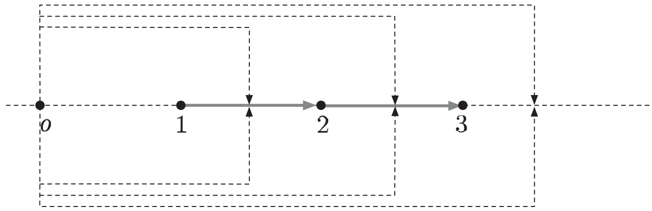


Figure 6.5. *Integration paths for the ODE* (6.7) *to compute* M_1, M_2, *and* M_3.

Because $A_1 + A_2 + A_3 = 0$, ∞ is a regular point and $\tilde{M}_3 \approx I$. It is clear that these can be obtained to any accuracy desired by reducing $\Delta|z|$.

Solving the RH problem: We proceed as if $\tilde{M}_k = M_k$ and leave the discussion of propagation of errors to a remark. Because $M_1 \neq I$ and generically $M_2 \neq M_1$ the RH problem in Problem 1.4.1 is not k-regular for any k. For the numerical method discussed here to have success we must convert it to a problem with smooth solutions.

We use local parametrices near $z = 1,2,3$ to remove singularities using the methodology of Problem 2.4.2. Let P_1 be a solution of $P_1^+(s) = P_1^-(s)M_1$ for $s \in (1,2)$, P_2 be a solution of $P_2^+(s) = P_2^-(s)M_2$ for $s \in (2,3)$, and P_{mid} be a solution of $P_{\text{mid}}^+(s) = P_{\text{mid}}^-(s)M_2^{-1}M_1$ for $s \in (1,2)$.

For our choice of A_k, we find positive eigenvalues for the monodromy matrices so that there is only one locally integrable solution of these problems and our parametrices are uniquely defined (up to multiplication on the left by a constant matrix).

The deformation process is described in Figure 6.6. As in Problem 1.4.1, Y is the solution of the undeformed problem. First, Y_1 is defined by a local redefinition of Y near $z = 2$ inside a circle of radius $0 < r < 1/2$; see Figure 6.6(a). This modifies the jumps to those given in Figure 6.6(b). This is essentially a lensing process, as described in Section 2.8.2. Using the local parametrices as shown in Figure 6.6(c) we find the jumps in Figure 6.6(d).

One can now easily check that all jump matrices are C^∞ smooth and satisfy the product condition to all orders. Furthermore, for our choice below, $\operatorname{tr} A_k = 0$ and therefore $\det Y(z) = 1$, and the determinant of all the jumps is constant by Liouville's formula. From here it follows that the determinants of all the jumps in Figure 6.6(d) are constant, and hence the index of the problem is zero; see Theorem 2.69. Therefore, the problem might be uniquely solvable. Indeed, because we have an invertible solution $Y(z)$, Lemma 2.67 shows that the associated singular integral operator is invertible.

Therefore, since the RH problem is uniquely solvable, the solution of the corresponding SIE $\mathscr{C}[G;\Gamma]u = G - I$ must have infinitely smooth solutions. The methodology of this chapter is readily applied to solve this RH problem with the normalization $Y_2(\infty) = I$. The circular contours in Figure 6.6 can also be replaced with piecewise linear ones.

To approximate $Y(z)$ and compute A_1, A_2, and A_3 we first calculate an approximation $Y_2^n(z) \approx Y_2(z)$ using collocation. We then renormalize at the origin via $Y(z) \approx Y_2^n(0)^{-1} Y_2^n(z) \triangleq Y_n(z)$ for z outside the circles in Figure 6.6(d). The first derivative $Y_n'(z)$ is also computed; see Section 4.3.2. Finally, the integrals

$$A_k \approx A_k^n \triangleq \int_{\partial B(k,2r)} Y_n'(z) Y_n(z)^{-1} đz$$

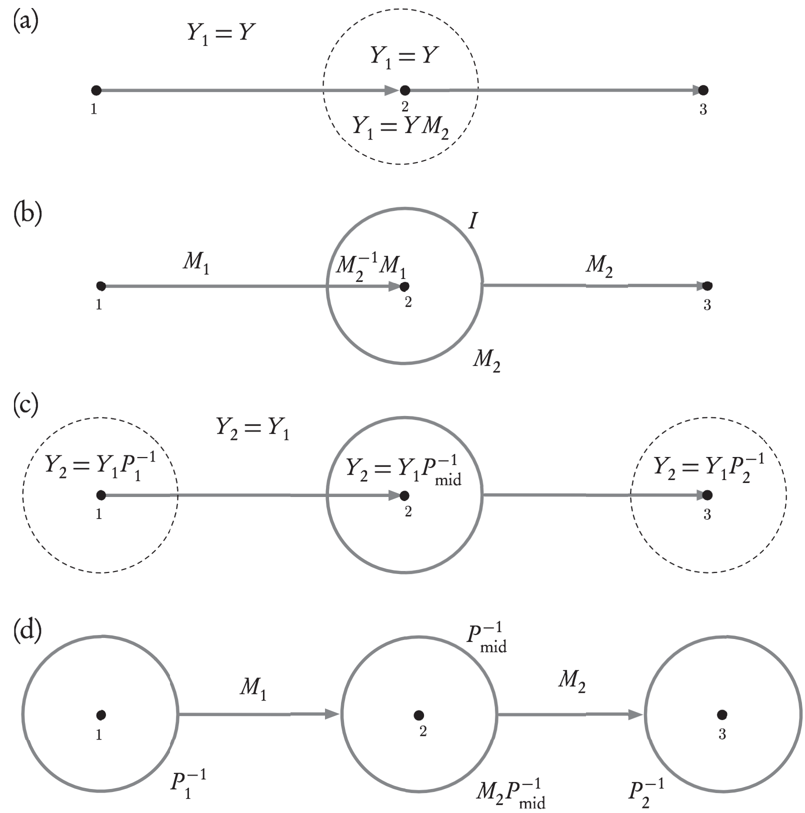


Figure 6.6. *The deformation of the monodromy RH problem for numerical purposes. All circular contours have radius $0 < r < 1/2$ with counterclockwise orientation.* (a) *A local lensing near $z = 2$ and the definition of Y_1.* (b) *The resulting jumps for Y_1.* (c) *The use of all three local parametrices to define Y_2.* (d) *The resulting jumps for Y_2. This RH problem has smooth solutions and is tractable numerically.*

for $k = 1,2,3$ are readily computed with the trapezoidal rule. Note that the integration contour has radius $2r$ so that the deformation regions are avoided.

Specifically, we consider the case

$$A_1 = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.2 \\ -0.6 & 0 \end{bmatrix}.$$

We find

$$\tilde{M}_1 = \begin{bmatrix} 1.20397 - 0.0707178i & -0.650764i \\ 0.698487i & 1.20397 + 0.0707178i \end{bmatrix},$$
$$\tilde{M}_2 = \begin{bmatrix} 8.92914 - 0.663479i & -6.16164i \\ 12.8488i & 8.92914 + 0.663479i \end{bmatrix},$$
$$\tilde{M}_3 \approx I,$$

with the step size $\Delta|z| = 0.0005$. Then, using 40 collocation points per contour to solve the RH problem for Y_2 (and 40 points for the trapezoidal rule) we find that $||A_k - A_k^n|| \approx 3 \times 10^{-14}$.

Remark 6.4.1. *Assume M_1 and M_2 have distinct eigenvalues and pick a normalization for the eigenvectors. It follows from the construction in Problem* 2.4.2 *that if $\|\tilde{M}_1 - M_1\| = \mathcal{O}(\epsilon) = \|\tilde{M}_2 - M_2\|$, then any derivative of the jumps in Figure* 6.6(d) *will differ from those with M_i replaced by $\tilde{M}_i$ by $\mathcal{O}(\epsilon)$ because the eigenvalues and eigenvectors must also converge. Lemma* 2.76 *demonstrates that the difference between any derivatives of the solutions of the two RH problems (one with M_i and the other with $\tilde{M}_i$) is $\mathcal{O}(\epsilon)$, measured appropriately.*