

# Conformal Mesh Mappings

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## Abstract

When performing shape optimization one encounters the problem of conformal mapping of a relatively simple onto a potentially very complicated domain. The Riemann mapping theorem guarantees the existence of such a conformal map, but does not provide any explicit construction of it. In practice, one has to resort to numerical methods to construct the conformal map. This text is intended to give a general overview of currently known numerical algorithms for this very problem, and to provide a comparison of their suitability in terms of input and output format as well as their performance in terms of accuracy and speed. One of the algorithms is then implemented in the context of a 2-dimensional shape optimization problem, and its performance is evaluated.

Keywords: numerical conformal mapping, Riemann mapping theorem, Schwarz-Christoffel mapping, conjugate function method, Fourier series representation, curvilinear triangular mesh, mesh generation, finite element method, Laplace-Beltrami equations, spectral conformal parameterization, Ricci flow, charge simulation method, RT-algorithm, CRDT (cross-ratios Delaunay triangulation)

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# **1 Introduction**

## **1.1 Problem Setting**

## 2 Theoretical Background

### 2.1 Conformal Mappings

### 2.2 Riemann Mapping Theorem

### 2.3 Riemann-Hilbert Problem

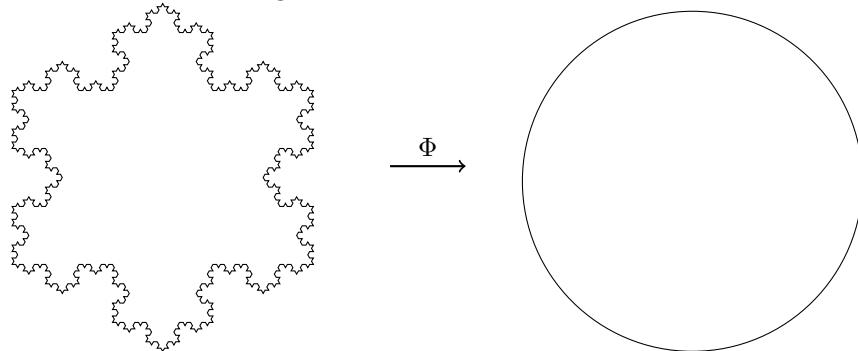
### 2.4 Hilbert Transform

The **Riemann mapping theorem** states that if  $U$  is a non-empty simply connected open subset of the complex number plane  $\mathbb{C}$  which is not all of  $\mathbb{C}$ , then there exists a biholomorphic mapping  $f$  (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from  $U$  onto the open unit disk

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

This mapping is known as a Riemann mapping.

The beauty of the Riemann mapping theorem lies in its weight of implications, i.e. the fact that it guarantees the existence of a conformal map between any two simply connected domains in the complex plane, provided they are not the entire plane. The existence of this Riemann map is a priori not obvious: Even relatively simple Riemann mappings (for example a map from the interior of a circle to the interior of a square) have no explicit formula using only elementary functions. Simply connected open sets in the plane can be highly complicated, for instance, the boundary can be a nowhere-differentiable fractal curve of infinite length, even if the set itself is bounded. One such example is the Koch curve.<sup>[8]</sup> The fact that such a set can be mapped in an angle-preserving manner to the nice and regular unit disc seems counter-intuitive.



**Definition 1.** A *conformal mapping*, also called a *conformal map*, *conformal transformation*, *angle-preserving transformation*, or *biholomorphic map*, is a transformation  $w=f(z)$  that preserves local angles. An analytic function is *conformal* at any point where it has a nonzero derivative.

Wegmann [Weg05] proved the following result.

**Theorem 1.** When the region  $G$  can be enclosed in a rectangle with sides  $a$  and  $b$ ,  $b \leq a$ , such that  $G$  touches both small sides (see Figure 4) then the conformal mapping  $\phi : D \rightarrow G$  satisfies

$$\|\phi'\|_D \geq b\psi(b/a)$$

with a function  $\psi(\tau)$  which behaves for small  $\tau$  like

$$\psi(\tau) \approx \frac{1}{2\pi\sqrt{\epsilon}} \exp\left(\frac{\pi}{2\tau}\right).$$

Crowding is cumbersome for all methods which work with grid points. On the other hand, methods which approximate the mapping functions by polynomials also face severe problems when the target region is elongated. It follows that, for the mapping of the disk to a region of aspect ratio 1:910 by a polynomial, the degree must be of several millions. In any case, the number of grid points and the degree of the approximating polynomials increase both like  $\exp(zr/2r)$  as the aspect ratio,  $r$ , tends to zero. DeLillo [37] has shown how crowding affects the accuracy of numerical computations. Crowding also limits the practical usefulness of conformal maps. This was demonstrated by DeLillo [36] for the Laplace equation. Crowding has also been observed for regions with elongated sections ("fingers"). For "pinched" regions, such as the interior of an inverted ellipse, ill conditioning occurs of a less severe, algebraic nature (DeLillo [37]).

#### 2.4.1 The operator $R$

In some conformal mapping methods, boundary value problems as follows occur,

$$\Psi(e^{it}) = B(t) + A(t)U(t)$$

where  $A, B : \mathbb{C} \rightarrow \mathbb{C}$  and  $U : \mathbb{C} \rightarrow \mathbb{R}$ . Multiplication with  $\bar{A}$  yields the RH problem

$$Im(A\bar{t})\Psi(e^{it}) = Im(A\bar{t})B(t)$$

This problem can be solved by the operator  $R_\beta$ , which is defined as follows:

This follows from the following theorem:

**Theorem 2.** There exists a function  $\Psi$  analytic in  $D$  with  $\Psi(0) = 0$  satisfying the boundary problem if and only if  $U$  is a solution of the Fredholm integral equation of the second kind

$$(I + R_\beta)U = g$$

with the right-hand side

$$g := -Re(e^{-i\beta}(I - iK + J)/B).$$

Where  $K$  is the conjugation operator and  $J$  the averaging operator.

## 2.5 Mapping from the region to the disk

### 2.5.1 Extremum Principles

Both the [Bergman and the Szegö norms](#) are very useful for characterizing the conformal mapping from  $G$  to a disk by extremum principles:

**Theorem 3.** *Let  $F$  be the conformal mapping from  $G$  to a disk normalized by  $f(0) = 0, f'(0) = 1$ . Then*

1. *(Principle of minimum area)  $F'$  is the unique function which minimizes  $\|f\|_B$  among all functions  $f \in B(G)$  satisfying  $f(O) = 1$ .*
2. *(Principle of minimum length)  $\sqrt{F'}$  is the unique function which minimizes  $\|f\|_S$  among all functions  $f \in S(G)$  satisfying  $f(O) = 1$ .*

### 2.5.2 Osculation Methods

The osculation method (Schmiegungsverfahren) of Koebe [ 140] approximates  $F$  by a composition of elementary maps. It is universally applicable, since it requires no hypotheses at all concerning the boundary  $OG$  of the region  $G$ .

## 2.6 Mapping from the disk to the region

### 3 Existing Methods

- Numerical stability of different conformal mapping methods
- Boundary discretization requirements (how smooth does  $\gamma$  need to be?)
- Mesh quality preservation - how does the mapping affect triangle quality?
- Computational complexity
- Input and output formats

## 4 Proposed Method

### 4.1 Choice/ Justification

criteria: - Accuracy for domains with sharp corners or high curvature - Speed for practical mesh sizes - Robustness - does it fail for certain domain shapes? - Implementation complexity given your timeline - Jacobian computation - analytical vs numerical differentiation

### 4.2 Implementation

- Separate modules for boundary parameterization, mapping computation, Jacobian eval, and mesh transformation - plot original vs. mapped grids (e.g., Matplotlib quiver for Jacobians) to spot issues early.

### 4.3 Numerical Experiments/ Testing

check angle preservation (e.g., via dot products on mapped vectors) and scale factors ( $\det(D\Phi) > 0$ ,  $|\frac{\partial \Phi}{\partial z}|$  constant in theory). - Test suite: Use known exact mappings (e.g., disk to square via Schwarz-Christoffel) for error metrics (L2 norm on boundary points). - Metrics: Runtime for N points, mesh quality post-mapping (e.g., min/max angles in triangles, shape regularity ratio). - Real-world applicability: Apply to a sample FEM problem (e.g., Poisson equation on  $\Omega$ ) and compare accuracy/speed vs. uniform mesh. - Robustness: Vary boundary complexity (smooth vs. corners), noise in Fourier coeffs, mesh resolutions. - Debugging: Use assertions for bijectivity (e.g., check injectivity numerically) - Error handling - what happens with degenerate inputs?

### 4.4 Results

## 5 Conclusion

## References

- [Weg05] Rudolf Wegmann. Chapter 9 - methods for numerical conformal mapping. In R. Kühnau, editor, *Geometric Function Theory*, volume 2 of *Handbook of Complex Analysis*, pages 351–477. North-Holland, 2005.