

# Conformal Mesh Mappings

Bachelor Thesis D-MATH ETHZ  
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## Abstract

Given a 2D triangular mesh  $\widehat{\mathcal{M}}$  of the unit disk  $\mathbb{D}$ , whose triangles are all nicely shape-regular (in the sense that the ratio of their diameter to the radius of the largest inscribed circle is uniformly bounded for all triangles), we are guaranteed the existence of a conformal mapping  $\psi$  from  $\mathbb{D}$  to any simply connected bounded domain  $\Omega$  by the Riemann mapping theorem. This allows for a sufficiently "nice" mesh  $\mathcal{M}$  on  $\Omega$  to be obtained as the image of  $\widehat{\mathcal{M}}$  under  $\psi$ , i.e.  $\mathcal{M} = \psi(\widehat{\mathcal{M}})$ . The challenge lies in the numerical construction/ approximation of this conformal mapping  $\psi$ . This text is intended to give a general overview of currently known numerical conformal mapping algorithms, and to provide a comparison in terms of accuracy, runtime, and the representation format of the resulting map, particularly emphasizing the efficiency of point evaluations for both the mapping  $\psi$  itself and its derivative (Jacobian)  $D\psi$ . Finally, we implement **Algorithm Name** in Python.

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# 1 Theoretical Background

## Conventions

Throughout this text, we identify  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$  and write

$$\mathbb{C} \ni z = x + iy \text{ where } x, y \in \mathbb{R}.$$

The task is to find an angle-preserving mapping  $\psi$  from the unit disk  $\mathbb{D}$  to  $\Omega$ , given a simply connected bounded domain  $\Omega \subset \mathbb{C}$  with boundary  $\Gamma$  and  $2\pi$ -periodic parametrization

$$\eta : [0, 2\pi] \rightarrow \Gamma \quad \text{with} \quad \eta \in C^0([0, 2\pi])$$

by its fourier series representation. **STIMMT DAS SO, DASS  $\eta$  HIER AUCH NUR EINDIMENSIONAL IST (ANDERS ALS IM PROJEKTBESCHRIEB), WENN WIR  $\mathbb{C}$  BENUTZEN STATT  $\mathbb{R}^2$ ?** We are also given a curvilinear triangular mesh  $\widehat{\mathcal{M}}$  on  $\mathbb{D}$  satisfying some regularity conditions (see chapter 1.8).

### Definition 1

We call a subset  $\Omega \subset \mathbb{C}$  **proper** if  $\emptyset \neq \Omega \neq \mathbb{C}$ .

## 1.1 Conformal Mappings

### Definition 2

A **conformal mapping**, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation  $f(z)$  that preserves local angles. An analytic function is conformal at any point where it has nonzero derivative.

### Definition 3

A **complex analytic function** or **holomorphic function** on an open subset  $U \subset \mathbb{C}$  is locally given by a convergent power series, i.e. for any  $z_0 \in U$  there exists a neighborhood  $V \subset U$  of  $z_0$  such that for all  $z \in V$  we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some complex coefficients  $a_n \in \mathbb{C}$ . The terms analytic and holomorphic are used interchangeably.

## 1.2 Riemann Mapping Theorem

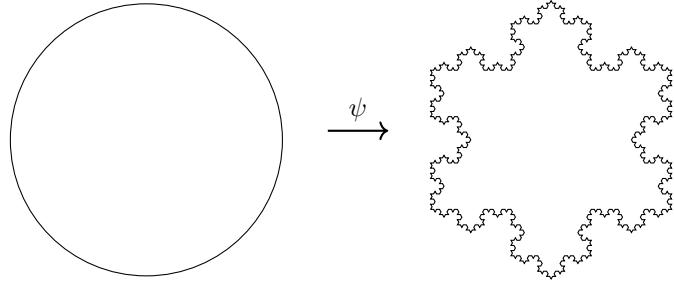
### Theorem 1.1 (Riemann Mapping Theorem)

If  $\Omega$  is a non-empty simply connected open proper subset of the complex plane  $\mathbb{C}$ , then there exists a biholomorphic mapping  $f$  (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from  $\Omega$  onto the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

This mapping is known as a Riemann mapping.

The beauty of the Riemann mapping theorem lies in its weight of implications, i.e. the fact that it guarantees the existence of a conformal map between any two simply connected domains in the complex plane, provided they are not the entire plane. The existence of this Riemann map is a priori not obvious: Even relatively simple Riemann mappings (for example a map from the interior of a circle to the interior of a square) have no explicit formula using only elementary functions [Weg05]. Simply connected open sets in the plane can be highly complicated, for instance, the boundary can be a nowhere-differentiable fractal curve of infinite length, even if the set itself is bounded. One such example is the Koch curve. The fact that such a set can be mapped in an angle-preserving manner from the nice and regular unit disc seems counter-intuitive.



We closely follow the proof by normal families in [SS03].

### 1.2.1 Preliminary Results

**Lemma 1.2** (Schwarz Lemma)

Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic with  $f(0) = 0$ . Then

1.  $|f(z)| \leq |z|$  for all  $z \in \mathbb{D}$ .
2. If for some  $z_0 \neq 0$  we have  $f(z_0) = z_0$  then  $f$  is a rotation.
3.  $|f'(0)| \leq 1$  and if equality holds, then  $f$  is a rotation.

*Proof.* See [SS03] page 218.

#### Definition 4

A family  $\mathcal{F}$  of holomorphic functions on a domain  $\Omega$  is called **normal** if every sequence in  $\mathcal{F}$  contains a subsequence that converges uniformly on any compact subset of  $\Omega$ .

#### Definition 5

A family  $\mathcal{F}$  is called **uniformly bounded on compact subsets** of  $\Omega$  if for every compact subset  $K \subset \Omega$  there exists a constant  $M_K$  such that  $|f(z)| \leq M_K$  for all  $z \in K$  and all  $f \in \mathcal{F}$ .

### Definition 6

A family  $\mathcal{F}$  of holomorphic functions on a domain  $\Omega$  is called **equicontinuous** if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $z \in \Omega$  with  $|z - z_0| < \delta$  we have  $|f(z) - f(z_0)| < \varepsilon$  for all  $f \in \mathcal{F}$ .

### Theorem 1.3 (Montel)

A family  $\mathcal{F}$  of holomorphic functions on  $\Omega$  that is uniformly bounded on compact subsets of  $\Omega$  is normal if and only if it is equicontinuous on compacta.

*Proof.* See [SS03] page 225.

### Theorem 1.4 (Hurwicz)

Let  $\Omega \subset \mathbb{C}$  be a connected open subset and let  $f_n : \Omega \rightarrow \mathbb{C}$  be a sequence of injective holomorphic functions that converges uniformly on compact subsets of  $\Omega$  to a holomorphic function  $f \neq 0$ . If  $f$  has a zero of order  $m$  at  $z_0$  then for every  $\varepsilon > 0$  and sufficiently large  $k = k(\varepsilon) \in \mathbb{N}$ ,  $f_k$  has precisely  $m$  zeros in  $B_\varepsilon(z_0)$  including multiplicities. Moreover, the zeros converge to  $z_0$  as  $k \rightarrow \infty$ .

We will use the following corollary of Hurwicz' theorem in the proof of 1.1:

**Corollary 1.5** (Uniform Convergence to Holomorphic Limit and Injectivity)  
 Let  $\Omega \subset \mathbb{C}$  be a connected open subset and let  $f_n : \Omega \rightarrow \mathbb{C}$  be a sequence of injective holomorphic functions that converges uniformly on compact subsets of  $\Omega$  to a holomorphic function  $f$ . Then  $f$  is either constant or injective.

### Proposition 1.6 (Cauchy Inequality)

Let  $f$  be holomorphic on an open set containing the closure of a ball  $B_R(z_0)$  centered at  $z_0$  of radius  $R$ . Then

$$|f^{(n)}(z_0)| \leq \frac{n! \|f\|_R}{R^n},$$

where  $\|f\|_R = \sup_{\|z\|=R} |f(z)|$  on the boundary circle  $\partial B_R(z_0)$ .

*Proof.* See [SS03] page 48.

### Theorem 1.7 (Maximum Modulus Principle)

Let  $\Omega \subset \mathbb{C}$  be a connected open bounded set and  $f : \Omega \rightarrow \mathbb{C}$  holomorphic. If  $z_0$  is a point in  $\Omega$  such that  $|f(z_0)| \geq |f(z)|$  for all  $z$  in a neighborhood of  $z_0$ , then  $f$  is constant on  $\Omega$ .

*Proof.* See [SS03] page 92.

### Theorem 1.8 (Implicit Mapping Theorem)

Let  $r > 0$  be a radius, and let  $x_0 \in \mathbb{R}^n$ ,  $y_0 \in \mathbb{R}^m$ . Consider the open set  $W = B_r(x_0) \times B_r(y_0) \subset \mathbb{R}^n \times \mathbb{R}^m$  defined as

$$W = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \|x - x_0\|_2 < r \text{ and } \|y - y_0\|_2 < r\}.$$

Let  $F : W \rightarrow \mathbb{R}^m$  be a continuous function satisfying the following conditions:

1.  $F(x_0, y_0) = 0$ .
2. The partial derivatives  $\partial_{y_k} F : W \rightarrow \mathbb{R}^m$  exist for all  $k \in \{1, \dots, m\}$  and are continuous on  $W$ .
3. The partial differential  $D_y F(x_0, y_0)$  (the differential of the map  $y \mapsto F(x_0, y)$  at  $y_0$ ) is invertible.

Then there exist radii  $\alpha, \beta \in (0, r)$  such that for the open balls  $U_0 = B_\alpha(x_0) \subset \mathbb{R}^n$  and  $V_0 = B_\beta(y_0) \subset \mathbb{R}^m$ , there exists a unique continuous function  $f : U_0 \rightarrow V_0$  satisfying:

- $f(x_0) = y_0$ .
- $\forall (x, y) \in U_0 \times V_0 : F(x, y) = 0 \iff y = f(x)$ .

*Proof.* See [EW22] page 573.

### 1.2.2 Proof of Riemann Mapping Theorem

*Step 1: Existence and injectivity*

Let  $\Omega$  be a simply connected open proper subset of  $\mathbb{C}$ . We show that  $\Omega$  is conformally equivalent to an open subset of the unit disk containing the origin. Indeed, choose  $\alpha \notin \Omega$  and consider the function

$$f(z) := \log(z - \alpha)$$

on  $\Omega$ , which is well-defined and holomorphic since  $z - \alpha$  never vanishes on  $\Omega$ . Note  $f$  is injective since  $e^{f(z)} = z - \alpha$  is ( $f(z_1) = f(z_2) \implies z_1 - \alpha = z_2 - \alpha$ ). Then for a point  $\omega \in \Omega$  we get  $f(z) \neq f(\omega) + 2\pi i \quad \forall z \in \Omega$  since otherwise we would find  $z = \omega$  again by exponentiating. In fact,  $f(z) \cap B_\varepsilon(f(\omega) + 2\pi i) = \emptyset$  for some  $\varepsilon > 0$  since otherwise we would find a sequence  $z_n \rightarrow \omega$  with  $f(z_n) \rightarrow f(\omega) + 2\pi i$ , contradicting the continuity of  $f$ . Finally, the function

$$g(z) := \frac{1}{f(z) - (f(\omega) + 2\pi i)}$$

is well-defined, holomorphic and injective on  $\Omega$  and maps  $\Omega$  to a bounded subset  $g(\Omega) \subset \mathbb{C}$ , so  $g$  is conformal. By boundedness of  $g(z) < \frac{1}{\varepsilon}$  we can scale and translate  $g(\Omega)$  to contain the origin and fit into the unit disk.

*Step 2: Existence of the extremal function, aka the maximum is attained*

By step 1 we can assume  $\Omega$  to be an open subset of the unit disk with  $0 \in \Omega$ . Consider the family

$$\mathcal{F} := \{f : \Omega \hookrightarrow \mathbb{D} \text{ holomorphic}, f(0) = 0\}$$

of all injective holomorphic functions which map the origin to itself. Note that  $\mathcal{F} \neq \emptyset$  since it contains the identity, and it is a uniformly bounded family by construction (maps into, hence bounded by the unit disk). In order to use the

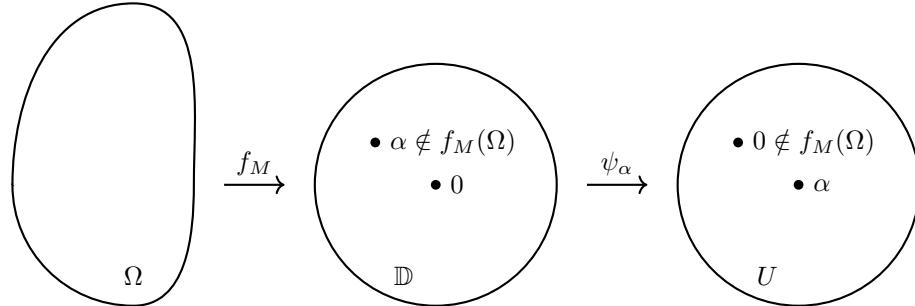
Hurwicz corollary and get injectivity, we now want to find a function  $f_M \in \mathcal{F}$  that maximizes  $|f'(0)|$  (to rule out constant functions yielded otherwise by the application of said corollary). Observe that by the Cauchy inequality 1.6, the derivatives  $|f'(0)|$  are uniformly bounded for all  $f$  in  $\mathcal{F}$ . Thus, we can define

$$s := \sup_{f \in \mathcal{F}} |f'(0)|.$$

and choose a sequence  $f_n \subset \mathcal{F}$  such that  $|f'_n(0)| \rightarrow s$  as  $n \rightarrow \infty$ . By Montel's theorem,  $f_n$  has a subsequence converging uniformly on compacta to a holomorphic  $f_M$  on  $\Omega$ . Since  $s \geq 1$  (the identity is in  $\mathcal{F}$ ),  $f_M$  is non-constant. Hence by the Hurwicz corollary 1.5,  $f_M$  must be injective. By continuity we have  $|f_M(z)| \leq 1$  for all  $z \in \Omega$ , and since  $f_M$  is non-constant, by the Maximum Modulus Principle 1.7 we get  $|f_M(z)| < 1$ . Finally and since  $f_M(0) = 0$ , we have  $f_M \in \mathcal{F}$  and  $|f'_M(0)| = s$ .

### Step 3: Surjectivity

By injectivity it suffices to show  $f_M$  is surjective. Suppose towards a contradiction that  $f_M$  is not surjective, we will construct a function in  $\mathcal{F}$  with derivative greater than  $s$  at the origin.



So let  $\alpha \in \mathbb{D}$  be such that  $\alpha \notin f_M(\Omega)$  and consider the automorphism of the unit disk that interchanges 0 and  $\alpha$ ,

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

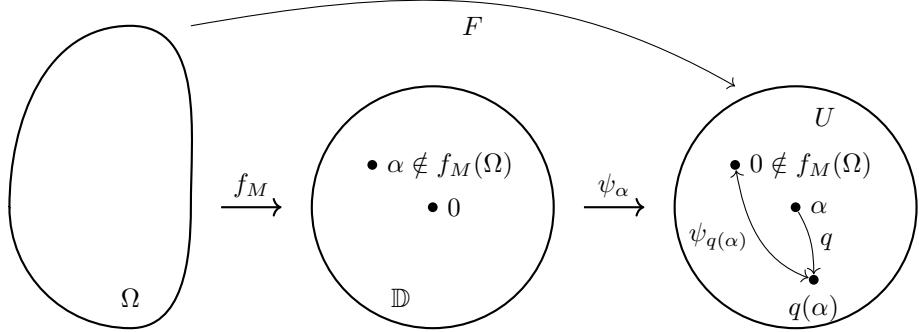
Since  $\Omega$  is simply connected and  $f_M$  and  $\psi_\alpha(\Omega)$  are continuous, the set

$$U := (\psi_\alpha \circ f_M)(\Omega)$$

is simply connected and does not contain the origin. Thus we can define a square root function on  $U$  by

$$q(w) := e^{\frac{1}{2} \log w}.$$

Next, consider the function  $F := \psi_{q(\alpha)} \circ q \circ \psi_\alpha \circ f_M$ .



Then  $F \in \mathcal{F}$  since  $F(0) = 0$  and  $F$  is holomorphic and injective since all the composing functions are. Also,  $F$  maps into the unit disk since all the composing functions do. But now if  $h$  denotes the square function  $h(w) = w^2$ , then we must have

$$f_M = \psi_{\alpha}^{-1} \circ h \circ \psi_{q(\alpha)}^{-1} \circ F = \phi \circ F.$$

But  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  satisfies  $\phi(0) = 0$  and is not injective since  $F$  is but  $h$  is not. By the Schwarz lemma 1.2 we get  $|\phi'(0)| < 1$  and hence

$$|f'_M(0)| = |\phi'(0)||F'(0)| < |F'(0)|,$$

contradicting maximality of  $|f'_M(0)|$  in  $\mathcal{F}$ .

Finally, multiplying  $f_M$  with a suitable unimodular complex number gives the desired conformal map  $\psi : \Omega \rightarrow \mathbb{D}$  with

$$\psi(0) = 0, \quad \psi'(0) > 0. \tag{1}$$

□

### Corollary 1.9

Any two simple connected open proper subsets of  $\mathbb{C}$  are conformally equivalent.

*Proof.* This follows directly from the Riemann mapping theorem by taking the unit disk as an intermediate step. □

### Theorem 1.10 (Carathéodory)

The conformal mapping  $\psi : \mathbb{D} \rightarrow \Omega$  can be extended to a continuous mapping  $\psi : \bar{\mathbb{D}} \rightarrow \bar{\Omega}$  if the boundary  $\Gamma = \partial\Omega$  consists of a closed curve. [Weg05], p. 357

*Proof.* See [Pom92, page 24].

Existence established, the problem now becomes the explicit construction of a conformal mapping. Once the boundaries of the domain and target region are conformally mapped, the interior can be deduced from the boundary by the Cauchy Integral Formula. Hence why the main focus of the task lies on mapping boundary to boundary.

### 1.3 Reformulation as Boundary Value Problem

When the region  $\Omega$  is bounded by a closed Jordan curve  $\Gamma$  the mapping

$$\psi : \mathbb{D} \rightarrow \Omega$$

can be extended continuously to the closure  $\overline{\mathbb{D}}$  by Carathéodory's Extension Theorem 1.10. Assume the boundary to be parametrized by a  $2\pi$ -periodic function  $\eta$  in counterclockwise direction (positive orientation, aka *normal representation* [Weg05, page 387]) and the mapping  $\psi$  to be determined by its boundary values

$$\psi(e^{it}) = \eta(S(t)) \quad (2)$$

for  $t \in [0, 2\pi]$ .  $S$  is called the **boundary correspondence function**. By the Riemann mapping theorem, we need to fix at least three points for uniqueness of the conformal mapping  $\psi$ , which we achieve through the boundary correspondence equation 2.

From Definition 2 we know that a complex differentiable function with non-vanishing complex derivative is conformal, and complex differentiability in two dimensions can be characterised by the Cauchy-Riemann equations (necessary and sufficient condition). Denoting  $z = x + iy$  and

$$J := \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}$$

we want to find a deformation  $\psi(z) = \psi(x + iy) = u + iv$  such that  $J\psi = 0$ .

#### Definition 7

A function  $f : \mathbb{C} \supseteq \Omega \rightarrow \mathbb{C}$  is called **harmonic** in  $\Omega$  if it has continuous second partial derivatives and

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \forall z = x + iy \in \Omega.$$

As a byproduct of the Cauchy-Riemann equations, both components of a conformal map are harmonic functions, i.e. they satisfy Laplace's equation  $\Delta u = 0 = \Delta v$  [Hen86, page 214]. Thus, one way to construct conformal maps is to solve Laplace's equation with suitable boundary conditions.

#### Definition 8

Two harmonic functions  $u(x, y)$  and  $v(x, y)$  on a domain are called **conjugate** if they satisfy the Cauchy-Riemann equations on the domain. E.g.  $v$  is a conjugate for  $u$ , and  $-u$  is a conjugate for  $v$ .

This directly leads to the conclusion that it suffices to find only one of the components of  $\psi = u + iv$ , since the other one can then be deduced as its conjugate harmonic function. This formulation is called the **Dirichlet problem** for the Laplace equation. It can be described as finding  $\psi$  analytic in  $\mathbb{D}$ , continuous in  $\overline{\mathbb{D}}$  and satisfying

$$\begin{aligned} u(e^{it}) &= \operatorname{Re}(\psi(e^{it})) = \eta(S(t)) && \text{on } \partial\mathbb{D} \\ \Delta u &= 0 && \text{in } \mathbb{D}^\circ, \end{aligned} \quad (3)$$

where  $\psi$  is  $2\pi$ -periodic and Hölder continuous [Weg05, page 362]. This problem has a unique solution up to an imaginary constant, which can be constructed using the conjugation operator

$$K\psi(s) := \frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot\left(\frac{s-t}{2}\right) dt, \quad (4)$$

which is also known as Hilbert Transform (details on this in [CR25]).

### Remark 1

In order to implement numerical methods efficiently via the Fast Fourier Transform later, we will prefer the functions' Fourier series representations. Hence on a grid of  $N = 2n$  equidistant why? FFT requires it? has crowding as an effect... points  $t_j = \frac{(j-1)2\pi}{N}$ ,  $\psi$  will be written as

$$\psi(t_j) = \sum_{k=-n+1}^n c_k e^{ikt_j} \text{ for } j \in [N]$$

and the conjugation operator can be approximated by the operator  $K_N$  defined as

$$K_N \psi(t_j) = \sum_{k=1}^{n-1} -ic_k e^{ikt_j} + ic_{-k} e^{-ilt_j}.$$

The function  $K_N$  is thus defined as a trigonometric polynomial obtained by interpolating  $\psi$  at the grid points. This approximation of the actual operator  $K$  satisfies

$$\|K\psi - K_N\psi\|_\infty \in O(n^{-\alpha+1/2})$$

for  $\psi \in C^\alpha$  Hölder continuous. If  $\psi$  is smoother, e.g.  $\psi \in C^{k-1}$ , the error decreases to

$$\|K\psi - K_N\psi\|_\infty \in O(n^{-k} \log(n) \|\psi^{(k)}\|_{inf ty})$$

[Weg05, page 362]. These considerations are of particular importance for all methods based on conjugation.

In principle, we have two options for finding conformal mappings, namely to solve linear or non-linear boundary value problems. The linear problems are naturally faster to solve numerically, however their kernels must be adapted to each new fixed boundary and thus these methods present complications when dealing with predefined boundaries. Non-linear methods on the other hand can be more flexible with respect to the boundary shape, but are computationally more expensive. But since we are mainly interested in solving the mapping problem for a fixed boundary shape, we will focus on linear methods in this article.

## 1.4 Boundary Conditions

### Remark 2

Note that the Cauchy-Riemann equations do not guarantee a solution for arbitrary boundary data. A holomorphic map from the boundary of the unit disk

onto some boundary of a convex set in  $\mathbb{C}$  satisfies Cauchy-Riemann equations if the boundary of the target set can be described by non-negative Fourier frequencies. Thus, the choice of parametrization of the target region's boundary is usually another challenge posed when solving conformal mapping problems, but for the scope of this article we assume to be given a suitable parametrization of  $\partial\Omega$ .

## 1.5 Solution of the Boundary Value Problem via Fredholm Integral Equations of the Second Kind

We often numerically solve BVPs by transforming them into integral equations. Integral equations are equations in which an unknown function appears under an integral sign. They are classified into two main types: Fredholm and Volterra integral equations, where Fredholm integral equations have fixed limits of integration, while Volterra integral equations have variable limits of integration. Both types can be further categorized into first kind and second kind, depending on whether the unknown function appears only under the integral sign or also outside of it.

### Definition 9

A **Fredholm integral equation of the second kind** is an equation of the form

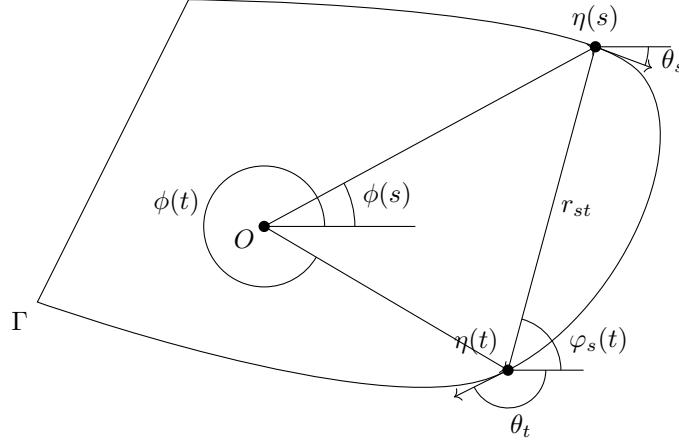
$$f(x) = \lambda \int_a^b K(x, t)\phi(t)dt + \phi(x) \quad (5)$$

where  $f$  is a known function,  $\phi$  is the unknown function to be solved for,  $K(x, t)$  is a given kernel function,  $\lambda \in \mathbb{R}$  is a constant, and  $x, a, b \in \mathbb{R}$ ,  $a \leq b$ .

There are several methods to solve Fredholm integral equations of the second kind, which we will explore in what follows.

### 1.5.1 Neumann Kernel

The Neumann kernel is named after Carl Neumann, who first introduced it as a tool for solving Integral equations corresponding to the Dirichlet problem [Gai64]. Let  $\eta : [0, 2\pi] \rightarrow \Gamma$  be a parametrization of the boundary and  $s, t \in [0, 2\pi]$ . Consider the straight line segment from  $\eta(t)$  to  $\eta(s)$  and denote by  $r_{st}$  its length. Moreover, let  $\theta(t), \theta(s)$  denote the tangent angles (measured with regard to the positive real axis) to  $\eta$  at  $s, t$  respectively, and  $\phi(s), \phi(t)$  the angles between the positive real axis and the line from the origin to  $\eta(s), \eta(t)$  respectively.



The Neumann kernel is defined for  $s \neq t$  as

$$K(s, t) = \frac{1}{\pi} \frac{\sin(\theta(t) - \varphi_s(t))}{r_{st}} \quad (6)$$

The integral in equation (5) can be interpreted as a Riemann or Riemann-Stieltjes integral, where the latter is more general and allows for evaluation of non-bounded integrands. This could be important if the boundary correspondence function  $S$  as seen in (2) is not continuously differentiable [Gai64, page 40]. These two interpretations lead to different methods of discretisation and different error estimates when solving the integral equation numerically.

### 1.5.2 Symm's equation

A potential theoretic formulation of the Dirichlet problem (3) leads to a Fredholm integral equation of the first kind called **Symm's equation**

$$\int_{\Gamma} \log |z - \zeta| \mu(\zeta) d\zeta = -\log |z| \quad \text{for } z \in \Omega, \zeta \in \Gamma \quad (7)$$

which has a kernel with logarithmic singularity. In this context, the conformal mapping problem reduces to finding a suitable source density function  $\mu(z) = g(z) + ih(z)$  on the boundary  $\Gamma$  where  $g$  and  $h$  are conjugate and  $g$  satisfies

$$\begin{aligned} \nabla^2 g(z) &= 0 & z \in \Omega \\ g(z) &= -\frac{1}{2} \log |z\bar{z}| & z \in \Gamma. \end{aligned} \quad (8)$$

It is more easily solvable by numerical methods than the above Fredholm integral equation of the second kind, where the kernel might have singularities near the boundary [Kyt98, chapter 9].

## 1.6 Sobolev Spaces

Let  $L^2$  be the space of all  $2\pi$ -periodic complex functions  $f$  which are square integrable over  $[0, 2\pi]$  equipped with the inner product

$$(f, g)_2 = \frac{1}{2\pi} \operatorname{Re} \int_0^{2\pi} f(t) \overline{g(t)} dt.$$

### Definition 10

The **Sobolev space**  $W$  is defined as the space of all absolutely continuous functions  $f \in L^2$  such that the derivative  $f'$  exists and is also in  $L^2$ . The inner product on  $W$  is defined as

$$(f, g)_W = (f, g)_2 + (f', g')_2.$$

This is a Hilbert space over  $\mathbb{R}$ . The subspaces of real functions are denoted  $L_{\mathbb{R}}^2$  and  $W_{\mathbb{R}}$  respectively. Note that we can decompose  $W$  into the direct sum of the subspaces  $W = W^+ \oplus W^-$  where  $f \in L^2$  is decomposed as follows into its Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} = \underbrace{\sum_{n=-\infty}^0 a_n e^{int}}_{=:f^- \in W^-} + i(\operatorname{Im}(a_1))e^{int} + (\operatorname{Re}(a_1))e^{int} + \underbrace{\sum_{n=2}^{\infty} a_n e^{int}}_{=:f^+ \in W^+}.$$

In this framework, the conformal mapping  $\psi$  we are looking for can be expressed as follows:

$$\psi(t) = \eta(t + \hat{u}(t)) \in W^+ \quad \forall t.$$

By (2)  $\hat{u}$  exists and by the implicit function theorem 1.8  $\hat{u}$  is continuously differentiable, hence  $\hat{u} \in W_{\mathbb{R}}$ . Thus,  $\psi$  lies in the intersection of a certain manifold  $M := \{u \in W_{\mathbb{R}} : \eta(t + u(t))\}$  with our space  $W^+$ . This formulation is used in chapter 2.1.1. **should I move this to the beginning of that chapter instead?**

## 1.7 Complex Analysis/ Function Theoretic Tools

### Definition 11

A **Möbius transform** is a function on the extended complex plane  $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  which is uniquely determined by where it sends three points. It has the form

$$f(z) = \frac{az + b}{cz + d}$$

with complex numbers  $a, b, c, d$  such that  $ad - bc \neq 0$ .

More explicitly, given three distinct points  $z_1, z_2, z_3 \in \mathbb{C}$  and three distinct points  $w_1, w_2, w_3 \in \mathbb{C}$  there exists a unique Möbius transform  $f$  such that  $f(z_i) = w_i$  for  $i = 1, 2, 3$ :

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

In particular, Möbius transforms are conformal and more general than affine maps, since they can e.g. map  $\infty$  to 0 and vice versa. **move this to beginning of zipper method instead?**

## 1.8 Discretisation

In order to numerically solve boundary value problems the domain is usually discretised by creating a mesh  $\mathcal{M}$  of  $N$  nodes (points) and  $M$  elements (cells) covering the domain  $\Omega$ . There are many methods to create such meshes, but for the scope of this article we will assume to be given a mesh on the domain and only briefly discuss some important properties of meshes and mesh generation methods. We highly recommend referring to [Bro90] for a more in-depth treatment of mesh generation techniques.

In two dimensions, the most common choice for mesh elements are triangles or quadrilaterals. Triangular meshes are more flexible in adapting to complex geometries, while quadrilateral meshes can provide better accuracy for certain types of problems. Hybrid meshes combining both types are also used in practice to combine advantages.

### 1.8.1 Mesh Structure

To ensure bijectivity and conformality of the mapping  $\psi$  acting on the discretised domain, the discretisation must be done intelligently, i.e. the mesh must satisfy certain regularity criteria. Ill-chosen meshes could lead to overlaps of the target mesh nodes, failing bijectivity, or even non-conformality of  $\psi$  due to the failure to satisfy the discrete geometric constraints which are required for angle preservation.

In order to keep the approximation error as low as possible while maintaining good solver performance, we typically want uniform meshes, e.g. where the triangulation is close to equilateral. The quality of a mesh is measured in terms of its individual cells where for a mesh  $\mathcal{M} := \{K\}$  of triangles  $K$  such that

$$\overline{\Omega} = \cup_{K \in \mathcal{M}} K$$

we define  $d(K)$  the diameter of the smallest  $K$ -circumscribing ball (aka diameter of  $K$ ) and  $\mu(K)$  the diameter of the largest ball inscribed in  $K$ . Then a measure [Wec19] for the quality of  $K$  is the ratio of these diameters

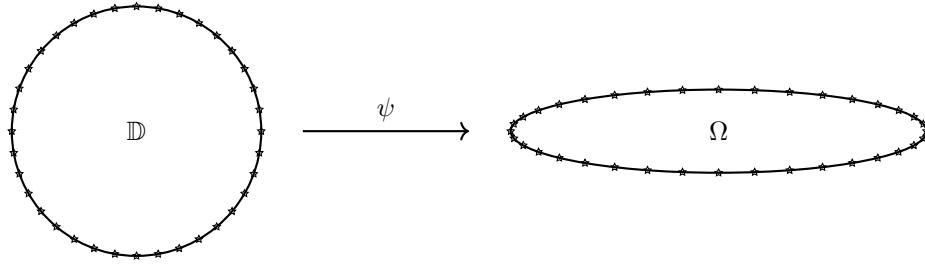
$$\rho(K) := \frac{d(K)}{\mu(K)} \in [1, \infty).$$

**to do. add a mesh quality analyser to the code**

## 1.9 Crowding

[Ban08] Crowding is the phenomenon that occurs when a set of more or less equally distributed points on the domain are mapped to a much more dense set

of points on the target region, thereby causing numerical issues due to the angles on the grid becoming very small. This typically occurs when the target region is elongated, i.e. has a high aspect ratio. The following illustration shows the concentration of points near the equator of the ellipse when mapping equidistant points from the unit disk.



Wegmann [Weg05] proved the following result.

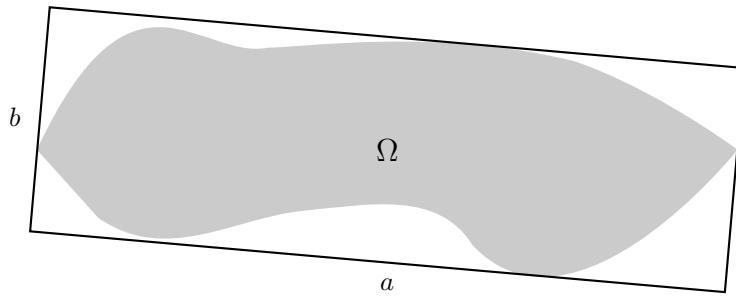
**Theorem 1.11**

When the region  $G$  can be enclosed in a rectangle with sides  $a$  and  $b$ ,  $b \leq a$ , such that  $G$  touches both small sides then the conformal mapping  $\psi : \mathbb{D} \rightarrow \Omega$  satisfies

$$\|\psi'\|_D \geq b\psi(b/a)$$

with a function  $\psi(\tau)$  which behaves for small  $\tau$  like

$$\psi(\tau) \approx \frac{1}{2\pi\sqrt{\epsilon}} \exp\left(\frac{\pi}{2\tau}\right).$$



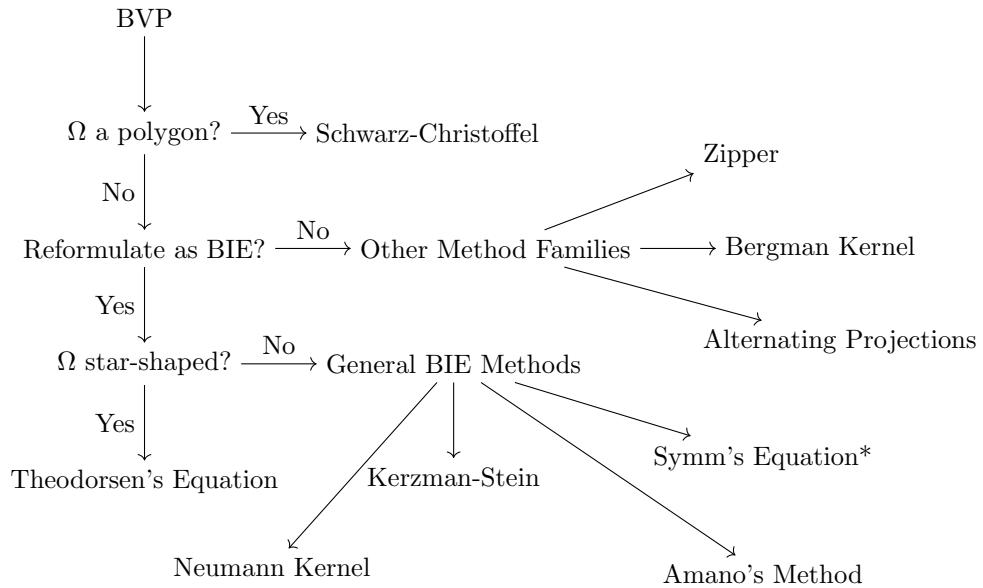
This means that crowding increases exponentially with the aspect ratio of the target region.

DeLillo has shown how crowding affects the accuracy of numerical computations.

## 2 Existing Methods

This chapter aims to give an overview and compare existing methods in terms of input/output format, suitability/ boundary requirements, computational complexity, numerical stability and mesh quality preservation/ accuracy.

We give an overview of some existing methods and how to choose each depending on the application before explaining each of them in more detail.



<sup>\*</sup>(Linear Fredholm 1<sup>st</sup> kind, generally ill-posed - $\zeta$ , not preferred)

### 2.1 Alternating Projections Method

Various methods for numerical construction of  $\psi$  essentially construct two sequences of functions, one of normalized analytic functions on the disk (using the operator  $K$  1.3) and one mapping the boundary of  $\mathbb{D}$  to the boundary  $\Gamma$ . The method of alternating projections uses both these sequences and alternates between them to find  $\psi$  [Weg89].

#### 2.1.1 Alternating Projections à la von Neumann

For two closed convex sets  $P, Q$  in a Hilbert space  $H$ , the method of alternating projections constructs a sequence  $(x_n)_n$  as follows: Starting from an arbitrary point  $x_0 \in H$ , we define

$$x_{n+1} := \begin{cases} \Pi_P(x_n) & n \equiv 0 \pmod{2} \\ \Pi_Q(x_n) & n \equiv 1 \pmod{2} \end{cases}$$

where  $\Pi_P(z) = \min_{x \in P} \|x - z\|^2$  and  $\Pi_Q(z) = \min_{x \in Q} \|x - z\|^2$  denote the orthogonal projections onto the sets  $P$  and  $Q$  respectively (though there is also a more general algorithm by Braun, Pokutta and Weismantel [BPW23] in case no such projections exist). It can be shown that the sequence  $(x_n)_n$  converges in norm to a point tuple  $(x^*, y^*)$  satisfying

$$\begin{cases} x^* = \Pi_P(y^*), \\ y^* = \Pi_Q(x^*), \\ d_H(x^*, y^*) = \min_{(x,y) \in P \times Q} \|x - y\|^2 \end{cases}$$

In particular,  $x^* = y^*$  if  $P \cap Q \neq \emptyset$ .

This exact idea is now applied to function spaces:

### 2.1.2 Algorithm

---

#### Algorithm 1 AP-Method

---

Start with a function  $U_0 \in W_{\mathbb{R}}$ .

Given  $U_k$  for  $k \geq 0$ ,

**for**  $n = 1, 0, -1, -2, \dots$  **do**

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \eta(t + U_k(t)) e^{-int} dt \quad [\text{Calculate Fourier coefficients}]$$

**end for**

$$U_{k+1}(t) := U_k(t) - \operatorname{Re} \frac{i(\operatorname{Im}(a_1)) e^{it} + \sum_{n=-\infty}^0 a_n e^{int}}{\dot{\eta}(t + U_k(t))} \quad [\text{Calculate the new iterate}]$$


---

### 2.1.3 Alternating Projections with Overrelaxation (OAP)

The OAP method is a variant of the AP method which introduces an overrelaxation parameter to speed up convergence [Weg89, p. 298]. The algorithm is the same except for a constant factor in the definition of  $U_{k+1}$ . This can be performed on a discretized grid of points on the boundary. In that case, the output is a set of Fourier coefficients, which allows for efficient point evaluations of  $\psi$  and  $D\psi$  using FFT.

## 2.2 Schwarz-Christoffel Method

One class of methods for finding the conformal mapping  $\Phi$  is given by the Schwarz-Christoffel equation, which relates the derivative of  $\Phi$  to an integral over the boundary of the target domain  $\Omega$  when  $\Omega$  is a polygon.

### 2.2.1 Preliminaries and Notation

#### Definition 12

A **Polygon** is a planar figure whose boundary is made of a chain of connected line segments which we will call arcs, connecting corner points.

The unit disk  $\mathbb{D}$  is in particular a polygon, and we parametrize the boundaries  $\partial\mathbb{D}$  and  $\partial\Omega = \Gamma$  by collections of arcs  $s_{\mathbb{D}}$  and  $s_{\Omega}$  respectively, in positive mathematical orientation. These arcs being smooth yields tangents with well-defined derivatives at every point of the boundary curves except for corners, and we denote the angles of these tangents with  $\theta_{\mathbb{D}}(s_{\mathbb{D}})$  and  $\theta_{\Omega}(s_{\Omega})$  respectively. If  $z_0$  is a corner it will have a turning angle of

$$\angle\theta_{\mathbb{D}}(z_0) = \theta_{\mathbb{D}}(z_0 + \varepsilon) - \theta_{\mathbb{D}}(z_0 - \varepsilon),$$

where  $\varepsilon \rightarrow 0$ . The same applies to corners of  $\Omega$ . Then,  $\frac{\partial\theta_{\mathbb{D}}}{\partial s_{\mathbb{D}}} = \angle\theta_{\mathbb{D}}(z_0)\delta(s_{\mathbb{D}} - z_0)$  for  $\delta$  the Dirac function, and similarly for  $\theta_{\Omega}$  and thus  $\theta_{\mathbb{D}}$  and  $\theta_{\Omega}$  are piecewise continuously differentiable functions with jump discontinuities at the corner points.

### 2.2.2 Green's Functions

A Green's function is a general concept for solving differential equations containing a linear operator.

#### Definition 13

A **Green's function** or **Green function**  $G(x, s)$  is any solution to

$$LG(x, s) = \delta(x - s)$$

where  $L = L(x)$  is a linear operator acting on distributions over  $\mathbb{R}^n$  and  $\delta$  is the Dirac delta function.

This definition can be exploited to solve inhomogeneous differential equations of the form  $Lu = f(x)$ . In the case of the SCE the Green's function are defined as  $G(z, z') : \mathbb{D} \rightarrow \mathbb{R}$  satisfying

$$\nabla^2 G(z, z') = 2\pi\delta(u - u')\delta(v - v')$$

inside the disk and

$$\frac{\partial G(z_B, z')}{\partial n} = \beta_i$$

where  $n$  is the outward normal vector at the boundary point  $z_B \in \partial\mathbb{D}$  and  $\beta_i$  is a real constant associated with the  $i$ 'th arc's length  $l_i$  such that  $\sum l_i \beta_i = 2\pi$ . It can be shown according to Floryan and Zemach that this defines a unique Green's function up to an additive constant [FZ87]. The SCE can be written explicitly whenever the Green's function is **obtainable in closed analytic form**, i.e. expressible via a finite number of elementary operations (is that what is meant on p348? i looked up wikipedia for closed analytic form).

### 2.2.3 Schwarz-Christoffel Equation Variants

Let  $\mathcal{G}(z, z'_B)$  be a complex extension of the above defined  $G(z, z'_B)$ , i.e.  $\mathcal{G}(z, z'_B)$  is analytic with real part  $G(z, z'_B)$ . A Schwarz-Christoffel equation for a confor-

mal mapping  $\Phi(z) : \mathbb{D} \rightarrow \Omega$  has the form

$$\log \frac{d\Phi}{dz} = C + \sum_i Q_i,$$

where  $C \in \mathbb{C}$  is a constant and the  $Q_i$  are the Green's function integrals over the boundary arcs of  $\mathbb{D}$ . Some parameters and constants have to be determined in order to get a unique  $\Phi$  for a particular given  $\Omega$ . The Riemann mapping theorem allows **HOW?** for the first three real parameters to be preassigned, for example to three boundary points in the case of simply connected bounded  $\Omega$ . The remaining degrees of freedom must be solved for using properties of the arcs  $s_\Omega(s_z)$  and  $s_{\mathbb{D}}(s_z)$ .

The most general form is called Schwarz-Christoffel equation with subtraction [FZ87]:

$$\log\left(\frac{d\Phi}{dz}\right) = C_0 - \frac{1}{2\pi} \int_{\partial\mathbb{D}} [\mathcal{G}(z, z'_B) - \mathcal{G}(z_0, z'_B)] \times [d\theta_\Omega(s'_z) - d\theta_{\partial\mathbb{D}}(s'_z)]$$

where  $C_0 \in \mathbb{C}$  is a constant,  $z_0$  is a fixed point in  $\mathbb{D}$  or del?,  $\mathcal{G}(z, z'_B)$  is the fundamental solution of the Laplace equation in  $\partial\mathbb{D}$  with singularity at  $z'_B \in \partial\mathbb{D}$  not at the boundary of omega?. However, we can use an unsubtracted form of this equation since the integrals converge separately and we know our  $\Omega$  is bounded (hence we need not care about behaviours at infinity), yielding the simpler form

$$\log\left(\frac{d\Phi}{dz}\right) = C - \frac{1}{2\pi} \int_{\partial\mathbb{D}} \mathcal{G}(z, z'_B) [d\theta_\Omega(s'_z) - d\theta_{\partial\mathbb{D}}(s'_z)]$$

#### 2.2.4 Implementation

[Bro90] [Tre80] [BT03] [Bez22] Crowding is a problem because it yields exponential derivative of  $\Phi$ , but can be fixed. [Ban08]

### 2.3 Theodorsen's Method **CHECK EQUATIONS**

We can reformulate the problem of finding the conformal mapping  $\psi : \mathbb{D} \rightarrow \Omega$  as solving Theodorsen's integral equation for the boundary correspondence function  $S(s)$  defined by

$$\psi(e^{is}) = \eta(S(s)) \text{ and } \int_0^{2\pi} S(s) ds = 2\pi^2 \quad (9)$$

where  $\eta$  is a parametrization of  $\Gamma = \partial\Omega$  and  $\psi$  is assumed to satisfy (1). Theodorsen's integral equation states that  $Y : s \mapsto S(s) - s$  is the conjugate periodic function of  $X : s \mapsto \log(\eta(S(s)))$ . After discretization, Theodorsen's integral equation becomes the fixed point equation

$$y = \psi(y) := K_\Sigma \log(\eta(s + y)) \quad (10)$$

for  $s := (\frac{k\pi}{N})_{k \in [2N-1]}$  where  $y$  approximates  $Y(s)$ . The product  $y$  can be computed efficiently using FFT (see chapter 1.3). If the discretization is based on trigonometric interpolation,  $K_\Sigma$  is called **Wittich's matrix** [Gut83]. By a permutation  $P$  the above components can be brought to the form

$$PK_\Sigma P = \begin{pmatrix} 0 & -L^T \\ L & 0 \end{pmatrix}, \quad Py = \begin{pmatrix} y'' \\ y' \end{pmatrix}, \quad Ps = \begin{pmatrix} s'' \\ s' \end{pmatrix}. \quad (11)$$

By Niethammer, (10) can be solved using nonlinear successive over-relaxation (SOR) for given  $y_0'', y_0'$  and  $\omega$  (relaxation factor) and  $m \geq 0$ :

$$\begin{aligned} y_{m+1}'' &:= (1 - \omega) y_m'' - \omega L^T \log(\eta(s' + y_m')) \\ y_{m+1}' &:= (1 - \omega) y_m' + \omega L \log(\eta(s'' + y_m'')) \end{aligned} \quad (12)$$

Alternatively, nonlinear Jacobi iteration with relaxation (JOR) can be used:

$$y_{m+1} := (1 - \omega) y_m + \omega \psi(y_m). \quad (13)$$

In case of  $\Gamma$  not being smooth, Theodorsen's method becomes inaccurate [Gut83]. In this case, the boundary curve can be smoothed first using preliminary maps (see osculation methods in chapter 2.9.1).

Theodorsen's method needs  $\Omega$  to be star-shaped in order to converge [Weg78], but has the advantage of being an easily computable fixed point equation in this case. In practice, the star-shape requirement can also be relaxed by smoothing the boundary curve first using preliminary maps [Gut83].

## 2.4 Zipper Method

This algorithm was found independently by Kühnau and Marshall in the 1980's and has the advantage of finding  $\psi$  and its inverse at the same time. The computed map is only approximately conformal, and is obtained as a composition of conformal maps onto slit halfplanes. Depending on the shape of the slits, the Zipper algorithm looks a bit different. In this section we will focus on the easiest version called the "geodesic algorithm" [MR06].

### 2.4.1 The Geodesic Algorithm

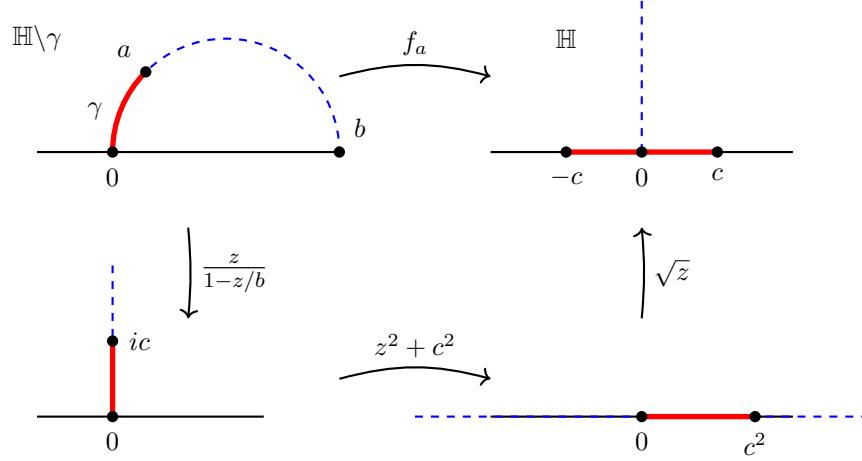
The most elementary version of this algorithm is based on a function

$$f_a : \mathbb{H} \setminus \gamma \rightarrow \mathbb{H}$$

where  $\mathbb{H}$  is the upper half plane and  $\gamma$  is a circular arc from 0 to  $a \in \mathbb{H}$  which is orthogonal to the real axis. The orthogonal circle also meets the real axis again at  $b = |a|^2 |\text{Im}(a)|$ . Then the map can be expressed in closed form as

$$f_a(z) = \sqrt{g_a \circ h_a(z)}$$

where  $g_a(z) = z^2 + c^2$ ,  $c = \frac{|a|^2}{\text{Im}(a)}$  is called the **conformal capacity** of  $\gamma$  and  $h_a(z) = \frac{z}{1 - z/b}$ .



Now suppose  $z_0, z_1, \dots, z_n$  are points arranged counterclockwise on a Jordan curve  $\Gamma$  in the upper half plane. The geodesic algorithm basically iterates over the arcs from  $z_i$  to  $z_{i+1}$  and "unzips" them one by one using the map  $f_{a_i}$  where  $a_i$  is the image of  $z_{i+1}$  under the composition of all previous maps. The original geodesic algorithm proposed by Marshall and Rohde constructs a conformal map from the upper half plane to the region bounded by  $\Gamma$ , but it can be adapted to map from the unit disk as well via a Möbius transformation mapping the half plane to the unit disk and back first.

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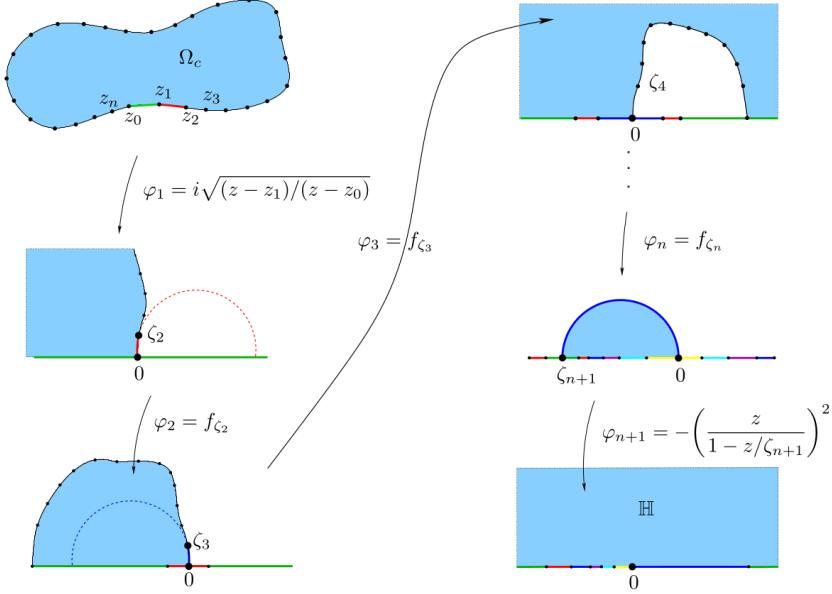
**Algorithm 2** Geodesic Zipper Algorithm

---

**Input:** Points  $z_0, z_1, \dots, z_n$  on a Jordan curve  $\Gamma$  in the upper half plane.  
**Output:**  $\psi$ : conformal map from  $\mathbb{H}$  to the region bounded by  $\Gamma$  and its inverse  $\psi^{-1}$ .

$\varphi_1(z) := i\sqrt{(z - z_1)/(z - z_0)}$   
 $\zeta_2 := \varphi_1(z_2)$   
 $\varphi_2(z) := f_{\zeta_2}(z)$   
**for** k in n **do**  
     $\zeta_k := \varphi_{k-1} \circ \dots \circ \varphi_1(z_k)$   
     $\varphi_k(z) := f_{\zeta_k}(z)$   
**end for**  
Finally,  $\zeta_{n+1} := \varphi_n \circ \dots \circ \varphi_1(z_0) \in \mathbb{R}$  and  $\varphi_{n+1}(z) := -(\frac{z}{1-z/\zeta_{n+1}})^2$   
Then  $\psi(z) := \varphi_1^{-1} \circ \varphi_2^{-1} \circ \dots \circ \varphi_{n+1}^{-1}(z)$  and  $\psi^{-1}(z) := \varphi_{n+1} \circ \dots \circ \varphi_2 \circ \varphi_1(z)$

---



#### 2.4.2 The Slit Algorithm

The above geodesic algorithm is only as accurate as the approximation of the boundary curve  $\Gamma$  by circular arcs between the points  $z_i$ . A more accurate version is given by the slit algorithm, which uses straight line segments instead of circular arcs. We therefore exchange the map  $f_a$  for a map  $g_a : \mathbb{H} \setminus L \rightarrow \mathbb{H}$  where  $L$  is the line segment from 0 to  $a \in \mathbb{H}$ . This map does not have a closed form expression, but can be computed numerically using Newton's method.

#### 2.4.3 The Zipper Algorithm

The approximation of  $\Gamma$  by circular arcs or straight line segments can be further improved by using circular arcs which meet  $\Gamma$  tangentially at the points  $z_i$ . Each arc is determined by the points  $z_i, z_{i+1}$  and  $z_{i+2}$ , hence we assume an even number of boundary points. The first arc is replaced by

$$\varphi_1(z) = \sqrt{\frac{(z - z_2)(z_1 - z_2)}{(z - z_0)(z_1 - z_2)}}.$$

At each subsequent step that circular arc through  $\zeta_k$  and  $\zeta_{k+1}$  is mapped onto a straight line segment by a Möbius transform, and then the Slit Algorithm is applied to unzip that segment. **This yields a sort of "quadratic approximation" of  $\partial\Omega$  instead of a linear one.**

## 2.5 Wegmann's Method

For smooth  $\partial\Omega$ , Integral equations of the second kind can also be solved by Newton's method [Weg78]. Given  $\eta$  the differentiable parametrization of  $\Gamma$  with  $\dot{\eta}(s) \neq 0 \forall s$ , we construct the conformal mapping by iterated correction of an initial guess  $S$  as follows:

$$\begin{array}{ccccc} & S & & \eta & \\ \partial\mathbb{D} & \xrightarrow{} & [0, 2\pi] & \xrightarrow{} & \Gamma \\ & \curvearrowright & & \curvearrowright & \\ & & \psi & & \end{array}$$

Assume that the approximation for  $\psi$  after  $k$  steps be given by  $\psi_k(\zeta) = \eta(S_k(\zeta))$ . We improve the approximation by moving the points along their tangents towards the boundary curve, i.e. by finding a correction function  $U_k : \partial\mathbb{D} \rightarrow [0, 2\pi]$  such that

$$\eta(S_k(\zeta)) + U_k(\zeta)\dot{\eta}(S_k(\zeta)) = h_{k+1}(\zeta) \quad (14)$$

where  $h$  is analytic on  $\overline{\Omega}$ . To uniquely solve for  $U_k$  numerically, we fix a boundary point  $z_0 \in \partial\mathbb{D}$  and its corresponding target parameter  $s_0 \in [0, 2\pi]$  which defines the target point  $\eta(s_0) = c_0 \in \Gamma$ . Note that since  $h_{k+1}(z_0)$  lies on the tangent through  $\eta(S_k(z_0))$  we cannot have  $h_{k+1}(z_0) = c_0$  unless  $\eta(S_k(z_0)) = c_0$ . Therefore we set

$$U_k(z_0) = s_0 - S_k(z_0). \quad (15)$$

Note also that  $h_{k+1}$  is an approximation of  $\psi$  but it does not generally take values on  $\Gamma$ . We get an approximation for the parametrization  $S$  by

$$S_{k+1}(\zeta) := S_k(\zeta) + U_k(\zeta)$$

which can be understood as "new guess equals last guess plus some correction".

### 2.5.1 Implementation

Since we assume  $\psi$  to map from the unit disk  $\mathbb{D}$ , the integral equations can be explicitly solved by discretization and trigonometric interpolation: Take  $N = 2n$  equidistant points  $\zeta_k = e^{i\theta_k}$  on the boundary of the disk, where

$$\theta_k = \theta_0 + \frac{\pi k}{N}, \quad k \in [N-1]$$

and write the integrals in terms of their Fourier transform

$$F(z) = \frac{1}{2\pi i} \int \frac{\sigma_n(\zeta)}{\zeta - z} d\zeta \quad (16)$$

where  $\sigma_n(\zeta_k) = \sum_{i=-n}^n c_k \zeta_k^i$ . This can be computed fast ( $\in \mathcal{O}(N \log N)$ ) via FFT.

## 2.6 Conjugate Function Method

Hakula, Quach and Rasila [HQR13] presented a new method in 2010 which is based on numerical solution of the Laplace equation subject to Dirichlet-Neumann mixed-type boundary conditions by using *hp*-FEM. In their paper they construct the mapping starting from a quadrilateral, but it can be tweaked to map from the unit disk as well. "It is well known that one can express the modulus of a quadrilateral  $Q$  in terms of the solution of the Dirichlet-Neumann mixed boundary value problem [Hen86, p.431]."

## 2.7 Amano's Method of Fundamental Solutions

A potential theoretic formulation of the conformal mapping problem leads to a Fredholm integral equation of the first kind, known as Symm's integral equation, which has a kernel with logarithmic singularity. Unlike Fredholm integral equations of the second kind like Theodorsen's equation, where the singularity of the kernel creates numerical instabilities, Symm's equation is easily solvable by numerical methods [Kyt98, p. 237]. One of these is Amano's method. Conceptually, a pair of conjugate harmonic functions are expressed by a complex logarithmic potential, and the mapping problems are reduced to singular Fredholm integral equations of the first kind. Gaier [Gai64] mathematically studied Symm's integral equation and proved the existence and uniqueness of the solution. These methods need  $\mathcal{O}(N^3)$  operations if the boundary is discretized at  $N$  points. Henrici showed that complexity of  $\mathcal{O}(N^2 \log N)$  can be achieved by using FFTs [Hen86].

### Definition 14

In two dimensions, the Laplace equation has a fundamental solution of the form  $\log(r)$  where  $r = |z - \zeta_k|$  is the modulus of the vector from any point  $z$  on the region to the boundary point  $\zeta_k$ ,  $k \in N$ . This function  $\log(|z - \zeta_k|)$  is called the **logarithmic potential**.

This is used in the so-called Charge Simulation Method which Amano's Method is based on.

### 2.7.1 Algorithm

The charge simulation method approximates the solution of the Laplace equation by a linear combination of fundamental solutions placed at so-called charge points outside the domain [Ama98] by

$$g(z) = \sum_{k=1}^N Q_k \log(|z - \zeta_k|), \quad (17)$$

where  $\zeta_1, \dots, \zeta_N \notin \bar{\Omega}$  are called **charge points** and placed outside the domain (**their placement is an art in and of itself**). The unknown constants  $Q_1, \dots, Q_N$  are called **charges** and determined to satisfy the boundary condition at the

**collocation points**  $z_1, \dots, z_N$  (fixed check points on the boundary  $\partial\Omega$ ). Hence,  $Q_k$  are found by plugging in the collocation points into the below **collocation condition** and solving the linear system:

$$\eta(z_i) = \sum_{k=1}^N Q_k \log(|z_i - \zeta_k|), \quad i = 1, 2, \dots, N. \quad (18)$$

Then the conformal map  $\psi(z) = g(z) + ih(z)$  is constructed, where  $h(z)$  is the harmonic conjugate of  $g(z)$ . If the boundary is analytic, this method can be shown to have exponential accuracy i.e. exponentially small error in the number of collocation points [Ama98].

## 2.8 Probabilistic Uniformization Method

In 2007, Binder, Braverman and Yampolsky proposed a method for finding a conformal map using a random walks solver to the general Dirichlet problem. They conjectured an upper bound of polynomial space and time for an algorithm with precision  $\frac{1}{n}$  pixels (for explicitly given  $\partial\Omega$ ; quadratic if  $\partial\Omega$  is given only approximately, via a so-called *oracle*, sort of a Dirac delta function). [BBY07]

## 2.9 Workarounds of Common Problems

Finally, we discuss some known workarounds for common problems in conformal mapping algorithms.

### 2.9.1 Osculation Methods

Osculation methods are preliminary conformal maps that smooth the boundary curve  $\Gamma$  of the domain  $\Omega$  before applying a more accurate conformal mapping algorithm such as Theodorsen's method.

### 2.9.2 Crowding

## 2.10 Comparison

We can directly rule out some of the options due to our project constraints: First we note that Schwarz-Christoffel only works on polygons and Theodorsen's method is restricted to star-shaped regions. Since we want to be able to work with more general shapes of  $\Omega$ , we have to disregard these two methods.

We refrain from using probabilistic methods due to their inherent randomness and the difficulty of guaranteeing a certain accuracy. Amano's Method of Fundamental Solutions seems good for fast point-evaluations of  $\psi$  itself, but the derivative could be tricky due to  $\psi$ 's form. Moreover, we cannot use our Fourier parametrization nor our mesh in this method, so it will probably not be the best choice for our specific problem. Note also that the accuracy and convergence of this method depend on the right choice of charge points, which poses another

difficulty, especially given the generality of our target domain. The Conjugate Function Method is very beautiful mathematically, but it relies only on  $hp$ -FEM for which there already are powerful libraries, so there would not be much value in half-assing an FEM solver given the scope of this thesis.

On the one hand, the Zipper method seems very well implementable and robust, but there is a small caveat to be aware of for the efficient point evaluations. Since the Zipper method essentially constructs  $\psi$  as a composition of many maps, the derivative is not straightforward to implement via a formula (chain rule). Also, trying a finite differences approach is slow and potentially numerically unstable, but there is hope in a technique called Forward Automatic Differentiation which is both exact and efficient [Wik25].

On the other hand, the Alternating Projections Method uses the boundary parametrization in its Fourier series form. The output of the discretized OAP is an interpolating polynomial, allowing for efficient point evaluations of  $\psi$  and  $D\psi$ . The AP Method converges linearly if the boundary parametrization is 3-Hölder and the initial approximation  $U_0$  is sufficiently close to the actual boundary correspondence function (2) [Weg89, p. 292]. The Alternating Projections Method is one of the simplest and most robust methods for conformal mapping. However, it is not very accurate for reasonably sized grids, and converges very slowly for finer meshes [Weg05, p. 389].

Lastly, Wegmann's Method (2.5) uses the boundary parametrization and its derivative in form of Fourier coefficients as well as the tangent angle as inputs. The output of the algorithm is an analytic function constructed from trigonometric polynomials. This allows for efficient point evaluations of both  $\psi$  and its derivative [Weg84]. Convergence is quadratic for analytic boundaries (as is known for Newton methods) and superlinear ( $\in \mathcal{O}(N^{1+\mu})$ ) for  $\eta \in C^{2+\mu}$  but depends on a "good enough" initial guess for the correspondence function [Weg84]. Discretizing the operator  $K$  (1.3) using **Wittich's method** yields the operator  $K_N$  which can be computed efficiently using FFT.

Wegmann also compared the accuracies of the AP, OAP, Theodorsen and Wegmann methods for the mapping from the disk to an inverted ellipse and found that OAP is most efficient for low accuracy and Newton methods are best for slower high accuracy calculations [Weg05, p. 415]. Note the computational costs of these last two methods are mainly determined by the FFTs and this parameter is dependent on the number of grid points.

Hence, Zipper, OAP and Wegmann's methods seem suitable for our problem, but **Wegmann's Method is more accurate with faster convergence while OAP is easier to implement.** COMPARISON TABLE HERE

### 3 Proposed Method

#### 3.1 Choice/ Justification

criteria: - Accuracy for domains with sharp corners or high curvature - Speed for practical mesh sizes - Robustness - does it fail for certain domain shapes? - Implementation complexity given your timeline - Jacobian computation - analytical vs numerical differentiation

#### 3.2 Implementation

- Separate modules for boundary parameterization, mapping computation, Jacobian eval, and mesh transformation - plot original vs. mapped grids (e.g., Matplotlib quiver for Jacobians) to spot issues early.

#### 3.3 Numerical Experiments/ Testing

check angle preservation (e.g., via dot products on mapped vectors) and scale factors ( $\det(D\Phi) > 0$ ,  $|\frac{\partial \Phi}{\partial z}|$  constant in theory). - Test suite: Use known exact mappings (e.g., disk to square via Schwarz-Christoffel) for error metrics (L2 norm on boundary points). - Metrics: Runtime for N points, mesh quality post-mapping (e.g., min/max angles in triangles, shape regularity ratio). - Real-world applicability: Apply to a sample FEM problem (e.g., Poisson equation on  $\Omega$ ) and compare accuracy/speed vs. uniform mesh. - Robustness: Vary boundary complexity (smooth vs. corners), noise in Fourier coeffs, mesh resolutions. - Debugging: Use assertions for bijectivity (e.g., check injectivity numerically) - Error handling - what happens with degenerate inputs?

#### 3.4 Results

## 4 Korrigenda

table of comparison OAP/ newton-wegmann? or prosa enough x note in the beginning thaty  $R^2=C$

eq 3  $u = \operatorname{Re}(\eta)$  nicht  $\operatorname{re}(\psi)=\eta$

This problem has a unique solution up to an imaginary constant falsch:  
realteil eindeutig, imaginärteil wird dann hergeleitet und ist bis auf const auch  
eindeutig

definition 8 falsch g sollte phi sein

check finite element mesh arvix

differentiability of zipper method construction? sollte gehen jacobi of zipper  
using back propagation?

während dem coden tests schreiben

check all methods are present in overview flowchart

put in each method's description what exactly it gets its restriction from,  
e.g. pinpoint the bottleneck

done: a deformation  $\psi$  von  $x,y$  klar angeben=  $u + iv$  such that  $J\psi = 0$ . As a byproduct of the Cauchy-Riemann equ  $\psi(z) = \psi(x,y)$  klar machen z  
komplex,  $x,y$  reell

## **5 Documentation and Implementation**

### **5.1 Testing (again?)**

VALIDATION AGAINST THE SCHWARZ-CHRISTOFFEL TOOLBOX!

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