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On the Approximate Conformal Mapping of the Unit Disk on a Simply Connected Domain

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Abstract—We present a method of constructing the analytic function realizing an approximate conformal mapping of the unit disk on an arbitrary simply connected domain with the given smooth parametrically defined boundary. The method is based on a new boundary parameterization. Solution of the problem is reduced to the Fredholm integral equation of the second kind. We present the examples of three ways to solve the integral equation.

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INTRODUCTION

This paper is devoted to construction of the approximate conformal mapping of the unit disc onto any simply connected domain with the smooth boundary. In Conclusion we discuss the case of the boundary with corners. There exist many approximate methods of conformal mapping construction. Some of them are based on integral equations [1–5]. In recent years due to appearance of high-performance computers numerical methods turned out to be actual. For example, the method presented in [6] is based on the arbitrary domain circular packing, which seems to be rather complicated task. Analytic methods clearly have certain advantages over numerical ones in the following matters: We can obtain sufficiently smooth solution and apply the differential calculus to the solution. Particularly, the polynomial representation of the solution makes it possible to find the analytical solutions of the basic elasticity theory for the plane domains. Here we present the method which is an analog of one introduced by Theodorsen–Garrick [1] but differs from it by the new parametric representation of the boundary of the domain. The solution to the problem is reduced to the Fredholm integral equation of the second kind. We give three solution methods illustrated by construction of three different mapping. Particularly, we give the Taylor series expansion of the conformal mapping which transforms the unit disk to the domain arbitrary close to elliptic one.

REDUCTION OF THE PROBLEM TO THE INTEGRAL EQUATION

Let the boundary of the simply connected domain D be the simple smooth closed curve L with its parametric representation

$$\{x = x(t),\ y = y(t),\ t \in [0, 2\pi]\},$$

here the functions $x(t)$ and $y(t)$ are 2π-periodic. So we represent these functions as Fourier series. Therefore the complex representation of the boundary L has the form

$$z(t) = x(t) + iy(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ikt}.$$

We assume that the boundary is traversed in the positive direction of the parameter t, otherwise we change the sign of t.

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Clearly, if in the representation given above $c_{-j}=0 \quad \forall j \in \mathbf{N}$, then the holomorphic function which maps the unit disk on the domain D has the Taylor expansion $f(\zeta)=\sum\limits_{k=0}^{+\infty} c_k\zeta^k$ up to the additional linear-fractional mapping of the unit disk to itself.

Assume now that there exist values $j \in \mathbf{N}$, such that $c_{-j}\neq 0$. We seek for a parametric representation $t=t(\theta)$, $\theta\in[0,2\pi]$ such that $x(t(\theta))+iy(t(\theta))$ has the complex Fourier series expansion with zero coefficients of $e^{ik\theta}$ for all $k<0$. We reduce this problem to the integral equation.

We also assume with no lost of generality that $0\in D$ and denote by $\zeta=\zeta(z)$ the function which gives the conformal univalent mapping of D to the unit disk, $\zeta(0)=0$. The function $\ln\frac{\zeta(z)}{z}$ is holomorphic in D and the boundary values

$$\operatorname{Re}\ln\frac{\zeta(z)}{z}|_{z\in L}=-\ln z(t),\ \ t\in[0,2\pi],$$

are known.

Let us introduce the notation $q(t)=\operatorname{Im}\ln\frac{\zeta(z)}{z}|_{z\in L}$. In accordance with [7] (P. 40) the boundary values of the analytic in D function $-\ln|z(t)|+iq(t)$, $t\in[0,2\pi]$, meet the relation

$$-\ln|z(t)|+iq(t)=\frac{1}{\pi i}\int_0^{2\pi}\frac{-\ln|z(\tau)|+iq(\tau)}{z(\tau)-z(t)}z'(\tau)\,d\tau.$$

We obtain the Fredholm integral equation of the second kind over $q(t)$ after we separate the imaginary parts of the previous equality:

$$q(t)=\frac{1}{\pi}\int_0^{2\pi} q(\tau)(\arg[z(\tau)-z(t)])'_\tau d\tau+\frac{1}{\pi}\int_0^{2\pi}\ln|z(\tau)|(\ln|z(\tau)-z(t)|)'_\tau d\tau. \tag{1}$$

Here the singular integral in the right part of Eq. (1) is the Cauchy principal value integral.

The similar integral equation with the kernel $(\arg[z(\tau)-z(t)])'_\tau$ appeared in the Fredholm solution of Dirichlet problem [8]. Eq. (1) was investigated in [9] where the inverse boundary-problem was studied. Clearly, for the case of the Lyapunov contour as boundary curve solution to Eq. (1) is the Hölder function.

Assertion. *In the case when the representation of boundary curve L has the form*

$$q(t)=\frac{1}{2\pi}\int_0^{2\pi}\ln|z(\tau)|\cot\frac{\tau-t}{2}d\tau+C,\ \ |z'(t)|\neq 0,$$

Eq. (1) *turns into the Hilbert formula*

$$z(t)=x(t)+iy(t)=\sum_{k=1}^{+\infty}c_k e^{ikt}.$$

Proof. Let us denote by

$$f(\zeta)=\sum_{k=1}^{+\infty}c_k\zeta^k$$

the function which gives the conformal mapping of the unit disk to the domain D. The expression

$$\ln\left(\frac{z(\tau)}{e^{i\tau}}\right)\left(\ln\frac{z(\tau)-z(t)}{e^{i\tau}-e^{it}}\right)'_\tau$$

gives the boundary values of the holomorphic in the unit disk function $\ln\big(\frac{f(\zeta)}{\zeta}\big)\zeta\big(\ln\frac{f(\zeta)-f(e^{it})}{\zeta-e^{it}}\big)'_\zeta i$, which vanishes for $\zeta=0$. Thus,

$$\int_0^{2\pi}\ln\left(\frac{z(\tau)}{e^{i\tau}}\right)\left(\ln\frac{z(\tau)-z(t)}{e^{i\tau}-e^{it}}\right)'_\tau d\tau=0.$$

We restrict ourselves only to the real part of the previous equality and obtain the following relation:

$$\int_0^{2\pi} \ln|z(\tau)|(\ln|z(\tau)-z(t)|)'_\tau \, d\tau + \int_0^{2\pi} q(\tau)[\arg(z(\tau)-z(t))]'_\tau \, d\tau$$
$$= \frac{1}{2}\int_0^{2\pi} \ln|z(\tau)| \cot\frac{\tau-t}{2}\, d\tau + \frac{1}{2}\int_0^{2\pi} q(\tau)\, d\tau.$$

We compare this relation with Eq. (1) and find that

$$q(t) = \frac{1}{2\pi}\int_0^{2\pi} \ln|z(\tau)| \cot\frac{\tau-t}{2}\, d\tau + \frac{1}{2\pi}\int_0^{2\pi} q(\tau)\, d\tau. \qquad \square$$

Let now the representation of the curve L contain the negative powers of e^{it}. We seek for the solution to the integral equation (1) in the form of Fourier series. The main idea of the presented method is separation of the summand $\cot(\frac{\tau-t}{2})$ in the kernel $(\ln|z(\tau)-z(t)|)'_\tau$ and the Hilbert formula application. The remaining term $K(\tau,t) = (\ln|z(\tau)-z(t)|)'_\tau - \cot(\frac{\tau-t}{2})$ is a continuous $2\pi-$periodic function which can be expanded to Fourier series on the interval $[0,2\pi]$ for any value of $t \in [0,2\pi]$.

Sometimes it seems apt to apply the Taylor expansion of $\ln(1+t)$, $|t|<1$, and obtain the Fourier coefficients of the kernels $K(\tau,t)$ and $(\arg[z(\tau)-z(t)])'_\tau$ for the cases when L is close to the circle. Example 1 given below demonstrates us the case when all the Fourier coefficients of $q(t)$ can be found.

Example 2 shows the possibility of reduction of solution to Eq. (1) to iterations.

Example 3 presents the case when the curve L is strongly dissimilar with the circle and when it is impossible to find the explicit form of the Fourier coefficients of of the kernels $K(\tau,t)$ and $(\arg[z(\tau)-z(t)])'_\tau$. Then we have to reduce solution to the infinite linear system over the coefficients to approximate solution to the finite linear system.

After we obtain the solution $q(t)$ to Eq. (1) we find the function

$$\theta(t) = \arg(\zeta(z))|_{z=z(t)\in L} = \arg z(t) + q(t).$$

Note that

$$\theta(t) = t - \frac{1}{2\pi}\int_0^{2\pi} \ln|z(\tau)| \cot\frac{\tau-t}{2} d\tau + \frac{1}{2\pi}\int_0^{2\pi} \ln|z(\tau)| \cot\frac{\tau-t}{2} d\tau + C = t + C$$

for the case of $z(t) = \sum\limits_{k=1}^{+\infty} c_k e^{ikt}$.

Now if we find the inverse function $t = t(\theta)$, $\theta \in [0,2\pi]$ and put it into the given representation $z = z(t(\theta))$ of the boundary L of the domain D, we have the boundary values $z(\zeta)|_{\zeta=e^{i\theta}}$ of the function $z(\zeta)$ which realizes the conformal mapping of the unit disk on the domain D. Hence the function $z(\zeta)$ can be restored with the help of the Cauchy integral formula

$$z(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z(t(\theta))e^{i\theta}\, d\theta}{e^{i\theta}-\zeta}.$$

Unfortunately, in general it is difficult to find the inverse to the monotonic function $\theta(t) = \arg z(t) + q(t)$. So one has to apply the Cauchy formula in the form

$$z(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{z(t)e^{i\theta(t)}\theta'(t)\, dt}{e^{i\theta(t)}-\zeta}.$$

It seems more practical to construct the function $z(\zeta)$, $|\zeta|<1$, in the form of the Fourier series: $z(\zeta) = \sum\limits_{j=1}^{\infty}(A_j + iB_j)\zeta^j$, where

$$\begin{aligned} A_j &= \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(j\theta(t))\theta'(t)\, dt = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(j\theta(t))\theta'(t)\, dt, \\ B_j &= -\frac{1}{\pi}\int_0^{2\pi} x(t)\sin(j\theta(t))\theta'(t)\, dt = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos(j\theta(t))\theta'(t)\, dt, \quad j \in \mathbf{N}. \end{aligned} \tag{1}$$

Obviously, it is impossible to calculate the coefficients A_j and B_j via formulas (2) if the function $q(t)$ has the form of some infinite Fourier series. So this infinite sum must be replaced by Fourier trigonometric polynomials. According to formula (2) it is possible to take sufficiently many Fourier summands for the function $q(t)$ so that the calculation result will not depend on the form of each of formulas. So we obtain the approximate function $z(\zeta)$, $|\zeta| < 1$, which maps the unit disk onto the domain close to D.

Example 1. Assume that $z(t) = e^{it} + be^{-it}$, $t \in [0, 2\pi]$, $0 < b < 1$. The corresponding domain D is the interior of the ellipse with the half-axes $1 + b$ and $1 - b$. Here we have

$$z(\tau) - z(t) = e^{i\frac{\tau+t}{2}} 2i \sin\frac{\tau - t}{2}(1 - be^{-i(\tau+t)}),$$

$$\begin{aligned}\ln[z(\tau) - z(t)] &= i\frac{\tau + t}{2} + \ln 2i + \ln\sin\frac{\tau - t}{2} + \ln(1 - be^{-i(\tau+t)})\\ &= i\frac{\tau + t}{2} + \ln 2i + \ln\sin\frac{\tau - t}{2} - \left(be^{-i(\tau+t)} + \frac{b^2e^{-2i(\tau+t)}}{2} + \frac{b^3e^{-3i(\tau+t)}}{3} + \cdots\right),\end{aligned}$$

$$(\ln[z(\tau) - z(t)])'_\tau = i\frac{1}{2} + \frac{1}{2}\cot\frac{\tau - t}{2} + i(be^{-i(\tau+t)} + b^2e^{-2i(\tau+t)} + b^3e^{-3i(\tau+t)} + \cdots).$$

Therefore the kernels of the integral operators in Eq. (1) have the following form:

$$\begin{aligned}(\arg[z(\tau) - z(t)])'_\tau &= \frac{1}{2} + b\cos(\tau + t) + b^2\cos(2(\tau + t)) + \cdots\\ &= \frac{1}{2} + \sum_{n=1}^{\infty} b^n(\cos n\tau\cos nt - \sin n\tau\sin nt),\end{aligned}$$

$$(\ln|z(\tau) - z(t)|)'_\tau = \frac{1}{2}\cot\frac{\tau - t}{2} + \sum_{n=1}^{\infty} b^n(\sin n\tau\cos nt - \cos n\tau\sin nt).$$

The function $\ln|z(\tau)|$ has the Fourier series expansion

$$\ln|z(\tau)| = \ln|1 + be^{-2i\tau}| = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}b^n}{n}\cos 2n\tau.$$

We seek for the solution to (1) (up to an arbitrary constant summand) in the following form:

$$q(t) = \sum_{n=1}^{\infty}\alpha_n\cos nt + \beta_n\sin nt.$$

We put this expression in (1) and obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha_n\cos nt + \beta_n\sin nt = \sum_{n=1}^{\infty} b^n(\alpha_n\cos nt - \beta_n\sin nt)&\\ - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}b^n}{n}\sin 2nt + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}b^{3n}}{n}\sin 2nt&.\end{aligned}$$

Clearly, $\alpha_n = 0 \ \ \forall n \in \mathbf{N}$, $\beta_{2k-1} = 0 \ \ \forall k \in \mathbf{N}$,

$$\beta_{2k} = \frac{(-1)^k b^k(1 - b^{2k})}{k(1 + b^{2k})}.$$

Hence

$$q(t) = \sum_{n=1}^{\infty} \frac{(-1)^n b^n (1-b^{2n})}{n(1+b^{2n})} \sin 2nt.$$

Finally we have

$$\theta(t) = \arg z(t) + q(t) = t + \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n} \sin 2n\tau \\ + \sum_{n=1}^{\infty} \frac{(-1)^n b^n (1-b^{2n})}{n(1+b^{2n})} \sin 2nt = t + \sum_{n=1}^{\infty} \frac{(-1)^n 2b^n}{n(1+b^{2n})} \sin 2nt.$$

Now application of (2) shows that the coefficients A_n and B_n of even indices vanish.

The map of the unit circle by the constructed function $z(\zeta) = \sum\limits_{j=0}^{N} (A_j + iB_j)\zeta^j$ for $b = 1/2$ was plotted with the help of MAXIMA. There remained 100 non-null terms in the expansions of the functions $q(t)$ and $z(\zeta)$. The resulting curve turned out to be close to the corresponding ellipse.

Example 2. Assume that $z(t) = e^{it} + be^{-2it}$, $t \in [0, 2\pi]$. Clearly, the corresponding domain D is 3-symmetric. Let the parameter b be positive. The contour L is convex for $0 < b < 1/4$ and is star-like with respect to $z = 0$ for $1/4 \le b < 1/2$. The contour L has three cusps for $b = 1/2$ and it fails to be a simple closed curve for $b > 1/2$.

In our case we have

$$\ln[z(\tau) - z(t)] = i\frac{\tau + t}{2} + \ln 2i + \ln\sin\frac{\tau - t}{2} - be^{-i\tau}e^{-2it} \\ - \sum_{n=1}^{\infty} \Big\{ e^{-2in\tau} \sum_{k=0}^{n} \frac{1}{2n-k} C_{2n-k}^{2n-2k} b^{2n-k} e^{-i(4n-3k)t} \\ + e^{-(2n+1)i\tau} \sum_{k=0}^{n} \frac{1}{2n+1-k} C_{2n+1-k}^{2n+1-2k} b^{2n+1-k} e^{-i(4n+2-3k)t} \Big\},$$

here C_a^c are the binomial coefficients.

So the kernels of the integral operators in (1) have the following forms:

$$(\arg[z(\tau) - z(t)])'_\tau = \frac{1}{2} + b(\cos\tau\cos 2t - \sin\tau\sin 2t) \\ + \sum_{n=1}^{\infty} \Big\{ \cos 2n\tau \sum_{k=0}^{n} \frac{2n}{2n-k} C_{2n-k}^{2n-2k} b^{2n-k} \cos(4n-3k)t \\ - \sin 2n\tau \sum_{k=0}^{n} \frac{2n}{2n-k} C_{2n-k}^{2n-2k} b^{2n-k} \sin(4n-3k)t \\ + \cos(2n+1)\tau \sum_{k=0}^{n} \frac{2n+1}{2n+1-k} C_{2n+1-k}^{2n+1-2k} b^{2n+1-k} \cos(4n+2-3k)t \\ - \sin(2n+1)\tau \sum_{k=0}^{n} \frac{2n+1}{2n+1-k} C_{2n+1-k}^{2n+1-2k} b^{2n+1-k} \sin(4n+2-3k)t \Big\},$$

$$(\ln|z(\tau) - z(t)|)'_\tau = \frac{1}{2}\cot\frac{\tau - t}{2} + b(\cos\tau\sin 2t - \sin\tau\cos 2t)$$

$$+\sum_{n=1}^{\infty}\Big\{\cos 2n\tau\sum_{k=0}^{n}\frac{2n}{2n-k}C_{2n-k}^{2n-2k}b^{2n-k}\sin(4n-3k)t$$
$$+\sin 2n\tau\sum_{k=0}^{n}\frac{2n}{2n-k}C_{2n-k}^{2n-2k}b^{2n-k}\cos(4n-3k)t$$
$$+\cos(2n+1)\tau\sum_{k=0}^{n}\frac{2n+1}{2n+1-k}C_{2n+1-k}^{2n+1-2k}b^{2n+1-k}\sin(4n+2-3k)t$$
$$+\sin(2n+1)\tau\sum_{k=0}^{n}\frac{2n+1}{2n+1-k}C_{2n+1-k}^{2n+1-2k}b^{2n+1-k}\cos(4n+2-3k)t\Big\}.$$

Also the singular integral operator density in (1) is as follows:

$$\ln|z(\tau)|=\ln|1+be^{-3i\tau}|=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}b^n}{n}\cos 3n\tau.$$

Put $q(t)=\sum\limits_{n=1}^{\infty}\alpha_n\cos nt+\beta_n\sin nt$ into integral equation (1) and compare the relative Fourier coefficients. It turns out that all Fourier coefficients of the function $q(t)$ vanish except for β_{3k}, $k\in\mathbf{N}$. Thus,

$$q(t)=\sum_{k=1}^{\infty}\beta_{3k}\sin 3kt.$$

In order to find coefficients β_{3k}, $k\in\mathbf{N}$ we consider the infinite linear system $(I+M)T=P$. Here T is the column matrix of the unknown coefficients, I is the infinite identity matrix, M is the infinite matrix with the finite number of non-zero elements at each row. Let us now describe the order and values of these elements.

Let the number of the row of M be even and equals $2t$, $t\in\mathbf{N}$, then all the nonzero elements reside at the columns from the tth till the $4t$th. Moreover, the element of the kth column, $t\le k\le 4t$, equals $3\frac{k}{k+2t}C_{k+2t}^{4t-k}b^{k+2t}$.

Let the index of the row of M be odd and equal $2t-1$, $t\in\mathbf{N}$. Then the nonzero elements are located at the columns from the tth till the $(4t-2)$th. Also the element of the kth column, $t\le k\le 4t-2$, equals $3\frac{k}{k+2t-1}C_{k+2t-1}^{4t-k-2}b^{k+2t-1}$.

The column matrix P contains at the kth row the expression

$$d_k=\frac{(-1)^kb^k}{k}-\sum_{l\ge k/4}^{k}\frac{3b^{4l+k}}{2l+k}C_{2l+k}^{2k-2l}+\sum_{l\ge(k+2)/4}^{k}\frac{3b^{4l+k-2}}{2l+k-1}C_{2l+k-1}^{2k-2l+1}.$$

Note that $\lim\limits_{k\to\infty}d_k=0$. Clearly,

$$|d_k|\le\frac{1}{k2^k}+\sum_{l\ge k/4}^{k}\frac{3}{(2l+k)2^{4l+k}}C_{2l+k}^{2k-2l}+\sum_{l\ge(k+2)/4}^{k}\frac{3}{(2l+k-1)2^{4l+k-2}}C_{2l+k-1}^{2k-2l+1}.$$

Let us show that the summands at the right-hand side of the inequality tend to zero as k grows. We will consider only the case of $k=4m$ and the summand

$$J_m=\sum_{l=m}^{4m}\frac{1}{(2l+4m)2^{4l+4m}}C_{2l+4m}^{8m-2l}=\frac{1}{2^{8m}}\sum_{j=0}^{3m}\frac{1}{(2j+6m)2^{4j}}C_{2j+6m}^{6m-2j}.$$

We apply the Stirling formula and expand the finite sum to an infinite series, so we have $J_m \le \frac{\alpha e^{\lambda m}}{2^{8m}}$, where α is a certain finite number, $0 < \lambda < 2$. Now it seems clear that $\lim\limits_{m\to\infty} J_m = 0$. Similarly we estimate all the other summands in the representation of d_k. Note that $|d_k| < 1 \ \ \forall k \in \mathbf{N}$.

Let us show that the linear transformation given by the matrix M is the contractive in the space l_∞ when $0 < b < 1/2$. Recall that the sufficient condition of the transformation given by an infinite matrix $[a_{ji}]$ to be contractive in the space l_∞ is as follows: $\max\limits_{j\in\mathbf{N}} \sum\limits_{i=1}^{\infty} |a_{ji}| < 1$.

It can be shown that the sum of the elements of each row of the matrix M equals 1 for $b = 1/2$, so the sufficient condition of contractive transformation for $0 < b < 1/2$ holds.

We prove this equality for the case of the even row, so we show that

$$3\sum_{k=t}^{4t} \frac{k}{k+2t} C_{k+2t}^{4t-k} (1/2)^{k+2t} = 3\sum_{k=0}^{3t} \frac{k+t}{k+3t} C_{k+3t}^{3t-k} (1/2)^{k+3t} = 1.$$

The proof is by induction. The formula for $t = 1$ can be easily verified. Now we apply the relation

$$3\sum_{k=0}^{3t} \frac{k+t}{k+3t} C_{k+3t}^{3t-k} (1/2)^{k+3t} = \sum_{k=0}^{3t} C_{k+3t}^{3t-k} (1/2)^{k+3t} + \sum_{k=0}^{3t-1} C_{k+3t}^{3t-k-1} (1/2)^{k+3t+1}.$$

We transform each of the summand at the right-hand side of this relation with the help of the identity $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$ and obtain

$$\begin{aligned}
&\sum_{k=0}^{3t} C_{k+3t}^{3t-k} (1/2)^{k+3t} + \sum_{k=0}^{3t-1} C_{k+3t}^{3t-k-1} (1/2)^{k+3t+1} \\
&\qquad = 64\sum_{k=0}^{3t-3} C_{k+3t-3}^{3t-k-3} (1/2)^{k+3t+3} + 32\sum_{k=0}^{3t-4} C_{k+3t-3}^{3t-k-4} (1/2)^{k+3t+3} \\
&\qquad\qquad = \sum_{k=0}^{3t-3} C_{k+3t-3}^{3t-k-3} (1/2)^{k+3t-3} + \sum_{k=0}^{3t-4} C_{k+3t-3}^{3t-k-4} (1/2)^{k+3t-2}.
\end{aligned}$$

Therefore, the value of $3\sum\limits_{k=0}^{3t} \frac{k+t}{k+3t} C_{k+3t}^{3t-k} (1/2)^{k+3t}$ does not depend on t, so it equals 1 as for the case $t = 1$.

Hence the column matrix of the coefficients T can be found for each b, $0 < b < 1/2$ with the help of iterations:

$$T = (I - M + M^2 - M^3 - \cdots)P.$$

Example 3. The curve L, defined by the parametric equation $z(t) = e^{it} + a(e^{-2it} + e^{2it})$, $t \in [0, 2\pi]$, is the simple closed curve for any value of $a > 0$, it is the boundary of the domain which is convex in the direction of the real axis. We have the following:

$$\ln[z(\tau) - z(t)] = i\frac{\tau + t}{2} + \ln\left(2i \sin\frac{\tau - t}{2}\right) + \ln[1 + a(e^{i\tau} + e^{it})(1 - e^{-2i(\tau+t)})].$$

The decomposition of Fourier series of kernels of the integral operators of Eq. (1) and comparison of Fourier coefficients similarly to the two previous examples is possible here for the values $0 < a < 1/4$, only.

We consider the case of $a = 1$. Now

$$(\ln|z(\tau) - z(t)|)'_\tau = \frac{1}{2}\cot\frac{\tau - t}{2} + K(\tau, t) = \frac{1}{2}\cot\frac{\tau - t}{2} + \frac{L(\tau, t)}{M(\tau, t)},$$

where

$$L(\tau,t) = -2\sin(\tau-t) - \sin\tau + 4\sin 2(\tau+t) + \sin(\tau+2t) + \sin(\tau+3t)$$
$$+ 2\sin(2\tau+t) + 3\sin(3\tau+t),$$
$$M(\tau,t) = 5 + 2\cos t + 4\cos(\tau-t) + 2\cos\tau - 4\cos 2(\tau+t) - 2\cos(\tau+2t)$$
$$- 2\cos(\tau+3t) - 2\cos(2\tau+t) - 2\cos(3\tau+t),$$

$$(\arg[z(\tau)-z(t)])'_\tau = \frac{1+2\cos t+4\cos\tau+2\cos(2\tau+t)}{2M(\tau,t)} = \frac{C(\tau,t)}{2M(\tau,t)}.$$

We consider the function $\ln|z(t)| = \ln|e^{4it}+e^{3it}-2e^{2it}+1|$ as the boundary value of the holomorphic function $|\zeta|<1$. We find the roots of the polynomial $\zeta^4+\zeta^3-2\zeta^2+1$. These are the numbers $\zeta_1=-0.67$, $\zeta_2 = 0.78+i0.40$, $\zeta_3=\overline{\zeta_2}$, $\zeta_4=-1.90$.

Since

$$\ln|z(t)| = \operatorname{Re}\left[\ln(-\zeta_4) + \ln\left(1-\frac{\zeta}{\zeta_4}\right) + \ln(1-\overline{\zeta_2}\zeta) + \ln(1-\overline{\zeta_3}\zeta) + \ln(1-\zeta_1\zeta)\right]\Big|_{\zeta=e^{it}}$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|z(\tau)|\cot\frac{\tau-t}{2}\,d\tau = \arg\left((1-\zeta_1e^{it})\left(1-\frac{e^{it}}{\zeta_4}\right)(1-\overline{\zeta_2}e^{it})(1-\overline{\zeta_3}e^{it})\right)$$
$$= \sum_{j=1}^{\infty}(\zeta_1^j + 2\operatorname{Re}(\zeta_2)^j + \zeta_4^{-j})\frac{\sin jt}{j} = G(t).$$

Therefore, Eq. (1) takes the form

$$q(t) = \frac{1}{2\pi}\int_0^{2\pi} q(\tau)\frac{C(\tau,t)}{M(\tau,t)}\,d\tau + G(t) + F(t),$$

here

$$F(t) = \frac{1}{2\pi}\int_0^{2\pi}\ln(7+2\cos 4\tau+2\cos 3\tau-8\cos 2\tau-2\cos\tau)\frac{L(\tau,t)}{M(\tau,t)}\,d\tau = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)K(\tau,t)\,d\tau.$$

Let us prove that the function $F(t)$ is an odd one. One can easily see that the function $(K(\tau,t)-K(\tau,-t))'_t$ is odd with respect to τ. So $\int\limits_{-\pi}^{\pi} f(\tau)\,(K(\tau,t)-K(\tau,-t))'_t\,d\tau = 0$ for any t because $f(\tau)$ is the even function. Hence, $(F(t)+F(-t))'_t = 0$ and $F(t)+F(-t)\equiv \text{const}$. We have $F(0)=0$, so $F(t)=-F(-t)$, and it proves the oddness of the function $F(t)$.

Therefore, the free term of the integral equation does not contain the summands of the $\cos kt$ type. And when we present the function $q(t)$ as the Fourier expansion, this decomposition does not contain the same summands. So the expansion of the function $q(t)$ has the form $q(t) = \sum\limits_{k=1}^{\infty}\beta_k\sin kt$.

Hence we have the infinite linear system over the coefficients β_k with the infinite matrix containing the elements $a_{ij} = \delta_{ij} - c_{ij}$, $i,j\in\mathbf{N}$, here δ_{ij} is the Kronecker δ-function and

$$c_{ij} = \frac{1}{2\pi^2}\int_0^{2\pi}\sin it\,dt\int_0^{2\pi}\sin j\tau\frac{C(\tau,t)}{M(\tau,t)}d\tau.$$

In this case the function $\frac{C(\tau,t)}{M(\tau,t)}$ is infinitely differentiable with respect to any of its variables. The formula of integration by parts now gives us that if there exists the continuous derivative $\frac{\partial^{n+m}}{\partial\tau^n\partial t^m}(\frac{C(\tau,t)}{M(\tau,t)})$ for $(\tau,t)\in[0,2\pi]\times[0,2\pi]$ then the coefficients c_{ij} can be estimated by the relation $c_{ij}\le\frac{D}{j^n i^m}$. So the coefficients c_{ij} rapidly decrease to 0 if any parameter increases.

Let us now write down the linear system with respect to coefficients $\beta = (\beta_1, \beta_2, \ldots, \beta_n, \ldots)$ in the vector form $(I - C)\beta = B$, here C is the infinite matrix of coefficients c_{ij}, $i, j \in \mathbf{N}$. Let us fix $N \in \mathbf{N}$. Then we have the vector β=$(\beta_0, \widetilde{\beta})$, here $\beta_0 = (\beta_1, \beta_2, \ldots, \beta_N)$ is an N-dimensional vector and $\widetilde{\beta} = (\beta_{N+1}, \beta_{N+2}, \ldots)$ is an infinite-dimensional vector. Consider the principal matrix of the system $I - C$ as follows:

$$I - C = \begin{pmatrix} I_0 - C_0 & -C' \\ -C'' & \widetilde{I} - \widetilde{C} \end{pmatrix}.$$

Here the matrix $I_0 - C_0$ is of dimension $N \times N$, matrix C' consists of N rows and infinite number of columns, matrix C'' consists of N columns and infinite number of rows and

$$\widetilde{I} - \widetilde{C} = \begin{pmatrix} 1 - c_{N+1\ N+1} & -c_{N+1\ N+2} & \ldots \\ -c_{N+1\ N+2} & 1 - c_{N+2\ N+2} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is an infinite matrix.

Column vector in the right-hand side of the system $(I - C)\beta = B$ can be represented in the form $B = (B_0, \widetilde{B})$, here the vector B_0 is N-dimensional and the vector $\widetilde{B}$ is infinite-dimensional. Now fix a number N so that $\|\widetilde{C}\| < 1$. Then starting with $(N + 1)$th equation of the system we can find the linear dependence of the infinite vector $\widetilde{\beta}$ coordinates through the vector β_0 coordinates:

$$\widetilde{\beta} = (\widetilde{I} - \widetilde{C})^{-1}(\widetilde{B} + C''\beta_0) = \left(\widetilde{I} + \sum_{k=1}^{\infty} \widetilde{C}^k\right)(\widetilde{B} + C''\beta_0). \tag{2}$$

Put relations (3) into the first N equations of the system $(I_0 - C_0)\beta_0 - C'\widetilde{\beta} = B_0$. This provides us with equation $(I_0 - C_0)\beta_0 - C'(\widetilde{I} - \widetilde{C})^{-1}(\widetilde{B} + C''\beta_0) = B_0$. Hence $(I_0 - C_0 - C'(\widetilde{I} - \widetilde{C})^{-1}C'')\beta_0 = B_0 + C'(\widetilde{I} - \widetilde{C})^{-1}\widetilde{B}$.

Assume now that by increasing if necessary the number N we find it so that the determinant of the square matrix $(I_0 - C_0 - C'(\widetilde{I} - \widetilde{C})^{-1}C'')$ is non-zero. Then

$$\beta_0 = (I_0 - C_0 - C'(\widetilde{I} - \widetilde{C})^{-1}C'')^{-1}(B_0 + C'(\widetilde{I} - \widetilde{C})^{-1}\widetilde{B}).$$

After we determine β_0 the infinite-dimensional vector $\widetilde{\beta}$ can be found according to formula (3).

In practice we are confined to an approximate solution to the system by reducing it to finite and replacing rows and columns with sufficiently large numbers with zeros. The approximate solution can be considered plausible if the resulting function $\theta(t) = \arg z(t) + q(t)$ is monotone and gives similar results after we compute the coefficients A_k and B_k according to formulas (2) in two different ways.

CONCLUSION

The presented analytic construction method of the approximate conformal mapping of the unit disc to a simply connected domain is applicable for any domain with the smooth boundary. The case of the domain D with the finite number of boundary corners can be reduced to the case of the domain with a smooth boundary if we apply the preliminary mappings of the given domain by several functions ϕ_j of $(z - z_j)^{\alpha_j}$ type in order to obtain the new domain $\widetilde{D}$ with the smooth boundary. After we construct the conformal mapping of the unit disk onto $\widetilde{D}$ we apply the corresponding inverse to ϕ_j functions and get the target map.

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