Conformal Mesh Mappings

Bachelor Thesis D-MATH ETHZ
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Abstract

Given a 2D triangular mesh $\widehat{\mathcal{M}}$ of the unit disk $\mathbb{D}$, whose triangles are all nicely shape-regular (in the sense that the ratio of their diameter to the radius of the largest inscribed circle is uniformly bounded for all triangles), we are guaranteed the existence of a conformal mapping Φ from $\mathbb{D}$ to any simply connected bounded domain Ω by the Riemann mapping theorem. This allows for a sufficiently "nice" mesh $\mathcal{M}$ on Ω to be obtained as the image of $\widehat{\mathcal{M}}$ under Φ, i.e. $\mathcal{M} = \Phi(\widehat{\mathcal{M}})$. The challenge lies in the numerical construction/ approximation of this conformal mapping Φ. This text is intended to give a general overview of currently known numerical conformal mapping algorithms, and to provide a comparison in terms of accuracy, runtime, and the representation format of the resulting map, particularly emphasizing the efficiency of point evaluations for both the mapping Φ itself and its derivative (Jacobian) $D\Phi$. Finally, we implement Algorithm Name.

Contents

1 Theoretical Background

1.1 Conformal Mappings

Definition 1
A **conformal mapping**, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation $f(z)$ that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative.

This type of mapping is useful as some mesh properties remain regular under such transformations. This ensures that cells do not become too stretched or overlap, which would cause numerical issues or even solver failure [Wec19]. In two dimensions, conformality can be achieved by enforcing that the mapping satisfies the Cauchy-Riemann equations.

Definition 2
We call a subset $\Omega \in \mathbb{C}$ **proper** if $\emptyset \neq \Omega \neq \mathbb{C}$.

Theorem 1.1 (Carathéodory)
The conformal mapping $\Phi : D \to G$ can be extended to a continuous mapping $\Phi : \bar{D} \to \bar{G}$ if the boundary Γ of G consists of a closed curve. [Weg05], p. 357

Definition 3
Ahlfors map

1.2 Riemann Mapping Theorem

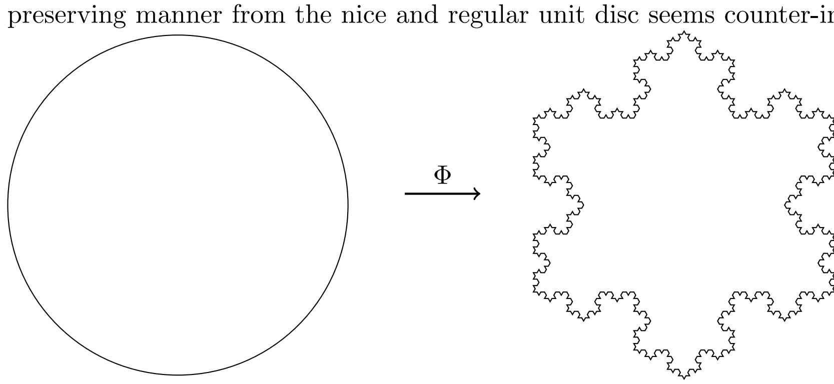
The reason why we use conformal mappings for shape optimisation in two dimensions is the following

Theorem 1.2 (Riemann Mapping Theorem)
If Ω is a non-empty simply connected open proper subset of the complex plane $\mathbb{C}$, then there exists a biholomorphic mapping f (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from Ω onto the open unit disk

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

This mapping is known as a Riemann mapping.

The beauty of the Riemann mapping theorem lies in its weight of implications, i.e. the fact that it guarantees the existence of a conformal map between any two simply connected domains in the complex plane, provided they are not the entire plane. The existence of this Riemann map is a priori not obvious: Even relatively simple Riemann mappings (for example a map from the interior of a circle to the interior of a square) have no explicit formula using only elementary functions. Simply connected open sets in the plane can be highly complicated, for instance, the boundary can be a nowhere-differentiable fractal curve of infinite length, even if the set itself is bounded. One such example is the Koch curve. The fact that such a set can be mapped in an angle-preserving manner from the nice and regular unit disc seems counter-intuitive.



The existence of a conformal map between any two simply connected, open proper subsets of $\mathbb{C}$ is what allows us to perform shape optimisation in two dimensions by solving the optimisation problem on a simple reference domain (e.g. the unit disk) and then mapping the solution to the complicated target domain. We closely follow the proof by normal families in [SS03].

1.2.1 Preliminary Results

Lemma 1.3 (Schwarz Lemma)
Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
2. If for some $z_0 \neq 0$ we have $f(z_0) = z_0$ then f is a rotation.
3. $|f'(0)| \leq 1$ and if equality holds, then f is a rotation.

Proof.?

Definition 4
A family $\mathcal{F}$ of holomorphic functions on a domain Ω is called **normal** if every sequence in $\mathcal{F}$ contains a subsequence that converges uniformly on any compact subset of Ω.

Definition 5
A family $\mathcal{F}$ is called **uniformly bounded on compact subsets** of Ω if for every compact subset $K \subset \Omega$ there exists a constant M_K such that $|f(z)| \leq M_K$ for all $z \in K$ and all $f \in \mathcal{F}$.

Definition 6
A family $\mathcal{F}$ of holomorphic functions on a domain Ω is called **equicontinuous** if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $z \in \Omega$ with $|z - z_0| < \delta$ we have $|f(z) - f(z_0)| < \epsilon$ for all $f \in \mathcal{F}$.

Theorem 1.4 (Montel)
A family $\mathcal{F}$ of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω is normal if and only if it is equicontinuous on compacta.

Proof.?

Proposition 1.5 (Uniform Convergence to Holomorphic Limit and Injectivity)
Let $\Omega \subset \mathbb{C}$ be a connected open subset and let $f_n : \Omega \to \mathbb{C}$ be a sequence of injective holomorphic functions that converges uniformly on compact subsets of Ω to a holomorphic function f. Then f is either constant or injective.

Proposition 1.6 (Cauchy Inequality)
Let f be holomorphic on an open set containing the closure of a ball $B_R(z_0)$ centered at z_0 of radius R. Then

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$ on the boundary circle C.

Proof.?

Theorem 1.7 (Implicit Mapping Theorem [EW22], p.573)
Let $r > 0$ be a radius, and let $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$. Consider the open set $W = B_r(x_0) \times B_r(y_0) \subset \mathbb{R}^n \times \mathbb{R}^m$ defined as

$$W = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \|x - x_0\|_2 < r \text{ and } \|y - y_0\|_2 < r\}.$$

Let $F : W \to \mathbb{R}^m$ be a continuous function satisfying the following conditions:

1. $F(x_0, y_0) = 0$.
2. The partial derivatives $\partial_{y_k} F : W \to \mathbb{R}^m$ exist for all $k \in \{1, \ldots, m\}$ and are continuous on W.
3. The partial differential $D_y F(x_0, y_0)$ (the differential of the map $y \mapsto F(x_0, y)$ at y_0) is invertible.

Then there exist radii $\alpha, \beta \in (0, r)$ such that for the open balls $U_0 = B_\alpha(x_0) \subset \mathbb{R}^n$ and $V_0 = B_\beta(y_0) \subset \mathbb{R}^m$, there exists a unique continuous function $f : U_0 \to V_0$ satisfying:

- $f(x_0) = y_0$.
- For all $(x, y) \in U_0 \times V_0$, we have

$$F(x, y) = 0 \quad \text{if and only if} \quad y = f(x).$$

1.2.2 Proof of Riemann Mapping Theorem

Step 1: Existence of a bounded injective conformal map to the unit disk
Let Ω be a simply connected open proper subset of $\mathbb{C}$. We show that Ω is conformally equivalent to an open subset of the unit disk containing the origin. Indeed, choose $\alpha \notin \Omega$ and consider the function $f(z) = log(z-\alpha)$ on Ω, which is well-posed and holomorphic since $z-\alpha$ never vanishes on Ω. Note f is injective since $e^{f(z)} = z - \alpha$ ($f(z_1) = f(z_2) \implies z_1 - \alpha = z_2 - \alpha$). Then for a point $\omega \in \Omega$ we get $f(z) \neq f(\omega) + 2\pi i \forall z \in \Omega$ since otherwise we would find $z = \omega$ again by exponentiating. In fact, $f(z) \cap B_\epsilon(f(\omega) + 2\pi i) = \emptyset$ for some $\epsilon > 0$ since otherwise we would find a sequence $z_n \to \omega$ with $f(z_n) \to f(\omega) + 2\pi i$, contradicting the continuity of f. Finally, the function $g(z) = \frac{1}{f(z)-(f(\omega)+2\pi i)}$ is well-defined, holomorphic and injective on Ω and maps Ω to a bounded subset $g(\Omega) \subset \mathbb{C}$, so g is conformal. By boundedness of $g(z) < \frac{1}{\epsilon}$ we can scale and translate $g(\Omega)$ to contain the origin and fit into the unit disk.

Step 2:
By step 1 we can assume Ω to be an open subset of the unit disk with $0 \in \Omega$. Consider the family $\mathcal{F}$ of all injective holomorphic functions $f : \Omega \to \mathbb{D}$ with $f(0) = 0$. Note that $\mathcal{F} \neq \emptyset$ since it contains the identity, and it is a uniformly bounded family by construction (maps into unit disk). We now want to find a function $f \in \mathcal{F}$ that maximizes $|f'(0)|$. (WHY?) Observe that by the Cauchy inequality 1.6 $|f'(0)|$ are uniformly bounded for f in $\mathcal{F}$. Next, let

$$s := \sup_{f \in \mathcal{F}} |f'(0)|.$$

and choose a sequence $f_n \subset \mathcal{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$. By Montel's theorem, f_n has a subsequence converging uniformly on compacta to a holomorphic f on Ω. Since $s \geq 1$ (the identity is in $\mathcal{F}$), f is non-constant and by the proposition on uniform convergence to holomorphic limit and injectivity, f is injective. By continuity we have $|f(z) \leq 1$ for all $z \in \Omega$, and by the maximum modulus principle $|f(z)| < 1$. Finally and since $f(0) = 0$, we have $f \in \mathcal{F}$ and $|f'(0)| = s$.

Step 3: f is a conformal map from Ω to $\mathbb{D}$.
By step 1 we have injectivity, hence it suffices to show f is surjective. Suppose towards a contradiction that f is not surjective, we will construct a function in $\mathcal{F}$ with derivative greater than s at the origin. So let $\alpha \in \mathbb{D}$ be such that $\alpha \notin f(\Omega)$ and consider the automorphism of the unit disk that interchanges 0 and α,

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}.$$

Since Ω is simply connected and by continuity of f and $\psi_\alpha(\Omega)$, the set $U := (\psi_\alpha \circ f)(\Omega)$ is simply connected and does not contain the origin. Thus we can define a square root function on U by

$$q(w) = e^{\frac{1}{2} logw}.$$

Next, consider the function

$$F = \psi_{q(\alpha)} \circ q \circ \psi_\alpha \circ f.$$

Then $F \in \mathcal{F}$ since $F(0) = 0$ and F is holomorphic and injective since all the composing functions are. Also, F maps into the unit disk since all the composing functions do. But now if h denotes the square function $h(w) = w^2$, then we must have

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{q(\alpha)}^{-1} \circ F = \Psi \circ F.$$

But $\Psi : \mathbb{D} \to \mathbb{D}$ satisfies $\Psi(0) = 0$ and is not injective since F is but h is not. By the Schwarz lemma 1.3 we get $|\Psi'(0)| < 1$ and hence

$$|f'(0)| = |\Psi'(0)||F'(0)| < |F'(0)|,$$

contradicting maximality of $|f'(0)|$ in $\mathcal{F}$.

Finally, multiplying f with a suitable unimodular complex number gives the desired conformal map from Ω to $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$. □

Corollary 1.8
Any two simple connected open proper subsets of $\mathbb{C}$ are conformally equivalent.

Proof. This follows directly from the Riemann mapping theorem by taking the unit disk as an intermediate step. □

This completes the theory for why shape optimisation of two dimensional simply connected regions works in the first place. Here follows some more theory on how to actually construct such conformal maps numerically.

Existence established, the problem now becomes the explicit construction of a conformal mapping. A priori there are infinitely many conformal mappings from $\mathbb{D}$ to any simply connected bounded region Ω, which is why the introduction of boundary conditions is needed to get uniqueness of Φ and thus well-posedness of the problem. This results in the problem being translated into solving a Boundary Value Problem (BVP) for analytic functions.

SOMEWHERE WE NEED TO ADD THE RESTRICTION OF $\Phi(0) = 0, \quad \Phi'(0) > 0$ FOR LATER REFERENCE

1.3 Boundary Value Problems

In Riemann's thesis, conformal mappings are a mere special case of the more general family of boundary value problems of analytic functions on the unit disk D or in its exterior $D^- := \{z : |z| > 1\}$. The simplest such problem can be described as finding Ψ analytic in D, continuous in $\bar{D}$ and satisfying

$$Re(\Psi(e^{it})) = \psi(t)$$

on the boundary of D, where ψ is 2π-periodic and Hölder continuous This problem has a unique solution up to an imaginary constant, which can be constructed using the conjugation operator

$$K\Psi(s) := \frac{1}{2\pi}\int_0^{2\pi} \Psi(t)cot(\frac{s-t}{2})dt,$$

which is also known as Hilbert Transform. In order to use numerical methods efficiently via the Fast Fourier Transform def? we will prefer the functions' Fourier series representations. Hence on a grid of $N = 2n$ equidistant why? FFT requires it? has crowding as an effect... points $t_j = \frac{(j-1)2\pi}{N}$, Ψ will be written as

$$\Psi(t_j) = \sum_{k=-n+1}^{n} c_k e^{ikt_j} \text{ for } j \in [N]$$

and the conjugation operator can be approximated by the operator K_N defined as

$$K_N\Psi(t_j) = \sum_{k=1}^{n-1} -ic_k e^{ikt_j} + ic_{-k}e^{-ilt_j}.$$

The function K_N is thus defined as a trigonometric polynomial obtained by interpolating Ψ at the grid points. This approximation of the actual operator K satisfies

$$\|K\Psi - K_N\Psi\|_\infty \in O(n^{-\alpha+1/2})$$

for $\Psi \in C^\alpha$ Hölder continuous. If Ψ is smoother, e.g. $\Psi \in C^{k-1}$, the error decreases to

$$\|K\Psi - K_N\Psi\|_\infty \in O(n^{-k}log(n)\|\Psi^{(k)}\|_{infty})$$

[Weg05] p. 362. add more details on this operator?

1.4 Hilbert Transform [CR25]

probably way too much detail

1.5 Mixed boundary conditions (Wec 4.3)(?)

1.6 Hilbert theory and different inner products

From complex analysis we know that a complex differentiable function with non-vanishing complex derivative is conformal, and complex differentiability in two dimensions can be caracterised by the Cauchy-Riemann equations (necessary and sufficient condition). Denoting

$$B := \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix}$$

we want to find a deformation Φ such that $B\Phi = 0$.

As a byproduct of the Cauchy-Riemann equations, both components of a conformal map are harmonic functions, i.e. they satisfy Laplace's equation

$\Delta u = 0$. Thus, one way to construct conformal maps is to solve Laplace's equation with suitable boundary conditions. 1.3

The Cauchy-Riemann equations however do not guarantee a solution for arbitrary boundary data. A holomorphic map from the boundary of the unit disk onto some boundary of a convex set in $\mathbb{R}^2$ satisfies Cauchy-Riemann equations if the boundary of the target set can be described by non-negative Fourier frequencies [PROOF]. Thus, the choice of parametrization of the target region's boundary is another challenge posed when solving conformal mapping problems, along with finding a holomorphic deformation Φ.

In order to keep the approximation error as low as possible while maintaining good solver performance, we typically want uniform meshes, e.g. where the triangulation is close to equilateral. The quality of a mesh is measured in terms of its individual cells where for a mesh $\mathcal{M} := \{K\}$ of triangles K such that

$$\overline{\Omega} = \cup_{K\in\mathcal{M}} K$$

we define $d(K)$ the diameter of the smallest K-circumscribing ball (aka diameter of K) and $\mu(K)$ the diameter of the largest ball inscribed in K. Then a measure [Wec19] for the quality of K is the ratio of these diameters

$$\rho(K) := \frac{d(K)}{\mu(K)} \in [1, \infty).$$

1.6.1 Sobolev Spaces

Let L^2 be the space of all 2π-periodic complex functions f which are square integrable over $[0, 2\pi]$ equipped with the inner product

$$(f, g)_2 = \frac{1}{2\pi} Re \int_0^{2\pi} f(f)\overline{g(t)} dt.$$

Definition 7
The **Sobolev space** W is defined as the space of all absolutely continuous functions $f \in L^2$ such that the derivative f' exists and is also in L^2. The inner product on W is defined as

$$(f, g)_W = (f, g)_2 + (f', g')_2.$$

This is a Hilbert space over $\mathbb{R}$. The subspaces of real functions are denoted $L^2_{\mathbb{R}}$ and $W_{\mathbb{R}}$ respectively. Note that we can decompose W into the direct sum of the subspaces $W = W^+ \oplus W^-$ where $f \in L^2$ is decomposed as follows into its Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} = \underbrace{\sum_{-\infty}^{0} a_n e^{int} + i(\mathrm{Im}(a_1))e^{int}}_{=:f^-\in W^-} + \underbrace{(\mathrm{Re}(a_1))e^{int} + \sum_{n=2}^{\infty} a_n e^{int}}_{=:f^+\in W^+}.$$

Recall the definition of Φ as the mapping from the boundary of the disk VIA ETA and subject to 1.2.2. This can now be expressed as $\Phi \in W^+$HUH. Then, since the boundary function 2 maps the unit circle to Γ there exists why a function $\hat{u}$ such that

$$\Phi(t) = \eta(t + \hat{u}(t)) \quad \forall t.$$

By the implicit function theorem 1.7 $\hat{u}$ is continuously differentiable, hence $\hat{u} \in W_{\mathbb{R}}$. This tells us that the function Φ we are looking for lies in the intersection of a certain manifold $M := \{u \in W_{\mathbb{R}} : \eta(t + u(t))\}$ with our space W^+. This formulation is used in chapter 1.7.1.

1.7 Numerical Construction of Conformal Mappings

1.7.1 Alternating Projections à la von Neumann

For two closed convex sets P, Q in a Hilbert space H, the method of alternating projections constructs a sequence $(x_n)_n$ as follows: Starting from an arbitrary point $x_0 \in H$, we define

$$x_{n+1} := \begin{cases} \Pi_P(x_n) & n \equiv 0 (\text{mod } 2) \\ \Pi_Q(x_n) & n \equiv 1 (\text{mod } 2) \end{cases}$$

where $\Pi_P(z) = \min_{x \in P} \|x - z\|^2$ and $\Pi_Q(z) = \min_{x \in Q} \|x - z\|^2$ denote the orthogonal projections onto the sets P and Q respectively. It can be shown that the sequence $(x_n)_n$ converges in norm to a point tuple $(x*, y*)$ satisfying

$$\begin{cases} x* = \Pi_P(y*), \\ y* = \Pi_Q(x*), \\ d_H(x*, y*) = min_{(x,y) \in P \times Q} \|x - y\|^2 \end{cases}$$

In particular, $x* = y*$ if $P \cap Q \neq \emptyset$. [BPW23]

1.8 Riemann-Hilbert Problems

XXXXXXXXXXXXXX

1.9 Crowding

Wegmann [Weg05] proved the following result.

Theorem 1.9
When the region G can be enclosed in a rectangle with sides a and b, $b \leq a$, such that G touches both small sides then the conformal mapping $\phi : D \to G$ satisfies

$$\|\phi'\|_D \geq b\psi(b/a)$$

with a function $\psi(\tau)$ which behaves for small τ like

$$\psi(\tau) \approx \frac{1}{2\pi\sqrt{\epsilon}} \exp\left(\frac{\pi}{2\tau}\right).$$

Crowding is cumbersome for all methods which work with grid points. On the other hand, methods which approximate the mapping functions by polynomials also face severe problems when the target region is elongated. It follows that, for the mapping of the disk to a region of aspect ratio 1 910 by a polynomial, the degree must be of several millions. In any case, the number of grid points and the degree of the approximating polynomials increase both like exp(zr/2r) as the aspect ratio, r, tends to zero. DeLillo [37] has shown how crowding affects the accuracy of numerical computations. Crowding also limits the practical usefulness of conformal maps. This was demonstrated by DeLillo [36] for the Laplace equation. Crowding has also been observed for regions with elongated sections (”fingers”). For ”pinched” regions, such as the interior of an inverted ellipse, ill conditioning occurs of a less severe, algebraic nature (DeLillo [37]).

1.9.1 The operator R

In some conformal mapping methods, boundary value problems as follows occur,

$$\Psi(e^{it}) = B(t) + A(t)U(t)$$

where $A, B : \mathbb{C} \to \mathbb{C}$ and $U : \mathbb{C} \to \mathbb{R}$. Multiplication with $\bar{A}$ yields the RH problem

$$Im(\bar{A(t)}\Psi(e^{it})) = Im(\bar{A(t)}B(t))$$

This problem can be solved by the operator R_β, which is defined as follows:

This follows from the following theorem:

Theorem 1.10
There exists a function Ψ analytic in D with $\Psi(0) = 0$ satisfying the boundary problem if and only if U is a solution of the Fredholm integral equation of the second kind

$$(I + R_\beta)U = g$$

with the right-hand side

$$g := -Re(e^{-i\beta}(I - iK + J)/B).$$

Where K is the conjugation operator and J the averaging operator.

2 Existing Methods

This chapter aims to give an overview and compare existing methods in terms of input/output format, suitability/ boundary requirements, computational complexity, numerical stability and mesh quality preservation/ accuracy. When the region G is bounded by a closed curve Γ the mapping $\Phi : D \to G$ can be extended continuously to the closure $\overline{D}$ by Carathéodory's Extension Theorem 1.1. Then the boundary can be parametrized by a 2π-periodic function $\eta(s)$ in counterclockwise direction (positive orientation, aka *normal representation* [Weg05], p. 387) and the mapping Φ is determined by its boundary values

$$\Phi(e^{is}) = \eta(s)$$

for $s \in [0, 2\pi)$. By the implicit mapping theorem 1.7, the conformal mapping Φ depends on the boundary curve of G by the boundary correspondence equation 2.

a rh problem arises when considering the change of the conformal mapping under changes to the boundary curve. weg388

2.1 Potential Theoretic Methods

The accuracy of the conformal mapping depends to a large extent on the smoothness of the boundary curve γ.

2.1.1 Bergman Kernel Method

2.2 Alternating Projections Method

Various methods for numerical construction of Φ essentially construct two sequences of functions, one of normalized analytic functions on the disk (using the operator K 1.3) and one mapping the boundary of $\mathbb{D}$ to the boundary Γ. The method of alternating projections uses both these sequences and alternates between them to find Φ [Weg89].

2.2.1 Prerequisites

For this method we suppose $\Gamma = \partial\Omega$ is a smooth Jordan curve parametrized by a 2π-periodic function $\eta(s)$ with continuous non-vanishing derivative $\eta'(s) \neq 0$ having winding number 1 on $[0, 2\pi)$, i.e. Γ surrounds Ω exactly once. Furthermore, we utilize the spaces and notation from chapter 1.7.1.

2.2.2 Algorithm

2.2.3 Convergence

The AP Method converges linearly if the boundary parametrization is 3-Hölder and the initial approximation U_0 is sufficiently close to the actual boundary correspondence function η?????? [Weg89] p.292. Fourier calculation can be done

Algorithm 1 AP-Method

Start with a function $U_0 \in W_{\mathbb{R}}$.
Given U_k for $k \geq 0$,
for $n = 1, 0, -1, -2, ...$ **do**
  $a_n = \frac{1}{2\pi}\int_0^{2\pi} \eta(t + U_k(t))e^{-int}dt$ [Calculate Fourier coefficients]
end for
$U_{k+1}(t) := U_k(t) - \mathrm{Re}\frac{i(\mathrm{Im}(a_1))e^{it} + \sum_{n=-\infty}^{0} a_n e^{int}}{\dot{\eta}(t+U_k(t))}$ [Calculate the new iterate]

very efficiently using FFT which makes AP one of the simplest and most robust methods for conformal mapping. However, it is also not very accurate for reasonably sized grids, and converges very slowly for finer meshes [Weg05] p. 389.

2.3 Newton Methods

2.4 Interpolation

2.5 Theodorsen's Method

[Son12]

2.6 Schwarz-Christoffel Method

One class of methods for finding the conformal mapping Φ is given by the Schwarz-Christoffel equation, which relates the derivative of Φ to an integral over the boundary of the target domain Ω when Ω is a polygon.

2.6.1 Preliminaries and Notation

Definition 8
A **Polygon** is a planar figure whose boundary is made of a chain of connected line segments which we will call arcs, connecting corner points.

The unit disk $\mathbb{D}$ is in particular a polygon, and we parametrize the boundaries $\partial\mathbb{D}$ and $\partial\Omega = \Gamma$ by collections of arcs $s_{\mathbb{D}}$ and s_Ω respectively, in positive mathematical orientation. These arcs being smooth yields tangents with well-defined derivatives at every point of the boundary curves except for corners, and we denote the angles of these tangents with $\theta_{\mathbb{D}}(s_{\mathbb{D}})$ and $\theta_\Omega(s_\Omega)$ respectively. If z_0 is a corner it will have a turning angle of

$$\measuredangle\theta_{\mathbb{D}}(z_0) = \theta_{\mathbb{D}}(z_0 + \varepsilon) - \theta_{\mathbb{D}}(z_0 - \varepsilon),$$

where $\varepsilon \to 0$. The same applies to corners of Ω. Then, $\frac{\partial\theta_{\mathbb{D}}}{\partial s_{\mathbb{D}}} = \measuredangle\theta_{\mathbb{D}}(z_0)\delta(s_{\mathbb{D}} - z_0)$ for δ the Dirac function, and similarly for θ_Ω and thus $\theta_{\mathbb{D}}$ and θ_Ω are piecewise continuously differentiable functions with jump discontinuities at the corner points.

2.6.2 the log derivative of Phi

2.6.3 Green's Functions

A Green's function is a general concept for solving differential equations containing a linear operator.

Definition 9
A **Green's function** or **Green function** $G(x, s)$ is any solution to

$$LG(x, s) = \delta(x - s)$$

where $L = L(x)$ is a linear operator acting on distributions over $\mathbb{R}^n$ and δ is the Dirac delta function.

This definition can be exploited to solve inhomogeneous differential equations of the form $Lu = f(x)$. In the case of the SCE the Green's function are defined as $G(z, z') : \mathbb{D} \to \mathbb{R}$ satisfying

$$\nabla^2 G(z, z') = 2\pi\delta(u - u')\delta(v - v')$$

inside the disk and

$$\frac{\partial G(z_B, z')}{\partial n} = \beta_i$$

where n is the outward normal vector at the boundary point $z_B \in \partial\mathbb{D}$ and β_i is a real constant associated with the i'th arc's length l_i such that $\sum l_i\beta_i = 2\pi$. It can be shown according to Floryan and Zemach that this defines a unique Green's function up to an additive constant [FZ87]. The SCE can be written explicitly whenever the Green's function is obtainable in closed analytic form, i.e. expressable via a finite number of elementary operations (is that what is meant on p348? i looked up wikipedia for closed analytic form).

2.6.4 Schwarz-Christoffel Equation Variants

Let $\mathcal{G}(z, z'_B)$ be a complex extension of the above defined $G(z, z'_B)$, i.e. $\mathcal{G}(z, z'_B)$ is analytic with real part $G(z, z'_B)$. A Schwarz-Christoffel equation for a conformal mapping $\Phi(z) : \mathbb{D} \to \Omega$ has the form

$$log\frac{d\Phi}{dz} = C + \sum_i Q_i,$$

where $C \in \mathbb{C}$ is a constant and the Q_i are the Green's function integrals over the boundary arcs of $\mathbb{D}$. Some parameters and constants have to be determined in order to get a unique Φ for a particular given Ω. The Riemann mapping theorem allows HOW? for the first three real parameters to be preassigned, for example to three boundary points in the case of simply connected bounded Ω. The remaining degrees of freedom must be solved for using properties of the arcs $s_\Omega(s_z)$ and $s_\mathbb{D}(s_z)$.

The most general form is called Schwarz-Christoffel equation with subtraction [FZ87]:

$$log(\frac{d\Phi}{dz}) = C_0 - \frac{1}{2\pi}\int_{\partial\mathbb{D}}[\mathcal{G}(z, z'_B) - \mathcal{G}(z_0, z'_B)] \times [d\theta_\Omega(s'_z) - d\theta_{\partial\mathbb{D}}(s'_z)]$$

where $C_0 \in \mathbb{C}$ is a constant, z_0 is a fixed point in $\mathbb{D}$or del?, $\mathcal{G}(z, z'_B)$ is the fundamental solution of the Laplace equation in $\partial\mathbb{D}$ with singularity at $z'_B \in \partial\mathbb{D}$not at the boundary of omega?. However, we can use an unsubtracted form of this equation since the integrals converge separately and we know our Ω is bounded (hence we need not care about behaviours at infinity), yielding the simpler form

$$log(\frac{d\Phi}{dz}) = C - \frac{1}{2\pi}\int_{\partial\mathbb{D}}\mathcal{G}(z, z'_B)[d\theta_\Omega(s'_z) - d\theta_{\partial\mathbb{D}}(s'_z)]$$

2.6.5 Convergence

[BT03]

2.6.6 Implementation

[Bro90] [Tre80] Crowding is a problem because it yields exponential derivative of Φ, but can be fixed. [Ban08]

2.7 Zipper Method

This algorithm was fonud independently by Kühnau and Marshall in the 1980's has the advantage of finding Φ and its inverse at the same time. The computed map is only approximately conformal, and is obtained as a composition of conformal maps onto slit halfplanes. Depending on the shape of the slits, the Zipper algorithm looks a bit different. In this section we will focus on the easiest version called the "geodesic algorithm" [MR06] which is quite beautiful from a geometric perspective.

2.7.1 The Geodesic Algorithm

The most elementary version of this algorithm is based on a function

$$f_a : \mathbb{H} \setminus \gamma \to \mathbb{H}$$

where $\mathbb{H}$ is the upper half plane and γ is a circular arc from 0 to $a \in \mathbb{H}$ which is orthogonal to the real axis. The orthogonal circle also meets the real axis again at $b = |a^2|/Im(a)$. Then the map can be expressed in closed form as

$$f_a(z) = \sqrt{g_a \circ h_a(z)}$$

where $g_a(z) = z^2 + c^2$ and $h_a(z) = \frac{z}{1-z/b}$.

INSERT IMAGE

Now suppose $z_0, z_1, ..., z_n$ are points arranged counterclockwise on a Jordan curve Γ in the upper half plane. The geodesic algorithm basically iterates over the arcs from z_i to z_{i+1} and "unzips" them one by one using the map f_{a_i} where a_i is the image of z_{i+1} under the composition of all previous maps. The original geodesic algorithm proposed by Marshall and Rohde constructs a conformal map from the upper half plane to the region bounded by Γ, but it can be adapted to map from the unit disk as well via a Möbius transformation mapping the half plane to the unit disk and back first (hopefully?).

Algorithm 2 Geodesic Zipper Algorithm

Input: Points $z_0, z_1, ..., z_n$ on a Jordan curve Γ in the upper half plane.
Output: Ψ: conformal map from $\mathbb{H}$ to the region bounded by Γ and its inverse Ψ^{-1}.
$\phi_1(z) := i\sqrt{(z - z_1)/(z - z_0)}$
$\zeta_2 := \phi_1(z_2)$
$\phi_2(z) := f_{\zeta_2}(z)$
for k in n **do**
  $\zeta_k := \phi_{k-1} \circ \ldots \circ \phi_1(z_k)$
  $\phi_k(z) := f_{\zeta_k}(z)$
end for
Finally, $\zeta_{n+1} := \phi_n \circ \ldots \circ \phi_1(z_0) \in \mathbb{R}$ and $\phi_{n+1}(z) := -(\frac{z}{1-z/\zeta_{n+1}})^2$
Then $\Psi(z) := \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n+1}^{-1}(z)$ and $\Psi^{-1}(z) := \phi_{n+1} \circ \ldots \circ \phi_2 \circ \phi_1(z)$

INSERT IMAGE

2.7.2 The Slit Algorithm

The above geodesic algorithm is only as accurate as the approximation of the boundary curve Γ by circular arcs between the points z_i. A more accurate version is given by the slit algorithm, which uses straight line segments instead of circular arcs. We therefore exchange the map f_a for a map $g_a : \mathbb{H} \setminus L \to \mathbb{H}$ where L is the line segment from 0 to $a \in \mathbb{H}$. This map does not have a closed form expression, but can be computed numerically using Newton's method.

2.7.3 The Zipper Algorithm

The approximation of Γ by circular arcs or straight line segments can be further improved by using circular arcs which meet Γ tangentially at the points z_i. Each arc is determined by the points z_i, z_{i+1} and z_{i+2}, hence we assume an even number of boundary points. The first arc is replaced by

$$\phi_1(z) = \sqrt{\frac{(z - z_2)(z_1 - z_2)}{(z - z_0)(z_1 - z_2)}}.$$

At each subsequent step that circular arc through ζ_k and ζ_{k+1} is mapped onto a straight line segment by a Möbius transform, and then the Slit Algorithm is applied to unzip that segment. This yields a sort of "quadratic approximation" of $\partial\Omega$ instead of a linear one.

2.7.4 Convergence

Marshall and Rohde [MR06] proved that if $\Gamma = \partial\Omega$ is a $C^{3/2}$ closed Jordan curve with points $\{z_i\} \subset \partial\Omega$ having mesh size $\mu = max|z_j - z_{j+1}|$ there is a constant C depending on the geometry of $\partial\Omega$ such that the conformal map Ψ satisfies accuracy of $\mathcal{O}(C\mu^{3/2})$. The geodesic algorithm works only with elementary functions, as a result its speed depends only on the number of points on the boundary (i.e. mesh size) and not on the shape of the region. The accuracy can be measured explicitly if the true conformal map is known, which is our case. ADD COMPUTATION AND IMAGES LIKE IN MnR'S PAPER ¡3 VERY GOOD PAPER FOR IMPLEMENTATION ESPECIALLY END NOTES

2.8 Shirokova's Method

[Shi14]

2.9 Conjugate Function Method

[HQR13]

2.10 Amano's Method

[Sak19]

2.11 Probabilistic Uniformization Method

In 2007, Binder, Braverman ans Yampolsky proposed the random walks solution to the general Dirichlet problem to produce a solution to the uniformization problem, yielding an algorithm with precision 2^{-n} pixels running in linear space (for explicitly given $\partial\Omega$; quadratic if $\partial\Omega$ is given only approximately, via a so-called *oracle*, sort of a Dirac delta function) and 2^n time. [BBY07]

2.12 Comparison

Wegmann [Weg05] section 2.5 proved that accuracy of methods based on function conjugation is bounded by the error of the operator K_N of the conjugation operator on the grid 1.3. Theodorsen's method is known [Gai64] add page/argument to have an error of order $O(R^{-n})$ for regions bounded by analytic curves, where $R > 1$ is the index of the largest disk centered at the origin to which the boundary parametrization η can be extended as a conformal map, and n is the order of the polynomials approximating η. Wegmann compared the accuracies of the AP, OAP, Theodorsen and Wegmann methods for the mapping from the disk to an inverted ellipse example 4? and found that OAP is most efficient for low accuracy and Newton methods are best for high accuracy calculations [Weg05] p.415.add figure? Note the computational costs are mainly determined by the FFTs and this parameter is dependent on the number of grid points.

Alternative approaches to numerical conformal mapping are enumerated in Henrici's book [8].

3 Proposed Method

3.1 Choice/ Justification

criteria: - Accuracy for domains with sharp corners or high curvature - Speed for practical mesh sizes - Robustness - does it fail for certain domain shapes? - Implementation complexity given your timeline - Jacobian computation - analytical vs numerical differentiation

3.2 Implementation

- Separate modules for boundary parameterization, mapping computation, Jacobian eval, and mesh transformation - plot original vs. mapped grids (e.g., Matplotlib quiver for Jacobians) to spot issues early.

3.3 Numerical Experiments/ Testing

check angle preservation (e.g., via dot products on mapped vectors) and scale factors ($det(D\Phi) > 0$, $|\frac{\partial \Phi}{\partial z}|$ constant in theory). - Test suite: Use known exact mappings (e.g., disk to square via Schwarz-Christoffel) for error metrics (L2 norm on boundary points). - Metrics: Runtime for N points, mesh quality post-mapping (e.g., min/max angles in triangles, shape regularity ratio). - Real-world applicability: Apply to a sample FEM problem (e.g., Poisson equation on Ω) and compare accuracy/speed vs. uniform mesh. - Robustness: Vary boundary complexity (smooth vs. corners), noise in Fourier coeffs, mesh resolutions. - Debugging: Use assertions for bijectivity (e.g., check injectivity numerically) - Error handling - what happens with degenerate inputs?

3.4 Results

4 Conclusion

References

[Ban08] L. Banjai. Revisiting the crowding phenomenon in schwarz–christoffel mapping. *SIAM Journal on Scientific Computing*, 30(2):618–636, 2008.

[BBY07] Ilia Binder, Mark Braverman, and Michael Yampolsky. On computational complexity of riemann mapping, 2007.

[BPW23] Gábor Braun, Sebastian Pokutta, and Robert Weismantel. Alternating linear minimization: Revisiting von neumann's alternating projections, 2023.

[Bro90] Philip Raymond Brown. *The use of the Schwarz-Christoffel transformation in finite element mesh generation.* Phd, University of Leicester, 1990.

[BT03] Lehel Banjai and L. N. Trefethen. A multipole method for schwarz–christoffel mapping of polygons with thousands of sides. *SIAM Journal on Scientific Computing*, 25(3):1042–1065, 2003.

[CR25] René E. Castillo and Edixon M. Rojas. Revisiting the hilbert transform of periodic functions. *Mathematische Semesterberichte*, 72:29–49, 2025.

[EW22] Manfred Einsiedler and Andreas Wieser. Analysis i und ii. Vorlesungsskript 2021/2022, 2022.

[FZ87] J.M Floryan and Charles Zemach. Schwarz-christoffel mappings: A general approach. *Journal of Computational Physics*, 72(2):347–371, 1987.

[Gai64] Dieter Gaier. *Konstruktive Methoden der konformen Abbildung*, volume 3 of *Springer Tracts in Natural Philosophy / Ergebnisse der angewandten Mathematik*. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1964.

[HQR13] Harri Hakula, Tri Quach, and Antti Rasila. Conjugate function method for numerical conformal mappings. *Journal of Computational and Applied Mathematics*, 237(1):340–353, January 2013.

[MR06] Donald E. Marshall and Steffen Rohde. Convergence of the zipper algorithm for conformal mapping, 2006.

[Sak19] Koya Sakakibara. Bidirectional numerical conformal mapping based on the dipole simulation method, 2019.

[Shi14] E. A. Shirokova. On the approximate conformal mapping of the unit disk on a simply connected domain. *Russian Mathematics*, 58(3):47–56, 2014.

[Son12] Eun-Jee Song. A study on the methods for solving the theodorsen equation for numerical conformal mapping. *Journal of the Korea Society of Computer and Information*, 17(3):11–18, March 2012.

[SS03] Elias M. Stein and Rami. Shakarchi. *Complex analysis*. Princeton lectures in analysis ; 2. Princeton University Press, Princeton, N.J, 2003.

[Tre80] Lloyd N. Trefethen. Numerical computation of the schwarz–christoffel transformation. *SIAM Journal on Scientific and Statistical Computing*, 1(1):82–102, 1980.

[Wec19] F Wechsung. *Shape optimisation and robust solvers for incompressible flow*. PhD thesis, University of Oxford, 2019.

[Weg89] Rudolf Wegmann. Conformal mapping by the method of alternating projections. *Numerische Mathematik*, 56:291–307, 1989.

[Weg05] Rudolf Wegmann. Chapter 9 - methods for numerical conformal mapping. In R. Kühnau, editor, *Geometric Function Theory*, volume 2 of *Handbook of Complex Analysis*, pages 351–477. North-Holland, 2005.