# COMPSCI-ECON206 Problem Set 1

Ai Zhou

September 2025

## 1 Part 1

## Subgame Perfect Nash Equilibrium

### Definition (Paraphrase)

Let an extensive-form game with perfect information be denoted by

$$\Gamma = \langle N, H, P, (A(h))_{h \in H}, (u_i)_{i \in N} \rangle,$$

where

- $N = \{1, \ldots, n\}$ is the finite set of players.
- $H$ is the set of histories (nodes), including the empty history $\emptyset$, with terminal histories $Z \subseteq H$.
- $P : H \setminus Z \to N$ assigns a player to each nonterminal history.
- $A(h)$ is the finite set of actions available after history $h$.
- Each player $i \in N$ has a payoff function $u_i : Z \to R$.
- A **strategy** $s_i$ for player $i$ is a complete contingent plan assigning an action in $A(h)$ for every history $h$ with $P(h) = i$.
- A **strategy profile** is $s = (s_1, \ldots, s_n)$.

A **subgame** of $\Gamma$ is defined as the restriction of $\Gamma$ to any history $h$ that constitutes a decision node not cutting across information sets.

[Subgame Perfect Nash Equilibrium] [pp. 93–95] Shoham2009, Rubinstein1994

A strategy profile $s^*$ is a *Subgame Perfect Nash Equilibrium (SPNE)* if and only if, for every subgame $\Gamma(h)$ of $\Gamma$, the restriction $s^*|_h$ induces a Nash equilibrium of that subgame:

$$u_i(s^*|_h) \;\geq\; u_i(s_i, s^*_{-i}|_h), \quad \forall i \in N,\, \forall s_i \in S_i(h),\, \forall h \in H.$$

That is, no player has a profitable deviation in any subgame, not only in the original game.

### Existence Theorem (Paraphrased)

For any finite extensive-form game with perfect information, there exists at least one strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ such that $s^*$ forms a subgame perfect Nash equilibrium. Formally,

$$\exists s^* \in S_1 \times \ldots \times S_n such that s^*|_h is a Nash equilibrium for every subgame \Gamma(h),\, \forall h \in H.$$

## Proof Idea

The proof uses **backward induction**:

1. Start from the terminal nodes and determine each player's optimal action at the last decision nodes.
2. Replace each subgame by the payoff vector resulting from optimal play.
3. Recursively move backward in the tree, selecting actions that maximize the player's payoff at each node.
4. The resulting strategy profile is optimal in every subgame and thus forms a SPNE.

Intuition: The finiteness of the game tree guarantees that this backward-induction construction produces at least one SPNE in pure strategies.

## Analytical Solution and Interpretation

### Analytical Solution

In this extensive-form game with perfect information, the Subgame Perfect Nash Equilibrium (SPNE) identifies strategies where each player optimally responds in every possible subgame. Conceptually, SPNE refines the standard Nash equilibrium by requiring that no player has an incentive to deviate, even after unexpected moves by others. We determine the equilibrium using backward induction: starting from the terminal nodes, each player chooses the action that maximizes their payoff given subsequent optimal decisions. This process is repeated recursively until reaching the initial node, producing a complete strategy profile that specifies an action for every decision point. For example, player 1's SPNE strategy can be written as $s_1^* = (a_1, a_2, \ldots)$, while player 2's is $s_2^* = (b_1, b_2, \ldots)$. This approach guarantees consistency across all subgames and provides a clear prediction for rational play.

The SPNE outcome is not necessarily socially optimal. From a Pareto perspective, some SPNE strategies may leave all players worse off than alternative cooperative outcomes. Similarly, total welfare (the sum of players' payoffs) might not be maximized, indicating suboptimal utilitarian efficiency. Sequential play can also generate unequal outcomes, raising fairness concerns: one player may consistently earn more than others, violating equity or proportionality principles. Nonetheless, SPNE serves as a normative benchmark: it shows what rational players would do if they perfectly anticipate others' behavior, even if the outcome is not efficient or fair.

### Interpretation

In practice, the Subgame Perfect Nash Equilibrium (SPNE) provides a strong prediction for how fully rational players with perfect information would behave in every subgame of a sequential interaction. However, its realism is limited because actual human decision-makers often face cognitive constraints and incomplete information, which may lead to deviations from SPNE predictions. Furthermore, many extensive-form games admit multiple SPNE, raising the question of equilibrium selection; observed behavior may depend on social norms, expectations, or pre-play communication. Refinements such as trembling-hand perfect equilibrium help address implausible strategies by eliminating those that rely on extremely unlikely mistakes. From a computational perspective, backward induction guarantees SPNE existence in finite games, but as the game tree grows in size or complexity, calculating SPNE becomes increasingly demanding, highlighting the practical relevance of algorithmic tools like GTE or NashPy. Thus, while SPNE offers a normative benchmark, its predictive accuracy is influenced by both bounded rationality and computational tractability.