Further Remarks on the Dual Negation in Team Semantics

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The 4th Tsinghua Interdisciplinary Workshop on Logic, Language and Meaning

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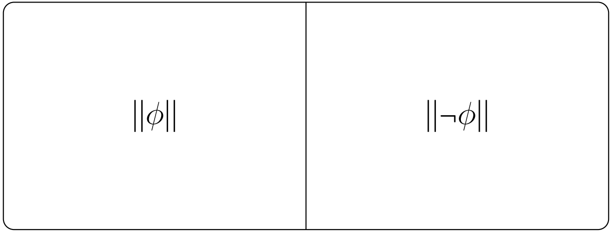

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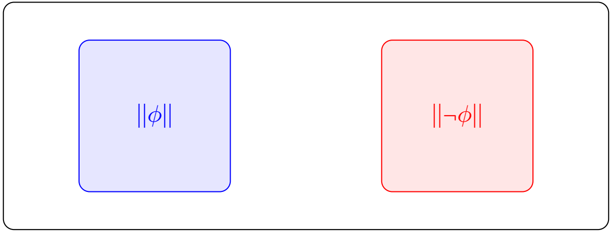

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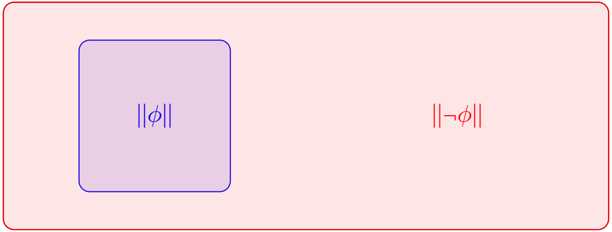

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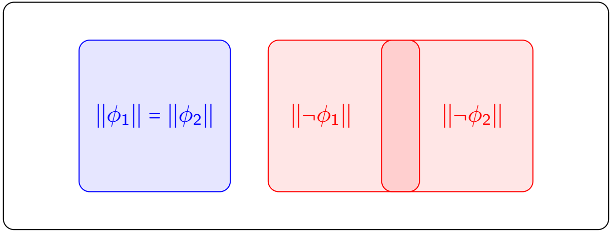

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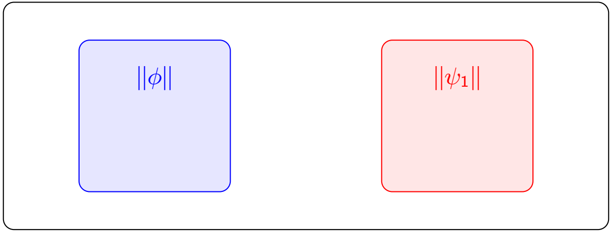

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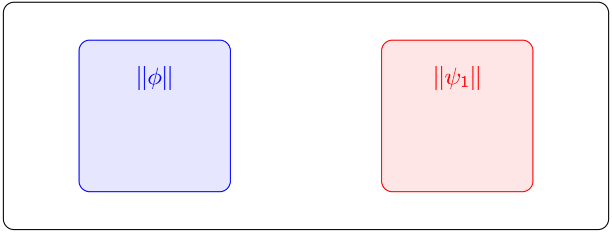

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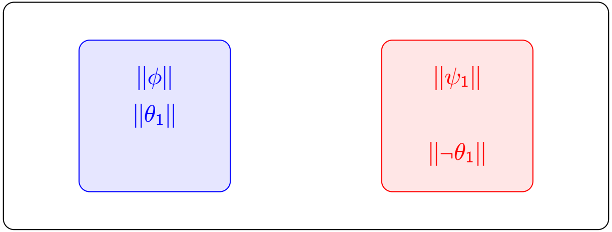

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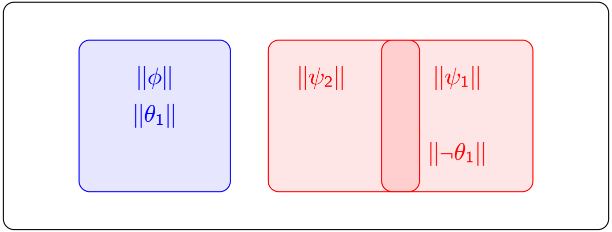

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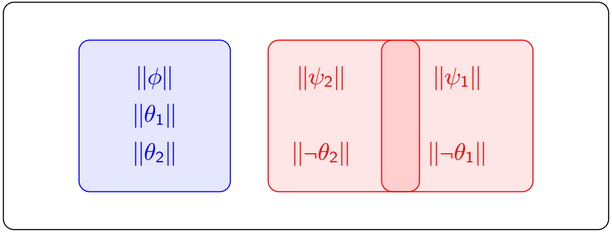

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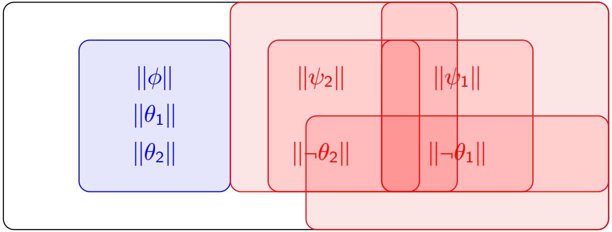

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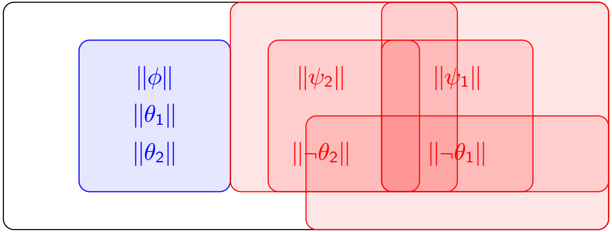

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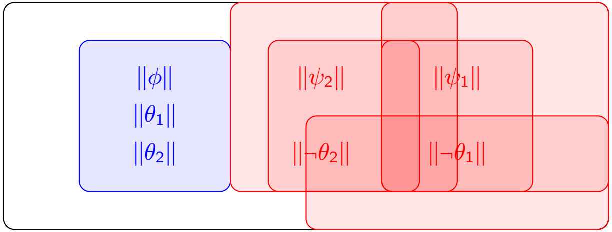

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E.g., $\phi \equiv$ "Jialiang is in Beijing" and $\neg\phi \equiv$ "Jialiang is in Amsterdam drinking coffee."

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3. Remarks on Burgess' interpretation of his remark

Dependence Logic

Henkin's branching quantifiers (1961), Hintikka and Sandu's independence-friendly logic (IF logic) (1989), and Väänänen's (2007) dependence logic each extend first-order logic (FO) with some means express dependence relations between variables which are not expressible in FO.

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Dependence logic (D) extends FO with dependence atoms $=(t_1, \ldots, t_n, t)$.

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$$\forall x_1 \exists x_2 \forall x_3 \exists x_4 (=(x_3, x_4) \wedge \phi),$$

x_4 depends only on x_3 (and not on x_1).

Team Semantics

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A (first-order) team t is a set of assignments.

	x	y	z
v_1	a	b	c
v_2	a	b	d
v_3	b	a	c
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In $t = \{v_1, v_2, v_3, v_4\}$, the value of x determines the value of y, but it does not determine the value of z. We have $t \models =\!(x, y)$ and $t \not\models =\!(x, z)$.

The Dual Negation

The dual negation $\neg$ arises naturally in the game-theoretic semantics for dependence logic (D): the rule corresponding to $\neg$ is simply the standard one whereby the players of the semantic game switch their verifier/falsifier roles (and indeed, the negation coincides with the first-order negation in the dependence atom-free fragment of D).

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In the team semantics for D, we let

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\mathcal{M} \models_X \neg\alpha \quad & :\Longleftrightarrow \quad \forall s \in X : \mathcal{M} \not\models_s \alpha \text{ for any first-order atom } \alpha \\
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The semantics for more complex negated formulas are determined by the equivalences:

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\neg\neg\phi &\equiv \phi \\
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\neg\exists x\phi &\equiv \forall x\neg\phi \\
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\neg\exists x\phi &\equiv \forall x\neg\phi \\
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$||\phi||$ fails to determine $||\neg\phi||$ because replacement of equivalents does not hold under negation: $\neg =\!(x, y) \equiv \bot$ but $\neg\neg =\!(x, y) \equiv =\!(x, y) \not\equiv \top \equiv \neg\bot$. I.e., $||\neg =\!(x, y)|| = ||\bot||$ but $||\neg\neg =\!(x, y)|| \neq ||\neg\bot||$.

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We assume $|dom(\mathcal{M})| \geq 2$ for all models $\mathcal{M}$.

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($\mathcal{M} \models \forall y =\!(y) \iff$ the value of y is constant in the team $\{(y, a) \mid a \in dom(\mathcal{M})\}$, which is false if $|dom(\mathcal{M})| \geq 2$.)

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($\mathcal{M} \models \neg\forall y = (y) \iff \mathcal{M} \models \exists y \neg = (y) \iff \exists a \in dom(\mathcal{M})$ s.t. some team $\{(y, a)\}$ is empty.)

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$$\begin{array}{ccccccccc} \phi_0 & \equiv & \phi \vee \theta_0 & \equiv & \phi \vee \bot & \equiv & \phi & & \\ \neg\phi_0 & \equiv & \neg(\phi \vee \theta_0) & \equiv & \neg\phi \wedge \neg\theta_0 & \equiv & \neg\phi \wedge \bot & \equiv & \bot \end{array}$$

Similarly $\psi_0 \equiv \psi$ and $\neg\psi_0 \equiv \bot$.

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Similarly $\psi_0 \equiv \psi$ and $\neg\psi_0 \equiv \perp$. By Separation let η be s.t. $\phi_0 \models \eta$ and $\psi_0 \models \neg\eta$.

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1. $||\phi||$ and $||\psi||$ are disjoint (i.e., $\mathcal{M} \models \phi$ iff $\mathcal{M} \not\models \psi$; i.e., $\phi, \psi \models \perp$).
2. There is a sentence θ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.

$1 \implies 2$: Let $\phi_0 := \phi \vee \theta_0$ and $\psi_0 := \psi \vee \theta_0$ with θ_0 from Lemma. Then:

$$\begin{array}{ccccccccc}
\phi_0 & \equiv & \phi \vee \theta_0 & \equiv & \phi \vee \perp & \equiv & \phi & & \\
\neg\phi_0 & \equiv & \neg(\phi \vee \theta_0) & \equiv & \neg\phi \wedge \neg\theta_0 & \equiv & \neg\phi \wedge \perp & \equiv & \perp
\end{array}$$

Similarly $\psi_0 \equiv \psi$ and $\neg\psi_0 \equiv \perp$. By Separation let η be s.t. $\phi_0 \models \eta$ and $\psi_0 \models \neg\eta$.
Let $\theta := \phi_0 \wedge (\neg\psi_0 \vee \eta)$.

Lemma: There is θ_0 s.t. $\theta_0 \equiv \bot \equiv \neg\theta_0$.
Separation Theorem: If $||\phi||$ and $||\psi||$ are disjoint, there is an η s.t. $\phi \models \eta$ and $\psi \models \neg\eta$.

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Let $\theta := \phi_0 \wedge (\neg\psi_0 \vee \eta)$. Then:

$$\begin{array}{lllllll} \theta & \equiv \phi_0 \wedge (\neg\psi_0 \vee \eta) & \equiv \phi_0 \wedge (\bot \vee \eta) & \equiv \phi_0 \wedge \eta & \equiv \phi_0 & \equiv \phi & \\ \neg\theta & \equiv \neg(\phi_0 \wedge (\neg\psi_0 \vee \eta)) & \equiv \neg\phi_0 \vee \neg(\neg\psi_0 \vee \eta) & \equiv \bot \vee (\neg\neg\psi_0 \wedge \neg\eta) & \equiv \psi_0 \wedge \neg\eta & \equiv \psi_0 & \equiv \psi \end{array}$$

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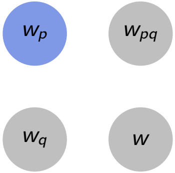
Kontinen and Väänänen (2011) later showed a similar result also holds for all formulas (not just sentences).

Modal team semantics

In modal team semantics, teams are sets of possible worlds:

single-world semantics

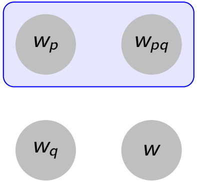
$M, w \models \phi$



$w_p \models p$

modal team semantics

$M, s \models \phi$



$\{w_p, w_{pq}\} \models p$

Bilateral State-Based Modal Logic

Aloni (2022) introduces a modal logic employing team semantics, Bilateral State-based Modal Logic to account for free choice inferences and related phenomena.

Free choice:	You may have coffee or tea.
	$\rightsquigarrow$ You may have coffee and you may have tea.

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Teams represent speakers' information states. BSML has a bilateral semantics with two primitive semantic relations, support $\models$ and anti-support $\dashv$. As with many other bilateral system, the bilateralism in BSML is linked with the view that both assertion and rejection conditions must figure in meanings:

$s \models \phi$	represents	"ϕ is assertable in state s"
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$$s \models \neg\phi \qquad \text{iff} \qquad s \dashv \phi$$

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The negation is designed to ensure one also gets predictions such as the following:

Dual prohibition:	You are not allowed to eat the cake or the ice cream.
	$\rightsquigarrow$ You are not allowed to eat either one.

Syntax and semantics:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Diamond \phi \mid \mathrm{NE}$$

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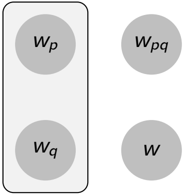
$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Diamond \phi \mid \mathrm{NE}$$

$$\begin{array}{lcl}
s \models p & \iff & \forall w \in s : w \in V(p) \\
s \dashv p & \iff & \forall w \in s : w \notin V(p) \\
 & & \\
s \models \neg\phi & \iff & s \dashv \phi \\
s \dashv \neg\phi & \iff & s \models \phi \\
 & & \\
s \models \phi \vee \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \models \phi \text{ and } t' \models \psi \\
s \dashv \phi \vee \psi & \iff & s \dashv \phi \text{ and } s \dashv \psi \\
 & & \\
s \models \phi \wedge \psi & \iff & s \models \phi \text{ and } s \models \psi \\
s \dashv \phi \wedge \psi & \iff & \exists t, t' : t \cup t' = s \text{ and } t \dashv \phi \text{ and } t' \dashv \psi \\
 & & \\
s \models \mathrm{NE} & \iff & s \neq \varnothing \\
s \dashv \mathrm{NE} & \iff & s = \varnothing \\
 & & \\
s \models \Diamond\phi & \iff & \forall w \in s : \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \models \phi \\
s \dashv \Diamond\phi & \iff & \forall w \in s : R[w] \dashv \phi
\end{array}$$

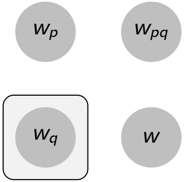
$$R[w] = \{v \mid wRv\}$$

Tensor disjunction $\vee$

$$
\begin{array}{llll}
s \models \phi \vee \psi & \iff & \exists t, t' : & t \cup t' = s \quad \text{and} \\
& & & t \models \phi \quad \text{and} \\
& & & t' \models \psi \\
s \mathrel{=\!\!\mid} \phi \vee \psi & \iff & s \mathrel{=\!\!\mid} \phi \quad \text{and} & s \mathrel{=\!\!\mid} \psi
\end{array}
$$



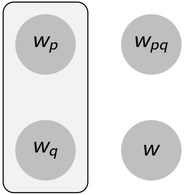
(a) $s \models p \vee q$



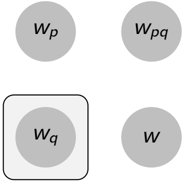
(b) $s \models p \vee q$

The non-emptiness atom NE

$$s \models \mathrm{NE} \iff s \neq \varnothing$$
$$s \mathrel{=\!\mid} \mathrm{NE} \iff s = \varnothing$$



(a) $s \models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$



(b) $s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

The non-emptiness atom NE

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\begin{array}{lcl}
s \models \mathrm{NE} & \iff & s \neq \varnothing \\
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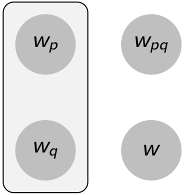

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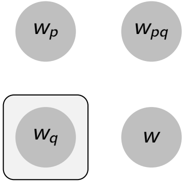

(b) $s \not\models (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$

Dependence logic has the following properties:

- Downwards closure property: $[s \models \phi \text{ and } t \subseteq s] \implies t \models \phi$
- Empty team property: $\varnothing \models \phi$ for all ϕ.

Due to NE, BSML clearly lacks both of these properties.

The following equivalences hold for the negation (where $\Box := \neg \Diamond \neg$):

$$\neg\neg\phi \equiv \phi \qquad \neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$$

$$\neg\mathrm{NE} \equiv p \wedge \neg p \qquad \neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$$

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We define the following abbreviations:

Weak contradiction $\bot := p \wedge \neg p$. $s \models \bot$ iff $s = \varnothing$.

Strong contradiction $⫫ := \bot \wedge \mathrm{NE}$. $s \models ⫫$ is never the case.

(Strong) tautology $\top := p \vee \neg p$. $s \models \top$ is always the case.

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As with dependence logic, replacement of equivalents does not hold under negation:
$\neg\mathrm{NE} \equiv \bot$ but $\neg\neg\mathrm{NE} \equiv \mathrm{NE} \not\equiv \top \equiv \neg\bot$.

Adapting Burgess' remark for BSML

The negation result for $\mathcal{D}$ employs the following notion of contradictoriness (reformulated for the modal setting):

$$\begin{array}{lcl} \phi \text{ and } \psi \text{ are } \textbf{contradictory}: & & \phi, \psi \models \bot \\ & \iff & t \models \phi \text{ and } t \models \psi \text{ implies } t = \varnothing \end{array}$$

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This does not work with BSML.
Take $\phi := (p \land \mathrm{NE})$ and $\psi := ((p \land \mathrm{NE}) \lor (\neg p \land \mathrm{NE}))$.

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The negation result for $\mathcal{D}$ employs the following notion of contradictoriness (reformulated for the modal setting):

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This does not work with BSML.
Take $\phi := (p \wedge \mathrm{NE})$ and $\psi := ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$.

Then $\phi, \psi \vDash \perp\!\!\!\perp \vDash \bot$.

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But one can show that there can be no θ s.t. $\theta \equiv \phi$ and $\neg\theta \equiv \psi$, so the negation result cannot hold with this notion of contradictoriness.

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Lemma: If $s \vDash \eta$ and $t \vDash \neg\eta$, then $s \cap t = \varnothing$.

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The negation result for $\mathcal{D}$ employs the following notion of contradictoriness (reformulated for the modal setting):

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$\{w_p\} \models \phi$ so $\{w_p\} \models \theta$.

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Consider the teams $\{w_p\}$ and $\{w_p, w_{\neg p}\}$.
$\{w_p\} \vDash \phi$ so $\{w_p\} \vDash \theta$.
$\{w_p, w_{\neg p}\} \vDash \psi$ so $\{w_p, w_{\neg p}\} \vDash \neg\theta$.
Therefore, by the lemma, $\{w_p\} \cap \{w_p, w_{\neg p}\} = \{w_p\} = \varnothing$, a contradiction.

When downwards closure and the empty team property fail, we can distinguish between multiple notions of contradictoriness.

Let $||\phi|| := \{(M, s) \mid M, s \models \phi\}$.

1. ϕ and ψ are weakly contradictory: $||\phi|| \cap ||\psi|| = \{(M, \varnothing) \mid M \text{ is a model}\}$
   (Abusing notation, $||\phi|| \cap ||\psi|| = \{\varnothing\}$.)

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4. ϕ and ψ are incompatible: $s \vDash \phi$ and $t \vDash \phi$ implies $s \cap t = \varnothing$
   (equivalently, and abusing notation, $\bigcup ||\phi|| \cap \bigcup ||\psi|| = \varnothing$)

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Let $||\phi|| := \{(M, s) \mid M, s \vDash \phi\}$.

1. ϕ and ψ are weakly contradictory: $||\phi|| \cap ||\psi|| = \{(M, \varnothing) \mid M$ is a model$\}$ (Abusing notation, $||\phi|| \cap ||\psi|| = \{\varnothing\}$.)
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For downward-closed formulas with the empty team property:
$1 \iff 3 \iff 4 \iff 5$, and 2 never holds.

Whereas in general: $5 \iff 4 \implies 3 \iff (1 \text{ or } 2)$; and none of $1, 2, 3$ imply 4.

Lemma: There is θ_0 s.t. $\theta_0 \equiv \bot \equiv \neg\theta_0$.
Let $\theta_0 := \Diamond(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp)$. Then:

$$
\begin{array}{ccccccccc}
 & & & & \Diamond(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp) & \equiv & \Diamond \perp\!\!\!\perp & \equiv & \bot \\
\neg \Diamond(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp) & \equiv & \Box \neg(\perp\!\!\!\perp \vee \neg \perp\!\!\!\perp) & \equiv & \Box(\neg \perp\!\!\!\perp \wedge \perp\!\!\!\perp) & \equiv & \Box \perp\!\!\!\perp & \equiv & \bot
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Separation Theorem: If $\bigcup ||\phi|| \cap \bigcup ||\psi|| = \varnothing$, then there is an η s.t. $\phi \models \eta$ and $\psi \models \neg\eta$.

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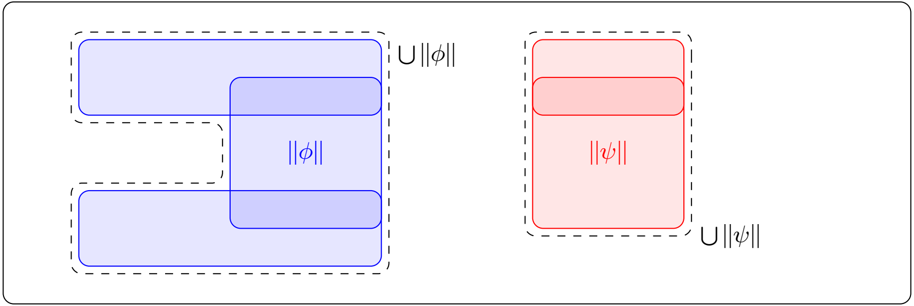
Separation Theorem: If $\bigcup\|\phi\| \cap \bigcup\|\psi\| = \varnothing$, then there is an η s.t. $\phi \vDash \eta$ and $\psi \vDash \neg\eta$.

Negation result for BSML

The following are equivalent:

1. $\bigcup\|\phi\| \cap \bigcup\|\psi\| = \varnothing$ (i.e., $[s \vDash \phi$ and $t \vDash \psi] \implies s \cap t = \varnothing$)
2. There is a θ such that $\phi \equiv \theta$ and $\psi \equiv \neg\theta$.



Burgess' assessment of his theorem:

> *In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety [dual negation], the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is.*

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But if we conceive of the dual negation as a bilateral negation as in BSML, the lack of determinacy is not unexpected or undesirable. Given a bilateralist viewpoint on which both assertion and rejection conditions must figure into meanings, the assertion conditions ought not determine the rejection conditions, or vice versa.

Moreover, as already observed by Hodges (1997), if one takes both assertion and rejection conditions into account by viewing the semantic value of a sentence ϕ to be the pair $(||\phi||, ||\neg\phi||)$, the value for $\neg\phi$ can be obtained from the value for ϕ by simply flipping the elements of the pair.

Indeed, the negation results are perhaps best thought of as *expressive completeness theorems*.

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Let $||BSML|| := \{||\phi|| \mid \phi \in \text{BSML}\}$.

The negation result says that BSML can express, via a formula and its negation, each pair (P, Q) of classes P and Q are expressible in BSML and P and Q are incompatible:

$$\{(||\phi||, ||\neg\phi||) \mid \phi \in \text{BSML}\} = \{(P, Q) \mid P, Q \in ||BSML|| \text{ and } \bigcup P \cap \bigcup Q = \varnothing\}$$

In simpler terms, BSML is complete for incompatible pairs.

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D is complete for incompatible pairs which are the same thing as contradictory/weakly contradictory pairs in the downwards-closed setting.

Further results

Propositional team-based logic with NE, the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, and the dual negation is complete for incompatible pairs.

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Propositional team-based logic with NE, the inquisitive disjunction $\mathbb{V}$, and the dual negation is complete for incompatible pairs.

Propositional dependence logic PD is not complete for incompatible pairs because for any formula ϕ of PD, $\cup||\neg\phi||$ is the complement of $\cup||\phi||$.
Open question: it complete for pairs (P, Q) s.t. $\bigcup P$ is the complement of $\bigcup Q$?

Thank you!

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