A Deep-Inference Sequent Calculus for a Propositional Team Logic

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TbiLLC 2025: Fifteenth International Tbilisi Symposium on Logic, Language and Computation

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- Goal: develop a sequent calculus for a team-based logic that is as simple as possible—departing minimally from a standard Gentzen-style system—and has nice proof-theoretic properties

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- Our approach: extend a standard Gentzen-style calculus (for the classical fragment of the logic we work with) with deep-inference rules (rules which apply to connectives at any depth within a formula) for nonclassical connectives.

- The resulting system is very simple, and standard proof-theoretic results can be shown as corollaries or extensions of the corresponding results for the classical Gentzen-style subsystem

The logic $PL(\mathbb{V})$ [YV16]

Syntax of classical propositional logic PL:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha$$

Syntax of propositional logic with the global/inquisitive disjunction $\mathbb{V}$ $PL(\mathbb{V})$

$$\phi ::= p \mid \bot \mid \neg\alpha \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathbb{V} \phi \quad \text{where } \alpha \in PL$$

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Relations to other team logics:

Propositional dependence logic [YV16] is $PL(\mathbb{V})$ without $\mathbb{V}$, and with dependence atoms (dependence atoms are definable in $PL(\mathbb{V})$, so one may view $PL(\mathbb{V})$ as an extension of propositional dependence logic).

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Propositional inquisitive logic [CR11] is $PL(\mathbb{V})$ without $\vee$ and $\neg$, and with the so-called intuitionistic implication $\rightarrow$.

All of these three logics are equivalent in expressive power.

Semantics

$$s \models p \quad \Longleftrightarrow \quad \forall v \in s : v(p) = 1$$

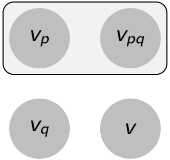
$$s \models \bot \quad \Longleftrightarrow \quad s = \varnothing$$

$$s \models \neg\alpha \quad \Longleftrightarrow \quad \forall v \in s : \{v\} \not\models \alpha$$

$$s \models \phi \vee \psi \quad \Longleftrightarrow \quad \exists t, t' : t \cup t' = s \ \& \ t \models \phi \ \& \ t' \models \psi$$

$$s \models \phi \wedge \psi \quad \Longleftrightarrow \quad s \models \phi \text{ and } s \models \psi$$

$$s \models \phi \mathbin{\mathbb{V}} \psi \quad \Longleftrightarrow \quad s \models \phi \text{ or } s \models \psi$$



(a) $s \models p \quad s \models \neg r$

Semantics

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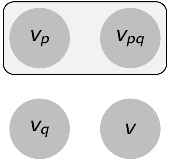

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\begin{array}{lcl}
s \models p & \iff & \forall v \in s : v(p) = 1 \\[1em]
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\color{red}{s \models \neg\alpha} & \color{red}{\iff} & \color{red}{\forall v \in s : \{v\} \not\models \alpha} \\[1em]
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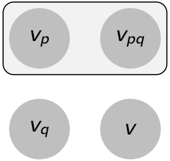

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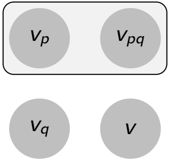

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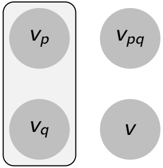
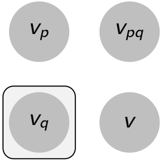


(b) $s \models p \vee q$



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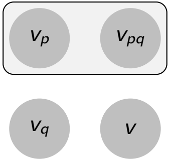

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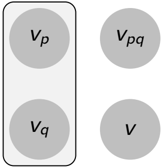

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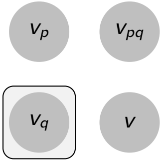

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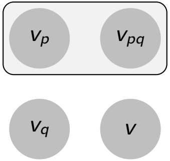

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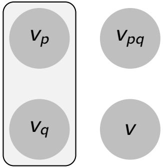

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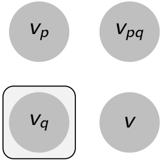

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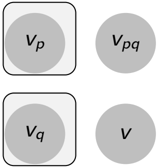

(d) $\{v_p\} \models p \mathbin{\mathbb{V}} \neg p$
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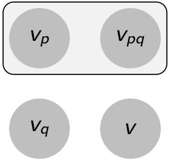

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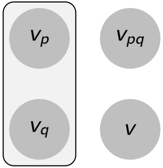

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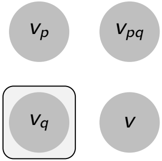

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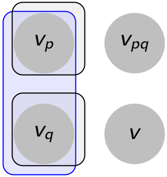

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$\{v_p, v_q\} \nvDash p \veebar \neg p$

Closure properties

ϕ is *downward closed*:	$[s \models \phi \text{ and } t \subseteq s] \implies t \models \phi$
ϕ is *union closed*:	$[s \models \phi \text{ for all } s \in S \neq \varnothing] \implies \bigcup S \models \phi$
ϕ has the *empty team property*:	$\varnothing \models \phi$
ϕ is *flat*:	$s \models \phi \iff \{v\} \models \phi \text{ for all } v \in s$

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Formulas in classical propositional logic *PL* (no $\mathbb{V}$) are flat, and their team semantics coincide with their standard semantics on singletons:

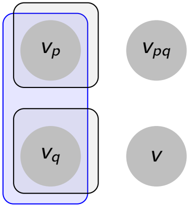
$$s \models \alpha \iff \forall w \in s : \{w\} \models \alpha \iff \forall w \in s : w \models \alpha$$

Therefore $PL(\mathbb{V})$ is a conservative extension of classical propositional logic:

$$\text{for } \Xi \cup \{\alpha\} \subseteq PL: \quad \Xi \models \alpha \text{ (in team semantics)} \iff \Xi \models \alpha \text{ (in standard semantics)}$$

The global/inquisitive disjunction $\mathbb{V}$

All formulas in $PL(\mathbb{V})$ are downward closed and have the empty team property, but formulas with $\mathbb{V}$ might not be union closed.



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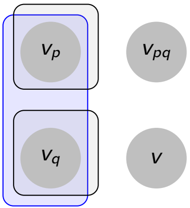

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\{v_p, v_{\neg p}\} & \models & (p \mathbin{\mathbb{V}} \neg p) \vee (p \mathbin{\mathbb{V}} \neg p)
\end{array}
$$

$PL(\mathbb{V})$ is not closed under uniform substitution.
E.g., $p \vee p \models p$ but $(p \mathbin{\mathbb{V}} \neg p) \vee (p \mathbin{\mathbb{V}} \neg p) \not\models p \mathbin{\mathbb{V}} \neg p$.

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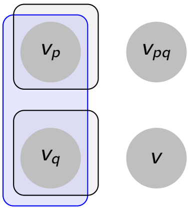

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$\wedge$, $\vee$, and $\mathbb{V}$ distribute over $\mathbb{V}$:

$$
\begin{array}{lcl}
\phi \wedge (\psi \mathbb{V} \chi) & \equiv & (\phi \wedge \psi) \mathbb{V} (\phi \wedge \chi) \\
\phi \vee (\psi \mathbb{V} \chi) & \equiv & (\phi \vee \psi) \mathbb{V} (\phi \vee \chi) \\
\phi \mathbb{V} (\psi \mathbb{V} \chi) & \equiv & (\phi \mathbb{V} \psi) \mathbb{V} (\phi \mathbb{V} \chi)
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Therefore, each $\phi \in PL(\mathbb{V})$ is equivalent to a $\mathbb{V}$-disjunction of classical formulas called the resolutions of ϕ: $\phi \equiv \mathbb{V} R(\phi)$ ($R(\phi) \subseteq PL$).

The global/inquisitive disjunction $\mathbin{\rotatebox[origin=c]{0}{$\vee$}\!\!\!\vee}$

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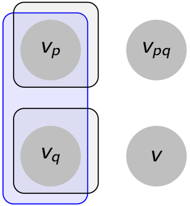

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Split property

For $\Xi \subseteq PL$:

$$\Xi \models \phi_1 \mathbb{V} \phi_2 \text{ iff } \Xi \models \phi_1 \text{ or } \Xi \models \phi_2.$$

A natural deduction system

α must be classical.

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\wedge\mathsf{I} \qquad \frac{\phi \wedge \psi}{\phi}\wedge\mathsf{E} \qquad \frac{\phi \wedge \psi}{\psi}\wedge\mathsf{E}$$

$$\begin{array}{c}[\alpha]\\ \vdots\\ \dfrac{\bot}{\neg\alpha}\neg\mathsf{I}\end{array} \qquad \frac{\alpha \qquad \neg\alpha}{\phi}\neg\mathsf{E} \qquad \begin{array}{c}[\neg\alpha]\\ \vdots\\ \dfrac{\bot}{\alpha}\mathsf{RAA}\end{array} \qquad \frac{\bot}{\phi}\mathsf{EF}$$

$$\frac{\phi}{\phi \vee\!\!\!\vee \psi}\vee\!\!\!\vee\mathsf{I} \qquad \frac{\phi}{\psi \vee\!\!\!\vee \phi}\vee\!\!\!\vee\mathsf{I} \qquad \frac{\phi \vee\!\!\!\vee \psi \qquad \begin{array}{c}[\phi]\\ \vdots\\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots\\ \chi\end{array}}{\chi}\vee\!\!\!\vee\mathsf{E}$$

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$$\dfrac{\phi \vee \psi}{\psi \vee \phi}\,\vee\mathsf{Com} \qquad \dfrac{\phi \vee \psi \qquad \begin{array}{c}[\phi]\\ \vdots \\ \chi\end{array}}{\chi \vee \psi}\,\vee\mathsf{Mon}$$

$$\dfrac{\phi}{\phi \Vee \psi}\,\Vee\mathsf{I} \qquad \dfrac{\phi}{\psi \Vee \phi}\,\Vee\mathsf{I} \qquad \dfrac{\phi \Vee \psi \qquad \begin{array}{c}[\phi]\\ \vdots \\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}\,\Vee\mathsf{E}$$

$$\dfrac{\phi \vee (\psi \Vee \chi)}{(\phi \vee \psi) \Vee (\phi \vee \chi)}\,\mathsf{Dstr}\vee\Vee$$

A naive sequent calculus translation of the ND-system

$$\Gamma, p \Rightarrow p, \Delta \quad At \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma, \neg\alpha \Rightarrow \Delta}\, L\neg \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\, R\neg$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\, L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \phi, \Xi \qquad \Gamma \Rightarrow \psi, \Xi}{\Gamma \Rightarrow \phi \wedge \psi, \Xi, \Delta}\, R\wedge$$

$$\frac{\Gamma, \phi \Rightarrow \Xi \qquad \Gamma, \psi \Rightarrow \Xi}{\Gamma, \phi \vee \psi \Rightarrow \Xi, \Delta}\, L\vee \qquad\qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\, R\vee$$

$$\frac{\Gamma, \phi_1 \Rightarrow \Delta \qquad \Gamma, \phi_2 \Rightarrow \Delta}{\Gamma, \phi_1 \bbvee \phi_2 \Rightarrow \Delta}\, L\bbvee \qquad\qquad \frac{\Gamma \Rightarrow \phi_i, \Delta}{\Gamma \Rightarrow \phi_1 \bbvee \phi_2, \Delta}\, R\bbvee$$

$$\frac{\Gamma, \phi \vee \psi_1 \Rightarrow \Delta \qquad \Gamma, \phi \vee \psi_2 \Rightarrow \Delta}{\Gamma, \phi \vee (\psi_1 \bbvee \psi_2) \Rightarrow \Delta}\, LDstr \qquad \frac{\Gamma \Rightarrow \phi \vee (\psi_1 \bbvee \psi_2), \Delta}{\Gamma \Rightarrow (\phi \vee \psi_1) \bbvee (\phi \vee \psi_2), \Delta}\, RDstr$$

α and Ξ must be classical. The interpretation of $\Gamma \Rightarrow \Delta$ is $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ (not $\bigwedge \Gamma \Rightarrow \bbvee \Delta$).

Problem 1

The distributivity rules are not strong enough if we do not have Cut—how would one give a cutfree proof of the following sequent in this system?

$$(((p \wedge x) \Vee (q \wedge x)) \vee (y \wedge x)) \vee (r \wedge x) \Rightarrow (((p \vee y) \vee r) \wedge x) \Vee (((q \vee y) \vee r) \wedge x)$$

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Problem 2

Problem 2: How does the cut elimination procedure work with the restricted rules?
If there are restrictions on the rules, we cannot, for instance, commute the cuts freely:

$$\dfrac{\dfrac{\dfrac{D_1'}{\Gamma, \eta \Rightarrow \phi, \Xi} \quad \dfrac{D_1'}{\Gamma, \xi \Rightarrow \phi, \Xi}}{\Gamma, \eta \vee \xi \Rightarrow \phi, \Xi, \Delta}\, L\vee \qquad \dfrac{D_2'}{\Pi, \phi \Rightarrow \Sigma}}{\Pi, \Gamma, \eta \vee \xi \Rightarrow \Xi, \Delta, \Sigma}\, \text{Cut}$$

would be transformed into

$$\dfrac{\dfrac{\dfrac{D_1'}{\Gamma, \eta \Rightarrow \phi, \Xi} \quad \dfrac{D_2'}{\Pi, \phi \Rightarrow \Sigma}}{\Pi, \Gamma, \eta \Rightarrow \Xi, \Sigma}\, \text{Cut} \qquad \dfrac{\dfrac{D_1'}{\Gamma, \xi \Rightarrow \phi, \Xi} \quad \dfrac{D_2'}{\Pi, \phi \Rightarrow \Sigma}}{\Pi, \Gamma, \xi \Rightarrow \Xi, \Sigma}\, \text{Cut}}{\Pi, \Gamma, \eta \vee \xi \Rightarrow \Xi, \Sigma, \Delta}\, \# L\vee$$

which contains an illegitimate application of $L\vee$ if Σ is not classical.

Possible approaches

Labelled systems

E.g., [San09; CM17; Mul22; LS25; BGYM24].

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$$\frac{w : p, w \subseteq \pi, \pi : p, \Gamma \Rightarrow \Delta}{w \subseteq \pi, \pi : p, \Gamma \Rightarrow \Delta} \; p_L$$

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Example of a General rule:

$$\frac{Z \vdash A; B}{Z \vdash A \mathbin{\mathbb{V}} B}$$

Example of a Flat rule:

$$\frac{\alpha, \beta \vdash \Gamma}{\alpha \sqcap \beta \vdash \Gamma}$$

Our approach: deep inference

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We obtain our calculus by generalizing and combining the $\mathbb{V}$- and distributivity rules to allow for the introduction of $\mathbb{V}$ in any non-negated context:

$$\frac{\Gamma, \chi\{\phi_1\} \Rightarrow \Delta \qquad \Gamma, \chi\{\phi_2\} \Rightarrow \Delta}{\Gamma, \chi\{\phi_1 \mathbin{\mathbb{V}} \phi_2\} \Rightarrow \Delta}\ L\mathbb{V} \qquad \frac{\Gamma \Rightarrow \chi\{\phi_i\}, \Delta}{\Gamma \Rightarrow \chi\{\phi_1 \mathbin{\mathbb{V}} \phi_2\}, \Delta}\ R\mathbb{V}$$

(Where $\cdot$ does not occur within the scope of a negation. Soundness follows from distributivity.)

Example:

$$\frac{p \wedge r \Rightarrow p \wedge r}{p \wedge r \Rightarrow p \wedge (r \mathbin{\mathbb{V}} q)}\ R\mathbb{V}$$

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(Where $\cdot$ does not occur within the scope of a negation. Soundness follows from distributivity.)

Example:

$$\frac{p \wedge r \Rightarrow p \wedge r}{p \wedge r \Rightarrow p \wedge (r \Vvee q)}\ R\Vvee$$

The result is a (limited) deep-inference system E.g., [Sch77; Gug99; Br9; Pog09] in that the rules of the calculus may introduce a connective which is not the main connective of the resulting formula.

The system GT for $PL(\mathbb{V})$

The system GT extends the system $G3cp$ for PL with deep-inference rules for $\mathbb{V}$:

Axioms

$$\Gamma, p \Rightarrow p, \Delta \quad At \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

Logical rules

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma, \neg\alpha \Rightarrow \Delta}\, L\neg \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\, R\neg$$

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\, L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \phi, \Xi \qquad \Gamma \Rightarrow \psi, \Xi}{\Gamma \Rightarrow \phi \wedge \psi, \Xi, \Delta}\, R\wedge$$

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$$\frac{\Gamma, \chi\{\phi_1\} \Rightarrow \Delta \qquad \Gamma, \chi\{\phi_2\} \Rightarrow \Delta}{\Gamma, \chi\{\phi_1 \mathbb{V} \phi_2\} \Rightarrow \Delta}\, L\mathbb{V} \qquad\qquad \frac{\Gamma \Rightarrow \chi\{\phi_i\}, \Delta}{\Gamma \Rightarrow \chi\{\phi_1 \mathbb{V} \phi_2\}, \Delta}\, R\mathbb{V}$$

The intended interpretation of $\Gamma \Rightarrow \Delta$ is $\bigwedge\Gamma \models \bigvee\Delta$.

α and Ξ must be classical.

$\cdot$ must not occur within the scope of a negation.

The full system GT also includes a standard Cut-rule. We call the cutfree system GT^-.

Example derivation

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
p \Rightarrow p, q \wedge \neg r
\qquad
\dfrac{
\dfrac{
q, \neg r \Rightarrow p, q
\qquad
\dfrac{
\dfrac{q, r \Rightarrow r, p}{q, \neg r, r \Rightarrow p}\; L\neg
}{q, \neg r \Rightarrow p, \neg r}\; R\neg
}{q, \neg r \Rightarrow p, q \wedge \neg r}\; R\wedge
}{q \wedge \neg r \Rightarrow p, q \wedge \neg r}\; L\wedge
}{p \vee (q \wedge \neg r) \Rightarrow p, q \wedge \neg r}\; L\vee
}{p \vee (q \wedge \neg r) \Rightarrow p \vee (q \wedge \neg r)}\; R\vee
}{p \vee (q \wedge \neg r) \Rightarrow (p \vee (q \wedge \neg r)) \mathbin{\mathbb{V}} (p \vee (q \wedge s))}\; R\mathbin{\mathbb{V}}
\qquad
\dfrac{
\dfrac{
\dfrac{
p \Rightarrow p, q \wedge s
\qquad
\dfrac{
\dfrac{q, s \Rightarrow p, q \qquad q, s \Rightarrow p, s}{q, s \Rightarrow p, q \wedge s}\; R\wedge
}{q \wedge s \Rightarrow p, q \wedge s}\; L\wedge
}{p \vee (q \wedge s) \Rightarrow p, q \wedge s}\; L\vee
}{p \vee (q \wedge s) \Rightarrow p \vee (q \wedge s)}\; R\vee
}{p \vee (q \wedge s) \Rightarrow (p \vee (q \wedge \neg r)) \mathbin{\mathbb{V}} (p \vee (q \wedge s))}\; R\mathbin{\mathbb{V}}
}{p \vee (q \wedge (\neg r \mathbin{\mathbb{V}} s)) \Rightarrow (p \vee (q \wedge \neg r)) \mathbin{\mathbb{V}} (p \vee (q \wedge s))}\; L\mathbin{\mathbb{V}}
$$

Alternative formulation

In our system, the structural rules (contraction, weakening) are absorbed into the logical rules. If we choose not to absorb the structural rules, we get the following rules for $\wedge$ and $\vee$:

Structural rules

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\; L\wedge \qquad \frac{\Gamma_1 \Rightarrow \phi, \Delta_1 \qquad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \phi \wedge \psi, \Delta_1, \Delta_2}\; R\wedge$$

$$\frac{\Gamma_1, \phi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \phi \vee \psi \Rightarrow \Delta_1, \Delta_2}\; L\vee \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\; R\vee$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\; LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\; RW$$

$$\frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\; LC \qquad \frac{\Gamma \Rightarrow \alpha, \alpha, \Delta}{\Gamma \Rightarrow \alpha, \Delta}\; RC$$

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Observations:

- These are the multiplicative rules for the conjunction and disjunction (as in linear logic)—the rules for $\wedge$ are those for the multiplicative conjunction (tensor) $\otimes$, and the rules for $\vee$ are those for the multiplicative disjunction (par) (cf. [AV09]).

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- Right weakening corresponds to the empty team property of ϕ.

  Right contraction corresponds to the union closure of α.

Properties of the calculus

The following properties/results follow for GT as natural extensions of the corresponding results for $G3cp$:

- Proof/countermodel search procedure
- Contraction, weakening and inversion lemmas
- Sequent interpolation theorem
- Cut elimination

Cutfree completeness & countermodel search

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The inverse direction of each *G3cp*-rule is sound.
E.g., for $L\vee$, $\bigwedge\Gamma \wedge (\phi \vee \psi) \vDash \bigvee\Delta$ implies $\bigwedge\Gamma \wedge \phi \vDash \bigvee\Delta$ and $\bigwedge\Gamma \wedge \psi \vDash \bigvee\Delta$.
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Similarly for $L\mathbb{V}$. As for $R\mathbb{V}$, we have by the split property that if $\bigwedge\Xi \models \chi\{\phi_1 \mathbb{V} \phi_2\} \vee \bigvee\Delta$, then either $\bigwedge\Xi \models \chi\{\phi_1\} \vee \bigvee\Delta$ or $\bigwedge\Xi \models \chi\{\phi_2\} \vee \bigvee\Delta$.
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Applying these facts, there is, for each $PL(\mathbb{V})$-sequent $\Gamma \Rightarrow \Delta$, a collection of atomic sequents $\Xi_{ijk} \Rightarrow \Lambda_{ijk}$ such that $\bigowedge_{i\in I} \bigovee_{j\in J} \bigowedge_{k\in K}(\Xi_{ijk} \Rightarrow \Lambda_{ijk}) \dashv\vdash_{GT^-} \Gamma \Rightarrow \Delta$.

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Countermodel search/cutfree completeness

There is a procedure that constructs a countermodel to $\Gamma \Rightarrow \Delta$ from countermodels to sequents only involving atomic formulas if there is such a countermodel. If there is no such countermodel, the procedure yields a cutfree proof of $\Gamma \Rightarrow \Delta$.

A visual example, with countermodels written above the sequent arrows (blue sequents hold; red sequents do not):

$$
\begin{prooftree}
\AxiomC{$p \Rightarrow p$}
\AxiomC{$p, p \overset{\{v_p\}}{\Rightarrow}$}
\RightLabel{$R\neg$}
\UnaryInfC{$p \overset{\{v_p\}}{\Rightarrow} \neg p$}
\RightLabel{$R\Vvee$}
\doubleLine
\BinaryInfC{$p \Rightarrow p \Vvee \neg p$}
\AxiomC{$p \Rightarrow p$}
\AxiomC{$\overset{\{v_{\overline{p}}\}}{\Rightarrow} p, p$}
\RightLabel{$L\neg$}
\UnaryInfC{$\neg p \overset{\{v_{\overline{p}}\}}{\Rightarrow} p$}
\RightLabel{$L\vee$}
\BinaryInfC{$p \vee \neg p \overset{\{v_{\overline{p}}\}}{\Rightarrow} p$}
\AxiomC{$p, p \overset{\{v_p\}}{\Rightarrow}$}
\RightLabel{$R\neg$}
\UnaryInfC{$p \overset{\{v_p\}}{\Rightarrow} \neg p$}
\AxiomC{$p \Rightarrow p$}
\RightLabel{$L\neg$}
\UnaryInfC{$\neg p, p \Rightarrow$}
\RightLabel{$R\neg$}
\UnaryInfC{$\neg p \Rightarrow \neg p$}
\RightLabel{$L\vee$}
\BinaryInfC{$p \vee \neg p \overset{\{v_p\}}{\Rightarrow} \neg p$}
\RightLabel{$R\Vvee$}
\doubleLine
\BinaryInfC{$p \vee \neg p \overset{\{v_p, v_{\overline{p}}\}}{\Rightarrow} p \Vvee \neg p$}
\RightLabel{$L\Vvee$}
\BinaryInfC{$p \Vvee (p \vee \neg p) \overset{\{v_p, v_{\overline{p}}\}}{\Rightarrow} p \Vvee \neg p$}
\end{prooftree}
$$

Here

$$
\begin{prooftree}
\AxiomC{$\Xi \Rightarrow \chi\{\phi_1\}, \Delta$}
\AxiomC{$\Xi \Rightarrow \chi\{\phi_2\}, \Delta$}
\RightLabel{$R\Vvee$}
\doubleLine
\BinaryInfC{$\Xi \Rightarrow \chi\{\phi_1 \Vvee \phi_2]\}, \Delta$}
\end{prooftree}
$$

denotes that $\Xi \Rightarrow \chi\{\phi_1 \Vvee \phi_2\}, \Delta$ holds iff either $\Xi \Rightarrow \chi\{\phi_1\}, \Delta$ or $\Xi \Rightarrow \chi\{\phi_2\}, \Delta$ holds (cf. the split property).

Depth-preserving weakening, contraction and inversion; Interpolation

$\vdash_n S$: S has a derivation of depth at most n. (Depth: the maximum length of branches in the derivation tree).

Weakening and contraction lemma

$$\text{If } \vdash_n \Gamma \Rightarrow \Delta \text{ then } \vdash_n \Gamma, \phi \Rightarrow \Delta$$
$$\text{If } \vdash_n \Gamma \Rightarrow \Delta \text{ then } \vdash_n \Gamma \Rightarrow \phi, \Delta$$
$$\text{If } \vdash_n \Gamma, \phi, \phi \Rightarrow \Delta \text{ then } \vdash_n \Gamma, \phi \Rightarrow \Delta$$
$$\text{If } \vdash_n \Gamma \Rightarrow \alpha, \alpha, \Delta \text{ then } \vdash_n \Gamma \Rightarrow \alpha, \Delta$$

Right contraction is not sound with respect to all formulas since, e.g., $(p \varovee \neg p) \vee (p \varovee \neg p) \not\models p \varovee \neg p$.

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Inversion lemma

All rules except $R\mathbb{V}$ are depth-preserving invertible.

E.g., (inverted $R\wedge$) $\vdash_n \Gamma \Rightarrow \phi \wedge \psi, \Delta$ implies $\vdash_n \Gamma \Rightarrow \phi, \Delta$ and $\vdash_n \Gamma \Rightarrow \psi, \Delta$.

Interpolation

We write $P^+(\phi)/P^-(\phi)$ for the set of propositional variables occurring positively/negatively in ϕ, and we let $P^i(\Gamma) := \bigcup_{\phi \in \Gamma} P^i(\phi)$, for $i \in \{+,-\}$.

Let $\Gamma_1;\Gamma_2$ be a partition of Γ and $\Delta_1;\Delta_2$ be a partition of Δ. I is a sequent interpolant of $\Gamma_1;\Gamma_2 \Rightarrow \Delta_1;\Delta_2$ if there are cutfree derivations of $\Gamma_1 \Rightarrow I, \Delta_1$ and $\Gamma_2, I \Rightarrow \Delta_2$, and $P(\phi)^i \subseteq (P^i(\Gamma_1) \cup P^j(\Delta_1)) \cap (P^j(\Gamma_2) \cup P^i(\Delta_2))$, for $i \in \{+,-\}$ and $j \in \{+,-\} \setminus \{i\}$.

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Maehara's interpolation for $G3cp$

Given a cutfree $G3cp$-derivation of $\Xi \Rightarrow \Lambda$, and a pair of partitions $\Xi_1;\Xi_2, \Lambda_1;\Lambda_2$ for $\Xi \Rightarrow \Lambda$, there is an effective procedure for constructing a sequent interpolant I of $\Xi_1;\Xi_2 \Rightarrow \Lambda_1;\Lambda_2$.

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Maehara's interpolation for *G3cp*

Given a cutfree *G3cp*-derivation of $\Xi \Rightarrow \Lambda$, and a pair of partitions $\Xi_1;\Xi_2, \Lambda_1;\Lambda_2$ for $\Xi \Rightarrow \Lambda$, there is an effective procedure for constructing a sequent interpolant I of $\Xi_1;\Xi_2 \Rightarrow \Lambda_1;\Lambda_2$.

The procedure for *GT* extends that for *G3cp*.

Maehara's interpolation for *GT*

Given a cutfree *GT*-derivation of $\Gamma \Rightarrow \Delta$, and a pair of partitions $\Gamma_1;\Gamma_2, \Lambda_1;\Delta_2$ for $\Gamma \Rightarrow \Delta$ (where Λ_1 is classical), there is an effective procedure for constructing a sequent interpolant I of $\Gamma_1;\Gamma_2 \Rightarrow \Lambda_1;\Delta_2$, and if Δ_2 is classical, then I is classical.

Normal form for cutfree derivations

A resolution Ξ for a multiset Γ ($\Xi \in R(\Gamma)$) is a multiset consisting of one resolution for each formula in Γ.

Theorem (Derivation normal form)

There is an effective procedure transforming any derivation witnessing $\vdash_{GT^-} \Gamma \Rightarrow \Delta$ *into a derivation witnessing*

$$\vdash_{G3cp^-} \bigwedge_{\Xi \in R(\Gamma)} (\Xi \Rightarrow f[\Xi]) \vdash_{R\mathbb{V}} \bigwedge_{\Xi \in R(\Gamma)} (\Xi \Rightarrow \Delta) \vdash_{L\mathbb{V}} \Gamma \Rightarrow \Delta,$$

where $f : R(\Gamma) \to R(\Delta)$. *We say that a derivation of* $\Gamma \Rightarrow \Delta$ *of the form above is in normal form.*

Cut elimination procedure

Given a cut

$$\dfrac{\begin{matrix} D_1 \\ \Gamma \Rightarrow \phi, \Delta \end{matrix} \qquad \begin{matrix} D_2 \\ \Pi, \phi \Rightarrow \Sigma \end{matrix}}{\Pi, \Gamma \Rightarrow \Delta, \Sigma} \text{ Cut}$$

where D_1 and D_2 are cutfree,

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where D_1 and D_2 are cutfree,

1. Apply the normal form theorem to D_1 and D_2 to obtain cutfree derivations in $G3cp$ involving the resolutions of Γ; ϕ, Δ; Π, ϕ; and Σ.

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1. Apply the normal form theorem to D_1 and D_2 to obtain cutfree derivations in $G3cp$ involving the resolutions of Γ; ϕ, Δ; Π, ϕ; and Σ.

2. Apply cut on the resolutions of ϕ to get derivations in $G3cp$ whose endsequents involve only resolutions of Γ; Δ; Π; and Σ.

Cut elimination procedure

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3, Apply cut elimination for the classical subsystem to get cutfree derivations of these endsequents.

4. Combine these derivations using the deep-inference rules to get a cutfree derivation of $\Gamma \Rightarrow \Delta$.

Going deeper: a Calculus of Structures-system for $PL(\mathbb{V})$

System *SKS* [Br6] for *PL*:

$$\frac{\eta\{\top\}}{\eta[\alpha,\overline{\alpha}]}\ i\downarrow \qquad \frac{\eta(\phi,\overline{\phi})}{\eta\{\bot\}}\ i\uparrow$$

$$\frac{\eta([\phi,\psi],\chi)}{\eta[(\phi,\chi),\psi]}\ s$$

$$\frac{\eta\{\bot\}}{\eta\{\phi\}}\ w\downarrow \qquad \frac{\eta\{\phi\}}{\eta\{\top\}}\ w\uparrow$$

$$\frac{\eta[\alpha,\alpha]}{\eta\{\alpha\}}\ c\downarrow \qquad \frac{\eta\{\phi\}}{\eta(\phi,\phi)}\ c\uparrow$$

$[\cdot]$-structures are disjunctions $\vee$ and $(\cdot)$-structures are conjunctions $\wedge$.

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$[\cdot]$-structures are disjunctions $\vee$ and $(\cdot)$-structures are conjunctions $\wedge$.
We can extend this with $\mathbb{V}$-structures $[\![\cdot]\!]$ and associated rules to get a system for $PL(\mathbb{V})$:

$$\frac{\eta[\![\phi,\phi]\!]}{\eta\{\phi\}}\ c\downarrow\mathbb{V} \quad \frac{\eta([\![\phi,\psi]\!],\chi)}{\eta[\![(\phi,\chi),\psi]\!]}\ s\mathbb{V} \quad \frac{\eta[\phi,[\![\psi,\chi]\!]]}{\eta[\![[\phi,\psi],[\phi,\chi]]\!]}\ Dstr$$

Conclusion

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Many proof-theoretic properties of GT and results concerning the calculus follow as natural extensions or corollaries of the corresponding proerties/results for $G3cp$.

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- A similar system for inquisitive logic

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Further work:

- A similar system for inquisitive logic
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- Extend GT with rules for modalities/quantifiers
- Investigate whether this approach works for other team logics; in particular those with different closure properties

References

[AIY25] Aleksi Anttila, Rosalie Iemhoff, and Fan Yang. A deep-inference sequent calculus for basic propositional team logic (without delving too deep). *ArXiv*, abs/2508.07509, 2025.

[AV09] Samson Abramsky and Jouko Väänänen. From If to Bi. *Synthese*, 167(2):207–230, 2009.

[BGYM24] Fausto Barbero, Marianna Girlando, Fan Yang, and Valentin Müller. Labelled prood systems for propositional inquisitive logic (with tensor), 2024. Manuscript.

[Br6] Kai Brünnler. Locality for classical logic. *Notre Dame J. Form. Log.*, 47(4):557–580, 2006.

[Br9] Kai Brünnler. Deep sequent systems for modal logic. *Arch. Math. Logic*, 48(6):551–577, 2009.

[CM17] Jinsheng Chen and Minghui Ma. Labelled sequent calculus for inquisitive logic. In Alexandru Baltag, Jeremy Seligman, and Tomoyuki Yamada, editors, *Logic, Rationality, and Interaction*, pages 526–540, Berlin, Heidelberg, 2017. Springer Berlin Heidelberg.

[CR11] Ivano Ciardelli and Floris Roelofsen. Inquisitive logic. *J. Philos. Logic*, 40(1):55–94, 2011.

[FGPY16] Sabine Frittella, Giuseppe Greco, Alessandra Palmigiano, and Fan Yang. Structural multi-type sequent calculus for inquisitive logic. *ArXiv*, abs/1604.00936, 2016.

[Gug99] Alessio Guglielmi. A system of interaction and structure. *CoRR*, cs.LO/9910023, 1999.

[LS25] Tadeusz Litak and Katsuhiko Sano. Bounded inquisitive logics: Sequent calculi and schematic validity. *ArXiv*, abs/2507.13946, 2025.

[Mul22] Valentin Muller. On the proof theory of inquisitive logic. Master's thesis, University of Amsterdam, The Netherlands, 2022.

[Pog09] Francesca Poggiolesi. The method of tree-hypersequents for modal propositional logic. In D. Makinson, J. Malinowski, and H. Wansing, editors, *Towards Mathematical Philosophy*, volume 28 of *Trends in Logic*, pages 31–51. Springer, 2009.

[San09] Katsuhiko Sano. Sound and complete tree-sequent calculus for inquisitive logic. In Hiroakira Ono, Makoto Kanazawa, and Ruy de Queiroz, editors, *Logic, Language, Information and Computation*, pages 365–378, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.

[Sch77] Kurt Schütte. *Proof Theory*. Springer-Verlag, 1977.

[YV16] Fan Yang and Jouko Väänänen. Propositional logics of dependence. *Annals of Pure and Applied Logic*, 167(7):557–589, July 2016.

Thank you!