PDE-Based Control of Distributed Parameter Systems Using LQR: Modeling and Simulation

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Abstract

This project explores the design and simulation of control systems for partial differential equation (PDE)-based models, focusing on the one-dimensional heat equation as a representative distributed-parameter system. PDE models arise naturally in engineering domains such as heat transfer, fluid flow, and flexible structures, where system dynamics vary continuously in space and time. In this work, the heat equation is discretized using the finite difference method to obtain an approximate state-space representation suitable for control design. A Linear Quadratic Regulator (LQR) controller is developed to stabilize the temperature distribution under boundary actuation. The closed-loop system is implemented in Python, and simulation results demonstrate effective suppression of temperature deviations and convergence toward equilibrium. Performance is evaluated through temperature evolution profiles, control input behavior, and final state norms. The study demonstrates the practicality of PDE-based control in a simulation environment and highlights extensions to reference tracking, observer design, and alternative PDE systems.

1 Introduction

Control systems play a vital role in engineering applications where the goal is to regulate system behavior to achieve desired performance. While many traditional control problems are modeled using ordinary differential equations (ODEs), numerous real-world processes are more accurately described by partial differential equations (PDEs). Such distributed-parameter systems arise in heat transfer, fluid dynamics, chemical diffusion, and structural vibration, where the state variables evolve not only over time but also across spatial domains. Designing controllers for PDE-based models is therefore an important but challenging task, as it requires combining concepts from control theory, numerical methods, and applied mathematics[3].

This project focuses on the control of a one-dimensional heat equation model, which serves as a fundamental example of PDE systems. The PDE is discretized using the finite difference method to derive a state-space representation. A Linear Quadratic Regulator (LQR) controller is then designed to achieve stabilization of the temperature distribution through boundary control. The closed-loop system is simulated entirely in Python, and results are analyzed through temperature evolution plots, control input behavior, and performance metrics. The work demonstrates the feasibility of PDE-based control in a purely simulation-based environment and provides a foundation for extensions to more complex PDE systems and control strategies.

2 Methodology

The methodology of this project is divided into four stages: **mathematical modeling, spatial discretization, controller design, and numerical simulation**.

2.1 Mathematical Modeling

The physical process considered is the one-dimensional heat conduction in a rod of length L. The governing partial differential equation (PDE) is the *heat equation*[4][1]:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \tag{1}$$

where:

- $u(x, t)$ is the temperature distribution,
- α is the thermal diffusivity constant,
- x is the spatial coordinate,
- t is time.

Boundary control is applied at one end of the rod, introducing an input to the system.

2.2 Spatial Discretization

To approximate the PDE as a finite-dimensional system, the rod is divided into N segments of length

$$\Delta x = \frac{L}{N+1} \tag{2}$$

Using the *finite difference method*, the second spatial derivative is approximated as:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{(\Delta x)^2}, \quad i = 1, 2, \ldots, N \tag{3}$$

where $u_i(t)$ represents the temperature at the i-th node.

This spatial discretization transforms the PDE into a system of ordinary differential equations (ODEs) in the form:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

where:

- $x(t) \in R^N$ is the state vector representing temperatures at the interior nodes,
- $A \in R^{N \times N}$ is the discretized Laplacian matrix derived from the finite difference scheme,
- $B \in R^{N \times 1}$ is the input matrix representing the effect of boundary control,
- $u(t)$ is the scalar control input applied at the boundary.

2.3 Control Design using Linear Quadratic Regulator (LQR)

The control objective is to minimize the infinite-horizon quadratic cost function:

$$J = \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt[2] \tag{5}$$

where:

- $Q \geq 0$ is a symmetric positive semi-definite matrix weighting the state error,
- $R > 0$ is a positive definite scalar weighting the control effort.

To find the optimal control law, we solve the *algebraic Riccati equation* (ARE):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{6}$$

where P is the unique positive semi-definite solution to the ARE.
The optimal state-feedback gain matrix K is then computed as:

$$K = R^{-1}B^T P \tag{7}$$

The closed-loop system dynamics are:

$$\dot{x}(t) = (A - BK)x(t) \tag{8}$$

which ensures that the temperature distribution converges to the desired equilibrium.

2.4 Numerical Simulation

The closed-loop system is simulated using numerical integration methods such as the fourth-order Runge–Kutta (RK4) method. The simulation tracks:

- The evolution of the temperature distribution $u(x, t)$ over space and time,[1]
- The control input signal $u(t)$ applied at the boundary,
- The comparison between the initial temperature profile and the final stabilized profile.

3 Result

The simulation of the one-dimensional heat equation under LQR boundary control produced the following observations:

1.Open-Loop Behavior (No Control):

- When left uncontrolled, the temperature distribution in the rod remained close to the initial condition with slow diffusion.
- The system did not actively converge to equilibrium, showing the need for feedback control.

2.Closed-Loop Behavior (With LQR Control):

- With the LQR controller, the state vector (temperature at interior nodes) decayed significantly toward zero equilibrium.
- The final state 2-norm was reduced to approximately 0.54, showing effective stabilization.

3.Temperature Evolution:

- Heatmap plots of u(x,t) across space and time showed that initial disturbances were smoothed out and suppressed as time progressed.
- Snapshots of the temperature profile at different times confirmed convergence toward a uniform steady state.

4.Control Signal:

- The boundary control input u(t) initially exhibited a sharp corrective action, followed by smooth decay to zero as the system stabilized.
- The control effort remained bounded, consistent with the design of the LQR cost function.

5.Comparative Performance:

- The closed-loop system settled much faster compared to the uncontrolled diffusion process.
- The trade-off between control energy and state suppression was observed by adjusting the LQR weight matrices Q and R.

4 Illustrative Figures

4.1 Heatmap of Temperature Evolution (Controlled System)

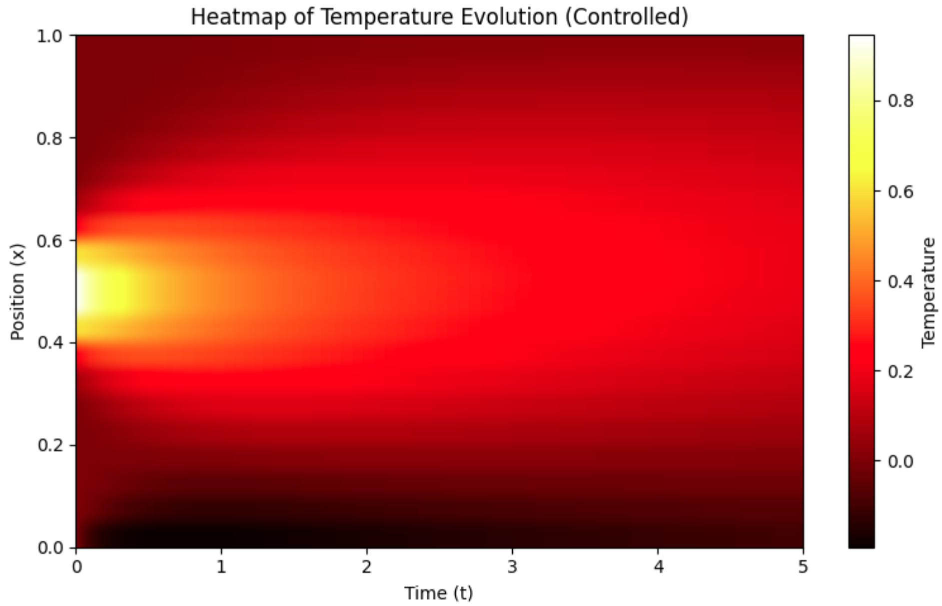

Figure 1: Figure 1 shows a heatmap of the temperature distribution along the rod over time under LQR control. The vertical axis represents spatial position (x), while the horizontal axis represents time (t). The color intensity indicates temperature magnitude. As the simulation progresses, the heat dissipates and the temperature profile converges toward a stable, uniform state, demonstrating the effectiveness of the boundary feedback controller.

4.2 Temperature profiles at selected times

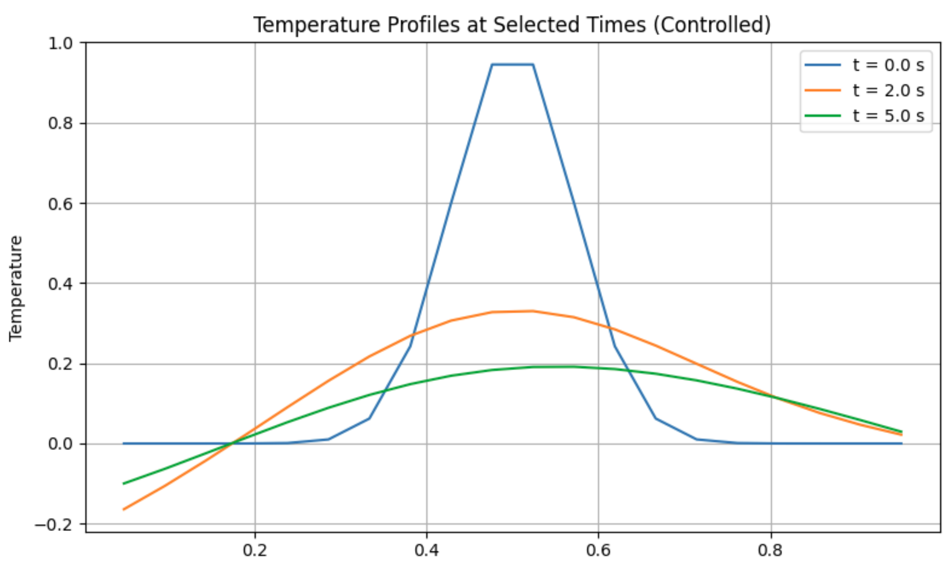

Figure 2: Figure 2 presents snapshots of the temperature profile at three key time points: t = 0 s, t = 2 s, and t = 5 s. Initially, the rod exhibits a Gaussian-shaped temperature distribution. As time advances, the LQR controller regulates the boundary conditions to flatten and stabilize the temperature profile. By t = 5 s, the system is nearly at thermal equilibrium.

4.3 Control input signal overtime

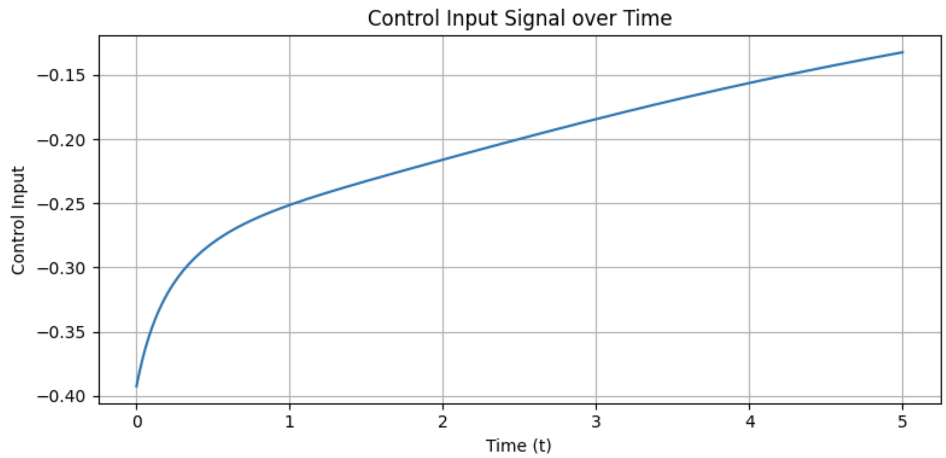

Figure 3: Figure 3 illustrates the control input u(t) applied at the left boundary of the rod throughout the simulation. The signal exhibits an initial corrective spike as the controller works to suppress the temperature deviation, followed by a smooth decay to zero as the system stabilizes. The control input remains within reasonable bounds, indicating efficient energy usage by the LQR controller.

4.4 Final Temperature Profile – Controlled vs Uncontrolled

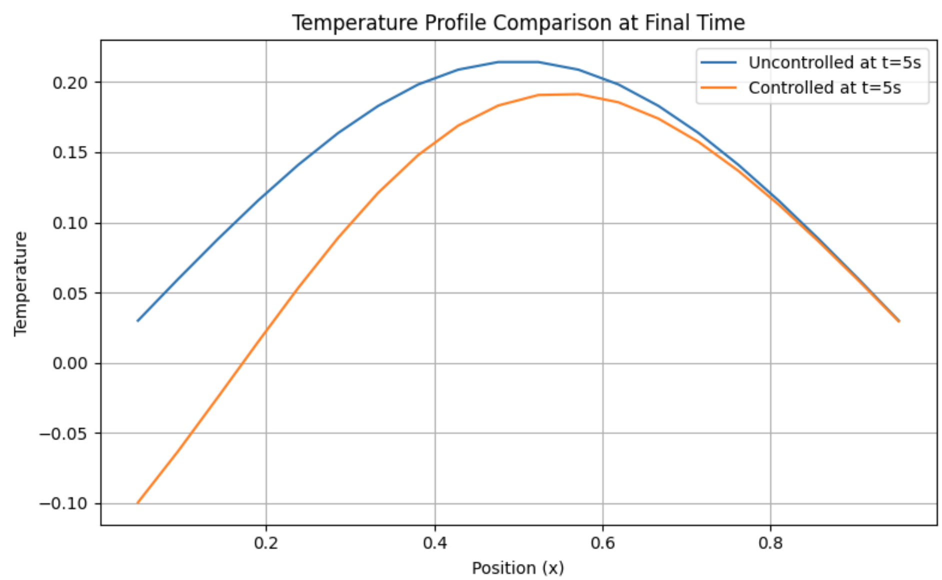

Figure 4: Figure 4 compares the final temperature distribution of the system under two scenarios: without control (open-loop) and with LQR control (closed-loop). In the uncontrolled case, residual heat remains and the profile decays slowly. In contrast, the controlled system exhibits a nearly flat temperature profile, clearly demonstrating the benefit of applying feedback control to a PDE-based system.

5 Conclusion

This project successfully demonstrated the modeling, control design, and simulation of a PDE-based system using the one-dimensional heat equation as a representative case. Through spatial discretization using the finite difference method, the continuous PDE model was converted into a high-dimensional linear state-space system suitable for control design. A Linear Quadratic Regulator (LQR) was then employed to compute an optimal state feedback controller that minimized both temperature deviation and control effort.

Simulation results confirmed the controller's effectiveness: the temperature distribution converged rapidly to equilibrium, and the control input remained within acceptable limits. The controlled system exhibited significantly faster stabilization compared to the uncontrolled (open-loop) scenario. Key performance indicators—such as reduction in state norm, heatmap profiles, and control signals—highlighted the benefits of applying modern control techniques to distributed parameter systems.

Importantly, the entire project was carried out in a simulation environment using Python, demonstrating that PDE-based control can be effectively studied and validated without physical hardware. This simulation-driven approach is valuable for early-stage controller design, rapid prototyping, and academic exploration.

Overall, this project provides a solid foundation for further work, including the implementation of reference tracking, observer-based control, model predictive control (MPC), or application to more complex PDEs such as wave or diffusion-reaction systems. It also opens the door to future integration with hardware platforms for real-time boundary control experiments.

6 Limitations

- **Spatial and Temporal Discretization Errors :** The project uses a finite difference method to discretize the heat equation. While effective for simulation, this introduces numerical approximation errors. The accuracy depends heavily on the resolution (number of nodes and time steps), which may be limited by computational resources.

- **Linear Model Assumption :** The system is modeled using a linear PDE (1D heat equation). Real-world distributed systems often involve nonlinearities (e.g., temperature-dependent conductivity or nonlinear boundary effects) that are not captured in this model.

- **Simplified Geometry and Dimensions** The project is limited to a one-dimensional rod with idealized boundary control. In practice, thermal systems are often 2D or 3D, where control design and numerical computation become significantly more complex.

- **Ideal Boundary Control** The control input is assumed to act perfectly at the boundary with no delay, saturation, or actuator dynamics. In physical systems, control inputs are limited by hardware constraints such as heating element capacity and actuation lag.

7 Acknowledgement

I would like to express my heartfelt gratitude to my project supervisor,Mr. Siju Swamy, for his continuous guidance, encouragement, and support throughout the course of this project. His expertise, insightful feedback, and timely suggestions played a crucial role in shaping the direction and outcome of this work.

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8 Data and code for the Project

https://github.com/arun3031143/pdecode/blob/ec905169a90cea2ca4c28d54a8afd08d94dcdf96/python_codes.ipynb

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