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Article

An Interval Temporal Logic for Time Series Specification and Data Integration

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Abstract: The analysis of temporal series—in particular, analysis of multisensor data—is a complex problem. It depends on the application domain, the way the data have to be used, and sensors available, among other factors. Various models, algorithms, and technologies have been designed for this goal. Temporal logics are used to describe temporal properties of systems. The properties may specify the occurrence and the order of events in time, recurring patterns, complex behaviors, and processes. In this paper, a new interval logic, called duration calculus for functions (DC4F), is proposed for the specification of temporal series corresponding to multisensor data. DC4F is a natural extension of the well-known duration calculus, an interval temporal logic for the specification of process duration. The adequacy of the proposed logic is analyzed in the case of multisensor data concerning volcanic eruption monitoring. It turns out that the relevant behavior concerns time intervals, not only accumulated history as it is described in other kinds of temporal logics. The examples analyzed demonstrate that a description language is required to specify time series of various kind relative to time intervals. The duration calculus cannot be successfully applied for this task. The proposed calculus allows one to specify temporal series and complex interval-dependent behaviors, and to evaluate the corresponding data within a unifying logical framework. It allows to formulate hypotheses concerning volcano eruption phenomena. However, the expressivity of DC4F comes at the cost of its decidability.

Keywords: duration calculus; data models; temporal logic; temporal series; data fusion; data evaluation; multisensor data; signal and data processing; interval logic



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1. Introduction

Remote sensing allows one to acquire information from a distance from cameras, sensors, microphones, and other external devices. The data may originate from satellites, aeroplanes, and sonar systems, among other sources. Satellite-based instruments are commonly used to monitor various parameters of the Earth's surface, such as temperatures in various infrared frequencies, and to take images. Global coverage is offered with frequency being as low as once per day. Stationary satellites are even capable of providing continuous monitoring of specific locations. The main problems associated with satellite data include their heterogeneity and large volumes. The data need to be analyzed on a daily basis, sometimes in real time, if timely response is needed.

In general, multisensor data are voluminous, heterogeneous, and sometimes incomplete. Prior to being analyzed, they need to be processed and represented in a specific form. They may be represented as discrete data by temporal series, i.e., by sequences of data usually taken in equal time intervals, or as continuous time-dependent functions and stochastic processes. Numerous algorithms have been developed, for example, concerning volcano monitoring (cf., e.g., [1]).

In the literature, heterogeneous means are used, such as means diagrams, functions, tables, and textual descriptions, to specify those heterogeneous types of phenomena. Dependencies between factors of different kind, such as similarities, regularities, and

periodicity, are quite often described with the use of text only, meaning that various problems stemming from such an approach, such as imprecision and the lack of a unifying framework for specification, validation, and reasoning need to be dealt with. Textual specifications are inherently imprecise, as this is the feature of natural languages. They do not rely on formal semantics in the form of comprehensive mathematical models. Consequently, they do not allow for precise evaluation or a formal reasoning, and thus the reasoning lacks due formality.

Logics provide uniform formal languages, models, and reasoning methods, and thus provide a solution enabling to address above mentioned problems. Logics are associated with well-defined classes of models constituting their semantics. They provide precise formal languages for the description of the models and capabilities for correct reasoning about properties of the models. In general, a logic is a system consisting of a formal language for specification, a class of models corresponding to the language, and a set of sound rules for a correct reasoning about the models.

Temporal logics are used to express change over time, properties of behaviors, and sequences of actions. Their languages provide temporal modalities for specifying future or past events. They are used also to define and synthesize system controllers. On the other hand, they provide rules for correct reasoning about temporal properties. Various kinds of temporal logics exist (see [2] for an overview).

Interval temporal logics (cf., e.g., [2,3]) are used for specifying time-dependent processes relative to time intervals. Duration calculus (DC) is an interval logic that is widely used for the specifying, modeling, and reasoning about discrete and continuous processes. It allows to specify propositional functions with Boolean values changing over time. There is an operator corresponding to the integral (cf., e.g., [4]) that measures how long such a propositional function remains true. It may be used to study periodicity of system states.

In this paper, we present duration calculus for functions (DC4F), a natural extension of the duration calculus for dealing with general integrable functions, not only Boolean-valued ones, as is the case of DC. The idea is simple; we take the integral operator on Riemann integrable functions and use it within a frame of an interval logic such as DC. Thus, the integral operator is used within a well-suited logical framework. DC4F, in addition to the expressive capabilities of DC, allows one to characterize the behavior of functions over time intervals in terms of their integrals. Consequently, we can characterize not only the duration of a certain property, as in the case of DC, but characterize the behavior of functions over time intervals in terms of integrals. The proposed extension is conservative in the logical sense: the DC part is unchanged meaning that its valid formulas remain valid and all its invalid formulas remain invalid. Even though the extension of DC is natural, we are, to our best knowledge, the first ones who propose it.

To evaluate the proposed logic, we investigated various phenomena, multisensor data, and facts concerning volcano monitoring, as this is a popular research topic and the degree of complexity of data is significant. Periodic degassing and temperature increases are common characteristic of active volcanoes. Distinct periodicity patterns concerning measurable parameters of volcanoes' activity have been widely identified. The timescales are ranging from seconds to weeks and months. The development of temperature and gas measurement techniques is aimed at enabling a robust quantification of high-frequency processes. Paper [5] presents an overview of the current state of knowledge regarding periodic volcanic degassing and evaluates the methods aiming at detecting periodicity. It summarizes and statistically analyzes published studies. Periodicity analysis of volcano activity (cf., e.g., [5]) is one of the challenges.

It turns out that such phenomena and their dependencies may be conveniently specified using DC4F, as it provides a expressive and uniform language for expressing various phenomena in a precise way, models for various data, and evaluation and reasoning capabilities. It provides a convenient specification language to express hypothesis concerning expected temporal properties. If data (in particular, multisensor data) are provided, then DC4F formulas may be validated for them and, thus, their truthfulness may be checked.

On the other hand, the reasoning rules provide for convenient reasoning possibilities. Thus, DC4F may be perceived as a unifying logical framework for data integration and for formulating hypotheses, their evaluation, and reasoning.

However, high expressivity of languages always comes at the cost of complexity of reasoning rules. The higher the expressivity, the more difficult the reasoning. More precisely, the question if a formula follows from a set of other formulas has high computational complexity. DC4F, like DC, is an expressive language and is, therefore, undecidable, i.e., there is no general algorithm for deciding the question mentioned above. Nonetheless, this does not hinder its use for specification, nor does it hinder the validation of formulas for concrete data.

The paper is organized as follows. In Section 2, we discuss related works. Section 3 contains a brief presentation of duration calculus and the way it is used. In Section 4, we present the main idea of DC4F and some examples of its application. In Section 5, we define its formal syntax and outline its semantics in an informal and exemplary way; we also list some of its properties and show how it applies to the multidimensional case. Section 6 is devoted to applications of DC4F in the area of volcano monitoring, in particular to the validation of the proposed concepts. In particular, we present an exemplary reasoning in DC4F. We conclude the paper with Section 7. The paper also contains an Appendix in which we define the formal semantics of DC4F.

2. Related Works

Data fusion means the integration of data and knowledge from different sources. It is a widely studied topic (cf. the overview papers [6,7] and the references there). Data fusion approaches may be classified in various ways—for example, data association, state estimation, and decision fusion. Data fusion has been studied in the context of temporal series as well. In paper [8], the authors study the fusion of long-term data in the form of dense time series from the Moderate Resolution Imaging Spectroradiometer (MODIS) and Landsat imagery. They investigate a spatiotemporal adaptive fusion algorithm in a regionalization study in which MODIS was used. They show the correlation of the time series achieved by different observation methods. In paper [8], the authors propose a data fusion method for producing high spatiotemporal resolution values for the normalized difference vegetation index in the case of time series.

We approach this topic from the perspective of temporal logic, rather than from the practical angle, as in the papers mentioned above and various others. Temporal logic (TL) describes behavior over time (cf., e.g., [2]). Their languages offer temporal modalities for specifying past or future events, as well as their temporal relations, such as: event A will happen or event A may happen in the future, property A has to be always present, property A holds until property B will be satisfied, and so on. They are also used to define and synthesize system controllers; the controllers are then guaranteed to monitor and control the underlying systems according to the specific requirements. On the other hand, they are used to ensure that conclusions drawn based on the assumptions made are correct, that the implementation satisfies the specifications, and so on.

There are different types of temporal logics (see [2] for the overview). The linear temporal logic describes behaviors by referring to linear sequences of events. The computation tree logic (CTL) describes the possibilities and, more precisely, the branching time structures. The interval temporal logic (ITL) (cf., e.g., [2,3]) describes behaviors relative to time intervals, stating, for example, that a property is valid during the entire time period or that it occurs within a certain subperiod. It is also possible to express the fact that a property holds within a given time period, which is then followed by another time period over which another property holds.

Duration calculus (DC) is a popular and widely studied kind of interval logic (cf., e.g., [4]). Duration calculus may be considered to be a form of ITL. In addition to the above-mentioned potential of ITL, DC contains an integral operator that allows one to express durations of properties in a quantitative manner.

In [9], automata-based semantics of DC have been defined covering data, real-time, and communication-related aspects. A model-checking algorithm has been presented for a subset of DC that may be model-checked. The algorithm has been implemented and its use demonstrated.

DC and ITL are expressive but not decidable, i.e., no procedure exists for figuring out whether a given formula is always valid or not. Thus, as usual, expressivity is at the cost of complexity—in this case, decidability. A restricted form of DC, known as RDC, has decidable inference relation [10]. In fact, formulas of this type may be reduced to the regular expressions. The problem of DC model checking is a topic of ongoing research (cf. e.g., [11] and the references there). In [12], the authors propose a method for solving minimal and maximal reachability problems for the multipriced timed automata. The automata are an extension of timed automata with multiple cost variables that may evolve according to specified rates.

The range of applications varies from the above-mentioned real-time systems (cf. e.g., [13] and the references there) to air traffic control (cf. e.g., [14]) and hybrid systems [15].

Duration calculus is often used for the synthesis of real-time system controllers (see, for example, [16]) and the references there). It is capable of specifying time constraints of dynamic systems. Various methods are used—for example, integer linear problem-solving and optimization problem-solving methods (cf. [16]). DC is used also for traffic system specification (cf., e.g., [13]). There is also a wide range of tools supporting DC (see, for example, [10] and the references there).

In [17], a variant of duration calculus was presented for system discounting, i.e., the idea that something happening earlier is more important than similar events happening later. The idea was introduced earlier into temporal logics such as LTL and CTL. The authors demonstrated decidability of the model-checking for timed automata and a dedicated fragment of discounted duration calculus.

We illustrate our ideas with data concerning volcano monitoring. In fact, volcano monitoring is one of the prime examples for the application of multisensor systems placed in satellites, aeroplanes, balloons, and on the ground. They provide huge amounts of data that have to be preprocessed, analyzed, evaluated, and reasoned about. Thus, it is a proper area to test ideas such as ours. Of particular interest are the Strombolian effects, i.e., periodic volcanic activity phenomena such as temperature picks, gas bursts, and lava eruptions (cf., e.g., [18–20]).

In [1], Koeppen et al. propose a fully automated interactive algorithm called MODVOLC for the analysis of thermal satellite time-series data. The algorithm is aimed at detecting and quantifying the excess energy radiated from the thermal anomalies such as active volcanoes. The algorithm enhances the previously developed approaches (see the references in [1]). It is characterized by the low rate of false positives (see [1,21]). It flags thermal anomalies—in particular, volcanic eruptions. The algorithm was tested for different localizations such as the Anatahan volcano, the Kīlauea volcano, and the Cantarell oil field in the Gulf of Mexico.

In [22], Laiolo et al. compared thermal satellite images of the Etna eruption that took place in 2018 with ground-based geophysical data of summit craters. The Moderate Resolution Imaging Spectroradiometer (MODIS) provided infrared images and helped to identify pixels including the possible hot spots.

Temporal series corresponding to Strombolian effects may be compared using similarity measures. In fact, various similarity measures are used to compare temporal series and functions in general (cf., e.g., [23] and the references there). One can use the dynamic time warping algorithm (cf., e.g., [24] and the references there). This algorithm is used in temporal series analysis for measuring similarity between temporal series that may vary in terms of speed. Algorithms of this type were applied to compare graphical data representing temporal series, but also in comparisons of audio and video materials (cf., e.g., [25,26]).

Dyea et al. [19] proposed a method of detecting Strombolian eruptions based on

training a convolutional neural network. The method automatically categorizes eruptions based on infrared images taken at the rim of a crater atop Mount Erebus. The authors show that machine learning may be effectively used to classify the characteristics of Strombolian eruptions, to facilitate the process of studying their origins, and to assess the hazards posed by volcanic eruptions.

3. Duration Calculus

In this section, we present the basic features of duration calculus (DC). We present the basic idea behind temporal specification relative to time intervals. We also list and explain the temporal modalities it provides and explain how they can be used.

3.1. Using Duration Calculus

In this subsection, we show the basic properties of duration calculus (DC) in its original form. DC allows to specify Boolean-valued functions, which play the role of time-dependent formulas, and to use the DC integral operator to measure the time, or equivalently the duration, when they hold. We also consider an operator for splitting intervals.

DC allows to treat durations of processes. More precisely, it expresses integrals of Boolean-valued functions with values 1 and 0 representing true and false, respectively. Such functions are called state functions since they indicate whether the system is in a given state or not. It should be noted that the properties are defined not in respect to specific points in time but in respect to intervals and in terms of durations.

We start with a simple example of system with states modeled by the $\sin(t)$ function and lasting for the time interval $[0, 2\pi]$ (see Figure 1). This function is first positive on $[0, \pi]$ subinterval and then it is negative on subinterval $[\pi, 2\pi]$. The fact that the function is first positive may be characterized in DC using the Boolean-valued function $sp(t) = 1_{\{x \mid 0 \leq \sin(x)\}}(t)$, which returns 1 if $t \in \{x \mid 0 \leq \sin(x)\}$ and 0 otherwise. In the first case, the value is 1, and in the second case, it equals 0; we identify *true* with 1 and *false* with 0. $sp(t)$ is called the characteristic function of the set $\{x \mid 0 \leq \sin(x)\}$. To describe that the value of \sin is negative, we can use the characteristic function $sn(t) = 1_{\{x \mid \sin(x) < 0\}}$. This function may be obtained from the first one using logical negation $sn = \neg sp$, which swaps 1 and 0. It should be noted that DC does not allow direct access to \sin and its integral, but only via Boolean-valued functions characterizing its values.

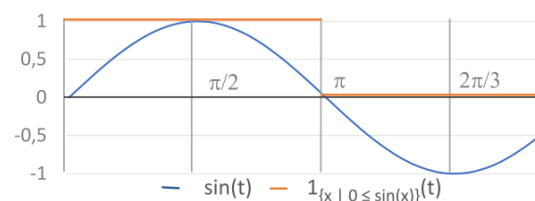


Figure 1. \sin -function and the corresponding state function $1_{\{x \mid 0 \leq \sin(x)\}}$.

Duration calculus enables one to express such properties as the duration of states—for example, the period of time over which \sin remains positive, with the help of state function sp . DC formulas are evaluated always in respect to an underlying time interval. Length function ℓ returns the length of the underlying interval. It is defined using the integration operator \int of DC: $\ell = \int 1$. The semantics of this operator is defined relatively to an underlying interval $[a, b]$ —in this case, $\int 1 = \int_a^b 1 dt$.

For a Boolean-valued function, the integral operator measures the time when the function is true. For intervals of length 2π , the length of subinterval where $\sin(t)$ is positive is equal to the length of the subinterval in which it is negative. This fact may be expressed using the length function ℓ :

$$\ell = 2\pi \Rightarrow \int sp = \int sn \quad (1)$$

In this formula, we use the integral $\int sp$, which, for a given interval $[a, b]$, returns the value $\int_a^b sp \, dt$, i.e., the value of the integral for the interval, and similarly for sn . The formula states that if the length of an interval equals to 2π , then the time when the sp is true equals to the time, the duration, when sn is true. We do not have to consider the fact that the function is equal 0 at the end of intervals, since the functions are characterized in terms of integrals and for subintervals of length 0, the integral value has no influence on the entire integral.

For interval $[0, 2\pi]$, it holds that $\int sp = \pi$, since the duration of sp being true is π . In fact, this expression is true for every interval of the form $[0, x]$, where $\pi \leq x \leq 2\pi$, but it is false for intervals of the form $[x, 2\pi]$. Similarly, we can express the fact that the duration of sn being negative is π : $\int sn = \pi$. This expression is true on every interval of the form $[y, 2\pi]$ where $0 \leq y \leq \pi$. In both cases, we have to use the auxiliary functions sp, sn to characterize sin in terms of being positive or negative, respectively.

3.2. Temporal Modalities

In this subsection, we present three temporal modalities definable in DC: the chop, the after-modality, the sometime-modality, and the always-modality. The chop is denoted by $\hat{}$ (cf., e.g., [10]); it applies to two neighboring intervals. It allows one to express the fact that a property holds in the first interval and another property holds in the second interval. The last two modalities occur in all kinds of modal and temporal logics (cf., e.g., [2]), but they have a specific meaning in DC. The sometime-modality is denoted by \diamond and requires a property to hold for some time interval. The always-modality is denoted by \Box and means that a property holds for all time periods.

Formally, for two formulas F, G , the formula $F \hat{} G$ means that for an interval $[a, b]$ in question, there exists a number x such that $a \leq x \leq b$, F holds for interval $[a, x]$ and G holds for interval $[x, b]$. For example, the fact that sin is initially positive and then negative (see the previous subsection) can be described in respect to two subintervals dividing the interval in question. We may express it by formula $(\int sp = \pi) \hat{} (\int sn = \pi)$. It is true on the interval $[0, 2\pi]$ as the interval can be split into two subintervals $[0, \pi]$ and $[\pi, 2\pi]$; thus, we chose $x = \pi$.

Note that the operator $\hat{}$ is associative; thus, the formula of the form $F \hat{} (G \hat{} H)$ is equivalent to formula $(F \hat{} G) \hat{} H$, since the formulas depend on splitting an underlying interval $[a, d]$ into three parts $[a, b], [b, c], [c, d]$ where the subformulas F, G, H have to hold, respectively. Consequently, we will drop the brackets when using chop.

Given a formula G and an interval $[a, b]$, the fact that $\diamond G$ holds for interval $[a, d]$ means that it has a subinterval $[b, c]$, i.e., $a \leq b \leq c \leq d$, such that G holds in $[b, c]$. The may-modality may be expressed using the chop operator $true \hat{} G \hat{} true$, which means that we can divide the underlying interval into three subintervals and that G holds for the middle subinterval, whereas we do not require anything from its adjacent subintervals on the left- and right-hand sides. For example, we have the following property:

$$\pi < \ell \Rightarrow \diamond 0 < \int sp = \ell \quad (2)$$

This formula says that if the length of the underlying interval is larger than π , then there is a subinterval of length larger than 0 where the property sp holds all the time, i.e., sin is positive.

The modal formula $\Box G$ means that for all subintervals of an underlying interval, the formula G holds. For example, for all subintervals of interval $[0, \pi]$, the function sp is true all the time: $\Box \int sp = \ell$. Modality \Box can be expressed using the sometime-modality \diamond : $\neg \diamond \neg G$; the formula means that no subinterval exists where G does not hold. Vice versa, $\diamond G$ can be expressed as $\neg \Box \neg G$. Thus, both modalities are dual. Both are definable in terms of the chop operator, but we will use them as syntactic sugar to facilitate readability of the formulas.

4. The Idea of Duration Calculus for Functions

In this section, we present the idea of duration calculus for functions (DC4F). The objective is to treat Riemann integrable functions with values of type real. Such functions can represent continuous as well as discrete time series. We informally present features of the proposed calculus and illustrate them with simple examples demonstrating its basic characteristics, the way it can be used, and the capabilities of the logic.

4.1. DC versus DC4F

DC allows one to treat Boolean-valued functions only and to reason about their durations. It does not allow to integrate general functions. In this subsection, we propose an extension of DC supporting all Riemann integrable functions. We present the basic features of the proposed logic. We also present examples that are specific to DC4F and, thus, cannot be expressed in DC.

In DC, a state function f , we can consider the time intervals, or periods, when this states holds. Such a case corresponds to considering the corresponding characteristic functions: $1_{\{y|0 \leq f(y)\}}(t)$. The integrals of the characteristic functions correspond to the duration of the characterized state or property. For example, as explained in the previous section, the fact that \sin is first negative and then positive can be expressed in DC only via the auxiliary characteristic functions sp and sn , but not directly. In DC, we can integrate function $sp(x)$ and obtain the duration over which it remains true, but we cannot integrate \sin or any other function that is not Boolean. Thus, for example, we can express the property that for intervals of length 2π , \sin is positive as equally long as it is negative $\ell = 2\pi \Rightarrow \int sp = \int sn$.

However, not all properties of integrals can be expressed in this way. For example, we cannot express properties such as $\ell = 2\pi \Rightarrow \int \sin = 0$, i.e., that if the length of the underlying interval is 2π , then the integral is equal to 0. Thus, DC is not expressive enough to treat time series in general.

DC4F is a natural extension of DC as allows all integrable functions. For example, we can consider $\int \sin$. In logical terms, it is a conservative extension of DC. Conservativity means that a formula of DC is a tautology in the sense of DC if, and only if, it is a tautology of DC4F; equivalently, it is satisfied in a DC model if, and only if, there is a DC4F extension of the model in which the formula is satisfied. In general, DC can be expressed in DC4F by restricting the set of integrable functions so that it contains Boolean-valued functions only.

4.2. Monotonicity

We formulate the definition of monotonicity in terms of intervals and integrals. The property that a function is monotone over an interval can be expressed in DC4F by the requirement that for two neighboring subintervals of the same length, the integral over the left one has smaller or equal value to the integral over the right one. It can be expressed by formula M of the form $\int f = x \wedge \ell = y \wedge \int f = z \wedge \ell = w \wedge y = w \Rightarrow x \leq z$. This property must hold for all subintervals of this kind; thus, we have the formula: $\Box M$. Figure 2 shows a monotonically increasing function f .

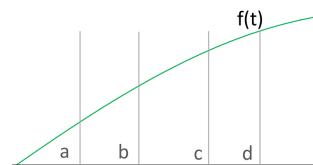


Figure 2. Monotone function $f(t)$.

Alternatively, we can define the monotonicity property in the following way:

$$\forall_{x,y} \Box \{(\ell > 0 \wedge \int f / \ell = x) \wedge (\ell > 0 \wedge \int f / \ell = y) \Rightarrow x \leq y\} \quad (3)$$

This formula, though it may seem complicated, is rather simple. It states that for a given interval and for all its adjacent subintervals, the normalized value of the integral for the left subinterval is lower than or equal to the normalized value of the integral for the right subinterval. It is illustrated in Figure 2. Intervals $[b, c]$ and $[c, d]$ are adjacent. The values of the integrals for these intervals have to be normalized by their length, as their lengths are different. We will abbreviate this property of function f as $MIncreasing(f)$. Analogously, we can define the property of monotonic decrease, $MDecreasing(f)$. We use a capital letter to indicate that both functions have a functional argument instead of a time value.

Although length operator ℓ occurs in the characteristics of the first and of the second interval, it returns independent values. Due to transitivity of smaller or equal relation \leq , the monotonicity property holds for all subintervals, not only for the adjacent ones. Thus, $\int_a^b f/\ell \, dt \leq \int_b^c f/\ell \, dt \leq \int_c^d f/\ell \, dt$ (see Figure 2).

4.3. Limits and Amplitude

In this subsection, we show how to define lower limits of functions and their minimal amplitudes. The definitions are based on temporal modalities. Upper limits and maximal amplitudes can be defined analogously. We use those definitions below when dealing with volcanic Strombolian effects.

We start with the case when a function $g(t)$ is an upper approximation of a function $f(t)$ or, equivalently, function $f(t)$ is a lower approximation of function $g(t)$ (see Figure 3). This property can be expressed by demanding that for each subinterval of a given interval, the integral of f is smaller than or equal to the integral of g :

$$\Box \int f \leq \int g \quad (4)$$

The property that g increases c times faster than c can be expressed in DC4F in a simple way: $\Box \int cf \leq g$. It should be noted that we cannot directly express it in DC.

The diagram presented in Figure 4 shows function h , which is above value a . The property that, for a given interval, values of function h are above a certain limit a all the time can be expressed as follows:

$$\Box \ell a \leq \int h \quad (5)$$

The integral of a is equal to the value ℓa . Consequently, we demand that it remains smaller than the integral of h for all subintervals of the underlying interval. The fact that function h exceeds threshold y for some time can be expressed as follows:

$$true \wedge (0 < \ell \wedge \Box \ell y \leq \int h) \wedge true \quad (6)$$

This formula states that for some subinterval of a nonzero length, the value of the integral is not lower than the value of a multiplied by the length of the subinterval. The function shown in Figure 4 extends beyond line y for a time period. Equivalently, the property can be expressed as follows:

$$\Diamond (0 < \ell \wedge \Box \ell y \leq \int h) \quad (7)$$

The fact that an amplitude of a function is at least d can be expressed as follows:

$$\exists x, y (\Diamond (0 < \ell \wedge \Box \int h \leq \ell x)) \wedge \Diamond (0 < \ell \wedge \Box \ell y \leq \int h) \wedge d \leq y - x \quad (8)$$

Formula (8) says that there exist two subintervals of nonzero length. The integral of h for the first one is smaller than or equal to ℓx , as it was expressed by formula (7). Analogously, the integral of h for the second one is larger than or equal to ℓy . Formula (8) requires also that the difference $y - x$ is equal to at least d .

In Figure 4, the intervals correspond to the parts of diagrams of h located below the x and above the y line, respectively, are indicated by bold lines. Below, we abbreviate

formula (8) by $Amplitude(h, d)$. Formula $PeakAbove(h, y)$ is a shorthand of formula (6) and expresses the fact that function h is above threshold y for some time period.

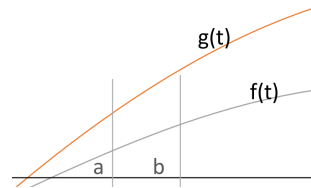


Figure 3. Functional limit.

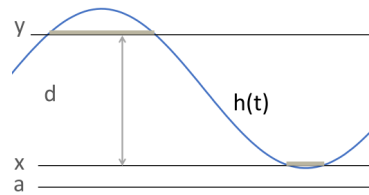


Figure 4. Constant limits and amplitude.

5. Syntax and Semantics of DC4F

In this section, we define the formal syntax of DC4F and describe its informal semantics. We list some basic properties of the proposed logic and show how to apply it in the multidimensional case, such as those involving two-dimensional images.

5.1. Syntax of DC4F

Every logic has its proper language, which has to be formally defined. In this subsection, we define the formal syntax of DC4F. This syntax is close to that of DC. The crucial difference consists in the fact that we allow all unary integrable functions, not only the Boolean-valued ones, and their integrals. Basically, there are constants, variables, and function symbols corresponding to the integrable functions, which can be added and multiplied by constants. There is also an integral operator. Formulas are either of the atomic form, when real numbers are compared, or are composed from other formulas.

To define the syntax formally, designated sets of variables, functions, and relation symbols are required. There are global variables, which remain unchanged. We assume that \mathbf{VSym} is an unbound set of real-valued variables denoted by letters x, y, z, \dots . We also assume that there is a set $\mathbf{CSym} = \{0, 1, \dots, true, false, \dots\}$ of constant symbols. The set \mathbf{FSym} contains unary functions symbols f, g, h, \dots corresponding to integrable functions. The set \mathbf{FISym} contains functional symbols X, Y, Z, \dots corresponding to functionals with arguments in the form of time intervals and values of type real \mathbb{R} (cf. Appendix A).

The syntax of the logic comprises three categories of expressions: functions Fn containing integrable functions, terms $Real$ containing expressions of type \mathbb{R} , and formulas Fo containing logical formulas in the proper sense of the word. We assume in the definition below that $f \in \mathbf{FSym}, C \in \mathbf{CSym}, x \in \mathbf{VSym}, X \in \mathbf{FISym}$.

- $Fn ::= f \mid C \mid C \cdot Fn \mid Fn \oplus Fn$, for $\oplus \in \{+, -, \vee\}$
- $Real ::= C \mid x \mid X \mid \int Fn \mid Real \oplus Real$, for $\oplus \in \{+, -, \cdot, \wedge\}$
- $Fo ::= true \mid false \mid Real \oplus Real \mid \neg Fo \mid Fo \vee Fo \mid \exists_x Fo \mid Fo \wedge Fo \mid Fo^*$, for $\oplus \in \{<, =\}$

The first category includes constants and functions of type real; the constants are treated here as functions with arity equal to 0. The functions can be multiplied by constants and added. There is also the maximum operator \vee ; it denotes their maximum and is a reminiscent of DC (cf., e.g., [4]). In DC, there is an indefinable negation operator \neg that applies to state functions (cf., e.g., [4]). It is characterized by the following property: $f(x) = 1$ if, and only if, $\neg f(x) = 0$. We can define the negation \neg on integrable functions such that it coincides with negation on state functions: $\neg f =_{def} 1 - f(x) \vee f(x) - 1$. Consequently, if

$f(x) = 1$, then $\neg f(x) =_{def} 1 - 1 \vee 1 - 1 = 0$. If $f(x) = 0$, then $\neg f(x) = 1 - 0 \vee 0 - 1 = 1$, since \vee is the maximum.

The second category includes constants, variables, and applications of the integral operator. There are also auxiliary operators X on intervals with values of type real. Complex terms can be constructed from simpler ones using arithmetic operations.

The third category comprises formulas as such. The atomic formulas include constants *true* and *false*; the atomic formulas are also formed from relational symbol applied to terms, such as $u < v$. Complex formulas are composed from other formulas by the application of logical operators: logical negation, alternative, existential quantification, concatenation, the chop operator, and iteration. Thus, if F and G are formulas, then the alternative $F \vee G$, the chop $F \wedge G$, the iteration F^* , and the quantification $\exists_x F$ are formulas as well.

As an example of the above syntax, we can present formula $\int \sin = 2 \wedge (\int \sin = -2)$, provided that \sin belongs to **FSym**. The formula is obtained by the application of chop to two atomic subformulas. The first subformula is obtained by the application of equation $=$ to terms $\int \sin$ and 2; the first term is obtained from function \sin by application of the integral operator. The other subformula is obtained in a similar way.

It should be noted that relations $u \leq v$ and $u > v$ can be defined in terms of $<$ and $=$ using logical operators \neg and \vee , e.g., in the first case, we have $u < v \vee u = v$. Similarly, the conjunction $F \wedge G$ is defined as $\neg(\neg F \vee \neg G)$. The implication $F \Rightarrow G$ is defined as $\neg F \vee G$. The general quantifier $\forall_x F$ is defined as $\neg \exists_x \neg F$. As mentioned in Section 3.2, modalities \Box , \Diamond are defined using chop and negation. $\Box F$ is defined as $\neg(\text{true} \wedge (\neg F \wedge \text{true}))$ and $\Diamond F$ as $\text{true} \wedge F \wedge \text{true}$. The length operator ℓ is defined as $\int 1$ (see Section 3.1). We assume that \wedge binds stronger than \vee and \vee and \wedge bind stronger than quantifiers \exists and \forall . Quantifiers bind stronger than implication \Rightarrow . Implication binds stronger than modalities \Diamond and \Box .

5.2. Informal Semantics of DC4F

In this subsection, we present the intuitive semantics of the previously defined syntax. The presentation is informal, but it can be easily formalized, as shown in the Appendix A. We discussed it for the three syntactic categories defined above: *Fn*, *Real* and *Fo*.

We assume here that the time domain \mathbb{T} is the set of non-negative real numbers \mathbb{R}_+ . In the case of discrete time modeled by natural numbers \mathbb{N} , for every discrete function f , we define the corresponding step function g on \mathbb{R}_+ by extending the values of f to the corresponding unit interval: $g(t) = f(\lfloor t \rfloor)$, where $\lfloor t \rfloor$ is the largest integer $n \in \mathbb{N}$ such that $n \leq t$. Thus, we can treat discrete functions, or discrete time series, as if they were defined for continuous time.

Elements belonging to category *Fn* are interpreted as time series, and more precisely as Riemann integrable functions with the domain \mathbb{R}_+ . The functions may be multiplied by constants, added, and subtracted; thus, the set corresponds to a linear space. The function $f(t) \vee g(t)$ returns the maximum of $f(t)$ and $g(t)$.

The category *Real* contains terms of type \mathbb{R} . The constants, e.g., 0 or 1, are interpreted as the corresponding real numbers. Variable symbols are evaluated by functions mapping them into the set \mathbb{R} . For $f \in \text{Fn}$ and an interval $[a, b]$, where $a, b \in \mathbb{R}_+$ and $a \leq b$, term of the form $\int f$ is interpreted as the integral $\int_a^b f(t) dt$. This operator is linear in respect to functions: $\int c f = c \int f$ and $\int f + g = \int f + \int g$.

A term t depends on variable $x \in \text{FSym}$ if t contains it. In general, the value of terms containing variables depend on the values of its variables. Valuations are mappings of variables into \mathbb{R} . We can, for example, define valuation val_1 as mapping of the form $x \mapsto 1, y \mapsto 2, z \mapsto 5$. Let term p be of the form $x + y + z + 2$. The value of p for val_1 is 10; it does not depend on an underlying interval. The value of a term depends on an underlying interval if contains operator \int . In the case of term q of the form $(\int 1) + z$, the value depends on the underlying interval and variable z ; for interval $I_1 = [0, 2]$ and val_1 , the value is 7.

The category of formulas *Fo* plays a crucial role, as formulas are the proper objects of logical reasoning. In fact, formulas can be understood as Boolean-valued terms. The satisfaction of formulas, i.e., the fact whether formulas are true or false, is defined in respect

to an underlying interval and a valuation of the variables they include. The values of these terms may be compared using $<$, $=$, given the underlying interval and the valuation. The result of this comparison is either true or false, depending on the value of the corresponding terms. For example, for interval I_1 and valuation val_1 , formula $q < p$ is true but formula $q = p$ is false.

Let $I = [a, b]$ be an interval and val be a valuation. Formula $\neg F$ is satisfied in interval I for valuation val , if formula F is false in interval I for val . Similarly, the alternative $F \vee G$ is satisfied if F or G is satisfied in I for val .

In case of chop, $F \wedge G$ is satisfied in interval I for valuation val if there exists $c \in [a, b]$ such that F is satisfied in $[a, c]$ for val and G is satisfied in $[c, b]$ for val . Iteration F^* of formula F is satisfied in I for val if there is an n such that we can split the interval into subintervals $[a, a_1], [a_1, a_2], \dots, [a_{n-1}, b]$ such that F is satisfied in each subinterval for val (cf. [4] Section 4). An example of the iteration formula is presented below and its formal semantics are presented in the Appendix A.

Quantified formula $\exists_x F$ is satisfied in interval I for valuation val if there exists a valuation val' which differs from val only for variable x such that F is satisfied in I for val' . The fact that val and val' differ only at x means that for every variable y different from variable x , it holds that $val(y) = val'(y)$. For example, $\exists_x x + y > 25$ is satisfied for valuation val_1 defined above; in fact, we can define val' as val_1 but $val'(x) = 30$.

5.3. Basic Properties

In this subsection, we list some basic properties of the logical operators, such as the linearity and monotonicity of the integral. We use them in an exemplary reasoning.

The DC4F calculus has the following logical properties:

1. $\int 0 = 0, \quad \int c = c \int 1 = c\ell$
2. $\int cf = c \int f, \quad \int f + g = \int f + \int g$
3. If $f(x) \leq g(x)$, for every $x \in \mathbb{R}_+$, then $\int f \leq \int g$
4. $\int f = x \wedge \int f = y \Rightarrow \int f = x + y$
5. $F \Rightarrow F \wedge (\ell = 0), \quad F \Rightarrow (\ell = 0) \wedge F$
6. $F \Rightarrow true, \quad (F \Rightarrow G) \Rightarrow (F \wedge H \Rightarrow G \wedge H), \quad F \wedge G \Rightarrow G \wedge F,$
7. $(F \Rightarrow G) \Rightarrow (F \wedge H \Rightarrow G \wedge H), \quad (F \Rightarrow G) \Rightarrow (H \wedge F \Rightarrow H \wedge G)$
8. If the formulas $F, F \Rightarrow G$ hold in an interval I for a valuation val , then G holds in an interval I for a valuation val .

Point (1) concerns integration of constants. Point (2) expresses the fact that integrals are linear operators, i.e., they commute with multiplication by constants and are distributive in respect to addition. Point (3) expresses the monotonicity of integral operators. If the antecedent inequality holds for all non-negative reals, then the consequent inequality holds for all time intervals. Property (4) can be expressed as $\int_a^b f dt + \int_b^c f dt = \int_a^c f dt$. The above-mentioned points are specific to DC4F because it allows integrable functions, not only propositional ones as in the case of DC. The following points are common with DC (cf. [4], Section 2). Property (5) states that every interval can be split into an interval of length 0 and the rest. Point (6) specifies three exemplary propositional tautologies (cf., e.g., [27]) that we use later in an exemplary derivation. The first tautology says that truth is implied by any formula. The second tautology expresses the monotonicity of the conjunction operator. The third one expresses the commutativity of conjunction. It should be noted that all propositional tautologies hold in case of DC as well as DC4F. Point (7) specifies the monotonicity of the chop operator. Property (8) is the modus ponens reasoning rule. It says that if the antecedent of an implication is true and the implication is true, then the consequent of the implication is true as well. The rule is used in many kinds of logic (cf., e.g., [2,27]).

5.4. Multidimensional Case

The functions considered in DC and DC4F are unary, i.e., they are of the form $f(t)$, where t is the time parameter. In this subsection, we consider integrals over multidimen-

sional sets and spaces. Such cases may occur when, for instance, images of an area are considered and measurements are performed. The development of values over time may concern specific areas or multidimensional spaces. In this case, the integrated function has a number of variables, apart from the time variable. We show that in fact such a multidimensional case can be reduced to the case of unary functions. In fact, we utilize such integrals in the examples presented in the following section.

An image may be represented by a number of pixels with two coordinates x, y . Thus, the area of an image may be considered as a set $A \subseteq \mathbb{R}^2$. In the multidimensional case, it has the form $A \subseteq \mathbb{R}^n$. If the images are used to measure certain values, then the measurement can be modeled by a function. The measurements may be time-dependent. Thus, we consider time-dependent integrable functions of the form $f(t, x_1, \dots, x_n)$ and multidimensional sets of the form $\mathbb{R}_+ \times A$. For an interval I , the corresponding integral has the form: $\int_{I \times A} f d(t, x_1, \dots, x_n)$, where $d(t, x_1, \dots, x_n)$ is a product measure on the space $\mathbb{R} \times \mathbb{R}^n$. Equivalently, we can present the integral using the characteristic function 1_A corresponding to set A , the function has value 1 for the elements of A and 0 for all other arguments: $\int 1_A f(t, x_1, \dots, x_n) d(t, x_1, \dots, x_n)$.

This integral can be presented also in the form $\int_I \int_A f d(x_1, \dots, x_n) dt$. Thus, the integral $\int_{I \times A} f d(t, x_1, \dots, x_n)$ has the form $\int_I g(t) dt$, where $g(t) = \int_A f(t, x_1, \dots, x_n) d(x_1, \dots, x_n)$. Consequently, it can be presented as an integral of unary function $g(t)$ depending on the time parameter t only.

6. Applications and Validation

In the case of multisensor systems, the data generated may be of various types, such as temperature, pressure, or density measurements, videos, or sound, among others. It is hard to bring them into a uniform and consistent model. Thus, quite often, textual specifications and informal validations are used, which is inherently imprecise. Specification, interpretation, comparison, and reasoning about such heterogeneous data pose a nontrivial problem. In this section, we apply the DC4F calculus to express measurements and complex behaviors. The goal is to illustrate the way DC4F can be applied and to demonstrate its capabilities for specifying temporal series. We use examples from the area of volcanic activity monitoring by multisensor systems because of their complexity and, thus, the challenge that they pose. We aim to show that the extension of DC that we propose in this paper provides a uniform language, models, and a reasoning system to integrate, describe, and reason about such data. The examples are used to show the practical applicability of DC4F and to demonstrate how it can be used. However, it is not our goal to adequately describe those phenomena per se. It should be also pointed out that the units of measurement do not play an essential role for the illustration. Our concepts concern mathematical properties and, thus, are independent of the units used.

6.1. Volcano Monitoring

To demonstrate the applicability of DC4F, we looked at several examples of signal processing and the corresponding temporal series. It turned out that volcano monitoring is an interesting and challenging task due to the complexity of the process, various monitoring methods used, and the heterogeneity of the acquired data. In this subsection, we present a brief presentation of volcano monitoring processes, as they are instrumental in illustrating our ideas.

As pointed out by Corradini et al. in [18,28], volcano monitoring processes require the correlation of satellite data, provided that satellites are present over the specific target. Satellite data are heterogeneous in nature and relate to temperatures measured in different wavelengths, among other measurements. Several phenomena, such as temperature picks, gas explosions, and magma eruptions, need to be considered (cf., e.g., [19,20]).

Strombolian volcano eruptions are a widely studied topic. They are characterized by specific cyclical patterns. Therefore, their specification is a challenging task. A Strombolian volcano eruption is characterized by regular, relatively mild blasts. It consists of

the ejection of fragments of solidified lava, material deposited by previous volcanic eruptions, and masses of molten rock to altitudes ranging from tens to hundreds of meters (cf., e.g., [18–20]). Eruptions of this type are named after the Stromboli volcano. They are commonly observed in volcanoes fed by low to moderate viscosity magma. The explosions are caused by the bursting of gas slugs, which rise to the surface faster than the surrounding magma. Quakes associated with such explosions occur at low depths (cf. e.g., [20]). Spectroscopic measurements performed on the Stromboli volcano were used to demonstrate that gas slugs originate from the volcano–crust interface at the depth of approximately 3 kilometers, which may promote slug coalescence. In the case of the Stromboli volcano, quantitative constraints for the depth of conglomerates of high-pressure gas bubbles inducing Strombolian activity can be identified on the basis of spectroscopic measurements of the magmatic gas phase driving the explosions.

6.2. Dealing with Estimates

Various parameters are considered when a volcano eruption is monitored. These parameters need to be measured and compared. In this subsection, we show that functional limits may be dealt with in CD4F.

Thermal satellite images of the Etna eruption that took place in 2018 were compared with ground-based geophysical data of summit craters using the Moderate Resolution Imaging Spectroradiometer (MODIS) (see [22]). The technology provides infrared images and makes the identification of hot spots including pixels possible. The thermal anomalies may be quantified in terms of volcanic radiative power (VRP), expressed in watts (see [22] and the references there). The VRP value is calculated as follows:

$$VRP_{pIX} = Ap_{IX} \times 18.7 \times (L_{4alert} - L_{4bk}) \quad (9)$$

where Ap_{IX} is the pixel area of approximately 10^6 m^2 , L_{4alert} is the recorded the middle infrared radiance ($\text{W m}^{-2}\text{sr}^{-1}\text{m}^{-1}$), and L_{4bk} are the background pixels. These values allow to estimate the average lava erupted volume, i.e., the time average discharge rate (TADR), through a unique parameter called radiant density c_{rad} (J m^3):

$$TADR = \frac{VRP}{c_{rad}} \quad (10)$$

This parameter represents capacity of the lava body to radiate heat (see [22] and the references there). It allows to estimate the erupted lava volume by assuming the appropriate parameter c_{rad} for the lava flow in question.

The fact that the actual lava flow volume $TADR$ is between limits $LApp$ and $UpApp$ (see Figure 5) can be specified by the following formula, which is analogous to Formula (5) from Section 4.3. This fact cannot be expressed in DC, since DC only allows to consider the duration properties on intervals but not the values of integrals:

$$\square \int LApp \leq \int TADR \leq \int UpApp \quad (11)$$

The inequalities express the desired property in terms of intervals. The formula means that the inequalities hold for every subinterval of the given interval (see Section 3.2).

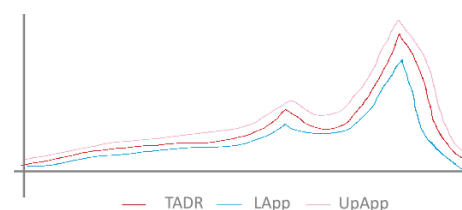


Figure 5. TADR with its lower and upper approximation.

6.3. Dealing with Two-Dimensional Spaces

There exist algorithms for detecting thermal anomalies in two-dimensional pictures related to volcano eruptions. In this subsection, we discuss the application of DC4F to such cases. We specify the units of measurement, but it should be noted that for the presentation, the units of measurement and exact numeric values obtained do not play a significant role in the presentation data, since in our framework we can deal with values of type \mathbb{R} independently of their interpretation.

MODVOLC is a fully automated algorithm for the analysis of thermal satellite time-series data [1]. The algorithm is based on spectral wavelengths centred at 4 and 11–12 μm emitted by high-temperature volcanic sources and the surrounding Earth surface. In the case of a volcanic eruption, the radiance is measured from the corresponding pixel. One pixel corresponds to approximately 4 km. Images are collected every 15 min. In pixels that show a thermal anomaly, the radiance measured in the mid-infrared (4 μm) is significantly higher than in surrounding pixels where no thermal anomalies are depicted [1,21].

Figure 6 shows an excerpt of Figure 6 from [1]. It contains results obtained by the MODVOLC algorithm detecting thermal anomalies for different thresholds 2.0, 2.6, and 3.0, respectively, in case of the Anatahan volcano monitoring data. The following formula presents the Local Index of Change of the Environment $\otimes_v(x, y, t)$ representing the degree to which each pixel deviates from its normal behavior normalized by its variability (cf. [1,29]).

$$\otimes_v(x, y, t) = \frac{V(x, y, t) - V_{REF}(x, y)}{\sigma_v(x, y)} \quad (12)$$

An individual observation is modeled by the function $V(x, y, t)$ specifying the state of a pixel (x, y) at time t . Its normal behavior is expressed by function $V_{REF}(x, y)$ and its variability by $\sigma_v(x, y)$. One can then identify pixels (x, y) for which the function exceeds a given threshold $Tr \leq \otimes_v(x, y, t)$. The number of such pixels can be then computed for the entire area of interest A using its characteristic function 1_A , i.e., $(x, y) \in A \Leftrightarrow 1_A(x, y) = 1$ in the discrete case when single pixels are considered and each pixel has measure 1. A thermal anomaly is a pixel with a temperature measurement value above a defined threshold. The total number of thermal anomalies occurring at time t with values above threshold Tr is modeled by function $anomalies(Tr, t)$ defined by the following integral:

$$anomalies(Tr, t) = \int_A 1_{\{(x, y) | Tr \leq \otimes_v(x, y, t)\}} dx dy = \int_{\mathbb{R} \times \mathbb{R}} 1_A 1_{\{(x, y) | Tr \leq \otimes_v(x, y, t)\}} dx dy \quad (13)$$

Thus, the number of anomalies is the integral of the product of the characteristic function of set A , corresponding to the area of interest, and of the characteristic function identifying points with temperature above threshold Tr . In the continuous case, the value of the integral is equal to the measure of the set of thermal anomalies in the area of interest.

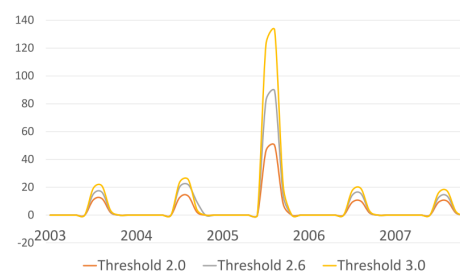


Figure 6. Number of thermal anomalies detected by the hybrid algorithm over MODVOLC using thresholds of 2.0, 2.6, and 3.0, respectively.

6.4. Specification of Strombolian Effects

Strombolian effects are characterized by cyclic periods of repetitive behaviors. Repetitive gas explosions and temperature peaks occurring in a periodic manner are one of

such aspects. These explosions are caused by deep slag-driven explosive activity (cf., e.g., [20]). Those effects include also consecutive magma eruptions. In this subsection, we demonstrate that these kinds of effects can be specified in DC4F. We utilize the temporal properties formulated in the preceding section. The data that we use were obtained on 9 April 2002 during hours of passive and explosive degassing on the Stromboli volcano that was most active at the time [20]. The spectrometer was placed at a distance of 240 m.

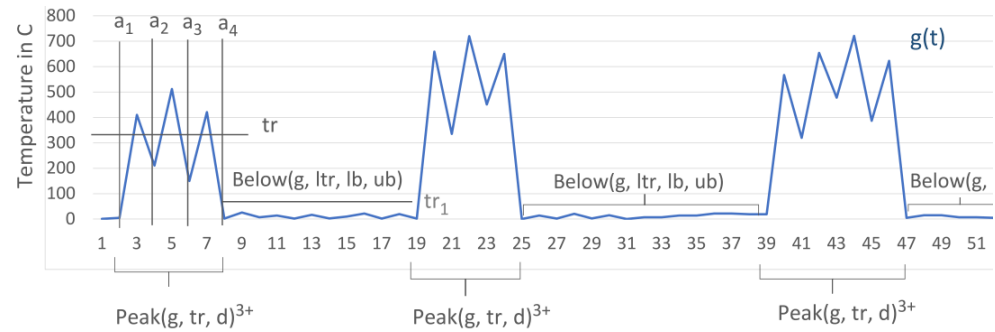


Figure 7. Periodic peaks in radiation corresponding to source temperature.

Repetitive temperature increases are shown in Figure 7. It shows the diagram presented in [20] (see Figure 1 and Figure S1 there) representing peaks in radiation corresponding to source temperature in Celsius. Figure 7 shows a function $g(t)$ with cyclic periods of consecutive value peaks and drops representing the level of radiation. The peaks can reach different maximal values, the drops can reach different minimal values, and their numbers are not fixed in a cycle. The time scale of Strombolian effects is usually hours and days (cf., e.g., [18,20,28]). In Figure 7 and below, we use only numeric time values, as the exact timescale does not play any role in the demonstration and may be changed in an arbitrary manner.

Each peak formation is characterized by a monotone increase of g followed by its monotone decrease. In consequence, a peak can be specified in DC4F by formula $MIncreasing(g); MDecreasing(g)$ (see Section 4.2). For example, Figure 7 shows a peak corresponding to interval $[a_1, a_2]$.

We can also require that the peak values be above a certain threshold tr and the amplitude be above the value of d . These requirements can be formulated as $PeakAbove(g, tr)$ and $Amplitude(g, d)$, respectively. The first formula states that function g reaches a value above threshold tr , and the second one states that its amplitude is at least equal to d (see Section 4.3). In Figure 7, the first peak is above 400 and the amplitude is above 100; of course, these parameters can be adjusted arbitrarily. The fact of reaching such a peak with the constraints concerning values of threshold and amplitude can be expressed by the formula $MIncreasing(g) \wedge MDecreasing(g) \wedge PeakAbove(g, tr) \wedge Amplitude(g, d)$. This formula is the logical conjunction of the previous formulas; we abbreviate it as $Peak(g, tr, d)$. We use capital letters to indicate that functions have functional arguments, not only arguments of type \mathbb{R} .

Now, the peaks and the corresponding drops form a repetitive behavior. The left-hand side of Figure 7 shows three peaks, modeling COS bursts, which correspond to the intervals $[a_i, a_{i+1}]$, for $i = 1, \dots, 3$. We can write the corresponding formula as $Peak(g, tr, d) \wedge Peak(g, tr, d) \wedge Peak(g, tr, d)$. It means that the interval may be split into three subintervals such that each satisfies the formula $Peak(g, tr, d)$. We abbreviate the formula as $Peak(g, tr, d)^3$.

In general, for a formula F , n repetitions of the form $F \wedge \dots \wedge F$ are abbreviated as F^n . This formula means that the interval may be split into n subintervals in which F is satisfied. If the number of repetitions is arbitrary, then we write F^* . If it equals at least n , then we write F^{n+} , which is equivalent to $F^n \wedge F^*$.

The peaks occur in cyclic series separated by periods of low volcanic activity. Let formula $Below(g, ltr, lb, ub)$ be the abbreviation of $(\exists g \leq ltr) \wedge lb \leq g \leq ub$. This means

that g is always below threshold ltr in a given time interval and that the length of the interval is between time bounds lb and ub (cf. Section 4.3). We can compose the formulas again to specify the behavior shown in Figure 7; for $m = 3$ and $n = 3$, the formula has the form:

$$\{Peak(g, tr, d)^{m+} \wedge Below(g, ltr, lb, ub)\}^n \quad (14)$$

This Strombolian activity consists of three cycles; each cycle is composed of at least three bursts followed by a period of a low volcanic activity. The three cycles of repetitive bursts are indicated in Figure 7 by brackets below the time axis. The periods of limited volcanic activity are indicated in a similar way.

Strombolian activities may be also detected using satellites equipped with thermal imaging cameras, which sometimes are the source of more reliable indicators than sensors on the ground (cf., e.g., [18,28]). However, if the satellites are not stationary or if visibility is reduced by clouds, then data from thermal imaging cameras cannot be used. We can specify the conditional visibility of thermal peaks by formula $\int visibility = \ell \Rightarrow Peak(f, tr', d')$. In this formula, *visibility* is a Boolean-valued time-dependent function modeling visibility; it is true if there is visibility and false otherwise. Function f models temperature measurements at a certain frequency. Parameter tr' is the corresponding temperature threshold and parameter d' is the corresponding amplitude. The formula is conditional and reads as follows: if there is visibility, then a thermal peak is observed. We abbreviate this formula by $CondPeak(f, tr', d')$. Analogously, we can define $CondBelow(b, ltr', lb', ub')$ for the case in which the thermal radiance measurement is below a certain threshold. If the peaks in thermal anomalies coincide with the peaks in COS bursts, then we can transform formula (14), taking the thermal anomalies into account as well:

$$\diamond \{CondPeak(g, tr, d)^{m+} \wedge CondBelow(g, ltr, lb, ub)\}^n \wedge \{CondPeak(f, tr', d')^{m+} \wedge CondBelow(b, ltr', lb', ub')\}^n \quad (15)$$

The sometime-modality is used here to specify that we do not require anything for the time periods before and after the characteristic behavior. We may also specify here the minimum and maximum lengths of the periods of low activity, if needed.

Of course, the formulas become more and more complicated as we specify more kinds of behaviors and data, e.g., multisensor data, and constraints. However, the point is that the formulas may be specified in DC4F. Thus, we may precisely formulate fine hypotheses concerning complex behaviors. Furthermore, if implemented, the hypotheses may be verified in respect to existing data. They may also be used to automatically mine the available data, historic or incoming.

6.5. Reasoning in DC4F

In the previous subsections, we presented the capabilities of DC4F for the specification of periodic behaviors and the hypothesis framing. In this subsection, we present a relatively simple example of reasoning in DC4F utilizing the properties of DC4F defined in Section 5.3. The example is somewhat artificial, as we did not find proper examples in the literature, and we do not claim that the formulas truly express actual volcanic behavior. We simplify the proof, as its full version would be rather long. As in the previous section, we skip units of measurement as they do not play any role for the illustration.

Suppose that we know that a Strombolian effect is characterized by two properties occurring within 24 h: a series of COS bursts and an observation of thermal anomalies. The data on COS burst is collected by sensors on the ground and the data on thermal anomalies are collected by a sensor placed on a satellite. The behavior is specified by functions $f_{COS}(t)$ and f_{anom} representing the emission level of COS and the number of observed anomalies, respectively. We use abbreviation $COSB$ for the formula obtained from formula (14) by substituting f_{COS} for g , 1 for n , and by setting other parameters somehow. We assume that $f_{anom}(t) = anomalies(t, Tr)$ for a certain threshold tr (cf. Section 6.3).

Let the Strombolian effect be characterized by at least 10 COS bursts and thermal anomalies of value 5 occurring within 24 h. Moreover, let it be an established fact that 12 COS bursts are always accompanied by thermal anomalies of value 7. The following formulas specify these two properties:

$$\ell \leq 24 \wedge \Diamond \text{COSB}^{10} \wedge 5 \leq \int f_{anom} \quad (16)$$

$$\ell \leq 24 \wedge \Diamond \text{COSB}^{12} \Rightarrow 7 \leq \int f_{anom} \quad (17)$$

Now, let the observation made by sensor on the ground be that within 24 h, there were 14 COS bursts, as specified by the following formula:

$$\ell \leq 24 \wedge \Diamond \text{COSB}^{14} \quad (18)$$

Suppose that the data on thermal anomalies are not available due to bad weather and, thus, the integral $\int f_{anom}$ cannot be computed. We will show that, despite that obstacle, the Strombolian behavior can be proved.

The formula $\Diamond \text{COSB}^{14}$ can be presented in the form $\text{true} \wedge \text{COSB}^{12} \wedge \text{COSB}^2 \wedge \text{true}$ (we apply here the definition of \Diamond and unfold the exponent). The formulas $\text{COSB}^2 \Rightarrow \text{true}$ and $\text{true} \wedge \text{true} \Rightarrow \text{true}$ are tautologies of the predicate calculus (see point (6) in Section 5.3). Due to these two tautologies and the monotonicity of the chop operator (see point (7) in Section 5.3), we deduce the implication

$$\text{true} \wedge \text{COSB}^{12} \wedge \text{COSB}^2 \wedge \text{true} \Rightarrow \text{true} \wedge \text{COSB}^{12} \wedge \text{true}$$

Further, applying the monotonicity property of conjunction (see point (6) two times in Section 5.3) to the above implications, we derive the following implications:

$$\ell \leq 24 \wedge \Diamond \text{COSB}^{14} \Rightarrow \ell \leq 24 \wedge \Diamond \text{COSB}^{12}, \ell \leq 24 \wedge \Diamond \text{COSB}^{12} \Rightarrow 7 \leq \int f_{anom}$$

Moreover, $7 \leq \int f_{anom} \Rightarrow 5 \leq \int f_{anom}$ and $\Diamond \text{COSB}^{12} \Rightarrow \Diamond \text{COSB}^{10}$. From the above implications, we derived the following implication:

$$\ell \leq 24 \wedge \Diamond \text{COSB}^{14} \Rightarrow \ell \leq 24 \wedge \Diamond \text{COSB}^{10} \wedge 5 \leq \int f_{anom}$$

Antecedent of this implication is the observation (18). The consequent of this implication is the formula to be proved. The consequent follows from the antecedent and the implication by the application of modus ponens rule (see point (8) in Section 5.3). Thus, observation (18) and fact (17) imply that the Strombolian property holds.

6.6. Detecting Similar Behaviors

In this subsection, we consider the methods used in different areas, volcano monitoring included, for detecting similarities between functions and, in particular, between temporal series. Various similarity measures can be used in this case (cf., e.g., [23–26]). Different types of volcanic activity may be characterized not only in absolute terms, i.e., by general characteristics satisfied by all behaviors, but also relative to one another, or in relation to known examples of behaviors. We consider neither the application of specific similarity measures to specific cases, nor their adequacy, as we are not aiming to identify similarities in volcanic behavior. We are not in the position to judge which similarity measures would be most suitable for the Strombolian eruption patterns. We merely intend to stress that such measures and the corresponding algorithms can be used within the framework of DC4F.

Similarity measures may be formalized by functionals **FISym**, which are interpreted as real-valued functions defined on intervals (see Section 5.1 and also the Appendix A). Similarity measures m can be seen as functionals that, for a given interval I and two functions f and g to be compared, return a real value being the measure of similarity. Thus,

we can consider the application of a measure functional m to the two measured functions f and g and an interval of interest I : $m(f, g, I)$. Consequently, the result can be presented as a real-valued function on intervals $X_{f,g} \in \mathbf{FISym}$, which returns a real value r for the interval I (see Section 5.1 and also Appendix A). The measured level of similarity can be then used in framing descriptions and hypotheses concerning the time series—in particular, concerning volcanic eruptions.

Similarly, the standard Pearson correlation coefficient used in statistics may be integrated into DC4F as well. The correlation coefficient $\rho_{f,g}$ of two functions f, g is specified by integral $\int (f - \int f)(g - \int g) / (\int (f - \int f)^2 \int (g - \int g)^2)$. The integral returns for an interval I a value of type real, provided that the denominator is different from zero. Thus, coefficient $\rho_{f,g}$ depends on the interval in question, and as was the case in the example presented above, it may be represented as a real-valued function on intervals.

In general, some of the similarity functions can be defined directly in DC4F. Others, such as algorithms computing the similarity measures, can be defined simply by the interval mapping operators corresponding to functionals \mathbf{FISym} . Thus, if there is an algorithm computing a similarity measure for intervals, then we can associate interval operator $X \in \mathbf{FISym}$ with the algorithm: $X([a, b]) = x$ if x is the value computed by the algorithm for interval $[a, b]$. The algorithm proposed in [19] can be treated along these lines.

7. Conclusions

In this paper, we proposed a duration calculus for integrable functions (DC4F). It is a natural, conservative extension of the well-known and widely used duration calculus. DC was used in numerous applications to specify various kinds of real-time and hybrid systems, to synthesize controllers, and so on. However, it allows to deal only with state functions, i.e., functions with Boolean values only. It did not aim to specify signals and measurements. The theoretical value of the proposed extension seems to be rather modest; however, its applications appear to be interesting. We are wondering why DC was not developed in this way from the beginning, as this would naturally broaden its application range and would be a consequent thing to do.

DC4F, like DC, is a type of interval logic providing a formal language, uniform mathematical models, and a reasoning system to deal with time series. Both calculi provide axioms and inference rules allowing to reason about properties of systems and to draw conclusions. However, they differ in expressivity, the scope of their applications, and, of course, maturity. Expressivity is achieved, in most cases, at the cost of complexity of the proof system and methods. This is the case of DC and DC4F as well. As the general form of DC is not decidable, i.e., there is no algorithm that decides whether a given formula is provable or not; the extended expressivity of DC4F relative to DC does not change much in that respect. DC, in contrary to DC4F, is aimed exclusively at the specification of durations of properties. The properties are specified by Boolean-valued functions. A great amount of research has been conducted about DC. Model-checking algorithms were created for its restricted versions and various tools were developed. It has been used to synthesize controllers. DC4F allows to integrate arbitrary Riemann integrable functions. Consequently, it provides a consistent general model for general form time series. The models can be handled using integral calculus, computer algebra, and so on. It allows one to specify several factors, parameters, and signals in one coherent language, to model them in a uniform manner, and to reason about them. Consequently, DC4F extends the ideas of DC to a qualitatively new area of applications.

We demonstrated that DC4F can be used to specify various temporal properties of time series, such as monotonicity, boundedness, and periodicity. We used it to specify different aspects of volcano monitoring activities, with a particular emphasis placed on Strombolian effects. DC4F proved to be a framework that is useful for integrating various types of data expressed in terms of time series and diagrams. Usually, in other papers, such types of data are informally related and reasoned about using informal textual descriptions in natural languages, such as English, and some mathematical formulas. This way of

handling data is, of course, imprecise and error-prone. Thus, DC4f provides a remedy for this problem in the form of a unifying logic.

DC4F can be used to formulate specifications and hypotheses, reason about them, and validate them. Informally speaking, the difference between validation and deduction is that validation is performed for a concrete data set, on a specific model, semantic deduction concerns all possible data sets and all concrete models, and syntactic deduction relies solely on the application of sound deduction rules. The proposed logic can be combined with other methods, such as algorithms, and in particular with trained neural networks and other methods used in artificial intelligence (cf., e.g., [19]).

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Appendix A. Formal Semantics

To make the paper self-contained, we present in this section the formal semantics of the proposed DC extension. The semantics were already outlined in Section 5.2, but the formal definitions, thought a bit tedious, offer full mathematical precision.

These semantics are closely related to the original semantics of DC. The main difference is that we use general Riemann integrable functions instead of functions with Boolean values. DC4F generalizes the scope of integral operator, but it does not add new logical operators to DC. From the logic point of view, DC4F is a conservative extension of DC. Thus, a formula of DC is a tautology of DC4F if, and only if, it is a tautology of DC. We define the semantics of the formulas defined in Section 5.1. The definition has three parts corresponding to the three syntactic categories defined there: functions F_n , terms $Real$, and formulas F_o .

The time domain $\mathbb{T}ime$ is the set \mathbb{R}_+ consisting of all non-negative real numbers. The discrete case, when the time domain is the set of natural numbers \mathbb{N} , can be dealt with as a special case of \mathbb{R}_+ , as was shown in Section 5.2. In particular, if $\mathbb{T}ime = \mathbb{N}$, then we assume that the set of $\mathbb{I}F\mathbb{u}n$ consists of functions that change their values at most at natural numbers and are constant elsewhere. Set $\mathbb{I}F\mathbb{u}n$ contains some unary Riemann integrable functions on positive reals. The symbols from the set \mathbf{FSym} are interpreted as functions from $\mathbb{I}F\mathbb{u}n$. For a function symbol $f \in \mathbf{FSym}$, the corresponding interpretation is an integrable function $I[f] \in \mathbb{I}F\mathbb{u}n$ of one variable of type \mathbb{R} .

The semantics of F_n is defined in a standard way. Interpretation function I is the key here. Let $f, f_1, f_2 \in \mathbf{FSym}$ and $c \in \mathbf{CSym}$. For the terms of category F_n , it is defined as follows:

- $I[0] = 0, I[1] = 1, I[true] = 1, I[c] \in \mathbb{R},$
- $I[f] \in \mathbb{I}F\mathbb{u}n$
- $I[cf] = I[c]I[f],$
- $I[g(f_1, f_2)] = I[g](I[f_1], I[f_2]),$ for $g \in \{+, -, \vee\}$; here, the operations $+, -, \vee$ are interpreted as addition and subtraction, respectively.

The set $\mathbb{I}nte$ consists of all intervals: $\mathbb{I}nte = \{[a, b] \mid a, b \in \mathbb{T}ime \wedge a \leq b\}$. The set $\mathbb{V}al = \{val \mid val : \mathbf{VSym} \mapsto \mathbb{R}\}$ contains all valuations of variables. Let $\mathbb{F}ls$ be a set containing functionals, i.e., functions that map time intervals to real numbers. The functional symbols \mathbf{FISym} (cf. Section 5.1) are interpreted as elements of $\mathbb{F}ls$. We assume that d is the usual measure on \mathbb{R} .

- $I[f](val, [a, b]) = \int_a^b f dx$
- $I[X](val, [a, b]) \in \mathbb{F}ls,$ for $X \in \mathbf{FISym}$

- $I[t_1 \oplus t_2](val, [a, b]) = I[\oplus](I[t_1](val, [a, b]), I[t_2](val, [a, b]))$, where t_1, t_2 are terms, $\oplus \in \{+, -, \vee, / \}$, and $I[\oplus]$ are interpreted as the addition, subtraction, maximum, or division, respectively.

We say that two valuations val, val' differ at most on variable x if $val(y) = val'(x)$ for every variable y is different from x . To define the semantics terms, the interpretation function Int needs pairs consisting of a valuation function $val \in \mathbb{Val}$ and an interval $[a, b] \in \mathbb{Inte}$.

We define now the satisfaction relation \models for the elements of category Fo . The relation has three arguments: a valuation, an interval, and a formula. It is defined as follows:

- $I(val, [a, b]) \models t_1 \oplus t_2$ iff the values $I[t_1](val, [a, b]), I[t_2](val, [a, b])$ are in relation $I[\oplus]$, where t_1, t_2 are terms, $\oplus \in \{<, =\}$ and $I[\oplus]$ is the inequality or equality, respectively
- $I(val, [a, b]) \models \neg Fo$ iff $I(val, [a, b])Fo$ does not hold
- $I(val, [a, b]) \models Fo_1 \vee Fo_2$ iff $I(val, [a, b]) \models Fo_1$ or $I(val, [a, b]) \models Fo_2$
- $I(val, [a, b]) \models Fo_1 \wedge Fo_2$ iff $I(val, [a, m]) \models Fo_1$ and $I(val, [m, b]) \models Fo_2$, for some $m \in [a, b]$
- $I(val, [a, b]) \models \exists_x Fo$ iff $I(val', [a, b]) \models Fo$ for some valuation val' which differs from valuation val at most on variable x
- $I[Fo^n](val, [a, b])$ if, and only if, either $n = 0$ and $a = b$ or $0 < n$ and there exist numbers $a_0, \dots, a_n \in \text{Time}$ such that $a = a_0 < a_1 < \dots < a_n = b$ and $I[Fo](val, [a_i, a_{i+1}])$, for $i = 0, \dots, n - 1$

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