Nomenclature :

— ligne double : remplacement d'une variable (e_1, γ_2,...) par sa définition (ou inversement)

— ligne simple : application d'une des règles de la sémantique opérationnelle

Simplifications d'écriture

— $e_1 = \texttt{fun}\ n\ \texttt{->}\ \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact\ (n-1)$

— $e_3 = \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact\ (n-1)$

— $\gamma_2 = \gamma :: \{fact \mapsto \langle \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 , \gamma \rangle\}$

— $\gamma_3 = \gamma_2 :: \{n \mapsto 2\}$

— $\gamma_4 = \gamma_2 :: \{n \mapsto 1\}$

— $\gamma_5 = \gamma_2 :: \{n \mapsto 0\}$

$$\dfrac{\dfrac{\gamma_2 \vdash 2 \Downarrow 2 \qquad \overset{[A]}{\gamma_2 \vdash fact \Downarrow \langle \texttt{fun}\ n\ \texttt{->}\ e_3 , \gamma_2 \rangle} \qquad \overset{[B]}{\gamma_2 :: \{n \mapsto 2\} \vdash e_3 \Downarrow 2}}{\gamma :: \{fact \mapsto \langle \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 , \gamma \rangle\} \vdash fact\ 2 \Downarrow 2}}{\gamma \vdash \texttt{letrec}\ fact\ \texttt{=}\ \texttt{fun}\ n\ \texttt{->}\ \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact\ (n-1)\ \texttt{in}\ fact\ 2 \Downarrow 2}$$

$[A]$: évaluation de $fact$

$$\dfrac{fact \in \gamma_2 \qquad \gamma_2(fact) = \langle \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 , \gamma \rangle \qquad \dfrac{\dfrac{\gamma_2 \vdash \texttt{fun}\ n\ \texttt{->}\ e_3 \Downarrow \langle \texttt{fun}\ n\ \texttt{->}\ e_3 , \gamma_2 \rangle}{\gamma :: \{fact \mapsto \langle \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 , \gamma \rangle\} \vdash e_1 \Downarrow \langle \texttt{fun}\ n\ \texttt{->}\ e_3 , \gamma_2 \rangle}}{\gamma \vdash \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 \Downarrow \langle \texttt{fun}\ n\ \texttt{->}\ e_3 , \gamma_2 \rangle}}{\gamma_2 \vdash fact \Downarrow \langle \texttt{fun}\ n\ \texttt{->}\ e_3 , \gamma_2 \rangle}$$

$[B]$: On déroule un appel récursif

$$\dfrac{\dfrac{\dfrac{\gamma_3 \vdash n \Downarrow 2 \quad \gamma_3 \vdash 0 \Downarrow 0 \quad 2 \times 0 \in dom(=) \quad \texttt{false} = (2 = 0)}{\gamma_3 \vdash n = 0 \Downarrow \texttt{false}} \qquad \dfrac{\gamma_3 \vdash n \Downarrow 2 \qquad \overset{[C]}{\gamma_3 \vdash fact\ (n-1) \Downarrow 1} \qquad 2 \times 1 \in dom(*) \qquad 2 = (2 * 1)}{\gamma_3 \vdash n * fact\ (n-1) \Downarrow 2}}{\gamma_3 \vdash \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact\ (n-1) \Downarrow 2}}{\gamma_2 :: \{n \mapsto 2\} \vdash e_3 \Downarrow 2}$$

$[C]$: Appel de fonction

$$\dfrac{\dfrac{\gamma_3 \vdash n \Downarrow 2 \qquad \gamma_3 \vdash 1 \Downarrow 1 \qquad 2 \times 1 \in dom(-) \qquad 1 = (2-1)}{\gamma_3 \vdash (n-1) \Downarrow 1} \qquad \overset{[A']}{\gamma_3 \vdash fact \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle} \qquad \overset{[B']}{\gamma_2 :: \{n \mapsto 1\} \vdash e_3 \Downarrow 1}}{\gamma_3 \vdash fact\ (n-1) \Downarrow 1}$$

$[A']$: évaluation de $fact$, presque identique à $[A]$

$$\dfrac{fact \in \gamma_3 \qquad \gamma_3(fact) = \langle\, \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1\,,\, \gamma \,\rangle \qquad \dfrac{\dfrac{\gamma_2 \vdash \texttt{fun}\ n\ \texttt{->}\ e_3 \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}{\gamma :: \{fact \mapsto \langle\, \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1\,,\, \gamma \,\rangle\} \vdash e_1 \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}}{\gamma \vdash \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}}{\gamma_3 \vdash fact \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}$$

$[B']$: On déroule un deuxième appel récursif

$$\dfrac{\dfrac{\dfrac{\gamma_4 \vdash n \Downarrow 1 \quad \gamma_4 \vdash 0 \Downarrow 0 \quad 1 \times 0 \in dom(=) \quad \texttt{false} = (1 = 0)}{\gamma_4 \vdash n = 0 \Downarrow \texttt{false}} \qquad \dfrac{\gamma_4 \vdash n \Downarrow 1 \qquad \overset{[C']}{\gamma_4 \vdash fact\ (n-1) \Downarrow 1} \qquad 1 \times 1 \in dom(*) \qquad 1 = (1 * 1)}{\gamma_3 \vdash n * fact\ (n-1) \Downarrow 1}}{\gamma_4 \vdash \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact\ (n-1) \Downarrow 1}}{\gamma_2 :: \{n \mapsto 1\} \vdash e_3 \Downarrow 1}$$

$[C']$: Appel de fonction

$$\dfrac{\dfrac{\gamma_4 \vdash n \Downarrow 1 \qquad \gamma_4 \vdash 1 \Downarrow 1 \qquad 1 \times 1 \in dom(-) \qquad 0 = (1-1)}{\gamma_4 \vdash (n-1) \Downarrow 0} \qquad \dfrac{[A”]}{\gamma_4 \vdash fact \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle} \qquad \dfrac{[B”]}{\gamma_2 :: \{n \mapsto 0\} \vdash e_3 \Downarrow 1}}{\gamma_4 \vdash fact\ (n-1) \Downarrow 1}$$

$[A”]$: évaluation de $fact$, presque identique à $[A]$ et $[A']$

$$\dfrac{fact \in \gamma_4 \qquad \gamma_4(fact) = \langle\, \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1\,,\, \gamma \,\rangle \qquad \dfrac{\dfrac{\gamma_2 \vdash \texttt{fun}\ n\ \texttt{->}\ e_3 \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}{\gamma :: \{fact \mapsto \langle\, \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1\,,\, \gamma \,\rangle\} \vdash e_1 \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}}{\gamma \vdash \texttt{letrec}\ fact\ \texttt{=}\ e_1\ \texttt{in}\ e_1 \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}}{\gamma_4 \vdash fact \Downarrow \langle\, \texttt{fun}\ n\ \texttt{->}\ e_3\,,\, \gamma_2 \,\rangle}$$

$[B”]$: On déroule un troisième appel récursif

$$\dfrac{\dfrac{\dfrac{\gamma_5 \vdash n \Downarrow 0 \qquad \gamma_5 \vdash 0 \Downarrow 0 \qquad 0 \times 0 \in dom(=) \qquad \texttt{true} = (0 = 0)}{\gamma_5 \vdash n = 0 \Downarrow \texttt{true}} \qquad \gamma_5 \vdash 1 \Downarrow 1}{\gamma_5 \vdash \texttt{if}\ n = 0\ \texttt{then}\ 1\ \texttt{else}\ n * fact\ (n-1) \Downarrow 1}}{\gamma_2 :: \{n \mapsto 0\} \vdash e_3 \Downarrow 1}$$