A.1.1 Introduction As discussed in the Chapter 9 on Sequences and Series, a sequence a1 , a2 , ..., an , ... having infinite number of terms is called infinite sequence and its indicated sum, i.e., a1 + a2 + a3 + ... + an + ... is called an infinte series associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e., a1 + a2 + a3 + . . . + an + . . . = 1 k k a ∞ = ∑ In this Chapter, we shall study about some special types of series which may be required in different problem situations. A.1.2 Binomial Theorem for any Index In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called Binomial Series. We illustrate few applications, by examples. We know the formula (1 + x) n = C0 n + C1 n x + . . . + C n n x n Here, n is non-negative integer. Observe that if we replace index n by negative integer or a fraction, then the combinations C n r do not make any sense. We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number. Theorem The formula ( ) ( ) ( )( ) 2 3 1 1 2 1 1 1 2 1 2 3 m m m m m m x mx x x ... . . . − − − + = + + + + holds whenever x < 1, i.e., – 1< x < 1 is necessary when m is negative integer or a fraction. For example, if we take x = – 2 and m = – 2, we obtain ( ) ( ) ( ) ( )( ) ( ) 2 2 3 2 1 2 1 2 2 2 ... 1.2 − − − − = + − − + − + or 1= 1 + 4 + 12 + . . . This is not possible 2. Note that there are infinite number of terms in the expansion of (1+ x) m , when m is a negative integer or a fraction Consider ( )m a b + = 1 1 m m b m b a a a a □ □ □ □ □ □ □ □ □ □ + = + □ □ □ □ □ □ □ □ = ( ) 2 1 1 ... 1.2 m b b m m a m a a □ − □ □ □ □ + + + □ □ □ □ □ □ □ □ □ = ( ) 1 2 2 1 ... 1.2 m m m m m a ma b a b − − − + + + This expansion is valid when < | a |. The general term in the expansion of (a + b) m is (1 2 ... 1)( ) ( ) 1.2.3... m r r m m m m r a b r − − − − + We give below certain particular cases of Binomial Theorem, when we assume x < 2. Rationalised 2023-24 316 MATHEMATICS Solution We have 1 2 1 2 x − □ □ □ □ − □ □ = 2 1 1 3 2 2 2 1 1 2 1 2 2 x x ... . □ □ □ □ □ □ □ □ □ □ □ □ − − − □ □ □ □ □ □ □ □ □ □ − + + − + □ □ □ □ □ □ □ □ = 2 3 1 ... 4 32 + + + x x A.1.3 Infinite Geometric Series From Chapter 9, Section 9.5, a sequence a1 , a2 , a3 , ..., an is called G.P., if k +1 k a a = r (constant) for k = 1, 2, 3, ..., n–1. Particularly, if we take a1 = a, then the resulting sequence a, ar, ar2 , ..., arn–1 is taken as the standard form of G.P., where a is first term and r, the common ratio of G.P. Earlier, we have discussed the formula to find the sum of finite series a + ar + ar2 + ... + arn – 1 which is given by (1) 1 n n a r S r − = − . In this section, we state the formula to find the sum of infinite geometric series a + ar + ar2 + ... + arn – 1 + ... and illustrate the same by examples. Let us consider the G.P. 2 4 1, , ,... 3 9 Here a = 1, r = 2 3 . We have 2 1 3 2 3 1 2 3 1 3 n n n S □ □ − □ □ □ □ □ □ □ □ = = − □ □ □ □ □ □ □ □ − □ □ ... (1) Let us study the behaviour of 2 3 n □ □ □ □ □ □ as n becomes larger and larger. Rationalised 2023-24 INFINITE SERIES 317 n 1 5 10 20 2 3 n □ □ □ □ □ □ 0.6667 0.1316872428 0.01734152992 0.00030072866 We observe that as n becomes larger and larger, 2 3 n □ □ □ □ □ □ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, 2 3 n □ □ □ □ □ □ becomes sufficiently small. In other words, as 2 , 0 3 n n □ □ → ∞ → □ □ □ □ . Consequently, we find that the sum of infinitely many terms is given by S = 3. Thus, for infinite geometric progression a, ar, ar2 , ..., if numerical value of common ratio r is less than 1, then n S = (1) 1 n a r r − − = 1 1 n a ar r r − − − In this case, 0 n r → as n → ∞ since | | 1 r < and then 0 1 n ar r → − . Therefore, 1 n a S r → − as n → ∞ . Symbolically, sum to infinity of infinite geometric series is denoted by S. Thus, we have S 1 a r = − For example (i) 2 3 1 1 1 1 1 ... 2 2 2 2 1 1 2 + + + + = = − (ii) 2 3 1 1 1 1 1 2 1 ... 2 2 2 1 1 3 1 1 2 2 − + − + = = = □ □ − − + □ □ □ □ Rationalised 2023-24 318 MATHEMATICS Example 2 Find the sum to infinity of the G.P. ; 5 5 5 , , ,.... 4 16 64 − − Solution Here 5 4 a − = and 1 4 r = − . Also | | 1 r < . Hence, the sum to infinity is 5 5 4 4 1 1 5 1 4 4 − − = = − + . A.1.4 Exponential Series Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number e in his calculus text in 1748. The number e is useful in calculus as π in the study of the circle. Consider the following infinite series of numbers 1 1 1 1 1 ... 1! 2! 3! 4! + + + + + ... (1) The sum of the series given in (1) is denoted by the number e Let us estimate the value of the number e. Since every term of the series (1) is positive, it is clear that its sum is also positive. Consider the two sums 1 1 1 1 ... ... 3! 4! 5! ! n + + + + + ... (2) and 2 3 4 1 1 1 1 1 .... 2 2 2 2n− + + + + + ... (3) Observe that 1 1 3! 6 = and 2 1 1 2 4 = , which gives 2 1 1 3! 2 < 1 1 4! 24 = and 3 1 1 2 8 = , which gives 3 1 1 4! 2 < 1 1 5! 120 = and 4 1 1 2 16 = , which gives 4 1 1 5! 2 < . Rationalised 2023-24 INFINITE SERIES 319 Therefore, by analogy, we can say that 1 1 1 ! 2 n n − < , when n > 2 We observe that each term in (2) is less than the corresponding term in (3), Therefore 2 3 4 1 1 1 1 1 1 1 1 1 ... ... 3! 4! 5! ! 2 2 2 2n n − + + + + < + + + + + + ... (4) Adding 1 1 1 1! 2! + + on both sides of (4), we get, 1 1 1 1 1 1 1 ... ... 1! 2! 3! 4! 5! ! n + + + + + + + + 2 3 4 1 1 1 1 1 1 1 1 ... ... 1! 2! 2 2 2 2n− < + + + + + + + + ... (5) = 2 3 4 1 1 1 1 1 1 1 1 ... ... 2 2 2 2 2n− + + + + + + + + = 1 1 1 2 1 1 2 + = + − = 3 Left hand side of (5) represents the series (1). Therefore e < 3 and also e > 2 and hence 2 < e < 3. Remark The exponential series involving variable x can be expressed as 2 3 1 ... ... 1! 2! 3! ! n x x x x x e n = + + + + + + Example 3 Find the coefficient of x 2 in the expansion of e 2x+3 as a series in powers of x. Solution In the exponential series x e = 2 3 1 1! 2! 3! x x x + + + + ... replacing x by (2x + 3), we get Rationalised 2023-24 320 MATHEMATICS 2 3 x e + = ( ) ( ) 2 2 3 2 3 1 ... 1! 2! x x + + + + + Here, the general term is (2 3) ! n x n + = (3+2) ! n x n . This can be expanded by the Binomial Theorem as ( ) ( ) ( ) 1 2 2 1 2 1 3 C 3 2 C 3 2 ... 2 . ! n n n n n n x x x n − − + + + + Here, the coefficient of x 2 is 2 2 C 3 2 2 ! n n− n . Therefore, the coefficient of x 2 in the whole series is 2 2 2 2 C 3 2 n n n n! ∞ − = ∑ = ( ) 2 2 1 3 2 ! n n n n n ∞ − = − ∑ = ( ) –2 2 3 2 2 ! n n n ∞ = ∑ − [using n! = n (n – 1) (n – 2)!] = 2 3 3 3 3 2 1 ... 1! 2! 3! + + + + = 2e 3 . Thus 2e 3 is the coefficient of x 2 in the expansion of e 2x+3 . Alternatively e 2x+3 = e 3 . e 2x = 2 3 3 2 (2) (2) 1 ... 1! 2! 3! x x x e + + + + Thus, the coefficient of x 2 in the expansion of e2x+3 is 2 3 2 3 . 2 2! e e = Example 4 Find the value of e 2 , rounded off to one decimal place. Solution Using the formula of exponential series involving x, we have 2 3 1 ... ... 1! 2! 3! ! n x x x x x e n = + + + + + + Rationalised 2023-24 INFINITE SERIES 321 Putting x = 2, we get 2 3 4 5 6 2 2 2 2 2 2 2 2 1 ... 1! 2! 3! 4! 5! 6! e = + + + + + + + + = 4 4 4 1 2 2 ... 3 3 15 45 2 + + + + + + + + ≥ the sum of first seven terms ≥ 7.355. On the other hand, we have 2 3 4 5 2 3 2 2 3 2 2 2 2 2 2 2 2 1 1 ... 1! 2! 3! 4! 5! 6 6 6 e < + + + + + + + + + = 2 4 1 1 7 1 ... 15 3 3 + + + + = 4 1 7 15 1 1 3 + − = 2 7 7.4 5 + = . Thus, e 2 lies between 7.355 and 7.4. Therefore, the value of e 2 , rounded off to one decimal place, is 7.4. A.1.5 Logarithmic Series Another very important series is logarithmic series which is also in the form of infinite series. We state the following result without proof and illustrate its application with an example. Theorem If | x | < 1, then ( ) 2 3 log 1 ... 2 3 e x x + = − + − x x The series on the right hand side of the above is called the logarithmic series. ANote The expansion of loge (1+x) is valid for x = 1. Substituting x = 1 in the expansion of loge (1+x), we get 1 1 1 log 2 1 – – ... 2 3 4 e = + + Rationalised 2023-24 322 MATHEMATICS Example 5 If α β, are the roots of the equation 2 x px q − + = 0 , prove that ( ) ( ) 2 2 3 3 2 2 3 log 1 ... 2 3 e px qx x x x α + β α + β + + = α + β − + − Solution Right hand side = 2 2 3 3 2 2 3 3 ... ... 2 3 2 3 x x x x x x α α β β α − + − + β − + − = log 1 log 1 e (+ α + + β x x) ( ) = (( )) 2 log 1 e + α + β + αβ x x = ( ) 2 log 1 e + + px qx = Left hand side. Here, we have used the facts α + β = p and αβ = q . We know this from the given roots of the quadratic equation. We have also assumed that both | | α x < 1 and | | β x < 1. — v — Rationalised 2023-24A.2.1 Introduction Much of our progress in the last few centuries has made it necessary to apply mathematical methods to real-life problems arising from different fields – be it Science, Finance, Management etc. The use of Mathematics in solving real-world problems has become widespread especially due to the increasing computational power of digital computers and computing methods, both of which have facilitated the handling of lengthy and complicated problems. The process of translation of a real-life problem into a mathematical form can give a better representation and solution of certain problems. The process of translation is called Mathematical Modelling. Here we shall familiaries you with the steps involved in this process through examples. We shall first talk about what a mathematical model is, then we discuss the steps involved in the process of modelling. A.2.2 Preliminaries Mathematical modelling is an essential tool for understanding the world. In olden days the Chinese, Egyptians, Indians, Babylonians and Greeks indulged in understanding and predicting the natural phenomena through their knowledge of mathematics. The architects, artisans and craftsmen based many of their works of art on geometric prinicples. Suppose a surveyor wants to measure the height of a tower. It is physically very difficult to measure the height using the measuring tape. So, the other option is to find out the factors that are useful to find the height. From his knowledge of trigonometry, he knows that if he has an angle of elevation and the distance of the foot of the tower to the point where he is standing, then he can calculate the height of the tower. So, his job is now simplified to find the angle of elevation to the top of the tower and the distance from the foot of the tower to the point where he is standing. Both of which are easily measurable. Thus, if he measures the angle of elevation as 40° and the distance as 450m, then the problem can be solved as given in Example 1. Appendix 2 MATHEMATICAL MODELLING 324 MATHEMATICS Example 1 The angle of elevation of the top of a tower from a point O on the ground, which is 450 m away from the foot of the tower, is 40°. Find the height of the tower. Solution We shall solve this in different steps. Step 1 We first try to understand the real problem. In the problem a tower is given and its height is to be measured. Let h denote the height. It is given that the horizontal distance of the foot of the tower from a particular point O on the ground is 450 m. Let d denotes this distance. Then d = 450m. We also know that the angle of elevation, denoted by θ, is 40°. The real problem is to find the height h of the tower using the known distance d and the angle of elevation θ. Step 2 The three quantities mentioned in the problem are height, distance and angle of elevation. So we look for a relation connecting these three quantities. This is obtained by expressing it geometrically in the following way (Fig 1). AB denotes the tower. OA gives the horizontal distance from the point O to foot of the tower. ∠AOB is the angle of elevation. Then we have tan θ = h d or h = d tan θ ... (1) This is an equation connecting θ, h and d. Step 3 We use Equation (1) to solve h. We have θ = 40°. and d = 450m. Then we get h = tan 40° × 450 = 450 × 0.839 = 377.6m Step 4 Thus we got that the height of the tower approximately 378m. Let us now look at the different steps used in solving the problem. In step 1, we have studied the real problem and found that the problem involves three parameters height, distance and angle of elevation. That means in this step we have studied the real-life problem and identified the parameters. In the Step 2, we used some geometry and found that the problem can be represented geometrically as given in Fig 1. Then we used the trigonometric ratio for the “tangent” function and found the relation as h = d tan θ So, in this step we formulated the problem mathematically. That means we found an equation representing the real problem. Fig 1 MATHEMATICAL MODELLING 325 In Step 3, we solved the mathematical problem and got that h = 377.6m. That is we found Solution of the problem. In the last step, we interpreted the solution of the problem and stated that the height of the tower is approximately 378m. We call this as Interpreting the mathematical solution to the real situation In fact these are the steps mathematicians and others use to study various reallife situations. We shall consider the question, “why is it necessary to use mathematics to solve different situations.” Here are some of the examples where mathematics is used effectively to study various situations. 1. Proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the body in humanbeings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and cause damages ranging from minor discomfort to sudden death. The problem is to find the relationship between blood flow and physiological characteristics of blood vessel. 2. In cricket a third umpire takes decision of a LBW by looking at the trajectory of a ball, simulated, assuming that the batsman is not there. Mathematical equations are arrived at, based on the known paths of balls before it hits the batsman’s leg. This simulated model is used to take decision of LBW. 3. Meteorology department makes weather predictions based on mathematical models. Some of the parameters which affect change in weather conditions are temperature, air pressure, humidity, wind speed, etc. The instruments are used to measure these parameters which include thermometers to measure temperature, barometers to measure airpressure, hygrometers to measure humidity, anemometers to measure wind speed. Once data are received from many stations around the country and feed into computers for further analysis and interpretation. 4. Department of Agriculture wants to estimate the yield of rice in India from the standing crops. Scientists identify areas of rice cultivation and find the average yield per acre by cutting and weighing crops from some representative fields. Based on some statistical techniques decisions are made on the average yield of rice. How do mathematicians help in solving such problems? They sit with experts in the area, for example, a physiologist in the first problem and work out a mathematical equivalent of the problem. This equivalent consists of one or more equations or inequalities etc. which are called the mathematical models. Then 326 MATHEMATICS solve the model and interpret the solution in terms of the original problem. Before we explain the process, we shall discuss what a mathematical model is. A mathematical model is a representation which comprehends a situation. An interesting geometric model is illustrated in the following example. Example 2 (Bridge Problem) Konigsberg is a town on the Pregel River, which in the 18th century was a German town, but now is Russian. Within the town are two river islands that are connected to the banks with seven bridges as shown in (Fig 2). People tried to walk around the town in a way that only crossed each bridge once, but it proved to be difficult problem. Leonhard Euler, a Swiss mathematician in the service of the Russian empire Catherine the Great, heard about the problem. In 1736 Euler proved that the walk was not possible to do. He proved this by inventing a kind of diagram called a network, that is made up of vertices (dots where lines meet) and arcs (lines) (Fig3). He used four dots (vertices) for the two river banks and the two islands. These have been marked A, B and C, D. The seven lines (arcs) are the seven bridges. You can see that 3 bridges (arcs) join to riverbank, A, and 3 join to riverbank B. 5 bridges (arcs) join to island C, and 3 join to island D. This means that all the vertices have an odd number of arcs, so they are called odd vertices (An even vertex would have to have an even number of arcs joining to it). Remember that the problem was to travel around town crossing each bridge only once. On Euler's network this meant tracing over each arc only once, visiting all the vertices. Euler proved it could not be done because he worked out that, to have an odd vertex you would have to begin or end the trip at that vertex. (Think about it). Since there can only be one beginning and one end, there can only be two odd vertices if you are to trace over each arc only once. Since the bridge problem has 4 odd vertices, it just not possible to do! Fig 2 Fig 3 MATHEMATICAL MODELLING 327 After Euler proved his Theorem, much water has flown under the bridges in Konigsberg. In 1875, an extra bridge was built in Konigsberg, joining the land areas of river banks A and B (Fig 4). Is it possible now for the Konigsbergians to go round the city, using each bridge only once? Here the situation will be as in Fig 4. After the addition of the new edge, both the vertices A and B have become even degree vertices. However, D and C still have odd degree. So, it is possible for the Konigsbergians to go around the city using each bridge exactly once. The invention of networks began a new theory called graph theory which is now used in many ways, including planning and mapping railway networks (Fig 4). A.2.3 What is Mathematical Modelling? Here, we shall define what mathematical modelling is and illustrate the different processes involved in this through examples. Definition Mathematical modelling is an attempt to study some part (or form) of the real-life problem in mathematical terms. Conversion of physical situation into mathematics with some suitable conditions is known as mathematical modelling. Mathematical modelling is nothing but a technique and the pedagogy taken from fine arts and not from the basic sciences. Let us now understand the different processes involved in Mathematical Modelling. Four steps are involved in this process. As an illustrative example, we consider the modelling done to study the motion of a simple pendulum. Understanding the problem This involves, for example, understanding the process involved in the motion of simple pendulum. All of us are familiar with the simple pendulum. This pendulum is simply a mass (known as bob) attached to one end of a string whose other end is fixed at a point. We have studied that the motion of the simple pendulum is periodic. The period depends upon the length of the string and acceleration due to gravity. So, what we need to find is the period of oscillation. Based on this, we give a precise statement of the problem as Statement How do we find the period of oscillation of the simple pendulum? The next step is formulation. Formulation Consists of two main steps. 1. Identifying the relevant factors In this, we find out what are the factors/ Fig 4 328 MATHEMATICS parameters involved in the problem. For example, in the case of pendulum, the factors are period of oscillation (T), the mass of the bob (m), effective length (l) of the pendulum which is the distance between the point of suspension to the centre of mass of the bob. Here, we consider the length of string as effective length of the pendulum and acceleration due to gravity (g), which is assumed to be constant at a place. So, we have identified four parameters for studying the problem. Now, our purpose is to find T. For this we need to understand what are the parameters that affect the period which can be done by performing a simple experiment. We take two metal balls of two different masses and conduct experiment with each of them attached to two strings of equal lengths. We measure the period of oscillation. We make the observation that there is no appreciable change of the period with mass. Now, we perform the same experiment on equal mass of balls but take strings of different lengths and observe that there is clear dependence of the period on the length of the pendulum. This indicates that the mass m is not an essential parameter for finding period whereas the length l is an essential parameter. This process of searching the essential parameters is necessary before we go to the next step. 2. Mathematical description This involves finding an equation, inequality or a geometric figure using the parameters already identified. In the case of simple pendulum, experiments were conducted in which the values of period T were measured for different values of l. These values were plotted on a graph which resulted in a curve that resembled a parabola. It implies that the relation between T and l could be expressed T 2 = kl ... (1) It was found that 2 4π k g = . This gives the equation T 2π l g = ... (2) Equation (2) gives the mathematical formulation of the problem. Finding the solution The mathematical formulation rarely gives the answer directly. Usually we have to do some operation which involves solving an equation, calculation or applying a theorem etc. In the case of simple pendulums the solution involves applying the formula given in Equation (2). MATHEMATICAL MODELLING 329 The period of oscillation calculated for two different pendulums having different lengths is given in Table 1 Table 1 l 225 cm 275cm T 3.04 sec 3.36 sec The table shows that for l = 225 cm, T = 3.04 sec and for l = 275 cm, T = 3.36 sec. Interpretation/Validation A mathematical model is an attempt to study, the essential characteristic of a real life problem. Many times model equations are obtained by assuming the situation in an idealised context. The model will be useful only if it explains all the facts that we would like it to explain. Otherwise, we will reject it, or else, improve it, then test it again. In other words, we measure the effectiveness of the model by comparing the results obtained from the mathematical model, with the known facts about the real problem. This process is called validation of the model. In the case of simple pendulum, we conduct some experiments on the pendulum and find out period of oscillation. The results of the experiment are given in Table 2. Table 2 Periods obtained experimentally for four different pendulums Mass (gms) Length (cms) Time (secs) 385 275 3.371 225 3.056 230 275 3.352 225 3.042 Now, we compare the measured values in Table 2 with the calculated values given in Table 1. The difference in the observed values and calculated values gives the error. For example, for l = 275 cm, and mass m = 385 gm, error = 3.371 – 3.36 = 0.011 which is small and the model is accepted. Once we accept the model, we have to interpret the model. The process of describing the solution in the context of the real situation is called interpretation of the model. In this case, we can interpret the solution in the following way: (a) The period is directly proportional to the square root of the length of the pendulum. 330 MATHEMATICS (b) It is inversely proportional to the square root of the acceleration due to gravity. Our validation and interpretation of this model shows that the mathematical model is in good agreement with the practical (or observed) values. But we found that there is some error in the calculated result and measured result. This is because we have neglected the mass of the string and resistance of the medium. So, in such situation we look for a better model and this process continues. This leads us to an important observation. The real world is far too complex to understand and describe completely. We just pick one or two main factors to be completely accurate that may influence the situation. Then try to obtain a simplified model which gives some information about the situation. We study the simple situation with this model expecting that we can obtain a better model of the situation. Now, we summarise the main process involved in the modelling as (a) Formulation (b) Solution (c) Interpretation/Validation The next example shows how modelling can be done using the techniques of finding graphical solution of inequality. Example 3 A farm house uses atleast 800 kg of special food daily. The special food is a mixture of corn and soyabean with the following compositions Table 3 Material Nutrients present per Kg Nutrients present per Kg Cost per Kg Protein Fibre Corn .09 .02 Rs 10 Soyabean .60 .06 Rs 20 The dietary requirements of the special food stipulate atleast 30% protein and at most 5% fibre. Determine the daily minimum cost of the food mix. Solution Step 1 Here the objective is to minimise the total daily cost of the food which is made up of corn and soyabean. So the variables (factors) that are to be considered are x = the amount of corn y = the amount of soyabean z = the cost Step 2 The last column in Table 3 indicates that z, x, y are related by the equation $z = 10x + 20y$ ... (1) The problem is to minimise z with the following constraints: MATHEMATICAL MODELLING 331 (a) The farm used atleast 800 kg food consisting of corn and soyabean i.e., $x + y \geq 800$ ... (2) (b) The food should have atleast 30% protein dietary requirement in the proportion as given in the first column of Table 3. This gives $0.09x + 0.6y \geq 0.3 (x + y)$ ... (3) (c) Similarly the food should have atmost 5% fibre in the proportion given in 2nd column of Table 3. This gives $0.02x + 0.06 y \leq 0.05 (x + y)$ ... (4) We simplify the constraints given in (2), (3) and (4) by grouping all the coefficients of x, y. Then the problem can be restated in the following mathematical form. Statement Minimise z subject to $x + y \geq 800$ $0.21x – .30y \leq 0$ $0.03x – .01y \geq 0$ This gives the formulation of the model. Step 3 This can be solved graphically. The shaded region in Fig 5 gives the possible solution of the equations. From the graph it is clear that the minimum value is got at the point (470.6,329.4) i.e., x = 470.6 and y = 329.4. Fig 5 This gives the value of z as $z = 10 \times 470.6 + 20 \times 329.4 = 11294$ This is the mathematical solution. 332 MATHEMATICS Step 4 The solution can be interpreted as saying that, “The minimum cost of the special food with corn and soyabean having the required portion of nutrient contents, protein and fibre is Rs 11294 and we obtain this minimum cost if we use 470.6 kg of corn and 329.4 kg of soyabean.” In the next example, we shall discuss how modelling is used to study the population of a country at a particular time. Example 4 Suppose a population control unit wants to find out “how many people will be there in a certain country after 10 years” Step 1 Formulation We first observe that the population changes with time and it increases with birth and decreases with deaths. We want to find the population at a particular time. Let t denote the time in years. Then t takes values 0, 1, 2, ..., t = 0 stands for the present time, t = 1 stands for the next year etc. For any time t, let p (t) denote the population in that particular year. Suppose we want to find the population in a particular year, say t0 = 2006. How will we do that. We find the population by Jan. 1st, 2005. Add the number of births in that year and subtract the number of deaths in that year. Let B(t) denote the number of births in the one year between t and t + 1 and D(t) denote the number of deaths between t and t + 1. Then we get the relation P (t + 1) = P (t) + B (t) – D (t) Now we make some assumptions and definitions 1. B ( ) P ( ) t t is called the birth rate for the time interval t to t + 1. 2. D (t) P (t) is called the death rate for the time interval t to t + 1. Assumptions 1. The birth rate is the same for all intervals. Likewise, the death rate is the same for all intervals. This means that there is a constant b, called the birth rate, and a constant d, called the death rate so that, for all t ≥ 0, B ( ) D ( ) and P ( ) P ( ) t t b d t t = = ... (1) 2. There is no migration into or out of the population; i.e., the only source of population change is birth and death. MATHEMATICAL MODELLING 333 As a result of assumptions 1 and 2, we deduce that, for $t \geq 0$, $P(t + 1) = P(t) + B(t) – D(t) = P(t) + bP(t) – dP(t) = (1 + b – d) P(t)$ ... (2) Setting t = 0 in (2) gives $P(1) = (1 + b – d)P(0)$ ... (3) Setting t = 1 in Equation (2) gives $P(2) = (1 + b – d) P(1) = (1 + b – d) (1 + b – d) P(0)$ (Using equation 3) $= (1 + b – d)^2 P(0)$ Continuing this way, we get $P(t) = (1 + b – d)^t P(0)$ ... (4) for t = 0, 1, 2, ... The constant 1 + b – d is often abbreviated by r and called the growth rate or, in more high-flown language, the Malthusian parameter, in honor of Robert Malthus who first brought this model to popular attention. In terms of r, Equation (4) becomes $P(t) = P(0)r^t$, t = 0, 1, 2, ... (5) P(t) is an example of an exponential function. Any function of the form cr^t, where c and r are constants, is an exponential function. Equation (5) gives the mathematical formulation of the problem. Step 2 – Solution Suppose the current population is 250,000,000 and the rates are b = 0.02 and d = 0.01. What will the population be in 10 years? Using the formula, we calculate P(10). $P(10) = (1.01)^{10}$ (250,000,000) = (1.104622125) (250,000,000) = 276,155,531.25 Step 3 Interpretation and Validation Naturally, this result is absurd, since one can’t have 0.25 of a person. So, we do some approximation and conclude that the population is 276,155,531 (approximately). Here, we are not getting the exact answer because of the assumptions that we have made in our mathematical model. The above examples show how modelling is done in variety of situations using different mathematical techniques. 334 MATHEMATICS Since a mathematical model is a simplified representation of a real problem, by its very nature, has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is a good one or not i.e., when the obtained results are interpreted physically whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new evaluation. Thus mathematical modelling can be a cycle of the modelling process as shown in the flowchart given below: — v — NO < < < STOP ↓ ↓ ↓ ↓ ↓ YES < ↓ START ASSUMPTIONS/AXIOMS VALIDATION INTERPRETATION SOLUTION FORMULATION SATISFIEDEXERCISE 1.1 1. (i), (iv), (v), (vi), (vii) and (viii) are sets. 2. (i) ∈ (ii) ∉ (iii) ∉ (vi) ∈ (v) ∈ (vi) ∉ 3. (i) A = {–3, –2, –1, 0, 1, 2, 3, 4, 5, 6 } (ii) B = {1, 2, 3, 4, 5} (iii) C = {17, 26, 35, 44, 53, 62, 71, 80} (iv) D = {2, 3, 5} (v) E = {T, R, I, G, O, N, M, E, Y} (vi) F = {B, E, T, R} 4. (i) { x : x = 3n, n∈N and 1 ≤ n ≤ 4 } (ii) { x : x = 2^n , n∈N and 1 ≤ n ≤ 5 } (iii) { x : x = 5^n , n∈N and 1 ≤ n ≤ 4 } (iv) { x : x is an even natural number} (v) { x : x = n^2 , n∈N and 1 ≤ n ≤ 10 } 5. (i) A = {1, 3, 5, . . . } (ii) B = {0, 1, 2, 3, 4 } (iii) C = { –2, –1, 0, 1, 2 } (iv) D = { L, O, Y, A } (v) E = { February, April, June, September, November } (vi) F = {b, c, d, f, g, h, j } 6. (i) ↔ (c) (ii) ↔ (a) (iii) ↔ (d) (iv) ↔ (b) EXERCISE 1.2 1. (i), (iii), (iv) 2. (i) Finite (ii) Infinite (iii) Finite (iv) Infinite (v) Finite 3. (i) Infinite (ii) Finite (iii) Infinite (iv) Finite (v) Infinite 4. (i) Yes (ii) No (iii) Yes (iv) No 5. (i) No (ii) Yes 6. B= D, E = G EXERCISE 1.3 1. (i) ⊂ (ii) ⊄ (iii) ⊂ (iv) ⊄ (v) ⊄ (vi) ⊂ (vii) ⊂ 2. (i) False (ii) True (iii) False (iv) True (v) False (vi) True 3. (i) as {3,4}∈A, (v) as 1∈A, (vii) as {1,2, 5}⊂A, (viii) as 3∉A, (ix) as φ ⊂ A, (xi) as φ ⊂ A, 4. (i) φ, { a } (ii) φ, { a }, { b }, { a, b } (iii) φ, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 }, { 1, 2, 3 } (iv) φ 5. (i) (– 4, 6] (ii) (– 12, –10) (iii) [0, 7) (iv) [3, 4] 6. (i) { x : x ∈ R, – 3 < x < 0 } (ii) { x : x ∈ R, 6 ≤ x ≤ 12 } (iii) { x : x ∈ R, 6 < x ≤ 12 } (iv) { x R : – 23 ≤ x < 5 } 8. (iii) ANSWERS Rationalised 2023-24 336 MATHEMATICS EXERCISE 1.4 1. (i) X ∪ Y = {1, 2, 3, 5 } (ii) A ∪ B = { a, b, c, e, i, o, u } (iii) A ∪ B = {x : x = 1, 2, 4, 5 or a multiple of 3 } (iv) A ∪ B = {x : 1 < x < 10, x ∈ N} (v) A ∪ B = {1, 2, 3 } 2. Yes, A ∪ B = { a, b, c } 3. B 4. (i) { 1, 2, 3, 4, 5, 6 } (ii) {1, 2, 3, 4, 5, 6, 7,8 } (iii) {3, 4, 5, 6, 7, 8 } (iv) {3, 4, 5, 6, 7, 8, 9, 10} (v) {1, 2, 3, 4, 5, 6, 7, 8 } (vi) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} (vii) { 3, 4, 5, 6, 7, 8, 9, 10 } 5. (i) X ∩ Y = { 1, 3 } (ii) A ∩ B = { a } (iii) { 3 } (iv) φ (v) φ 6. (i) { 7, 9, 11 } (ii) { 11, 13 } (iii) φ (iv) { 11 } (v) φ (vi) { 7, 9, 11 } (vii) φ (viii) { 7, 9, 11 } (ix) {7, 9, 11 } (x) { 7, 9, 11, 15 } 7. (i) B (ii) C (iii) D (iv) φ (v) { 2 } (vi) { x : x is an odd prime number } 8. (iii) 9. (i) {3, 6, 9, 15, 18, 21} (ii) {3, 9, 15, 18, 21 } (iii) {3, 6, 9, 12, 18, 21} (iv) {4, 8, 16, 20 } (v) { 2, 4, 8, 10, 14, 16 } (vi) { 5, 10, 20 } (vii) { 20 } (viii) { 4, 8, 12, 16 } (ix) { 2, 6, 10, 14} (x) { 5, 10, 15 } (xi) {2, 4, 6, 8, 12, 14, 16} (xii) {5, 15, 20} 10. (i) { a, c } (ii) {f, g } (iii) { b , d } 11. Set of irrational numbers 12. (i) F (ii) F (iii) T (iv) T

EXERCISE 1.5 1. (i) { 5, 6, 7, 8, 9} (ii) {1, 3, 5, 7, 9 } (iii) {7, 8, 9 } (iv) { 5, 7, 9 } (v) { 1, 2, 3, 4 } (vi) { 1, 3, 4, 5, 6, 7, 9 } 2. (i) { d, e, f, g, h} (ii) { a, b, c, h } (iii) { b, d , f, h } (iv) { b, c, d, e } 3. (i) { x : x is an odd natural number } (ii) { x : x is an even natural number } (iii) { x : x ∈ N and x is not a multiple of 3 } Rationalised 2023-24 ANSWERS 337 (iv) { x : x is a positive composite number or x = 1 } (v) { x : x is a positive integer which is not divisible by 3 or not divisible by 5} (vi) { x : x ∈ N and x is not a perfect square } (vii) { x : x ∈ N and x is not a perfect cube } (viii) { x : x ∈ N and x ≠ 3 } (ix) { x : x ∈ N and x ≠ 2 } (x) { x : x ∈ N and x < 7 } (xi) { x : x ∈ N and x ≤ 9 2 } 6. A' is the set of all equilateral triangles. 7. (i) U (ii) A (iii) ϕ (iv) ϕ Miscellaneous Exercise on Chapter 1 1. A ⊂ B, A ⊂ C, B ⊂ C, D ⊂ A, D ⊂ B, D ⊂ C 2. (i) False (ii) False (iii) True (iv) False (v) False (vi) True 10. We may take A = { 1, 2 }, B = { 1, 3 }, C = { 2 , 3 } EXERCISE 2.1 1. x = 2 and y = 1 2. The number of elements in A × B is 9. 3. G × H = {(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)} H × G = {(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)} 4. (i) False P × Q = {(m, n), (m, m), (n, n), (n, m)} (ii) True (iii) True 5. A × A = {(– 1, – 1), (– 1, 1), (1, – 1), (1, 1)} A × A × A = {(–1, –1, –1), (–1, –1, 1), (–1, 1, –1), (–1, 1, 1), (1, –1, –1), (1, –1, 1), (1, 1, –1), (1, 1, 1)} 6. A = {a, b}, B = {x, y} 8. A × B = {(1, 3), (1, 4), (2, 3), (2, 4)} A × B will have 24 = 16 subsets. 9. A = {x, y, z} and B = {1,2} 10. A = {–1, 0, 1}, remaining elements of A × A are (–1, –1), (–1, 1), (0, –1), (0, 0), (1, –1), (1, 0), (1, 1) Rationalised 2023-24 338 MATHEMATICS EXERCISE 2.2 1. R = {(1, 3), (2, 6), (3, 9), (4, 12)} Domain of R = {1, 2, 3, 4} Range of R = {3, 6, 9, 12} Co domain of R = {1, 2, ..., 14} 2. R = {(1, 6), (2, 7), (3, 8)} Domain of R = {1, 2, 3} Range of R = {6, 7, 8} 3. R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)} 4. (i) R = {(x, y) : y = x – 2 for x = 5, 6, 7} (ii) R = {(5,3), (6,4), (7,5)}. Domain of R = {5, 6, 7}, Range of R = {3, 4, 5} 5. (i) R = {(1, 1), (1,2), (1, 3), (1, 4), (1, 6), (2 4), (2, 6), (2, 2), (4, 4), (6, 6), (3, 3), (3, 6)} (ii) Domain of R = {1, 2, 3, 4, 6} (iii) Range of R = {1, 2, 3, 4, 6} 6. Domain of R = {0, 1, 2, 3, 4, 5,} 7. R = {(2, 8), (3, 27), (5, 125), (7, 343)} Range of R = {5, 6, 7, 8, 9, 10} 8. No. of relations from A into B = 26 9. Domain of R = Z Range of R = Z EXERCISE 2.3 1. (i) yes, Domain = {2, 5, 8, 11, 14, 17}, Range = {1} (ii) yes, Domain = (2, 4, 6, 8, 10, 12, 14}, Range = {1, 2, 3, 4, 5, 6, 7} (iii) No. 2. (i) Domain = R, Range = (– ∞, 0] (ii) Domain of function = {x : –3 ≤ x ≤ 3} Range of function = {x : 0 ≤ x ≤ 3} 3. (i) f (0) = –5 (ii) f (7) = 9 (iii) f (–3) = –11 4. (i) t (0) = 32 (ii) t (28) = 412 5 (iii) t (–10) = 14 (iv) 100 5. (i) Range = (– ∞, 2) (ii) Range = [2, ∞) (iii) Range = R Rationalised 2023-24 ANSWERS 339 Miscellaneous Exercise on Chapter 2 2. 2.1 3. Domain of function is set of real numbers except 6 and 2. 4. Domain = [1, ∞), Range = [0, ∞) 5. Domain = R, Range = non-negative real numbers 6. Range = [0, 1) 7. (f + g) x = 3x – 2 8. a = 2, b = – 1 9. (i) No (ii) No (iii) No (f – g) x = – x + 4 1 3 , 2 3 2 f x x x g x ⎛ ⎞ + ⎛ ⎞ = ≠ ⎜ ⎟ – 10. (i) Yes, (ii) No 11. No 12. Range of f = {3, 5, 11, 13 } EXERCISE 3.1 1. (i) 5π 36 (ii) 19π 72 – (iii) 4π 3 (iv) 26π 9 2. (i) 39° 22′ 30″ (ii) –229° 5′ 27″ (iii) 300° (iv) 210° 3. 12π 4. 12° 36′ 5. 20π 3 6. 5 : 4 7. (i) 2 15 (ii) 1 5 (iii) 7 25 EXERCISE 3.2 1. 3 2 1 sin cosec sec 2 tan 3 cot 2 3 3 x , x – , x , x , x = – = = = – = = 2. 5 4 5 3 4 cosec cos sec tan cot 3 5 4 4 3 x , x – , x , x , x = = = = – = – = – = – 3. 4 5 3 5 4 sin cosec cos sec tan 5 4 5 3 3 x , x – , x , x , x = – = = = – = – = = 4. 12 13 5 12 5 sin cosec cos tan cot 13 12 13 5 12 x , x – , x , x , x = – = = = = – = – Rationalised 2023-24 340 MATHEMATICS 5. 5 13 12 13 12 sin cosec cos sec cot 13 5 13 12 5 x , x , x , x , x = = = – = – = – 6. 1 2 7. 2 8. 3 9. 3 2 10. 1 EXERCISE 3.3 5. (i) 3 1 2 2 + (ii) 2 – 3 Miscellaneous Exercise on Chapter 3 8. 2 5 5 1 5 5 2 , , 9. 6 3 2 3 3 , – , – 10. 8 2 15 8 2 15 4 15 4 4 , , + – + EXERCISE 4.1 1. 3 + i0 2. 0 + i 0 3. 0+i 1 4. 14 + 28 i 5. 2 – 7 i 6. 19 21 5 10 i – – 7. 17 5 3 3 + i 8. - 4 + i 0 9. 242 26 27 – – i 10. 22 107 3 27 i – – 11. 4 3 25 25 + i 12. 5 3 14 14 – i 13. 0 + i1 14. 0 – i 7 2 2 Rationalised 2023-24 ANSWERS 341 Miscellaneous Exercise on Chapter 4 1. 2 – 2 i 3. 307 599 442 + i 5. 2 7. 2 (i) (ii) 0 5 , – 8. x = 3, y = – 3 9. 2 11. 1 12. 0 14. 4 EXERCISE 5.1 1. (i) {1, 2, 3, 4} (ii) {... – 3, – 2, –1, 0, 1,2,3,4,} 2. (i) No Solution (ii) {... – 4, – 3} 3. (i) {... – 2, – 1, 0, 1} (ii) (–∞, 2) 4. (i) {–1, 0, 1, 2, 3, ...} (ii) (–2, ∞) 5. (–4, ∞) 6. (– ∞, –3) 7. (–∞, –3] 8. (–∞, 4] 9. (– ∞, 6) 10. (–∞, –6) 11. (–∞, 2] 12. (– ∞, 120] 13. (4, ∞) 14. (–∞, 2] 15. (4, ∞) 16. (–∞, 2] 17. (– ∞, 3), 18. [–1, ∞), 19. (–1, ∞), 20. 2 7 , ⎡ ⎞ – ∞⎟ ⎢ ⎣, 21. 35 22. 82 23. (5,7), (7,9) 24. (6,8), (8,10), (10,12) 25. 9 cm 26. Greater than or equal to 8cm but less than or equal to 22cm Rationalised 2023-24 342 MATHEMATICS Miscellaneous Exercise on Chapter 5 1. [2, 3] 2. (0, 1] 3. [– 4, 2] 4. (– 23, 2] 5. 80 10 3 3 – – , ⎛ ⎤ ⎜ ⎥ ⎝ ⎦ 6. 11 1 3 , ⎡ ⎤ ⎢ ⎥ ⎣ ⎦ 7. (–

5, 5) 8. (–1, 7) 9. (5, ∞) 10. [– 7, 11] 11. Between 20°C and 25°C 12. More than 320 litres but less than 1280 litres. 13. More than 562.5 litres but less than 900 litres. 14. 9.6 ≤ MA ≤ 16.8 EXERCISE 6.1 1. (i) 125, (ii) 60. 2. 108 3. 5040 4. 336 5. 8 6. 20 EXERCISE 6.2 1. (i) 40320, (ii) 18 2. 30, No 3. 28 4. 64 5. (i) 30, (ii) 15120 EXERCISE 6.3 1. 504 2. 4536 3. 60 4. 120, 48 5. 56 6. 9 7. (i) 3, (ii) 4 8. 40320 Rationalised 2023-24 ANSWERS 343 9. (i) 360, (ii) 720, (iii) 240 10. 33810 11. (i) 1814400, (ii) 2419200, (iii) 25401600 EXERCISE 6.4 1. 45 2. (i) 5, (ii) 6 3. 210 4. 40 5. 2000 6. 778320 7. 3960 8. 200 9. 35 Miscellaneous Exercise on Chapter 6 1. 3600 2. 1440 3. (i) 504, (ii) 588, (iii) 1632 4. 907200 5. 120 6. 50400 7. 420 8. 4C1 × 48C4 9. 2880 10. 22C7 + 22C10 11. 151200 EXERCISE 7.1 1. 1–10x + 40x 2 – 80x 3 + 80x 4 – 32x 5 2. 5 3 5 3 32 40 20 5 5 8 32 x x x x x x – + – + – 3. 64 x 6 –576 x 5 + 2160 x 4 – 4320 x 3 + 4860 x 2 – 2916 x + 729 4. 5 3 3 5 5 10 10 5 1 243 81 27 9 3 x x x x x x + + + + + 5. 6 4 2 2 4 6 15 6 1 x x x 6 15 20 x x x + + + + + + 6. 884736 7. 11040808032 8. 104060401 9. 9509900499 10. (1.1)10000 > 1000 11. 8(a 3b + ab3); 40 6 12. 2(x 6 + 15x 4 + 15x 2 + 1), 198 Miscellaneous Exercise on Chapter 7 2. 396 6 3. 2a 8 + 12a 6 – 10a 4 – 4a 2 + 2 4. 0.9510 5. 2 3 4 2 3 4 16 8 32 16 4 5 2 2 16 x x x x x x x x + – + – + + + – 6. 27x 6 – 54ax5 + 117a 2x 4 – 116a 3x 3 + 117a 4x 2 – 54a 5x + 27a 6 Rationalised 2023-24 344 MATHEMATICS EXERCISE 8.1 1. 3, 8, 15, 24, 35 2. 1 2 3 4 5 2 3 4 5 6 , , , , 3. 2, 4, 8, 16 and 32 4. 1 1 1 5 7 and 6 6 2 6 6 – , , , 5. 25, –125, 625, –3125, 15625 6. 3 9 21 75 21and 2 2 2 2 , , , 7. 65, 93 8. 49 128 9. 729 10. 360 23 11. 3, 11, 35, 107, 323; 3 + 11 + 35 + 107 + 323 + ... 12. 1 1 1 1 1 1 1 1 1 1 2 6 24 120 2 6 24 120 , , , , ; – ... – – – – – – – – � � � � � � � � – + + + + + � � � � � � � � � � � � � � � � � 13. 2, 2, 1, 0, –1; 2 + 2 + 1 + 0 + (–1) + ... 14. 3 5 8 1 2 and 2 3 5 , , , EXERCISE 8.2 1. 20 5 5 2 2n , 2. 3072 4. – 2187 5. (a) 13th , (b) 12th, (c) 9th 6. ± 1 7. ( ) 1 20 1 0 1 6 . � � – � � 8. ( ) 2 7 3 1 3 1 2 n � � + – � � � � � � 9. 1 ( ) 1 n a a � � – – � � + 10. ( ) 3 2 2 1 1 n x x x – – 11. ( ) 3 11 22 3 1 2 + – 12. 5 2 2 5 5 2 or ; Terms are 1 or 1 2 5 5 2 2 5 r = , , , , 13. 4 14. ( ) 16 16 ;2; 2 1 7 7 n – 15. 2059 or 463 Rationalised 2023-24 ANSWERS 345 16. 4 8 16 or 4 8 16 32 64 3 3 3 , , ,... , , , , ,.. – – – – – 18. ( ) 80 8 10 1 81 9 n – – n 19. 496 20. rR 21. 3, –6, 12, –24 26. 9 and 27 27. 1 2 – n = 30. 120, 480, 30 (2n) 31. Rs 500 (1.1)10 32. x 2 –16x + 25 = 0 Miscellaneous Exercise on Chapter 8 1. 4 2. 160; 6 3. ± 3 4. 8, 16, 32 5. 4 11. (i) ( ) 50 5 10 1 81 9 n n – – , (ii) ( ) 2 2 1 10 3 27 n –n – – 12. 1680 13. Rs 16680 14. Rs 39100 15. Rs 43690 16. Rs 17000; 20,000 17. Rs 5120 18. 25 days EXERCISE 9.1 1. 121square unit. 2 2. (0, a), (0, – a) and (– 3 0 a,) or (0, a), (0, – a), and (3 0 a,) 3. (i) 2 1 y y , – (ii) 2 1 x x – 4. 15 0 2 , � � � � � � 5. 1 2 – 7. – 3 9. 135° 10. 1 and 2, or 1 2 and 1, or – 1 and –2, or 1 2 – and – 1 EXERCISE 9.2 1. y = 0 and x = 0 2. x – 2y + 10 = 0 3. y = mx 4. (3 1 3 1 4 3 1 + – – =) x y – ( ) ( ) 5. 2x + y + 6 = 0 6. x y – + = 3 2 3 0 7. 5x + 3y + 2 = 0 Rationalised 2023-24 346 MATHEMATICS 8. 3x – 4y + 8 = 0 9. 5x – y + 20 = 0 10. (1 + n)x + 3(1 + n)y = n +11 11. x + y = 5 12. x + 2y – 6 = 0, 2x + y – 6 = 0 13. 3 2 0 and 3 2 0 x y + – = + + = x y 14. 2x – 9y + 85 = 0 15. ( ) 192 L C 20 124 942 90 . = – + . 16. 1340 litres. 18. 2kx + hy = 3kh. EXERCISE 9.3 1. (i) 1 1 0 0; 7 7 y x , , = – + – (ii) 5 5 2 2 ; 3 3 y x , , = – + – (iii) y = 0x + 0, 0, 0 2. (i) 1 4 6 4 6 x y + = , , ; (ii) 3 1 2; 3 2 2 2 x y + = – , , – (iii) 2 3 y , = – intercept with y-axis = 2 3 – and no intercept with x-axis. 3. 5 units 4. (– 2, 0) and (8, 0) 5. (i) 65 1 units, (ii) units. 17 2 p r l + 6. 3x – 4y + 18 = 0 7. y + 7x = 21 8. 30° and 150° 9. 22 9 11. (3 2 2 3 1 8 3 1 + + = +) x – y ( ) or 3 2 1 2 3 1 8 3 (– + + = +) x y – ( ) 12. 2x + y = 5 13. 68 49 25 25 , � � – � � � � 14. 1 5 2 2 m ,c = = 16. y – x = 1, 2 Miscellaneous Exercise on Chapter 9 1. (a) 3, (b) ± 2, (c) 6 or 1 2. 2 3 6 3 2 6 x y , x y – = – + = 3. 8 32 0 0 3 3 , , , � � � � – � � � � � � � � Rationalised 2023-24 ANSWERS 347 4. Cos 2 φ θ– 5. 5 22 x = – 6. 2x – 3y + 18 = 0 7. k 2 square units 8. 5 10. 3x – y = 7, x + 3y = 9 11. 13x + 13y = 6 13. 1 : 2 14. 23 5 units 18 15. The line is parallel to x - axis or parallel to y-axis 16. x = 1, y = 1. or x = – 4, y = 3 17. (–1, – 4). 18. 1 5 2 7 ± 20. 18x + 12y + 11 = 0 21. 13 0 5 , � � � � � � 23. 119x + 102y = 125 EXERCISE 10.1 1. x 2 + y 2 – 4y = 0 2. x 2 + y2 + 4x – 6y –3 = 0 3. 36x 2 + 36y 2 – 36x – 18y + 11 = 0 4. x 2 + y 2 – 2x – 2y = 0 5. x 2 + y 2 + 2ax + 2by + 2b 2 = 0 6. c(–5, 3), r = 6 7. c(2, 4), r = 65 8. c(4, – 5), r = 53 9. c (1 4 , 0) ; r = 1 4 10. x 2 + y 2 – 6x – 8y + 15 = 0 11. x 2 + y 2 – 7x + 5y – 14 = 0 12. x 2 + y 2 + 4x – 21 = 0 & x2 + y 2 – 12x + 11 = 0 13. x 2 + y 2 – ax – by = 0 14. x 2 + y 2 – 4x – 4y = 5 15. Inside the circle; since the distance of the point to the centre of the circle is less than the radius of the circle. EXERCISE 10.2 1. F

(3, 0), axis - x - axis, directrix x = – 3, length of the Latus rectum = 12 2. F (0, 3 2), axis - y - axis, directrix y = – 3 2 , length of the Latus rectum = 6 3. F (–2, 0), axis - x - axis, directrix x = 2, length of the Latus rectum = 8 Rationalised 2023-24 348 MATHEMATICS 4. F (0, –4), axis - y - axis, directrix y = 4, length of the Latus rectum = 16 5. F (5 2 , 0) axis - x - axis, directrix x = – 5 2 , length of the Latus rectum = 10 6. F (0, 9 4 –) , axis - y - axis, directrix y = 9 4 , length of the Latus rectum = 9 7. y 2 = 24x 8. x 2 = – 12y 9. y 2 = 12x 10. y 2 = –8x 11. 2y 2 = 9x 12. 2x 2 = 25y EXERCISE 10.3 1. F (± 20 ,0); V (± 6, 0); Major axis = 12; Minor axis = 8 , e = 20 6 , Latus rectum = 16 3 2. F (0, ± 21); V (0, ± 5); Major axis = 10; Minor axis = 4 , e = 21 5 ; Latus rectum = 8 5 3. F (± 7 , 0); V (± 4, 0); Major axis = 8; Minor axis = 6 , e = 7 4 ; Latus rectum = 9 2 4. F (0, ± 75); V (0,± 10); Major axis = 20; Minor axis = 10 , e = 3 2 ; Latus rectum = 5 5. F (± 13 ,0); V (± 7, 0); Major axis =14 ; Minor axis = 12 , e = 13 7 ; Latus rectum = 72 7 6. F (0, ±10 3); V (0,± 20); Major axis =40 ; Minor axis = 20 , e = 3 2 ; Latus rectum = 10 Rationalised 2023-24 ANSWERS 349 7. F (0, ± 4 2); V (0,± 6); Major axis =12 ; Minor axis = 4 , e = 2 2 3 ; Latus rectum = 4 3 8. F 0 15 (,±) ; V (0,± 4); Major axis = 8 ; Minor axis = 2 , e = 15 4 ; Latus rectum = 1 2 9. F (± 5 ,0); V (± 3, 0); Major axis = 6 ; Minor axis = 4 , e = 5 3 ; Latus rectum = 8 3 10. 2 2 1 25 9 x y + = 11. 2 2 1 144 169 x y + = 12. 2 2 1 36 20 x y + = 13. 2 2 1 9 4 x y + = 14. 2 2 1 1 5 x y + = 15. 2 2 1 169 144 x y + = 16. 2 2 1 64 100 x y + = 17. 2 2 1 16 7 x y + = 18. 2 2 1 25 9 x y + = 19. 2 2 1 10 40 x y + = 20. x 2 + 4y 2 = 52 or 2 2 1 52 13 x y + = EXERCISE 10.4 1. Foci (± 5, 0), Vertices (± 4, 0); e = 4 5 ; Latus rectum = 9 2 2. Foci (0 ± 6), Vertices (0, ± 3); e = 2; Latus rectum = 18 3. Foci (0, ± 13), Vertices (0, ± 2); e = 13 2 ; Latus rectum = 9 4. Foci (± 10, 0), Vertices (± 6, 0); e = 5 3 ; Latus rectum 64 3 = Rationalised 2023-24 350 MATHEMATICS 5. Foci (0,± 2 14 5), Vertices (0,± 6 5); e = 14 3 ; Latus rectum 4 5 3 = 6. Foci (0, ± 65), Vertices (0, ± 4); e = 65 4 ; Latus rectum 49 2 = 7. 2 2 1 4 5 x y − = 8. 2 2 1 25 39 y x − = 9. 2 2 1 9 16 y x − = 10. 2 2 1 16 9 x y − = 11. 2 2 1 25 144 y x − = 12. 2 2 1 25 20 x y − = 13. 2 2 1 4 12 x y − = 14. 2 2 9 1 49 343 x y − = 15. 2 2 1 5 5 y x − = Miscellaneous Exercise on Chapter 10 1. Focus is at the mid-point of the given diameter. 2. 2.23 m (approx.) 3. 9.11 m (approx.) 4. 1.56m (approx.) 5. 2 2 1 81 9 x y + = 6. 18 sq units 7. 2 2 1 25 9 x y + = 8. 8 3a EXERCISE 11.1 1. y and z - coordinates are zero 2. y - coordinate is zero 3. I, IV, VIII, V, VI, II, III, VII 4. (i) XY - plane (ii) (x, y, 0) (iii) Eight EXERCISE 11.2 1. (i) 2 5 (ii) 43 (iii) 2 26 (iv) 2 5 4. x – 2z = 0 5. 9x 2 + 25y 2 + 25z 2 – 225 = 0 Miscellaneous Exercise on Chapter 11 1. (1, – 2, 8) 2. 7 34 7 , , 3. a = – 2, b = 16 3 − , c = 2 4. 2 2 2 2 109 2 7 2 2 k – x y z x y z + + − − + = Rationalised 2023-24 ANSWERS 351 EXERCISE 12.1 1. 6 2. 22 π 7 − 3. π 4. 19 2 5. 1 2 − 6. 5 7. 11 4 8. 108 7 9. b 10. 2 11. 1 12. 1 4 − 13. a b 14. a b 15. 1 π 16. 1 π 17. 4 18. a 1 b + 19. 0 20. 1 21. 0 22. 2 23. 3, 6 24. Limit does not exist at x = 1 25. Limit does not exist at x = 0 26. Limit does not exist at x = 0 27. 0 28. a = 0, b = 4 29. 1 lim x a → f (x) = 0 and lim x a → f (x) = (a – a1) (a – a2) ... (a – ax) 30. lim x a → f (x) exists for all a ≠ 0. 31. 2 32. For 0 lim x→ f (x) to exists, we need m = n; 1 lim x→ f (x) exists for any integral value of m and n. EXERCISE 12.2 1. 20 2. 1 3. 99 4. (i) 3x 2 (ii) 2x – 3 (iii) 3 2 x − (iv) ( )2 2 x 1 − − 6. 1 2 2 3 1 (1) (2) n n n n nx a n x a n x ... a − − − − + − + − + + 7. (i) 2x a b − − (ii) ( ) 2 4ax ax b + (iii) ( )2 a b x b − − Rationalised 2023-24 352 MATHEMATICS 8 . ( ) 1 2 n n n n nx anx x a x a − − − + − 9 . (i) 2 (ii) 20x 3 – 15x 2 + 6x – 4 (iii) ( ) 4 3 5 2x x − + (iv) 15x 4 + 5 24 x (v) 5 10 –12 36 x x + (vi) ( )2 2 2 (3 2) 1 (3 1) – x x – − x + x − 10. – sin x 11. (i) cos 2x (ii) sec x tan x (iii) 5sec x tan x – 4sin x (iv) – cosec x cot x (v) – 3cosec2 x – 5 cosec x cot x (vi) 5cos x+ 6sin x (vii) 2sec2 x – 7sec x tan x Miscellaneous Exercise on Chapter 12 1. (i) – 1 (ii) 2 1 x (iii) cos (x + 1) (iv) π sin 8 x − − 2. 1 3. 2 qr ps x − + 4. 2c (ax+b) (cx + d) + a (cx + d) 2 5. ( )2 ad bc cx d − + 6. ( )2 2 0 1 1 , x , x − − ≠ 7. ( ) ( )2 2 2ax b ax bx c − + + + 8. ( ) 2 2 2 apx bpx ar bq 2 px qx r − − + − + + 9. ( ) 2 2 apx bpx bq ar 2 ax b + + − + 10. 5 3 4 2 sin − + − a b x x x 11. 2 x 12. ( )n 1 na ax b − + 13. ( ) ( ) ( ) ( ) n 1 m 1 ax b cx d mc ax b na cx d − − + + + + + 14. cos (x+a) 15. – cosec3 x – cosec x cot2 x 16. 1 1 sin − + x 17. ( )2 2 sin cos − x x − 18. ( )2 2sec tan sec 1+ x x x 19. n sinn–1 x cos x Rationalised 2023-24 ANSWERS 353 20. ( )2 cos sin cos + + + bc x ad x bd c d x 21. 2 cos cos a x 22. ( ) 3 x x x x x x x 5 cos 3 sin 20 sin 12cos + + − 23. 2 − − + x x x x x x sin sin 2 cos 24. ( ) ( )( ) 2 − + + + q sin x ax x p q x a x x sin cos 2 cos + 25. ( ) ( )( ) 2 − + + − − tan cos tan 1 sin x x x x x x 26. ( )2 35 15 cos 28 cos 28 sin 15sin 3 7 cos + + + − + x x x x x x x x x 27. ( ) 2 π cos 2 sin cos 4 sin x x x x − x 28. ( ) 2 2 1 tan sec 1 tan x x x x + − + 29. ( )( ) ( ) ( ) 2 x x x x x . x x + − + − + sec 1 sec tan 1 sec tan 30. 1 sin cos sin + − n x n x x x EXERCISE 13.1 1. 3 2. 8.4 3. 2.33 4. 7 5. 6.32 6. 16 7. 3.23 8. 5.1 9. 157.92 10. 11.28 11. 10.34 12. 7.35 EXERCISE 13.2 1. 9, 9.25 2. 2 1 1 2 12 n n , + − 3. 16.5, 74.25 4. 19, 43.4 5. 100, 29.09 6. 64, 1.69 7. 107, 2276 8. 27, 132 9. 93, 105.58, 10.27 10. 5.55, 43.5 Rationalised 2023-24 354 MATHEMATICS Miscellaneous Exercise on Chapter 13 1. 4, 8 2. 6, 8 3. 24, 12 5. (i) 10.1, 1.99 (ii) 10.2, 1.98 6. 20, 3.036 EXERCISE 14.1 1. No. 2. (i) {1, 2, 3, 4, 5, 6} (ii) ϕ (iii) {3, 6} (iv) {1, 2, 3} (v) {6} (vi) {3, 4, 5, 6}, A∪B = {1, 2, 3, 4, 5, 6}, A∩B = ϕ, B∪C = {3, 6}, E∩F = {6}, D∩E = ϕ, A – C = {1, 2,4,5}, D – E = {1,2,3}, E∩ F′ = ϕ, F′ = {1, 2} 3. A = {(3,6), (4,5), (5, 4), (6,3), (4,6), (5,5), (6,4), (5,6), (6,5), (6,6)} B = {(1,2), (2,2), (3, 2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)} C ={(3,6), (6,3), (5, 4), (4,5), (6,6)} A and B, B and C are mutually exclusive. 4. (i) A and B; A and C; B and C; C and D (ii) A and C (iii) B and D 5. (i) “Getting at least two heads”, and “getting at least two tails” (ii) “Getting no heads”, “getting exactly one head” and “getting at least two heads” (iii) “Getting at most two tails”, and “getting exactly two tails” (iv) “Getting exactly one head” and “getting exactly two heads” (v) “Getting exactly one tail”, “getting exactly two tails”, and getting exactly three tails” ANote There may be other events also as answer to the above question. 6. A = {(2, 1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} B = {(1, 1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)} C = {(1, 1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)} (i) A′ = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)} = B (ii) B′ = {(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} = A (iii) A∪B = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (2,1), (2,2), (2,3), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} = S Rationalised 2023-24 ANSWERS 355 (iv) A ∩ B = ϕ (v) A – C = {(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} (vi) B ∪ C = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)} (vii) B ∩ C = {(1,1), (1,2), (1,3), (1,4), (3,1), (3,2)} (viii) A ∩ B′ ∩ C′ = {(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} 7. (i) True (ii) True (iii) True (iv) False (v) False (vi) False EXERCISE 14.2 1. (a) Yes (b) Yes (c) No (d) No (e) No 2. 3 4 3. (i) 1 2 (ii) 2 3 (iii) 1 6 (iv) 0 (v) 5 6 4. (a) 52 (b) 1 52 (c) 1 1 (i) (ii) 13 2 5. 1 1 (i) (ii) 12 12 6. 3 5 7. Rs 4.00 gain, Rs 1.50 gain, Re 1.00 loss, Rs 3.50 loss, Rs 6.00 loss. P (Winning Rs 4.00) 1 16 = , P(Winning Rs 1.50) 1 4 = , P (Losing Re. 1.00) 3 8 = P (Losing Rs 3.50) 1 4 = , P (Losing Rs 6.00) 1 16 = . 8. (i) 1 8 (ii) 3 8 (iii) 1 2 (iv) 7 8 (v) 1 8 (vi) 1 8 (vii) 3 8 (viii) 1 8 (ix) 7 8 9. 9 11 10. 6 7 (i) (ii) 13 13 11. 1 38760 12. (i) No, because P(A∩B) must be less than or equal to P(A) and P(B), (ii) Yes 13. 7 (i) (ii) 0.5 (iii) 0.15 15 14. 4 5 15. 5 3 (i) (ii) 8 8 16. No 17. (i) 0.58 (ii) 0.52 (iii) 0.74 Rationalised 2023-24 356 MATHEMATICS 18. 0.6 19. 0.55 20. 0.65 21. 19 11 2 (i) (ii) (iii) 30 30 15 Miscellaneous Exercise on Chapter 14 1. (i) 20 5 60 5 C C (ii) 30 5 60 5 C 1 C − 2. 13 13 3 1 52 4 C . C C 3. 1 1 5 (i) (ii) (iii) 2 2 6 4. 9990 2 10000 2 999 C (a) (b) 1000 C 9990 10 10000 10 C (c) C 5. 17 16 (a) (b) 33 33 6. 2 3 7. (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34 8. 4 5 9. 33 3 (i) (ii) 83 8 10. 1 5040 Rationalised 2023-24MATHEMATICS Textbook for Class XI Rationalised 2023-24 First Edition February 2006 Phalguna 1927 Reprinted October 2006, November 2007, December 2008, December 2009, January 2011, February 2012, December 2012, November 2013, December 2014, May 2016, December 2016, December 2017, January 2019, August 2019, January 2021 and November 2021 Revised Edition November 2022 Agrahayan 1944 PD 500T BS © National Council of Educational Research and Training, 2006, 2022 ` 210.00 ALL RIGHTS RESERVED q No part of this publication may be reproduced, stored in a retrieval system or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise without the prior permission of the publisher. q This book is sold subject to the condition that it shall not, by way of trade, be lent, re-sold, hired out or otherwise disposed of without the publisher’s consent, in any form of binding or cover other than that in which

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This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986). The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge. These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience. The National Council of Educational Research and Training (NCERT) appreciates the hard work done by the Textbook Development Committee responsible for this Rationalised 2023-24 iv book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. We are especially grateful to the members of the National Monitoring Committee, appointed by the Department of Secondary and Higher Education, Ministry of Human Resource Development under the Chairpersonship of Professor Mrinal Miri and Professor G.P. Deshpande, for their valuable time and contribution. As an organisation committed to the systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement. Director New Delhi National Council of Educational 20 December 2005 Research and Training Rationalised 2023-24 v Rationalisation of Content in the Textbooks In view of the COVID-19 pandemic, it is imperative to reduce content load on students. The National Education Policy 2020, also emphasises reducing the content load and providing opportunities for experiential learning with creative mindset. In this background, the NCERT has undertaken the exercise to rationalise the textbooks across all classes. Learning Outcomes already developed by the NCERT across classes have been taken into consideration in this exercise. Contents of the textbooks have been rationalised in view of the following: • Overlapping with similar content included in other subject areas in the same class • Similar content included in the lower or higher class in the same subject • Difficulty level • Content, which is easily accessible to students without much interventions from teachers and can be learned by children through self-learning or peerlearning • Content, which is irrelevant in the present context This present edition, is a reformatted version after carrying out the changes given above. Rationalised 2023-24 Rationalised 2023-24 Textbook Development Committee CHAIRPERSON, ADVISORY GROUP IN SCIENCE AND MATHEMATICS J.V. 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College Azamgarh (U.P.); Sumat Kumar Jain, Lecturer, K.L. Jain Inter College Sasni Hathras (U.P.); R.P. Gihare, Lecturer (BRC), Janpad Shiksha Kendra Chicholi Distt. Betul (M.P.); Sangeeta Arora, P.G.T., A.P.J. School Saket, New Delhi; P.N. Malhotra, ADE (Sc.), Directorate of Education, Delhi; D.R. Sharma, P.G.T., J.N.V. Mungespur, Delhi; Saroj, P.G.T. Government Girls Sr. Secondary School, No. 1, Roop Nagar, Delhi, Manoj Kumar Thakur, P.G.T., D.A.V. Public School, Rajender Nagar, Sahibabad, Ghaziabad (U.P.) and R.P. Maurya, Reader, DESM, NCERT, New Delhi. Acknowledgements are due to Professor M. Chandra, Head, Department of Education in Science and Mathematics for her support. The Council acknowledges the efforts of the Computer Incharge, Deepak Kapoor; Rakesh Kumar, Kamlesh Rao and Sajjad Haider Ansari, D.T.P. Operators; Kushal Pal Singh Yadav, Copy Editor and Proof Readers, Mukhtar Hussain and Kanwar Singh. 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 Rationalised 2023-24SUPPLEMENTARY MATERIAL 357 SUPPLEMENTARY MATERIAL CHAPTER 8 8.6 Infinite G.P. and its Sum G.P. of the form a, ar, ar2 , ar3 , ... is called infinite G.P. Now, to find the formulae for finding sum to infinity of a G.P., we begin with an example. Let us consider the G.P., 2 4 1 , , , .. . 3 9 Here a = 1, 2 3 r = . We have 2 1 3 2 S 3 1 2 3 1 3 n n n   −   = = −   −   Let us study the behaviour of 2 3 n   as n becomes larger and larger: n 1 5 10 20 2 3 n   0.6667 0.1316872428 0.01734152992 0.00030072866 We observe that as n becomes larger and larger, 2 3 n   becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, 2 3 n   becomes sufficiently small. In other words as 2 , 0. 3 n n   → ∞ →   Consequently, we find that the sum of infinitely many terms is given by S 3. ∞ = 358 MATHEMATICS Now, for a geometric progression, a, ar, ar2 , ..., if numerical value of common ratio r is less than 1, then (1) S (1) 1 1 n n n a r a ar r r r − = = − − − − In this case as , n →∞ r n → 0 since |r| < 1. Therefore 1 n a S r → − Symbolically sum to infinity is denoted by S or S. ∞ Thus, we have S 1 – a r = . For examples, (i) 2 3 1 1 1 1 1 ... 2 . 2 2 2 1 1 2 + + + + = = − (ii) 2 3 1 1 1 1 1 2 1 .. . 2 2 2 1 1 3 1 1 2 2 − + − + = = =   − − +   Exercise 8.3 Find the sum to infinity in each of the following Geometric Progression. 1. 1 1 1, , ,... 3 9 (Ans. 1.5) 2. 6, 1.2, .24, ... (Ans. 7.5) 3. 5, 2 0 8 0 , , ... 7 4 9 (Ans. 35 3) 4. 3 3 3 , , ,... 4 16 64 − − (Ans. 3 5 −) 5. Prove that 1 1 1 2 4 8 3 3 3 .. . 3 × × = 6. Let x = 1 + a + a 2 + ... and y = 1 + b + b 2 + ..., where |a| < 1 and |b| < 1. Prove that 1 + ab + a 2b 2 + ... = 1 xy x y + − SUPPLEMENTARY MATERIAL 359 CHAPTER 12 12.6 Limits Involving Exponential and Logarithmic Functions Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly. Leonhard Euler (1707–1783), the great Swiss mathematician introduced the number e whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as f (x) = e x , x ∈ R. Its domain is R, range is the set of positive real numbers. The graph of exponential function, i.e., y = e x is as given in Fig.13.11. Fig. 13.11 Similarly, the logarithmic function expressed as loge R+ → R is given by loge x = y, if and only if e y = x. Its domain is R+ which is the set of all positive real numbers and range is R. The graph of logarithmic function y = loge x is shown in Fig.13.12. 360 MATHEMATICS Fig. 13.12 In order to prove the result 0 – 1 lim 1 x x e → x = , we make use of an inequality involving the expression x e x −1 which runs as follows: 1 1 x ≤ + – 1 x e x ≤ 1 + (e – 2) |x| holds for all x in [–1, 1] ~ {0}. Theorem 6 Prove that 0 1 lim 1 x x e → x − = Proof Using above inequality, we get 1 1 ≤ + x 1 x e x − ≤ 1 + | x| (e – 2), x Î [–1, 1] ~ {0} Also 0 0 1 1 1 lim 1 x 1 1 lim 1 0 x → x x → = = = + + + SUPPLEMENTARY MATERIAL 361 and 0 0 lim 1 (2) 1 (2) lim 1 (2)0 1 x x e x e x e → → + − = + − = + − =   Therefore, by Sandwich theorem, we get 0 1 lim 1 x x e → x − = Theorem 7 Prove that 0 log (1) lim 1 e x x → x + = Proof Let log (1) e x y x + = . Then log (1) e + = x xy 1 xy ⇒ + = x e 1 1 xy e x − ⇒ = or 1 . 1 xy e y xy − = 0 0 1 lim lim 1 (sin ce 0 g iv es 0) x y xy x e y x xy → → xy − ⇒ = → → 0 0 1 lim 1 a s lim 1 x y x x y e y → → x y − ⇒ = =   0 lo g (1) lim 1 e x x → x + ⇒ = Example 5 Compute 3 0 1 l i m x x e → x − Solution We have 3 3 0 3 0 0 1 1 lim lim 3 3 1 3 lim , w here 3 3.1 3 x x x x y y e e x x e y x y → → → − − = ⋅ − =   =   = = 362 MATHEMATICS Example 6 Compute 0 sin 1 lim x x e x → x − − Solution We have 0 0 sin 1 1 sin lim lim x x x x e x e x → x → x x − − −   = −   0 0 1 sin lim lim 1 1 0 x x x e x → x x → − = − = − = Example 7 Evaluate 1 log lim 1 e x x → x − Solution Put x = 1 + h, then as x h → ⇒ → 1 0. Therefore, 1 0 log log (1) lim lim 1 e e x h x h → x → h + = − 0 log (1) 1 since lim 1 e x x → x + = =   . Exercise 13.2 Evaluate the following limits, if exist 1. 4 0 1 lim x x e → x − (Ans. 4) 2. 2 2 0 lim x x e e x + → − (Ans. e 2) 3. 5 5 lim 5 x x e e → x − − (Ans. e5) 4. sin 0 1 lim x x e → x − (Ans. 1) 5. 3 3 lim 3 x x e e → x − − (Ans. e 3) 6. 0 (1) lim 1 cos x x x e → x − − (Ans. 2) 7. 0 log (1 2) lim e x x → x + (Ans. 2) 8. 3 3 0 log (1) limx sin x → x + (Ans. 1) SUPPLEMENTARY MATERIAL 363 Notes 364 MATHEMATICSSUPPLEMENTARY MATERIAL 357 SUPPLEMENTARY MATERIAL CHAPTER 8 8.6 Infinite G.P. and its Sum G.P. of the form a, ar, ar2 , ar3 , ... is called infinite G.P. Now, to find the formulae for finding sum to infinity of a G.P., we begin with an example. Let us consider the G.P., 2 4 1 , , , .. . 3 9 Here a = 1, 2 3 r = . We have 2 1 3 2 S 3 1 2 3 1 3 n n n −   = = −   −   Let us study the behaviour of 2 3 n   as n becomes larger and larger: n 1 5 10 20 2 3 n   0.6667 0.1316872428 0.01734152992 0.00030072866 We observe that as n becomes larger and larger, 2 3 n   becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, 2 3 n   becomes sufficiently small. In other words as 2 , 0. 3 n n   → ∞ →   Consequently, we find that the sum of infinitely many terms is given by S 3. ∞ = 358 MATHEMATICS Now, for a geometric progression, a, ar, ar2 , ..., if numerical value of common ratio r is less than 1, then (1) S (1) 1 1 n n n a r a ar r r r − = = − − − − In this case as , n →∞ r n → 0 since |r| < 1. Therefore 1 n a S r → − Symbolically sum to infinity is denoted by S or S. ∞ Thus, we have S 1 – a r = . For examples, (i) 2 3 1 1 1 1 1 ... 2 . 2 2 2 1 1 2 + + + + = = − (ii) 2 3 1 1 1 1 1 2 1 .. . 2 2 2 1 1 3 1 1 2 2 − + − + = = =   − − +   Exercise 8.3 Find the sum to infinity in each of the following Geometric Progression. 1. 1 1 1, , ,... 3 9 (Ans. 1.5) 2. 6, 1.2, .24, ... (Ans. 7.5) 3. 5, 2 0 8 0 , , ... 7 4 9 (Ans. 35 3) 4. 3 3 3 , , ,... 4 16 64 − − (Ans. 3 5 −) 5. Prove that 1 1 1 2 4 8 3 3 3 .. . 3 × × = 6. Let x = 1 + a + a 2 + ... and y = 1 + b + b 2 + ..., where |a| < 1 and |b| < 1. Prove that 1 + ab + a 2b 2 + ... = 1 xy x y + − SUPPLEMENTARY MATERIAL 359 CHAPTER 12 12.6 Limits Involving Exponential and Logarithmic Functions Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly. Leonhard Euler (1707–1783), the great Swiss mathematician introduced the number e whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as f (x) = e x , x ∈ R. Its domain is R, range is the set of positive real numbers. The graph of exponential function, i.e., y = e x is as given in Fig.13.11. Fig. 13.11 Similarly, the logarithmic function expressed as loge R+ → R is given by loge x = y, if and only if e y = x. Its domain is R+ which is the set of all positive real numbers and

range is R. The graph of logarithmic function y = loge x is shown in Fig.13.12. 360 MATHEMATICS Fig. 13.12 In order to prove the result 0 – 1 lim 1 x x e → x = , we make use of an inequality involving the expression x e x −1 which runs as follows: 1 1 x ≤ + – 1 x e x ≤ 1 + (e – 2) |x| holds for all x in [–1, 1] ~ {0}. Theorem 6 Prove that 0 1 lim 1 x x e → x − = Proof Using above inequality, we get 1 1 ≤ + x 1 x e x − ≤ 1 + | x| (e – 2), x Î [–1, 1] ~ {0} Also 0 0 1 1 1 lim 1 x 1 1 lim 1 0 x → x x → = = = + + + SUPPLEMENTARY MATERIAL 361 and 0 0 lim 1 (2) 1 (2) lim 1 (2)0 1 x x e x e x e → → + − = + − = + − = Therefore, by Sandwich theorem, we get 0 1 lim 1 x x e → x − = Theorem 7 Prove that 0 log (1) lim 1 e x x → x + = Proof Let log (1) e x y x + = . Then log (1) e + = x xy 1 xy ⇒ + = x e 1 1 xy e x − ⇒ = or 1 . 1 xy e y xy − = 0 0 1 lim lim 1 (sin ce 0 g iv es 0) x y xy x e y x xy → → xy − ⇒ = → → 0 0 1 lim 1 a s lim 1 x y x x y e y → → x y − ⇒ = = 0 lo g (1) lim 1 e x x → x + ⇒ = Example 5 Compute 3 0 1 l i m x x e → x − Solution We have 3 3 0 3 0 0 1 1 lim lim 3 3 1 3 lim , w here 3 3.1 3 x x x x y y e e x x e y x y → → → − − = · − = = = = 362 MATHEMATICS Example 6 Compute 0 sin 1 lim x x e x → x − − Solution We have 0 0 sin 1 1 sin lim lim x x x x e x e x → x → x x − − − = − 0 0 1 sin lim lim 1 1 0 x x x e x → x x → − = − = − = Example 7 Evaluate 1 log lim 1 e x x → x − Solution Put x = 1 + h, then as x h → ⇒ → 1 0. Therefore, 1 0 log log (1) lim lim 1 e e x h x h → x → h + = − 0 log (1) 1 since lim 1 e x x → x + = = . Exercise 13.2 Evaluate the following limits, if exist 1. 4 0 1 lim x x e → x − (Ans. 4) 2. 2 2 0 lim x x e e x + → − (Ans. e 2) 3. 5 5 lim 5 x x e e → x − − (Ans. e5) 4. sin 0 1 lim x x e → x − (Ans. 1) 5. 3 3 lim 3 x x e e → x − − (Ans. e 3) 6. 0 (1) lim 1 cos x x x e → x − − (Ans. 2) 7. 0 log (1 2) lim e x x → x + (Ans. 2) 8. 3 3 0 log (1) limx sin x → x + (Ans. 1) SUPPLEMENTARY MATERIAL 363 Notes 364 MATHEMATICSvMathematics is the indispensable instrument of all physical research. – BERTHELOT v 2.1 Introduction Much of mathematics is about finding a pattern – a recognisable link between quantities that change. In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student. In mathematics also, we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order. In this Chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. Finally, we will learn about special relations which will qualify to be functions. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correspondence between one quantity with the other. 2.2 Cartesian Products of Sets Suppose A is a set of 2 colours and B is a set of 3 objects, i.e., A = {red, blue}and B = {b, c, s}, where b, c and s represent a particular bag, coat and shirt, respectively. How many pairs of coloured objects can be made from these two sets? Proceeding in a very orderly manner, we can see that there will be 6 distinct pairs as given below: (red, b), (red, c), (red, s), (blue, b), (blue, c), (blue, s). Thus, we get 6 distinct objects (Fig 2.1). Let us recall from our earlier classes that an ordered pair of elements taken from any two sets P and Q is a pair of elements written in small Fig 2.1 Chapter 2 RELATIONS AND FUNCTIONS G. W. Leibnitz (1646–1716) Rationalised 2023-24 RELATIONS AND FUNCTIONS 25 brackets and grouped together in a particular order, i.e., (p,q), p ∈ P and q ∈ Q . This leads to the following definition: Definition 1 Given two non-empty sets P and Q. The cartesian product P × Q is the set of all ordered pairs of elements from P and Q, i.e., P × Q = { (p,q) : p ∈ P, q ∈ Q } If either P or Q is the null set, then P × Q will also be empty set, i.e., P × Q = ϕ From the illustration given above we note that A × B = {(red,b), (red,c), (red,s), (blue,b), (blue,c), (blue,s)}. Again, consider the two sets: A = {DL, MP, KA}, where DL, MP, KA represent Delhi, Madhya Pradesh and Karnataka, respectively and B = {01,02, 03}representing codes for the licence plates of vehicles issued by DL, MP and KA . If the three states, Delhi, Madhya Pradesh and Karnataka were making codes for the licence plates of vehicles, with the restriction that the code begins with an element from set A, which are the pairs available from these sets and how many such pairs will there be (Fig 2.2)? The available pairs are:(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03) and the product of set A and set B is given by A × B = {(DL,01),

(DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03)}. It can easily be seen that there will be 9 such pairs in the Cartesian product, since there are 3 elements in each of the sets A and B. This gives us 9 possible codes. Also note that the order in which these elements are paired is crucial. For example, the code (DL, 01) will not be the same as the code (01, DL). As a final illustration, consider the two sets A= {a1 , a2 } and B = {b1 , b2 , b3 , b4 } (Fig 2.3). A × B = {(a1 , b1), (a1 , b2), (a1 , b3), (a1 , b4), (a2 , b1), (a2 , b2), (a2 , b3), (a2 , b4)}. The 8 ordered pairs thus formed can represent the position of points in the plane if A and B are subsets of the set of real numbers and it is obvious that the point in the position (a1 , b2) will be distinct from the point in the position (b2 , a1). Remarks (i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal. DL MP KA 03 02 01 Fig 2.2 Fig 2.3 Rationalised 2023-24 26 MATHEMATICS (ii) If there are p elements in A and q elements in B, then there will be pq elements in A × B, i.e., if n(A) = p and n(B) = q, then n(A × B) = pq. (iii) If A and B are non-empty sets and either A or B is an infinite set, then so is A × B. (iv) A × A × A = {(a, b, c) : a, b, c ∈ A}. Here (a, b, c) is called an ordered triplet. Example 1 If (x + 1, y – 2) = (3,1), find the values of x and y. Solution Since the ordered pairs are equal, the corresponding elements are equal. Therefore x + 1 = 3 and y – 2 = 1. Solving we get x = 2 and y = 3. Example 2 If P = {a, b, c} and Q = {r}, form the sets P × Q and Q × P. Are these two products equal? Solution By the definition of the cartesian product, P × Q = {(a, r), (b, r), (c, r)} and Q × P = {(r, a), (r, b), (r, c)} Since, by the definition of equality of ordered pairs, the pair (a, r) is not equal to the pair (r, a), we conclude that P × Q ≠ Q × P. However, the number of elements in each set will be the same. Example 3 Let A = {1,2,3}, B = {3,4} and C = {4,5,6}. Find (i) A × (B ∩ C) (ii) (A × B) ∩ (A × C) (iii) A × (B ∪ C) (iv) (A × B) ∪ (A × C) Solution (i) By the definition of the intersection of two sets, (B ∩ C) = {4}. Therefore, A × (B ∩ C) = {(1,4), (2,4), (3,4)}. (ii) Now (A × B) = {(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)} and (A × C) = {(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)} Therefore, (A × B) ∩ (A × C) = {(1, 4), (2, 4), (3, 4)}. (iii) Since, (B ∪ C) = {3, 4, 5, 6}, we have A × (B ∪ C) = {(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)}. (iv) Using the sets A × B and A × C from part (ii) above, we obtain (A × B) ∪ (A × C) = {(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)}. Rationalised 2023-24 RELATIONS AND FUNCTIONS 27 Example 4 If P = {1, 2}, form the set P × P × P. Solution We have, P× P× P = {(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)}. Example 5 If R is the set of all real numbers, what do the cartesian products R × R and R × R × R represent? Solution The Cartesian product R × R represents the set R × R={(x, y) : x, y ∈ R} which represents the coordinates of all the points in two dimensional space and the cartesian product R × R × R represents the set R × R × R ={(x, y, z) : x, y, z ∈ R} which represents the coordinates of all the points in three-dimensional space. Example 6 If A × B ={(p, q),(p, r), (m, q), (m, r)}, find A and B. Solution A = set of first elements = {p, m} B = set of second elements = {q, r}. EXERCISE 2.1 1. If 2 5 1 1 3 3 3 3 x ,y – , ⎛ ⎞ ⎛ ⎞ + = ⎜ ⎟ ⎜ ⎟ ⎝ ⎠ ⎝ ⎠, find the values of x and y. 2. If the set A has 3 elements and the set B = {3, 4, 5}, then find the number of elements in (A×B). 3. If G = {7, 8} and H = {5, 4, 2}, find G × H and H × G. 4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly. (i) If P = {m, n} and Q = { n, m}, then P × Q = {(m, n),(n, m)}. (ii) If A and B are non-empty sets, then A × B is a non-empty set of ordered pairs (x, y) such that x ∈ A and y ∈ B. (iii) If A = {1, 2}, B = {3, 4}, then A × (B ∩ ϕ) = ϕ. 5. If A = {–1, 1}, find A × A × A. 6. If A × B = {(a, x),(a , y), (b, x), (b, y)}. Find A and B. 7. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify that (i) A × (B ∩ C) = (A × B) ∩ (A × C). (ii) A × C is a subset of B × D. 8. Let A = {1, 2} and B = {3, 4}. Write A × B. How many subsets will A × B have? List them. 9. Let A and B be two sets such that n(A) = 3 and n(B) = 2. If (x, 1), (y, 2), (z, 1) are in A × B, find A and B, where x, y and z are distinct elements. Rationalised 2023-24 28 MATHEMATICS 10. The Cartesian product A × A has 9 elements among which are found (–1, 0) and (0,1). Find the set A and the remaining elements of A × A. 2.3 Relations Consider the two sets P = {a, b, c} and Q = {Ali, Bhanu, Binoy, Chandra, Divya}. The cartesian product of P and Q has 15 ordered pairs which can be listed as P × Q = {(a, Ali), (a,Bhanu), (a, Binoy),

..., (c, Divya)}. We can now obtain a subset of P × Q by introducing a relation R between the first element x and the second element y of each ordered pair (x, y) as R= { (x,y): x is the first letter of the name y, x ∈ P, y ∈ Q}. Then R = {(a, Ali), (b, Bhanu), (b, Binoy), (c, Chandra)} A visual representation of this relation R (called an arrow diagram) is shown in Fig 2.4. Definition 2 A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B. The second element is called the image of the first element. Definition 3 The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R. Definition 4 The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the codomain of the relation R. Note that range ⊂ codomain. Remarks (i) A relation may be represented algebraically either by the Roster method or by the Set-builder method. (ii) An arrow diagram is a visual representation of a relation. Example 7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by R = {(x, y) : y = x + 1 } (i) Depict this relation using an arrow diagram. (ii) Write down the domain, codomain and range of R. Solution (i) By the definition of the relation, R = {(1,2), (2,3), (3,4), (4,5), (5,6)}. Fig 2.4 Rationalised 2023-24 RELATIONS AND FUNCTIONS 29 The corresponding arrow diagram is shown in Fig 2.5. (ii) We can see that the domain ={1, 2, 3, 4, 5,} Similarly, the range = {2, 3, 4, 5, 6} and the codomain = {1, 2, 3, 4, 5, 6}. Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range? Solution It is obvious that the relation R is "x is the square of y". (i) In set-builder form, R = {(x, y): x is the square of y, x ∈ P, y ∈ Q} (ii) In roster form, R = {(9, 3), (9, –3), (4, 2), (4, –2), (25, 5), (25, –5)} The domain of this relation is {4, 9, 25}. The range of this relation is {– 2, 2, –3, 3, –5, 5}. Note that the element 1 is not related to any element in set P. The set Q is the codomain of this relation. ANote The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B. If n(A) = p and n(B) = q, then n (A × B) = pq and the total number of relations is 2 pq . Example 9 Let A = {1, 2} and B = {3, 4}. Find the number of relations from A to B. Solution We have, A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}. Since n (A×B) = 4, the number of subsets of A×B is 24 . Therefore, the number of relations from A into B will be 24 . Remark A relation R from A to A is also stated as a relation on A. EXERCISE 2.2 1. Let A = {1, 2, 3,...,14}. Define a relation R from A to A by R = {(x, y) : 3x – y = 0, where x, y ∈ A}. Write down its domain, codomain and range. Fig 2.5 Fig 2.6 Rationalised 2023-24 30 MATHEMATICS 2. Define a relation R on the set N of natural numbers by R = {(x, y) : y = x + 5, x is a natural number less than 4; x, y ∈N}. Depict this relationship using roster form. Write down the domain and the range. 3. A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by R = {(x, y): the difference between x and y is odd; x ∈ A, y ∈ B}. Write R in roster form. 4. The Fig2.7 shows a relationship between the sets P and Q. Write this relation (i) in set-builder form (ii) roster form. What is its domain and range? 5. Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by {(a, b): a , b ∈A, b is exactly divisible by a}. (i) Write R in roster form (ii) Find the domain of R (iii) Find the range of R. 6. Determine the domain and range of the relation R defined by R = {(x, x + 5) : x ∈ {0, 1, 2, 3, 4, 5}}. 7. Write the relation R = {(x, x3) : x is a prime number less than 10} in roster form. 8. Let A = {x, y, z} and B = {1, 2}. Find the number of relations from A to B. 9. Let R be the relation on Z defined by R = {(a,b): a, b ∈ Z, a – b is an integer}. Find the domain and range of R. 2.4 Functions In this Section, we study a special type of relation called function. It is one of the most important concepts in mathematics. We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'map' or 'mapping' used to denote a function. Definition 5 A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element. If f is a function from A to B and (a, b) ∈ f, then f (a) = b, where b is called the image of a under f and a is called the preimage of b under f. Fig 2.7 Rationalised 2023-24 RELATIONS AND FUNCTIONS 31 The function f from A to B is denoted by f: A ‡ B. Looking at the previous examples, we can easily see that the relation in Example 7 is not a function because the element 6 has no image. Again, the relation in Example 8 is not a function because the elements in the domain are connected to more than one images. Similarly, the relation in Example 9 is also not a function. (Why?) In the examples given below, we will see many more relations some of which are functions and others are not. Example 10 Let N be the set of natural numbers and the relation R be defined on N such that R = {(x, y) : y = 2x, x, y ∈ N}. What is the domain, codomain and range of R? Is this relation a function? Solution The domain of R is the set of natural numbers N. The codomain is also N. The range is the set of even natural numbers. Since every natural number n has one and only one image, this relation is a function. Example 11 Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not? (i) R = {(2,1),(3,1), (4,2)}, (ii) R = {(2,2),(2,4),(3,3), (4,4)} (iii) R = {(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)} Solution (i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function. (ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function. (iii) Since every element has one and only one image, this relation is a function. Definition 6 A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, it is called a real function. Example 12 Let N be the set of natural numbers. Define a real valued function f : N‡ N by f (x) = 2x + 1. Using this definition, complete the table given below. x 1 2 3 4 5 6 7 y f (1) = ... f (2) = ... f (3) = ... f (4) = ... f (5) = ... f (6) = ... f (7) = ... Solution The completed table is given by x 1 2 3 4 5 6 7 y f (1) = 3 f (2) = 5 f (3) = 7 f (4) = 9 f (5) = 11 f (6) = 13 f (7) =15 Rationalised 2023-24 32 MATHEMATICS 2.4.1 Some functions and their graphs (i) Identity function Let R be the set of real numbers. Define the real valued function f : R → R by y = f(x) = x for each x ∈ R. Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line as shown in Fig 2.8. It passes through the origin. Fig 2.9 Fig 2.8 (ii) Constant function Define the function f: R → R by y = f (x) = c, x ∈ R where c is a constant and each x ∈ R. Here domain of f is R and its range is {c}. Rationalised 2023-24 RELATIONS AND FUNCTIONS 33 The graph is a line parallel to x-axis. For example, if f(x)=3 for each x∈R, then its graph will be a line as shown in the Fig 2.9. (iii) Polynomial function A function f : R → R is said to be polynomial function if for each x in R, y = f (x) = a0 + a1 x + a2 x 2 + ...+ an x n , where n is a non-negative integer and a0 , a1 , a2 ,...,an∈R. The functions defined by f(x) = x3 – x2 + 2, and g(x) = x4 + 2 x are some examples of polynomial functions, whereas the function h defined by h(x) = 2 3 x + 2x is not a polynomial function.(Why?) Example 13 Define the function f: R → R by y = f(x) = x2 , x ∈ R. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f. x – 4 –3 –2 –1 0 1 2 3 4 y = f(x) = x2 Solution The completed Table is given below: x – 4 –3 –2 –1 0 1 2 3 4 y = f (x) = x2 16 9 4 1 0 1 4 9 16 Domain of f = {x : x∈R}. Range of f = {x 2 : x ∈ R}. The graph of f is given by Fig 2.10 Fig 2.10 Rationalised 2023-24 34 MATHEMATICS Example 14 Draw the graph of the function f :R → R defined by f (x) = x 3 , x∈R. Solution We have f(0) = 0, f(1) = 1, f(–1) = –1, f(2) = 8, f(–2) = –8, f(3) = 27; f(–3) = –27, etc. Therefore, f = {(x,x 3): x∈R}. The graph of f is given in Fig 2.11. Fig 2.11 (iv) Rational functions are functions of the type ( ) ( ) f x g x , where f(x) and g(x) are polynomial functions of x defined in a domain, where g(x) ≠ 0. Example 15 Define the real valued function f : R – {0} → R defined by 1 f x( ) = x , x ∈ R –{0}. Complete the Table given below using this definition. What is the domain and range of this function? x –2 –1.5 –1 –0.5 0.25 0.5 1 1.5 2 y = 1 x ... ... Solution The completed Table is given by x –2 –1.5 –1 –0.5 0.25 0.5 1 1.5 2 y = 1 x – 0.5 – 0.67 –1 – 2 4 2 1 0.67 0.5 Rationalised 2023-24 RELATIONS AND FUNCTIONS 35 The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig 2.12. Fig 2.13 (v) The Modulus function The function f: R→R defined by f(x) = |x| for each x ∈R is called modulus function. For each non-negative value of x, f(x) is equal to x. But for negative values of x, the value of f(x) is the negative of the value of x, i.e., 0 ( ) 0 x,x f x x,x

 ≥ = – < The graph of the modulus function is given in Fig 2.13. (vi) Signum function The function f:R→R defined by 1 if 0 ( ) 0 if 0 1 if 0 , x f x , x , x > = =   – < Fig 2.12 Rationalised 2023-24 36 MATHEMATICS is called the signum function. The domain of the signum function is R and the range is the set {–1, 0, 1}. The graph of the signum function is given by the Fig 2.14. Fig 2.14 (vii) Greatest integer function The function f: R → R defined by f(x) = [x], x ∈R assumes the value of the greatest integer, less than or equal to x. Such a function is called the greatest integer function. From the definition of [x], we can see that [x] = –1 for –1 ≤ x < 0 [x] = 0 for 0 ≤ x < 1 [x] = 1 for 1 ≤ x < 2 [x] = 2 for 2 ≤ x < 3 and so on. The graph of the function is shown in Fig 2.15. 2.4.2 Algebra of real functions In this Section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another. (i) Addition of two real functions Let f : X → R and g : X → R be any two real functions, where X ⊂ R. Then, we define (f + g): X → R by (f + g) (x) = f (x) + g (x), for all x ∈ X. Fig 2.15 Rationalised 2023-24 RELATIONS AND FUNCTIONS 37 (ii) Subtraction of a real function from another Let f : X → R and g: X → R be any two real functions, where X ⊂ R. Then, we define (f – g) : X→R by (f–g) (x) = f(x) –g(x), for all x ∈ X. (iii) Multiplication by a scalar Let f : X→R be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product α f is a function from X to R defined by (α f) (x) = α f (x), x ∈X. (iv) Multiplication of two real functions The product (or multiplication) of two real functions f:X→R and g:X→R is a function fg:X→R defined by (fg) (x) = f(x) g(x), for all x ∈ X. This is also called pointwise multiplication. (v) Quotient of two real functions Let f and g be two real functions defined from X→R, where X ⊂ R. The quotient of f by g denoted by f g is a function defined by , ( ) ( ) ( ) f f x x g g x   =   , provided g(x) ≠ 0, x ∈ X Example 16 Let f(x) = x 2 and g(x) = 2x + 1 be two real functions.Find (f + g) (x), (f –g) (x), (fg) (x), ( ) f x g   . Solution We have, (f + g) (x) = x 2 + 2x + 1, (f –g) (x) = x 2 – 2x – 1, (fg) (x) = x 2 (2x + 1) = 2x 3 + x 2 , ( ) f x g   = 2 2 1 x x + , x ≠ 1 2 − Example 17 Let f(x) = x and g(x) = x be two functions defined over the set of nonnegative real numbers. Find (f + g) (x), (f – g) (x), (fg) (x) and f g   (x). Solution We have (f + g) (x) = x + x, (f – g) (x) = x – x , (fg) x = 3 2 x(x) x = and ( ) f x g   1 2 0 x – x , x x = = ≠ Rationalised 2023-24 38 MATHEMATICS EXERCISE 2.3 1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range. (i) {(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)} (ii) {(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)} (iii) {(1,3), (1,5), (2,5)}. 2. Find the domain and range of the following real functions: (i) f(x) = – x (ii) f(x) = 2 9 − x . 3. A function f is defined by f(x) = 2x –5. Write down the values of (i) f (0), (ii) f (7), (iii) f (–3). 4. The function 't' which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by t(C) = 9C 5 + 32. Find (i) t(0) (ii) t(28) (iii) t(–10) (iv) The value of C, when t(C) = 212. 5. Find the range of each of the following functions. (i) f (x) = 2 – 3x, x ∈ R, x > 0. (ii) f (x) = x 2 + 2, x is a real number. (iii) f (x) = x, x is a real number. Miscellaneous Examples Example 18 Let R be the set of real numbers. Define the real function f: R→R by f(x) = x + 10 and sketch the graph of this function. Solution Here f(0) = 10, f(1) = 11, f(2) = 12, ..., f(10) = 20, etc., and f(–1) = 9, f(–2) = 8, ..., f(–10) = 0 and so on. Therefore, shape of the graph of the given function assumes the form as shown in Fig 2.16. Remark The function f defined by f(x) = mx + c , x ∈ R, is called linear function, where m and c are constants. Above function is an example of a linear function. Fig 2.16 Rationalised 2023-24 RELATIONS AND FUNCTIONS 39 Example 19 Let R be a relation from Q to Q defined by R = {(a,b): a,b ∈ Q and a – b ∈ Z}. Show that (i) (a,a) ∈ R for all a ∈ Q (ii) (a,b) ∈ R implies that (b, a) ∈ R (iii) (a,b) ∈ R and (b,c) ∈ R implies that (a,c) ∈R Solution (i) Since, a – a = 0 ∈ Z, if follows that (a, a) ∈ R. (ii) (a,b) ∈ R implies that a – b ∈ Z. So, b – a ∈ Z. Therefore, (b, a) ∈ R (iii) (a, b) and (b, c) ∈ R implies that a – b ∈ Z. b – c ∈ Z. So, a – c = (a – b) + (b – c) ∈ Z. Therefore, (a,c) ∈ R Example 20 Let f = {(1,1), (2,3), (0, –1), (–1, –3)} be a linear function from Z into Z. Find f(x). Solution Since f is a linear function, f (x) = mx + c. Also, since (1, 1), (0, – 1) ∈ R, f (1) = m + c = 1 and f (0) = c = –1. This gives m = 2 and f(x) = 2x – 1. Example 21 Find the domain of the function 2 2 3 5 ( ) 5 4 x x f x x x + + = − + Solution Since x 2 –5x +

4 = (x – 4) (x –1), the function f is defined for all real numbers except at x = 4 and x = 1. Hence the domain of f is R – {1, 4}. Example 22 The function f is defined by f (x) = 1 0 1 0 1 0 x, x , x x , x ⎧ − < ⎪ = ⎨ ⎪ + > Draw the graph of f (x). Solution Here, f(x) = 1 – x, x < 0, this gives f(– 4) = 1 – (– 4)= 5; f(– 3) =1 – (– 3) = 4, f(– 2) = 1 – (– 2)= 3 f(–1) = 1 – (–1) = 2; etc, and f(1) = 2, f (2) = 3, f (3) = 4 f(4) = 5 and so on for f(x) = x + 1, x > 0. Thus, the graph of f is as shown in Fig 2.17 Fig 2.17 Rationalised 2023-24 40 MATHEMATICS Miscellaneous Exercise on Chapter 2 1. The relation f is defined by 2 0 3 ( ) = 3 3 10 x , x f x x, x ⎧ ≤ ≤ ⎨ ≤ ≤ The relation g is defined by 2 , 0 2 ( ) 3 , 2 10 x x g x x x ⎧ ≤ ≤ =⎨ ≤ ≤ Show that f is a function and g is not a function. 2. If f (x) = x 2 , find (1 1) (1) (1 1 1) f . – f . – . 3. Find the domain of the function f (x) 2 2 2 1 8 12 x x x – x + + = + . 4. Find the domain and the range of the real function f defined by f (x) = (1) x − . 5. Find the domain and the range of the real function f defined by f (x) = x –1 . 6. Let 2 2 , : 1 x f x x x ⎛ ⎞ ⎛ ⎞ = ⎜ ⎟ ⎜ ∈ ⎟ ⎝ + ⎠ R be a function from R into R. Determine the range of f. 7. Let f, g : R→R be defined, respectively by f(x) = x + 1, g(x) = 2x – 3. Find f + g, f – g and f g . 8. Let f = {(1,1), (2,3), (0,–1), (–1, –3)} be a function from Z to Z defined by f(x) = ax + b, for some integers a, b. Determine a, b. 9. Let R be a relation from N to N defined by R = {(a, b) : a, b ∈N and a = b2 }. Are the following true? (i) (a,a) ∈ R, for all a ∈ N (ii) (a,b) ∈ R, implies (b,a) ∈ R (iii) (a,b) ∈ R, (b,c) ∈ R implies (a,c) ∈ R. Justify your answer in each case. 10. Let A ={1,2,3,4}, B = {1,5,9,11,15,16} and f = {(1,5), (2,9), (3,1), (4,5), (2,11)} Are the following true? (i) f is a relation from A to B (ii) f is a function from A to B. Justify your answer in each case. Rationalised 2023-24 RELATIONS AND FUNCTIONS 41 11. Let f be the subset of Z × Z defined by f = {(ab, a + b) : a, b ∈ Z}. Is f a function from Z to Z? Justify your answer. 12. Let A = {9,10,11,12,13} and let f : A→N be defined by f (n) = the highest prime factor of n. Find the range of f. Summary In this Chapter, we studied about relations and functions.The main features of this Chapter are as follows: Æ Ordered pair A pair of elements grouped together in a particular order. Æ Cartesian product A × B of two sets A and B is given by A × B = {(a, b): a ∈ A, b ∈ B} In particular R × R = {(x, y): x, y ∈ R} and R × R × R = {(x, y, z): x, y, z ∈ R} Æ If (a, b) = (x, y), then a = x and b = y. Æ If n(A) = p and n(B) = q, then n(A × B) = pq. Æ A × φ = φ Æ In general, A × B ≠ B × A. Æ Relation A relation R from a set A to a set B is a subset of the cartesian product A × B obtained by describing a relationship between the first element x and the second element y of the ordered pairs in A × B. Æ The image of an element x under a relation R is given by y, where (x, y) ∈ R, Æ The domain of R is the set of all first elements of the ordered pairs in a relation R. Æ The range of the relation R is the set of all second elements of the ordered pairs in a relation R. Æ Function A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B. We write f: A→B, where f(x) = y. Æ A is the domain and B is the codomain of f. Rationalised 2023-24 42 MATHEMATICS Æ The range of the function is the set of images. Æ A real function has the set of real numbers or one of its subsets both as its domain and as its range. Æ Algebra of functions For functions f : X → R and g : X → R, we have (f + g) (x) = f (x) + g(x), x ∈ X (f – g) (x) = f (x) – g(x), x ∈ X (f.g) (x) = f (x) .g (x), x ∈ X (kf) (x) = k (f (x)), x ∈ X, where k is a real number. ( ) f x g ⎛ ⎞ ⎜ ⎟ ⎝ ⎠ = ( ) ( ) f x g x , x ∈ X, g(x) ≠ 0 Historical Note The word FUNCTION first appears in a Latin manuscript "Methodus tangentium inversa, seu de fuctionibus" written by Gottfried Wilhelm Leibnitz (1646-1716) in 1673; Leibnitz used the word in the non-analytical sense. He considered a function in terms of "mathematical job" – the "employee" being just a curve. On July 5, 1698, Johan Bernoulli, in a letter to Leibnitz, for the first time deliberately assigned a specialised use of the term function in the analytical sense. At the end of that month, Leibnitz replied showing his approval. Function is found in English in 1779 in Chambers' Cyclopaedia: "The term function is used in algebra, for an analytical expression any way compounded of a variable quantity, and of numbers, or constant quantities". — v — Rationalised 2023-24vA mathematician knows how to solve a problem, he can not solve it. – MILNE v 3.1 Introduction The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving

triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas. In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this Chapter, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties. 3.2 Angles Angle is a measure of rotation of a given ray about its initial point. The original ray is Chapter 3 TRIGONOMETRIC FUNCTIONS Arya Bhatt (476-550) Fig 3.1 Vertex Rationalised 2023-24 44 MATHEMATICS called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative (Fig 3.1). The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. one complete revolution from the position of the initial side as indicated in Fig 3.2. This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure. 3.2.1 Degree measure If a rotation from the initial side to terminal side is th 1 360 ������ of a revolution, the angle is said to have a measure of one degree, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds . One sixtieth of a degree is called a minute, written as 1′, and one sixtieth of a minute is called a second, written as 1″. Thus, 1° = 60′, 1′ = 60″ Some of the angles whose measures are 360°,180°, 270°, 420°, – 30°, – 420° are shown in Fig 3.3. Fig 3.2 Fig 3.3 Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 45 3.2.2 Radian measure There is another unit for measurement of an angle, called the radian measure. Angle subtended at the centre by an arc of length 1 unit in a unit circle (circle of radius 1 unit) is said to have a measure of 1 radian. In the Fig 3.4(i) to (iv), OA is the initial side and OB is the terminal side. The figures show the angles whose measures are 1 radian, –1 radian, 1 1 2 radian and –1 1 2 radian. (i) (ii) (iii) Fig 3.4 (i) to (iv) (iv) We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian. More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length l will subtend an angle whose measure is l r radian. Thus, if in a circle of radius r, an arc of length l subtends an angle θ radian at the centre, we have θ = l r or l = r θ. Rationalised 2023-24 46 MATHEMATICS 3.2.3 Relation between radian and real numbers Consider the unit circle with centre O. Let A be any point on the circle. Consider OA as initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle. Consider the line PAQ which is tangent to the circle at A. Let the point A represent the real number zero, AP represents positive real number and AQ represents negative real numbers (Fig 3.5). If we rope the line AP in the anticlockwise direction along the circle, and AQ in the clockwise direction, then every real number will correspond to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same. 3.2.4 Relation between degree and radian Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that 2π radian = 360° or π radian = 180° The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as 22 7 , we have 1 radian = 180 π ° = 57° 16′ approximately. Also 1° = π 180 radian = 0.01746 radian approximately. The relation between degree measures and radian measure of some common angles are given in the following table: A O 1 P 1 2 −1 −2 Q 0 Fig 3.5 Degree 30° 45° 60° 90° 180° 270° 360° Radian π 6 π 4 π 3 π 2 π 3π 2 2π Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 47 Notational Convention Since angles are measured either in degrees or in radians, we adopt the convention that whenever we write angle θ°, we mean the angle whose degree measure is θ and whenever we write angle β, we mean the angle whose radian measure is β. Note that when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, π π 180 and 45 4 = ° = ° are written with the understanding that π and π 4 are radian measures. Thus, we can say that Radian measure = π 180 × Degree measure Degree measure = 180 π ×Radian measure Example 1 Convert 40° 20′ into radian measure. Solution We know that 180° = π radian. Hence 40° 20′ = 40 1 3 degree = π 180 × 121 3 radian = 121π 540 radian. Therefore 40° 20′ = 121π 540 radian. Example 2 Convert 6 radians into degree measure. Solution We know that π radian = 180°. Hence 6 radians = 180 π × 6 degree = 1080 7 22 × degree = 343 7 11degree = 343° + 7 60 11 × minute [as 1° = 60′] = 343° + 38′ + 2 11 minute [as 1′ = 60″] = 343° + 38′ + 10.9″ = 343°38′ 11″ approximately. Hence 6 radians = 343° 38′ 11″ approximately. Example 3 Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use 22 π 7 =). Rationalised 2023-24 48 MATHEMATICS Solution Here l = 37.4 cm and θ = 60° = 60π π radian = 180 3 Hence, by r = θ l , we have r = 37.4×3 37.4×3×7 = π 22 = 35.7 cm Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use π = 3.14). Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through 2 3 of a revolution. Therefore, 2 θ = × 360° 3 or 4π 3 radian. Hence, the required distance travelled is given by l = r θ = 1.5 × 4π 3 cm = 2π cm = 2 × 3.14 cm = 6.28 cm. Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii. Solution Let r 1 and r 2 be the radii of the two circles. Given that θ1 = 65° = π 65 180 × = 13π 36 radian and θ2 = 110° = π 110 180 × = 22π 36 radian Let l be the length of each of the arc. Then l = r 1 θ1 = r2 θ2 , which gives 13π 36 × r 1 = 22π 36 × r 2 , i.e., 1 2 r r = 22 13 Hence r 1 : r 2 = 22 : 13. EXERCISE 3.1 1. Find the radian measures corresponding to the following degree measures: (i) 25° (ii) – 47°30′ (iii) 240° (iv) 520° Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 49 2 . Find the degree measures corresponding to the following radian measures (Use 22 π 7 =). (i) 11 16 (ii) – 4 (iii) 5π 3 (iv) 7π 6 3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second? 4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use 22 π 7 =). 5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord. 6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii. 7. Find the angle in radian through which a pendulum swings if its length is 75 cm and th e tip describes an arc of length (i) 10 cm (ii) 15 cm (iii) 21 cm 3.3 Trigonometric Functions In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions. Consider a unit circle with centre at origin of the coordinate axes. Let P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x (Fig 3.6). We define cos x = a and sin x = b Since ΔOMP is a right triangle, we have OM2 + MP2 = OP2 or a 2 + b 2 = 1 Thus, for every point on the unit circle, we have a 2 + b 2 = 1 or cos2 x + sin2 x = 1 Since one complete revolution subtends an angle of 2π radian at the centre of the circle, ∠AOB = π 2 , Fig 3.6 Rationalised 2023-24 50 MATHEMATICS ∠AOC = π and ∠AOD = 3π 2 . All angles which are integral multiples of π 2 are called quadrantal angles. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have cos 0° = 1 sin 0° = 0, cos π 2 = 0 sin π 2 = 1 cosπ = − 1 sinπ = 0 cos 3π 2 = 0 sin 3π 2 = –1 cos 2π = 1 sin 2π = 0 Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus, sin (2nπ + x) = sin x, n ∈ Z , cos (2nπ + x) = cos x , n ∈ Z Further, sin x = 0, if x = 0, ± π, ± 2π , ± 3π, ..., i.e., when x is an integral multiple of π and cos x = 0, if x = ± π 2 , ± 3π 2 , ± 5π 2 , ... i.e., cos x vanishes when x is an odd multiple of π 2 . Thus sin x = 0 implies x = nπ, where n is any integer cos x = 0 implies x = (2n + 1) π 2 , where n is any integer. We now define other trigonometric functions in terms of sine and cosine functions: cosec x = 1 sin x , x ≠ nπ, where n is any integer. sec x = 1 cos x , x ≠ (2n + 1) π 2 , where n is any integer. tan x = sin cos x x , x ≠ (2n +1) π 2 , where n is any integer. cot x = cos sin x x , x ≠ n π, where n is any integer. Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 51 not defined not defined We have shown that for all real x, sin2 x + cos2 x = 1 It follows that 1 + tan2 x = sec2 x (why?) 1 + cot2 x = cosec2 x (why?) In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table: 0° π 6 π 4 π 3 π 2 π 3π 2 2π sin 0 1 2 1 2 3 2 1 0 – 1 0 cos 1 3 2 1 2 1 2 0 – 1 0 1 tan 0 1 3 1 3 0 0 The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively. 3.3.1 Sign of trigonometric functions Let P (a, b) be a point on the unit circle with centre at the origin such that ∠AOP = x. If ∠AOQ = – x, then the coordinates of the point Q will be (a, –b) (Fig 3.7). Therefore cos (– x) = cos x and sin (– x) = – sin x Since for every point P (a, b) on the unit circle, – 1 ≤ a ≤ 1 and Fig 3.7 Rationalised 2023-24 52 MATHEMATICS – 1 ≤ b ≤ 1, we have – 1 ≤ cos x ≤ 1 and –1 ≤ sin x ≤ 1 for all x. We have learnt in previous classes that in the first quadrant (0 < x < π 2) a and b are both positive, in the second quadrant (π 2 < x < x < 3π 2) a and b are both negative and in the fourth quadrant (3π 2 < x < 2π) a is positive and b is negative. Therefore, sin x is positive for 0 < x < π, and negative for π < x < 2π. Similarly, cos x is positive for 0 < x < π 2 , negative for π 2 < x < 3π 2 and also positive for 3π 2 < x < 2π. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table. I II III IV sin x + + – – cos x + – – + tan x + – + – cosec x + + – – sec x + – – + cot x + – + – 3.3.2 Domain and range of trigonometric functions From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x, – 1 ≤ sin x ≤ 1 and – 1 ≤ cos x ≤ 1 Thus, domain of y = sin x and y = cos x is the set of all real numbers and range is the interval [–1, 1], i.e., – 1 ≤ y ≤ 1. Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 53 Since cosec x = 1 sin x , the domain of y = cosec x is the set { x : x ∈ R and x ≠ n π, n ∈ Z} and range is the set {y : y ∈ R, y ≥ 1 or y ≤ – 1}. Similarly, the domain of y = sec x is the set {x : x ∈ R and x ≠ (2n + 1) π 2 , n ∈ Z} and range is the set {y : y ∈ R, y ≤ – 1or y ≥ 1}. The domain of y = tan x is the set {x : x ∈ R and x ≠ (2n + 1) π 2 , n ∈ Z} and range is the set of all real numbers. The domain of y = cot x is the set {x : x ∈ R and x ≠ n π, n ∈ Z} and the range is the set of all real numbers. We further observe that in the first quadrant, as x increases from 0 to π 2 , sin x increases from 0 to 1, as x increases from π 2 to π, sin x decreases from 1 to 0. In the third quadrant, as x increases from π to 3π 2 , sin x decreases from 0 to –1and finally, in the fourth quadrant, sin x increases from –1 to 0 as x increases from 3π 2 to 2π. Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table: Remark In the above table, the statement tan x increases from 0 to ∞ (infinity) for 0 < x < π 2 simply means that tan x increases as x increases for 0 < x < π 2 and I quadrant II quadrant III quadrant IV quadrant sin increases from 0 to 1 decreases from 1 to 0 decreases from 0 to –1 increases from –1 to 0 cos decreases from 1 to 0 decreases from 0 to – 1 increases from –1 to 0 increases from 0 to 1 tan increases from 0 to ∞ increases from –∞to 0 increases from 0 to ∞ increases from –∞to 0 cot decreases from ∞ to 0 decreases from 0 to–∞ decreases from ∞ to 0 decreases from 0to –∞ sec increases from 1 to ∞ increases from –∞to–1 decreases from –1to–∞ decreases from ∞ to 1 cosec decreases from ∞ to 1 increases from 1 to ∞ increases from –∞to–1 decreases from–1to–∞ Rationalised 2023-24 54 MATHEMATICS Fig 3.10 Fig 3.11 Fig 3.8 Fig 3.9 assumes arbitraily large positive values as x approaches to π 2 . Similarly, to say that cosec x decreases from –1 to – ∞ (minus infinity) in the fourth quadrant means that cosec x decreases for x ∈ (3π 2 , 2π) and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and – ∞ simply specify certain types of behaviour of functions and variables. We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 55 shall see in the next section that tan (π + x) = tan x. Hence, values of tan x will repeat after an interval of π. Since cot x is reciprocal of tan x, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometic functions, we can sketch the graph of these functions. The graph of these functions are given above: Example 6 If cos x = – 3 5 , x lies in the third quadrant, find the values of other five trigonometric functions. Solution Since cos x = 3 5 − , we have sec x = 5 3 − Now sin2 x + cos2 x = 1, i.e., sin2 x = 1 – cos2 x or sin2 x = 1 – 9 25 = 16 25 Hence sin x = ± 4 5 Since x lies in third quadrant, sin x is negative. Therefore sin x = – 4 5 which also gives cosec x = – 5 4 Fig 3.12 Fig 3.13 Rationalised 2023-24 56 MATHEMATICS Further, we have tan x = sin cos x x = 4 3 and cot x = cos sin x x = 3 4 . Example 7 If cot x = – 5 12 , x lies in second quadrant, find the values of other five trigonometric functions. Solution Since cot x = – 5 12 , we have tan x = – 12 5 Now sec2 x = 1 + tan2 x = 1 + 144 25 = 169 25 Hence sec x = ± 13 5 Since x lies in second quadrant, sec x will be negative. Therefore sec x = – 13 5 , which also gives 5 cos 13 x = − Further, we have sin x = tan x cos x = (– 12 5) ×(– 5 13) = 12 13 and cosec x = 1 sin x = 13 12 . Example 8 Find the value of sin 31π 3 . Solution We know that values of sin x repeats after an interval of 2π. Therefore sin 31π 3 = sin (10π + π 3) = sin π 3 = 3 2 . Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 57 Example 9 Find the value of cos (–1710°). Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, cos (–1710°) = cos (–1710° + 5 × 360°) = cos (–1710° + 1800°) = cos 90° = 0. EXERCISE 3.2 Find the values of other five trigonometric functions in Exercises 1 to 5. 1. cos x = – 1 2 , x lies in third quadrant. 2. sin x = 3 5 , x lies in second quadrant. 3. cot x = 4 3 , x lies in third quadrant. 4. sec x = 13 5 , x lies in fourth quadrant. 5. tan x = – 5 12 , x lies in second quadrant. Find the values of the trigonometric functions in Exercises 6 to 10. 6. sin 765° 7. cosec (– 1410°) 8. tan 19π 3 9. sin (– 11π 3) 10. cot (– 15π 4) 3.4 Trigonometric Functions of Sum and Difference of Two Angles In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called trigonometric identities. We have seen that 1. sin (– x) = – sin x 2. cos (– x) = cos x We shall now prove some more results: Rationalised 2023-24 58 MATHEMATICS 3. cos (x + y) = cos x cos y – sin x sin y Consider the unit circle with centre at the origin. Let x be the angle P4OP1 and y be the angle P1OP2 . Then (x + y) is the angle P4OP2 . Also let (– y) be the angle P4OP3 . Therefore, P1 , P2 , P3 and P4 will have the coordinates P1 (cos x, sin x), P2 [cos (x + y), sin (x + y)], P3 [cos (– y), sin (– y)] and P4 (1, 0) (Fig 3.14). Consider the triangles P1OP3 and P2OP4 . They are congruent (Why?). Therefore, P1 P3 and P2 P4 are equal. By using distance formula, we get P1 P3 2 = [cos x – cos (– y)]2 + [sin x – sin(–y] 2 = (cos x – cos y) 2 + (sin x + sin y) 2 = cos2 x + cos2 y – 2 cos x cos y + sin2 x + sin2 y + 2sin x sin y = 2 – 2 (cos x cos y – sin x sin y) (Why?) Also, P2 P4 2 = [1 – cos (x + y)] 2 + [0 – sin (x + y)]2 = 1 – 2cos (x + y) + cos2 (x + y) + sin2 (x + y) = 2 – 2 cos (x + y) Fig 3.14 Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 59 Since P1 P3 = P2 P4 , we have P1 P3 2 = P2 P4 2 . Therefore, 2 –2 (cos x cos y – sin x sin y) = 2 – 2 cos (x + y). Hence cos (x + y) = cos x cos y – sin x sin y 4 . cos (x – y) = cos x cos y + sin x sin y Replacing y by – y in identity 3, we get cos (x + (– y)) = cos x cos (– y) – sin x sin (– y) or cos (x – y) = cos x cos y + sin x sin y 5. cos (x π – 2) = sin x If we replace x by π 2 and y by x in Identity (4), we get cos (π 2 − x) = cos π 2 cos x + sin π 2 sin x = sin x. 6. sin (x π – 2) = cos x Using the Identity 5, we have sin (π 2 − x) = cos π π 2 2 x   − −   = cos x. 7. sin (x + y) = sin x cos y + cos x sin y We know that sin (x + y) = cos π ( ) 2 x y   − +   = cos π ( ) 2 x y   − −   = cos (π 2 − x) cos y + sin π ( ) 2 − x sin y = sin x cos y + cos x sin y 8. sin (x – y) = sin x cos y – cos x sin y If we replace y by –y, in the Identity 7, we get the result. 9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results: cos x π (+) 2 = – sin x sin x π (+) 2 = cos x cos (π – x) = – cos x sin (π – x) =

sin x Rationalised 2023-24 60 MATHEMATICS cos (π + x) = – cos x sin (π + x) = – sin x cos (2π – x) = cos x sin (2π – x) = – sin x Similar results for tan x, cot x, sec x and cosec x can be obtained from the results of sin x and cos x. 10. If none of the angles x, y and (x + y) is an odd multiple of π 2 , then tan (x + y) = x y x y tan + tan 1 – tan tan Since none of the x, y and (x + y) is an odd multiple of π 2 , it follows that cos x, cos y and cos (x + y) are non-zero. Now tan (x + y) = sin( ) cos( ) x y x y + + = sin cos cos sin cos cos sin sin x y x y x y x y + − . Dividing numerator and denominator by cos x cos y, we have tan (x + y) = yx yx yx yx yx yx yx yx coscos sinsin coscos coscos coscos sincos coscos cossin − + = tan tan 1 – tan tan x y x y + 11. tan (x – y) = x y x y tan – tan 1 + tan tan If we replace y by – y in Identity 10, we get tan (x – y) = tan [x + (– y)] = tan tan ( ) 1 tan tan ( ) x y x y + − − − − = tan tan 1 tan tan x y x y − + 12. If none of the angles x, y and (x + y) is a multiple of π, then cot (x + y) = x y y x cot cot – 1 cot +cot Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 61 Since, none of the x, y and (x + y) is multiple of π, we find that sin x sin y and sin (x + y) are non-zero. Now, cot (x + y)= cos ( ) cos cos – sin sin sin ( ) sin cos cos sin x y x y x y x y x y x y + = + + Dividing numerator and denominator by sin x sin y, we have cot (x + y) = cot cot –1 cot cot x y y x + 13. cot (x – y)= x y y x cot cot + 1 cot – cot if none of angles x, y and x–y is a multiple of π If we replace y by –y in identity 12, we get the result 14. cos 2x = cos2x – sin2 x = 2 cos2 x – 1 = 1 – 2 sin 2 x = x x 2 2 1 – tan 1 + tan We know that cos (x + y) = cos x cos y – sin x sin y Replacing y by x, we get cos 2x = cos2x – sin2 x = cos2 x – (1 – cos2 x) = 2 cos2x – 1 Again, cos 2x = cos2 x – sin2 x = 1 – sin2 x – sin2 x = 1 – 2 sin2 x. We have cos 2x = cos2 x – sin 2 x = 2 2 2 2 cos sin cos sin x x x x − + Dividing numerator and denominator by cos2 x, we get cos 2x = 2 2 1 – tan 1+ tan x x , π π 2 x n ≠ + , where n is an integer 15. sin 2x = 2 sinx cos x = x x 2 2tan 1 + tan π π 2 x n ≠ + , where n is an integer We have sin (x + y) = sin x cos y + cos x sin y Replacing y by x, we get sin 2x = 2 sin x cos x. Again sin 2x = 2 2 2sin cos cos sin x x x x + Rationalised 2023-24 62 MATHEMATICS Dividing each term by cos2 x, we get sin 2x = 2 2tan 1 tan x + x 16. tan 2x = x x 2 2tan 1 – tan if π 2 π 2 x n ≠ + , where n is an integer We know that tan (x + y) = tan tan 1 tan tan x y – x y + Replacing y by x , we get 2 2 tan tan 2 1 tan x x x = − 17. sin 3x = 3 sin x – 4 sin3 x We have, sin 3x = sin (2x + x) = sin 2x cos x + cos 2x sin x = 2 sin x cos x cos x + (1 – 2sin2 x) sin x = 2 sin x (1 – sin2 x) + sin x – 2 sin3 x = 2 sin x – 2 sin3 x + sin x – 2 sin3 x = 3 sin x – 4 sin3 x 18. cos 3x= 4 cos3 x – 3 cos x We have, cos 3x = cos (2x +x) = cos 2x cos x – sin 2x sin x = (2cos2 x – 1) cos x – 2sin x cos x sin x = (2cos2 x – 1) cos x – 2cos x (1 – cos2 x) = 2cos3 x – cos x – 2cos x + 2 cos3 x = 4cos3 x – 3cos x. 19. = x x x x 3 2 3 tan – tan tan3 1– 3tan if π 3 π 2 x n ≠ + , where n is an integer We have tan 3x =tan (2x + x) = tan 2 tan 1 tan 2 tan x x – x x + 2 2 2tan tan 1 tan 2tan tan 1 1 tan x x – x x . x – – x + = Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 63 3 3 2 2 2 2tan tan tan 3 tan tan 1 tan 2tan 1 3tan x x – x x – x – x – x – x + = = 20. (i) cos x + cos y = x y x y + – 2cos cos 2 2 (ii) cos x – cos y = – x y x y + – 2sin sin 2 2 (iii) sin x + sin y = x y x y + – 2sin cos 2 2 (iv) sin x – sin y = x y x y + – 2cos sin 2 2 We know that cos (x + y) = cos x cos y – sin x sin y ... (1) and cos (x – y) = cos x cos y + sin x sin y ... (2) Adding and subtracting (1) and (2), we get cos (x + y) + cos(x – y) = 2 cos x cos y ... (3) and cos (x + y) – cos (x – y) = – 2 sin x sin y ... (4) Further sin (x + y) = sin x cos y + cos x sin y ... (5) and sin (x – y) = sin x cos y – cos x sin y ... (6) Adding and subtracting (5) and (6), we get sin (x + y) + sin (x – y) = 2 sin x cos y ... (7) sin (x + y) – sin (x – y) = 2cos x sin y ... (8) Let x + y = θ and x – y = ϕ. Therefore θ θ and 2 2 x y   +ϕ −ϕ =   =   Substituting the values of x and y in (3), (4), (7) and (8), we get cos θ + cos ϕ = 2 cos θ θ cos 2 2   +ϕ −ϕ   cos θ – cos ϕ = – 2 sin θ θ sin 2   + ϕ ϕ –   2 sin θ + sin ϕ = 2 sin θ θ cos 2 2   +ϕ −ϕ   Rationalised 2023-24 64 MATHEMATICS sin θ – sin ϕ = 2 cos θ θ sin 2 2   +ϕ −ϕ   Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y. Thus, we get cos x + cos y = 2 cos cos 2 2 x y x y + − ; cos x – cos y = – 2 sin sin 2 2 x y x y + − , sin x + sin y = 2 sin cos 2 2 x y x y + − ; sin x – sin y = 2 cos sin 2 2 x y x y + − . Remark As a part of identities given in 20, we can prove the following results: 21. (i) 2 cos x cos y = cos (x + y) + cos (x – y) (ii) –2 sin x sin y = cos (x + y) – cos (x – y) (iii) 2 sin x cos y = sin (x + y) + sin (x – y) (iv) 2 cos x sin y = sin (x + y) – sin (x – y). Example 10 Prove that 5 3sin sec 4sin cot 1 6 3 6 4 π π π π − = Solution We have L.H.S. = 5 3sin sec 4sin cot 6 3 6 4 π π π π − = 3 × 1 2 × 2 – 4 sin 6 □ □ π π− □ □ □ □ × 1 = 3 – 4 sin 6 π = 3 – 4 × 1 2 = 1 = R.H.S. Example 11 Find the value of sin 15°. Solution We have sin 15° = sin (45° – 30°) = sin 45° cos 30° – cos 45° sin 30° = 1 3 1 1 3 1 2 2 2 2 2 2 2 – × − × = . Example 12 Find the value of tan 13 12 π . Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 65 Solution We have tan 13 12 π = tan 12 □ □ π □ □ π + □ □ = tan tan 12 4 6 π π π □ □ = − □ □ □ □ = tan tan 4 6 1 tan tan 4 6 π π − π π + = 1 1 3 3 1 2 3 1 3 1 1 3 − − = = − + + Example 13 Prove that sin ( ) tan tan sin ( ) tan tan x y x y x y x y + + = − − . Solution We have L.H.S. sin ( ) sin cos cos sin sin ( ) sin cos cos sin x y x y x y x y x y x y + + = = − − Dividing the numerator and denominator by cos x cos y, we get sin ( ) tan tan sin ( ) tan tan x y x y x y x y + + = − − . Example 14 Show that tan 3 x tan 2 x tan x = tan 3x – tan 2 x – tan x Solution We know that 3x = 2x + x Therefore, tan 3x = tan (2x + x) or tan 2 tan tan3 1– tan 2 tan x x x x x + = or tan 3x – tan 3x tan 2x tan x = tan 2x + tan x or tan 3x – tan 2x – tan x = tan 3x tan 2x tan x or tan 3x tan 2x tan x = tan 3x – tan 2x – tan x. Example 15 Prove that cos cos 2 cos 4 4 x x x □ □ □ □ π π □ □ □ □ + + − = □ □ □ □ Solution Using the Identity 20(i), we have Rationalised 2023-24 66 MATHEMATICS L.H.S. cos cos 4 4 x x □ □ □ □ π π = + + − □ □ □ □ □ □ □ □ ( ) 4 4 4 4 2cos cos 2 2 x x x – x □ π π π π □ □ □ + + − + − □ □ □ □ = □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ = 2 cos 4 π cos x = 2 × 1 2 cos x = 2 cos x = R.H.S. Example 16 Prove that cos 7 cos 5 cot sin 7 – sin 5 x x x x x + = Solution Using the Identities 20 (i) and 20 (iv), we get L.H.S. = 7 5 7 5 2cos cos 2 2 7 5 7 5 2cos sin 2 2 x x x x x x x x + − + − = cos sin cot x x = x = R.H.S. Example 17 Prove that sin 5 2sin 3 sin tan cos5 cos x x x x x x − + = = − Solution We have L.H.S. sin 5 2sin3 sin cos5 cos x x x x x − + = − sin 5 sin 2sin 3 cos5 cos x x x x x + − = − 2sin3 cos 2 2sin3 – 2sin3 sin 2 x x x x x − = sin 3 (cos2 1) sin 3 sin 2 x x – x x − = 2 1 cos 2 2sin sin 2 2sin cos x x x x x − = = = tan x = R.H.S. Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 67 EXERCISE 3.3 Prove that: 1. sin2 π 6 + cos2 3 π – tan2 1 – 4 2 π = 2. 2sin2 6 π + cosec2 7 3 2 cos 6 3 2 π π = 3. 2 5 2 cot cosec 3tan 6 6 6 6 π π π + + = 4. 2 3 2 2 2sin 2cos 2sec 10 4 4 3 π π π + + = 5. Find the value of: (i) sin 75° (ii) tan 15° Prove the following: 6. cos cos sin sin sin( ) 4 4 4 4 x y x y x y □ □ □ □ □ □ □ □ π π π π − − − − − = + □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ 7. 2 π tan 4 1 tan π 1 tan tan 4 x x x x □ □ □ □ + □ □ □ □ + = □ □ □ □ □ □ − □ □ − □ □ 8. cos ( ) cos ( ) 2 cot sin ( ) cos 2 x x x x x x π + − = □ □ π π − + □ □ □ □ 9. 3π 3π cos cos (2π) cot cot (2π) 1 2 2 x x x x □ □ □ □ □ □ + + − + + = □ □ □ □ □ □ □ □ □ □ □ □ 10. sin (n + 1)x sin (n + 2)x + cos (n + 1)x cos (n + 2)x = cos x 11. 3 3 cos cos 2 sin 4 4 x x x □ □ □ □ π π + − − = − □ □ □ □ □ □ □ □ 12. sin2 6x – sin2 4x = sin 2x sin 10x 13. cos2 2x – cos2 6x = sin 4x sin 8x 14. sin2 x + 2 sin 4x + sin 6x = 4 cos2 x sin 4x 15. cot 4x (sin 5x + sin 3x) = cot x (sin 5x – sin 3x) 16. cos cos sin sin sin cos 9 5 17 3 2 10 x x x x x x − − = − 17. sin sin cos cos tan 5 3 5 3 4 x x x x x x + + = 18. sin sin cos cos tan x y x y − x y + = − 2 19. sin sin cos cos tan x x x x x + + = 3 3 2 20. sin sin sin cos sin x x x x x − − = 3 2 2 2 21. cos cos cos sin sin sin cot 4 3 2 4 3 2 3 x x x x x x x + + + + = Rationalised 2023-24 68 MATHEMATICS 22. cot x cot 2x – cot 2x cot 3x – cot 3x cot x = 1 23. 2 2 4 4tan (1 tan) tan 4 1 6 tan tan x x x x x − = − + 24. cos 4x = 1 – 8sin2 x cos2 x 25. cos 6x = 32 cos6 x – 48cos4 x + 18 cos2 x – 1 Miscellaneous Examples Example 18 If sin x = 3 5 , cos y = − 12 13 , where x and y both lie in second quadrant, find the value of sin (x + y). Solution We know that sin (x + y) = sin x cos y + cos x sin y ... (1) Now cos2 x = 1 – sin2 x = 1 – 9 25 = 16 25 Therefore cos x = ± 4 5 . Since x lies in second quadrant, cos x is negative. Hence cos x = − 4 5 Now sin2 y = 1 – cos2 y = 1 – 144 169 25 169 = i.e. sin y = ± 5 13 . Since y lies in second quadrant, hence sin y is positive. Therefore, sin y = 5 13 . Substituting the values of sin x, sin y, cos x and cos y in (1), we get sin( )x y + = × − □ □ □ □ □ □+ − □ □ □ □ □ □× 3 5 12 13 4 5 5 13 = 36 20 56 65 65 65 − − = − . Example 19 Prove that 9 5 cos 2 cos cos 3 cos sin 5 sin 2 2 2 x x x x − x = x . Solution We have Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 69 L.H.S. = 1 9 2cos 2 cos 2cos cos 3 2 2 2 x x x x □ □ − □ □ □ □ = 1 9 9 cos 2 cos 2 cos 3 cos 3 2 2 2 2 2 x x x x x x x x □ □ □ □ □ □ □ □ □□ □ □ □ □ □ □ □ □ □ + + − − + − − □ □ □ □ □ □ □ □ □ □ □ □□ = 1 2 5 2 3 2 15 2 3 2 cos cos cos cos x x x x + − − □ □ □ □ □ □ □ = 1 2 5 2 15 2 cos cos x x − □ □ □ □ □ □ □ = 5 15 5 15 1 2 2 2 2 2sin sin 2 2 2 □ □ □ □ □ x x x x □ + − □ □ □ □ □□ □− □ □ □ □□ □ □ □ □ □□ □□ □ □ □ □□□ = − − □ □ □ □ □ sin sin sin sin 5 □ = 5 2 5 5 2 x x x x = R.H.S. Example 20 Find the value of tan π 8 . Solution Let π 8 x = . Then π 2 4 x = . Now tan tan tan 2 2 1 2 x x x = − or 2 π 2tan π 8 tan 4 π 1 tan 8 = − Let y = tan π 8 .

Then 1 = 2 1 2 y y − or y2 + 2y – 1 = 0 Therefore y = − ± = − ± 2 2 2 2 1 2 Rationalised 2023-24 70 MATHEMATICS Since π 8 lies in the first quadrant, y = tan π 8 is positve. Hence π tan 2 1 8 = − . Example 21 If 3 3π tan = , π < < 4 2 x x , find the value of sin x 2 , cos x 2 and tan x 2 . Solution Since 3π π 2 < < < . Therefore, sin x 2 is positive and cos x 2 is negative. Now sec2 x = 1 + tan2 x = 1 9 16 25 16 + = Therefore cos2 x = 16 25 or cos x = 4 5 – (Why?) Now 2 2 2 sin x = 1 – cos x = 1 4 5 9 5 + = . Therefore sin2 x 2 = 9 10 or sin x 2 = 3 10 (Why?) Again 2cos2 x 2 = 1+ cos x = 1 4 5 1 5 − = Therefore cos2 x 2 = 1 10 or cos x 2 = − 1 10 (Why?) Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 71 Hence tan x 2 = sin cos x x 2 2 3 10 10 1 = × − = – 3. Example 22 Prove that cos2 x + cos2 π 2 π 3 cos 3 3 2 x x + + − = . Solution We have L.H.S. = 2π 2π 1 cos 2 1 cos 2 1 cos 2 3 3 2 2 2 x x x + + + − + + + . = 1 2π 2π 3 cos 2 cos 2 cos 2 2 3 3 x x x + + + + − = 1 2π 3 cos 2 2cos 2 cos 2 3 x x + + = 1 π 3 cos 2 2cos 2 cos π 2 3 x x + + − = 1 π 3 cos 2 2cos 2 cos 2 3 x x + − = [ ] 1 3 3 cos 2 cos 2 2 2 + − = x x = R.H.S. Miscellaneous Exercise on Chapter 3 Prove that: 1. 0 13 5π cos 13 3π cos 13 9π cos 13 π cos2 =++ 2. (sin 3x + sin x) sin x + (cos 3x – cos x) cos x = 0 3. (cos x + cos y) 2 + (sin x – sin y) 2 = 4 cos2 2 x y + Rationalised 2023-24 72 MATHEMATICS 4. (cos x – cos y) 2 + (sin x – sin y) 2 = 4 sin2 2 x y − 5. sin x + sin 3x + sin 5x + sin 7x = 4 cos x cos 2x sin 4x 6 . (sin7 sin5) (sin 9 sin3) tan 6 (cos7 cos5) (cos9 cos3) x x x x x x x x x + + + = + + + 7. sin 3x + sin 2x – sin x = 4sin x cos x 2 cos 3 2 x Find sin x 2 , cos x 2 and tan x 2 in each of the following : 8. tan x = − 4 3 , x in quadrant II 9. cos x = − 1 3 , x in quadrant III 10. sin x = 4 1 , x in quadrant II Summary ÆIf in a circle of radius r, an arc of length l subtends an angle of θ radians, then l = r θ ÆRadian measure = π 180 × Degree measure ÆDegree measure = 180 π × Radian measure Æcos2 x + sin2 x = 1 Æ1 + tan2 x = sec2 x Æ1 + cot2 x = cosec2 x Æcos (2nπ + x) = cos x Æsin (2nπ + x) = sin x Æsin (– x) = – sin x Æcos (– x) = cos x Æcos (x + y) = cos x cos y – sin x sin y Æcos (x – y) = cos x cos y + sin x sin y Æcos (π 2 − x) = sin x Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 73 Æsin (π 2 − x) = cos x Æsin (x + y) = sin x cos y + cos x sin y Æsin (x – y) = sin x cos y – cos x sin y Æcos π + 2 x = – sin x sin π + 2 x = cos x cos (π – x) = – cos x sin (π – x) = sin x cos (π + x) = – cos x sin (π + x) = – sin x cos (2π – x) = cos x sin (2π – x) = – sin x ÆIf none of the angles x, y and (x ± y) is an odd multiple of π 2 , then tan (x + y) = tan tan tan tan x y x y + 1− Ætan (x – y) = tan tan tan tan x y x y − 1+ ÆIf none of the angles x, y and (x ± y) is a multiple of π, then cot (x + y) = cot cot 1 cot cot x y y x − + Æcot (x – y) = cot cot 1 cot cot x y y x − + Æcos 2x = cos2 x – sin2 x = 2cos2 x – 1 = 1 – 2 sin2 x 2 2 1 tan 1 tan − x x = + Æsin 2x = 2 sin x cos x 2 2 tan 1 tan x x = + Ætan 2x = 2 2tan 1 tan x − x Æsin 3x = 3sinx – 4sin3 x Æcos 3x = 4cos3 x – 3cosx Rationalised 2023-24 74 MATHEMATICS Ætan 3x = 3 2 3tan tan 1 3tan x x x − − Æ (i) cos x + cos y = 2cos cos 2 2 x y x y + − (ii) cos x – cos y = – 2sin sin 2 2 x y x y + − (iii) sin x + sin y = 2 sin cos 2 2 x y x y + − (iv) sin x – sin y = 2cos sin 2 2 x y x y + − Æ (i) 2cos x cos y = cos (x + y) + cos (x – y) (ii) – 2sin x sin y = cos (x + y) – cos (x – y) (iii) 2sin x cos y = sin (x + y) + sin (x – y) (iv) 2 cos x sin y = sin (x + y) – sin (x – y). Historical Note The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476), Brahmagupta (598), Bhaskara I (600) and Bhaskara II (1114) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world. In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents the main contribution of the siddhantas (Sanskrit astronomical works) to the history of mathematics. Bhaskara I (about 600) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work Yuktibhasa (period) contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II. Rationalised 2023-24 TRIGONOMETRIC FUNCTIONS 75 — v — The symbols sin–1 x, cos–1 x, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The names of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios: H S h s = = tan (sun's altitude) Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works. Rationalised 2023-2476 MATHEMATICS Chapter COMPLEX NUMBERS AND QUADRATIC EQUATIONS W. R. Hamilton (1805-1865) vMathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS v 4.1 Introduction In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation x 2 + 1 = 0 has no real solution as x 2 + 1 = 0 gives x 2 = – 1 and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation x 2 = – 1. In fact, the main objective is to solve the equation ax2 + bx + c = 0, where D = b 2 – 4ac < 0, which is not possible in the system of real numbers. 4.2 Complex Numbers Let us denote −1 by the symbol i. Then, we have 2 i = −1 . This means that i is a solution of the equation x 2 + 1 = 0. A number of the form a + ib, where a and b are real numbers, is defined to be a complex number. For example, 2 + i3, (– 1) + i 3 , 1 4 11 i ⍰ ⍰ – + ⍰ ⍰ ⍰ ⍰ are complex numbers. For the complex number z = a + ib, a is called the real part, denoted by Re z and b is called the imaginary part denoted by Im z of the complex number z. For example, if z = 2 + i5, then Re z = 2 and Im z = 5. Two complex numbers z 1 = a + ib and z 2 = c + id are equal if a = c and b = d. 4 Rationalised 2023-24 COMPLEX NUMBERS AND QUADRATIC EQUATIONS 77 Example 1 If 4x + i(3x – y) = 3 + i (– 6), where x and y are real numbers, then find the values of x and y. Solution We have 4x + i (3x – y) = 3 + i (–6) ... (1) Equating the real and the imaginary parts of (1), we get 4x = 3, 3x – y = – 6, which, on solving simultaneously, give 3 4 x = and 33 4 y = . 4.3 Algebra of Complex Numbers In this Section, we shall develop the algebra of complex numbers. 4.3.1 Addition of two complex numbers Let z 1 = a + ib and z 2 = c + id be any two complex numbers. Then, the sum z 1 + z 2 is defined as follows: z 1 + z2 = (a + c) + i (b + d), which is again a complex number. For example, (2 + i3) + (– 6 +i5) = (2 – 6) + i (3 + 5) = – 4 + i 8 The addition of complex numbers satisfy the following properties: (i) The closure law The sum of two complex numbers is a complex number, i.e., z 1 + z 2 is a complex number for all complex numbers z 1 and z 2 . (ii) The commutative law For any two complex numbers z 1 and z 2 , z 1 + z 2 = z 2 + z1 (iii) The associative law For any three complex numbers z 1 , z 2 , z 3 , (z 1 + z 2) + z 3 = z 1 + (z 2 + z 3). (iv) The existence of additive identity There exists the complex number 0 + i 0 (denoted as 0), called the additive identity or the zero complex number, such that, for every complex number z, z + 0 = z. (v) The existence of additive inverse To every complex number z = a + ib, we have the complex number – a + i(– b) (denoted as – z), called the additive inverse or negative of z. We observe that z + (–z) = 0 (the additive identity). 4.3.2 Difference of two complex numbers Given any two complex numbers z 1 and z 2 , the difference z 1 – z 2 is defined as follows: z 1 – z 2 = z 1 + (– z2). For example, (6 + 3i) – (2 – i) = (6 + 3i) + (– 2 + i) = 4 + 4i and (2 – i) – (6 + 3i) = (2 – i) + (– 6 – 3i) = – 4 – 4i Rationalised 2023-24 78 MATHEMATICS 4.3.3 Multiplication of two complex numbers Let z 1 = a + ib and z 2 = c + id be any two complex numbers. Then, the product z 1 z 2 is defined as follows: z 1 z 2 = (ac – bd) + i(ad + bc) For example, (3 + i5) (2 + i6) = (3 × 2 – 5 × 6) + i(3 × 6 + 5 × 2) = – 24 + i28 The multiplication of complex numbers possesses the following properties, which we state without proofs. (i) The closure law The product of two complex numbers is a complex number, the product z 1 z 2 is a complex number for all complex numbers z 1 and z 2 . (ii) The commutative law For any two complex numbers z 1 and z 2 , z 1 z 2 = z 2 z 1 . (iii) The associative law For any three complex numbers z 1 , z 2 , z 3 , (z 1 z 2) z 3 = z 1 (z 2 z 3). (iv) The existence of multiplicative identity There exists the complex number 1 + i 0 (denoted as 1), called the multiplicative identity such that z.1 = z, for every complex number z. (v) The existence of multiplicative inverse For every non-zero complex number z = a + ib or a + bi(a ≠ 0, b ≠ 0), we have the complex number 2 2 2 2 a –b i a b a b + + +

(denoted by 1 z or z –1), called the multiplicative inverse of z such that 1 z. 1 z = (the multiplicative identity). (vi) The distributive law For any three complex numbers z 1 , z 2 , z 3 , (a) z1 (z 2 + z 3) = z 1 z 2 + z 1 z 3 (b) (z 1 + z 2) z 3 = z 1 z 3 + z 2 z 3 4.3.4 Division of two complex numbers Given any two complex numbers z 1 and z 2 , where 2 z ≠ 0 , the quotient 1 2 z z is defined by 1 1 2 2 z 1 z z z = For example, let z 1 = 6 + 3i and z 2 = 2 – i Then 1 2 1 (6 3) 2 z i z i   = + ×   − = (6 3 + i) ( ) ( ) ( ) 2 2 2 2 2 1 2 1 2 1 2 1 i − −   +   + − + −   Rationalised 2023-24 COMPLEX NUMBERS AND QUADRATIC EQUATIONS 79 = ( ) 2 6 3 5 i i   + +   = ( ) ( ) 1 1 12 3 6 6 9 12 5 5 − + + = + i i   4.3.5 Power of i we know that ( ) 3 2 i i i i i = = − = − 1 , ( ) ( ) 2 4 2 2 i i = = − = 1 1 ( ) ( ) 2 5 2 2 i i i i i = = − = 1 , ( ) ( ) 3 6 2 3 i i = = − = − 1 1 , etc. Also, we have 1 2 2 1 1 1 , 1, 1 1 i i i i i i i i − − = × = = − = = = − − − 3 4 3 4 1 1 1 1 , 1 1 1 i i i i i i i i i i − − = = × = = = = = − In general, for any integer k, i 4k = 1, i 4k + 1 = i, i4k + 2 = –1, i 4k + 3 = – i 4.3.6 The square roots of a negative real number Note that i2 = –1 and (– i) 2 = i2 = – 1 Therefore, the square roots of – 1 are i, – i. However, by the symbol −1, we would mean i only. Now, we can see that i and –i both are the solutions of the equation x 2 + 1 = 0 or x 2 = –1. Similarly ( ) ( ) 2 2 3 3 i = i 2 = 3 (– 1) = – 3 ( ) 2 − 3i = ( ) 2 − 3 i 2 = – 3 Therefore, the square roots of –3 are 3 i and − 3i . Again, the symbol −3 is meant to represent 3 i only, i.e., −3 = 3 i . Generally, if a is a positive real number, −a = a −1 = a i , We already know that a b × = ab for all positive real number a and b. This result also holds true when either a > 0, b < 0 or a < 0, b > 0. What if a < 0, b < 0? Let us examine. Note that Rationalised 2023-24 80 MATHEMATICS ( ) ( ) 2 i = − − = − − 1 1 1 1 (by assuming a b × = ab for all real numbers) = 1 = 1, which is a contradiction to the fact that = − 2 i 1. Therefore, a b ab × ≠ if both a and b are negative real numbers. Further, if any of a and b is zero, then, clearly, a b ab × = = 0. 4.3.7 Identities We prove the following identity ( )2 2 2 z z z z z z 1 2 1 2 1 2 + = + + 2 , for all complex numbers z 1 and z 2 . Proof We have, (z 1 + z 2) 2 = (z 1 + z 2) (z 1 + z 2), = (z 1 + z 2) z 1 + (z 1 + z 2) z 2 (Distributive law) = 2 2 1 2 1 1 2 2 z z z z z z + + + (Distributive law) = 2 2 1 1 2 1 2 2 z z z z z z + + + (Commutative law of multiplication) = 2 2 1 1 2 2 z z z z + + 2 Similarly, we can prove the following identities: (i) ( )2 2 2 z z z z z z 1 2 1 1 2 2 − = − + 2 (ii) ( )3 3 2 2 3 z z z z z z z z 1 2 1 1 2 1 2 2 + = + + + 3 3 (iii) ( )3 3 2 2 3 z z z z z z z z 1 2 1 1 2 1 2 2 − = − + − 3 3 (iv) ( ) ( ) 2 2 1 2 1 2 1 2 z − z z z z − z = + In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers. Example 2 Express the following in the form of a + bi: (i) ( ) 1 5 8 i i   −   (ii) (–i i) (2) 3 1 8 i   −   Solution (i) ( ) 1 5 8 i i   −   = 5 2 8 i − = ( ) 5 1 8 − − = 5 8 = 5 0 8 + i (ii) ( ) ( ) 3 1 2 8 i i i   − −   = 1 5 2 8 8 8 × ×i × × = ( )2 1 2 256 i 1 256 i i = . Rationalised 2023-24 COMPLEX NUMBERS AND QUADRATIC EQUATIONS 81 Example 3 Express (5 – 3i) 3 in the form a + ib. Solution We have, (5 – 3i) 3 = 5 3 – 3 × 52 × (3i) + 3 × 5 (3i) 2 – (3i) 3 = 125 – 225i – 135 + 27i = – 10 – 198i. Example 4 Express (− + − − 3 2 2 3)(i)in the form of a + ib Solution We have, (− + − − 3 2 2 3) (i) = (− + − 3 2 2 3 i i) ( ) = 2 − + + − 6 3 2 6 2 i i i = (− + + + 6 2 3 1 2 2) ( )i 4.4 The Modulus and the Conjugate of a Complex Number Let z = a + ib be a complex number. Then, the modulus of z, denoted by | z |, is defined to be the non-negative real number 2 2 a b + , i.e., | z | = 2 2 a b + and the conjugate of z, denoted as z , is the complex number a – ib, i.e., z = a – ib. For example, 2 2 3 3 1 10 + = + = i , 2 2 2 5 2 (5) 29 − = + − = i , and 3 3 + = − i i , 2 5 2 5 − = + i i , − − 3 5 i = 3i – 5 Observe that the multiplicative inverse of the non-zero complex number z is given by z –1 = 1 a ib + = 2 2 2 2 a b i a b a b − + + + = 2 2 a ib a b − + = 2 z z or z 2 z z = Furthermore, the following results can easily be derived. For any two compex numbers z 1 and z 2 , we have (i) 1 2 1 2 z z z z = (ii) 1 1 2 2 z z z z = provided 2 z ≠ 0 (iii) 1 2 1 2 z z z z = (iv) 1 2 1 2 z z z z ± = ± (v) 1 1 2 2 z z z z   =   provided z 2 ≠ 0. Rationalised 2023-24 82 MATHEMATICS Example 5 Find the multiplicative inverse of 2 – 3i. Solution Let z = 2 – 3i Then z = 2 + 3i and 2 2 2 z = + − = 2 (3) 13 Therefore, the multiplicative inverse of 2 3 − i is given by z –1 2 2 3 2 3 13 13 13 z i i z + = = = + The above working can be reproduced in the following manner also, z –1 = 1 2 3 2 3 (2 3)(2 3) i i i i + = − − + = 2 2 2 3 2 3 2 3 2 (3) 13 13 13 i i i i + + = = + − Example 6 Express the following in the form a + ib (i) 5 2 1 2 i i + − (ii) i–35 Solution (i) We have, 5 2 5 2 1 2 1 2 1 2 1 2 i i i i i i + + + = × − − + ( )2 5 5 2 2 2 1 2 i i i + + − = − = 3 6 2 3(1 2 2) 1 2 3 +

+ i i = + = 1 2 2 + i . (ii) ( ) 35 35 17 2 1 1 1 i i i i i i i − = = = × − = 2 i i i = − EXERCISE 4.1 Express each of the complex number given in the Exercises 1 to 10 in the form a + ib. 1. ( ) 3 5 5 i i −   2. i i 9 19 + 3. i −39 Rationalised 2023-24 COMPLEX NUMBERS AND QUADRATIC EQUATIONS 83 Fig 4.1 4. 3(7 + i7) + i (7 + i7) 5. (1 – i) – (–1 + i6) 6. 1 2 5 4 5 5 2 i i   + − +   7. 1 7 1 4 4 3 3 3 3 i i i   + + + − − +   8. (1 – i) 4 9. 3 1 3 3 i   +   10. 3 1 2 3 i   − −   Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13. 11. 4 – 3i 12. 5 3 + i 13. – i 14. Express the following expression in the form of a + ib : ( ) ( ) ( ) ( ) 3 5 3 5 3 2 3 2 i i i i + − + − − 4.5 Argand Plane and Polar Representation We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XYplane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number x + iy which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point P(x, y) in the XY-plane and vice-versa. Some complex numbers such as 2 + 4i, – 2 + 3i, 0 + 1i, 2 + 0i, – 5 –2i and 1 – 2i which correspond to the ordered pairs (2, 4), (– 2, 3), (0, 1), (2, 0), (–5, –2), and (1, – 2), respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 4.1. The plane having a complex number assigned to each of its point is called the complex plane or the Argand plane. Rationalised 2023-24 84 MATHEMATICS Obviously, in the Argand plane, the modulus of the complex number x + iy = 2 2 x y + is the distance between the point P(x, y) and the origin O (0, 0) (Fig 4.2). The points on the x-axis corresponds to the complex numbers of the form a + i 0 and the points on the y-axis corresponds to the complex numbers of the form Fig 4.2 Fig 4.3 0 + i b. The x-axis and y-axis in the Argand plane are called, respectively, the real axis and the imaginary axis. The representation of a complex number z = x + iy and its conjugate z = x – iy in the Argand plane are, respectively, the points P (x, y) and Q (x, – y). Geometrically, the point (x, – y) is the mirror image of the point (x, y) on the real axis (Fig 4.3). Rationalised 2023-24 COMPLEX NUMBERS AND QUADRATIC EQUATIONS 85 Miscellaneous Examples Example 7 Find the conjugate of (3 2)(2 3) (1 2)(2) i i i i − + + − . Solution We have , (3 2)(2 3) (1 2)(2) i i i i − + + − = 6 9 4 6 2 4 2 i i i i + − + − + + = 12 5 4 3 4 3 4 3 i i i i + − × + − = 48 36 20 15 63 16 16 9 25 − + + − i i i = + = 63 16 25 25 − i Therefore, conjugate of (3 2)(2 3) 63 16 is (1 2)(2) 25 25 i i i i i − + + + − . Example 8 If x + iy = a ib a ib + − , prove that x 2 + y 2 = 1. Solution We have, x + iy = ( )( ) ( )( ) a ib a ib a ib a ib + + − + = 2 2 2 2 a b abi 2 a b − + + = 2 2 2 2 2 2 a b ab 2 i a b a b − + + + So that, x – iy = 2 2 2 2 2 2 a b ab 2 i a b a b − − + + Therefore, x 2 + y 2 = (x + iy) (x – iy) = 2 2 2 2 2 2 2 2 2 2 2 2 ( ) 4 ( ) ( ) a b a b a b a b − + + + = 2 2 2 2 2 2 ( ) ( ) a b a b + + = 1 Miscellaneous Exercise on Chapter 4 1. Evaluate: 3 25 18 1 i i   +   . 2. For any two complex numbers z 1 and z 2 , prove that Re (z 1 z 2) = Re z 1 Re z 2 – Imz 1 Imz 2 . Rationalised 2023-24 86 MATHEMATICS 3. Reduce 1 2 3 4 1 4 1 5 i i i i   −   −   − + + to the standard form . 4. If a ib x iy c id − − = − prove that ( ) 2 2 2 2 2 2 2 a b x y c d + + = + . 5. If z 1 = 2 – i, z 2 = 1 + i, find 1 2 1 2 1 – 1 z z z z + + + . 6. If a + ib = 2 2 ( ) 2 1 x i x + + , prove that a2 + b 2 = ( ) 2 2 2 2 (1) 2 1 x x + + . 7. Let z 1 = 2 – i, z 2 = –2 + i. Find (i) 1 2 1 Re z z z   , (ii) 1 1 1 Im z z   . 8. Find the real numbers x and y if (x – iy) (3 + 5i) is the conjugate of –6 – 24i. 9. Find the modulus of 1 1 1 1 i i i i + − − − + . 10. If (x + iy) 3 = u + iv, then show that 2 2 4(–) u v x y x y + = . 11. If α and β are different complex numbers with β 1 = , then find β α 1 αβ − − . 12. Find the number of non-zero integral solutions of the equation 1 2 x x – i = . 13. If (a + ib) (c + id) (e + if) (g + ih) = A + iB, then show that (a 2 + b 2) (c 2 + d 2) (e 2 + f 2) (g 2 + h 2) = A2 + B2 14. If 1 1 1 m i – i   +   =   , then find the least positive integral value of m. Rationalised 2023-24 COMPLEX NUMBERS AND QUADRATIC EQUATIONS 87 Summary ÆA number of the form a + ib, where a and b are real numbers, is called a complex number, a is called the real part and b is called the imaginary part of the complex number. ÆLet z 1 = a + ib and z 2 = c + id. Then (i) z 1 + z 2 = (a + c) + i (b + d) (ii) z 1 z 2 = (ac – bd) + i (ad + bc) ÆFor any non-zero complex number z = a + ib (a ≠ 0, b ≠ 0), there exists the complex number 2 2 2 2 a b i a b a b − + + + , denoted by 1 z or z –1, called the multiplicative inverse of z such that (a + ib) 2 2

2 2 − + + + a b i a b a b = 1 + i0 =1 ÆFor any integer k, i 4k = 1, i 4k + 1 = i, i 4k + 2 = – 1, i 4k + 3 = – i ÆThe conjugate of the complex number z = a + ib, denoted by z , is given by z = a – ib. Historical Note The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician Mahavira (850) who first stated this difficulty clearly. "He mentions in his work 'Ganitasara Sangraha' as in the nature of things a negative (quantity) is not a square (quantity)', it has, therefore, no square root". Bhaskara, another Indian mathematician, also writes in his work Bijaganita, written in 1150. "There is no square root of a negative quantity, for it is not a square." Cardan (1545) considered the problem of solving x + y = 10, xy = 40. Rationalised 2023-24 88 MATHEMATICS — v — He obtained x = 5 + −15 and y = 5 – −15 as the solution of it, which was discarded by him by saying that these numbers are 'useless'. Albert Girard (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. Euler was the first to introduce the symbol i for −1 and W.R. Hamilton (about 1830) regarded the complex number a + ib as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called 'imaginary numbers'. Rationalised 2023-24Chapter 5 vMathematics is the art of saying many things in many different ways. – MAXWELLv 5.1 Introduction In earlier classes, we have studied equations in one variable and two variables and also solved some statement problems by translating them in the form of equations. Now a natural question arises: 'Is it always possible to translate a statement problem in the form of an equation? For example, the height of all the students in your class is less than 160 cm. Your classroom can occupy atmost 60 tables or chairs or both. Here we get certain statements involving a sign '' (greater than), '≤' (less than or equal) and ≥ (greater than or equal) which are known as inequalities. In this Chapter, we will study linear inequalities in one and two variables. The study of inequalities is very useful in solving problems in the field of science, mathematics, statistics, economics, psychology, etc. 5.2 Inequalities Let us consider the following situations: (i) Ravi goes to market with `200 to buy rice, which is available in packets of 1kg. The price of one packet of rice is ` 30. If x denotes the number of packets of rice, which he buys, then the total amount spent by him is ` 30x. Since, he has to buy rice in packets only, he may not be able to spend the entire amount of ` 200. (Why?) Hence 30x < 200 ... (1) Clearly the statement (i) is not an equation as it does not involve the sign of equality. (ii) Reshma has ` 120 and wants to buy some registers and pens. The cost of one register is ` 40 and that of a pen is ` 20. In this case, if x denotes the number of registers and y, the number of pens which Reshma buys, then the total amount spent by her is ` (40x + 20y) and we have 40x + 20y ≤ 120 ... (2) LINEAR INEQUALITIES Rationalised 2023-24 90 MATHEMATICS Since in this case the total amount spent may be upto ` 120. Note that the statement (2) consists of two statements 40x + 20y < 120 ... (3) and 40x + 20y = 120 ... (4) Statement (3) is not an equation, i.e., it is an inequality while statement (4) is an equation. Definition 1 Two real numbers or two algebraic expressions related by the symbol '', '≤' or '≥' form an inequality. Statements such as (1), (2) and (3) above are inequalities. 3 < 5; 7 > 5 are the examples of numerical inequalities while x < 5; y > 2; x ≥ 3, y ≤ 4 are some examples of literal inequalities. 3 < 5 < 7 (read as 5 is greater than 3 and less than 7), 3 < x < 5 (read as x is greater than or equal to 3 and less than 5) and 2 < y < 4 are the examples of double inequalities. Some more examples of inequalities are: ax + b < 0 ... (5) ax + b > 0 ... (6) ax + b ≤ 0 ... (7) ax + b ≥ 0 ... (8) ax + by < c ... (9) ax + by > c ... (10) ax + by ≤ c ... (11) ax + by ≥ c ... (12) ax2 + bx + c ≤ 0 ... (13) ax2 + bx + c > 0 ... (14) Inequalities (5), (6), (9), (10) and (14) are strict inequalities while inequalities (7), (8), (11), (12), and (13) are slack inequalities. Inequalities from (5) to (8) are linear inequalities in one variable x when a ≠ 0, while inequalities from (9) to (12) are linear inequalities in two variables x and y when a ≠ 0, b ≠ 0. Inequalities (13) and (14) are not linear (in fact, these are quadratic inequalities in one variable x when a ≠ 0). In this Chapter, we shall confine ourselves to the study of linear inequalities in one and two variables only. Rationalised 2023-24 LINEAR INEQUALITIES 91 5.3 Algebraic Solutions of Linear Inequalities in One Variable and their

Graphical Representation Let us consider the inequality (1) of Section 6.2, viz, 30x < 200 Note that here x denotes the number of packets of rice. Obviously, x cannot be a negative integer or a fraction. Left hand side (L.H.S.) of this inequality is 30x and right hand side (RHS) is 200. Therefore, we have For x = 0, L.H.S. = 30 (0) = 0 < 200 (R.H.S.), which is true. For x = 1, L.H.S. = 30 (1) = 30 < 200 (R.H.S.), which is true. For x = 2, L.H.S. = 30 (2) = 60 < 200, which is true. For x = 3, L.H.S. = 30 (3) = 90 < 200, which is true. For x = 4, L.H.S. = 30 (4) = 120 < 200, which is true. For x = 5, L.H.S. = 30 (5) = 150 < 200, which is true. For x = 6, L.H.S. = 30 (6) = 180 < 200, which is true. For x = 7, L.H.S. = 30 (7) = 210 < 200, which is false. In the above situation, we find that the values of x, which makes the above inequality a true statement, are 0,1,2,3,4,5,6. These values of x, which make above inequality a true statement, are called solutions of inequality and the set {0,1,2,3,4,5,6} is called its solution set. Thus, any solution of an inequality in one variable is a value of the variable which makes it a true statement. We have found the solutions of the above inequality by trial and error method which is not very efficient. Obviously, this method is time consuming and sometimes not feasible. We must have some better or systematic techniques for solving inequalities. Before that we should go through some more properties of numerical inequalities and follow them as rules while solving the inequalities. You will recall that while solving linear equations, we followed the following rules: Rule 1 Equal numbers may be added to (or subtracted from) both sides of an equation. Rule 2 Both sides of an equation may be multiplied (or divided) by the same non-zero number. In the case of solving inequalities, we again follow the same rules except with a difference that in Rule 2, the sign of inequality is reversed (i.e., ‘’, ≤’ becomes ‘≥’ and so on) whenever we multiply (or divide) both sides of an inequality by a negative number. It is evident from the facts that 3 > 2 while – 3 < – 2, – 8 < – 7 while (– 8) (– 2) > (– 7) (– 2) , i.e., 16 > 14. Rationalised 2023-24 92 MATHEMATICS Thus, we state the following rules for solving an inequality: Rule 1 Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality. Rule 2 Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is reversed. Now, let us consider some examples. Example 1 Solve 30 x < 200 when (i) x is a natural number, (ii) x is an integer. Solution We are given 30 x < 200 or 30 200 30 30 x < (Rule 2), i.e., x < 20 / 3. (i) When x is a natural number, in this case the following values of x make the statement true. 1, 2, 3, 4, 5, 6. The solution set of the inequality is {1,2,3,4,5,6}. (ii) When x is an integer, the solutions of the given inequality are ..., – 3, –2, –1, 0, 1, 2, 3, 4, 5, 6 The solution set of the inequality is {...,–3, –2,–1, 0, 1, 2, 3, 4, 5, 6} Example 2 Solve 5x – 3 < 3x +1 when (i) x is an integer, (ii) x is a real number. Solution We have, 5x –3 < 3x + 1 or 5x –3 + 3 < 3x +1 +3 (Rule 1) or 5x < 3x +4 or 5x – 3x < 3x + 4 – 3x (Rule 1) or 2x < 4 or x < 2 (Rule 2) (i) When x is an integer, the solutions of the given inequality are ..., – 4, – 3, – 2, – 1, 0, 1 (ii) When x is a real number, the solutions of the inequality are given by x < 2, i.e., all real numbers x which are less than 2. Therefore, the solution set of the inequality is x ∈ (– ∞, 2). We have considered solutions of inequalities in the set of natural numbers, set of integers and in the set of real numbers. Henceforth, unless stated otherwise, we shall solve the inequalities in this Chapter in the set of real numbers. Rationalised 2023-24 LINEAR INEQUALITIES 93 Example 3 Solve 4x + 3 < 6x +7. Solution We have, 4x + 3 < 6x + 7 or 4x – 6x < 6x + 4 – 6x or – 2x < 4 or x > – 2 i.e., all the real numbers which are greater than –2, are the solutions of the given inequality. Hence, the solution set is (–2, ∞). Example 4 Solve 5 2 5 3 6 – x x ≤ – . Solution We have 5 2 5 3 6 – x x ≤ – or 2 (5 – 2x) ≤ x – 30. or 10 – 4x ≤ x – 30 or – 5x ≤ – 40, i.e., x ≥ 8 Thus, all real numbers x which are greater than or equal to 8 are the solutions of the given inequality, i.e., x ∈ [8, ∞). Example 5 Solve 7x + 3 < 5x + 9. Show the graph of the solutions on number line. Solution We have 7x + 3 < 5x + 9 or 2x < 6 or x < 3 The graphical representation of the solutions are given in Fig 5.1. Fig 5.1 Example 6 Solve 3 4 1 1 2 4 x x − + ≥ − . Show the graph of the solutions on number line. Solution We have 3 4 1 1 2 4 x x − + ≥ − or 3 4 3 2 4 x x − − ≥ or 2 (3x – 4) ≥ (x – 3) Rationalised 2023-24 94 MATHEMATICS or 6x – 8 ≥ x – 3 or 5x ≥ 5 or x ≥ 1 The graphical representation of solutions is given in Fig 5.2. Fig 5.2 Example 7 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks. Solution Let x be the marks obtained by student in the annual examination. Then 62 48 60 3 + + x ≥ or 110 + x ≥ 180 or x ≥ 70 Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks. Example 8 Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40. Solution Let x be the smaller of the two consecutive odd natural number, so that the other one is x +2. Then, we should have x > 10 ... (1) and x + (x + 2) < 40 ... (2) Solving (2), we get 2x + 2 < 40 i.e., x < 19 ... (3) From (1) and (3), we get 10 < x < 19 Since x is an odd number, x can take the values 11, 13, 15, and 17. So, the required possible pairs will be (11, 13), (13, 15), (15, 17), (17, 19) Rationalised 2023-24 LINEAR INEQUALITIES 95 EXERCISE 5.1 1. Solve 24x < 100, when (i) x is a natural number. (ii) x is an integer. 2. Solve – 12x > 30, when (i) x is a natural number. (ii) x is an integer. 3. Solve 5x – 3 < 7, when (i) x is an integer. (ii) x is a real number. 4. Solve 3x + 8 >2, when (i) x is an integer. (ii) x is a real number. Solve the inequalities in Exercises 5 to 16 for real x. 5. 4x + 3 < 5x + 7 6. 3x – 7 > 5x – 1 7. 3(x – 1) ≤ 2 (x – 3) 8. 3 (2 – x) ≥ 2 (1 – x) 9. 11 2 3 x x x + + < 10. 1 3 2 x x > + 11. 3(2) 5(2) 5 3 x − − x ≤ 12. 1 3 1 4 (6) 2 5 3 x x �� �� + ≥ − �� 13. 2 (2x + 3) – 10 < 6 (x – 2) 14. 37 – (3x + 5) > 9x – 8 (x – 3) 15. (5 2) (7 3) 4 3 5 x x x − − < − 16. (2 1) (3 2) (2) 3 4 5 x x x − − − ≥ − Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line 17. 3x – 2 < 2x + 1 18. 5x – 3 > 3x – 5 19. 3 (1 – x) < 2 (x + 4) 20. (5 – 2) (7 –3) – 2 3 5 x x x ≥ 21. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks. 22. To receive Grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course. 23. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11. 24. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23. Rationalised 2023-24 96 MATHEMATICS 25. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side. 26. A man wants to cut three lengths from a single piece of board of length 91cm. The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second? [Hint: If x is the length of the shortest board, then x , (x + 3) and 2x are the lengths of the second and third piece, respectively. Thus, x + (x + 3) + 2x ≤ 91 and 2x ≥ (x + 3) + 5]. Miscellaneous Examples Example 9 Solve – 8 ≤ 5x – 3 < 7. Solution In this case, we have two inequalities, – 8 ≤ 5x – 3 and 5x – 3 < 7, which we will solve simultaneously. We have – 8 ≤ 5x –3 < 7 or –5 ≤ 5x < 10 or –1 ≤ x < 2 Example 10 Solve – 5 ≤ 5 3 2 – x ≤ 8. Solution We have – 5 ≤ 5 3 2 – x ≤ 8 or –10 ≤ 5 – 3x ≤ 16 or – 15 ≤ – 3x ≤ 11 or 5 ≥ x ≥ – 11 3 which can be written as –11 3 ≤ x ≤ 5 Example 11 Solve the system of inequalities: 3x – 7 < 5 + x ... (1) 11 – 5 x ≤ 1 ... (2) and represent the solutions on the number line. Solution From inequality (1), we have 3x – 7 < 5 + x or x < 6 ... (3) Also, from inequality (2), we have 11 – 5 x ≤ 1 or – 5 x ≤ – 10 i.e., x ≥ 2 ... (4) Rationalised 2023-24 LINEAR INEQUALITIES 97 If we draw the graph of inequalities (3) and (4) on the number line, we see that the values of x, which are common to both, are shown by bold line in Fig 5.3. Fig 5.3 Thus, solution of the system are real numbers x lying between 2 and 6 including 2, i.e., 2 ≤ x < 6 Example 12 In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by C = 5 9 (F – 32), where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively. Solution It is given that 30 < C < 35. Putting C = 5 9 (F – 32), we get 30 < 5 9 (F – 32) < 35, or 9 5 × (30) < (F – 32) < 9 5 × (35) or 54 < (F – 32) < 63 or 86 < F < 95. Thus, the required range of temperature is between 86° F and 95° F. Example 13 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%? Solution Let x litres of 30% acid solution is required to be added. Then Total mixture = (x + 600) litres Therefore 30% x + 12% of 600 > 15% of (x + 600) and 30% x + 12% of 600 < 18% of (x + 600) or 30 100 x + 12 100 (600) > 15 100 (x + 600) Rationalised 2023-24 98 MATHEMATICS and 30 100 x + 12 100 (600) < 18 100 (x + 600) or 30x + 7200 > 15x + 9000 and 30x + 7200 < 18x + 10800 or 15x > 1800 and 12x < 3600 or x > 120 and x < 300, i.e. 120 < x < 300 Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres. Miscellaneous Exercise on Chapter 5 Solve the inequalities in Exercises 1 to 6. 1. 2 ≤ 3x – 4 ≤ 5 2. 6 ≤ – 3 (2x – 4) < 12 3. 7 3 4 18 2 x – ≤ – ≤ 4. 3 2 15 0 5 (x) – – < ≤ 5. 3 12 4 2 5 x – < – ≤ – 6. 3 11 7 11 2 (x) + ≤ ≤ . Solve the inequalities in Exercises 7 to 10 and represent the solution graphically on number line. 7. 5x + 1 > – 24, 5x – 1 < 24 8. 2 (x – 1) < x + 5, 3 (x + 2) > 2 – x 9. 3x – 7 > 2 (x – 6) , 6 – x > 11 – 2x 10. 5 (2x – 7) – 3 (2x + 3) ≤ 0 , 2x + 19 ≤ 6x + 47 . 11. A solution is to be kept between 68° F and 77° F. What is the range in temperature in degree Celsius (C) if the Celsius / Fahrenheit (F) conversion formula is given by F = 9 5 C + 32 ? 12. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added? Rationalised 2023-24 LINEAR INEQUALITIES 99 — v — 13. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content? 14. IQ of a person is given by the formula IQ = MA CA × 100, where MA is mental age and CA is chronological age. If 80 ≤ IQ ≤ 140 for a group of 12 years old children, find the range of their mental age. Summary ÆTwo real numbers or two algebraic expressions related by the symbols , ≤ or ≥ form an inequality. ÆEqual numbers may be added to (or subtracted from) both sides of an inequality. ÆBoth sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied (or divided) by a negative number, then the inequality is reversed. ÆThe values of x, which make an inequality a true statement, are called solutions of the inequality. ÆTo represent x < a (or x > a) on a number line, put a circle on the number a and dark line to the left (or right) of the number a. ÆTo represent x ≤ a (or x ≥ a) on a number line, put a dark circle on the number a and dark the line to the left (or right) of the number x. Rationalised 2023-24100 MATHEMATICS vEvery body of discovery is mathematical in form because there is no other guidance we can have – DARWINv 6.1 Introduction Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques. 6.2 Fundamental Principle of Counting Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are 3 × 2 = 6 pairs of a pant and a shirt. Chapter 6 PERMUTATIONS AND COMBINATIONS Jacob Bernoulli (1654-1705) Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 101 Let us name the three pants as P1 , P2 , P3 and the two shirts as S1 , S2 . Then, these six possibilities can be illustrated in the Fig. 6.1. Let us consider another problem of the same type. Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each). A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are 2 × 3 = 6 pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are 6 × 2 = 12 different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B1 , B2 , the three tiffin boxes as T1 , T2 , T3 and the two water bottles as W1 , W2 , these possibilities can be illustrated in the Fig. 6.2. Fig 6.1 Fig 6.2 Rationalised 2023-24 102 MATHEMATICS In fact, the problems of the above types are solved by applying the following principle known as the fundamental principle of counting, or, simply, the multiplication principle, which states that "If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n." The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows: 'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is m × n × p." In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession: (i) the event of choosing a pant (ii) the event of choosing a shirt. In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession: (i) the event of choosing a school bag (ii) the event of choosing a tiffin box (iii) the event of choosing a water bottle. Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order. Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed. Solution There are as many words as there are ways of filling in 4 vacant places by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is 4 × 3 × 2 × 1 = 24. Hence, the required number of words is 24. Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 103 ANote If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = 4 × 4 × 4 × 4 = 256. Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other? Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = 4 × 3 = 12. Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated? Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2 × 5, i.e., 10. Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available. Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers. There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is 5 × 4 = 20. Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags. Rationalised 2023-24 104 MATHEMATICS The number of ways is 5 × 4 × 3 = 60. Continuing the same way, we find that The number of 4 flag signals = 5 × 4 × 3 × 2 = 120 and the number of 5 flag signals = 5 × 4 × 3 × 2 × 1 = 120 Therefore, the required no of signals = 20 + 60 + 120 + 120 = 320. EXERCISE 6.1 1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that (i) repetition of the digits is allowed? (ii) repetition of the digits is not allowed? 2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated? 3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated? 4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once? 5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there? 6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other? 6.3 Permutations In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a permutation of 4 different letters taken all at a time. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words = 6 × 5 × 4 = 120 (by using multiplication principle). If the repetition of the letters was allowed, the required number of words would be 6 × 6 × 6 = 216. Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 105 Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time. In the following sub-section, we shall obtain the formula needed to answer these questions immediately. 6.3.1 Permutations when all the objects are distinct Theorem 1 The number of permutations of n different objects taken r at a time, where 0 < r ≤ n and the objects do not repeat is n (n – 1) (n – 2). . .(n – r + 1), which is denoted by nPr . Proof There will be as many permutations as there are ways of filling in r vacant places . . . by ← r vacant places → the n objects. The first place can be filled in n ways; following which, the second place can be filled in (n – 1) ways, following which the third place can be filled in (n – 2) ways,..., the rth place can be filled in (n – (r – 1)) ways. Therefore, the number of ways of filling in r vacant places in succession is n(n – 1) (n – 2) . . . (n – (r – 1)) or n (n – 1) (n – 2) ... (n – r + 1) This expression for nPr is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol n! (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually n! means. 6.3.2 Factorial notation The notation n! represents the product of first n natural numbers, i.e., the product 1 × 2 × 3 × . . . × (n – 1) × n is denoted as n!. We read this symbol as 'n factorial'. Thus, 1 × 2 × 3 × 4 . . . × (n – 1) × n = n ! 1 = 1 ! 1 × 2 = 2 ! 1× 2 × 3 = 3 ! 1 × 2 × 3 × 4 = 4 ! and so on. We define 0 ! = 1 We can write 5 ! = 5 × 4 ! = 5 × 4 × 3 ! = 5 × 4 × 3 × 2 ! = 5 × 4 × 3 × 2 × 1! Clearly, for a natural number n n ! = n (n – 1) ! = n (n – 1) (n – 2) ! [provided (n ≥ 2)] = n (n – 1) (n – 2) (n – 3) ! [provided (n ≥ 3)] and so on. Rationalised 2023-24 106 MATHEMATICS Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5! Solution (i) 5 ! = 1 × 2 × 3 × 4 × 5 = 120 (ii) 7 ! = 1 × 2 × 3 × 4 × 5 × 6 ×7 = 5040 and (iii) 7 ! – 5! = 5040 – 120 = 4920. Example 6 Compute (i) 7! 5! (ii) ( ) 12! 10! (2!) Solution (i) We have 7! 5! = 7 6 5! 5! × × = 7 × 6 = 42 and (ii) ( ) ( ) 12! 10! 2! = ( ) ( ) ( ) 12 11 10! 10! 2 × × × = 6 × 11 = 66. Example 7 Evaluate ( ) ! ! ! n r n r − , when n = 5, r = 2. Solution We have to evaluate ( ) 5! 2! 5 2 ! − (since n = 5, r = 2) We have ( ) 5! 2 ! 5 2 ! − = 5! 5 4 10 2! 3! 2 × = =

× . Example 8 If 1 1 8! 9! 10! x + = , find x. Solution We have 1 1 8! 9 8! 10 9 8! x + = × × × Therefore 1 1 9 10 9 x + = × or 10 9 10 9 x = × So x = 100. EXERCISE 6.2 1. Evaluate (i) 8 ! (ii) 4 ! – 3 ! Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 107 2. Is 3 ! + 4 ! = 7 ! ? 3. Compute 8! 6! 2! × 4. If 1 1 6! 7! 8! x + = , find x 5. Evaluate ( ) ! ! n n r − , when (i) n = 6, r = 2 (ii) n = 9, r = 5. 6.3.3 Derivation of the formula for nPr ( ) ! P ! n r n n r − = , 0 ≤ r ≤ n Let us now go back to the stage where we had determined the following formula: nPr = n (n – 1) (n – 2) . . . (n – r + 1) Multiplying numerator and denomirator by (n – r) (n – r – 1) . . . 3 × 2 × 1, we get ( ) ( ) ( )( )( ) ( )( ) 1 2 1 1 3 2 1 P 1 3 2 1 n r n n n ... n r n r n r ... n r n r ... − − − + − − − × × = − − − × × = ( ) ! ! n n r − , Thus ( ) ! P ! n r n n r = − , where 0 < r ≤n This is a much more convenient expression for nPr than the previous one. In particular, when r = n, ! P ! 0! n n n = = n Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have n P0 = 1 = ! ! ! (0)! = − n n n n ... (1) Therefore, the formula (1) is applicable for r = 0 also. Thus ( ) ! P 0 ! n r n , r n n r = ≤ ≤ − . Rationalised 2023-24 108 MATHEMATICS Theorem 2 The number of permutations of n different objects taken r at a time, where repetition is allowed, is n r . Proof is very similar to that of Theorem 1 and is left for the reader to arrive at. Here, we are solving some of the problems of the pervious Section using the formula for nPr to illustrate its usefulness. In Example 1, the required number of words = 4P4 = 4! = 24. Here repetition is not allowed. If repetition is allowed, the required number of words would be 44 = 256. The number of 3-letter words which can be formed by the letters of the word NUMBER = 6 3 6! P 3! = = 4 × 5 × 6 = 120. Here, in this case also, the repetition is not allowed. If the repetition is allowed,the required number of words would be 63 = 216. The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person can not hold more than one position, clearly 12 2 12! P 11 12 10! = = × = 132. 6.3.4 Permutations when all the objects are not distinct objects Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O1 and O2 . The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO1O2 T. Corresponding to this permutation,we have 2 ! permutations RO1O2 T and RO2O1 T which will be exactly the same permutation if O1 and O2 are not treated as different, i.e., if O1 and O2 are the same O at both places. Therefore, the required number of permutations = 4! 3 4 12 2! = × = . Permutations when O1 , O2 are Permutations when O1 , O2 are different. the same O. 1 2 2 1 RO O T RO O T ��� R O O T 1 2 2 1 T O O R T O O R ��� T O O R Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 109 1 2 2 1 R O T O R O T O ��� R O T O 1 2 2 1 T O R O T O R O ��� T O R O 1 2 2 1 R T O O R T O O ��� R T O O 1 2 2 1 T R O O T R O O ��� T R O O 1 2 2 1 O O R T O O T R ��� O O R T 1 2 2 1 O R O T O R O T ��� O R O T 1 2 2 1 O T O R O T O R ��� O T O R 1 2 2 1 O R T O O R T O ��� O R T O 1 2 2 1 O T R O O T R O ��� O T R O 1 2 2 1 O O T R O O T R ��� O O T R Let us now find the number of ways of rearranging the letters of the word INSTITUTE. In this case there are 9 letters, in which I appears 2 times and T appears 3 times. Temporarily, let us treat these letters different and name them as I1 , I2 , T1 , T2 , T3 . The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, I1 NT1 SI2 T2 U E T3 . Here if I1 , I2 are not same Rationalised 2023-24 110 MATHEMATICS and T1 , T2 , T3 are not same, then I1 , I2 can be arranged in 2! ways and T1 , T2 , T3 can be arranged in 3! ways. Therefore, 2! × 3! permutations will be just the same permutation corresponding to this chosen permutation I1NT1 SI2 T2UET3 . Hence, total number of different permutations will be 9! 2! 3! We can state (without proof) the following theorems: Theorem 3 The number of permutations of n objects, where p objects are of the same kind and rest are all different = ! ! n p . In fact, we have a more general theorem. Theorem 4 The number of permutations of n objects, where p1 objects are of one kind, p2 are of second kind, ..., pk are of k th kind and the rest, if any, are of different kind is 1 2 ! ! ! !k n p p ... p . Example 9 Find the number of permutations of the letters of the word ALLAHABAD. Solution Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different. Therefore, the required number of arrangements = 9! 5 6 7 8 9 4!2! 2 × × × × = = 7560 Example 10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed? Solution Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time. Therefore, the required 4 digit numbers ( ) 9 4 9! 9! = P = = 9 – 4 ! 5! = 9 × 8 × 7 × 6 = 3024. Example 11 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed? Solution Every number between 100 and 1000 is a 3-digit number. We, first, have to Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 111 count the permutations of 6 digits taken 3 at a time. This number would be 6P3 . But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from 6P3 to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is 5P2 . So The required number 6 5 3 2 6! 5! = P P 3! 3! − = − = 4 × 5 × 6 – 4 ×5 = 100 Example 12 Find the value of n such that (i) P 42 P 4 5 3 n n = > , n (ii) 4 –1 4 P 5 = P 3 n n , n > 4 Solution (i) Given that P 42 P 5 3 n n = or n (n – 1) (n – 2) (n – 3) (n – 4) = 42 n(n – 1) (n – 2) Since n > 4 so n(n – 1) (n – 2) ≠ 0 Therefore, by dividing both sides by n(n – 1) (n – 2), we get (n – 3 (n – 4) = 42 or n 2 – 7n – 30 = 0 or n 2 – 10n + 3n – 30 or (n – 10) (n + 3) = 0 or n – 10 = 0 or n + 3 = 0 or n = 10 or n = – 3 As n cannot be negative, so n = 10. (ii) Given that 4 –1 4 P 5 P 3 n n = Therefore 3n (n – 1) (n – 2) (n – 3) = 5(n – 1) (n – 2) (n – 3) (n – 4) or 3n = 5 (n – 4) [as (n – 1) (n – 2) (n – 3) ≠ 0, n > 4] or n = 10. Rationalised 2023-24 112 MATHEMATICS Example 13 Find r, if 5 4Pr = 6 5Pr–1 . Solution We have 4 5 1 5 P 6 P r r = − or ( ) ( ) 4! 5! 5 6 4 ! 5 1 ! r r × = × − − + or ( ) ( ) ( )( ) 5! 6 5! 4 ! 5 1 5 5 1 ! r r r r × = − − + − − − or (6 – r) (5 – r) = 6 or r 2 – 11r + 24 = 0 or r 2 – 8r – 3r + 24 = 0 or (r – 8) (r – 3) = 0 or r = 8 or r = 3. Hence r = 8, 3. Example 14 Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that (i) all vowels occur together (ii) all vowels do not occur together. Solution (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be 6P6 = 6!. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations = 6 ! × 3 ! = 4320. (ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together. Therefore, the required number 8 ! – 6 ! × 3 ! = 6 ! (7×8 – 6) = 2 × 6 ! (28 – 3) = 50 × 6 ! = 50 × 720 = 36000 Example 15 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ? Solution Total number of discs are 4 + 3 + 2 = 9. Out of 9 discs, 4 are of the first kind Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 113 (red), 3 are of the second kind (yellow) and 2 are of the third kind (green). Therefore, the number of arrangements 9! =1260 4! 3! 2! . Example 16 Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements, (i) do the words start with P (ii) do all the vowels always occur together (iii) do the vowels never occur together (iv) do the words begin with I and end in P? Solution There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore The required number of arrangements 12! 1663200 3! 4! 2! = = (i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters.

Therefore, the required number of words starting with P 11! 138600 3! 2! 4! = = . (ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object EEEEI for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in 8! 3! 2! ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E and I can be rearranged in 5! 4! ways. Therefore, by multiplication principle, the required number of arrangements 8! 5! = 16800 3! 2! 4! × = (iii) The required number of arrangements = the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together. Rationalised 2023-24 114 MATHEMATICS = 1663200 – 16800 = 1646400 (iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters. Hence, the required number of arrangements = 10! 3! 2! 4! = 12600 EXERCISE 6.3 1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated? 2. How many 4-digit numbers are there with no digit repeated? 3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated? 4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even? 5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position? 6. Find n if n – 1P3 : nP4 = 1 : 9. 7. Find r if (i) 5 6 P 2 P r r = −1 (ii) 5 6 P P r r = −1 . 8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once? 9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if. (i) 4 letters are used at a time, (ii) all letters are used at a time, (iii) all letters are used but first letter is a vowel? 10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together? 11. In how many ways can the letters of the word PERMUTATIONS be arranged if the (i) words start with P and end with S, (ii) vowels are all together, (iii) there are always 4 letters between P and S? 6.4 Combinations Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed. Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 115 These are XY, YZ and ZX (Fig 6.3). Here, each selection is called a combination of 3 different objects taken 2 at a time. In a combination, the order is not important. Now consider some more illustrations. Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time. Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time. Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by nCr .. Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., 4C2 = 6. Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations = 4C2 × 2!. On the other hand, the number of permutations of 4 different things taken 2 at a time = 4P2 . Therefore 4P2 = 4C2 × 2! or ( ) 4 2 4! C 4 2 ! 2! = − Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these 5C3 combinations, there are 3! permutations, because, the three objects in each combination can be Fig. 6.3 Rationalised

2023-24 116 MATHEMATICS rearranged in 3 ! ways. Therefore, the total of permutations = 5C 3! 3 × Therefore 5P3 = 5C3 × 3! or ( ) 5 3 5! C 5 3 ! 3! = − These examples suggest the following theorem showing relationship between permutaion and combination: Theorem 5 P C ! n n r r = r , 0 < r ≤ n. Proof Corresponding to each combination of nCr , we have r ! permutations, because r objects in every combination can be rearranged in r ! ways. Hence, the total number of permutations of n different things taken r at a time is nCr × r!. On the other hand, it is P n r . Thus P C ! n n r r = × r , 0 < ≤r n . Remarks 1. From above ( ) ! C ! ! n r n r n r = × − , i.e., ( ) ! C ! ! n r n r n r = − . In particular, if r n= , ! C 1 ! 0! n n n n = = . 2. We define nC0 = 1, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define nC0 = 1. 3. As ( ) 0 ! 1 C 0! 0 ! n n n = = − , the formula ( ) ! C ! ! n r n r n r = − is applicable for r = 0 also. Hence ( ) ! C ! ! n r n r n r = − , 0 ≤ r ≤ n. 4. ( ) ( ) ( ) ! C ! ! n n r n n r n n r − = − − − = ( ) ! ! ! n n r r − = C n r , Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 117 i.e., selecting r objects out of n objects is same as rejecting (n – r) objects. 5. nCa = nCb ⇒ a = b or a = n – b, i.e., n = a + b Theorem 6 1 C C C 1 n n n r r r + + = − Proof We have ( ) ( ) ( ) 1 ! ! C C ! ! 1 ! 1 ! n n r r n n r n r r r n r + = + − − − − + = ( ) ( ) ! 1 ! ! n r r n r × − − + ( ) ( ) ( ) ! 1 ! 1 ! n r n r n r − − + − = ( ) ( ) ! 1 ! ! n r n r − − 1 1 r n r 1 ⎡ ⎤ + ⎢ ⎥ ⎣ ⎦ − + = ( ) ( ) ( ) ! 1 1 ! ! 1 n n r r r n r r n r − + + × − − − + = ( ) ( ) 1 1 ! C ! 1 ! n r n r n r + + = + − Example 17 If C C 9 8 n n = , find C17 n . Solution We have C C 9 8 n n = i.e., ( ) ( ) ! ! 9! 9 ! 8 ! 8! n n n n = − − or 1 1 9 8 n = − or n – 8 = 9 or n = 17 Therefore 17 C C 1 17 17 n = = . Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women? Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = 5 3 5! 4 5 C 10 3! 2! 2 × = = = . Now, 1 man can be selected from 2 men in 2C1 ways and 2 women can be selected from 3 women in 3C2 ways. Therefore, the required number of committees Rationalised 2023-24 118 MATHEMATICS = 2 3 1 2 2! 3! C C 6 1! 1! 2! 1! × = × = . Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these (i) four cards are of the same suit, (ii) four cards belong to four different suits, (iii) are face cards, (iv) two are red cards and two are black cards, (v) cards are of the same colour? Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore The required number of ways = 52 4 52! 49 50 51 52 C 4! 48! 2 3 4 × × × = = × × = 270725 (i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are 13C4 ways of choosing 4 diamonds. Similarly, there are 13C4 ways of choosing 4 clubs, 13C4 ways of choosing 4 spades and 13C4 ways of choosing 4 hearts. Therefore The required number of ways = 13C4 + 13C4 + 13C4 + 13C4 . = 13! 4 2860 4! 9! × = (ii) There are13 cards in each suit. Therefore, there are 13C1 ways of choosing 1 card from 13 cards of diamond, 13C1 ways of choosing 1 card from 13 cards of hearts, 13C1 ways of choosing 1 card from 13 cards of clubs, 13C1 ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways = 13C1 × 13C1 × 13C1 × 13C1 = 134 (iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in 12C4 ways. Therefore, the required number of ways = 12! 495 4! 8! = . Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 119 (iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = 26C2 × 26C2 = ( ) 2 26! 2 325 2! 24! ⎛ ⎞ ⎜ ⎟ ⎝ ⎠ = = 105625 (v) 4 red cards can be selected out of 26 red cards in 26C4 ways. 4 black cards can be selected out of 26 black cards in 26C4ways. Therefore, the required number of ways = 26C4 + 26C4 = 26! 2 4! 22! × = 29900. EXERCISE 6.4 1. If nC8 = nC2 , find nC2 . 2. Determine n if (i) 2nC3 : nC3 = 12 : 1 (ii) 2nC3 : nC3 = 11 : 1 3. How many chords can be drawn through 21 points on a circle? 4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour. 6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination. 7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers? 8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. 9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student? Miscellaneous Examples Example 20 How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE ? Solution In the word INVOLUTE, there are 4 vowels, namely, I,O,E,Uand 4 consonants, namely, N, V, L and T. Rationalised 2023-24 120 MATHEMATICS The number of ways of selecting 3 vowels out of 4 = 4C3 = 4. The number of ways of selecting 2 consonants out of 4 = 4C2 = 6. Therefore, the number of combinations of 3 vowels and 2 consonants is 4 × 6 = 24. Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5 ! ways. Therefore, the required number of different words is 24 × 5 ! = 2880. Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ? Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in 7C5 ways. Therefore, the required number of ways = 7 5 7! 6 7 C 21 5! 2! 2 × = = = (ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of (a) 1 boy and 4 girls (b) 2 boys and 3 girls (c) 3 boys and 2 girls (d) 4 boys and 1 girl. 1 boy and 4 girls can be selected in 7C1 × 4C4 ways. 2 boys and 3 girls can be selected in 7C2 × 4C3 ways. 3 boys and 2 girls can be selected in 7C3 × 4C2 ways. 4 boys and 1 girl can be selected in 7C4 × 4C1 ways. Therefore, the required number of ways = 7C1 × 4C4 + 7C2 × 4C3 + 7C3 × 4C2 + 7C4 × 4C1 = 7 + 84 + 210 + 140 = 441 (iii) Since, the team has to consist of at least 3 girls, the team can consist of (a) 3 girls and 2 boys, or (b) 4 girls and 1 boy. Note that the team cannot have all 5 girls, because, the group has only 4 girls. 3 girls and 2 boys can be selected in 4C3 × 7C2 ways. 4 girls and 1 boy can be selected in 4C4 × 7C1 ways. Therefore, the required number of ways = 4C3 × 7C2 + 4C4 × 7C1 = 84 + 7 = 91 Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 121 Example 22 Find the number of words with or without meaning which can be made using all the letters of the word AGAIN. If these words are written as in a dictionary, what will be the 50th word? Solution There are 5 letters in the word AGAIN, in which A appears 2 times. Therefore, the required number of words = 5! 60 2! = . To get the number of words starting with A, we fix the letter A at the extreme left position, we then rearrange the remaining 4 letters taken all at a time. There will be as many arrangements of these 4 letters taken 4 at a time as there are permutations of 4 different things taken 4 at a time. Hence, the number of words starting with A = 4! = 24. Then, starting with G, the number of words 4! 2! = = 12 as after placing G at the extreme left position, we are left with the letters A, A, I and N. Similarly, there are 12 words starting with the next letter I. Total number of words so far obtained = 24 + 12 + 12 =48. The 49th word is NAAGI. The 50th word is NAAIG. Example 23 How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4? Solution Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the numbers to be counted will be 7-digit only. Also, the numbers have to be greater than 1000000, so they can begin either with 1, 2 or 4. The number of numbers beginning with 1 = 6! 4 5 6 3! 2! 2 × × = = 60, as when 1 is fixed at the extreme left position, the remaining digits to be rearranged will be 0, 2, 2, 2, 4, 4, in which there are 3, 2s and 2, 4s. Total numbers begining with 2 = 6! 3 4 5 6 2! 2! 2 × × × = = 180 and total numbers begining with 4 6! 4 5 6 3! = = × × = 120 Rationalised 2023-24 122 MATHEMATICS Therefore, the required number of numbers = 60 + 180 + 120 = 360. Alternative Method The number of 7-digit arrangements, clearly, 7! 420 3! 2! = . But, this will include those numbers also, which have 0 at the extreme left position. The number of such arrangements 6! 3! 2!

(by fixing 0 at the extreme left position) = 60. Therefore, the required number of numbers = 420 – 60 = 360. ANote If one or more than one digits given in the list is repeated, it will be understood that in any number, the digits can be used as many times as is given in the list, e.g., in the above example 1 and 0 can be used only once whereas 2 and 4 can be used 3 times and 2 times, respectively. Example 24 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together? Solution Let us first seat the 5 girls. This can be done in 5! ways. For each such arrangement, the three boys can be seated only at the cross marked places. × G × G × G × G × G ×. There are 6 cross marked places and the three boys can be seated in 6P3 ways. Hence, by multiplication principle, the total number of ways = 5! × 6P3 = 6! 5!× 3! = 4 × 5 × 2 × 3 × 4 × 5 × 6 = 14400. Miscellaneous Exercise on Chapter 6 1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ? 2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together? 3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of: (i) exactly 3 girls ? (ii) atleast 3 girls ? (iii) atmost 3 girls ? 4. If the different permutations of all the letter of the word EXAMINATION are Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 123 listed as in a dictionary, how many words are there in this list before the first word starting with E ? 5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated ? 6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ? 7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions ? 8. Determine the number of 5-card combinations out of a deck of 52 cards if each selection of 5 cards has exactly one king. 9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ? 10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen ? 11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ? Summary ÆFundamental principle of counting If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m × n. ÆThe number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by nPr and is given by nPr = ! ( )! n n r − , where 0 ≤ r ≤ n. Æn! = 1 × 2 × 3 × ...×n Æn! = n × (n – 1) ! ÆThe number of permutations of n different things, taken r at a time, where repeatition is allowed, is n r . ÆThe number of permutations of n objects taken all at a time, where p1 objects Rationalised 2023-24 124 MATHEMATICS are of first kind, p2 objects are of the second kind, ..., pk objects are of the k th kind and rest, if any, are all different is 1 2 ! ! ! !k n p p ... p . ÆThe number of combinations of n different things taken r at a time, denoted by nCr , is given by nCr = ! ! ! n r (n r) = − , 0 ≤ r ≤ n. Historical Note The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject matter as a self-contained topic in mathematics, under the name Vikalpa. Among the Jains, Mahavira, (around 850) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations. In the 6th century B.C., Sushruta, in his medicinal work, Sushruta Samhita, asserts that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc. Pingala, a Sanskrit scholar around third century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his work Chhanda Sutra. Bhaskaracharya (born 1114) treated the subject matter of permutations and combinations under the name Anka Pasha in his famous work

Lilavati. In addition to the general formulae for nCr and nPr already provided by Mahavira, Bhaskaracharya gives several important theorems and results concerning the subject. Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I–King (Book of changes). It is difficult to give the approximate time of this work, since in 213 B.C., the emperor had ordered all books and manuscripts in the country to be burnt which fortunately was not completely carried out. Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations. Some Arabic and Hebrew writers used the concepts of permutations and combinations in studying astronomy. Rabbi ben Ezra, for instance, determined the number of combinations of known planets taken two at a time, three at a time and so on. This was around 1140. It appears that Rabbi ben Ezra did not know Rationalised 2023-24 PERMUTATIONS AND COMBINATIONS 125 the formula for nCr . However, he was aware that nCr = nCn–r for specific values n and r. In 1321, Levi Ben Gerson, another Hebrew writer came up with the formulae for nPr , nPn and the general formula for nCr . The first book which gives a complete treatment of the subject matter of permutations and combinations is Ars Conjectandi written by a Swiss, Jacob Bernoulli (1654 – 1705), posthumously published in 1713. This book contains essentially the theory of permutations and combinations as is known today. — v — Rationalised 2023-24126 MATHEMATICS vMathematics is a most exact science and its conclusions are capable of absolute proofs. – C.P. STEINMETZv 7.1 Introduction In earlier classes, we have learnt how to find the squares and cubes of binomials like a + b and a – b. Using them, we could evaluate the numerical values of numbers like (98)2 = (100 – 2)2 , (999)3 = (1000 – 1)3 , etc. However, for higher powers like (98)5 , (101)6 , etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand (a + b) n , where n is an integer or a rational number. In this Chapter, we study binomial theorem for positive integral indices only. 7.2 Binomial Theorem for Positive Integral Indices Let us have a look at the following identities done earlier: (a+ b) 0 = 1 a + b ≠ 0 (a+ b) 1 = a + b (a+ b) 2 = a 2 + 2ab + b 2 (a+ b) 3 = a 3 + 3a 2b + 3ab2 + b 3 (a+ b) 4 = (a + b) 3 (a + b) = a 4 + 4a 3b + 6a 2b 2 + 4ab3 + b 4 In these expansions, we observe that (i) The total number of terms in the expansion is one more than the index. For example, in the expansion of (a + b) 2 , number of terms is 3 whereas the index of (a + b) 2 is 2. (ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms. (iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of a + b. Chapter 7 Blaise Pascal (1623-1662) BINOMIALTHEOREM Rationalised 2023-24 BINOMIAL THEOREM 127 We now arrange the coefficients in these expansions as follows (Fig 7.1): Do we observe any pattern in this table that will help us to write the next row? Yes we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our interest. We can extend the pattern given in Fig 7.2 by writing a few more rows. Pascal's Triangle The structure given in Fig 7.2 looks like a triangle with 1 at the top vertex and running down the two slanting sides. This array of numbers is known as Pascal's triangle, after the name of French mathematician Blaise Pascal. It is also known as Meru Prastara by Pingla. Expansions for the higher powers of a binomial are also possible by using Pascal's triangle. Let us expand (2x + 3y) 5 by using Pascal's triangle. The row for index 5 is 1 5 10 10 5 1 Using this row and our observations (i), (ii) and (iii), we get (2x + 3y) 5 = (2x) 5 + 5(2x) 4 (3y) + 10(2x) 3 (3y) 2 +10 (2x) 2 (3y) 3 + 5(2x)(3y) 4 +(3y) 5 = 32x 5 + 240x 4y + 720x 3y 2 + 1080x 2y 3 + 810xy4 + 243y 5 . Fig 7.1 Fig 7.2 Rationalised 2023-24 128 MATHEMATICS Now, if we want to find the expansion of (2x + 3y) 12, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe, will become more difficult, if we need the expansions involving still larger powers. We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, that come before the row of the desired index. For this, we make use of the concept of combinations studied earlier to rewrite the numbers in the Pascal's triangle. We know that ! C !( )! n r n r n – r = , 0 ≤ r ≤ n and n is a non-negative integer. Also, nC0 = 1 = nCn The Pascal's triangle can now be rewritten as (Fig 7.3) Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example, for the index 7 the row would be 7C0 7C1 7C2 7C3 7C4 7C5 7C6 7C7. Thus, using this row and the observations (i), (ii) and (iii), we have (a + b) 7 = 7C0 a 7 + 7C1 a 6b + 7C2 a 5b 2 + 7C3 a 4b 3 + 7C4 a 3b 4 + 7C5 a 2b 5 + 7C6 ab6 + 7C7 b 7 An expansion of a binomial to any positive integral index say n can now be visualised using these observations. We are now in a position to write the expansion of a binomial to any positive integral index. Fig 7.3 Pascal's triangle Rationalised 2023-24 BINOMIAL THEOREM 129 7.2.1 Binomial theorem for any positive integer n, (a + b) n = nC0 a n + nC1 a n–1b + nC2 a n–2 b 2 + ...+ nCn – 1a.b n–1 + nCn b n Proof The proof is obtained by applying principle of mathematical induction. Let the given statement be P(n) : (a + b) n = nC0 a n + nC1 a n – 1b + nC2 a n – 2b 2 + ...+ nCn–1a.b n – 1 + nCn b n For n = 1, we have P (1) : (a + b) 1 = 1C0 a 1 + 1C1 b 1 = a + b Thus, P (1) is true. Suppose P (k) is true for some positive integer k, i.e. (a + b) k = kC0 a k + kC1 a k – 1b + kC2 a k – 2b 2 + ...+ kCk b k ... (1) We shall prove that P(k + 1) is also true, i.e., (a + b) k + 1 = k + 1C0 a k + 1 + k + 1C1 a kb + k + 1C2 a k – 1b 2 + ...+ k + 1Ck+1 b k + 1 Now, (a + b) k + 1 = (a + b) (a + b) k = (a + b) (kC0 a k + kC1 a k – 1 b + kC2 a k – 2 b 2 +...+ kCk – 1 abk – 1 + kCk b k) [from (1)] = kC0 a k + 1 + kC1 a kb + kC2 a k – 1b 2 +...+ kCk – 1 a 2b k – 1 + kCk abk +kC0 a kb + kC1 a k – 1b 2 + kC2 a k – 2b 3+...+ kCk-1abk + kCk b k + 1 [by actual multiplication] = kC0 a k + 1 + (kC1 + kC0) akb + (kC2 + kC1)a k – 1b 2 + ... + (kCk + kCk–1) abk + kCk b k + 1 [grouping like terms] = k + 1C0 a k + 1 + k + 1C1 a kb + k + 1C2 a k – 1b 2 +...+ k + 1Ck abk + k + 1Ck + 1 b k +1 (by using k + 1C0 =1, kCr + kCr–1 = k + 1Cr and kCk = 1= k + 1Ck + 1) Thus, it has been proved that P (k + 1) is true whenever P(k) is true. Therefore, by principle of mathematical induction, P(n) is true for every positive integer n. We illustrate this theorem by expanding (x + 2)6 : (x + 2)6 = 6C0 x 6 + 6C1 x 5 .2 + 6C2 x 42 2 + 6C3 x 3 .23 + 6C4 x 2 .24 + 6C5 x.25 + 6C6 .26 . = x 6 + 12x 5 + 60x 4 + 160x 3 + 240x 2 + 192x + 64 Thus (x + 2)6 = x 6 + 12x 5 + 60x 4 + 160x 3 + 240x 2 + 192x + 64. Rationalised 2023-24 130 MATHEMATICS Observations 1. The notation ∑= − n k kkn k n a b 0 C stands for nC0 a nb 0 + nC1 a n–1b 1 + ...+ nCr a n–rb r + ...+nCn a n–nb n , where b 0 = 1 = an–n . Hence the theorem can also be stated as ∑= − =+ n k kkn k n n a b ba 0 C)(. 2. The coefficients nCr occuring in the binomial theorem are known as binomial coefficients. 3. There are (n+1) terms in the expansion of (a+b) n , i.e., one more than the index. 4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, (n–1) in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term. 5. In the expansion of (a+b) n , the sum of the indices of a and b is n + 0 = n in the first term, (n – 1) + 1 = n in the second term and so on 0 + n = n in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion. 7.2.2 Some special cases In the expansion of (a + b) n , (i) Taking a = x and b = – y, we obtain (x – y)n = [x + (–y)]n = nC0 x n + nC1 x n – 1(–y) + nC2 x n–2(–y) 2 + nC3 x n–3(–y) 3 + ... + nCn (–y) n = nC0 x n – nC1 x n – 1y + nC2 x n – 2y 2 – nC3 x n – 3y 3 + ... + (–1)n nCn y n Thus (x–y) n = nC0 x n – nC1 x n – 1 y + nC2 x n – 2 y 2 + ... + (–1)n nCn y n Using this, we have (x–2y) 5 = 5C0 x 5 – 5C1 x 4 (2y) + 5C2 x 3 (2y) 2 – 5C3 x 2 (2y) 3 + 5C4 x(2y) 4 – 5C5 (2y) 5 = x 5 –10x 4y + 40x 3y 2 – 80x 2y 3 + 80xy4 – 32y 5 . (ii) Taking a = 1, b = x, we obtain (1 + x) n = nC0 (1)n + nC1 (1)n – 1x + nC2 (1)n – 2 x 2 + ... + nCn x n = nC0 + nC1 x + nC2 x 2 + nC3 x 3 + ... + nCn x n Thus (1 + x) n = nC0 + nC1 x + nC2 x 2 + nC3 x 3 + ... + nCn x n Rationalised 2023-24 BINOMIAL THEOREM 131 In particular, for x = 1, we have 2 n = nC0 + nC1 + nC2 + ... + nCn . (iii) Taking a = 1, b = – x, we obtain (1– x) n = nC0 – nC1 x + nC2 x 2 – ... + (– 1)n nCn x n In particular, for x = 1, we get 0 = nC0 – nC1 + nC2 – ... + (–1)n nCn Example 1 Expand 4 2 3 x x + , x ≠ 0 Solution By using binomial theorem, we have 4 2 3 x x + = 4C0 (x 2) 4 + 4C1 (x 2) 3 ⍰ ⍰ ⍰ ⍰ ⍰ ⍰ x 3 + 4C2 (x 2) 2 2 3 ⍰ ⍰ ⍰ ⍰ ⍰ ⍰ x + 4C3 (x 2) 3 3 ⍰ ⍰ ⍰ ⍰ ⍰ ⍰ x + 4C4 4 3 ⍰ ⍰ ⍰ ⍰ ⍰ ⍰ x = x 8 + 4.x 6 . x 3 + 6.x 4 . 2 9 x + 4.x 2 . 3 27 x + 4 81 x = x 8 + 12x 5 + 54x 2 + 4 81108 x x + . Example 2 Compute (98)5 . Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem. Write 98 = 100 – 2 Therefore, (98)5 = (100 – 2)5 = 5C0 (100)5 – 5C1 (100)4 .2 + 5C2 (100)32 2 – 5C3 (100)2 (2)3 + 5C4 (100) (2)4 – 5C5 (2)5 = 10000000000 – 5 × 100000000 × 2 + 10 × 1000000 × 4 – 10 ×10000 × 8 + 5 × 100 × 16 – 32 = 10040008000 – 1000800032 = 9039207968. Example 3 Which is larger (1.01)1000000 or 10,000? Solution Splitting 1.01 and using binomial theorem to write the first few terms we have Rationalised 2023-24 132 MATHEMATICS (1.01)1000000 = (1 + 0.01)1000000 = 1000000C0 + 1000000C1 (0.01) + other positive terms = 1 + 1000000 × 0.01 + other positive terms = 1 + 10000 + other positive terms > 10000 Hence (1.01)1000000 > 10000 Example 4 Using binomial theorem, prove that 6n–5n always leaves remainder 1 when divided by 25. Solution For two numbers a and b if we can find numbers q and r such that a = bq + r, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that 6n – 5n leaves remainder 1 when divided by 25, we prove that 6 n – 5n = 25k + 1, where k is some natural number. We have (1 + a) n = nC0 + nC1 a + nC2 a 2 + ... + nCn a n For a = 5, we get (1 + 5)n = nC0 + nC1 5 + nC2 5 2 + ... + nCn 5 n i.e. (6)n = 1 + 5n + 52 . nC2 + 53 . nC3 + ... + 5n i.e. 6 n – 5n = 1+52 (nC2 + nC3 5 + ... + 5n-2) or 6 n – 5n = 1+ 25 (nC2 + 5 . nC3 + ... + 5n-2) or 6 n – 5n = 25k+1 where k = nC2 + 5 . nC3 + ... + 5n–2 . This shows that when divided by 25, 6n – 5n leaves remainder 1. EXERCISE 7.1 Expand each of the expressions in Exercises 1 to 5. 1. (1–2x) 5 2. 5 2 2 x – x ⍰ ⍰ ⍰ ⍰ ⍰ ⍰ 3. (2x – 3)6 Rationalised 2023-24 BINOMIAL THEOREM 133 4. 5 1 3 x x ⍰ ⍰ + ⍰ ⍰ ⍰ ⍰ 5. 6 1 ⍰ ⍰ ⍰ ⍰ ⍰ ⍰ + x x Using binomial theorem, evaluate each of the following: 6. (96)3 7. (102)5 8. (101)4 9. (99)5 10. Using Binomial Theorem, indicate which number is larger (1.1)10000 or 1000. 11. Find (a + b) 4 – (a – b) 4 . Hence, evaluate 4 +)23(– 4)2–3(. 12. Find (x + 1)6 + (x – 1)6 . Hence or otherwise evaluate (2 + 1)6 + (2 – 1)6 . 13. Show that 9n+1 – 8n – 9 is divisible by 64, whenever n is a positive integer. 14. Prove that ∑= = n r nr n r 0 4C3 . Miscellaneous Exercise on Chapter 7 1. If a and b are distinct integers, prove that a – b is a factor of a n – b n , whenever n is a positive integer. [Hint write a n = (a – b + b) n and expand] 2. Evaluate ( ) ( ) 6 6 3 2 3 2 + – – . 3. Find the value of ( ) ( ) 4 4 2 2 2 2 a a a a + – + – – 1 1 . 4. Find an approximation of (0.99)5 using the first three terms of its expansion. 5. Expand using Binomial Theorem 4 2 1 0 2 x , x x ⍰ ⍰ ⍰ ⍰ + – ≠ ⍰ ⍰ . 6. Find the expansion of (3x 2 – 2ax + 3a 2) 3 using binomial theorem. Summary ÆThe expansion of a binomial for any positive integral n is given by Binomial Theorem, which is (a + b) n = nC0 a n + nC1 a n – 1b + nC2 a n – 2b 2 + ...+ nCn – 1a.b n – 1 + nCn b n . ÆThe coefficients of the expansions are arranged in an array. This array is called Pascal’s triangle. Rationalised 2023-24 134 MATHEMATICS Historical Note The ancient Indian mathematicians knew about the coefficients in the expansions of (x + y) n , 0 ≤ n ≤ 7. The arrangement of these coefficients was in the form of a diagram called Meru-Prastara, provided by Pingla in his book Chhanda shastra (200B.C.). This triangular arrangement is also found in the work of Chinese mathematician Chu-shi-kie in 1303. The term binomial coefficients was first introduced by the German mathematician, Michael Stipel (1486-1567) in approximately 1544. Bombelli (1572) also gave the coefficients in the expansion of (a + b) n , for n = 1,2 ...,7 and Oughtred (1631) gave them for n = 1, 2,..., 10. The arithmetic triangle, popularly known as Pascal’s triangle and similar to the MeruPrastara of Pingla was constructed by the French mathematician Blaise Pascal (1623-1662) in 1665. The present form of the binomial theorem for integral values of n appeared in Trate du triange arithmetic, written by Pascal and published posthumously in 1665. — v — Rationalised 2023-24vNatural numbers are the product of human spirit. – DEDEKINDv 8.1 Introduction In mathematics, the word, “sequence” is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. The amount of money deposited in a bank, over a number of years form a sequence. Depreciated values of certain commodity occur in a sequence. Sequences have important applications in several spheres of human activities. Sequences, following specific patterns are called progressions. In previous class, we have studied about arithmetic progression (A.P). In this Chapter, besides discussing more about A.P.; arithmetic mean, geometric mean, relationship between A.M. and G.M., special series in forms of sum to n terms of consecutive natural numbers, sum to n terms of squares of natural numbers and sum to n terms of cubes of natural numbers will also be studied. 8.2 Sequences Let us consider the following examples: Assume that there is a generation gap of 30 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 300 years. Here, the total number of generations = 300 10 30 = Fibonacci (1175-1250) Chapter SEQUENCES AND SERIES 8 Rationalised 2023-24 136 MATHEMATICS The number of person's ancestors for the first, second, third, …, tenth generations are 2, 4, 8, 16, 32, …, 1024. These numbers form what we call a sequence. Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division. In this process we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The various numbers occurring in a sequence are called its terms. We denote the terms of a sequence by a1 , a2 , a3 , …, an , …, etc., the subscripts denote the position of the term. The n th term is the number at the n th position of the sequence and is denoted by an. The n th term is also called the generalterm of the sequence. Thus, the terms of the sequence of person's ancestors mentioned above are: a1 = 2, a2 = 4, a3 = 8, …, a10 = 1024. Similarly, in the example of successive quotients a1 = 3, a2 = 3.3, a3 = 3.33, …, a6 = 3.33333, etc. A sequence containing finite number of terms is called a finite sequence. For example, sequence of ancestors is a finite sequence since it contains 10 terms (a fixed number). A sequence is called infinite, if it is not a finite sequence. For example, the sequence of successive quotients mentioned above is an infinite sequence, infinite in the sense that it never ends. Often, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Consider for instance, the sequence of even natural numbers 2, 4, 6, … Here a1 = 2 = 2 × 1 a2 = 4 = 2 × 2 a3 = 6 = 2 × 3 a4 = 8 = 2 × 4 .... a23 = 46 = 2 × 23, a24 = 48 = 2 × 24, and so on. In fact, we see that the n th term of this sequence can be written as an = 2n, where n is a natural number. Similarly, in the sequence of odd natural numbers 1,3,5, …, the n th term is given by the formula, an = 2n – 1, where n is a natural number. In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is generated by the recurrence relation given by a1 = a2 = 1 a3 = a1 + a2 an = an – 2 + an – 1, n > 2 This sequence is called Fibonacci sequence. Rationalised 2023-24 SEQUENCES AND SERIES 137 In the sequence of primes 2,3,5,7,…, we find that there is no formula for the n th prime. Such sequence can only be described by verbal description. In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms a1 , a2 , a3 ,…,an ,… in succession. In view of the above, a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it. Sometimes, we use the functional notation a(n) for an . 8.3 Series Let a1 , a2 , a3 ,…,an , be a given sequence. Then, the expression a1 + a2 + a3 +,…+ an + ... is called the series associated with the given sequence .The series is finite or infinite according as the given sequence is finite or infinite. Series are often represented in compact form, called sigma notation, using the Greek letter ∑ (sigma) as means of indicating the summation involved. Thus, the series a1+ a2 + a3 + ... + an is abbreviated as 1 n k k a = ∑ . Remark When the series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "sum of a series," we will mean the number that results from adding the terms, the sum of the series is 16. We now consider some examples. Example 1 Write the first three terms in each of the following sequences defined by the following: (i) an = 2n + 5, (ii) an = 3 4 n − .

Solution (i) Here an = 2n + 5 Substituting n = 1, 2, 3, we get a1 = 2(1) + 5 = 7, a2 = 9, a3 = 11 Therefore, the required terms are 7, 9 and 11. (ii) Here an = 3 4 n − . Thus, 1 2 3 1 3 1 1 0 4 2 4 a , a , a − = = − = − = Rationalised 2023-24 138 MATHEMATICS Hence, the first three terms are 1 1 2 4 – , – and 0. Example 2 What is the 20th term of the sequence defined by an = (n – 1) (2 – n) (3 + n) ? Solution Putting n = 20 , we obtain a20 = (20 – 1) (2 – 20) (3 + 20) = 19 × (– 18) × (23) = – 7866. Example 3 Let the sequence an be defined as follows: a1 = 1, an = an – 1 + 2 for n ≥ 2. Find first five terms and write corresponding series. Solution We have a1 = 1, a2 = a1 + 2 = 1 + 2 = 3, a3 = a2 + 2 = 3 + 2 = 5, a4 = a3 + 2 = 5 + 2 = 7, a5 = a4 + 2 = 7 + 2 = 9. Hence, the first five terms of the sequence are 1,3,5,7 and 9. The corresponding series is 1 + 3 + 5 + 7 + 9 +... EXERCISE 8.1 Write the first five terms of each of the sequences in Exercises 1 to 6 whose nth terms are: 1. an = n (n + 2) 2. an = 1 n n + 3. an = 2 n 4. an = 2 3 6 n − 5. an = (–1)n–1 5n+1 6. an 2 5 4 n n + = . Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n th terms are: 7. an = 4n – 3; a17, a24 8. an = 2 7 ; 2 n n a 9. an = (–1)n – 1n 3 ; a9 10. 20 (– 2) ; 3 n n n a a n = + . Rationalised 2023-24 SEQUENCES AND SERIES 139 Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series: 11. a1 = 3, an = 3an – 1 + 2 for all n > 1 12. a1 = – 1, an = n 1 a n − , n ≥ 2 13. a1 = a2 = 2, an = an – 1–1, n > 2 14. The Fibonacci sequence is defined by 1 = a1 = a2 and an = an – 1 + an – 2, n > 2. Find n 1 n a a + , for n = 1, 2, 3, 4, 5 8.4 Geometric Progression (G. P.) Let us consider the following sequences: (i) 2,4,8,16,..., (ii) 1 1 1 1 9 27 81 243 − – , , , ... (iii) .01,.0001,.000001,... In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order. In (i), we have a a a a a a 1 2 1 3 2 4 3 = = = = 2 2 2 2 , , , and so on. In (ii), we observe, a a a a a a 1 2 1 3 2 4 3 1 9 1 3 1 3 1 3 = = = = , , , ⍰ ⍰ ⍰ and so on. Similarly, state how do the terms in (iii) progress? It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding it. In (i), this constant ratio is 2; in (ii), it is – 1 3 and in (iii), the constant ratio is 0.01. Such sequences are called geometric sequence or geometric progression abbreviated as G.P. A sequence a1 , a2 , a3 , …, an , … is called geometric progression, if each term is non-zero and a a k k + 1 = r (constant), for k ≥ 1. By letting a1 = a, we obtain a geometric progression, a, ar, ar2 , ar3 ,…., where a is called the first term and r is called the common ratio of the G.P. Common ratio in geometric progression (i), (ii) and (iii) above are 2, – 1 3 and 0.01, respectively. Rationalised 2023-24 140 MATHEMATICS As in case of arithmetic progression, the problem of finding the n th term or sum of n terms of a geometric progression containing a large number of terms would be difficult without the use of the formulae which we shall develop in the next Section. We shall use the following notations with these formulae: a = the first term, r = the common ratio, l = the last term, n = the numbers of terms, Sn = the sum of first n terms. 8.4.1 General term of a G.P. Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus a2 = ar. Similarly, third term is obtained by multiplying a2 by r. Thus, a3 = a2 r = ar2 , and so on. We write below these and few more terms. 1 st term = a1 = a = ar1–1, 2nd term = a2 = ar = ar2–1, 3rd term = a3 = ar2 = ar3–1 4 th term = a4 = ar3 = ar4–1, 5th term = a5 = ar4 = ar5–1 Do you see a pattern? What will be 16th term? a16 = ar16–1 = ar15 Therefore, the pattern suggests that the n th term of a G.P. is given by an = arn–1 . Thus, a, G.P. can be written as a, ar, ar2 , ar3 , … arn – 1; a, ar, ar2 ,...,arn – 1 ...;according as G.P. is finite or infinite, respectively. The series a + ar + ar2 + ... + arn–1 or a + ar + ar2 + ... + arn–1 +...are called finite or infinite geometric series, respectively. 8.4.2. Sum to n terms of a G.P. Let the first term of a G.P. be a and the common ratio be r. Let us denote by Sn the sum to first n terms of G.P. Then Sn = a + ar + ar2 +...+ arn–1 ... (1) Case 1 If r = 1, we have Sn = a + a + a + ... + a (n terms) = na Case 2 If r ≠ 1, multiplying (1) by r, we have rSn = ar + ar2 + ar3 + ... + arn ... (2) Subtracting (2) from (1), we get (1 – r) Sn = a – arn = a(1 – r n) This gives or (1) S 1 n n a r r − = − Example 4 Find the 10th and n th terms of the G.P. 5, 25,125,… . Solution Here a = 5 and r = 5. Thus, a10 = 5(5)10–1 = 5(5)9 = 510 and an = arn–1 = 5(5)n–1 = 5n . Rationalised 2023-24 SEQUENCES AND SERIES 141 Example 5 Which term of the G.P., 2,8,32, ... up to n terms is 131072? Solution Let 131072 be the n th term of the given G.P. Here a = 2 and r = 4. Therefore 131072 = an = 2(4)n – 1 or 65536 = 4n – 1 This gives 4 8 = 4n – 1. So that n – 1 = 8, i.e., n = 9. Hence, 131072 is the 9th term of the G.P. Example 6 In a G.P., the 3rd term is 24 and the 6th term is 192.Find the 10th term. Solution Here, a ar 3 2 = = 24 ... (1) and a ar 6 5 = = 192 ... (2) Dividing (2) by (1), we get r = 2. Substituting r = 2 in (1), we get a = 6. Hence a10 = 6 (2)9 = 3072. Example 7 Find the sum of first n terms and the sum of first 5 terms of the geometric series 2 4 1 3 9 + + +... Solution Here a = 1 and r = 2 3 . Therefore Sn = 2 1 3 (1) 1 2 1 3 n n a r r 􀀂 􀀃 􀀂 􀀃 􀀄 􀀅 − 􀀆 􀀇 − 􀀈 􀀉 􀀊 􀀋 = − − = 2 3 1 3 n 􀀌 􀀍 􀀌 􀀍 􀀎 􀀏 −􀀐 􀀑 􀀒 􀀓 􀀔 􀀕 In particular, 5 5 2 S 3 1 3 􀀖 􀀗 􀀖 􀀗 = − 􀀘 􀀙 􀀚 􀀛 􀀜 􀀝 􀀞 􀀟 􀀠 􀀡 = 211 3 243 × = 211 81 . Example 8 How many terms of the G.P. 3 3 3 2 4 , , ,... are needed to give the sum 3069 512 ? Solution Let n be the number of terms needed. Given that a = 3, r = 1 2 and 3069 S 512 n = Since (1) S 1 n n a – r r = − Rationalised 2023-24 142 MATHEMATICS Therefore 1 3(1) 3069 1 2 6 1 512 1 2 1 2 n n − 􀀂 􀀃 = = − 􀀄 􀀅 􀀆 􀀇 − or 3069 3072 = 1 1 2 n − or 1 2 n = 3069 1 3072 − 3 1 3072 1024 = = or 2 n = 1024 = 210 , which gives n = 10. Example 9 The sum of first three terms of a G.P. is 13 12 and their product is – 1. Find the common ratio and the terms. Solution Let a r , a, ar be the first three terms of the G.P. Then a ar a r + + = 13 12 ... (1) and ( ) ( ) 1 a a ar – r 􀀂 􀀃 􀀄 􀀅 = 􀀆 􀀇 ... (2) From (2), we get a 3 = – 1, i.e., a = – 1 (considering only real roots) Substituting a = –1 in (1), we have 1 13 1 12 – – – r r = or 12r 2 + 25r + 12 = 0. This is a quadratic in r, solving, we get 3 4 or 4 3 r – – = . Thus, the three terms of G.P. are : 4 3 –3 3 4 –4 , 1, for = and , 1, for = 3 4 4 4 3 3 – r – r , Example10 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms. Solution This is not a G.P., however, we can relate it to a G.P. by writing the terms as Sn = 7 + 77 + 777 + 7777 + ... to n terms Rationalised 2023-24 SEQUENCES AND SERIES 143 = 7 [9 99 999 9999 to term] 9 + + + + ... n = 7 2 3 4 [(10 1) (10 1) (10 1) (10 1) terms] 9 − + − + − + − + ...n = 7 2 3 [(10 10 10 terms) (1+1+1+... terms)] 9 + + +...n – n = 7 10(10 1) 7 10 (10 1) 9 10 1 9 9 n n n n 􀀂 − 􀀃 􀀄 − 􀀅 􀀆 − = 􀀇 􀀈 − 􀀉 − 􀀊 􀀋 􀀌 􀀍 . Example 11 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own. Solution Here a = 2, r = 2 and n = 10 Using the sum formula Sn = (1) 1 n a r r − − We have S10 = 2(210 – 1) = 2046 Hence, the number of ancestors preceding the person is 2046. 8.4.3 Geometric Mean (G.M.) The geometric mean of two positive numbers a and b is the number ab . Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers. Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P. Let G1 , G2 ,..., Gn be n numbers between positive numbers a and b such that a,G1 ,G2 ,G3 ,...,Gn ,b is a G.P. Thus, b being the (n + 2)th term,we have n 1 b ar + = , or 1 b n 1 r a 􀀂 􀀃 + = 􀀄 􀀅 􀀆 􀀇 . Hence 1 1 G1 b n ar a a 􀀂 􀀃 + = = 􀀄 􀀅 􀀆 􀀇 , 2 1 2 G2 b n ar a a 􀀈 􀀉 + = = 􀀊 􀀋 􀀌 􀀍 , 3 1 3 G3 b n ar a a 􀀎 􀀏 + = = 􀀐 􀀑 􀀒 􀀓 , 1 G n n n n b ar a a 􀀔 􀀕 + = = 􀀖 􀀗 􀀘 􀀙 Rationalised 2023-24 144 MATHEMATICS Example12 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P. Solution Let G1 , G2 ,G3 be three numbers between 1 and 256 such that 1, G1 ,G2 ,G3 ,256 is a G.P. Therefore 256 = r 4 giving r = ± 4 (Taking real roots only) For r = 4, we have G1 = ar = 4, G2 = ar2 = 16, G3 = ar3 = 64 Similarly, for r = – 4, numbers are – 4,16 and – 64. Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P. 8.5 Relationship Between A.M. and G.M. Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then A and G 2 a b ab + = = Thus, we have A – G = 2 a b ab + − = 2 2 a b ab + − = ( ) 2 0 2 a b − ≥ ... (1) From (1), we obtain the relationship A ≥ G. Example 13 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers. Solution Given that A.M. 10 2 a b + = = ... (1) and G.M. 8 = = ab ... (2) From (1) and (2), we get a + b = 20 ... (3) ab = 64 ... (4) Putting the value of a and b from (3), (4) in the identity (a – b) 2 = (a + b) 2 – 4ab, we get (a – b) 2 = 400 – 256 = 144 or a – b = ± 12 ... (5) Rationalised 2023-24 SEQUENCES AND SERIES 145 Solving (3) and (5), we obtain a = 4, b = 16 or a = 16, b = 4 Thus, the numbers a and b are 4, 16 or 16, 4 respectively. EXERCISE 8.2 1. Find the 20th and n th terms of the G.P. 5 5 5 2 4 8 , , , ... 2. Find the

12th term of a G.P. whose 8th term is 192 and the common ratio is 2. 3. The 5th, 8th and 11 th terms of a G.P. are p, q and s, respectively. Show that q 2 = ps. 4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term. 5. Which term of the following sequences: (a) 2 2 2 4 is 128 ? , , ,... (b) 3 3 3 3 is729 ? , , ,... (c) 1 1 1 1 is 3 9 27 19683 , , ,... ? 6. For what values of x, the numbers 2 7 7 2 – , x, – are in G.P.? Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10: 7. 0.15, 0.015, 0.0015, ... 20 terms. 8. 7 , 21 , 3 7 , ... n terms. 9. 1, – a, a 2 , – a3 , ... n terms (if a ≠ – 1). 10. x 3 , x 5 , x 7 , ... n terms (if x ≠ ± 1). 11. Evaluate 11 1 (2 3) k k= ∑ + . 12. The sum of first three terms of a G.P. is 39 10 and their product is 1. Find the common ratio and the terms. 13. How many terms of G.P. 3, 32 , 33 , … are needed to give the sum 120? 14. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P. 15. Given a G.P. with a = 729 and 7th term 64, determine S7 . Rationalised 2023-24 146 MATHEMATICS 16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term. 17. If the 4th, 10th and 16th terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P. 18. Find the sum to n terms of the sequence, 8, 88, 888, 8888… . 19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, 1 2 . 20. Show that the products of the corresponding terms of the sequences a, ar, ar2 , …arn – 1 and A, AR, AR2 , … ARn – 1 form a G.P, and find the common ratio. 21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18. 22. If the p th , q th and r th terms of a G.P. are a, b and c, respectively. Prove that a q – r br – pc P – q = 1. 23. If the first and the n th term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that P2 = (ab) n . 24. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from (n + 1)th to (2n) th term is 1 n r . 25. If a, b, c and d are in G.P. show that (a 2 + b 2 + c 2) (b 2 + c 2 + d 2) = (ab + bc + cd) 2 . 26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P. 27. Find the value of n so that a b a b n n n n + + + + 1 1 may be the geometric mean between a and b. 28. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio (3 2 2 : 3 2 2 + −) ( ). 29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are A A G A G ± + − ( )( ) . 30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2 nd hour, 4th hour and n th hour ? Rationalised 2023-24 SEQUENCES AND SERIES 147 31. What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually? 32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation. Miscellaneous Examples Example 14 If a, b, c, d and p are different real numbers such that (a 2 + b 2 + c 2)p 2 – 2(ab + bc + cd) p + (b 2 + c2 + d2) ≤ 0, then show that a, b, c and d are in G.P. Solution Given that (a 2 + b 2 + c 2) p 2 – 2 (ab + bc + cd) p + (b 2 + c2 + d2) ≤ 0 ... (1) But L.H.S. = (a 2p 2 – 2abp + b2) + (b 2p 2 – 2bcp + c2) + (c 2p 2 – 2cdp + d 2), which gives (ap – b) 2 + (bp – c) 2 + (cp – d) 2 ≥ 0 ... (2) Since the sum of squares of real numbers is non negative, therefore, from (1) and (2), we have, (ap – b) 2 + (bp – c) 2 + (cp – d) 2 = 0 or ap – b = 0, bp – c = 0, cp – d = 0 This implies that b c d p a b c = = = Hence a, b, c and d are in G.P. Miscellaneous Exercise On Chapter 8 1. If f is a function satisfying f (x +y) = f(x) f(y) for all x, y ∈ N such that f(1) = 3 and 1 ( ) 120 n x f x = ∑ = , find the value of n. 2. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms. 3. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P. 4. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers. 5. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio. Rationalised 2023-24 148 MATHEMATICS 6. If a bx a bx b cx b cx c dx c dx x + − = + − = + − ( ) ,0 ≠ then show that a, b, c and d are in G.P. 7. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that P2R n = Sn . 8. If a, b, c, d are in G.P, prove that (a n + b n), (b n + c n), (c n + d n) are in G.P. 9. If a and b are the roots of x 2 – 3x + p = 0 and c, d are roots of x 2 – 12x + q = 0, where a, b, c, d form a G.P. Prove that (q + p) : (q – p) = 17:15. 10. The ratio of the A.M. and G.M. of two positive numbers a and b, is m : n. Show that ( ) ( ) 2 2 2 2 2 a b m m – n : m – m – n : = + . 11. Find the sum of the following series up to n terms: (i) 5 + 55 +555 + … (ii) .6 +. 66 +. 666+… 12. Find the 20th term of the series 2 × 4 + 4 × 6 + 6 × 8 + ... + n terms. 13. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him? 14. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him? 15. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed. 16. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years. 17. A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years. 18. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed. Rationalised 2023-24 SEQUENCES AND SERIES 149 Summary ÆBy a sequence, we mean an arrangement of number in definite order according to some rule. Also, we define a sequence as a function whose domain is the set of natural numbers or some subsets of the type {1, 2, 3, ...k}. A sequence containing a finite number of terms is called a finite sequence. A sequence is called infinite if it is not a finite sequence. ÆLet a1 , a2 , a3 , ... be the sequence, then the sum expressed as a1 + a2 + a3 + ... is called series. A series is called finite series if it has got finite number of terms. ÆA sequence is said to be a geometric progression or G.P., if the ratio of any term to its preceding term is same throughout. This constant factor is called the common ratio. Usually, we denote the first term of a G.P. by a and its common ratio by r. The general or the n th term of G.P. is given by an = arn – 1 . The sum Sn of the first n terms of G.P. is given by (– 1 1–) ( ) S 1 1 1 – n n n a r a r = or , if r r – r ≠ ÆThe geometric mean (G.M.) of any two positive numbers a and b is given by ab i.e., the sequence a, G, b is G.P. Historical Note Evidence is found that Babylonians, some 4000 years ago, knew of arithmetic and geometric sequences. According to Boethius (510), arithmetic and geometric sequences were known to early Greek writers. Among the Indian mathematician, Aryabhatta (476) was the first to give the formula for the sum of squares and cubes of natural numbers in his famous work Aryabhatiyam, written around 499. He also gave the formula for finding the sum to n terms of an arithmetic sequence starting with p th term. Noted Indian mathematicians Brahmgupta Rationalised 2023-24 150 MATHEMATICS — v — (598), Mahavira (850) and Bhaskara (1114-1185) also considered the sum of squares and cubes. Another specific type of sequence having important applications in mathematics, called Fibonacci sequence, was discovered by Italian mathematician Leonardo Fibonacci (1170-1250). Seventeenth century witnessed the classification of series into specific forms. In 1671 James Gregory used the term infinite series in connection with infinite sequence. It was only through the rigorous development of algebraic and set theoretic tools that the concepts related to sequence and series could be formulated suitably. Rationalised 2023-24vG eometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit. – H. FREUDENTHALv 9.1 Introduction We are familiar with two-dimensional coordinate geometry from earlier classes. Mainly, it is a combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book ‘La Géométry, published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as analytical geometry. In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formulae, etc. All these concepts are the basics of coordinate geometry. Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 9.1. We may note that the point (6, – 4) is at 6 units distance from the y-axis measured along the positive x-axis and at 4 units distance from the x-axis measured along the negative y-axis. Similarly, the point (3, 0) is at 3 units distance from the y-axis measured along the positive x-axis and has zero distance from the x-axis. We also studied there following important formulae: Chapter 9 STRAIGHT LINES René Descartes (1596 - 1650) Fig 9.1 Rationalised 2023-24 152 MATHEMATICS I. Distance between the points P (x1, y1) and Q (x2 , y2) is ( ) 1 ( ) 2 2 PQ 2 2 1 = + x – x y – y For example, distance between the points (6, – 4) and (3, 0) is ( ) ( ) 2 2 3 6 0 4 9 16 5 − + + = + = units. II. The coordinates of a point dividing the line segment joining the points (x1, y1) and (x2 , y2) internally, in the ratio m: n are 🯄 🯄 🯄 🯄 🯄 🯄 🯄 🯄 + + + + nm m y n y nm m x n x12 12 , . For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by 1 (3) 3 1 0 1 3 . . x − + = = + and 1.9 + 3. –3 ( ) = = 0. 1 + 3 y III. In particular, if m = n, the coordinates of the mid-point of the line segment joining the points (x1, y1) and (x2 , y2) are 🯄 🯄 🯄 🯄 🯄 🯄 + + 2 , 2 21 21 xx yy . IV. Area of the triangle whose vertices are (x1, y1), (x2 , y2) and (x3 , y3) is 1( ) ( ) 2 3 2 3 1 3 ( ) 1 2 1 2 x x x y y y y y y − + − + − . For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is 1 54 4(2 16) 3(16 4) (3)(4 2) 27. 2 2 − − − + − + − + = = Remark If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear. In the this Chapter, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – straight line. Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which slope is most essential. 9.2 Slope of a Line A line in a coordinate plane forms two angles with the x-axis, which are supplementary. Rationalised 2023-24 STRAIGHT LINES 153 The angle (say) θ made by the line l with positive direction of x-axis and measured anti clockwise is called the inclination of the line. Obviously 0° ≤ θ ≤ 180° (Fig 9.2). We observe that lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°. Definition 1 If θ is the inclination of a line l, then tan θ is called the slope or gradient of the line l. The slope of a line whose inclination is 90° is not defined. The slope of a line is denoted by m. Thus, m = tan θ, θ ≠ 90° It may be observed that the slope of x-axis is zero and slope of y-axis is not defined. 9.2.1 Slope of a line when coordinates of any two points on the line are given We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line. Let P(x 1 , y 1) and Q(x 2 , y 2) be two points on non-vertical line l whose inclination is θ. Obviously, x 1 ≠ x 2 , otherwise the line will become perpendicular to x-axis and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take these two cases. Draw perpendicular QR to x-axis and PM perpendicular to RQ as shown in Figs. 9.3 (i) and (ii). Case 1 When angle θ is acute: In Fig 9.3 (i), ∠MPQ = θ. ... (1) Therefore, slope of line l = m = tan θ. But in ΔMPQ, we have 2 1 2 1 MQ tanθ . MP y y x x − = = − ... (2) Fig 9.2 Fig 9. 3 (i) Rationalised 2023-24 154 MATHEMATICS From equations (1) and (2), we have 2 1 2 1 . y y m x x − = − Case II When angle θ is obtuse: In Fig 9.3 (ii), we have ∠MPQ = 180° – θ. Therefore, θ = 180° – ∠MPQ. Now, slope of the line l m = tan θ = tan (180° – ∠MPQ) = – tan ∠MPQ = 2 1 1 2 MQ MP y y x x − − = − − = 2 1 2 1 y y . x x − − Consequently, we see that in both the cases the slope m of the line through the points (x 1 , y 1) and (x 2 , y 2) is given by 2 1 2 1 y y m x x − = − . 9.2.2 Conditions for parallelism and perpendicularity of lines in terms of their slopes In a coordinate plane, suppose that non-vertical lines l1 and l2 have slopes m1 and m2 , respectively. Let their inclinations be α and β, respectively. If the line l 1 is parallel to l 2 (Fig 9.4), then their inclinations are equal, i.e., α = β, and hence, tan α = tan β Therefore m1 = m2 , i.e., their slopes are equal. Conversely, if the slope of two lines l 1 and l 2 is same, i.e., m1 = m2 . Then tan α = tan β. By the property of tangent function (between 0° and 180°), α = β. Therefore, the lines are parallel. Fig 9. 3 (ii) Fig 9. 4 Rationalised 2023-24 STRAIGHT LINES 155 Hence, two non vertical lines l1 and l2 are parallel if and only if their slopes are equal. If the lines l1 and l 2 are perpendicular (Fig 9.5), then β = α + 90°. Therefore, tan β = tan (α + 90°) = – cot α = 1 tanα − i.e., m2 = 1 1 m − or m1 m2 = – 1 Conversely, if m1 m2 = – 1, i.e., tan α tan β = – 1. Then tan α = – cot β = tan (β + 90°) or tan (β – 90°) Therefore, α and β differ by 90°. Thus, lines l1 and l 2 are perpendicular to each other. Hence, two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other, i.e., m2 = 1 1 m − or, m1 m2 = – 1. Let us consider the following example. Example 1 Find the slope of the lines: (a) Passing through the points (3, – 2) and (–1, 4), (b) Passing through the points (3, – 2) and (7, – 2), (c) Passing through the points (3, – 2) and (3, 4), (d) Making inclination of 60° with the positive direction of x-axis. Solution (a) The slope of the line through (3, – 2) and (– 1, 4) is 4 (2) 6 3 1 3 4 2 m − − = = = − − − − . (b) The slope of the line through the points (3, – 2) and (7, – 2) is –2 – (–2) 0 = = = 0 7 – 3 4 m . (c) The slope of the line through the points (3, – 2) and (3, 4) is Fig 9. 5 Rationalised 2023-24 156 MATHEMATICS 4 – (–2) 6 = = 3 – 3 0 m , which is not defined. (d) Here inclination of the line α = 60°. Therefore, slope of the line is m = tan 60° = 3 . 9.2.3 Angle between two lines When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes. Let L1 and L2 be two non-vertical lines with slopes m1 and m2 ,respectively. If α1 and α2 are the inclinations of lines L1 and L2 , respectively. Then m1 = 1 andαtan m2 = αtan 2 . We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and φ be the adjacent angles between the lines L1 and L2 (Fig 9.6). Then θ = α2 – α1 and α1 , α2 ≠ 90°. Therefore tan θ = tan (α2 – α1) 2 1 2 1 1 2 1 2 tan tan 1 tan tan 1 m m m m α α α α − − = = + + (as 1 + m1m2 ≠ 0) and φ = 180° – θ so that tan φ = tan (180° – θ) = – tan θ = 2 1 1 2 – 1 m m m m − + , as 1 + m1m2 ≠ 0 Fig 9. 6 Now, there arise two cases: Rationalised 2023-24 STRAIGHT LINES 157 Case I If 2 1 1 1 2 m m– + m m is positive, then tan θ will be positive and tan φ will be negative, which means θ will be acute and φ will be obtuse. Case II If 2 1 1 1 2 m m– + m m is negative, then tan θ will be negative and tan φ will be positive, which means that θ will be obtuse and φ will be acute. Thus, the acute angle (say θ) between lines L1 and L2 with slopes m1 and m2 , respectively, is given by 2 1 1 2 1 2 tan θ , as 1 0 1 m m m m m m − = + ≠ + ... (1) The obtuse angle (say φ) can be found by using φ =1800 – θ. Example 2 If the angle between two lines is π 4 and slope of one of the lines is 1 2 , find the slope of the other line. Solution We know that the acute angle θ between two lines with slopes m1 and m2 is given by 2 1 1 2 tan θ 1 m m m m − = + ... (1) Let m1 = 2 1 , m2 = m and θ = π 4 . Now, putting these values in (1), we get 1 1 π 2 2 tan or 1 4 1 1 1 1 2 2 m m , m m − − = = + + which gives 1 1 2 2 1 or 1 1 1 1 2 2 m m – . m m − − = = + + Rationalised 2023-24 158 MATHEMATICS Fig 9.7 1 Therefore 3 or 3 m m . = = − Hence, slope of the other line is 3 or 1 3 − . Fig 9.7 explains the reason of two answers. Example 3 Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x. Solution Slope of the line through the points (– 2, 6) and (4, 8) is ( ) 1 8 6 2 1 4 2 6 3 m − = = = − − Slope of the line through the points (8, 12) and (x, 24) is 2 24 12 12 8 8 m x x − = = − − Since two lines are perpendicular, m1 m2 = –1, which gives 1 12 1 or = 4 3 8 x x × = − − . EXERCISE 9.1 1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area. 2. The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle. Rationalised 2023-24 STRAIGHT LINES

159 3. Find the distance between P (x_1 , y_1) and Q (x_2 , y_2) when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis. 4. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4). 5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0). 6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle. 7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise. 8. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram. 9. Find the angle between the x-axis and the line joining the points (3,–1) and (4,–2). 10. The slope of a line is double of the slope of another line. If tangent of the angle between them is 3 1 , find the slopes of the lines. 11. A line passes through (x_1 , y_1) and (h, k). If slope of the line is m, show that k – y_1 = m (h – x_1). 9.3 Various Forms of the Equation of a Line We know that every line in a plane contains infinitely many points on it. This relationship between line and points leads us to find the solution of the following problem: How can we say that a given point lies on the given line? Its answer may be that for a given line we should have a definite condition on the points lying on the line. Suppose P (x, y) is an arbitrary point in the XY-plane and L is the given line. For the equation of L, we wish to construct a statement or condition for the point P that is true, when P is on L, otherwise false. Of course the statement is merely an algebraic equation involving the variables x and y. Now, we will discuss the equation of a line under different conditions. 9.3.1 Horizontal and vertical lines If a horizontal line L is at a distance a from the xaxis then ordinate of every point lying on the line is either a or – a [Fig 9.8 (a)]. Therefore, equation of the line L is either y = a or y = – a. Choice of sign will depend upon the position of the line according as the line is above or below the y-axis. Similarly, the equation of a vertical line at a distance b from the y-axis is either x = b or x = – b [Fig 9.8(b)]. Rationalised 2023-24 160 MATHEMATICS Example 4 Find the equations of the lines parallel to axes and passing through (– 2, 3). Solution Position of the lines is shown in the Fig 9.9. The y-coordinate of every point on the line parallel to x-axis is 3, therefore, equation of the line parallel tox-axis and passing through (– 2, 3) is y = 3. Similarly, equation of the line parallel to y-axis and passing through (– 2, 3) is x = – 2. 9.3.2 Point-slope form Suppose that P_0 (x_0 , y_0) is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Fig 9.10). Then, by the definition, the slope of L is given by y y m( ) x x x x y y m 0 0 0 0 ,i.e., –=– – – = ...(1) Since the point P_0 (x_0 , y_0) along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L. Fig 9.8 Fig 9.10 Fig 9.9 Rationalised 2023-24 STRAIGHT LINES 161 Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0 , y_0), if and only if, its coordinates satisfy the equation y – y_0 = m (x – x_0) Example 5 Find the equation of the line through (– 2, 3) with slope – 4. Solution Here m = – 4 and given point (x_0 , y_0) is (– 2, 3). By slope-intercept form formula (1) above, equation of the given line is y – 3 = – 4 (x + 2) or 4x + y + 5 = 0, which is the required equation. 9.3.3 Two-point form Let the line L passes through two given points P_1 (x_1 , y_1) and P_2 (x_2 , y_2). Let P (x, y) be a general point on L (Fig 9.11). The three points P_1 , P_2 and P are collinear, therefore, we have slope of P_1 P = slope of P_1 P_2 i.e., 1 2 1 2 1 1 1 1 2 1 2 1 or y y y y y y , y y (x). x x x x x x x – – – = – = – – – – Thus, equation of the line passing through the points (x_1 , y_1) and (x_2 , y_2) is given by)(1 12 12 1 xx xx yy y y – – – =– ... (2) Example 6 Write the equation of the line through the points (1, –1) and (3, 5). Solution Here x_1 = 1, y_1 = – 1, x_2 = 3 and y_2 = 5. Using two-point form (2) above for the equation of the line, we have ( ) ( ) ( ) 5 – –1 – –1 = – 1 3 – 1 y x or –3x + y + 4 = 0, which is the required equation. 9.3.4 Slope-intercept form Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines. Fig 9.11 Rationalised 2023-24 162 MATHEMATICS Fig 9.12 Case I Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Fig 9.12). The distance c is called the yintercept of the line L. Obviously, coordinates of the point where the line meet the y-axis are (0, c). Thus, L has slope m and passes through a fixed point (0, c). Therefore, by point-slope form, the

equation of L is y c m(x) y mx c − = − = + 0 or Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if y = mx +c ...(3) Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively. Case II Suppose line L with slope m makes x-intercept d. Then equation of L is y = m(x – d) ... (4) Students may derive this equation themselves by the same method as in Case I. Example 7 Write the equation of the lines for which tan θ = 2 1 , where θ is the inclination of the line and (i) y-intercept is 3 2 – (ii) x-intercept is 4. Solution (i) Here, slope of the line is m = tan θ = 2 1 and y - intercept c = – 2 3 . Therefore, by slope-intercept form (3) above, the equation of the line is 032or 2 3 2 1 −= xyxy =+− , which is the required equation. (ii) Here, we have m = tan θ = 2 1 and d = 4. Therefore, by slope-intercept form (4) above, the equation of the line is 042or)4(2 1 xyxy =+−−= , which is the required equation. Rationalised 2023-24 STRAIGHT LINES 163 9.3.5 Intercept - form Suppose a line L makes x-intercept a and y-intercept b on the axes. Obviously L meets x-axis at the point (a, 0) and y-axis at the point (0, b) (Fig .9.13). By two-point form of the equation of the line, we have 0 0 ( ) or 0 b y x a ay bx ab a − − = − = − + − , i.e., =+ 1 b y a x . Thus, equation of the line making intercepts a and b on x-and y-axis, respectively, is =+ 1 b y a x ... (5) Example 8 Find the equation of the line, which makes intercepts –3 and 2 on the x- and y-axes respectively. Solution Here a = –3 and b = 2. By intercept form (5) above, equation of the line is 1 or 2 3 6 0 3 2 x y + = − + = x y − . Any equation of the form Ax + By + C = 0, where A and B are not zero simultaneously is called general linear equation or general equation of a line. EXERCISE 9.2 In Exercises 1 to 8, find the equation of the line which satisfy the given conditions: 1. Write the equations for the x-and y-axes. 2. Passing through the point (– 4, 3) with slope 2 1 . 3. Passing through (0, 0) with slope m. 4. Passing through (32,2)and inclined with the x-axis at an angle of 75o . 5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2. 6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30o with positive direction of the x-axis. Fig 9.13 Rationalised 2023-24 164 MATHEMATICS 7. Passing through the points (–1, 1) and (2, – 4). 8. The vertices of Δ PQR are P (2, 1), Q (–2, 3) and R (4, 5). Find equation of the median through the vertex R. 9. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6). 10. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: n. Find the equation of the line. 11. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3). 12. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9. 13. Find equation of the line through the point (0, 2) making an angle 2π 3 with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin. 14. The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line. 15. The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C. 16. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre? 17. P (a, b) is the mid-point of a line segment between axes. Show that equation of the line is =+ 2 b y a x . 18. Point R (h, k) divides a line segment between the axes in the ratio 1: 2. Find equation of the line. 19. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear. 9.4 Distance of a Point From a Line The distance of a point from a line is the length of the perpendicular drawn from the point to the line. Let L : Ax + By + C = 0 be a line, whose distance from the point P (x1 , y1) is d. Draw a perpendicular PM from the point P to the line L (Fig 9.14). If the Rationalised 2023-24 STRAIGHT LINES 165 line meets the x-and y-axes at the points Q and R, respectively. Then, coordinates of the points are Q C 0 A , � � � � − � � and R C 0 B , � � � � − � �. Thus, the area of the triangle PQR is given by area 1 (PQR) PM.QR 2 Δ = , which gives 2 area (ΔPQR) PM = QR ... (1) Also, area 1 ( ) 1 1 1 C C C (ΔPQR) 0 0 0 2 B A

B x y y ⎜ ⎟ ⎜ ⎟ ⎝ ⎠⎝ ⎠ = + + − − − + − ⎛ ⎞ ⎛ ⎞ ⎜⎟ ⎜ ⎟ ⎜ ⎟ ⎜ ⎟ ⎝⎠ ⎝ 2 1 1 1 C C C 2 B A AB = + + x y or 1 1 C 2 area (ΔPQR) A B C and AB = + + . , x y ( ) 2 2 C C C 2 2 QR 0 0 A B A B AB ⎛ ⎞ = + = + ⎜ ⎟ + − ⎝ ⎠ Substituting the values of area (ΔPQR) and QR in (1), we get 1 1 2 2 A B C PM A B x + + y = + Fig 9.14 Rationalised 2023-24 166 MATHEMATICS or 1 1 2 2 A B C A B x y d + + = + . Thus, the perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x1 , y1) is given by 1 1 2 2 A B C A B x y d + + = + . 9.4.1 Distance between two parallel lines We know that slopes of two parallel lines are equal. Therefore, two parallel lines can be taken in the form y = mx + c1 ... (1) and y = mx + c 2 ... (2) Line (1) will intersect x-axis at the point A 1 0 c , m ⎛ ⎞ ⎜ ⎟ − ⎝ ⎠ as shown in Fig 9.15. Distance between two lines is equal to the length of the perpendicular from point A to line (2). Therefore, distance between the lines (1) and (2) is ( ) ( ) 1 2 1 2 2 2 or = 1 1 c m c m c c d m m ⎛ ⎞ − − + − ⎜ ⎟ ⎝ ⎠ − + + . Thus, the distance d between two parallel lines 1 y mx c = + and 2 y mx c = + is given by 1 2 2 = 1 c c d m − + . If lines are given in general form, i.e., Ax + By + C1 = 0 and Ax + By + C2 = 0, then above formula will take the form 1 2 2 2 C C A B d − = + Students can derive it themselves. Fig 9.15 Rationalised 2023-24 STRAIGHT LINES 167 Example 9 Find the distance of the point (3, – 5) from the line 3x – 4y –26 = 0. Solution Given line is 3x – 4y –26 = 0 ... (1) Comparing (1) with general equation of line Ax + By + C = 0, we get A = 3, B = – 4 and C = – 26. Given point is (x1 , y1) = (3, –5). The distance of the given point from given line is ( )( ) ( ) 1 1 2 2 2 2 A B C 3 3 4 5 26 3 . A B 5 3 4 x y . − − − d – + + + = = = + + Example 10 Find the distance between the parallel lines 3x – 4y +7 = 0 and 3x – 4y + 5 = 0 Solution Here A = 3, B = –4, C1 = 7 and C2 = 5. Therefore, the required distance is ( )2 2 7 5 2 . 5 3 4 – d – = = + EXERCISE 9.3 1. Reduce the following equations into slope - intercept form and find their slopes and the y - intercepts. (i) x + 7y = 0, (ii) 6x + 3y – 5 = 0, (iii) y = 0. 2. Reduce the following equations into intercept form and find their intercepts on the axes. (i) 3x + 2y – 12 = 0, (ii) 4x – 3y = 6, (iii) 3y + 2 = 0. 3. Find the distance of the point (–1, 1) from the line 12(x + 6) = 5(y – 2). 4. Find the points on the x-axis, whose distances from the line 1 3 4 x y + = are 4 units. 5. Find the distance between parallel lines (i) 15x + 8y – 34 = 0 and 15x + 8y + 31 = 0 (ii) l (x + y) + p = 0 and l (x + y) – r = 0. 6. Find equation of the line parallel to the line 3 4 2 0 x y − + = and passing through the point (–2, 3). 7. Find equation of the line perpendicular to the line x – 7y + 5 = 0 and having x intercept 3. 8. Find angles between the lines yxyx =+=+ .13and13 9. The line through the points (h, 3) and (4, 1) intersects the line 7 9 19 0 x y . − − = at right angle. Find the value of h. Rationalised 2023-24 168 MATHEMATICS 10. Prove that the line through the point (x1 , y1) and parallel to the line Ax + By + C = 0 is A (x –x1) + B (y – y1) = 0. 11. Two lines passing through the point (2, 3) intersects each other at an angle of 60o . If slope of one line is 2, find equation of the other line. 12. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2). 13. Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line 3x – 4y – 16 = 0. 14. The perpendicular from the origin to the line y = mx + c meets it at the point (–1, 2). Find the values of m and c. 15. If p and q are the lengths of perpendiculars from the origin to the lines − = kyx θ2cosθsinθcos and x sec θ + y cosec θ = k, respectively, prove that p 2 + 4q 2 = k 2 . 16. In the triangle ABC with vertices A (2, 3), B (4, –1) and C (1, 2), find the equation and length of altitude from the vertex A. 17. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that . 111 2 22 p ba += Miscellaneous Examples Example 11 If the lines 2 3 0 5 3 0 x y , x ky + − = + − = and 3 2 0 x y − − = are concurrent, find the value of k. Solution Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are 2x + y – 3 = 0 ... (1) 5x + ky – 3 = 0 ... (2) 3x – y – 2 = 0 ... (3) Solving (1) and (3) by cross-multiplication method, we get 1 = = or = 1, = 1 –2 – 3 –9 + 4 –2 – 3 x y x y . Therefore, the point of intersection of two lines is (1, 1). Since above three lines are concurrent, the point (1, 1) will satisfy equation (2) so that 5.1 + k .1 – 3 = 0 or k = – 2. Rationalised 2023-24 STRAIGHT LINES 169 Example 12 Find the distance of the line 4x – y = 0 from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis. Solution Given line is 4x – y = 0 ... (1) In order to find the distance of the line (1) from the point P (4, 1) along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Fig 9.16). Slope of second line is tan 135° = –1. Equation of the line with slope – 1 through the point P (4, 1) is y – 1 = – 1 (x – 4) or x + y – 5 = 0 ... (2) Solving (1) and (2), we get x = 1 and y = 4 so that point of intersection of the two lines is Q (1, 4). Now, distance of line (1) from the point P (4, 1) along the line (2) = the distance between the points P (4, 1) and Q (1, 4). = ( ) ( ) 2 2 1 4 4 1 3 2 units − + − = . Example 13 Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line x – 3y + 4 = 0. Solution Let Q (h, k) is the image of the point P (1, 2) in the line x – 3y + 4 = 0 ... (1) Therefore, the line (1) is the perpendicular bisector of line segment PQ (Fig 9.17). Fig 9.16 (1, 4) Fig 9.17 Rationalised 2023-24 170 MATHEMATICS Hence Slope of line PQ = 1 Slope of line 3 4 0 x y − − + = , so that 2 1 or 3 5 1 1 3 k h k h − − = + = − ... (2) and the mid-point of PQ, i.e., point ￼ ￼ ￼ ￼ ￼ ￼ ++ 2 2 , 2 1 kh will satisfy the equation (1) so that 33or04 2 2 3 2 1 ￼ −=−=+ ￼ ￼ ￼ ￼ ￼ + − + kh kh ... (3) Solving (2) and (3), we get h = 5 6 and k = 5 7 . Hence, the image of the point (1, 2) in the line (1) is 6 7 5 5 , ￼ ￼ ￼ ￼ ￼ ￼ . Example 14 Show that the area of the triangle formed by the lines y = m1 x + c1 , y = m2 x + c2 and x = 0 is ( ) 2 1 2 2 1 2 cc m m − − . Solution Given lines are y = m1 x + c1 ... (1) y = m2 x + c 2 ... (2) x = 0 ... (3) We know that line y = mx + c meets the line x = 0 (y-axis) at the point (0, c). Therefore, two vertices of the triangle formed by lines (1) to (3) are P (0, c 1) and Q (0, c 2) (Fig 9.18). Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get Fig 9.18 Rationalised 2023-24 STRAIGHT LINES 171 ( ) ( ) ( ) ( ) 2 1 1 2 2 1 1 2 1 2 and c c m c m c x y m m m m − − = = − − Therefore, third vertex of the triangle is R ( ) ( ) ( ) ( ) 2 1 1 2 2 1 1 2 1 2 c c m c m c , m m m m ￼ − − ￼ ￼ ￼ ￼ ￼ − − ￼ ￼ . Now, the area of the triangle is ( ) ( ) 2 2 1 1 2 2 1 2 1 1 2 2 1 2 2 1 1 1 2 1 2 1 2 1 2 1 0 0 2 2 c c m c m c c c m c m c c c c c c m m m m m m m m ￼ − ￼ − ￼ − ￼ = ￼ − + − + − ￼ ￼ ￼ = ￼ − ￼ − ￼ − ￼ − − Example 15 A line is such that its segment between the lines 5x – y + 4 = 0 and 3x + 4y – 4 = 0 is bisected at the point (1, 5). Obtain its equation. Solution Given lines are 5x – y + 4 = 0 ... (1) 3x + 4y – 4 = 0 ... (2) Let the required line intersects the lines (1) and (2) at the points, (α1 , β1) and (α2 , β2), respectively (Fig 9.19). Therefore 5α1 – β1 + 4 = 0 and 3 α2 + 4 β2 – 4 = 0 or β1 = 5α1 + 4 and 2 2 4 – 3α β 4 = . We are given that the mid point of the segment of the required line between (α1 , β1) and (α2 , β2) is (1, 5). Therefore α1 + α2 2 β + β 1 = 1 and = 5 2 2 , or 2 1 1 2 4 – 3α 5α + 4 + 4 α + = 2 and α = 5, 2 or α1 + α2 = 2 and 20α1 – 3α2 = 20 ... (3) Solving equations in (3) for α1 and α2 , we get Fig 9.19 Rationalised 2023-24 172 MATHEMATICS 1 26 α = 23 and 2 20 α = 23 and hence, 23 222 4 23 26 β .5 1 =+= . Equation of the required line passing through (1, 5) and (α1 , β1) is)1(α 1 β 5 5 1 1 − − − y =− x or 222 5 23 5 (1) 26 1 23 y x − − = − − or 107x – 3y – 92 = 0, which is the equation of required line. Example 16 Show that the path of a moving point such that its distances from two lines 3x – 2y = 5 and 3x + 2y = 5 are equal is a straight line. Solution Given lines are 3x – 2y = 5 … (1) and 3x + 2y = 5 … (2) Let (h, k) is any point, whose distances from the lines (1) and (2) are equal. Therefore 523523or 49 523 49 523 −+=−− + −+ = + −− khkh khkh , which gives 3h – 2k – 5 = 3h + 2k – 5 or – (3h – 2k – 5) = 3h + 2k – 5. Solving these two relations we get k = 0 or h = 3 5 . Thus, the point (h, k) satisfies the equations y = 0 or x = 3 5 , which represent straight lines. Hence, path of the point equidistant from the lines (1) and (2) is a straight line. Miscellaneous Exercise on Chapter 9 1. Find the values of k for which the line (k–3) x – (4 – k 2) y + k 2 –7k + 6 = 0 is (a) Parallel to the x-axis, (b) Parallel to the y-axis, (c) Passing through the origin. 2. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and – 6, respectively. Rationalised 2023-24 STRAIGHT LINES 173 3. What are the points on the y-axis whose distance from the line 1 3 4 x y + = is 4 units. 4. Find perpendicular distance from the origin to the line joining the points (cosθ, sin θ) and (cos ϕ, sin ϕ). 5. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines x – 7y + 5 = 0 and 3x + y = 0. 6. Find the equation of a line drawn perpendicular to the line 1 4 6 =+ yx through the point, where it meets the y-axis. 7. Find the area of the triangle formed by the lines y – x = 0, x + y = 0 and x – k = 0. 8. Find the value of p so that the three lines 3x + y – 2 = 0, px + 2 y – 3 = 0 and 2x – y – 3 = 0 may intersect at one point. 9. If three lines whose equations are y = m1 x + c1 , y = m2 x + c2 and y = m3 x + c3 are concurrent, then show that m1 (c2 – c3) + m2 (c3 – c1) + m3 (c1 – c2) = 0. 10. Find the equation of the lines through the point (3, 2) which make an angle of 45o with the line x – 2y = 3. 11. Find the equation of the line passing through the point of intersection of the lines 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0 that has equal intercepts on the axes. 12. Show that the equation of the line passing through the origin and making an angle θ with the line y mx c is y x m m = + = ± tan , ∓ tan ,1 . 13. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line x + y = 4? 14. Find the distance of the line 4x + 7y + 5 = 0 from the point (1, 2) along the line 2x – y = 0. 15. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line x + y = 4 may be at a distance of 3 units from this point. 16. The hypotenuse of a right angled triangle has its ends at the points (1, 3) and (– 4, 1). Find an equation of the legs (perpendicular sides) of the triangle which are parallel to the axes. 17. Find the image of the point (3, 8) with respect to the line x +3y = 7 assuming the line to be a plane mirror. 18. If the lines y = 3x +1 and 2y = x + 3 are equally inclined to the line y = mx + 4, find the value of m. 19. If sum of the perpendicular distances of a variable point P (x, y) from the lines x + y – 5 = 0 and 3x – 2y +7 = 0 is always 10. Show that P must move on a line. Rationalised 2023-24 174 MATHEMATICS 20. Find equation of the line which is equidistant from parallel lines 9x + 6y – 7 = 0 and 3x + 2y + 6 = 0. 21. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A. 22. Prove that the product of the lengths of the perpendiculars drawn from the points ( ) 2 2 a b , − 0 and ( ) 2 2 − − a b ,0 to the line 2 cosθ sin θ 1is x y b a b + = . 23. A person standing at the junction (crossing) of two straight paths represented by the equations 2x – 3y + 4 = 0 and 3x + 4y – 5 = 0 wants to reach the path whose equation is 6x – 7y + 8 = 0 in the least time. Find equation of the path that he should follow. Summary ÆSlope (m) of a non-vertical line passing through the points (x 1 , y1) and (x 2 , y2) is given by 2 1 1 2 1 2 2 1 1 2 y y y y m , . x x x x x x − − = = ≠ − − Æ If a line makes an angle á with the positive direction of x-axis, then the slope of the line is given by m = tan α, α ≠ 90°. ÆSlope of horizontal line is zero and slope of vertical line is undefined. Æ An acute angle (say θ) between lines L1 and L2 with slopes m1 and m2 is given by 2 1 1 2 1 2 tanθ 1 0 1 m – m , m m m m = + ≠ + . ÆTwo lines are parallel if and only if their slopes are equal. ÆTwo lines are perpendicular if and only if product of their slopes is –1. ÆThree points A, B and C are collinear, if and only if slope of AB = slope of BC. ÆEquation of the horizontal line having distance a from the x-axis is either y = a or y = – a. ÆEquation of the vertical line having distance b from the y-axis is either x = b or x = – b. ÆThe point (x, y) lies on the line with slope m and through the fixed point (xo , yo), if and only if its coordinates satisfy the equation y – yo = m (x – xo). Æ Equation of the line passing through the points (x1 , y1) and (x2 , y2) is given by).(1 12 12 1 x x xx yy y y − − − =− Rationalised 2023-24 STRAIGHT LINES 175 ÆThe point (x, y) on the line with slope m and y-intercept c lies on the line if and only if y = mx + c. ÆIf a line with slope m makes x-intercept d. Then equation of the line is y = m (x – d). ÆEquation of a line making intercepts a and b on the x-and y-axis, respectively, is =+ 1 b y a x . ÆAny equation of the form Ax + By + C = 0, with A and B are not zero, simultaneously, is called the general linear equation or general equation of a line. ÆThe perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x1 , y1) is given by 1 1 2 2 A B C A B x y d + + = + . ÆDistance between the parallel lines Ax + By + C1 = 0 and Ax + By + C2 = 0, is given by 1 2 2 2 C C A B d − = + . Rationalised 2023-24176 MATHEMATICS vLet the relation of knowledge to real life be very visible to your pupils and let them understand how by knowledge the world could be transformed. – BERTRAND RUSSELL v 10.1 Introduction In the preceding Chapter 10, we have studied various forms of the equations of a line. In this Chapter, we shall study about some other curves, viz., circles, ellipses, parabolas and hyperbolas. The names parabola and hyperbola are given by Apollonius. These curves are in fact, known as conic sections or more commonly conics because they can be obtained as intersections of a plane with a double napped right circular cone. These curves have a very wide range of applications in fields such as planetary motion, design of telescopes and antennas, reflectors in flashlights and automobile headlights, etc. Now, in the subsequent sections we will see how the intersection of a plane with a double napped right circular cone results in different types of curves. 10.2 Sections of a Cone Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α (Fig10.1). Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as Apollonius (262 B.C. -190 B.C.) Chapter 10 Fig 10. 1 CONIC SECTIONS Rationalised 2023-24 CONIC SECTIONS 177 Fig 10. 2 Fig 10. 3 cone and extending indefinitely far in both directions (Fig10.2). The point V is called the vertex; the line l is the axis of the cone. The rotating line m is called a generator of the cone. The vertex separates the cone into two parts called nappes. If we take the intersection of a plane with a cone, the section so obtained is called a conic section. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane. We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone (Fig10.3). The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex. 10.2.1 Circle, ellipse, parabola and hyperbola When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations: (a) When β = 90o , the section is a circle (Fig10.4). (b) When α < β < 90o , the section is an ellipse (Fig10.5). (c) When β = α; the section is a parabola (Fig10.6). (In each of the above three situations, the plane cuts entirely across one nappe of the cone). (d) When 0 ≤ β < α; the plane cuts through both the nappes and the curves of intersection is a hyperbola (Fig10.7). Rationalised 2023-24 178 MATHEMATICS Fig 10. 4 10.2.2 Degenerated conic sections When the plane cuts at the vertex of the cone, we have the following different cases: (a) When α < β ≤ 90o , then the section is a point (Fig10.8). (b) When β = α, the plane contains a generator of the cone and the section is a straight line (Fig10.9). It is the degenerated case of a parabola. (c) When 0 ≤ β < α, the section is a pair of intersecting straight lines (Fig10.10). It is the degenerated case of a hyperbola. Fig 10. 6 Fig 10. 7 Fig 10. 5 Rationalised 2023-24 CONIC SECTIONS 179 In the following sections, we shall obtain the equations of each of these conic sections in standard form by defining them based on geometric properties. Fig 10. 8 Fig 10. 9 Fig 10. 10 10.3 Circle Definition 1 A circle is the set of all points in a plane that are equidistant from a fixed point in the plane. The fixed point is called the centre of the circle and the distance from the centre to a point on the circle is called the radius of the circle (Fig 10.11). Rationalised 2023-24 180 MATHEMATICS The equation of the circle is simplest if the centre of the circle is at the origin. However, we derive below the equation of the circle with a given centre and radius (Fig 10.12). Given C (h, k) be the centre and r the radius of circle. Let P(x, y) be any point on the circle (Fig10.12). Then, by the definition, | CP | = r . By the distance formula, we have 2 2 ( ) ( ) x – h y – k r + = i.e. (x – h) 2 + (y – k) 2 = r 2 This is the required equation of the circle with centre at (h,k) and radius r . Example 1 Find an equation of the circle with centre at (0,0) and radius r. Solution Here h = k = 0. Therefore, the equation of the circle is x 2 + y 2 = r 2 . Example 2 Find the equation of the circle with centre (–3, 2) and radius 4. Solution Here h = –3, k = 2 and r = 4. Therefore, the equation of the required circle is (x + 3)2 + (y –2)2 = 16 Example 3 Find the centre and the radius of the circle x 2 + y2 + 8x + 10y – 8 = 0 Solution The given equation is (x 2 + 8x) + (y 2 + 10y) = 8 Now, completing the squares within the parenthesis, we get (x 2 + 8x + 16) + (y 2 + 10y + 25) = 8 + 16 + 25 i.e. (x + 4)2 + (y + 5)2 = 49 i.e. {x – (– 4)}2 + {y – (–5)}2 = 72 Therefore, the given circle has centre at (– 4, –5) and radius 7. Fig 10. 11 Fig 10. 12 Rationalised 2023-24 CONIC SECTIONS 181 Example 4 Find the equation of the circle which passes through the points (2, – 2), and (3,4) and whose centre lies on the line x + y = 2. Solution Let the equation of the circle be (x – h) 2 + (y – k) 2 = r 2 . Since the circle passes through (2, – 2) and

(3,4), we have (2 – h) 2 + (–2 – k) 2 = r 2 ... (1) and (3 – h) 2 + (4 – k) 2 = r 2 ... (2) Also since the centre lies on the line x + y = 2, we have h + k = 2 ... (3) Solving the equations (1), (2) and (3), we get h = 0.7, k = 1.3 and r 2 = 12.58 Hence, the equation of the required circle is (x – 0.7)2 + (y – 1.3)2 = 12.58. EXERCISE 10.1 In each of the following Exercises 1 to 5, find the equation of the circle with 1. centre (0,2) and radius 2 2. centre (–2,3) and radius 4 3. centre (4 1 , 2 1) and radius 12 1 4. centre (1,1) and radius 2 5. centre (–a, –b) and radius 22 − ba . In each of the following Exercises 6 to 9, find the centre and radius of the circles. 6. (x + 5)2 + (y – 3)2 = 36 7. x 2 + y 2 – 4x – 8y – 45 = 0 8. x 2 + y 2 – 8x + 10y – 12 = 0 9. 2x 2 + 2y 2 – x = 0 10. Find the equation of the circle passing through the points (4,1) and (6,5) and whose centre is on the line 4x + y = 16. 11. Find the equation of the circle passing through the points (2,3) and (–1,1) and whose centre is on the line x – 3y – 11 = 0. 12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2,3). 13. Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes. 14. Find the equation of a circle with centre (2,2) and passes through the point (4,5). 15. Does the point (–2.5, 3.5) lie inside, outside or on the circle x 2 + y 2 = 25? Rationalised 2023-24 182 MATHEMATICS Fig 10. 13 Fig 10.14 10.4 Parabola Definition 2 A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line) in the plane. The fixed line is called the directrix of the parabola and the fixed point F is called the focus (Fig 10.13). (‘Para’ means ‘for’ and ‘bola’ means ‘throwing’, i.e., the shape described when you throw a ball in the air). ANote If the fixed point lies on the fixed line, then the set of points in the plane, which are equidistant from the fixed point and the fixed line is the straight line through the fixed point and perpendicular to the fixed line. We call this straight line as degenerate case of the parabola. A line through the focus and perpendicular to the directrix is called the axis of the parabola. The point of intersection of parabola with the axis is called the vertex of the parabola (Fig10.14). 10.4.1 Standard equations of parabola The equation of a parabola is simplest if the vertex is at the origin and the axis of symmetry is along the x-axis or y-axis. The four possible such orientations of parabola are shown below in Fig10.15 (a) to (d). Rationalised 2023-24 CONIC SECTIONS 183 We will derive the equation for the parabola shown above in Fig 10.15 (a) with focus at (a, 0) a > 0; and directricx x = – a as below: Let F be the focus and l the directrix. Let FM be perpendicular to the directrix and bisect FM at the point O. Produce MO to X. By the definition of parabola, the mid-point O is on the parabola and is called the vertex of the parabola. Take O as origin, OX the x-axis and OY perpendicular to it as the y-axis. Let the distance from the directrix to the focus be 2a. Then, the coordinates of the focus are (a, 0), and the equation of the directrix is x + a = 0 as in Fig10.16. Let P(x, y) be any point on the parabola such that PF = PB, ... (1) where PB is perpendicular to l. The coordinates of B are (– a, y). By the distance formula, we have PF = 2 2 ( ) x – a y + and PB = 2 ( ) x a + Since PF = PB, we have 2 2 2 ( ) ( ) x – a y x a + = + i.e. (x – a) 2 + y 2 = (x + a) 2 or x 2 – 2ax + a 2 + y 2 = x 2 + 2ax + a 2 or y 2 = 4ax (a > 0). Fig 10.15 (a) to (d) Fig 10.16 Rationalised 2023-24 184 MATHEMATICS Hence, any point on the parabola satisfies y 2 = 4ax. ... (2) Conversely, let P(x, y) satisfy the equation (2) PF = 2 2 ( ) x – a y + = 2 ( ) 4 x – a ax + = 2 ( ) x a + = PB ... (3) and so P(x,y) lies on the parabola. Thus, from (2) and (3) we have proved that the equation to the parabola with vertex at the origin, focus at (a,0) and directrix x = – a is y 2 = 4ax. Discussion In equation (2), since a > 0, x can assume any positive value or zero but no negative value and the curve extends indefinitely far into the first and the fourth quadrants. The axis of the parabola is the positive x-axis. Similarly, we can derive the equations of the parabolas in: Fig 11.15 (b) as y 2 = – 4ax, Fig 11.15 (c) as x 2 = 4ay, Fig 11.15 (d) as x 2 = – 4ay, These four equations are known as standard equations of parabolas. ANote The standard equations of parabolas have focus on one of the coordinate axis; vertex at the origin and thereby the directrix is parallel to the other coordinate axis. However, the study of the equations of parabolas with focus at any point and any line as directrix is beyond the scope here. From the standard equations of the parabolas, Fig10.15, we have the following observations: 1. Parabola is symmetric with respect to the axis of the parabola.If the equation has a y^2 term, then the axis of symmetry is along the x-axis and if the equation has an x^2 term, then the axis of symmetry is along the y-axis. 2. When the axis of symmetry is along the x-axis the parabola opens to the (a) right if the coefficient of x is positive, (b) left if the coefficient of x is negative. 3. When the axis of symmetry is along the y-axis the parabola opens (c) upwards if the coefficient of y is positive. (d) downwards if the coefficient of y is negative. Rationalised 2023-24 CONIC SECTIONS 185 10.4.2 Latus rectum Definition 3 Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola (Fig10.17). To find the Length of the latus rectum of the parabola $y^2 = 4ax$ (Fig 10.18). By the definition of the parabola, AF = AC. But AC = FM = 2a Hence AF = 2a. And since the parabola is symmetric with respect to x-axis AF = FB and so AB = Length of the latus rectum = 4a. Fig 10.17 Fig 10.18 Example 5 Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$. Solution The given equation involves y^2 , so the axis of symmetry is along the x-axis. The coefficient of x is positive so the parabola opens to the right. Comparing with the given equation $y^2 = 4ax$, we find that a = 2. Thus, the focus of the parabola is (2, 0) and the equation of the directrix of the parabola is x = – 2 (Fig 10.19). Length of the latus rectum is 4a = 4 × 2 = 8. Fig 10.19 Rationalised 2023-24 186 MATHEMATICS Example 6 Find the equation of the parabola with focus (2,0) and directrix x = – 2. Solution Since the focus (2,0) lies on the x-axis, the x-axis itself is the axis of the parabola. Hence the equation of the parabola is of the form either $y^2 = 4ax$ or $y^2 = – 4ax$. Since the directrix is x = – 2 and the focus is (2,0), the parabola is to be of the form $y^2 = 4ax$ with a = 2. Hence the required equation is $y^2 = 4(2)x = 8x$ Example 7 Find the equation of the parabola with vertex at (0, 0) and focus at (0, 2). Solution Since the vertex is at (0,0) and the focus is at (0,2) which lies on y-axis, the y-axis is the axis of the parabola. Therefore, equation of the parabola is of the form $x^2 = 4ay$. thus, we have $x^2 = 4(2)y$, i.e., $x^2 = 8y$. Example 8 Find the equation of the parabola which is symmetric about the y-axis, and passes through the point (2,–3). Solution Since the parabola is symmetric about y-axis and has its vertex at the origin, the equation is of the form $x^2 = 4ay$ or $x^2 = – 4ay$, where the sign depends on whether the parabola opens upwards or downwards. But the parabola passes through (2,–3) which lies in the fourth quadrant, it must open downwards. Thus the equation is of the form $x^2 = – 4ay$. Since the parabola passes through (2,–3), we have $2^2 = – 4a (–3)$, i.e., a = 1 3 Therefore, the equation of the parabola is x 2 = 1 4 3 �� – ���� y, i.e., $3x^2 = – 4y$. EXERCISE 10.2 In each of the following Exercises 1 to 6, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum. 1. $y^2 = 12x$ 2. $x^2 = 6y$ 3. $y^2 = – 8x$ 4. $x^2 = – 16y$ 5. $y^2 = 10x$ 6. $x^2 = – 9y$ In each of the Exercises 7 to 12, find the equation of the parabola that satisfies the given conditions: Rationalised 2023-24 CONIC SECTIONS 187 Fig 10.20 Fig 10.21 Fig 10.22 We denote the length of the major axis by 2a, the length of the minor axis by 2b and the distance between the foci by 2c. Thus, the length of the semi major axis is a and semi-minor axis is b (Fig10.22). 7. Focus (6,0); directrix x = – 6 8. Focus (0,–3); directrix y = 3 9. Vertex (0,0); focus (3,0) 10. Vertex (0,0); focus (–2,0) 11. Vertex (0,0) passing through (2,3) and axis is along x-axis. 12. Vertex (0,0), passing through (5,2) and symmetric with respect to y-axis. 10. 5 Ellipse Definition 4 An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant. The two fixed points are called the foci (plural of 'focus') of the ellipse (Fig10.20). ANote The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points. The mid point of the line segment joining the foci is called the centre of the ellipse. The line segment through the foci of the ellipse is called the major axis and the line segment through the centre and perpendicular to the major axis is called the minor axis. The end points of the major axis are called the vertices of the ellipse(Fig 10.21). Rationalised 2023-24 188 MATHEMATICS 10.5.1 Relationship between semi-major axis, semi-minor axis and the distance of the focus from the centre of the ellipse (Fig 10.23). Take a point P at one end of the major axis. Sum of the distances of the point P to the foci is F1 P + F2 P = F1O + OP + F2 P (Since, F1 P = F1O + OP) = c + a + a – c = 2a Take a point Q at one end of the minor axis. Sum of the distances from the point Q to the foci is F1Q + F2Q = 22 22 +++ cbcb = 22 2 + cb Since both P and Q lies on the ellipse. By the definition of ellipse, we have 2 22 + cb = 2a, i.e., a = 22 + cb or a 2 = b 2 + c 2 , i.e., c = 22 − ba . 10.5.2 Eccentricity Definition 5 The eccentricity of an ellipse is the ratio of the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse (eccentricity is denoted by e) i.e., c e a = . Then since the focus is at a distance of c from the centre, in terms of the eccentricity the focus is at a distance of ae from the centre. 10.5.3 Standard equations of an ellipse The equation of an ellipse is simplest if the centre of the ellipse is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig 10.24. We will derive the equation for the ellipse shown above in Fig 10.24 (a) with foci on the x-axis. Fig 10.23 Rationalised 2023-24 CONIC SECTIONS 189 Fig 10.24 (a) Let F1 and F2 be the foci and O be the mid-point of the line segment F1 F2 . Let O be the origin and the line from O through F2 be the positive x-axis and that through F1 as the negative x-axis. Let, the line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F1 be (– c, 0) and F2 be (c, 0) (Fig 10.25). Let P(x, y) be any point on the ellipse such that the sum of the distances from P to the two foci be 2a so given PF1 + PF2 = 2a. ... (1) Using the distance formula, we have 22 22)( )(+−+++ ycxycx = 2a i.e., 22)(++ ycx = 2a – 22)(+− ycx Squaring both sides, we get (x + c) 2 + y 2 = 4a 2 – 4a 22 22)()(+−++− ycxycx Fig 10.25 2 2 2 2 1 x y a b + = Rationalised 2023-24 190 MATHEMATICS which on simplification gives x a c aycx −=+− 22)(Squaring again and simplifying, we get 22 2 2 2 ca y a x − + = 1 i.e., 2 2 2 2 b y a x + = 1 (Since c 2 = a 2 – b 2) Hence any point on the ellipse satisfies 2 2 2 2 b y a x + = 1. ... (2) Conversely, let P (x, y) satisfy the equation (2) with 0 < c < a. Then y 2 = b 2 ⎟ ⎟ ⎠ ⎞ ⎜ ⎜ ⎝ ⎛ − 2 2 1 a x Therefore, PF1 = 2 2 ( ) x c y + + = ⎟ ⎟ ⎠ ⎞ ⎜ ⎜ ⎝ ⎛ − ++ 2 22 22)(a xa bcx = 2 2 2 2 2 2 ( ) ( ) a x x c a c a ⎛ ⎞ − + + − ⎜ ⎟ ⎝ ⎠ (since b 2 = a 2 – c 2) = 2 cx c a a x a a ⎛ ⎞ + = + ⎜ ⎟ ⎝ ⎠ Similarly PF2 = c a x a − Hence PF1 + PF2 = 2 c c a x a – x a a a + + = ... (3) Rationalised 2023-24 CONIC SECTIONS 191 So, any point that satisfies 2 2 2 2 b y a x + = 1, satisfies the geometric condition and so P(x, y) lies on the ellipse. Hence from (2) and (3), we proved that the equation of an ellipse with centre of the origin and major axis along the x-axis is 2 2 2 2 x y a b + = 1. Discussion From the equation of the ellipse obtained above, it follows that for every point P (x, y) on the ellipse, we have 2 2 2 2 1 b y a x −= ≤ 1, i.e., x 2 ≤ a 2 , so – a ≤ x ≤ a. Therefore, the ellipse lies between the lines x = – a and x = a and touches these lines. Similarly, the ellipse lies between the lines y = – b and y = b and touches these lines. Similarly, we can derive the equation of the ellipse in Fig 10.24 (b) as 2 2 2 2 1 x y b a + = . These two equations are known as standard equations of the ellipses. ANote The standard equations of ellipses have centre at the origin and the major and minor axis are coordinate axes. However, the study of the ellipses with centre at any other point, and any line through the centre as major and the minor axes passing through the centre and perpendicular to major axis are beyond the scope here. From the standard equations of the ellipses (Fig10.24), we have the following observations: 1. Ellipse is symmetric with respect to both the coordinate axes since if (x, y) is a point on the ellipse, then (– x, y), (x, –y) and (– x, –y) are also points on the ellipse. 2. The foci always lie on the major axis. The major axis can be determined by finding the intercepts on the axes of symmetry. That is, major axis is along the x-axis if the coefficient of x 2 has the larger denominator and it is along the y-axis if the coefficient of y 2 has the larger denominator. Rationalised 2023-24 192 MATHEMATICS 10.5.4 Latus rectum Definition 6 Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse (Fig 10.26). To find the length of the latus rectum of the ellipse 1 x y a b 2 2 2 2 + = Let the length of AF2 be l. Then the coordinates of A are (c, l),i.e., (ae, l) Since A lies on the ellipse 2 2 2 2 1 x y a b + = , we have 2 2 2 2 ( ) 1 ae l a b + = ⇒ l2 = b 2 (1 – e 2) But 2 2 2 2 2 2 2 2 1 c a – b b e – a a a = = = Therefore l 2 = 4 2 b a , i.e., 2 b l a = Since the ellipse is symmetric with respect to y-axis (of course, it is symmetric w.r.t. both the coordinate axes), AF2 = F2B and so length of the latus rectum is 2 2b a . Example 9 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse 2 2 1 25 9 x y + = Solution Since denominator of 2 25 x is larger than the denominator of 2 9 y , the major Fig 10. 26 Rationalised 2023-24 CONIC SECTIONS 193 axis is along the x-axis. Comparing the given equation with 2 2 2 2 1 x y a b + = , we get a = 5 and b = 3. Also 2 2 c a – b – = = = 25 9 4 Therefore, the coordinates of the foci are (– 4,0) and (4,0), vertices are (– 5, 0) and (5, 0). Length of the major axis is 10 units length of the minor axis 2b is 6 units and the eccentricity is 4 5 and latus rectum is 2 2 18 5 b a = . Example 10 Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse 9x 2 + 4y 2 = 36. Solution The given equation of the ellipse can be written in standard form as 2 2 1 4 9 x y + = Since the denominator of 2 9 y is larger than the denominator of 2 4 x , the major axis is along the y-axis. Comparing the given equation with the standard equation 2 2 2 2 1 x y b a + = , we have b = 2 and a = 3. Also c = 2 2 a – b = 9 4 5 – = and 5 3 c e a = = Hence the foci are (0, 5) and (0, – 5), vertices are (0,3) and (0, –3), length of the major axis is 6 units, the length of the minor axis is 4 units and the eccentricity of the ellipse is 5 3 . Example 11 Find the equation of the ellipse whose vertices are (± 13, 0) and foci are (± 5, 0). Solution Since the vertices are on x-axis, the equation will be of the form 2 2 2 2 1 x y a b + = , where a is the semi-major axis. Rationalised 2023-24 194 MATHEMATICS Given that a = 13, c = ± 5. Therefore, from the relation c 2 = a 2 – b 2 , we get 25 = 169 – b 2 , i.e., b = 12 Hence the equation of the ellipse is 2 2 1 169 144 x y + = . Example 12 Find the equation of the ellipse, whose length of the major axis is 20 and foci are (0, ± 5). Solution Since the foci are on y-axis, the major axis is along the y-axis. So, equation of the ellipse is of the form 2 2 2 2 1 x y b a + = . Given that a = semi-major axis 20 10 2 = = and the relation c 2 = a2 – b 2 gives 5 2 = 102 – b 2 i.e., b 2 = 75 Therefore, the equation of the ellipse is 2 2 1 75 100 x y + = Example 13 Find the equation of the ellipse, with major axis along the x-axis and passing through the points (4, 3) and (– 1,4). Solution The standard form of the ellipse is 2 2 2 2 b y a x + = 1. Since the points (4, 3) and (–1, 4) lie on the ellipse, we have 1 916 22 =+ ba ... (1) and 22 161 ba + = 1 ….(2) Solving equations (1) and (2), we find that 2 247 7 a = and 2 247 15 b = . Hence the required equation is Rationalised 2023-24 CONIC SECTIONS 195 2 2 1 247 247 7 15 x y + = �����, i.e., 7x 2 + 15y 2 = 247. EXERCISE 10.3 In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse. 1. 2 2 1 36 16 x y + = 2. 2 2 1 4 25 x y + = 3. 2 2 1 16 9 x y + = 4. 2 2 1 25 100 x y + = 5. 2 2 1 49 36 x y + = 6. 400100 2 2 yx + = 1 7. 36x 2 + 4y 2 = 144 8. 16x 2 + y 2 = 16 9. 4x 2 + 9y 2 = 36 In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions: 10. Vertices (± 5, 0), foci (± 4, 0) 11. Vertices (0, ± 13), foci (0, ± 5) 12. Vertices (± 6, 0), foci (± 4, 0) 13. Ends of major axis (± 3, 0), ends of minor axis (0, ± 2) 14. Ends of major axis (0, ± 5), ends of minor axis (± 1, 0) 15. Length of major axis 26, foci (± 5, 0) 16. Length of minor axis 16, foci (0, ± 6). 17. Foci (± 3, 0), a = 4 18. b = 3, c = 4, centre at the origin; foci on the x axis. 19. Centre at (0,0), major axis on the y-axis and passes through the points (3, 2) and (1,6). 20. Major axis on the x-axis and passes through the points (4,3) and (6,2). 10.6 Hyperbola Definition 7 A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant. Rationalised 2023-24 196 MATHEMATICS The term "difference" that is used in the definition means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the centre of the hyperbola. The line through the foci is called the transverse axis and the line through the centre and perpendicular to the transverse axis is called the conjugate axis. The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola (Fig 10.27). We denote the distance between the two foci by 2c, the distance between two vertices (the length of the transverse axis) by 2a and we define the quantity b as b = 2 2 c – a Also 2b is the length of the conjugate axis (Fig 10.28). To find the constant P1 F2 – P1 F1 : By taking the point P at A and B in the Fig 10.28, we have BF1 – BF2 = AF2 – AF1 (by the definition of the hyperbola) BA +AF1 – BF2 = AB + BF2 – AF1 i.e., AF1 = BF2 So that, BF1 – BF2 = BA + AF1 – BF2 = BA = 2a Fig 10.27 Fig 10.28 Rationalised 2023-24 CONIC SECTIONS 197 10.6.1 Eccentricity Definition 8 Just like an ellipse, the ratio e = c a is called the eccentricity of the hyperbola. Since c ≥ a, the eccentricity is never less than one. In terms of the eccentricity, the foci are at a distance of ae from the centre. 10.6.2 Standard equation of Hyperbola The equation of a hyperbola is simplest if the centre of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig10.29. We will derive the equation for the hyperbola shown in Fig 10.29(a) with foci on the x-axis. Let F1 and F2 be the foci and O be the mid-point of the line segment F1 F2 . Let O be the origin and the line through O through F2 be the positive x-axis and that through F1 as the negative x-axis. The line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F1 be (– c,0) and F2 be (c,0) (Fig 10.30). Let P(x, y) be any point on the hyperbola such that the difference of the distances from P to the farther point minus the closer point be 2a. So given, PF1 – PF2 = 2a Fig 10.29 (a) (b) Fig 10.30 Rationalised 2023-24 198 MATHEMATICS Using the distance formula, we have 2 2 2 2 ( ) ( ) 2 x c y – x – c y a + + + = i.e., 2 2 2 2 ( ) 2 ( ) x c y a x – c y + + = + + Squaring both side, we get (x + c) 2 + y2 = 4a 2 + 4a 2 2 ( ) x – c y + + (x – c) 2 + y2 and on simplifying, we get a cx – a = 2 2 ( ) x – c y + On squaring again and further simplifying, we get 2 2 2 2 2 1 x y – a c – a = i.e., 2 2 2 2 1 x y – a b = (Since c 2 – a 2 = b 2) Hence any point on the hyperbola satisfies 2 2 2 2 1 x y – a b = 1. Conversely, let P(x, y) satisfy the above equation with 0 < a < c. Then y 2 = b 2 2 2 2 x – a a 󰀨 󰀩 󰀨 󰀩 󰀪 󰀫 Therefore, PF1 = + 2 2 ( ) x c y + + = + 2 2 2 2 2 ( ) x – a x c b a 󰀨 󰀩 + + 󰀪 󰀫 󰀬 󰀭 = a + x a c Similarly, PF2 = a – a c x In hyperbola c > a; and since P is to the right of the line x = a, x > a, c a x > a.Therefore, a – c a x becomes negative. Thus, PF2 = c a x – a. Rationalised 2023-24 CONIC SECTIONS 199 Therefore PF1 – PF2 = a + c a x – cx a + a = 2a Also, note that if P is to the left of the line x = – a, then PF1 c – a x a 󰀨 󰀩 = + 󰀪 󰀫 󰀬 󰀭, PF2 = a – c x a . In that case P F2 – PF1 = 2a. So, any point that satisfies 2 2 2 2 1 x y – a b = , lies on the hyperbola. Thus, we proved that the equation of hyperbola with origin (0,0) and transverse axis along x-axis is 2 2 2 2 1 x y – a b = . ANote A hyperbola in which a = b is called an equilateral hyperbola. Discussion From the equation of the hyperbola we have obtained, it follows that, we have for every point (x, y) on the hyperbola, 2 2 2 2 1 x y a b = + ≥ 1. i.e, a x ≥ 1, i.e., x ≤ – a or x ≥ a. Therefore, no portion of the curve lies between the lines x = + a and x = – a, (i.e. no real intercept on the conjugate axis). Similarly, we can derive the equation of the hyperbola in Fig 11.31 (b) as 2 2 2 2 y x a b − = 1 These two equations are known as the standard equations of hyperbolas. ANote The standard equations of hyperbolas have transverse and conjugate axes as the coordinate axes and the centre at the origin. However, there are hyperbolas with any two perpendicular lines as transverse and conjugate axes, but the study of such cases will be dealt in higher classes. From the standard equations of hyperbolas (Fig10.27), we have the following observations: 1. Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then (– x, y), (x, – y) and (– x, – y) are also points on the hyperbola. Rationalised 2023-24 200 MATHEMATICS 2. The foci are always on the transverse axis. It is the positive term whose denominator gives the transverse axis. For example, 2 2 1 9 16 x y – = has transverse axis along x-axis of length 6, while 2 2 1 25 16 y x – = has transverse axis along y-axis of length 10. 10.6.3 Latus rectum Definition 9 Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola. As in ellipse, it is easy to show that the length of the latus rectum in hyperbola is 2 2b a . Example 14 Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas: (i) 2 2 1 9 16 x y – = , (ii) y 2 – 16x 2 = 16 Solution (i) Comparing the equation 2 2 1 9 16 x y – = with the standard equation 2 2 2 2 1 x y – a b = Here, a = 3, b = 4 and c = 2 2 a b + = + = 9 16 5 Therefore, the coordinates of the foci are (± 5, 0) and that of vertices are (± 3, 0).Also, The eccentricity e = 5 3 c a = . The latus rectum 2 2 32 3 b a = = (ii) Dividing the equation by 16 on both sides, we have 2 2 1 16 1 y x – = Comparing the equation with the standard equation 2 2 2 2 1 y x – a b = , we find that a = 4, b = 1 and 2 2 c a b = + =

+ = 16 1 17 . Rationalised 2023-24 CONIC SECTIONS 201 Therefore, the coordinates of the foci are (0, ± 17) and that of the vertices are (0, ± 4). Also, The eccentricity 17 4 c e a = = . The latus rectum 2 2 1 2 b a = = . Example 15 Find the equation of the hyperbola with foci (0, ± 3) and vertices (0, ± 11 2). Solution Since the foci is on y-axis, the equation of the hyperbola is of the form 2 2 2 2 1 y x – a b = Since vertices are (0, ± 11 2), a = 11 2 Also, since foci are (0, ± 3); c = 3 and b 2 = c 2 – a 2 = 25 4 . Therefore, the equation of the hyperbola is 2 2 11 25 4 4 y x – = 1, i.e., 100 y 2 – 44 x 2 = 275. Example 16 Find the equation of the hyperbola where foci are (0, ±12) and the length of the latus rectum is 36. Solution Since foci are (0, ± 12), it follows that c = 12. Length of the latus rectum = 36 2 2 = a b or b 2 = 18a Therefore c2 = a 2 + b 2 ; gives 144 = a 2 + 18a i.e., a 2 + 18a – 144 = 0, So a = – 24, 6. Since a cannot be negative, we take a = 6 and so b 2 = 108. Therefore, the equation of the required hyperbola is 2 2 1 36 108 y x – = , i.e., 3y 2 – x 2 = 108 Rationalised 2023-24 202 MATHEMATICS Fig 10.31 EXERCISE 10.4 In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas. 1. 2 2 1 16 9 x y – = 2. 2 2 1 9 27 y x – = 3. 9y 2 – 4x 2 = 36 4. 16x 2 – 9y 2 = 576 5. 5y 2 – 9x 2 = 36 6. 49y 2 – 16x 2 = 784. In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions. 7. Vertices (± 2, 0), foci (± 3, 0) 8. Vertices (0, ± 5), foci (0, ± 8) 9. Vertices (0, ± 3), foci (0, ± 5) 10. Foci (± 5, 0), the transverse axis is of length 8. 11. Foci (0, ±13), the conjugate axis is of length 24. 12. Foci (± 3 5 , 0), the latus rectum is of length 8. 13. Foci (± 4, 0), the latus rectum is of length 12 14. vertices (± 7,0), e = 3 4 . 15. Foci (0, ± 10), passing through (2,3) Miscellaneous Examples Example 17 The focus of a parabolic mirror as shown in Fig 10.31 is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB (Fig 10.31). Solution Since the distance from the focus to the vertex is 5 cm. We have, a = 5. If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is y 2 = 4 (5) x = 20 x Note that x = 45. Thus y 2 = 900 Therefore y = ± 30 Hence AB = 2y = 2 × 30 = 60 cm. Example 18 A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there Rationalised 2023-24 CONIC SECTIONS 203 is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm? Solution Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig 10.32. Fig 10.32 The equation of the parabola takes the form x 2 = 4ay. Since it passes through 3 6 100 , , we have (6)2 = 4a 3 100 , i.e., a = 36 100 12 × = 300 m Let AB be the deflection of the beam which is 1 100 m. Coordinates of B are (x, 2 100). Therefore x 2 = 4 × 300 × 2 100 = 24 i.e. x = 24 = 2 6 metres Example 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point P(x, y) is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse. Solution Let AB be the rod making an angle θ with OX as shown in Fig 10.33 and P (x, y) the point on it such that AP = 6 cm. Since AB = 15 cm, we have PB = 9 cm. From P draw PQ and PR perpendiculars on y-axis and x-axis, respectively. Fig 10.33 Rationalised 2023-24 204 MATHEMATICS From Δ PBQ, cos θ = 9 x From Δ PRA, sin θ = 6 y Since cos2 θ + sin2 θ = 1 2 2 1 9 6 x y + = or 2 2 1 81 36 x y + = Thus the locus of P is an ellipse. Miscellaneous Exercise on Chapter 10 1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus. 2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola? 3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle. 4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end. 5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis. 6.

Find the area of the triangle formed by the lines joining the vertex of the parabola x 2 = 12y to the ends of its latus rectum. 7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man. 8. An equilateral triangle is inscribed in the parabola y 2 = 4 ax, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle. Rationalised 2023-24 CONIC SECTIONS 205 Summary In this Chapter the following concepts and generalisations are studied. ÆA circle is the set of all points in a plane that are equidistant from a fixed point in the plane. ÆThe equation of a circle with centre (h, k) and the radius r is (x – h) 2 + (y – k) 2 = r 2 . ÆA parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane. ÆThe equation of the parabola with focus at (a, 0) a > 0 and directrix x = – a is y 2 = 4ax. ÆLatus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola. ÆLength of the latus rectum of the parabola y 2 = 4ax is 4a. ÆAn ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant. ÆThe equation of an ellipse with foci on the x-axis is 2 2 2 2 1 x y + = a b . ÆLatus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse. ÆLength of the latus rectum of the ellipse 2 2 2 2 + = 1 x y a b is 2 2b a . ÆThe eccentricity of an ellipse is the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse. ÆA hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant. ÆThe equation of a hyperbola with foci on the x-axis is : 2 2 2 2 1 x y a b − = Rationalised 2023-24 206 MATHEMATICS ÆLatus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola. ÆLength of the latus rectum of the hyperbola : 2 2 2 2 1 x y a b − = is : 2 2b a . ÆThe eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola. Historical Note Geometry is one of the most ancient branches of mathematics. The Greek geometers investigated the properties of many curves that have theoretical and practical importance. Euclid wrote his treatise on geometry around 300 B.C. He was the first who organised the geometric figures based on certain axioms suggested by physical considerations. Geometry as initially studied by the ancient Indians and Greeks, who made essentially no use of the process of algebra. The synthetic approach to the subject of geometry as given by Euclid and in Sulbasutras, etc., was continued for some 1300 years. In the 200 B.C., Apollonius wrote a book called 'The Conic' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries. Modern analytic geometry is called 'Cartesian' after the name of Rene Descartes (1596-1650) whose relevant 'La Geometrie' was published in 1637. But the fundamental principle and method of analytical geometry were already discovered by Pierre de Fermat (1601-1665). Unfortunately, Fermats treatise on the subject, entitled Ad Locus Planos et So LIDOS Isagoge (Introduction to Plane and Solid Loci) was published only posthumously in 1679. So, Descartes came to be regarded as the unique inventor of the analytical geometry. Isaac Barrow avoided using cartesian method. Newton used method of undetermined coefficients to find equations of curves. He used several types of coordinates including polar and bipolar. Leibnitz used the terms 'abscissa', 'ordinate' and 'coordinate'. L' Hospital (about 1700) wrote an important textbook on analytical geometry. Clairaut (1729) was the first to give the distance formula although in clumsy form. He also gave the intercept form of the linear equation. Cramer (1750) Rationalised 2023-24 CONIC SECTIONS 207 made formal use of the two axes and gave the equation of a circle as (y – a) 2 + (b – x) 2 = r He gave the best exposition of the analytical geometry of his time. Monge (1781) gave the modern 'point-slope' form of equation of a line as y – y' = a (x – x') and the condition of perpendicularity of two lines as aa' + 1 = 0. S.F. Lacroix (1765–1843) was a prolific textbook writer, but his contributions to analytical geometry are found scattered. He gave the ‘two-point’ form of equation of a line as β β β = (α) α α – y – x – – ’ ’ and the length of the perpendicular from (α, β) on y = ax + b as 2 (β) 1 – a – b + a . His formula for finding angle between two lines was tan θ 1 a – a aa   ’ =   + ’   . It is, of course, surprising that one has to wait for more than 150 years after the invention of analytical geometry before finding such essential basic formula. In 1818, C. Lame, a civil engineer, gave mE + m’E’ = 0 as the curve passing through the points of intersection of two loci E = 0 and E’ = 0. Many important discoveries, both in Mathematics and Science, have been linked to the conic sections. The Greeks particularly Archimedes (287–212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles. — v — Rationalised 2023-24208 MATHEMATICS vMathematics is both the queen and the hand-maiden of all sciences – E.T. BELLv 11.1 Introduction You may recall that to locate the position of a point in a plane, we need two intersecting mutually perpendicular lines in the plane. These lines are called the coordinate axes and the two numbers are called the coordinates of the point with respect to the axes. In actual life, we do not have to deal with points lying in a plane only. For example, consider the position of a ball thrown in space at different points of time or the position of an aeroplane as it flies from one place to another at different times during its flight. Similarly, if we were to locate the position of the lowest tip of an electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, we will not only require the perpendicular distances of the point to be located from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. The three numbers representing the three distances are called the coordinates of the point with reference to the three coordinate planes. So, a point in space has three coordinates. In this Chapter, we shall study the basic concepts of geometry in three dimensional space.* * For various activities in three dimensional geometry one may refer to the Book, “A Hand Book for designing Mathematics Laboratory in Schools”, NCERT, 2005. Leonhard Euler (1707-1783) Chapter 11 INTRODUCTION TO THREE DIMENSIONAL GEOMETRY Rationalised 2023-24 INTRODUCTION TO THREE DIMENSIONAL GEOMETRY 209 11.2 Coordinate Axes and Coordinate Planes in Three Dimensional Space Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other (Fig 11.1). These three planes intersect along the lines X’OX, Y’OY and Z’OZ, called the x, y and z-axes, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the rectangular coordinate system. The planes XOY, YOZ and ZOX, called, respectively the XY-plane, YZ-plane and the ZX-plane, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line Z’OZ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line Z’OZ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ’ are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY’ as negative, in front of the YZ-plane along OX as positive and to the back of it along OX’ as negative. The point O is called the origin of the coordinate system. The three coordinate planes divide the space into eight parts known as octants. These octants could be named as XOYZ, X’OYZ, X’OY’Z, XOY’Z, XOYZ’, X’OYZ’, X’OY’Z’ and XOY’Z’. and denoted by I, II, III, ..., VIII , respectively. 11.3 Coordinates of a Point in Space Having chosen a fixed coordinate system in the space, consisting of coordinate axes, coordinate planes and the origin, we now explain, as to how, given a point in the space, we associate with it three coordinates (x,y,z) and conversely, given a triplet of three numbers (x, y, z), how, we locate a point in the space. Given a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular (Fig 11.2). Then, from the point M, we draw a perpendicular ML to the x-axis, meeting it at L. Let OL be x, LM be y and MP be z. Then x,y and z are called the x, y and z coordinates, respectively, of the point P in the space. In Fig 11.2, we may note that the point P (x, y, z) lies in the octant XOYZ and so all x, y, z are positive. If P was in any other octant, the signs of x, y and z would change Fig 11.1 Fig 11.2 Rationalised 2023-24 210 MATHEMATICS accordingly. Thus, to each point P in the space there corresponds an ordered triplet (x, y, z) of real numbers. Conversely, given any triplet (x, y, z), we would first fix the point L on the x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the coordinates of the point M in the XY-plane. Note that LM is perpendicular to the x-axis or is parallel to the y-axis. Having reached the point M, we draw a perpendicular MP to the XY-plane and locate on it the point P corresponding to z. The point P so obtained has then the coordinates (x, y, z). Thus, there is a one to one correspondence between the points in space and ordered triplet (x, y, z) of real numbers. Alternatively, through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the x-axis, y-axis and z-axis in the points A, B and C, respectively (Fig 11.3). Let OA = x, OB = y and OC = z. Then, the point P will have the coordinates x, y and z and we write P (x, y, z). Conversely, given x, y and z, we locate the three points A, B and C on the three coordinate axes. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of interesection of these three planes, namely, ADPF, BDPE and CEPF is obviously the point P, corresponding to the ordered triplet (x, y, z). We observe that if P (x, y, z) is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively. ANote The coordinates of the origin O are (0,0,0). The coordinates of any point on the x-axis will be as (x,0,0) and the coordinates of any point in the YZ-plane will be as (0, y, z). Remark The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants. Table 11.1 Fig 11.3 I II III IV V VI VII VIII x + – – + + – – + y + + – – + + – – z + + + + – – – – Octants Coordinates Rationalised 2023-24 INTRODUCTION TO THREE DIMENSIONAL GEOMETRY 211 Example 1 In Fig 11.3, if P is (2,4,5), find the coordinates of F. Solution For the point F, the distance measured along OY is zero. Therefore, the coordinates of F are (2,0,5). Example 2 Find the octant in which the points (–3,1,2) and (–3,1,– 2) lie. Solution From the Table 11.1, the point (–3,1, 2) lies in second octant and the point (–3, 1, – 2) lies in octant VI. EXERCISE 11.1 1. A point is on the x-axis. What are its y-coordinate and z-coordinates? 2. A point is in the XZ-plane. What can you say about its y-coordinate? 3. Name the octants in which the following points lie: (1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (– 4, 2, 5), (–3, –1, 6) (– 2, – 4, –7). 4. Fill in the blanks: (i) The x-axis and y-axis taken together determine a plane known as_______. (ii) The coordinates of points in the XY-plane are of the form _______. (iii) Coordinate planes divide the space into ______ octants. 11.4 Distance between Two Points We have studied about the distance between two points in two-dimensional coordinate system. Let us now extend this study to three-dimensional system. Let P(x1 , y1 , z 1) and Q (x2 , y2 , z 2) be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal PQ (Fig 11.4). Now, since ∠PAQ is a right angle, it follows that, in triangle PAQ, PQ2 = PA2 + AQ2 ... (1) Also, triangle ANQ is right angle triangle with ∠ANQ a right angle. Fig 11.4 Rationalised 2023-24 212 MATHEMATICS Therefore AQ2 = AN2 + NQ2 ... (2) From (1) and (2), we have PQ2 = PA2 + AN2 + NQ2 Now PA = y2 – y1 , AN = x2 – x1 and NQ = z 2 – z 1 Hence PQ2 = (x2 – x1) 2 + (y2 – y1) 2 + (z 2 – z 1) 2 Therefore PQ = 2 12 2 12 2 12 −+−+− zzyyxx)()()(This gives us the distance between two points (x1 , y1 , z 1) and (x2 , y2 , z 2). In particular, if x1 = y1 = z 1 = 0, i.e., point P is origin O, then OQ = 2 2 2 2 2 2 ++ zyx , which gives the distance between the origin O and any point Q (x2 , y2 , z 2). Example 3 Find the distance between the points P(1, –3, 4) and Q (– 4, 1, 2). Solution The distance PQ between the points P (1,–3, 4) and Q (– 4, 1, 2) is PQ = 2 2 2 −+++−−)42()31()14(= ++ 41625 = 45 = 3 5 units Example 4 Show that the points P (–2, 3, 5), Q (1, 2, 3) and R (7, 0, –1) are collinear. Solution We know that points are said to be collinear if they lie on a line. Now, PQ =

14419)53()32()21(2 2 2 =++=−+−++ QR = 1425616436)31()20()17(2 2 2 ==++=−−+−+− and PR = 14312636981)51()30()27(2 2 2 ==++=−−+−++ Thus, PQ + QR = PR. Hence, P, Q and R are collinear. Example 5 Are the points A (3, 6, 9), B (10, 20, 30) and C (25, – 41, 5), the vertices of a right angled triangle? Solution By the distance formula, we have AB2 = (10 – 3)2 + (20 – 6)2 + (30 – 9)2 = 49 + 196 + 441 = 686 BC2 = (25 – 10)2 + (– 41 – 20)2 + (5 – 30)2 = 225 + 3721 + 625 = 4571 Rationalised 2023-24 INTRODUCTION TO THREE DIMENSIONAL GEOMETRY 213 CA2 = (3 – 25)2 + (6 + 41)2 + (9 – 5)2 = 484 + 2209 + 16 = 2709 We find that CA2 + AB2 ≠ BC2 . Hence, the triangle ABC is not a right angled triangle. Example 6 Find the equation of set of points P such that PA2 + PB2 = 2k 2 , where A and B are the points (3, 4, 5) and (–1, 3, –7), respectively. Solution Let the coordinates of point P be (x, y, z). Here PA2 = (x – 3)2 + (y – 4)2 + (z – 5)2 PB2 = (x + 1)2 + (y – 3)2 + (z + 7)2 By the given condition PA2 + PB2 = 2k 2 , we have (x – 3)2 + (y – 4)2 + (z – 5)2 + (x + 1)2 + (y – 3)2 + (z + 7)2 = 2k 2 i.e., 2x 2 + 2y 2 + 2z 2 – 4x – 14y + 4z = 2k 2 – 109. EXERCISE 11.2 1. Find the distance between the following pairs of points: (i) (2, 3, 5) and (4, 3, 1) (ii) (–3, 7, 2) and (2, 4, –1) (iii) (–1, 3, – 4) and (1, –3, 4) (iv) (2, –1, 3) and (–2, 1, 3). 2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear. 3. Verify the following: (i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle. (ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle. (iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram. 4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1). 5. Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10. Miscellaneous Examples Example 7 Show that the points A (1, 2, 3), B (–1, –2, –1), C (2, 3, 2) and D (4, 7, 6) are the vertices of a parallelogram ABCD, but it is not a rectangle. Solution To show ABCD is a parallelogram we need to show opposite side are equal Note that. Rationalised 2023-24 214 MATHEMATICS AB = 2 2 2 −−+−−+−−)31()22()11(= 4 16 16 + + = 6 BC = 2 2 2 +++++)12()23()12(= ++ 9259 = 43 CD = 2 2 2 −+−+−)26()37()24(= =++ 616164 DA = 2 2 2 −+−+−)63()72()41(= =++ 439259 Since AB = CD and BC = AD, ABCD is a parallelogram. Now, it is required to prove that ABCD is not a rectangle. For this, we show that diagonals AC and BD are unequal. We have AC = 3111)32()23()12(2 2 2 =++=−+−+− BD = 155498125)16()27()14(2 2 2 =++=+++++ . Since AC ≠ BD, ABCD is not a rectangle. ANote We can also show that ABCD is a parallelogram, using the property that diagonals AC and BD bisect each other. Example 8 Find the equation of the set of the points P such that its distances from the points A (3, 4, –5) and B (– 2, 1, 4) are equal. Solution If P (x, y, z) be any point such that PA = PB. Now 2 2 2 2 2 2 zyx zyx −+−++=++−+−)4()1()2()5()4()3(or 2 2 2 2 2 2 zyx zyx −+−++=++−+−)4()1()2()5()4()3(or 10 x+ 6y – 18z – 29 = 0. Example 9 The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, – 6), respectively, find the coordinates of the point C. Solution Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then Rationalised 2023-24 INTRODUCTION TO THREE DIMENSIONAL GEOMETRY 215 x + − = 3 1 3 1, i.e., x = 1; y − + = 5 7 3 1, i.e., y = 1; z + − = 7 6 3 1, i.e., z = 2. Hence, coordinates of C are (1, 1, 2). Miscellaneous Exercise on Chapter 11 1. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex. 2. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0,4, 0) and (6, 0, 0). 3. If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (– 4, 3b, –10) and R(8, 14, 2c), then find the values of a, b and c. 4. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that PA2 + PB2 = k 2 , where k is a constant. Summary ÆIn three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z-axes. ÆThe three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX-planes. ÆThe three coordinate planes divide the space into eight parts known as octants. ÆThe coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the YZ, ZX and XY-planes. Æ (i) Any point on x-axis is of the form (x, 0, 0) (ii) Any point on y-axis is of the form (0, y, 0) (iii) Any point on z-axis is of the form (0, 0, z). ÆDistance between two points P(x1 , y1 , z 1) and Q (x2 , y2 , z 2) is given by 2 2 2 PQ 2 1 2 1 2 1 = − + − + − (x x) (y y) (z z) Rationalised 2023-24 216 MATHEMATICS — v — Historical Note Rene' Descartes (1596–1650), the father of analytical geometry, essentially dealt with plane geometry only in 1637. The same is true of his co-inventor Pierre Fermat (1601-1665) and La Hire (1640-1718). Although suggestions for the three dimensional coordinate geometry can be found in their works but no details. Descartes had the idea of coordinates in three dimensions but did not develop it. J.Bernoulli (1667-1748) in a letter of 1715 to Leibnitz introduced the three coordinate planes which we use today. It was Antoinne Parent (1666-1716), who gave a systematic development of analytical solid geometry for the first time in a paper presented to the French Academy in 1700. L.Euler (1707-1783) took up systematically the three dimensional coordinate geometry, in Chapter 5 of the appendix to the second volume of his "Introduction to Geometry" in 1748. It was not until the middle of the nineteenth century that geometry was extended to more than three dimensions, the well-known application of which is in the Space-Time Continuum of Einstein's Theory of Relativity. Rationalised 2023-24vWith the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature – WHITEHEAD v 12.1 Introduction This chapter is an introduction to Calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change. First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits. Then we come back to a definition of derivative and study some algebra of derivatives. We also obtain derivatives of certain standard functions. 12.2 Intuitive Idea of Derivatives Physical experiments have confirmed that the body dropped from a tall cliff covers a distance of 4.9t 2 metres in t seconds, i.e., distance s in metres covered by the body as a function of time t in seconds is given by s = 4.9t 2 . The adjoining Table 13.1 gives the distance travelled in metres at various intervals of time in seconds of a body dropped from a tall cliff. The objective is to find the veloctiy of the body at time t = 2 seconds from this data. One way to approach this problem is to find the average velocity for various intervals of time ending at t = 2 seconds and hope that these throw some light on the velocity at t = 2 seconds. Average velocity between t = t 1 and t = t2 equals distance travelled between t = t1 and t = t2 seconds divided by (t 2 – t 1). Hence the average velocity in the first two seconds Chapter 12 LIMITS AND DERIVATIVES Sir Issac Newton (1642-1727) Rationalised 2023-24 218 MATHEMATICS = 2 1 2 1 Distance travelled between 2 0 Time interval ( ) t and t t t = = − = ( ) ( ) 19.6 0 9.8 / 2 0 m m s s − = − . Similarly, the average velocity between t = 1 and t = 2 is ( ) ( ) 19.6 – 4.9 2 1 m − s = 14.7 m/s Likewise we compute the average velocitiy between t = t 1 and t = 2 for various t 1 . The following Table 13.2 gives the average velocity (v), t = t 1 seconds and t = 2 seconds. Table 12.2 t 1 0 1 1.5 1.8 1.9 1.95 1.99 v 9.8 14.7 17.15 18.62 19.11 19.355 19.551 From Table 12.2, we observe that the average velocity is gradually increasing. As we make the time intervals ending at t = 2 smaller, we see that we get a better idea of the velocity at t = 2. Hoping that nothing really dramatic happens between 1.99 seconds and 2 seconds, we conclude that the average velocity at t = 2 seconds is just above 19.551m/s. This conclusion is somewhat strengthened by the following set of computation. Compute the average velocities for various time intervals starting at t = 2 seconds. As before the average velocity v between t = 2 seconds and t = t 2 seconds is = 2 2 Distance travelled between 2 seconds and seconds 2 t t − = 2 2 Distance travelled in seconds Distance travelled in 2 seconds 2 t t − − t s 0 0 1 4.9 1.5 11.025 1.8 15.876 1.9 17.689 1.95 18.63225 2 19.6 2.05 20.59225 2.1 21.609 2.2 23.716 2.5 30.625 3 44.1 4 78.4 Table 12.1 Rationalised 2023-24 LIMITS AND DERIVATIVES 219 = 2 2 Distance travelled in seconds 19.6 2 t t − − The following Table 12.3 gives the average velocity v in metres per second between t = 2 seconds and t 2 seconds. Table 12.3 t 2 4 3 2.5 2.2 2.1 2.05 2.01 v 29.4 24.5 22.05 20.58 20.09 19.845 19.649 Here again we note that if we take smaller time intervals starting at t = 2, we get better idea of the velocity at t = 2. In the first set of computations, what we have done is to find average velocities in increasing time intervals ending at t

= 2 and then hope that nothing dramatic happens just before t = 2. In the second set of computations, we have found the average velocities decreasing in time intervals ending at t = 2 and then hope that nothing dramatic happens just after t = 2. Purely on the physical grounds, both these sequences of average velocities must approach a common limit. We can safely conclude that the velocity of the body at t = 2 is between 19.551m/s and 19.649 m/s. Technically, we say that the instantaneous velocity at t = 2 is between 19.551 m/s and 19.649 m/s. As is well-known, velocity is the rate of change of displacement. Hence what we have accomplished is the following. From the given data of distance covered at various time instants we have estimated the rate of change of the distance at a given instant of time. We say that the derivative of the distance function s = 4.9t 2 at t = 2 is between 19.551 and 19.649. An alternate way of viewing this limiting process is shown in Fig 12.1. This is a plot of distance s of the body from the top of the cliff versus the time t elapsed. In the limit as the sequence of time intervals h1 , h2 , ..., approaches zero, the sequence of average velocities approaches the same limit as does the sequence of ratios Fig 12.1 Rationalised 2023-24 220 MATHEMATICS 1 1 2 2 3 3 1 2 3 C B C B C B , , AC AC AC , ... where C1B1 = s 1 – s 0 is the distance travelled by the body in the time interval h1 = AC1 , etc. From the Fig 12.1 it is safe to conclude that this latter sequence approaches the slope of the tangent to the curve at point A. In other words, the instantaneous velocity v(t) of a body at time t = 2 is equal to the slope of the tangent of the curve s = 4.9t 2 at t = 2. 12.3 Limits The above discussion clearly points towards the fact that we need to understand limiting process in greater clarity. We study a few illustrative examples to gain some familiarity with the concept of limits. Consider the function f(x) = x 2 . Observe that as x takes values very close to 0, the value of f(x) also moves towards 0 (See Fig 2.10 Chapter 2). We say ( ) 0 lim 0 x f x → = (to be read as limit of f(x) as x tends to zero equals zero). The limit of f(x) as x tends to zero is to be thought of as the value f (x) should assume at x = 0. In general as x → a, f (x) → l, then l is called limit of the function f (x) which is symbolically written as lim ( ) x a f x l → = . Consider the following function g(x) = |x|, x ≠ 0. Observe that g(0) is not defined. Computing the value of g(x) for values of x very near to 0, we see that the value of g(x) moves towards 0. So, 0 lim x→ g(x) = 0. This is intuitively clear from the graph of y = |x| for x ≠ 0. (See Fig 2.13, Chapter 2). Consider the following function. ( ) 2 4 , 2 2 x h x x x − = ≠ − . Compute the value of h(x) for values of x very near to 2 (but not at 2). Convince yourself that all these values are near to 4. This is somewhat strengthened by considering the graph of the function y = h(x) given here (Fig 12.2). Fig 12.2 Rationalised 2023-24 LIMITS AND DERIVATIVES 221 In all these illustrations the value which the function should assume at a given point x = a did not really depend on how is x tending to a. Note that there are essentially two ways x could approach a number a either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a. This naturally leads to two limits – the right hand limit and the left hand limit. Right hand limit of a function f(x) is that value of f(x) which is dictated by the values of f(x) when x tends to a from the right. Similarly, the left hand limit. To illustrate this, consider the function ( ) 1, 0 2, 0 x f x x □ ≤ = □ □ > Graph of this function is shown in the Fig 12.3. It is clear that the value of f at 0 dictated by values of f(x) with x ≤ 0 equals 1, i.e., the left hand limit of f (x) at 0 is 0 lim ( ) 1 x f x → = . Similarly, the value of f at 0 dictated by values of f (x) with x > 0 equals 2, i.e., the right hand limit of f (x) at 0 is 0 lim ( ) 2 x f x → + = . In this case the right and left hand limits are different, and hence we say that the limit of f (x) as x tends to zero does not exist (even though the function is defined at 0). Summary We say lim x a → – f(x) is the expected value of f at x = a given the values of f near x to the left of a. This value is called the left hand limit of f at a. We say lim ( ) x a f x → + is the expected value of f at x = a given the values of f near x to the right of a. This value is called the right hand limit of f(x) at a. If the right and left hand limits coincide, we call that common value as the limit of f(x) at x = a and denote it by lim x a → f(x). Illustration 1 Consider the function f(x) = x + 10. We want to find the limit of this function at x = 5. Let us compute the value of the function f(x) for x very near to 5. Some of the points near and to the left of 5 are 4.9, 4.95, 4.99, 4.995. . ., etc. Values of the function at these points are tabulated below. Similarly, the real number 5.001, Fig 12.3 Rationalised 2023-24 222 MATHEMATICS 5.01, 5.1 are also points near and to the right of 5. Values of the function at these points are also given in the Table 12.4. Table 12.4 From the Table 12.4, we deduce that value of f(x) at x = 5 should be greater than 14.995 and less than 15.001 assuming nothing dramatic happens between x = 4.995 and 5.001. It is reasonable to assume that the value of the f(x) at x = 5 as dictated by the numbers to the left of 5 is 15, i.e., ( ) – 5 lim 15 x f x → = . Similarly, when x approaches 5 from the right, f(x) should be taking value 15, i.e., ( ) 5 lim 15 x f x → + = . Hence, it is likely that the left hand limit of f(x) and the right hand limit of f(x) are both equal to 15. Thus, ( ) ( ) ( ) 5 5 5 lim lim lim 15 x x x f x f x f x → → − + → = = = . This conclusion about the limit being equal to 15 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.16, Chapter 2. In this figure, we note that as x approaches 5 from either right or left, the graph of the function f(x) = x +10 approaches the point (5, 15). We observe that the value of the function at x = 5 also happens to be equal to 15. Illustration 2 Consider the function f(x) = x3 . Let us try to find the limit of this function at x = 1. Proceeding as in the previous case, we tabulate the value of f(x) at x near 1. This is given in the Table 12.5. Table 12.5 From this table, we deduce that value of f(x) at x = 1 should be greater than 0.997002999 and less than 1.003003001 assuming nothing dramatic happens between x 0.9 0.99 0.999 1.001 1.01 1.1 f(x) 0.729 0.970299 0.997002999 1.003003001 1.030301 1.331 x 4.9 4.95 4.99 4.995 5.001 5.01 5.1 f(x) 14.9 14.95 14.99 14.995 15.001 15.01 15.1 Rationalised 2023-24 LIMITS AND DERIVATIVES 223 x = 0.999 and 1.001. It is reasonable to assume that the value of the f(x) at x = 1 as dictated by the numbers to the left of 1 is 1, i.e., ( ) 1 lim 1 x f x → − = . Similarly, when x approaches 1 from the right, f(x)should be taking value 1, i.e., ( ) 1 lim 1 x f x → + = . Hence, it is likely that the left hand limit of f(x) and the right hand limit of f(x) are both equal to 1. Thus, ( ) ( ) ( ) 1 1 1 lim lim lim 1 x x x f x f x f x → → − + → = = = . This conclusion about the limit being equal to 1 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.11, Chapter 2. In this figure, we note that as x approaches 1 from either right or left, the graph of the function f(x) = x3 approaches the point (1, 1). We observe, again, that the value of the function at x = 1 also happens to be equal to 1. Illustration 3 Consider the function f(x) = 3x. Let us try to find the limit of this function at x = 2. The following Table 12.6 is now self-explanatory. Table 12.6 x 1.9 1.95 1.99 1.999 2.001 2.01 2.1 f(x) 5.7 5.85 5.97 5.997 6.003 6.03 6.3 As before we observe that as x approaches 2 from either left or right, the value of f(x) seem to approach 6. We record this as ( ) ( ) ( ) 2 2 2 lim lim lim 6 x x x f x f x f x → → − + → = = = Its graph shown in Fig 12.4 strengthens this fact. Here again we note that the value of the function at x = 2 coincides with the limit at x = 2. Illustration 4 Consider the constant function f(x) = 3. Let us try to find its limit at x = 2. This function being the constant function takes the same Fig 12.4 Rationalised 2023-24 224 MATHEMATICS value (3, in this case) everywhere, i.e., its value at points close to 2 is 3. Hence ( ) ( ) ( ) 2 2 2 lim lim lim 3 x x x f x f x f x → → + → = = = Graph of f(x) = 3 is anyway the line parallel to x-axis passing through (0, 3) and is shown in Fig 2.9, Chapter 2. From this also it is clear that the required limit is 3. In fact, it is easily observed that lim 3 ( ) x a f x → = for any real number a. Illustration 5 Consider the function f(x) = x2 + x. We want to find ( ) 1 lim x f x → . We tabulate the values of f(x) near x = 1 in Table 12.7. Table 12.7 x 0.9 0.99 0.999 1.01 1.1 1.2 f(x) 1.71 1.9701 1.997001 2.0301 2.31 2.64 From this it is reasonable to deduce that ( ) ( ) ( ) 1 1 1 lim lim lim 2 x x x f x f x f x → → − + → = = = . From the graph of f(x) = x2 + x shown in the Fig 12.5, it is clear that as x approaches 1, the graph approaches (1, 2). Here, again we observe that the 1 lim x→ f (x) = f (1) Now, convince yourself of the following three facts: 2 1 1 1 lim 1, lim 1 and lim 1 2 x x x x x x → → → = = + = Then 2 2 1 1 1 lim lim 1 1 2 lim x x x x x x x → → → + = + = = + � � � � . Also ( ) ( ) 2 1 1 1 1 lim . lim 1 1.2 2 lim 1 lim x x x x x x x x x x → → → → + = = = + = + � � � � � � � � . Fig 12.5 Rationalised 2023-24 LIMITS AND DERIVATIVES 225 Illustration 6 Consider the function f(x) = sin x. We are interested in 2 lim sin x x π → , where the angle is measured in radians. Here, we tabulate the (approximate) value of f(x) near 2 π (Table 12.8). From this, we may deduce that ( ) ( ) ( ) 2 2 2 lim lim

lim 1 x x x f x f x f x π − π + π → → → = = = . Further, this is supported by the graph of f(x) = sin x which is given in the Fig 3.8 (Chapter 3). In this case too, we observe that 2 lim x π → sin x = 1. Table 12.8 x 0.1 2 π − 0.01 2 π − 0.01 2 π + 0.1 2 π + f(x) 0.9950 0.9999 0.9999 0.9950 Illustration 7 Consider the function f(x) = x + cos x. We want to find the 0 lim x→ f (x). Here we tabulate the (approximate) value of f(x) near 0 (Table 12.9). Table 12.9 From the Table 13.9, we may deduce that ( ) ( ) ( ) 0 0 0 lim lim lim 1 x x x f x f x f x → → − + → = = = In this case too, we observe that 0 lim x→ f (x) = f (0) = 1. Now, can you convince yourself that [ ] 0 0 0 lim cos lim lim cos x x x x x x x → → → + = + is indeed true? x – 0.1 – 0.01 – 0.001 0.001 0.01 0.1 f(x) 0.9850 0.98995 0.9989995 1.0009995 1.00995 1.0950 Rationalised 2023-24 226 MATHEMATICS Illustration 8 Consider the function ( ) 2 1 f x x = for x > 0 . We want to know 0 lim x→ f (x). Here, observe that the domain of the function is given to be all positive real numbers. Hence, when we tabulate the values of f(x), it does not make sense to talk of x approaching 0 from the left. Below we tabulate the values of the function for positive x close to 0 (in this table n denotes any positive integer). From the Table 12.10 given below, we see that as x tends to 0, f(x) becomes larger and larger. What we mean here is that the value of f(x) may be made larger than any given number. Table 12.10 x 1 0.1 0.01 10–n f(x) 1 100 10000 102n Mathematically, we say ( ) 0 lim x f x → = +∞ We also remark that we will not come across such limits in this course. Illustration 9 We want to find ( ) 0 lim x f x → , where ( ) 2, 0 0 , 0 2, 0 x x f x x x x − < = =   + > As usual we make a table of x near 0 with f(x). Observe that for negative values of x we need to evaluate x – 2 and for positive values, we need to evaluate x + 2. Table 12.11 From the first three entries of the Table 12.11, we deduce that the value of the function is decreasing to –2 and hence. ( ) 0 lim 2 x f x → − = − x – 0.1 – 0.01 – 0.001 0.001 0.01 0.1 f(x) – 2.1 – 2.01 – 2.001 2.001 2.01 2.1 Rationalised 2023-24 LIMITS AND DERIVATIVES 227 From the last three entires of the table we deduce that the value of the function is increasing from 2 and hence ( ) 0 lim 2 x f x → + = Since the left and right hand limits at 0 do not coincide, we say that the limit of the function at 0 does not exist. Graph of this function is given in the Fig12.6. Here, we remark that the value of the function at x = 0 is well defined and is, indeed, equal to 0, but the limit of the function at x = 0 is not even defined. Illustration 10 As a final illustration, we find ( ) 1 lim x f x → , where ( ) 2 1 0 1 x x f x x + ≠ =   = Table 12.12 x 0.9 0.99 0.999 1.001 1.01 1.1 f(x) 2.9 2.99 2.999 3.001 3.01 3.1 As usual we tabulate the values of f(x) for x near 1. From the values of f(x) for x less than 1, it seems that the function should take value 3 at x = 1., i.e., ( ) 1 lim 3 x f x → − = . Similarly, the value of f(x)should be 3 as dictated by values of f(x) at x greater than 1. i.e. ( ) 1 lim 3 x f x → + = . But then the left and right hand limits coincide and hence ( ) ( ) ( ) 1 1 1 lim lim lim 3 x x x f x f x f x → → − + → = = = . Graph of function given in Fig 12.7 strengthens our deduction about the limit. Here, we Fig 12.6 Fig 12.7 Rationalised 2023-24 228 MATHEMATICS note that in general, at a given point the value of the function and its limit may be different (even when both are defined). 12.3.1 Algebra of limits In the above illustrations, we have observed that the limiting process respects addition, subtraction, multiplication and division as long as the limits and functions under consideration are well defined. This is not a coincidence. In fact, below we formalise these as a theorem without proof. Theorem 1 Let f and g be two functions such that both lim x a → f(x) and lim x a → g(x) exist. Then (i) Limit of sum of two functions is sum of the limits of the functions, i.e., lim x a → [f(x) + g (x)] = lim x a → f(x) + lim x a → g(x). (ii) Limit of difference of two functions is difference of the limits of the functions, i.e., lim x a → [f(x) – g(x)] = lim x a → f(x) – lim x a → g(x). (iii) Limit of product of two functions is product of the limits of the functions, i.e., lim x a → [f(x) . g(x)] = lim x a → f(x). lim x a → g(x). (iv) Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e., ( ) ( ) ( ) ( ) lim lim lim x a x a x a f x f x g x g x → → → = ANote In particular as a special case of (iii), when g is the constant function such that g(x) = λ , for some real number λ , we have lim . .lim ( ) ( ) ( ) x a x a f x f x → →   λ = λ   . In the next two subsections, we illustrate how to exploit this theorem to evaluate limits of special types of functions. 12.3.2 Limits of polynomials and rational functions A function f is said to be a polynomial function of degree n f(x) = a0 + a1 x + a2 x 2 +. . . + an x n , where ai s are real numbers such that an ≠ 0 for some natural number n. We know that lim x a → x = a. Hence Rationalised 2023-24 LIMITS AND DERIVATIVES 229 ( ) 2 2 lim lim . lim .lim . x a x a x a x a x x x x x x a a a → → → → = = = = An easy exercise in induction on n tells us that lim n n x a x a → = Now, let ( ) 2 0 1 2 ... n n f x a a x a x a x = + + + + be a polynomial function. Thinking of each of 2 0 1 2 , , ,..., n n a a x a x a x as a function, we have lim ( ) x a f x → = 2 0 1 2 lim ... n n x a a a x a x a x → □ + + + + □ □ □ = 2 0 1 2 lim lim lim ... lim n n x a x a x a x a a a x a x a x → → → → + + + + = 2 0 1 2 lim lim ... lim n n x a x a x a a a x a x a x → → → + + + + = 2 0 1 2 ... n n a a a a a a a a + + + + = f a( ) (Make sure that you understand the justification for each step in the above!) A function f is said to be a rational function, if f(x) = ( ) ( ) g x h x , where g(x) and h(x) are polynomials such that h(x) ≠ 0. Then ( ) ( ) ( ) ( ) ( ) ( ) ( ) lim lim lim lim x a x a x a x a g x g x g a f x h x h x h a → → → → = = = However, if h(a) = 0, there are two scenarios – (i) when g(a) ≠ 0 and (ii) when g(a) = 0. In the former case we say that the limit does not exist. In the latter case we can write g(x) = (x – a) k g1 (x), where k is the maximum of powers of (x – a) in g(x) Similarly, h(x) = (x – a) l h1 (x) as h (a) = 0. Now, if k > l, we have ( ) ( ) ( ) ( ) ( ) ( ) ( ) 1 1 lim lim lim lim lim k x a x a l x a x a x a g x x a g x f x h x x a h x → → → → → − = = − Rationalised 2023-24 230 MATHEMATICS = ( )( ) ( ) ( ) ( ) ( ) 1 1 1 1 lim 0. 0 lim k l x a x a x a g x g a h x h a − → → − = = If k < l, the limit is not defined. Example 1 Find the limits: (i) 3 2 1 lim 1 x x x → □ □ − + □ □ (ii) ( ) 3 lim 1 x x x → □ □ + □ □ (iii) 2 10 1 lim 1 ... x x x x →− □ + + + + □ □ □ . Solution The required limits are all limits of some polynomial functions. Hence the limits are the values of the function at the prescribed points. We have (i) 1 lim x→ [x 3 – x 2 + 1] = 13 – 12 + 1 = 1 (ii) ( ) ( ) ( ) 3 lim 1 3 3 1 3 4 12 x x x → □ □ + = + = = □ □ (iii) 2 10 1 lim 1 ... x x x x →− □ + + + + □ □ □ ( ) ( ) ( ) 2 10 = + − + − + + − 1 1 1 ... 1 = − + + = 1 1 1... 1 1. Example 2 Find the limits: (i) 2 1 1 lim x 100 x → x □ □ + □ □ + □ □ (ii) 3 2 2 2 4 4 lim x 4 x x x → x □ − + □ □ □ □ − □ (iii) 2 3 2 2 4 lim x 4 4 x → x x x □ − □ □ □ □ − + □ (iv) 3 2 2 2 2 lim x 5 6 x x → x x □ □ − □ □ □ □ − + (v) 2 3 2 1 2 1 lim x 3 2 x → x x x x x □ − □ − □ □ □ − − + □ . Solution All the functions under consideration are rational functions. Hence, we first evaluate these functions at the prescribed points. If this is of the form 0 0 , we try to rewrite the function cancelling the factors which are causing the limit to be of the form 0 0 . Rationalised 2023-24 LIMITS AND DERIVATIVES 231 (i) We have 2 2 1 1 1 1 2 lim x 100 1 100 101 x → x + + = = + + (ii) Evaluating the function at 2, it is of the form 0 0 . Hence 3 2 2 2 4 4 lim x 4 x x x → x − + − = ( ) ( )( ) 2 2 2 lim x 2 2 x x → x x − + − = ( ) ( ) 2 2 lim as 2 x 2 x x x → x − ≠ + = 2 2 2 ( ) 0 0 2 2 4 − = = + . (iii) Evaluating the function at 2, we get it of the form 0 0 . Hence 2 3 2 2 4 lim x 4 4 x → x x x − − + = ( )( ) ( )2 2 2 2 lim 2 x x x x x → + − − = ( ) ( ) ( ) 2 2 2 2 4 lim x 2 2 2 2 0 x → x x + + = = − − which is not defined. (iv) Evaluating the function at 2, we get it of the form 0 0 . Hence 3 2 2 2 2 lim x 5 6 x x → x x − − + = ( ) ( ) ( ) 2 2 2 lim x 2 3 x x → x x − − − = ( ) ( )2 2 2 2 4 lim 4 x 3 2 3 1 x → x = = = − − − − . Rationalised 2023-24 232 MATHEMATICS (v) First, we rewrite the function as a rational function. 2 3 2 2 1 3 2 x x x x x x □ − □ − □ □ □ − − + □ = ( ) ( ) 2 2 1 1 3 2 x x x x x x □ □ − □ − □ □ − − + □ □ □ = ( ) ( ) ( ) 2 1 1 1 2 x x x x x x □ − □ □ − □ □□ − − − □□ = ( ) ( ) 2 4 4 1 1 2 x x x x x x □ − + − □ □ □ □□ − − □□ = ( ) ( ) 2 4 3 1 2 x x x x x x − + − − Evaluating the function at 1, we get it of the form 0 0 . Hence 2 2 3 2 1 2 1 lim x 3 2 x → x x x x x □ − □ □ − □ □ − − + □ = ( )( ) 2 1 4 3 lim x 1 2 x x → x x x − + − − = ( )( ) ( ) ( ) 1 3 1 lim x 1 2 x x → x x x − − − − = ( ) 1 3 lim x 2 x → x x − − = ( ) 1 3 1 1 2 − − = 2. We remark that we could cancel the term (x – 1) in the above evaluation because x ≠ 1 . Evaluation of an important limit which will be used in the sequel is given as a theorem below. Theorem 2 For any positive integer n, 1 lim n n n x a x a na x a − → − = − . Remark The expression in the above theorem for the limit is true even if n is any rational number and a is positive. Rationalised 2023-24 LIMITS AND DERIVATIVES 233 Proof Dividing (x n – a n) by (x – a), we see that x n – a n = (x–a) (x n–1 + x n–2 a + x n–3 a 2 + ... + x an–2 + a n–1) Thus, lim lim n n x a x a x a → x a → − = − (x n–1 + x n–2 a + x n–3 a 2 + ... + x an–2 + a n–1) = a n – l + a an–2 +. . . + a n–2 (a) +an–l = a n–1 + a n – 1 +...+a n–1 + a n–1 (n terms) = n 1 na − Example 3 Evaluate: (i) 15 10 1 1 lim x 1 x → x − − (ii) 0 1 1 lim x x → x + − Solution (i) We have 15 10 1 1 lim x 1 x → x − − = 15 10 1 1 1 lim x 1

1 x x → x x − −   ÷ − −   = 15 10 1 1 1 1 lim lim x 1 x 1 x x → x → x   − −   ÷ − −   = 15 (1)14 ÷ 10(1)9 (by the theorem above) = 15 ÷ 10 3 2 = (ii) Put y = 1 + x, so that y →1 as x → 0. Then 0 1 1 lim x x → x + − = 1 1 lim y –1 y → y − = 1 1 2 2 1 1 lim y 1 y → y − − = 1 1 2 1 (1) 2 − (by the remark above) = 1 2 Rationalised 2023-24 234 MATHEMATICS 12.4 Limits of Trigonometric Functions The following facts (stated as theorems) about functions in general come in handy in calculating limits of some trigonometric functions. Theorem 3 Let f and g be two real valued functions with the same domain such that f (x) ≤ g(x) for all x in the domain of definition, For some a, if both lim x a → f(x) and lim x a → g(x) exist, then lim x a → f(x) ≤ lim x a → g(x). This is illustrated in Fig 12.8. Theorem 4 (Sandwich Theorem) Let f, g and h be real functions such that f(x) ≤ g(x) ≤ h(x) for all x in the common domain of definition. For some real number a, if lim x a → f(x) = l = lim x a → h(x), then lim x a → g(x) = l. This is illustrated in Fig 12.9. Given below is a beautiful geometric proof of the following important inequality relating trigonometric functions. sin cos 1 x x x < < for π 0 2 < < x (*) Fig 12.8 Fig 12.9 Rationalised 2023-24 LIMITS AND DERIVATIVES 235 Proof We know that sin (– x) = – sin x and cos(– x) = cos x. Hence, it is sufficient to prove the inequality for π 0 2 < < x < π 2 . Line segments B A and CD are perpendiculars to OA. Further, join AC. Then Area of ∆OAC < Area of sector OAC < Area of ∆ OAB. i.e., 1 2 1 OA.CD .π.(OA) OA.AB 2 2π 2 x < < . i.e., CD < x . OA < AB. From ∆ OCD, sin x = CD OA (since OC = OA) and hence CD = OA sin x. Also tan x = AB OA and hence AB = OA. tan x. Thus OA sin x < OA. x < OA. tan x. Since length OA is positive, we have sin x < x < tan x. Since 0 < x < π 2 , sinx is positive and thus by dividing throughout by sin x, we have 1< 1 sin cos x x x < . Taking reciprocals throughout, we have sin cos 1 x x x < < which complete the proof. Theorem 5 The following are two important limits. (i) 0 sin lim 1 x x → x = . (ii) 0 1 cos lim 0 x x → x − = . Proof (i) The inequality in (*) says that the function sin x x is sandwiched between the function cos x and the constant function which takes value 1. Fig 12.10 Rationalised 2023-24 236 MATHEMATICS Further, since 0 lim x→ cos x = 1, we see that the proof of (i) of the theorem is complete by sandwich theorem. To prove (ii), we recall the trigonometric identity 1 – cos x = 2 sin2 2   x   . Then 0 1 cos lim x x → x − = 2 0 0 2sin sin 2 2 lim lim .sin 2 2 x x x x x → x → x   =   = 0 0 sin 2 lim .limsin 1.0 0 2 2 x x x x → x →   = =   Observe that we have implicitly used the fact that x → 0 is equivalent to 0 2 x → . This may be justified by putting y = 2 x . Example 4 Evaluate: (i) 0 sin 4 lim x sin 2 x → x (ii) 0 tan lim x x → x Solution (i) 0 sin 4 lim x sin 2 x → x 0 sin 4 2 lim . .2 x 4 sin 2 x x → x x   =   = 0 sin 4 sin 2 2.lim x 4 2 x x → x x   ÷   = 4 0 2 0 sin 4 sin 2 2. lim lim x x 4 2 x x → x → x   ÷   = 2.1.1 = 2 (as x → 0, 4x → 0 and 2x → 0) Rationalised 2023-24 LIMITS AND DERIVATIVES 237 (ii) We have 0 tan lim x x → x = 0 sin lim x cos x → x x = 0 0 sin 1 lim . lim x x cos x → x x → = 1.1 = 1 A general rule that needs to be kept in mind while evaluating limits is the following. Say, given that the limit ( ) ( ) lim x a f x → g x exists and we want to evaluate this. First we check the value of f (a) and g(a). If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write f(x) = f 1 (x) f 2 (x) so that f 1 (a) = 0 and f 2 (a) ≠ 0. Similarly, we write g(x) = g1 (x) g2 (x), where g1 (a) = 0 and g2 (a) ≠ 0. Cancel out the common factors from f(x) and g(x) (if possible) and write ( ) ( ) ( ) ( ) f x p x g x q x = , where q(x) ≠ 0. Then ( ) ( ) ( ) ( ) lim x a f x p a → g x q a = . EXERCISE 12.1 Evaluate the following limits in Exercises 1 to 22. 1. 3 lim 3 x x → + 2. π 22 lim x 7 x →   −   3. 2 1 limπ r r → 4. 4 4 3 lim x 2 x → x + − 5. 10 5 1 1 lim x 1 x x → − x + + − 6. ( )5 0 1 1 lim x x → x + − 7. 2 2 2 3 10 lim x 4 x x → x − − − 8. 4 2 3 81 lim x 2 5 3 x → x x − − − 9. 0 lim x 1 ax b → cx + + 10. 1 3 1 1 6 1 lim 1 z z z → − − 11. 2 2 1 lim , 0 x ax bx c a b c → cx bx a + + + + ≠ + + 12. 2 1 1 2 lim x 2 x →− x + + 13. 0 sin lim x ax → bx 14. 0 sin lim , , 0 x sin ax a b → bx ≠ Rationalised 2023-24 238 MATHEMATICS 15. ( ) ( ) π sin π lim x π π x → x − − 16. 0 cos lim x π x → − x 17. 0 cos 2 1 lim x cos 1 x → x − − 18. 0 cos lim x sin ax x x → b x + 19. 0 lim sec x x x → 20. 0 sin lim , , 0 x sin ax bx a b a b → ax bx + + ≠ + , 21. 0 lim (cosec cot) x x x → − 22. π 2 tan 2 lim π 2 x x → x − 23. Find ( ) 0 lim x f x → and ( ) 1 lim x f x → , where ( ) ( ) 2 3, 0 3 1 , 0 x x f x x x + ≤ = + > 24. Find ( ) 1 lim x f x → , where ( ) 2 2 1, 1 1, 1 x x f x x x   − ≤ = − − > 25. Evaluate ( ) 0

lim x f x → , where ( ) | | , 0 0, 0 x x f x x x   ≠ =   = 26. Find ( ) 0 lim x f x → , where ( ) , 0 | | 0, 0 x x f x x x   ≠ =   = 27. Find ( ) 5 lim x f x → , where f x x ( ) = − | | 5 28. Suppose ( ) , 1 4, 1 , 1 a bx x f x x b ax x + < = = − > and if 1 lim x→ f (x) = f (1) what are possible values of a and b? Rationalised 2023-24 LIMITS AND DERIVATIVES 239 29. Let a1 , a2 , . . ., an be fixed real numbers and define a function f x x a x a x a ( ) = − − − (1 2) ( )...(n) . What is 1 lim x a → f (x) ? For some a ≠ a1 , a2 , ..., an , compute limx a → f (x). 30. If ( ) 1, 0 0, 0 1, 0 x x f x x x x + < = =   − > . For what value (s) of a does lim x a → f (x) exists? 31. If the function f(x) satisfies ( ) 2 1 2 lim π x 1 f x → x − = − , evaluate ( ) 1 lim x f x → . 32. If ( ) 2 3 , 0 , 0 1 , 1 mx n x f x nx m x nx m x + < = + ≤ ≤   + > . For what integers m and n does both ( ) 0 lim x f x → and ( ) 1 lim x f x → exist? 12.5 Derivatives We have seen in the Section 13.2, that by knowing the position of a body at various time intervals it is possible to find the rate at which the position of the body is changing. It is of very general interest to know a certain parameter at various instants of time and try to finding the rate at which it is changing. There are several real life situations where such a process needs to be carried out. For instance, people maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time, Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times. Financial institutions need to predict the changes in the value of a particular stock knowing its present value. In these, and many such cases it is desirable to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is derivative of a function at a given point in its domain of definition. Rationalised 2023-24 240 MATHEMATICS Definition 1 Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by ( ) ( ) 0 lim h f a h f a → h + − provided this limit exists. Derivative of f (x) at a is denoted by f′(a). Observe that f′(a) quantifies the change in f(x) at a with respect to x. Example 5 Find the derivative at x = 2 of the function f(x) = 3x. Solution We have f′ (2) = ( ) ( ) 0 2 2 lim h f h f → h + − = ( ) ( ) 0 3 2 3 2 lim h h → h + − = 0 0 0 6 3 6 3 lim lim lim3 3 h h h h h → h h → → + − = = = . The derivative of the function 3x at x = 2 is 3. Example 6 Find the derivative of the function f(x) = 2x 2 + 3x – 5 at x = –1. Also prove that f ′ (0) + 3f ′ (–1) = 0. Solution We first find the derivatives of f(x) at x = –1 and at x = 0. We have f ' 1 (−) = ( ) ( ) 0 1 1 lim h f h f → h − + − − = ( ) ( ) ( ) ( ) 2 2 0 2 1 3 1 5 2 1 3 1 5 lim h h h → h − + + − + − − − + − −   = ( ) ( ) 2 0 0 2 lim lim 2 1 2 0 1 1 h h h h h → h → − = − = − = − and f ' 0( ) = ( ) ( ) 0 0 0 lim h f h f → h + − = ( ) ( ) ( ) ( ) 2 2 0 2 0 3 0 5 2 0 3 0 5 lim h h h → h + + + − − + −   Rationalised 2023-24 LIMITS AND DERIVATIVES 241 = ( ) ( ) 2 0 0 2 3 lim lim 2 3 2 0 3 3 h h h h h → h → + = + = + = Clearly f f ' 0 3 ' 1 0 ( ) + − = ( ) Remark At this stage note that evaluating derivative at a point involves effective use of various rules, limits are subjected to. The following illustrates this. Example 7 Find the derivative of sin x at x = 0. Solution Let f(x) = sin x. Then f ′(0) = ( ) ( ) 0 0 0 lim h f h f → h + − = ( ) ( ) 0 sin 0 sin 0 lim h h → h + − = 0 sin lim 1 h h → h = Example 8 Find the derivative of f(x) = 3 at x = 0 and at x = 3. Solution Since the derivative measures the change in function, intuitively it is clear that the derivative of the constant function must be zero at every point. This is indeed, supported by the following computation. f ' 0( ) = ( ) ( ) 0 0 0 0 0 3 3 0 lim lim lim 0 h h h f h f → h → h h → + − − = = = . Similarly f ' 3( ) = ( ) ( ) 0 0 3 3 3 3 lim lim 0 h h f h f → h → h + − − = = . We now present a geometric interpretation of derivative of a function at a point. Let y = f(x) be a function and let P = (a, f(a)) and Q = (a + h, f(a + h) be two points close to each other on the graph of this function. The Fig 12.11 is now self explanatory. Fig 12.11 Rationalised 2023-24 242 MATHEMATICS We know that ( ) ( ) ( ) 0 lim h f a h f a f a → h + − ′ = From the triangle PQR, it is clear that the ratio whose limit we are taking is precisely equal to tan(QPR) which is the slope of the chord PQ. In the limiting process, as h tends to 0, the point Q tends to P and we have ( ) ( ) 0 Q P QR lim lim h PR f a h f a → h → + − = This is equivalent to the fact that the chord PQ tends to the tangent at P of the curve y = f(x). Thus the limit turns out to be equal to the slope of the tangent. Hence f a ′( ) = tan ψ . For a given function f we can find the derivative at every point. If the derivative exists at every point, it defines a new function called the derivative of f . Formally, we define derivative of a function as follows. Definition 2 Suppose f is a real valued function, the function defined by ( ) ( ) 0 lim h f x h f x → h + − wherever the limit exists is defined to be the derivative of f at x and is denoted by f′(x). This definition of derivative is also called the first principle of derivative. Thus ( ) ( ) ( ) 0 ' lim h f x h f x f x → h + − = Clearly the domain of definition of f′ (x) is wherever the above limit exists. There are different notations for derivative of a function. Sometimes f′(x) is denoted by ( ) ( ) d f x dx or if y = f(x), it is denoted by dy dx . This is referred to as derivative of f(x) or y with respect to x. It is also denoted by D (f (x)). Further, derivative of f at x = a is also denoted by ( ) or a a d df f x dx dx or even x a df dx = ⎛ ⎞ ⎜ ⎟ ⎝ ⎠ . Example 9 Find the derivative of f(x) = 10x. Solution Since f′ (x) = ( ) ( ) 0 lim h f x h f x → h + − Rationalised 2023-24 LIMITS AND DERIVATIVES 243 = ( ) ( ) 0 10 10 lim h x h x → h + − = 0 10 lim h h → h = ( ) 0 lim 10 10 h→ = . Example 10 Find the derivative of f(x) = x 2 . Solution We have, f ′(x) = ( ) ( ) 0 lim h f x h f x → h + − = ( ) ( ) 2 2 0 lim h x h x → h + − = ( ) 0 lim 2 2 h h x x → + = Example 11 Find the derivative of the constant function f (x) = a for a fixed real number a. Solution We have, f ′(x) = ( ) ( ) 0 lim h f x h f x → h + − = 0 0 0 lim lim 0 h h a a → h h → − = = as h ≠ 0 Example 12 Find the derivative of f(x) = 1 x Solution We have f ′(x) = ( ) ( ) 0 lim h f x h f x → h + − = 0 1 1 – ( ) lim h x h x → h + = ( ) ( ) 0 1 lim h x x h → h x x h ⎡ ⎤ − + ⎢ ⎥ ⎣ ⎦ + ⎣ ⎦ = ( ) 0 1 lim h h → h x x h ⎡ ⎤ − ⎢ ⎥ ⎣ ⎦ + ⎣ ⎦ = ( ) 0 1 lim h→ x x h − + = 2 1 x − Rationalised 2023-24 244 MATHEMATICS 12.5.1 Algebra of derivative of functions Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules for derivatives to follow closely that of limits. We collect these in the following theorem. Theorem 5 Let f and g be two functions such that their derivatives are defined in a common domain. Then (i) Derivative of sum of two functions is sum of the derivatives of the functions. ( ) ( ) ( ) ( ) d d d f x g x f x g x dx dx dx ⎡ ⎤ + = + ⎣ ⎦ . (ii) Derivative of difference of two functions is difference of the derivatives of the functions. ( ) ( ) ( ) ( ) d d d f x g x f x g x dx dx dx ⎡ − = − ⎣ ⎦ ⎤ . (iii) Derivative of product of two functions is given by the following product rule. ( ) ( ) . ( ) . ( ) ( ) . ( ) d d d f x g x f x g x f x g x dx dx dx ⎡ ⎤ = + ⎣ ⎦ (iv) Derivative of quotient of two functions is given by the following quotient rule (whenever the denominator is non–zero). ( )2 ( ) . ( ) ( ) ( ) ( ) ( ) ( ) d d f x g x f x g x d f x dx dx dx g x g x − ⎛ ⎞ ⎜ ⎟ = ⎝ ⎠ The proofs of these follow essentially from the analogous theorem for limits. We will not prove these here. As in the case of limits this theorem tells us how to compute derivatives of special types of functions. The last two statements in the theorem may be restated in the following fashion which aids in recalling them easily: Let u f x = ( ) and v = g (x). Then ( ) uv ′ = u v uv ′ ′ + This is referred to a Leibnitz rule for differentiating product of functions or the product rule. Similarly, the quotient rule is Rationalised 2023-24 LIMITS AND DERIVATIVES 245 u v ′ ⎛ ⎞ ⎜ ⎟ ⎝ ⎠ = 2 u v uv v ′ ′ − Now, let us tackle derivatives of some standard functions. It is easy to see that the derivative of the function f(x) = x is the constant function 1. This is because f x ′( ) = ( ) ( ) 0 lim h f x h f x → h + − = 0 lim h x h x → h + − = 0 lim1 1 h→ = . We use this and the above theorem to compute the derivative of f(x) = 10x = x + ... + x (ten terms). By (i) of the above theorem df x( ) dx = d dx (x x + + ...) (ten terms) = . . . d d x x dx dx + + (ten terms) = 1 ... 1 + + (ten terms) = 10. We note that this limit may be evaluated using product rule too. Write f(x) = 10x = uv, where u is the constant function taking value 10 everywhere and v(x) = x. Here, f(x) = 10x = uv we know that the derivative of u equals 0. Also derivative of v(x) = x equals 1. Thus by the product rule we have f x ′( ) = ( ) ( ) 10x uv u v uv x 0. 10.1 10 ′ ′ = = + = + = ′ ′ On similar lines the derivative of f(x) = x 2 may be evaluated. We have f(x) = x 2 = x .x and hence df dx = ( ) . . . ( ) ( ) d d d x x x x x x dx dx dx = + = 1. .1 2 x x x + = . More generally, we have the following theorem. Theorem 6 Derivative of f(x) = x n is nxn – 1 for any positive integer n. Proof By definition of the derivative function, we have ( ) ( ) ( ) 0 ' lim h f x h f x f x → h + − = = ( ) 0 lim n n h x h x → h + − . Rationalised 2023-24 246 MATHEMATICS Binomial theorem tells that (x + h) n = ( ) ( ) ( ) 1 C C ... C 0 1 n n n n n n n x x h h − + + + and hence (x + h) n – x n = h(nxn – 1 +... + h n – 1). Thus df x( ) dx = ( ) 0 lim n n h x h x → h + − = ( ) 1 1 0 .... lim n n h h nx h h − − → + + = ( ) 1 1 0 lim ... n n h nx h − − → + + = n 1 nx − . Alternatively, we may also prove this by induction on n and the product rule as follows. The result is true for n = 1, which has been proved earlier. We have ( ) d n x dx = ( ) 1 . d n x x dx − = ( ) ( ) ( ) 1 1 . . d n d n x x x x dx dx − − + (by product rule) = (( )) 1 2 1. . 1 n n x x n x − − + − (by induction hypothesis) = ( ) 1 1 1 1 n n n x n x nx − − − + − = . Remark The above theorem is true for all powers of x, i.e., n can be any real number (but we will not prove it here). 12.5.2 Derivative of polynomials and trigonometric functions We start with the following theorem which tells us the derivative of a polynomial function. Theorem 7 Let f(x) = 1 1 1 0 .... n n n n a x a x a x a − + + + + − be a polynomial function, where ai s are all real numbers and an ≠ 0. Then, the derivative function is given by ( ) 1 2 1 ( ) 1 ... n x n n df x na x n a x dx − − = + − + + − 2 1 2a x a + . Proof of this theorem is just putting together part (i) of Theorem 5 and Theorem 6. Example 13 Compute the derivative of 6x 100 – x 55 + x. Solution A direct application of the above theorem tells that the derivative of the above function is 99 54 600 55 1 x x − + . Rationalised 2023-24 LIMITS AND DERIVATIVES 247 Example 14 Find the derivative of f(x) = 1 + x + x 2 + x 3 +... + x 50 at x = 1. Solution A direct application of the above Theorem 6 tells that the derivative of the above function is 1 + 2x + 3x 2 + . . . + 50x 49 . At x = 1 the value of this function equals 1 + 2(1) + 3(1)2 + .. . + 50(1)49 = 1 + 2 + 3 + . . . + 50 = (50 51)( ) 2 = 1275. Example 15 Find the derivative of f(x) = x 1 x + Solution Clearly this function is defined everywhere except at x = 0. We use the quotient rule with u = x + 1 and v = x. Hence u′ = 1 and v′ = 1. Therefore df x d x d u ( ) 1 dx dx x dx v   + = =   ( ) ( ) 2 2 2 u v uv 1 1 1 x x 1 v x x ′ ′ − − + = = = − Example 16 Compute the derivative of sin x. Solution Let f(x) = sin x. Then df x( ) dx = ( ) ( ) ( ) ( ) 0 0 sin sin lim lim h h f x h f x x h x → h → h + − + − = = 0 2 2cos sin 2 2 lim h x h h → h   +   (using formula for sin A – sin B) = 0 0 sin 2 lim cos .lim cos .1 cos 2 2 h h h h x x x → → h   + = =   . Example 17 Compute the derivative of tan x. Solution Let f(x) = tan x. Then df x( ) dx = ( ) ( ) ( ) ( ) 0 0 tan tan lim lim h h f x h f x x h x → h → h + − + − = = ( ) ( ) 0 1 sin sin lim h cos cos x h x → h x h x +   −   +   Rationalised 2023-24 248 MATHEMATICS = ( ) ( ) ( ) 0 sin cos cos sin lim h cos cos x h x x h x → h x h x + − +   +   = ( ) ( ) 0 sin lim h cos cos x h x → h x h x + − + (using formula for sin (A + B)) = ( ) 0 0 sin 1 lim .lim h h cos cos h → h x h x → + = 2 2 1 1. sec cos x x = . Example 18 Compute the derivative of f(x) = sin2 x. Solution We use the Leibniz product rule to evaluate this. ( ) ( ) sin sin df x d x x dx dx = ( ) sin sin sin sin x x x x ( ) ′ ′ = + = (cos sin sin cos x x x x) + ( ) = = 2sin cos sin 2 x x x . EXERCISE 12.2 1. Find the derivative of x 2 – 2 at x = 10. 2. Find the derivative of x at x = 1. 3. Find the derivative of 99x at x = l00. 4. Find the derivative of the following functions from first principle. (i) 3 x − 27 (ii) (x x − − 1 2)( ) (iii) 2 1 x (iv) 1 1 x x + − 5. For the function ( ) 100 99 2 . . . 1 100 99 2 x x x f x = + + + + +x . Rationalised 2023-24 LIMITS AND DERIVATIVES 249 Prove that f f ′ ′ (1 100 0) = ( ). 6. Find the derivative of 1 2 2 1 . . . n n n n n x ax a x a x a − − − + + + + + for some fixed real number a. 7. For some constants a and b, find the derivative of (i) (x a x b − −) ( ) (ii) ( ) 2 2 ax b + (iii) x a x b − − 8. Find the derivative of n n x a x a − − for some constant a. 9. Find the derivative of (i) 3 2 4 x − (ii) ( ) ( ) 3 5 3 1 1 x x x + − − (iii) ( ) 3 x x 5 3 − + (iv) ( ) 5 9 x x 3 6 − − (v) ( ) 4 5 x x 3 4 − − − (vi) 2 2 1 3 1 x x x − + − 10. Find the derivative of cos x from first principle. 11. Find the derivative of the following functions: (i) sin cos x x (ii) sec x (iii) 5sec 4cos x x + (iv) cosec x (v) 3cot 5cosec x x + (vi) 5sin 6cos 7 x x − + (vii) 2tan 7sec x x − Miscellaneous Examples Example 19 Find the derivative of f from the first principle, where f is given by (i) f (x) = 2 3 2 x x + − (ii) f (x) = 1 x x + Solution (i) Note that function is not defined at x = 2. But, we have ( ) ( ) ( ) ( ) 0 0 2 3 2 3 2 2 lim lim h h x h x f x h f x x h x f x → h → h + + + − + − + − − ′ = = Rationalised 2023-24 250 MATHEMATICS = ( )( ) ( )( ) ( ) ( ) 0 2 2 3 2 2 3 2 lim h 2 2 x h x x x h → h x x h + + − − + + − − + − = ( )( ) ( ) ( )( ) ( ) ( )( ) 0 2 3 2 2 2 2 3 2 2 3 lim h 2 2 x x h x x x h x → h x x h + − + − − + − − + − + − = ( ) ( ) ( )2 0 –7 7 lim 2 2 2 h x x h x → = − − + − − Again, note that the function f ′ is also not defined at x = 2. (ii) The function is not defined at x = 0. But, we have f x ′( ) = ( ) ( ) 0 0 1 1 lim lim h h x h x f x h f x x h x → h → h   + + − + −   +   = = 0 1 1 1 lim h h → h x h x   + −   + = ( ) ( ) 0 0 1 1 1 lim lim 1 h h x x h h h → h x x h h x x h → − −   + = −   + +   = ( ) 2 0 1 1 lim 1 1 h→ x x h x ⎛ ⎞ ⎜ ⎟ − = − ⎜ ⎟ + ⎝ ⎠ Again, note that the function f ′ is not defined at x = 0. Example 20 Find the derivative of f(x) from the first principle, where f(x) is (i) sin cos x x + (ii) x x sin Solution (i) we have f x '( ) = f x h f x ( ) ( ) h + − = ( ) ( ) 0 sin cos sin cos lim h x h x h x x → h + + + − − = 0 sin cos cos sin cos cos sin sin sin cos lim h x h x h x h x h x x → h + + − − − Rationalised 2023-24 LIMITS AND DERIVATIVES 251 = ( ) ( ) ( ) 0 sin cos sin sin cos 1 cos cos 1 lim h h x x x h x h → h − + − + − = ( ) ( ) 0 0 sin cos 1 lim cos sin lim sin h h h h x x x → h → h − − + ( ) 0 cos 1 lim cos h h x → h − + = cos sin x x − (ii) f x '( ) = ( ) ( ) ( ) ( ) 0 0 sin sin lim lim h h f x h f x x h x h x x → h → h + − + + − = = ( )( ) 0 sin cos sin cos sin lim h x h x h h x x x → h + + − = ( ) ( ) 0 sin cos 1 cos sin sin cos sin cos lim h x x h x x h h x h h x → h − + + + = ( ) 0 0 sin cos 1 sin lim lim cos h h x x h h x x h h → → − + ( ) 0 lim sin cos sin cos h x h h x → + + = x x x cos sin + Example 21 Compute derivative of (i) f(x) = sin 2x (ii) g(x) = cot x Solution (i) Recall the trigonometric formula sin 2x = 2 sin x cos x. Thus df x( ) dx = ( ) 2sin cos 2 sin cos ( ) d d x x x x dx dx = 2 sin cos sin cos ( ) x x x x ( ) ⎡ ′ ′⎤ = + ⎣ ⎦ ⎡ ⎤ = 2 cos cos sin sin ⎣(x x x x) + −( )⎦ ( ) 2 2 = − 2 cos sin x x (ii) By definition, g(x) = cos cot sin x x x = . We use the quotient rule on this function wherever it is defined. cos (cot) sin dg d d x x dx dx dx x ⎛ ⎞ = = ⎜ ⎟ ⎝ ⎠ Rationalised 2023-24 252 MATHEMATICS = 2 (cos) (sin) (cos)(sin) (sin) x x x x x ′ − ′ = 2 (sin)(sin) (cos)(cos) (sin) x x x x x − − = 2 2 2 2 sin cos cosec sin x x x x + − = − Alternatively, this may be computed by noting that 1 cot tan x x = . Here, we use the fact that the derivative of tan x is sec2 x which we saw in Example 17 and also that the derivative of the constant function is 0. dg dx = 1 (cot) tan d d x dx dx x ⎛ ⎞ = ⎜ ⎟ ⎝ ⎠ = 2 (1) (tan) (1)(tan) (tan) x x x ′ − ′ = 2 2 (0)(tan) (sec) (tan) x x x − = 2 2 2 sec cosec tan x x x − = − Example 22 Find the derivative of (i) 5 cos sin x x x − (ii) cos tan x x x + Solution (i) Let 5 cos ( ) sin x x h x x − = . We use the quotient rule on this function wherever it is defined. 5 5 2 (cos) sin (cos)(sin) ( ) (sin) x x x x x x h x x − − − ′ ′ ′ = Rationalised 2023-24 LIMITS AND DERIVATIVES 253 = 4 5 2 (5 sin)sin (cos)cos sin x x x x x x x x + − − = 5 4 2 cos 5 sin 1 (sin) x x x x x − + + (ii) We use quotient rule on the function cos tan x x x + wherever it is defined. h x ′( ) = 2 (cos) tan (cos)(tan) (tan) x x x x x x x x + − + ′ ′ = 2 2 (1 sin) tan (cos)sec (tan) x x x x x x − − + Miscellaneous Exercise on Chapter 12 1. Find the derivative of the following functions from first principle: (i) −x (ii) 1 ( ) x − − (iii) sin (x + 1) (iv) cos (x − π 8) Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): 2. (x + a) 3. (px + q) r s x ⎛ ⎞ + ⎜ ⎟ ⎝ ⎠ 4. ( ) ( )2 ax b cx d + + 5. ax b cx d + + 6. 1 1 1 1 x x + − 7. 2 1 ax bx c + + 8. 2 ax b px qx r + + + 9. 2 px qx r ax b + + + 10. 4 2 cos a b x x x − + 11. 4 2 x − 12. ( )n ax b + 13. ( ) ( ) n m ax b cx d + + 14. sin (x + a) 15. cosec x cot x 16. cos 1 sin x + x Rationalised 2023-24 254 MATHEMATICS 17. sin cos sin cos x x x x + − 18. sec 1 sec 1 x x − + 19. sinn x 20. sin cos a b x c d x + + 21. sin( ) cos x a x + 22. 4 x x x (5sin 3cos) − 23. ( ) 2 x x +1 cos 24. ( )( ) 2 ax x p q x + + sin cos 25. (x x x x + − cos tan) ( ) 26. 4 5sin 3 7cos x x x x + + 27. 2 cos 4 sin x x ⎛ ⎞ π ⎜ ⎟ ⎝ ⎠ 28. 1 tan x + x 29. (x x x x + − sec tan) ( ) 30. sinn x x Summary ÆThe expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit. ÆLimit of a function at a point is the common value of the left and right hand limits, if they coincide. ÆFor a function f and a real number a, lim x a → f(x) and f (a) may not be same (In fact, one may be defined and not the other one). ÆFor functions f and g the following holds: lim ( ) ( ) lim ( ) lim ( ) [ ] x a x a x a f x g x f x g x → → → ± = ± lim ( ). ( ) lim ( ).lim ( ) [ ] x a x a x a f x g x f x g x → → → = lim ( ) ( ) lim ( ) lim ( ) x a x a x a f x f x g x g x → → → ⎡ ⎤ = ⎢ ⎥ ⎣ ⎦ ÆFollowing are some of the standard limits 1 lim n n n x a x a na x a − → − = − Rationalised 2023-24 LIMITS AND DERIVATIVES 255 0 sin lim 1 x x → x = 0 1 cos lim 0 x x → x − = ÆThe derivative of a function f at a is defined by 0 ( ) ( ) ( ) lim h f a h f a f a → h + − ′ = ÆDerivative of a function f at any point x is defined by 0 ( ) ( ) ( ) ( ) lim h df x f x h f x f x dx → h + − ′ = = ÆFor functions u and v the following holds: ( ) u v u v ± = ± ′ ′ ′ ( ) uv u v uv ′ ′ ′ = + 2 u u v uv v v ′ ⎛ ⎞ ′ ′ − = ⎜ ⎟ ⎝ ⎠ provided all are defined. ÆFollowing are some of the standard derivatives. 1 ( ) d n n x nx dx − = (sin) cos d x x dx = (cos) sin d x x dx = − Historical Note In the history of mathematics two names are prominent to share the credit for inventing calculus, Issac Newton (1642 – 1727) and G.W. Leibnitz

(1646 – 1717). Both of them independently invented calculus around the seventeenth century. After the advent of calculus many mathematicians contributed for further development of calculus. The rigorous concept is mainly attributed to the great Rationalised 2023-24 256 MATHEMATICS mathematicians, A.L. Cauchy, J.L.Lagrange and Karl Weierstrass. Cauchy gave the foundation of calculus as we have now generally accepted in our textbooks. Cauchy used D'Alembert's limit concept to define the derivative of a function. Starting with definition of a limit, Cauchy gave examples such as the limit of sinα α for α = 0. He wrote ( ) ( ) , y f x i f x x i Δ + − = Δ and called the limit for i → 0, the "function derive'e, y′ for f ′ (x)". Before 1900, it was thought that calculus is quite difficult to teach. So calculus became beyond the reach of youngsters. But just in 1900, John Perry and others in England started propagating the view that essential ideas and methods of calculus were simple and could be taught even in schools. F.L. Griffin, pioneered the teaching of calculus to first year students. This was regarded as one of the most daring act in those days. Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological Sciences are enjoying the fruits of calculus. — v — Rationalised 2023-24v"Statistics may be rightly called the science of averages and their estimates." – A.L.BOWLEY & A.L. BODDINGTON v 13.1 Introduction We know that statistics deals with data collected for specific purposes. We can make decisions about the data by analysing and interpreting it. In earlier classes, we have studied methods of representing data graphically and in tabular form. This representation reveals certain salient features or characteristics of the data. We have also studied the methods of finding a representative value for the given data. This value is called the measure of central tendency. Recall mean (arithmetic mean), median and mode are three measures of central tendency. A measure of central tendency gives us a rough idea where data points are centred. But, in order to make better interpretation from the data, we should also have an idea how the data are scattered or how much they are bunched around a measure of central tendency. Consider now the runs scored by two batsmen in their last ten matches as follows: Batsman A : 30, 91, 0, 64, 42, 80, 30, 5, 117, 71 Batsman B : 53, 46, 48, 50, 53, 53, 58, 60, 57, 52 Clearly, the mean and median of the data are Batsman A Batsman B Mean 53 53 Median 53 53 Recall that, we calculate the mean of a data (denoted by x) by dividing the sum of the observations by the number of observations, i.e., Chapter 13 STATISTICS Karl Pearson (1857-1936) Rationalised 2023-24 258 MATHEMATICS 1 1 n i i x x n = = ∑ Also, the median is obtained by first arranging the data in ascending or descending order and applying the following rule. If the number of observations is odd, then the median is th 1 2 n + observation. If the number of observations is even, then median is the mean of th 2 n and th 1 2 n + observations. We find that the mean and median of the runs scored by both the batsmen A and B are same i.e., 53. Can we say that the performance of two players is same? Clearly No, because the variability in the scores of batsman A is from 0 (minimum) to 117 (maximum). Whereas, the range of the runs scored by batsman B is from 46 to 60. Let us now plot the above scores as dots on a number line. We find the following diagrams: For batsman A For batsman B We can see that the dots corresponding to batsman B are close to each other and are clustering around the measure of central tendency (mean and median), while those corresponding to batsman A are scattered or more spread out. Thus, the measures of central tendency are not sufficient to give complete information about a given data. Variability is another factor which is required to be studied under statistics. Like 'measures of central tendency' we want to have a single number to describe variability. This single number is called a 'measure of dispersion'. In this Chapter, we shall learn some of the important measures of dispersion and their methods of calculation for ungrouped and grouped data. Fig 13.1 Fig 13.2 Rationalised 2023-24 STATISTICS 259 13.2 Measures of Dispersion The dispersion or scatter in a data is measured on the basis of the observations and the types of the measure of central tendency, used there. There are following measures of dispersion: (i) Range, (ii) Quartile deviation, (iii) Mean deviation, (iv) Standard deviation. In this Chapter, we shall study all of these measures of dispersion except the quartile deviation. 13.3 Range Recall that, in the example of runs scored by two batsmen A and B, we had some idea of variability in the scores on the basis of minimum and maximum runs in each series. To obtain a single number for this, we find the difference of maximum and minimum values of each series. This difference is called the 'Range' of the data. In case of batsman A, Range = 117 – 0 = 117 and for batsman B, Range = 60 – 46 = 14. Clearly, Range of A > Range of B. Therefore, the scores are scattered or dispersed in case of A while for B these are close to each other. Thus, Range of a series = Maximum value – Minimum value. The range of data gives us a rough idea of variability or scatter but does not tell about the dispersion of the data from a measure of central tendency. For this purpose, we need some other measure of variability. Clearly, such measure must depend upon the difference (or deviation) of the values from the central tendency. The important measures of dispersion, which depend upon the deviations of the observations from a central tendency are mean deviation and standard deviation. Let us discuss them in detail. 13.4 Mean Deviation Recall that the deviation of an observation x from a fixed value 'a' is the difference x – a. In order to find the dispersion of values of x from a central value 'a' , we find the deviations about a. An absolute measure of dispersion is the mean of these deviations. To find the mean, we must obtain the sum of the deviations. But, we know that a measure of central tendency lies between the maximum and the minimum values of the set of observations. Therefore, some of the deviations will be negative and some positive. Thus, the sum of deviations may vanish. Moreover, the sum of the deviations from mean (x) is zero. Also Mean of deviations Sum of deviations 0 0 Number of observations n = = = Thus, finding the mean of deviations about mean is not of any use for us, as far as the measure of dispersion is concerned. Rationalised 2023-24 260 MATHEMATICS Remember that, in finding a suitable measure of dispersion, we require the distance of each value from a central tendency or a fixed number 'a'. Recall, that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Thus, to find the measure of dispersion from a fixed number 'a' we may take the mean of the absolute values of the deviations from the central value. This mean is called the 'mean deviation'. Thus mean deviation about a central value 'a' is the mean of the absolute values of the deviations of the observations from 'a'. The mean deviation from 'a' is denoted as M.D. (a). Therefore, M.D.(a) = Sum of absolute values of deviations from ' ' Number of observations a . Remark Mean deviation may be obtained from any measure of central tendency. However, mean deviation from mean and median are commonly used in statistical studies. Let us now learn how to calculate mean deviation about mean and mean deviation about median for various types of data 13.4.1 Mean deviation for ungrouped data Let n observations be x1 , x2 , x3 , ..., xn . The following steps are involved in the calculation of mean deviation about mean or median: Step 1 Calculate the measure of central tendency about which we are to find the mean deviation. Let it be 'a'. Step 2 Find the deviation of each x i from a, i.e., x 1 – a, x 2 – a, x 3 – a,. . . , x n – a Step 3 Find the absolute values of the deviations, i.e., drop the minus sign (–), if it is there, i.e., axaxaxax1 2 3 ....,,,, n −−−− Step 4 Find the mean of the absolute values of the deviations. This mean is the mean deviation about a, i.e., 1 M.D.( ) n i i x a a n = − = ∑ Thus M.D. (x) = 1 1 n i i x x n = ∑ − , where x = Mean and M.D. (M) = 1 1 M n i i x n = ∑ − , where M = Median Rationalised 2023-24 STATISTICS 261 ANote In this Chapter, we shall use the symbol M to denote median unless stated otherwise.Let us now illustrate the steps of the above method in following examples. Example 1 Find the mean deviation about the mean for the following data: 6, 7, 10, 12, 13, 4, 8, 12 Solution We proceed step-wise and get the following: Step 1 Mean of the given data is 6 7 10 12 13 4 8 12 72 9 8 8 x + + + + + + + = = = Step 2 The deviations of the respective observations from the mean x, i.e., xi – x are 6 – 9, 7 – 9, 10 – 9, 12 – 9, 13 – 9, 4 – 9, 8 – 9, 12 – 9, or –3, –2, 1, 3, 4, –5, –1, 3 Step 3 The absolute values of the deviations, i.e., i x x − are 3, 2, 1, 3, 4, 5, 1, 3 Step 4 The required mean deviation about the mean is M.D. (x) = 8 1 8 i i x x = ∑ − = 3 2 1 3 4 5 1 3 22 2 75 8 8 . + + + + + + + = = ANote Instead of carrying out the steps every time, we can carry on calculation, step-wise without referring to steps. Example 2

Find the mean deviation about the mean for the following data : 12, 3, 18, 17, 4, 9, 17, 19, 20, 15, 8, 17, 2, 3, 16, 11, 3, 1, 0, 5 Solution We have to first find the mean (x) of the given data 20 1 1 20 i i x x = = ∑ = 20 200 = 10 Rationalised 2023-24 262 MATHEMATICS The respective absolute values of the deviations from mean, i.e., xxi − are 2, 7, 8, 7, 6, 1, 7, 9, 10, 5, 2, 7, 8, 7, 6, 1, 7, 9, 10, 5 Therefore 20 1 124 i i x x = ∑ − = and M.D. (x) = 124 20 = 6.2 Example 3 Find the mean deviation about the median for the following data: 3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21. Solution Here the number of observations is 11 which is odd. Arranging the data into ascending order, we have 3, 3, 4, 5, 7, 9, 10, 12, 18, 19, 21 Now Median = th 11 1 2 ⎛ ⎞ + ⎜ ⎟ ⎝ ⎠ or 6th observation = 9 The absolute values of the respective deviations from the median, i.e., xi − M are 6, 6, 5, 4, 2, 0, 1, 3, 9, 10, 12 Therefore 11 1 M 58 i i x = ∑ − = and ( ) 11 1 1 1 M.D. M M 58 5.27 11 11 i i x = = − = × = ∑ 13.4.2 Mean deviation for grouped data We know that data can be grouped into two ways : (a) Discrete frequency distribution, (b) Continuous frequency distribution. Let us discuss the method of finding mean deviation for both types of the data. (a) Discrete frequency distribution Let the given data consist of n distinct values x 1 , x 2 , ..., x n occurring with frequencies f 1 , f 2 , ..., f n respectively. This data can be represented in the tabular form as given below, and is called discrete frequency distribution: x : x1 x2 x3 ... xn f : f 1 f2 f 3 ... f n Rationalised 2023-24 STATISTICS 263 (i) Mean deviation about mean First of all we find the mean x of the given data by using the formula 1 1 1 1 N n i i n i n i i i i i x f x x f f = = = = = ∑ ∑ ∑ , where ∑= n i ii fx 1 denotes the sum of the products of observations xi with their respective frequencies f i and ∑= = n i i f 1 N is the sum of the frequencies. Then, we find the deviations of observations xi from the mean x and take their absolute values, i.e., xxi − for all i =1, 2,..., n. After this, find the mean of the absolute values of the deviations, which is the required mean deviation about the mean. Thus 1 1 M.D. ( ) n i i i n i i f x x x f = = − = ∑ ∑ = xxf i n i ∑ i − N =1 1 (ii) Mean deviation about median To find mean deviation about median, we find the median of the given discrete frequency distribution. For this the observations are arranged in ascending order. After this the cumulative frequencies are obtained. Then, we identify the observation whose cumulative frequency is equal to or just greater than N 2 , where N is the sum of frequencies. This value of the observation lies in the middle of the data, therefore, it is the required median. After finding median, we obtain the mean of the absolute values of the deviations from median.Thus, 1 1 M.D.(M) M N n i i i f x = = ∑ − Example 4 Find mean deviation about the mean for the following data : xi 2 5 6 8 10 12 f i 2 8 10 7 8 5 Rationalised 2023-24 264 MATHEMATICS Solution Let us make a Table 13.1 of the given data and append other columns after calculations. Table 13.1 xi f i f i xi xxi − f i xxi − 2 2 4 5.5 11 5 8 40 2.5 20 6 10 60 1.5 15 8 7 56 0.5 3.5 10 8 80 2.5 20 12 5 60 4.5 22.5 40 300 92 N 40 6 1 ∑ == i= i f , 300 6 1 ∑ = i= ii xf , 92 6 1 ∑ =− = xxf i i i Therefore 6 1 1 1 300 7.5 N 40 i i i x f x = = = × = ∑ and 6 1 1 1 M. D. ( ) 92 2.3 N 40 i i i x f x x = = ∑ − = × = Example 5 Find the mean deviation about the median for the following data: xi 3 6 9 12 13 15 21 22 f i 3 4 5 2 4 5 4 3 Solution The given observations are already in ascending order. Adding a row corresponding to cumulative frequencies to the given data, we get (Table 13.2). Table 13.2 x i 3 6 9 12 13 15 21 22 f i 3 4 5 2 4 5 4 3 c.f. 3 7 12 14 18 23 27 30 Now, N=30 which is even. Rationalised 2023-24 STATISTICS 265 Median is the mean of the 15th and 16th observations. Both of these observations lie in the cumulative frequency 18, for which the corresponding observation is 13. th th 15 observation 16 observation 13 13 Therefore, Median M 13 2 2 + + = = = Now, absolute values of the deviations from median, i.e., xi − M are shown in Table 13.3. Table 13.3 xi − M 10 7 4 1 0 2 8 9 f i 3 4 5 2 4 5 4 3 f i xi − M 30 28 20 2 0 10 32 27 We have 8 8 1 1 i 30 and M 149 i i i i f f x = = ∑ ∑ = − = Therefore 8 1 1 M. D. (M) M N i i i f x = = ∑ − = 1 149 30 × = 4.97. (b) Continuous frequency distribution A continuous frequency distribution is a series in which the data are classified into different class-intervals without gaps alongwith their respective frequencies. For example, marks obtained by 100 students are presented in a continuous frequency distribution as follows : Marks obtained 0-10 10-20 20-30 30-40 40-50 50-60 Number of Students 12 18 27 20 17 6 (i) Mean deviation about mean While calculating the mean of a continuous frequency distribution, we had made the assumption that the frequency in each class is centred at its mid-point. Here also, we write the mid-point of each given class and proceed further as for a discrete frequency distribution to find the mean deviation. Let us take the following example. Rationalised 2023-24 266 MATHEMATICS Example 6 Find the mean deviation about the mean for the following data. Marks obtained 10-20 20-30 30-40 40-50 50-60 60-70 70-80 Number of students 2 3 8 14 8 3 2 Solution We make the following Table 13.4 from the given data : Table 13.4 Marks Number of Mid-points f i x i xxi − f i xxi − obtained students f i xi 10-20 2 15 30 30 60 20-30 3 25 75 20 60 30-40 8 35 280 10 80 40-50 14 45 630 0 0 50-60 8 55 440 10 80 60-70 3 65 195 20 60 70-80 2 75 150 30 60 40 1800 400 Here 7 7 7 1 1 1 N 40, 1800, i i i i i 400 i i i f f x f x x = = = = = = ∑ ∑ ∑ − = Therefore 7 1 1 1800 45 N 40 i i i x f x = = = = ∑ and ( ) 7 1 1 1 M.D. 400 10 N 40 i i i x f x x = = − = × = ∑ Shortcut method for calculating mean deviation about mean We can avoid the tedious calculations of computing x by following step-deviation method. Recall that in this method, we take an assumed mean which is in the middle or just close to it in the data. Then deviations of the observations (or mid-points of classes) are taken from the Rationalised 2023-24 STATISTICS 267 assumed mean. This is nothing but the shifting of origin from zero to the assumed mean on the number line, as shown in Fig 13.3 If there is a common factor of all the deviations, we divide them by this common factor to further simplify the deviations. These are known as step-deviations. The process of taking step-deviations is the change of scale on the number line as shown in Fig 13.4 The deviations and step-deviations reduce the size of the observations, so that the computations viz. multiplication, etc., become simpler. Let, the new variable be denoted by h ax d i i − = , where ‘a’ is the assumed mean and h is the common factor. Then, the mean x by step-deviation method is given by 1 N n f d i i i x a h ∑ = = + × Let us take the data of Example 6 and find the mean deviation by using stepdeviation method. Fig 13.3 Fig 13.4 Rationalised 2023-24 268 MATHEMATICS Number of students Marks obtained Take the assumed mean a = 45 and h = 10, and form the following Table 13.5. Table 13.5 Mid-points 45 10 i i x d − = i i f d xxi − f i xxi − f i x i 10-20 2 15 – 3 – 6 30 60 20-30 3 25 – 2 – 6 20 60 30-40 8 35 – 1 – 8 10 80 40-50 14 45 0 0 0 0 50-60 8 55 1 8 10 80 60-70 3 65 2 6 20 60 70-80 2 75 3 6 30 60 40 0 400 Therefore 7 1 N f d i i i x a h ∑ = = + × = 0 45 10 45 40 + × = and 7 1 1 400 M D ( ) 10 N 40 i i i x f x x = . . = ∑ − = = ANote The step deviation method is applied to compute x . Rest of the procedure is same. (ii) Mean deviation about median The process of finding the mean deviation about median for a continuous frequency distribution is similar as we did for mean deviation about the mean. The only difference lies in the replacement of the mean by median while taking deviations. Let us recall the process of finding median for a continuous frequency distribution. The data is first arranged in ascending order. Then, the median of continuous frequency distribution is obtained by first identifying the class in which median lies (median class) and then applying the formula Rationalised 2023-24 STATISTICS 269 frequency N C Median 2 l h f − = + × where median class is the class interval whose cumulative frequency is just greater than or equal to N 2 , N is the sum of frequencies, l, f, h andC are,respectively the lower limit , the frequency, the width of the median class and C the cumulative frequency of the class just preceding the median class. After finding the median, the absolute values of the deviations of mid-point x i of each class from the median i.e., xi − M are obtained. Then 1 M.D. (M) M N 1 n f x i i i = − ∑ = The process is illustrated in the following example: Example 7 Calculate the mean deviation about median for the following data : Class 0-10 10-20 20-30 30-40 40-50 50-60 Frequency 6 7 15 16 4 2 Solution Form the following Table 13.6 from the given data : Table 13.6 Class Frequency Cumulative Mid-points x Med. i − f i x Med. i − f i (c.f.) xi 0-10 6 6 5 23 138 10-20 7 13 15 13 91 20-30 15 28 25 3 45 30-40 16 44 35 7 112 40-50 4 48 45 17 68 50-60 2 50 55 27 54 50 508 Rationalised 2023-24 270 MATHEMATICS The class interval containing th N 2 or 25th item is 20-30. Therefore, 20–30 is the median class. We know that Median = N C 2 l h f − + × Here l = 20, C = 13, f = 15, h = 10 and N = 50 Therefore, Median 25 13 20 10 15 − = + × = 20 + 8 = 28 Thus, Mean deviation about median is given by M.D. (M) = 6 1 1 M N i i i f x = ∑ − = 1 508 50 × = 10.16 EXERCISE 13.1 Find the mean deviation about the mean for the data in Exercises 1 and 2. 1. 4, 7, 8, 9, 10, 12, 13, 17 2. 38, 70, 48, 40, 42, 55, 63, 46, 54, 44 Find the mean deviation about the median for the data in Exercises 3 and 4. 3. 13, 17, 16, 14, 11, 13, 10, 16, 11, 18, 12, 17 4. 36, 72, 46, 42, 60, 45, 53, 46, 51, 49 Find the mean deviation about the mean for the data in Exercises 5 and 6. 5. xi 5 10 15 20 25 f i 7 4 6 3 5 6. xi 10 30 50 70 90 f i 4 24 28 16 8 Find the mean deviation about the median for the data in Exercises 7 and 8. 7. xi 5 7 9 10 12 15 f i 8 6 2 2 2 6 8. xi 15 21 27 30 35 f i 3 5 6 7 8 Rationalised 2023-24 STATISTICS 271 Find the mean deviation about the mean for the data in Exercises 9 and 10. 9. Income per 0-100 100-200 200-300 300-400 400-500 500-600 600-700 700-800 day in ` Number 4 8 9 10 7 5 4 3 of persons 10. Height 95-105 105-115 115-125 125-135 135-145 145-155 in cms Number of 9 13 26 30 12 10 boys 11. Find the mean deviation about median for the following data : Marks 0-10 10-20 20-30 30-40 40-50 50-60 Number of 6 8 14 16 4 2 Girls 12. Calculate the mean deviation about median age for the age distribution of 100 persons given below: Age 16-20 21-25 26-30 31-35 36-40 41-45 46-50 51-55 (in years) Number 5 6 12 14 26 12 16 9 [Hint Convert the given data into continuous frequency distribution by subtracting 0.5 from the lower limit and adding 0.5 to the upper limit of each class interval] 13.4.3 Limitations of mean deviation In a series, where the degree of variability is very high, the median is not a representative central tendency. Thus, the mean deviation about median calculated for such series can not be fully relied. The sum of the deviations from the mean (minus signs ignored) is more than the sum of the deviations from median. Therefore, the mean deviation about the mean is not very scientific.Thus, in many cases, mean deviation may give unsatisfactory results. Also mean deviation is calculated on the basis of absolute values of the deviations and therefore, cannot be subjected to further algebraic treatment. This implies that we must have some other measure of dispersion. Standard deviation is such a measure of dispersion. 13.5 Variance and Standard Deviation Recall that while calculating mean deviation about mean or median, the absolute values of the deviations were taken. The absolute values were taken to give meaning to the mean deviation, otherwise the deviations may cancel among themselves. Another way to overcome this difficulty which arose due to the signs of deviations, is to take squares of all the deviations. Obviously all these squares of deviations are Rationalised 2023-24 272 MATHEMATICS non-negative. Let x1 , x2 , x3 , ..., xn be n observations and x be their mean. Then 2 2 2 2 2 1 1 ( ) ( ) .... ( ) ( ) n n i i x x x x x x x x x = − + − + + − = − . If this sum is zero, then each xx)(i − has to be zero. This implies that there is no dispersion at all as all observations are equal to the mean x . If ∑= − n i i xx 1 2)(is small , this indicates that the observations x1 , x2 , x3 ,...,xn are close to the mean x and therefore, there is a lower degree of dispersion. On the contrary, if this sum is large, there is a higher degree of dispersion of the observations from the mean x . Can we thus say that the sum ∑= − n i i xx 1 2)(is a reasonable indicator of the degree of dispersion or scatter? Let us take the set A of six observations 5, 15, 25, 35, 45, 55. The mean of the observations is x = 30. The sum of squares of deviations from x for this set is ∑= − 6 1 2)(i i xx = (5–30)2 + (15–30)2 + (25–30)2 + (35–30)2 + (45–30)2 +(55–30)2 = 625 + 225 + 25 + 25 + 225 + 625 = 1750 Let us now take another set B of 31 observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45. The mean of these observations is y = 30 Note that both the sets A and B of observations have a mean of 30. Now, the sum of squares of deviations of observations for set B from the mean y is given by ∑= − 31 1 2)(i i yy = (15–30)2 +(16–30)2 + (17–30)2 + ...+ (44–30)2 +(45–30)2 = (–15)2 +(–14)2 + ...+ (–1)2 + 02 + 12 + 22 + 32 + ...+ 142 + 152 = 2 [152 + 142 + ... + 12] = 15 (15 1) (30 1) 2 6 × + + × = 5 × 16 × 31 = 2480 (Because sum of squares of first n natural numbers = (1) (2 1) 6 n n n + + . Here n = 15) Rationalised 2023-24 STATISTICS 273 If ∑= − n i i xx 1 2)(is simply our measure of dispersion or scatter about mean, we will tend to say that the set A of six observations has a lesser dispersion about the mean than the set B of 31 observations, even though the observations in set A are more scattered from the mean (the range of deviations being from –25 to 25) than in the set B (where the range of deviations is from –15 to 15). This is also clear from the following diagrams. For the set A, we have For the set B, we have Thus, we can say that the sum of squares of deviations from the mean is not a proper measure of dispersion. To overcome this difficulty we take the mean of the squares of the deviations, i.e., we take ∑= − n i i xx n 1 2)(1 . In case of the set A, we have 1 Mean 6 = × 1750 = 291.67 and in case of the set B, it is 1 31× 2480 = 80. This indicates that the scatter or dispersion is more in set A than the scatter or dispersion in set B, which confirms with the geometrical representation of the two sets. Thus, we can take ∑ − 2)(1 xx n i as a quantity which leads to a proper measure of dispersion. This number, i.e., mean of the squares of the deviations from mean is called the variance and is denoted by 2 σ (read as sigma square). Therefore, the variance of n observations x1 , x2 ,..., xn is given by Fig 13.5 Fig 13.6 Rationalised 2023-24 274 MATHEMATICS Deviations from mean (xi – x) ∑= −= n i i xx n 1 2 2)(1 σ 13.5.1 Standard Deviation In the calculation of variance, we find that the units of individual observations xi and the unit of their mean x are different from that of variance, since variance involves the sum of squares of (x i – x). For this reason, the proper measure of dispersion about the mean of a set of observations is expressed as positive square-root of the variance and is called standard deviation. Therefore, the standard deviation, usually denoted by σ , is given by ∑= = − n i i xx n 1 2)(1 σ ... (1) Let us take the following example to illustrate the calculation of variance and hence, standard deviation of ungrouped data. Example 8 Find the variance of the following data: 6, 8, 10, 12, 14, 16, 18, 20, 22, 24 Solution From the given data we can form the following Table 13.7. The mean is calculated by step-deviation method taking 14 as assumed mean. The number of observations is n = 10 Table 13.7 xi 14 2 i i x d − = (xi – x) 6 –4 –9 81 8 –3 –7 49 10 –2 –5 25 12 –1 –3 9 14 0 –1 1 16 1 1 1 18 2 3 9 20 3 5 25 22 4 7 49 24 5 9 81 5 330 Rationalised 2023-24 STATISTICS 275 Therefore Mean x = assumed mean + h n d n i i × ∑=1 = 5 14 2 15 10 + × = and Variance (2 σ) = 10 2 1 1) i i (x x n = ∑ − = 1 330 10 × = 33 Thus Standard deviation (σ) = 33 5 74 = . 13.5.2 Standard deviation of a discrete frequency distribution Let the given discrete frequency distribution be x : x1 , x2 , x3 ,. . . , xn f : f 1 , f2 , f 3 ,. . . , f n In this case standard deviation ( ) 2 1 1 ( ) N n i i i σ f x x = = − ∑ ... (2) where 1 N n i i f = =∑ . Let us take up following example. Example 9 Find the variance and standard deviation for the following data: x i 4 8 11 17 20 24 32 f i 3 5 9 5 4 3 1 Solution Presenting the data in tabular form (Table 13.8), we get Table 13.8 xi f i f i xi xi – x 2 xx)(i − f i 2 xx)(i – 4 3 12 –10 100 300 8 5 40 –6 36 180 11 9 99 – 3 9 81 17 5 85 3 9 45 20 4 80 6 36 144 24 3 72 10 100 300 32 1 32 18 324 324 30 420 1374 Rationalised 2023-24 276 MATHEMATICS N = 30, ( ) 7 7 2 1 1 420, 1374 i i i i i i f x f x x = = ∑ ∑ = − = Therefore 7 1 1 420 14 N 30 i i i f x x = = = × = ∑ Hence variance 2 ( ) σ = 7 2 1 1 ( ) N i i i f x x = ∑ − = 1 30 × 1374 = 45.8 and Standard deviation σ = 8.45)(= 6.77 13.5.3 Standard deviation of a continuous frequency distribution The given continuous frequency distribution can be represented as a discrete frequency distribution by replacing each class by its mid-point. Then, the standard deviation is calculated by the technique adopted in the case of a discrete frequency distribution. If there is a frequency distribution of n classes each class defined by its mid-point xi with frequency f i , the standard deviation will be obtained by the formula 2 1 1 ( ) N n i i i σ f x x = = − ∑ , where x is the mean of the distribution and 1 N n i i f = =∑ . Another formula for standard deviation We know that Variance 2 ( ) σ = 2 1 1 ( ) N n i i i f x x = ∑ − = 2 2 1 1 (2) N n i i i i f x x x x = ∑ + − = 2 2 1 1 1 1 2 N n n n i i i i i i i i f x x f x f x = = = ⎛ ⎞ + − ⎜ ⎟ ⎝ ⎠ ∑ ∑ ∑ = 2 2 1 1 1 1 2 N n n n i i i i i i i i f x x f x x f = = = ⎛ ⎞ + − ⎜ ⎟ ⎝ ⎠ ∑ ∑ ∑ Rationalised 2023-24 STATISTICS 277 = 2 2 1 1 N 2 . N N = + − n i i i f x x x x 1 1 1 Here or N N n n i i i i i i x f x x f x = = ⎛ ⎞ = = ⎜ ⎟ ⎝ ⎠ ∑ ∑ = 2 2 2 1 2 1 N n i i i f x x x = ∑ + − 2 2 1 1 N n i i i f x x = = ∑ − or 2 σ = 2 2 2 =1 2 2 1 1 =1 1 1 N N N N n n i i n n i i i i i i i i i i f x f x f x f x − = ⎛ ⎞ ⎛ ⎞ ⎜ ⎟ ⎜ ⎟ ⎝ ⎠ ⎝ ⎠ − = ⎡ − ⎛ ⎞ ⎤ ⎢ ⎜ ⎟ ⎥ ⎢ ⎝ ⎠ ⎥ ⎣ ⎦ ∑ ∑ ∑ ∑ Thus, standard deviation ( ) 2 2 1 =1 1 N N n n i i i i i i σ f x f x = = − ⎛ ⎞ ⎜ ⎟ ⎝ ⎠ ∑ ∑ ... (3) Example 10 Calculate the mean, variance and standard deviation for the following distribution : Class 30-40 40-50 50-60 60-70 70-80 80-90 90-100 Frequency 3 7 12 15 8 3 2 Solution From the given data, we construct the following Table 13.9. Table 13.9 Class Frequency Mid-point f i xi (xi – x) 2 f i (xi – x) 2 (f i) (xi) 30-40 3 35 105 729 2187 40-50 7 45 315 289 2023 50-60 12

55 660 49 588 60-70 15 65 975 9 135 70-80 8 75 600 169 1352 80-90 3 85 255 529 1587 90-100 2 95 190 1089 2178 50 3100 10050 Rationalised 2023-24 278 MATHEMATICS Thus 7 1 1 3100 Mean 62 N 50 i i i x f x = = ∑ = = Variance ( ) 2 σ = 7 2 1 1 ( ) N i i i f x x = ∑ − = 1 10050 50 × = 201 and Standard deviation (σ) = = 201 14 18 . Example 11 Find the standard deviation for the following data : x i 3 8 13 18 23 f i 7 10 15 10 6 Solution Let us form the following Table 13.10: Table 13.10 xi f i f i xi xi 2 f i xi 2 3 7 21 9 63 8 10 80 64 640 13 15 195 169 2535 18 10 180 324 3240 23 6 138 529 3174 48 614 9652 Now, by formula (3), we have σ = ( )2 1 2 N N i i i i ∑ ∑ f x f x − = 1 2 48 9652 (614) 48 × − = 1 463296 376996 48 − Rationalised 2023-24 STATISTICS 279 = 1 293 77 48 × . = 6.12 Therefore, Standard deviation (σ) = 6.12 13.5.4. Shortcut method to find variance and standard deviation Sometimes the values of xi in a discrete distribution or the mid points xi of different classes in a continuous distribution are large and so the calculation of mean and variance becomes tedious and time consuming. By using step-deviation method, it is possible to simplify the procedure. Let the assumed mean be 'A' and the scale be reduced to h 1 times (h being the width of class-intervals). Let the step-deviations or the new values be yi . i.e. i A i x y h − = or xi = A + hyi ... (1) We know that 1 N n i i i f x x = = ∑ ... (2) Replacing x i from (1) in (2), we get x = 1 A) N n i i i f (hy = ∑ + = 1 1 1 A N n n i i i i i f h f y = = + ⎛ ⎞ ⎜ ⎟ ⎝ ⎠ ∑ ∑ = 1 1 1 A N i n n i i i i f h f y = = ⎛ ⎞ + ⎛ ⎞ ∑ ∑ = N 1 A N N n i i i f y . h = + ∑ 1 because N n i i f = ⎛ ⎞ = ⎛ ⎞ ∑ Thus x = A + h y ... (3) Now Variance of the variable x, 2 2 1 1) N n x i i i σ f (x x = = − ∑ = 2 1 1 (A A) N n i i i f hy h y = ∑ + − − (Using (1) and (3)) Rationalised 2023-24 280 MATHEMATICS = 2 2 1 1 ( ) N n i i i f h y y = ∑ − = 2 2 1 ( ) N n i i i h f y y = ∑ − = h 2 × variance of the variable yi i.e. 2 σ x = 2 2 h σ y or σ x = hσ y ... (4) From (3) and (4), we have σ x = 2 2 1 1 N N n n i i i i i i h f y f y = = ⎛ ⎞ − ⎛ ⎞ ⎝ ⎠ ∑ ∑ ... (5) Let us solve Example 11 by the short-cut method and using formula (5) Examples 12 Calculate mean, variance and standard deviation for the following distribution. Classes 30-40 40-50 50-60 60-70 70-80 80-90 90-100 Frequency 3 7 12 15 8 3 2 Solution Let the assumed mean A = 65. Here h = 10 We obtain the following Table 13.11 from the given data : Table 13.11 Class Frequency Mid-point yi = 65 10 i x − yi 2 f i yi f i yi 2 f i x i 30-40 3 35 – 3 9 – 9 27 40-50 7 45 – 2 4 – 14 28 50-60 12 55 – 1 1 – 12 12 60-70 15 65 0 0 0 0 70-80 8 75 1 1 8 8 80-90 3 85 2 4 6 12 9 0-100 2 95 3 9 6 18 N=50 – 15 105 Rationalised 2023-24 STATISTICS 281 Therefore x = 15 A 65 10 62 50 50 i i f y + × = − × = ∑ h Variance 2 σ = ( ) 2 2 2 N 2 N i i h f y f yi i ⎛ ⎞ ∑ ∑ − ⎛ ⎞ ⎝ ⎠ = ( )2 10 2 50 105 (–15) 2 (50) ⎛ ⎞ × − ⎛⎞ ⎝⎠ = 1 [5250 225] 201 25 − = and standard deviation (σ) = 201 = 14.18 EXERCISE 13.2 Find the mean and variance for each of the data in Exercies 1 to 5. 1. 6, 7, 10, 12, 13, 4, 8, 12 2. First n natural numbers 3. First 10 multiples of 3 4. x i 6 10 14 18 24 28 30 f i 2 4 7 12 8 4 3 5. xi 92 93 97 98 102 104 109 f i 3 2 3 2 6 3 3 6. Find the mean and standard deviation using short-cut method. xi 60 61 62 63 64 65 66 67 68 f i 2 1 12 29 25 12 10 4 5 Find the mean and variance for the following frequency distributions in Exercises 7 and 8. 7. Classes 0-30 30-60 60-90 90-120 120-150 150-180 180-210 Frequencies 2 3 5 10 3 5 2 Rationalised 2023-24 282 MATHEMATICS 8. Classes 0-10 10-20 20-30 30-40 40-50 Frequencies 5 8 15 16 6 9. Find the mean, variance and standard deviation using short-cut method Height 70-75 75-80 80-85 85-90 90-95 95-100 100-105105-110 110-115 in cms No. of 3 4 7 7 15 9 6 6 3 children 10. The diameters of circles (in mm) drawn in a design are given below: Diameters 33-36 37-40 41-44 45-48 49-52 No. of circles 15 17 21 22 25 Calculate the standard deviation and mean diameter of the circles. [Hint First make the data continuous by making the classes as 32.5-36.5, 36.5-40.5, 40.5-44.5, 44.5 - 48.5, 48.5 - 52.5 and then proceed.] Miscellaneous Examples Example 13 The variance of 20 observations is 5. If each observation is multiplied by 2, find the new variance of the resulting observations. Solution Let the observations be x 1 , x 2 , ..., x 20 and x be their mean. Given that variance = 5 and n = 20. We know that Variance ( ) 2 20 2 1 1 ( ) i i x x n σ = = − ∑ , i.e., 20 2 1 1 5 ( ) 20 i i x x = = − ∑ or 20 2 1 ( ) i i x x = ∑ − = 100 ... (1) If each observation is multiplied by 2, and the new resulting observations are yi , then yi = 2xi i.e., xi = i y 2 1 Rationalised 2023-24 STATISTICS 283 Therefore 20 20 1 1 1 1 2 20 i i i i y y x n = = = = ∑ ∑ = 20 1 1 2 20 i i . x = ∑ i.e. y = 2 x or x = y 2 1 Substituting the values of xi and x in (1), we get 2 20 1 1 1 100 2 2 i i y y = ⎛ ⎞ − = ⎛

    ∑ , i.e., ∑= =− 20 1 2 400)(i i yy Thus the variance of new observations = 1 2 400 20 2 5 20 × = = × ANote The reader may note that if each observation is multiplied by a constant k, the variance of the resulting observations becomes k 2 times the original variance. Example14 The mean of 5 observations is 4.4 and their variance is 8.24. If three of the observations are 1, 2 and 6, find the other two observations. Solution Let the other two observations be x and y. Therefore, the series is 1, 2, 6, x, y. Now Mean x = 4.4 = 1 2 6 5 + + + +x y or 22 = 9 + x + y Therefore x + y = 13 ... (1) Also variance = 8.24 = 2 5 1)(1 xx n i ∑ i = − i.e. 8.24 = ( ) ( ) ( ) ( ) 1 2 2 2 2 2 2 3 4 2 4 1 6 2 4 4 ( ) 2 4 4 5 . . . x y . x y . + + + + − × + + ×   or 41.20 = 11.56 + 5.76 + 2.56 + x 2 + y 2 –8.8 × 13 + 38.72 Therefore x 2 + y 2 = 97 ... (2) But from (1), we have x 2 + y 2 + 2xy = 169 ... (3) From (2) and (3), we have 2xy = 72 ... (4) Subtracting (4) from (2), we get Rationalised 2023-24 284 MATHEMATICS x 2 + y 2 – 2xy = 97 – 72 i.e. (x – y) 2 = 25 or x – y = ± 5 ... (5) So, from (1) and (5), we get x = 9, y = 4 when x – y = 5 or x = 4, y = 9 when x – y = – 5 Thus, the remaining observations are 4 and 9. Example 15 If each of the observation x1 , x2 , ...,xn is increased by 'a', where a is a negative or positive number, show that the variance remains unchanged. Solution Let x be the mean of x1 , x2 , ...,xn . Then the variance is given by 2 σ1 = 2 1 1 ( ) n i i x x n = ∑ − If 'a is added to each observation, the new observations will be yi = xi + a ... (1) Let the mean of the new observations be y . Then y = 1 1 1 1 ( ) n n i i i i y x a n n = = ∑ ∑ = + = 1 1 1 n n i i i x a n = =   +   ∑ ∑ = ax n na x n n i ∑ i +=+ =1 1 i.e. y = x + a ... (2) Thus, the variance of the new observations 2 σ 2 = 2 1 1 ( ) n i i y y n = ∑ − = 2 ( ) 1 1 axax n n i ∑ i −−+ = [Using (1) and (2)] = 2 1 1 ( ) n i i x x n = ∑ − = 2 σ1 Thus, the variance of the new observations is same as that of the original observations. ANote We may note that adding (or subtracting) a positive number to (or from) each observation of a group does not affect the variance. Example 16 The mean and standard deviation of 100 observations were calculated as 40 and 5.1, respectively by a student who took by mistake 50 instead of 40 for one observation. What are the correct mean and standard deviation? Rationalised 2023-24 STATISTICS 285 Solution Given that number of observations (n) = 100 Incorrect mean (x) = 40, Incorrect standard deviation (σ) = 5.1 We know that ∑= = n i i x n x 1 1 i.e. 100 1 1 40 100 i i x = = ∑ or 100 1 i i x = ∑ = 4000 i.e. Incorrect sum of observations = 4000 Thus the correct sum of observations = Incorrect sum – 50 + 40 = 4000 – 50 + 40 = 3990 Hence Correct mean = correct sum 3990 100 100 = = 39.9 Also Standard deviation σ = 2 2 2 1 1 1 1 n n i i i i x x n = = n   −   ∑ ∑ = ( )2 1 1 2 xx n n i ∑ i − = i.e. 5.1 = 2 2 1 1 Incorrect (40) 100 n i i x = × ∑ − or 26.01 = 2 1 1 Incorrect 100 n i i x = × ∑ – 1600 Therefore Incorrect 2 1 n i i x = ∑ = 100 (26.01 + 1600) = 162601 Now Correct 2 1 n i i x = ∑ = Incorrect ∑= n i i x 1 2 – (50)2 + (40)2 = 162601 – 2500 + 1600 = 161701 Therefore Correct standard deviation Rationalised 2023-24 286 MATHEMATICS = 2 Correct 2 (Correct mean) i x n − ∑ = 161701 2 (39 9) 100 − . = 1617 01 1592 01 . . − = 25 = 5 Miscellaneous Exercise On Chapter 13 1. The mean and variance of eight observations are 9 and 9.25, respectively. If six of the observations are 6, 7, 10, 12, 12 and 13, find the remaining two observations. 2. The mean and variance of 7 observations are 8 and 16, respectively. If five of the observations are 2, 4, 10, 12, 14. Find the remaining two observations. 3. The mean and standard deviation of six observations are 8 and 4, respectively. If each observation is multiplied by 3, find the new mean and new standard deviation of the resulting observations. 4. Given that x is the mean and σ 2 is the variance of n observations x1 , x2 , ...,xn . Prove that the mean and variance of the observations ax1 , ax2 , ax3 , ..., axn are a x and a 2 σ 2 , respectively, (a ≠ 0). 5. The mean and standard deviation of 20 observations are found to be 10 and 2, respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases: (i) If wrong item is omitted. (ii) If it is replaced by 12. 6. The mean and standard deviation of a group of 100 observations were found to be 20 and 3, respectively. Later on it was found that three observations were incorrect, which were recorded as 21, 21 and 18. Find the mean and standard deviation if the incorrect observations are omitted. Summary ÆMeasures of dispersion Range, Quartile deviation, mean deviation, variance, standard deviation are measures of dispersion. Range =

Maximum Value – Minimum Value ÆMean deviation for ungrouped data M M.D. ( ) M.D. (M) i i x – x x – x , n n = ∑ = ∑ Rationalised 2023-24 STATISTICS 287 ÆMean deviation for grouped data M.D. N M.D. M M N ( ) x , ( ) , where N f x x f x f i i i i = = = i ∑ ∑ ∑ ÆVariance and standard deviation for ungrouped data 2 1 2 ( ) i x – x n σ = ∑ , 1 2 (–) i x x n σ = ∑ ÆVariance and standard deviation of a discrete frequency distribution ( ) ( ) 2 1 1 2 2 , N N i i i i σ = ∑ f x x − = σ ∑ f x x − ÆVariance and standard deviation of a continuous frequency distribution ( ) ( ) 2 2 2 1 1 2 , N N N i i i i i i σ = ∑ f x x − = σ ∑ ∑ f x f x − ÆShortcut method to find variance and standard deviation. ( ) 2 2 2 2 2 N N i i i i h σ f y f y   = −   ∑ ∑   , ( )2 2 N N i i i i h σ = ∑ ∑ f y f y − , where i A i x y h − = Historical Note 'Statistics' is derived from the Latin word 'status' which means a political state. This suggests that statistics is as old as human civilisation. In the year 3050 B.C., perhaps the first census was held in Egypt. In India also, about 2000 years ago, we had an efficient system of collecting administrative statistics, particularly, during the regime of Chandra Gupta Maurya (324-300 B.C.). The system of collecting data related to births and deaths is mentioned in Kautilya's Arthshastra (around 300 B.C.) A detailed account of administrative surveys conducted during Akbar's regime is given in Ain-I-Akbari written by Abul Fazl. Captain John Graunt of London (1620-1674) is known as father of vital statistics due to his studies on statistics of births and deaths. Jacob Bernoulli (1654-1705) stated the Law of Large numbers in his book "Ars Conjectandi', published in 1713. Rationalised 2023-24 288 MATHEMATICS — v — The theoretical development of statistics came during the mid seventeenth century and continued after that with the introduction of theory of games and chance (i.e., probability). Francis Galton (1822-1921), an Englishman, pioneered the use of statistical methods, in the field of Biometry. Karl Pearson (1857-1936) contributed a lot to the development of statistical studies with his discovery of Chi square test and foundation of statistical laboratory in England (1911). Sir Ronald A. Fisher (1890-1962), known as the Father of modern statistics, applied it to various diversified fields such as Genetics, Biometry, Education, Agriculture, etc. Rationalised 2023-24vWhere a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle in your hand. – JOHN ARBUTHNOT v 14.1 Event We have studied about random experiment and sample space associated with an experiment. The sample space serves as an universal set for all questions concerned with the experiment. Consider the experiment of tossing a coin two times. An associated sample space is S = {HH, HT, TH, TT}. Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = { HT, TH} We know that the set E is a subset of the sample space S . Similarly, we find the following correspondence between events and subsets of S. Description of events Corresponding subset of 'S' Number of tails is exactly 2 A = {TT} Number of tails is atleast one B = {HT, TH, TT} Number of heads is atmost one C = {HT, TH, TT} Second toss is not head D = { HT, TT} Number of tails is atmost two S = {HH, HT, TH, TT} Number of tails is more than two ϕ The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows. Definition Any subset E of a sample space S is called an event. Chapter 14 PROBABILITY Rationalised 2023-24 290 MATHEMATICS 14.1.1 Occurrence of an event Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that event E has occurred Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that ω ∈ E. If the outcome ω is such that ω ∉ E, we say that the event E has not occurred. 14.1.2 Types of events Events can be classified into various types on the basis of the elements they have. 1. Impossible and Sure Events The empty set ϕ and the sample space S describe events. In fact ϕ is called an impossible event and S, i.e., the whole sample space is called the sure event. To understand these let us consider the experiment of rolling a die. The associated sample space is S = {1, 2, 3, 4, 5, 6} Let E be the event “ the number appears on the die is a multiple of 7”. Can you write the subset associated with the event E? Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event E = ϕ is an impossible event. Now let us take up another event F “the number turns up is odd or even”. Clearly F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event F = S is a sure event. 2. Simple Event If an event E has only one sample point of a sample space, it is called a simple (or elementary) event. In a sample space containing n distinct elements, there are exactly n simple events. For example in the experiment of tossing two coins, a sample space is S={HH, HT, TH, TT} There are four simple events corresponding to this sample space. These are E1 = {HH}, E2 ={HT}, E3 = { TH} and E4 ={TT}. Rationalised 2023-24 PROBABILITY 291 3. Compound Event If an event has more than one sample point, it is called a Compound event. For example, in the experiment of “tossing a coin thrice” the events E: ‘Exactly one head appeared’ F: ‘Atleast one head appeared’ G: ‘Atmost one head appeared’ etc. are all compound events. The subsets of S associated with these events are E={HTT,THT,TTH} F={HTT,THT, TTH, HHT, HTH, THH, HHH} G= {TTT, THT, HTT, TTH} Each of the above subsets contain more than one sample point, hence they are all compound events. 14.1.3 Algebra of events In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations. Let A, B, C be events associated with an experiment whose sample space is S. 1. Complementary Event For every event A, there corresponds another event A′ called the complementary event to A. It is also called the event ‘not A’. For example, take the experiment ‘of tossing three coins’. An associated sample space is S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} Let A={HTH, HHT, THH} be the event ‘only one tail appears’ Clearly for the outcome HTT, the event A has not occurred. But we may say that the event ‘not A’ has occurred. Thus, with every outcome which is not in A, we say that ‘not A’ occurs. Thus the complementary event ‘not A’ to the event A is A′ = {HHH, HTT, THT, TTH, TTT} or A′ = {ω : ω ∈ S and ω ∉A} = S – A. 2. The Event ‘A or B’ Recall that union of two sets A and B denoted by A ∪ B contains all those elements which are either in A or in B or in both. When the sets A and B are two events associated with a sample space, then ‘A ∪ B’ is the event ‘either A or B or both’. This event ‘A∪ B’ is also called ‘A or B’. Therefore Event ‘A or B’ = A ∪ B = {ω : ω ∈ A or ω ∈ B} Rationalised 2023-24 292 MATHEMATICS 3. The Event ‘A and B’ We know that intersection of two sets A ∩ B is the set of those elements which are common to both A and B. i.e., which belong to both ‘A and B’. If A and B are two events, then the set A ∩ B denotes the event ‘A and B’. Thus, A ∩ B = {ω : ω ∈ A and ω ∈ B} For example, in the experiment of ‘throwing a die twice’ Let A be the event ‘score on the first throw is six’ and B is the event ‘sum of two scores is atleast 11’ then A = {(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}, and B = {(5,6), (6,5), (6,6)} so A ∩ B = {(6,5), (6,6)} Note that the set A ∩ B = {(6,5), (6,6)} may represent the event ‘the score on the first throw is six and the sum of the scores is atleast 11’. 4. The Event ‘A but not B’ We know that A–B is the set of all those elements which are in A but not in B. Therefore, the set A–B may denote the event ‘A but not B’.We know that A – B = A ∩ B′ Example 1 Consider the experiment of rolling a die. Let A be the event ‘getting a prime number’, B be the event ‘getting an odd number’. Write the sets representing the events (i) Aor B (ii) A and B (iii) A but not B (iv) ‘not A’. Solution Here S = {1, 2, 3, 4, 5, 6}, A = {2, 3, 5} and B = {1, 3, 5} Obviously (i) ‘A or B’ = A ∪ B = {1, 2, 3, 5} (ii) ‘A and B’ = A ∩ B = {3,5} (iii) ‘A but not B’ = A – B = {2} (iv) ‘not A’ = A′ = {1,4,6} 14.1.4 Mutually exclusive events In the experiment of rolling a die, a sample space is S = {1, 2, 3, 4, 5, 6}. Consider events, A ‘an odd number appears’ and B ‘an even number appears’ Clearly the event A excludes the event B and vice versa. In other words, there is no outcome which ensures the occurrence of events A and B simultaneously. Here A = {1, 3, 5} and B = {2, 4, 6} Clearly A ∩ B = ϕ, i.e., A and B are disjoint sets. In general, two events A and B are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously. In this case the sets A and B are disjoint. Rationalised 2023-24 PROBABILITY 293 Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears' Obviously A = {1, 3, 5} and B = {1, 2, 3} Now 3 ∈ A as well as 3 ∈ B Therefore, A and B are not mutually exclusive events. Remark Simple events of a sample space are always mutually exclusive. 14.1.5 Exhaustive events Consider the experiment of throwing a die. We have S = {1, 2, 3, 4, 5, 6}. Let us define the following events A: 'a number less than 4 appears', B: 'a number greater than 2 but less than 5 appears' and C: 'a number greater than 4 appears'. Then A = {1, 2, 3}, B = {3,4} and C = {5, 6}. We observe that A ∪ B ∪ C = {1, 2, 3} ∪ {3, 4} ∪ {5, 6} = S. Such events A, B and C are called exhaustive events. In general, if E1 , E2 , ..., En are n events of a sample space S and if 1 2 3 1 E E E E E S n n i i ... = ∪ ∪ ∪ ∪ = ∪ = then E1 , E2 , ..., En are called exhaustive events.In other words, events E1 , E2 , ..., En are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed. Further, if Ei ∩ Ej = ϕ for i ≠ j i.e., events Ei and Ej are pairwise disjoint and SE 1 =∪ = i n i , then events E1 , E2 , ..., En are called mutually exclusive and exhaustive events. We now consider some examples. Example 2 Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with this experiment A: 'the sum is even'. B: 'the sum is a multiple of 3'. C: 'the sum is less than 4'. D: 'the sum is greater than 11'. Which pairs of these events are mutually exclusive? Rationalised 2023-24 294 MATHEMATICS Solution There are 36 elements in the sample space S = {(x, y): x, y = 1, 2, 3, 4, 5, 6}. Then A = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)} B = {(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)} C = {(1, 1), (2, 1), (1, 2)} and D = {(6, 6)} We find that A ∩ B = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)} ≠ ϕ Therefore, A and B are not mutually exclusive events. Similarly A ∩ C ≠ ϕ, A ∩ D ≠ ϕ, B ∩ C ≠ ϕ and B ∩ D ≠ ϕ. Thus, the pairs of events, (A, C), (A, D), (B, C), (B, D) are not mutually exclusive events. Also C ∩ D = ϕ and so C and D are mutually exclusive events. Example 3 A coin is tossed three times, consider the following events. A: 'No head appears', B: 'Exactly one head appears' and C: 'Atleast two heads appear'. Do they form a set of mutually exclusive and exhaustive events? Solution The sample space of the experiment is S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} and A = {TTT}, B = {HTT, THT, TTH}, C = {HHT, HTH, THH, HHH} Now A ∪ B ∪ C = {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH} = S Therefore, A, B and C are exhaustive events. Also, A ∩ B = ϕ, A ∩ C = ϕ and B ∩ C = ϕ Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive. Hence, A, B and C form a set of mutually exclusive and exhaustive events. EXERCISE 14.1 1. A die is rolled. Let E be the event "die shows 4" and F be the event "die shows even number". Are E and F mutually exclusive? 2. A die is thrown. Describe the following events: (i) A: a number less than 7 (ii) B: a number greater than 7 (iii) C: a multiple of 3 (iv) D: a number less than 4 (v) E: an even number greater than 4 (vi) F: a number not less than 3 Also find A ∪ B, A ∩ B, B ∪ C, E ∩ F, D ∩ E, A – C, D – E, E ∩ F', F' Rationalised 2023-24 PROBABILITY 295 3. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events: A: the sum is greater than 8, B: 2 occurs on either die C: the sum is at least 7 and a multiple of 3. Which pairs of these events are mutually exclusive? 4. Three coins are tossed once. Let A denote the event 'three heads show", B denote the event "two heads and one tail show", C denote the event" three tails show and D denote the event 'a head shows on the first coin". Which events are (i) mutually exclusive? (ii) simple? (iii) Compound? 5. Three coins are tossed. Describe (i) Two events which are mutually exclusive. (ii) Three events which are mutually exclusive and exhaustive. (iii) Two events, which are not mutually exclusive. (iv) Two events which are mutually exclusive but not exhaustive. (v) Three events which are mutually exclusive but not exhaustive. 6. Two dice are thrown. The events A, B and C are as follows: A: getting an even number on the first die. B: getting an odd number on the first die. C: getting the sum of the numbers on the dice ≤ 5. Describe the events

(i) A′ (ii) not B (iii) A or B (iv) A and B (v) A but not C (vi) B or C (vii) B and C (viii) A ∩ B′ ∩ C′ 7. Refer to question 6 above, state true or false: (give reason for your answer) (i) A and B are mutually exclusive (ii) A and B are mutually exclusive and exhaustive (iii) A = B′ (iv) A and C are mutually exclusive (v) A and B′ are mutually exclusive. (vi) A′, B′, C are mutually exclusive and exhaustive. 14.2 Axiomatic Approach to Probability In earlier sections, we have considered random experiments, sample space and events associated with these experiments. In our day to day life we use many words about the chances of occurrence of events. Probability theory attempts to quantify these chances of occurrence or non occurrence of events. Rationalised 2023-24 296 MATHEMATICS In earlier classes, we have studied some methods of assigning probability to an event associated with an experiment having known the number of total outcomes. Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities. Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0,1] satisfying the following axioms (i) For any event E, P (E) ≥ 0 (ii) P(S) = 1 (iii) If E and F are mutually exclusive events, then P(E ∪ F) = P(E) + P(F). It follows from (iii) that P(ϕ) = 0. To prove this, we take F = ϕ and note that E and ϕ are disjoint events. Therefore, from axiom (iii), we get P (E ∪ ϕ) = P (E) + P (ϕ) or P(E) = P(E) + P (ϕ) i.e. P (ϕ) = 0. Let S be a sample space containing outcomes 1 2 , ,..., ω ω ωn , i.e., S = {ω1 , ω2 , ..., ωn } It follows from the axiomatic definition of probability that (i) 0 ≤ P (ωi) ≤ 1 for each ωi ∈ S (ii) P (ω1) + P (ω2) + ... + P (ωn) = 1 (iii) For any event A, P(A) = ∑ P(ωi), ωi ∈ A. ANote It may be noted that the singleton {ωi } is called elementary event and for notational convenience, we write P(ωi) for P({ωi }). For example, in ‘a coin tossing’ experiment we can assign the number 1 2 to each of the outcomes H and T. i.e. P(H) = 1 2 and P(T) = 1 2 (1) Clearly this assignment satisfies both the conditions i.e., each number is neither less than zero nor greater than 1 and P(H) + P(T) = 2 1 + 2 1 = 1 Therefore, in this case we can say that probability of H = 2 1 , and probability of T = 2 1 If we take P(H) = 4 1 and P(T) = 4 3 ... (2) Rationalised 2023-24 PROBABILITY 297 Does this assignment satisfy the conditions of axiomatic approach? Yes, in this case, probability of H = 1 4 and probability of T = 4 3 . We find that both the assignments (1) and (2) are valid for probability of H and T. In fact, we can assign the numbers p and (1 – p) to both the outcomes such that 0 ≤ p ≤ 1 and P(H) + P(T) = p + (1 – p) = 1 This assignment, too, satisfies both conditions of the axiomatic approach of probability. Hence, we can say that there are many ways (rather infinite) to assign probabilities to outcomes of an experiment. We now consider some examples. Example 4 Let a sample space be S = {ω1 , ω2 ,..., ω6 }.Which of the following assignments of probabilities to each outcome are valid? Outcomes ω 1 ω 2 ω 3 ω 4 ω 5 ω 6 (a) 6 1 6 1 6 1 6 1 6 1 6 1 (b) 1 0 0 0 0 0 (c) 8 1 3 2 3 1 3 1 4 1 − 3 1 − (d) 12 1 12 1 6 1 6 1 6 1 2 3 (e) 0.1 0.2 0.3 0.4 0.5 0.6 Solution (a) Condition (i): Each of the number p(ωi) is positive and less than one. Condition (ii): Sum of probabilities = 1 6 1 6 1 6 1 6 1 6 1 6 1 =+++++ Therefore, the assignment is valid (b) Condition (i): Each of the number p(ωi) is either 0 or 1. Condition (ii) Sum of the probabilities = 1 + 0 + 0 + 0 + 0 + 0 = 1 Therefore, the assignment is valid (c) Condition (i) Two of the probabilities p(ω5) and p(ω6) are negative, the assignment is not valid (d) Since p(ω6) = 3 2 > 1, the assignment is not valid Rationalised 2023-24 298 MATHEMATICS (e) Since, sum of probabilities = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1, the assignment is not valid. 14.2.1 Probability of an event Let S be a sample space associated with the experiment ‘examining three consecutive pens produced by a machine and classified as Good (non-defective) and bad (defective)’. We may get 0, 1, 2 or 3 defective pens as result of this examination. A sample space associated with this experiment is S = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}, where B stands for a defective or bad pen and G for a non – defective or good pen. Let the probabilities assigned to the outcomes be as follows Sample point: BBB BBG BGB GBB BGG GBG GGB GGG Probability: 8 1 8 1 8 1 8 1 8 1 8 1 8 1 8 1 Let event A: there is exactly one defective pen and event B: there are atleast two defective pens. Hence A = {BGG, GBG, GGB} and B = {BBG, BGB, GBB, BBB} Now P(A) = ∑ ∀ ∈ P(ω), ω A i i = P(BGG) + P(GBG) + P(GGB) = 8 3 8 1 8 1 8 1

=++ and P(B) = ∑ ∀ ∈ P(ω), ω B i i = P(BBG) + P(BGB) + P(GBB) + P(BBB) = 2 1 8 4 8 1 8 1 8 1 8 1 ==+++ Let us consider another experiment of 'tossing a coin "twice" The sample space of this experiment is S = {HH, HT, TH, TT} Let the following probabilities be assigned to the outcomes P(HH) = 4 1 , P(HT) = 7 1 , P(TH) = 7 2 , P(TT) = 28 9 Clearly this assignment satisfies the conditions of axiomatic approach. Now, let us find the probability of the event E: 'Both the tosses yield the same result'. Here E = {HH, TT} Now P(E) = Σ P(wi), for all wi ∈ E Rationalised 2023-24 PROBABILITY 299 = P(HH) + P(TT) = 7 4 28 9 4 1 =+ For the event F: 'exactly two heads', we have F = {HH} and P(F) = P(HH) = 1 4 14.2.2 Probabilities of equally likely outcomes Let a sample space of an experiment be S = {ω1 , ω2 ,..., ωn }. Let all the outcomes are equally likely to occur, i.e., the chance of occurrence of each simple event must be same. i.e. P(ωi) = p, for all ωi ∈ S where 0 ≤ p ≤ 1 Since 1 P(ω) 1 n i i= ∑ = i.e., p + p + ... + p (n times) = 1 or np = 1 i.e., p = 1 n Let S be a sample space and E be an event, such that n(S) = n and n(E) = m. If each out come is equally likely, then it follows that P(E) m n = = Number of outcomes favourable to E Total possible outcomes 14.2.3 Probability of the event 'A or B' Let us now find the probability of event 'A or B', i.e., P (A ∪ B) Let A = {HHT, HTH, THH} and B = {HTH, THH, HHH} be two events associated with 'tossing of a coin thrice' Clearly A ∪ B = {HHT, HTH, THH, HHH} Now P (A ∪ B) = P(HHT) + P(HTH) + P(THH) + P(HHH) If all the outcomes are equally likely, then ( ) 1 1 1 1 4 1 P A B 8 8 8 8 8 2 ∪ = + + + = = Also P(A) = P(HHT) + P(HTH) + P(THH) = 3 8 Rationalised 2023-24 300 MATHEMATICS and P(B) = P(HTH) + P(THH) + P(HHH) = 3 8 Therefore P(A) + P(B) = 3 3 6 8 8 8 + = It is clear that P(A ∪ B) ≠ P(A) + P(B) The points HTH and THH are common to both A and B. In the computation of P(A) + P(B) the probabilities of points HTH and THH, i.e., the elements of A ∩B are included twice. Thus to get the probability P(A∪ B) we have to subtract the probabilities of the sample points in A ∩ B from P(A) + P(B) i.e. P(A B) ∪ = P(A) P(B) P(ω) ω A B i i + − ∑ ∀ ∈ ∩ , = P A P B P A B ( ) ( ) ( ) + − ∩ Thus we observe that, ∪ = + − ∩)BA(P)B(P)A(P)BA(P In general, if A and B are any two events associated with a random experiment, then by the definition of probability of an event, we have P A B (∪ = ∑ ∀ ∈ ∪) p , (ω ω A B i i) . Since A B = (A–B) (A B) (B–A) ∪ ∪ ∩ ∪ , we have P(A ∪ B) = [P(ω) ω (A–B) P(ω) ω A B + i] [i i] ∑ ∀ ∈ + ∑ ∀ ∈ ∩ i [∑ ∀ ∈ P(ω) ω B – A i i] (because A–B, A ∩ B and B – A are mutually exclusive) ... (1) Also P(A) P(B) (+ = ∑ ∀ ∈ ∑ ∀ ∈ [p ω) A + (ω) ω B i i ω] [p i i] = [∑ ∀ ∈ ∪ ∩ P(ω) ω (A–B) (A B) + i i] [∑ ∀ ∈ ∪ ∩ P(ω) ω (B – A) (A B) i i] = [∑ ∀ ∈ ∑ ∀ ∈ ∩ P(ω) ω (A – B) + P(ω) ω (A B) i i] [i i] + [∑ ∀ ∈ P(ω) ω (B–A) i i] + [∑ ∀ ∈ ∩ P(ω) ω (A B) i i] = P(A B) P(∪ + ∑ ∀ ∈ ∩ [ω) ω A B i i] [using (1)] = P(A B)+ P(A B) ∪ ∩ . Hence P(A B) P (A) +P(B) – P(A B) ∪ = ∩ . Alternatively, it can also be proved as follows: A ∪ B = A ∪ (B – A), where A and B – A are mutually exclusive, and B = (A ∩ B) ∪ (B – A), where A ∩ B and B – A are mutually exclusive. Using Axiom (iii) of probability, we get Rationalised 2023-24 PROBABILITY 301 P (A ∪B) = P (A) + P (B – A) ... (2) and P(B) = P (A ∩ B) + P (B – A) ... (3) Subtracting (3) from (2) gives P (A ∪ B) – P(B) = P(A) – P (A ∩ B) or P(A ∪ B) = P(A) + P (B) – P (A ∩ B) The above result can further be verified by observing the Venn Diagram (Fig 14.1) If A and B are disjoint sets, i.e., they are mutually exclusive events, then A ∩ B = ϕ Therefore P(A B) = P ( ) = 0 ∩ ϕ Thus, for mutually exclusive events A and B, we have ∪ = +)B(P)A(P)BA(P , which is Axiom (iii) of probability. 14.2.4 Probability of event 'not A' Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10} If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability of each outcome is 10 1 Now P(A) = P(2) + P(4) + P(6) + P(8) = 1 1 1 1 4 2 10 10 10 10 10 5 + + + = = Also event 'not A' = A′ = {1, 3, 5, 7, 9, 10} Now P(A′) = P(1) + P(3) + P(5) + P(7) + P(9) + P(10) Fig 14.1 Rationalised 2023-24 302 MATHEMATICS = 6 3 10 5 = Thus, P(A′) = 3 5 =)A(P1 5 2 1 −=− Also, we know that A′ and A are mutually exclusive and exhaustive events i.e., A ∩ A′ = ϕ and A ∪ A′ = S or P(A ∪ A′) = P(S) Now P(A) + P(A′) = 1, by using axioms (ii) and (iii). or P(A′) = P(not A) = 1 – P(A) We now consider some examples and exercises having equally likely outcomes unless stated otherwise. Example 5 One card is drawn from a well shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be (i) a diamond (ii) not an ace (iii) a black card (i.e., a club or, a spade) (iv) not a diamond (v) not a black card. Solution When a card is drawn from a well shuffled deck of 52 cards, the number of possible outcomes is 52. (i) Let A be the event 'the card drawn is a diamond' Clearly the number of elements in set A is 13. Therefore, P(A) = 13 1 52 4 = i.e. probability of a diamond card = 1 4 (ii) We assume that the event 'Card drawn is an ace' is B Therefore 'Card drawn is not an ace' should be B′. We know that P(B′) = 1 – P(B) = 13 12 13 1 1 52 4 1 =–=– (iii) Let C denote the event 'card drawn is black card' Therefore, number of elements in the set C = 26 i.e. P(C) = 2 1 52 26 = Rationalised 2023-24 PROBABILITY 303 Thus, probability of a black card = 2 1 . (iv) We assumed in (i) above that A is the event 'card drawn is a diamond', so the event 'card drawn is not a diamond' may be denoted as A' or 'not A' Now P(not A) = 1 – P(A) = 4 3 4 1 1 =– (v) The event 'card drawn is not a black card' may be denoted as C′ or 'not C'. We know that P(not C) = 1 – P(C) = 2 1 2 1 1 =– Therefore, probability of not a black card = 2 1 Example 6 A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be (i) red, (ii) yellow, (iii) blue, (iv) not blue, (v) either red or blue. Solution There are 9 discs in all so the total number of possible outcomes is 9. Let the events A, B, C be defined as A: 'the disc drawn is red' B: 'the disc drawn is yellow' C: 'the disc drawn is blue'. (i) The number of red discs = 4, i.e., n (A) = 4 Hence P(A) = 9 4 (ii) The number of yellow discs = 2, i.e., n (B) = 2 Therefore, P(B) = 9 2 (iii) The number of blue discs = 3, i.e., n(C) = 3 Therefore, P(C) = 3 1 9 3 = (iv) Clearly the event 'not blue' is 'not C'. We know that P(not C) = 1 – P(C) Rationalised 2023-24 304 MATHEMATICS Therefore P(not C) = 3 2 3 1 1 =– (v) The event 'either red or blue' may be described by the set 'A or C' Since, A and C are mutually exclusive events, we have P(A or C) = P (A ∪ C) = P(A) + P(C) = 9 7 3 1 9 4 =+ Example 7Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that (a) Both Anil and Ashima will not qualify the examination. (b) Atleast one of them will not qualify the examination and (c) Only one of them will qualify the examination. Solution Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that P(E) = 0.05, P(F) = 0.10 and P(E ∩ F) = 0.02. Then (a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as E´ ∩ F´. Since, E´ is 'not E', i.e., Anil will not qualify the examination and F´ is 'not F', i.e., Ashima will not qualify the examination. Also E´ ∩ F´ = (E ∪ F)´ (by Demorgan's Law) Now P(E ∪ F) = P(E) + P(F) – P(E ∩ F) or P(E ∪ F) = 0.05 + 0.10 – 0.02 = 0.13 Therefore P(E´ ∩ F´) = P(E ∪ F)´ = 1 – P(E ∪ F) = 1 – 0.13 = 0.87 (b) P (atleast one of them will not qualify) = 1 – P(both of them will qualify) = 1 – 0.02 = 0.98 (c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima Rationalised 2023-24 PROBABILITY 305 will qualify) i.e., E ∩ F´ or E´ ∩ F, where E ∩ F´ and E´ ∩ F are mutually exclusive. Therefore, P(only one of them will qualify) = P(E ∩ F´ or E´ ∩ F) = P(E ∩ F´) + P(E´ ∩ F) = P (E) – P(E ∩ F) + P(F) – P (E ∩ F) = 0.05 – 0.02 + 0.10 – 0.02 = 0.11 Example 8 A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men? Solution The total number of persons = 2 + 2 = 4. Out of these four person, two can be selected in 4C2 ways. (a) No men in the committee of two means there will be two women in the committee. Out of two women, two can be selected in 2C 1 2 = way. Therefore ( ) 2 2 4 2 C 1 2 1 1 P no man C 4 3 6 × × = = = × (b) One man in the committee means that there is one woman. One man out of 2 can be selected in 2C1 ways and one woman out of 2 can be selected in 2C1 ways. Together they can be selected in 2 2 C C 1 1 × ways. Therefore ( ) 2 2 1 1 4 2 C C 2 2 2 P One man C 2 3 3 × × = = = × (c) Two men can be selected in 2C2 way. Hence ( ) 2 2 4 4 2 2 C 1 1 P Two men C C 6 = = = EXERCISE 14.2 1. Which of the following can not be valid assignment of probabilities for outcomes of sample Space S = {ω ω ω ω ω ω ω 1 2 3 4 5 6 7 , , , , , , } Rationalised 2023-24 306 MATHEMATICS Assignment ω 1 ω 2 ω 3 ω 4 ω 5 ω 6 ω 7 (a) 0.1 0.01 0.05

0.03 0.01 0.2 0.6 (b) 7 1 7 1 7 1 7 1 7 1 7 1 7 1 (c) 0.1 0.2 0.3 0.4 0.5 0.6 0.7 (d) – 0.1 0.2 0.3 0.4 – 0.2 0.1 0.3 (e) 14 1 14 2 14 3 14 4 14 5 14 6 14 15 2. A coin is tossed twice, what is the probability that atleast one tail occurs? 3. A die is thrown, find the probability of following events: (i) A prime number will appear, (ii) A number greater than or equal to 3 will appear, (iii) A number less than or equal to one will appear, (iv) A number more than 6 will appear, (v) A number less than 6 will appear. 4. A card is selected from a pack of 52 cards. (a) How many points are there in the sample space? (b) Calculate the probability that the card is an ace of spades. (c) Calculate the probability that the card is (i) an ace (ii) black card. 5. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. find the probability that the sum of numbers that turn up is (i) 3 (ii) 12 6. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman? 7. A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up. From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts. 8. Three coins are tossed once. Find the probability of getting (i) 3 heads (ii) 2 heads (iii) atleast 2 heads (iv) atmost 2 heads (v) no head (vi) 3 tails (vii) exactly two tails (viii) no tail (ix) atmost two tails 9. If 11 2 is the probability of an event, what is the probability of the event 'not A'. 10. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant Rationalised 2023-24 PROBABILITY 307 11. In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game? [Hint order of the numbers is not important.] 12. Check whether the following probabilities P(A) and P(B) are consistently defined (i) P(A) = 0.5, P(B) = 0.7, P(A ∩ B) = 0.6 (ii) P(A) = 0.5, P(B) = 0.4, P(A ∪ B) = 0.8 13. Fill in the blanks in following table: P(A) P(B) P(A ∩ B) P(A ∪ B) (i) 1 3 1 5 1 15 . . . (ii) 0.35 . . . 0.25 0.6 (iii) 0.5 0.35 . . . 0.7 14. Given P(A) = 5 3 and P(B) = 5 1 . Find P(A or B), if A and B are mutually exclusive events. 15. If E and F are events such that P(E) = 4 1 , P(F) = 2 1 and P(E and F) = 8 1 , find (i) P(E or F), (ii) P(not E and not F). 16. Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive. 17. A and B are events such that P(A) = 0.42, P(B) = 0.48 and P(A and B) = 0.16. Determine (i) P(not A), (ii) P(not B) and (iii) P(A or B) 18. In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology. 19. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both? 20. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Hindi examination? Rationalised 2023-24 308 MATHEMATICS 21. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that (i) The student opted for NCC or NSS. (ii) The student has opted neither NCC nor NSS. (iii) The student has opted NSS but not NCC. Miscellaneous Examples Example 9 On her vacations Veena visits four cities (A, B, C and D) in a random order. What is the probability that she visits (i) A before B? (ii) A before B and B before C? (iii) A first and B last? (iv) A either first or second? (v) A just before B? Solution The number of arrangements (orders) in which Veena can visit four cities A, B, C, or D is 4! i.e., 24.Therefore, n (S) = 24. Since the number of elements in the sample space of the experiment is 24 all of these outcomes are considered to be equally likely. A sample space for the experiment is S = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB BACD, BADC, BDAC, BDCA, BCAD, BCDA CABD, CADB, CBDA, CBAD, CDAB, CDBA DABC, DACB, DBCA, DBAC, DCAB, DCBA} (i) Let the event 'she visits A before B' be denoted by E

Therefore,E = {ABCD, CABD, DABC, ABDC, CADB, DACB ACBD, ACDB, ADBC, CDAB, DCAB, ADCB} Thus ( ) ( ) ( ) E 12 1 P E S 24 2 n n = = = (ii) Let the event 'Veena visits A before B and B before C' be denoted by F. Here F = {ABCD, DABC, ABDC, ADBC} Therefore, ( ) ( ) ( ) F 4 1 P F S 24 6 n n = = = Students are advised to find the probability in case of (iii), (iv) and (v). Rationalised 2023-24 PROBABILITY 309 Example 10 Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all Kings (ii) 3 Kings (iii) atleast 3 Kings. Solution Total number of possible hands = 52C7 (i) Number of hands with 4 Kings = 4 48 C C 4 3 × (other 3 cards must be chosen from the rest 48 cards) Hence P (a hand will have 4 Kings) = 4 48 4 3 52 7 C C 1 C 7735 × = (ii) Number of hands with 3 Kings and 4 non-King cards = 4 48 C C 3 4 × Therefore P (3 Kings) = 4 48 3 4 52 7 C C 9 C 1547 × = (iii) P(atleast 3 King) = P(3 Kings or 4 Kings) = P(3 Kings) + P(4 Kings) = 9 1 46 1547 7735 7735 + = Example 11 If A, B, C are three events associated with a random experiment, prove that P A B C (∪ ∪) = P A P B +P C P A B P A C ( ) + ( ) ( ) − ∩ − ∩ ( ) ( ) – P (B ∩ C) + P (A ∩ B ∩ C) Solution Consider E = B ∪ C so that P (A ∪ B ∪ C) = P (A ∪ E) = P A P E P ( ) + − ( ) ( ) A E ∩ ... (1) Now P E P B C ( ) = ∪ ( ) = + − ∩ P B P C P B C ( ) ( ) ( ) ... (2) Also A E A B C ∩ = ∩ ∪ ( ) = (A B A C ∩ ∪ ∩) ( ) [using distribution property of intersection of sets over the union]. Thus P A E P A B P A C (∩ = ∩ + ∩) ( ) ( ) – P A B A C (∩ ∩ ∩) ( ) Rationalised 2023-24 310 MATHEMATICS = P A B P A C (∩ + ∩) ( ) – P A B C [∩ ∩] ... (3) Using (2) and (3) in (1), we get P A B C P A P B P C P B C [∪ ∪ = + + − ∩] ( ) ( ) ( ) ( ) – P A B P A C P A B C (∩ − ∩ + ∩ ∩) ( ) ( ) Example 12 In a relay race there are five teams A, B, C, D and E. (a) What is the probability that A, B and C finish first, second and third, respectively. (b) What is the probability that A, B and C are first three to finish (in any order) (Assume that all finishing orders are equally likely) Solution If we consider the sample space consisting of all finishing orders in the first three places, we will have 5 P3 , i.e., ( ) 5! 5 3 ! − = 5 × 4 × 3 = 60 sample points, each with a probability of 1 60 . (a) A, B and C finish first, second and third, respectively. There is only one finishing order for this, i.e., ABC. Thus P(A, B and C finish first, second and third respectively) = 1 60 . (b) A, B and C are the first three finishers. There will be 3! arrangements for A, B and C. Therefore, the sample points corresponding to this event will be 3! in number. So P (A, B and C are first three to finish) 3! 6 1 60 60 10 = = = Miscellaneous Exercise on Chapter 14 1. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that (i) all will be blue? (ii) atleast one will be green? 2. 4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade? Rationalised 2023-24 PROBABILITY 311 3. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine (i) P(2) (ii) P(1 or 3) (iii) P(not 3) 4. In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets. 5. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that (a) you both enter the same section? (b) you both enter the different sections? 6. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope. 7. A and B are two events such that P(A) = 0.54, P(B) = 0.69 and P(A ∩ B) = 0.35. Find (i) P(A ∪ B) (ii) P(A′ ∩ B′) (iii) P(A ∩ B′) (iv) P(B ∩ A′) 8. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. Particulars of five persons are as follows: S. No. Name Sex Age in years 1. Harish M 30 2. Rohan M 33 3. Sheetal F 46 4. Alis F 28 5. Salim M 41 A person is selected at random from this group to act as a spokesperson. What is the probability that the spokesperson will be either male or over 35 years? 9. If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when, (i) the digits are repeated? (ii) the repetition of digits is not allowed? 10. The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e., from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase? Rationalised 2023-24 312 MATHEMATICS Summary In this Chapter, we studied about the axiomatic approach of probability. The main features of this Chapter are as follows: ÆEvent: A subset of the sample space ÆImpossible event : The empty set ÆSure event: The whole sample space ÆComplementary event or 'not event' : The set A′ or S – A ÆEvent A or B: The set A ∪ B ÆEvent A and B: The set A ∩ B ÆEvent A and not B: The set A – B ÆMutually exclusive event: A and B are mutually exclusive if A ∩ B = φ ÆExhaustive and mutually exclusive events: Events E1 , E2 ,..., En are mutually exclusive and exhaustive if E1 ∪ E2 ∪ ...∪ En = S and Ei ∩ Ej = φ V i ≠ j ÆProbability: Number P (ωi) associated with sample point ω i such that (i) 0 ≤ P (ωi) ≤ 1 (ii) ∑P(ωi) for all ωi ∈ S = 1 (iii) P(A) = ∑P(ωi) for all ωi ∈A. The number P (ωi) is called probability of the outcome ωi . ÆEqually likely outcomes: All outcomes with equal probability ÆProbability of an event: For a finite sample space with equally likely outcomes Probability of an event (A) P(A) (S) n n = , where n(A) = number of elements in the set A, n(S) = number of elements in the set S. ÆIf A and B are any two events, then P(A or B) = P(A) + P(B) – P(A and B) equivalently, P(A ∪ B) = P(A) + P(B) – P(A ∩ B) ÆIf A and B are mutually exclusive, then P(A or B) = P(A) + P(B) ÆIf A is any event, then P(not A) = 1 – P(A) Rationalised 2023-24 PROBABILITY 313 — v — Historical Note Probability theory like many other branches of mathematics, evolved out of practical consideration. It had its origin in the 16th century when an Italian physician and mathematician Jerome Cardan (1501–1576) wrote the first book on the subject "Book on Games of Chance" (Biber de Ludo Aleae). It was published in 1663 after his death. In 1654, a gambler Chevalier de Metre approached the well known French Philosopher and Mathematician Blaise Pascal (1623–1662) for certain dice problem. Pascal became interested in these problems and discussed with famous French Mathematician Pierre de Fermat (1601–1665). Both Pascal and Fermat solved the problem independently. Besides, Pascal and Fermat, outstanding contributions to probability theory were also made by Christian Huygenes (1629– 1665), a Dutchman, J. Bernoulli (1654–1705), De Moivre (1667–1754), a Frenchman Pierre Laplace (1749–1827), the Russian P.L Chebyshev (1821– 1897), A. A Markov (1856–1922) and A. N Kolmogorove (1903–1987). Kolmogorov is credited with the axiomatic theory of probability. His book 'Foundations of Probability' published in 1933, introduces probability as a set function and is considered a classic. Rationalised 2023-24