Name: 肖涵溥
StudentID:31360164

SI 140
Probability and Statistics
Semester Spring 2019
Assignment 1

Problem 1. If two sets have identical complements, then they are themselves identical. Show this in two ways:(i) by verbal definition, (ii) by using formula$(A^c)^c$

Solution:
(i) It means for any ω, satisfy $I_A(\omega) = I_B(\omega)$, obviously A and B are identical.
(ii) $I_{(A^C)^C} = 1 - I_{A^C} = 1 - (1 - I_A) = I_A$, so A and B are identical. □

Problem 2. Show that

$$(A \cup B) \cap C \neq A \cup (B \cap C)$$

but also give some special cases where there is equality.

Solution:
inequality example:

$$\begin{cases} A &= \{1, 2, 3\} \\ B &= \{1, 2\} \\ C &= \{1\} \end{cases}$$

In this case, $(A \cup B) \cap C = \{1\}$, but $A \cup (B \cap C) = \{1, 2, 3\}$.
equality examples:

$$A = B = C$$

$$A = \emptyset$$

$$C = A \cup B$$

□

Problem 3. Show that A $\subset$ B if and only if AB = A; or A$\cup$ B = B. (So the relation of inclusion can be defined through identity and the operations.)

Solution:
$A \subset B \iff$ for any $\omega \in A$, satisfy $\omega \in B$.
$A \cup B = B \iff 1 - I_{(A \cup B)^C} = 1 - I_{A^C} I_{B^C} = I_A + I_B - I_A I_B = I_B \iff I_A I_B = I_A \iff AB = A$

For $\omega_0 \in A$, suppose $A \subset B$, then $\omega_0 \in A$, so that: $I_A I_B = I_A = 1$.
So $A \subset B \implies$ AB = A or A$\cup$ B = B

For $I_A I_B = I_A = 1, \omega \in A$ and $\omega \in B$
For $I_A I_B = I_A = 0, \omega \notin A$
So, if $\omega \in A$, it must have $\omega \in B$, which means that $AB = A \implies A \subset B$.

$\square$

Problem 4. Show that there is a distributive law also for difference:

$$(A \backslash B) \cap C = (A \cap C) \setminus (B \cap C).$$

Is the dual

$$(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$$

also true?

Solution:
Accroding to $A \backslash B = A - A \cap B$, it follows $I_{A \backslash B} = I_A - I_A I_B$,
(1)

$$\begin{aligned} (A \backslash B) \cap C &= (I_A - I_A I_B) I_C \\ (A \cap C) \setminus (B \cap C) &= I_A I_C - I_A I_C I_B I_C = I_A I_C - I_A I_B I_C \end{aligned}$$

obviously $(I_A - I_A I_B) I_C = I_A I_C - I_A I_B I_C$.
(2) Yes.

$$\begin{aligned} (A \cap B) \backslash C &= I_A I_B - I_A I_B I_C \\ (A \backslash C) \cap (B \backslash C) &= (I_A - I_A I_C)(I_B - I_B I_C) = I_A I_B - 2 I_A I_B I_C + I_A I_B I_C = I_A I_B - I_A I_B I_C \end{aligned}$$

$\square$

Problem 5. Show that A $\subset$ B if and only if $I_A \leq I_B$; and A $\cap$ B $= \emptyset$ if and only if $I_A I_B$ =0.

Solution:
(1)
Suppose that $A \subset B, I_A = 1 > I_B = 0$, then there exist ω, s.t. $\omega \in A$ and $\omega \notin B$, that's impossible. Thus $A \subset B \implies I_A \leq I_B$
Suppose that $I_A \leq I_B$, and there exist $\omega_0 \in A, \omega_0 \notin B$. Then $I_A(\omega) = 1 > I_B(\omega) = 0$, that's impossible. Thus $I_A \leq I_B \implies A \subset B$
(2)

Name: 肖涵溥
StudentID:31360164

$A \cap B = \emptyset$ follows that, if $\omega \in A\,(I_A = 1)$, then $\omega \notin B\,(I_B = 0)$, and if $\omega \in B\,(I_B = 1)$, then $\omega \notin A\,(I_A = 0)$.
Thus $I_A\,(\omega_0)\,I_B\,(\omega_0) = 0$. $\square$

Problem 6. Given n events $A_1, A_2, ..., A_n$ and indicators $I_j, j = 1, ..., n$ ($I_j = 1$ if A_j occur, else $I_j = 0$). Let $X = \sum_{j=1}^{n} I_j$ be the number of events that occur. You need to find the number of pairs of distinct events that occur: (i) Write your answer in terms of X. (ii) Write your answer in terms of indicators.

Solution:
(i)

$$P = \binom{X}{2} = \frac{X\,(X-1)}{2}$$

(ii)

$$P = \sum_{i>j} I_i I_j$$

$\square$

Problem 7. Express $I_{A\cup B\cup C}$ as a polynomial of I_A, I_B, I_C.

Solution:

$$\begin{aligned} I_{A\cup B\cup C} &= 1 - I_{(A\cup B\cup C)^C} \\ &= 1 - I_{A^C} I_{B^C} I_{C^C} \\ &= 1 - (1 - I_A)\,(1 - I_B)\,(1 - I_C) \\ &= I_A + I_B + I_C - I_A I_B - I_A I_C - I_B I_C + I_A I_B I_C \end{aligned}$$

$\square$

Problem 8. Show that

$$I_{ABC} = I_A + I_B + I_C - I_{A\cup B} - I_{A\cup C} - I_{B\cup C} + I_{A\cup B\cup C}$$

You can verify this directly, but it is nicer to derive it from problem 7 by duality.

Name: 肖涵薄
StudentID:31360164

Solution:

$$
\begin{aligned}
& I_A + I_B + I_C - I_{A\cup B} - I_{A\cup C} - I_{B\cup C} + I_{A\cup B\cup C} \\
= \;& I_A + I_B + I_C - 3 + I_{(A\cup B)^C} + I_{(A\cup C)^C} + + I_{(B\cup C)^C} + I_{A\cup B\cup C} \\
= \;& I_A + I_B + I_C - 2I_A - 2I_B - 2I_C + I_A I_B + I_A I_C + I_B I_C + I_{A\cup B\cup C} \\
& (\text{substitute } \; I_{A\cup B\cup C} \text{ in problem 7}) \\
= \;& I_A I_B I_C \\
= \;& I_{ABC}
\end{aligned}
$$

□

Problem 9. Prove that the set of all rational numbers is countable.

Solution: Write rational numbers for the following rule:

$$
\begin{array}{cccc}
\frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \cdots \\
\frac{2}{1}, & \frac{2}{2}, & \frac{2}{3}, & \cdots \\
\frac{3}{1}, & \frac{3}{2}, & \frac{3}{3}, & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

Sort start from column 1, row 1,
then column 1, row 2,
then colsum 2, row 1,
then column 3, row 1,
and so on. Following the snake shape track. Every element on the table will be counted.

□

Problem 10. Let A be the set of all sequences whose elements are the digits 0 and 1. For example, the following sequence is a element of A.

$$1, 0, 1, 0, 0, 0, 1, 1, ...$$

Prove that set A is uncountable. (Hint: You can prove it by using Cantor's diagonal process.)

Solution:

If A is countable, sort elements in A, and let A_i be i^{th} sequence of A.
Let sequence b satisfy that:

Name: 肖涵溥
StudentID:31360164

If i^{th} number in A_i is 1, then i^{th} number in b is 0, else i^{th} number in b is 1.

So, for any i, i^{th} number in A_i is different from b, which means A_i has at least one different number from b.

Thus b is different from all A_i, A is uncountable. $\square$