SI 140
Probability and Statistics
Semester Spring 2019
Assignment 5

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Problem 1. On a flight from Urbana to Paris my luggage did not arrive with me. It had been transferred three times and the probabilities that the transfer was not done in time were estimated to be 4/10, 2/10, 1/10, respectively, in the order of transfer. What is the probability that the first airline goofed?

Solution: 令三次转移没有成功事件分别为 A.B.C. 最后没有成功为事件 D.

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{4/10}{P(D)}$$

$$P(D) = 1 - P(A^C B^C C^C) = 1 - 0.6 * 0.8 * 0.9 = 0.568$$

Hence $P(A|D) = 0.704225$ □

Problem 2. Suppose that the probability that both twins are boys is α, and that both are girls β; suppose also that when the twins are of different sexes the probability of the first born being a girl is 1/2. If the first born of twins is a girl, what is the probability that the second is also a girl?

Solution: 设第一次和第二次生 girl 是 A_1, A_2.

$$P(A_1A_2) = \beta$$

$$P(A_1^c A_2^c) = \alpha \implies 1 + P(A_1A_2) - P(A_1) - P(A_2) = \alpha$$

$$\implies \alpha + P(A_1) + P(A_2) = \beta + 1$$

$$P(A_1A_2^C|(A_1A_2^C + A_1^C A_2)) = \frac{1}{2} \implies P(A_1A_2^c) = P(A_1^c A_2) \implies P(A_1) = P(A_2)$$

$$\implies \alpha + 2P(A) = \beta + 1$$

所要求的是 $P(A_1A_2|A_1)$.

$$P(A_1A_2|A_1) = \frac{P(A_1|A_1A_2)P(A_1A_2)}{P(A_2)} = \frac{\beta}{P(A_2)} = \frac{\beta}{\frac{\beta+1-\alpha}{2}} = \frac{2\beta}{\beta+1-\alpha}$$

□

Problem 3. A line of 100 airline passengers is waiting to board a plane. They each hold a ticket to one of the 100 seats on that flight. Unfortunately, the first person in line is crazy, and will ignore the seat number on their ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. What is the probability that the last (100th) person to board the plane will sit in their proper seat

Solution: 如果有 2 个人, 则概率为 1/2.
如果有 3 个人, 则概率为 $\frac{1}{3} + \frac{1}{3*2}$.
如果有 4 个人, 则概率为 $\frac{1}{4} + \frac{1}{3*4} + \frac{1}{4*3*2} + \frac{1}{4*2}$.
以此类推, 当有 100 个人时, 概率为

$$P = \frac{1}{100} + \sum_{2}^{99} \frac{1}{n(n+1)} = \frac{1}{2}$$

□

Problem 4. Prove the sure-thing principle: if

$$P(A|C) \geq P(B|C)$$

$$P(A|C^c) \geq P(B|C^c)$$

then $P(A) \geq P(B)$.

Solution: 显然

$$P(A|C) + P(A|C^c) = P(A)$$
$$P(B|C) + P(B|C^c) = P(B)$$

因此有

$$P(A) - P(A|C^c) \geq P(B) - P(B|C^c)$$
$$P(A|C^c) \geq P(B|C^c)$$

二式相加:

$$P(A) \geq P(B)$$

□

Problem 5. i).Wang's Family has ten children, and we know that at least 9 of them are boys, show the prob that the rest is also a boy.
ii).Wang's Family has ten children, You come to his house and see nine boys, show the probability of the remaining one to be a boy.

Solution:

i)
设男孩概率为 $P(A_i) = 1/2$, 且互相独立

$$P = \frac{P(\prod_1^{10} A_i)}{P(\prod_1^{10} A_i) + \sum_{j=1}^{10} \left(P(A_j^c) P(\prod_{i \neq j}^{10} A_i) \right)}$$

$$P(\prod_{1}^{10} A_i) = 1/2^{10}$$

$$P(A_j^c)P(\prod_{i\neq j}^{10} A_i) = 1/2^{10}$$

$$\implies P = 1/11$$

ii)

$$P(\prod_{1}^{10} A_i | \prod_{1}^{9} A_i) = P(A) = 1/2$$

□

Problem 6. A hat contains 100 coins, where at least 99 are fair, but there may be one that is double-headed(always landing Heads); if there is no such coin, then all 100 are fair. Let D be the event that there is such a coin, and suppose that $P(D) = 1/2$. A coin is chosen uniformly at random. The chosen coin is flipped 7 times, and it lands Heads all 7 times.

(i). Given this information, what is the probability that one of the coins is double headed?

(ii). Given this information, what is the probability that the chosen coin is double-headed?

Solution:

(i)

设题述 7 次为正面事件为事件 A. 当 D 发生. 连扔 7 次为 heads 的概率是

$$P(A|D) = \frac{1}{100} + (1 - \frac{1}{100})\frac{1}{2^7}$$

当 D 未发生, 概率为:

$$P(A|D^c) = \frac{1}{2^7}$$

因此 $P(A) = P(AD) + P(AD^c) = \frac{1}{2}(\frac{1}{100} + (1 - \frac{1}{100})\frac{1}{2^7}) + \frac{1}{2^7}\frac{1}{2}$

所要求的是 $P(D|A)$

$$P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{(\frac{1}{100} + (1 - \frac{1}{100})\frac{1}{2^7}) \times \frac{1}{2}}{\frac{1}{2}(\frac{1}{100} + (1 - \frac{1}{100})\frac{1}{2^7}) + \frac{1}{2^7}\frac{1}{2}}$$

$$\implies P(D|A) = P(D|A) = \frac{P(A|D)P(D)}{P(A)} = \frac{(\frac{1}{100} + (1 - \frac{1}{100})\frac{1}{2^7})}{(\frac{1}{100} + (1 - \frac{1}{100})\frac{1}{2^7}) + \frac{1}{2^7}} \approx 69.4\%$$

(ii)

此题所要求的是

$$P(\text{choose THE coin}|AD)P(D|A)$$

$P(\text{choose THE coin}|AD)$ 即为有一颗硬币 double-headed, 且丢了七次都是 heads 的概率. 设此时 D 为全集, 显然:

$$P(\text{choose THE coin}|AD) = \frac{P(A|\text{choose THE coin}, D)}{P(A|D)}P(\text{choose THE coin}|D)$$

可得:

$$P(\text{choose THE coin}|AD)P(D|A) = 0.564 * 0.694 = 0.39$$

□

Problem 7. An urn contains red, green, and blue balls. Let r, g, b be the proportions of red, green, blue balls, respectively ($r + g + b = 1$).

(i). Balls are drawn randomly with replacement. Find the probability that the first time a green ball is drawn is before the first time a blue ball is drawn.

Hint: Explain how this relates to finding the probability that a draw is green, given that it is either green or blue.

(ii). Balls are drawn randomly without replacement. Find the probability that the first time a green ball is drawn is before the first time a blue ball is drawn. Is the answer the same or different than the answer in (i)?

Hint: Imagine the balls all lined up, in the order in which they will be drawn. Note that where the red balls are standing in this line is irrelevant.

(iii). Generalize the result from (i) to the following setting. Independent trials are performed, and the outcome of each trial is classified as being exactly one of type 1, type 2,..., or type n, with probabilities $p_1, p_2, ..., p_n$, respectively. Find the probability that the first trial to result in type i comes before the first trial to result in type j, for $i \neq j$.

Solution:

(i)

如果全为绿球, 蓝球, 那么必然第一个就抽到绿球. 即 $P = g$. 若存在红球, 设第 n 次拿到第一个绿球, 则之前的 n-1 次必然全为红球. 则概率为:

$$P = g + rg + r^2g + \cdots = \frac{g}{1-r}$$

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(ii)

设一共有 N 个球, 那么绿球和蓝球的组合方法有 $\binom{Ng+Nb}{Ng}$

绿球在蓝球前面的组合方法有 $\binom{Ng+Nb-1}{Ng-1}$

因此概率为

$$P = \frac{\binom{Ng+Nb-1}{Ng-1}}{\binom{Ng+Nb}{Ng}} = \frac{g}{g+b} = \frac{g}{1-r}$$

与 (i) 中相同.

(iii)

有三种可能: i, j, 既不是 i 也不是 j, 这三种可能分别对应 g, b, r 三色. 因此

$$P = \frac{g}{g+b} = \frac{p_i}{p_i+p_j}$$

□

Problem 8. Consider four nonstandard dice (the Efron dice), whose sides are labeled as follows (he 6 sides on each die are equally likely).

A: 4,4,4,4,0,0

B: 3,3,3,3,3,3

C: 6,6,2,2,2,2

D: 5,5,5,1,1,1

These four dice are each rolled once. Let A be the result for die A, B be the result for die B, etc.

(i). Find $P(A > B)$, $P(B > C)$, $P(C > D)$ and $P(D > A)$.

(ii). Is the event $A > B$ independent of the event $B > C$? Is the event $B > C$ independent of the event $C > D$? Explain.

Solution:

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(i)
恒有 $B = 3$. 因此 $P(A > B) = P(A = 4) = 4/6 = 2/3$.
同理 $P(B > C) = P(C = 2) = 4/6 = 2/3$.
$P(C > D) = P(C = 6) + P(C = 2, D = 1) = 2/6 + 2/6 = 2/3$.
$P(D > A) = P(D = 5) + P(D = 1)P(A = 0) = 1/2 + 1/6 = 2/3$.

(ii)
$P(A > B > C) = P(A = 4, C = 2) = (2/3)^2$. 由第一问, $P(A > B > C) = P(A > B)P(B > C)$. 因此独立.
$P(B > C > D) = P(C = 2, D = 1) = 1/3$. 由第一问, $P(B > C)P(C > D) = 4/9$. 二者不相等, 因此不独立. □

Problem 9. A family has two children. Let C be a characteristic that a child can have, and assume that each child has characteristic C with probability p, independently of each other and of gender. Find that the probability that both children are girls given that at least one is a girl with characteristic C.

Solution: 至少有一个女孩有性格 C 设为事件 A. 两个都是女孩为事件 B.

$$P(A) = \frac{1}{4}(1-(1-p)^2) + \frac{1}{2}p = p - \frac{p^2}{4}$$

$$P(B) = \frac{1}{4}$$

.

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} = \frac{\frac{1}{4}(1-(1-p)^2)}{p - \frac{p^2}{4}} = \frac{2-p}{4-p}$$

□

Problem 10. Alice is trying to communicate with Bob, by sending a message (encoded in binary) across a channel.

(i). Suppose for this part that she sends only one bit (a 0 or 1), with equal probabilities. If she sends a 0, there is a 5% chance of an error occurring, resulting in Bob receiving a 1; if she sends a 1, there is a 10% chance of an error occurring, resulting in Bob receiving a 0. Given that Bob receives a 1, what is the probability that Alice actually sent a 1?

(ii). To reduce the chance of miscommunication, Alice and Bob decide to use a repetition code. Again Alice wants to convey a 0 or a 1, but this time she repeats it two more times, so that she sends 000 to convey 0 and 111 to convey 1. Bob will decode the message by going with what the majority of the bits were. Assume that the error probabilities are as in (i), with error events for different bits independent of each other. Given that Bob receives 110, what is the probability that Alice intended to convey a 1?

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Solution:

(i)

设收到的数字为 $0_r, 1_r$. 发出的为 $0_s, 1_s$.

$$P(1_r) = 0.05P(0_s) + 0.9P(1_s) = 0.475$$

$$P(1_s|1_r) = \frac{P(1_r|1_s)P(1_s)}{P(1_r)} = \frac{0.9 * 0.5}{0.475} = 0.947$$

(ii)

$$P(110_r) = 0.9^2 * 0.1P(111_s) + 0.05 * 0.05 * 0.95P(000_s) = 0.0416875$$

$$P(110_r|111_s) = 0.9^2 * 0.1 = 0.081$$

$$P(111_s|110_r) = \frac{P(110_r|111_s)P(111_s)}{P(110_r)} = \frac{0.081 * 0.5}{0.0416875} = 0.97$$

□