数理方法 II 第三次作业

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Q1

(1)

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+(2+\lambda/x)\,y=0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)-(-2)\,y+\frac{\lambda}{x}y=0$$

即

$$k(x)=x, q(x)=-2, \rho(x)=\frac{1}{x}$$

(2)

原式化为 $y''+\frac{a-bx}{x-x^2}y'-\frac{\lambda}{x-x^2}y=0$

$\exp\left(\int\frac{a-bx}{x-x^2}\,\mathrm{d}x\right)=\exp\left(a\int\frac{1}{x}\,\mathrm{d}x+(a-b)\int\frac{1}{1-x}\,\mathrm{d}x\right)=\exp\left(a\ln x-(a-b)\ln(1-x)\right)=\dfrac{x^a}{(1-x)^{a-b}}$

则最后可化为标准形式:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^a}{(1-x)^{a-b}}y'\right)+\lambda\left(\frac{1}{x(x-1)}\frac{x^a}{(1-x)^{a-b}}\right)y=0$$

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设 $y_m, y_n, n\neq m$ 是函数不同本征值的两个解.

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}x}\left(py_m'\right)+\left(\lambda_m\rho-q\right)y_m=0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(py_n'\right)+\left(\lambda_n\rho-q\right)y_n=0\end{cases}$$

两式分别乘以 y_n, y_m，相减, $y_n \frac{\mathrm{d}}{\mathrm{d}x}(py_m') - y_m \frac{\mathrm{d}}{\mathrm{d}x}(py_n') + (\lambda_m - \lambda_n) y_m y_n = 0$. 求区间 $[a,b]$ 积分,

$$\int_a^b \left[y_n \frac{d}{dx}(py_m') - y_m \frac{d}{dx}(py_n')\right] dx + \int_a^b \left(py_m' \frac{d}{dx} y_n - py_n' \frac{d}{dx} y_m\right) dx + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx$$

$$= \int_a^b \frac{d}{dx}(py_n y_m' - py_m y_n') dx + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx$$

$$= (py_n y_m' - py_m y_n')|_{x=b} - (py_n y_m' - py_m y_n')|_{x=a} + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx$$

由于 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, 有 $y_n(b) y_m'(b) = y_m(b) y_n'(b)$, $y_n(a) y_m'(a) = y_m(a) y_n'(a)$

$$= (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx = 0$$

当 $\lambda_m \neq \lambda_n$, $\int_a^b \rho y_m y_n dx = 0$.

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(1)

根据定义, $\int \delta(\boldsymbol{r} - \boldsymbol{r}_0) \mathrm{d}r^3 = 1$, 在球坐标下即为

$$\int_0^\infty \mathrm{d}r \int_0^{2\pi} r \,\mathrm{d}\varphi \int_0^\pi r \sin\varphi \delta(\boldsymbol{r} - \boldsymbol{r}_0) \mathrm{d}\theta = 1$$

$$\Longrightarrow \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\cos\varphi \int_0^\pi r^2 \delta(\boldsymbol{r} - \boldsymbol{r}_0) \mathrm{d}\theta = 1$$

根据直角坐标下形式, 可知

$$\delta(r - r_0) \delta(\cos\theta - \cos\theta_0) \delta(\varphi - \varphi_0) = r^2 \delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

移项即得

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \frac{1}{r^2} \delta(r - r_0) \delta(\cos\theta - \cos\theta_0) \delta(\varphi - \varphi_0)$$

(2)

$$\nabla^2 \frac{1}{|r - r_0|} = -\nabla \cdot \frac{1}{(r - r_0)^2}$$

由高斯定理可知

$$-\int \nabla \cdot \frac{1}{(r-r_0)^2}\,\mathrm{d}V = -\int_\Omega \frac{1}{(r-r_0)^2}\,\mathrm{d}S$$

取积分面为 $r - r_0 = a$ 的球壳, a 为任意常数.

$$-\int_\Omega \frac{1}{(r-r_0)^2}\,\mathrm{d}S = -4\pi a^2 \frac{1}{a^2} = -4\pi$$

即

$$\int \nabla^2 \frac{1}{|r-r_0|}\,\mathrm{d}r^3 = -4\pi$$

根据定义,

$$\nabla^2 \frac{1}{|r-r_0|} = -4\pi\delta\left(\boldsymbol{r}-\boldsymbol{r}_0\right)$$

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(1) 先计算 $\mathcal{F}\left(e^{-a|t|}\right)$

$$\mathcal{F}\left(e^{-a|t|}\right) = \int_{-\infty}^{0} e^{at-iwt}\,\mathrm{d}t + \int_0^\infty e^{-at-iwt}\,\mathrm{d}t$$

$$= \int_0^\infty \left(e^{-(a+iw)t} + e^{-(a-iw)t}\right)\mathrm{d}t = \frac{2a}{a^2+w^2}$$

则 $\mathcal{F}^{-1}\left(\frac{2a}{a^2+w^2}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+w^2}e^{iwt}\,\mathrm{d}w = e^{-a|t|}$, 两边求实部:

$$\int_{-\infty}^{\infty} \frac{1}{a^2+w^2}\cos wt\,\mathrm{d}w = \frac{\pi}{a}e^{-a|t|}$$

(2)

$$\int \frac{1}{r}e^{ikr\cos\theta}\,\mathrm{d}r = \int_0^\infty \frac{1}{r}r^2\,\mathrm{d}r \int_1^{-1} e^{ikr\cos\theta}\,\mathrm{d}\cos\theta \int_0^{2\pi}\mathrm{d}\varphi$$

$$= 2\pi \int_0^\infty \frac{1}{r}r^2\frac{1}{ikr}\left(e^{ikr}-e^{-ikr}\right)\mathrm{d}r = \frac{2\pi}{ik}\int_0^\infty \left(e^{ikr}-e^{-ikr}\right)\mathrm{d}r$$

$$= \frac{2\pi}{ik}\left(2i\int_0^\infty \sin kr\,\mathrm{d}r\right) = \frac{4\pi}{k}\int_0^\infty \sin kr\,\mathrm{d}r$$

$$\begin{aligned}
\int_0^\infty \sin kr\,\mathrm{d}r &= \lim_{\varepsilon\to 0^+}\operatorname{Im}\left(\int_0^\infty e^{\varepsilon r}e^{ikr}\,\mathrm{d}r\right) \\
&= \lim_{\varepsilon\to 0^+}\operatorname{Im}\left(\int_0^\infty e^{(\varepsilon+ik)r}\,\mathrm{d}r\right) \\
&= \lim_{\varepsilon\to 0^+}\operatorname{Im}\left(\frac{e^{(\varepsilon+ik)\infty}-1}{\varepsilon+ik}\right) \\
&= \frac{1}{k}
\end{aligned}$$

则

$$\mathcal{F}\left(\frac{1}{r}\right)=\frac{1}{(2\pi)^{3/2}}\int\frac{1}{r}e^{ikr\cos\theta}\,\mathrm{d}r=\frac{\sqrt{2}}{\sqrt{\pi}k}\int_0^\infty\sin kr\,\mathrm{d}r=\frac{\sqrt{2}}{\sqrt{\pi}k^2}$$

(ii)

$$\int\frac{\delta(r-a)}{r}e^{-ikr\cos\theta}\,\mathrm{d}r=\int_0^\infty\frac{\delta(r-a)}{r}r^2\,\mathrm{d}r\int_1^{-1}e^{-ikr\cos\theta}\,\mathrm{d}\cos\theta\int_0^{2\pi}\mathrm{d}\varphi$$
$$=-2\pi\int_0^\infty\frac{\delta(r-a)}{ik}\left(e^{ikr}-e^{-ikr}\right)\mathrm{d}r=-\frac{2\pi}{ik}\left(e^{ika}-e^{-ika}\right)=\frac{\pi}{k}\sin ka$$

则可得到

$$\mathcal{F}^{-1}\left(\frac{\sin ak}{k}\right)=\sqrt{\frac{\pi}{2}}\frac{\delta(r-a)}{r}$$

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根据周期性有:

$$\mathcal{L}\left(f\left(t-a\right)\right)=\mathcal{L}\left(f\left(t\right)u\left(t-a\right)\right)$$

又由于

$$\mathcal{L}\left(f\left(t-a\right)\right)=\int_0^\infty\frac{f\left(t-a\right)}{e^{ap}}e^{-pt+ap}\,\mathrm{d}t=e^{-ap}F\left(p\right)$$
$$\mathcal{L}\left(f\left(t\right)u\left(t-a\right)\right)=F\left(p\right)-\int_0^a f\left(t\right)e^{-pt}\,\mathrm{d}t$$

上面两式相等即可得到

$$\left(1-e^{-ap}\right)F\left(p\right)=\int_0^a f\left(t\right)e^{-pt}\,\mathrm{d}t\implies F\left(p\right)=\frac{1}{1-e^{-ap}}\int_0^a f\left(t\right)e^{-pt}dt$$

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(1)

$$pU\left(x,p\right)-u\left(x,0\right)=a^2\frac{\partial^2U}{\partial x^2}+F\left(x,p\right)$$
$$\implies pU\left(x,p\right)-\varphi\left(x\right)=a^2\frac{\partial^2U}{\partial x^2}+F\left(x,p\right)$$

先求齐次方程的通解:

$$pU=a^2\frac{\partial^2U}{\partial x^2}$$
$$U=A\sinh\frac{a}{\sqrt{p}}x+B\cosh\frac{a}{\sqrt{p}}x$$

再求非齐次方程特解: 设 $U=\sum a_i\sin n\pi x$