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Problem 1. Let X be the total from rolling 6 fair dice, and let $X_1, ..., X_6$ be the individual rolls. What is $P(X = 18)$?

Solution:

$$g_{X_i}(z) = \frac{1}{6}(z + \cdots + z^6)$$
$$g_X(z) = \prod g_{X_i}(z) = \frac{1}{6^6}(z + \cdots + z^6)^6$$

$P(X = 18)$ 为 z^{18} 的系数, 为 $\frac{3431}{6^6} \approx 0.0735$ □

Problem 2. Find the MGF of $X \sim Unif(a, b)$ and $Y \sim Expo(\lambda)$.

Solution:

$$M(t) = E\left[e^{-tX}\right] = \int_{-\infty}^{\infty} e^{-tx} f(x) dx$$

For $Unif(a, b)$, $f(x) = \frac{1}{b-a}$, then $M(t) = \frac{1}{b-a}\int_a^b e^{-tx}\,\mathrm{d}x = \frac{e^{-tb}-e^{-ta}}{t(a-b)}$.
For $Expo(\lambda)$, $f(y) = \lambda e^{-\lambda y}$, then $M(t) = -\lambda \int_0^{\infty} e^{-(t+\lambda)y}\,\mathrm{d}y = \frac{\lambda}{t+\lambda}(t < \lambda)$. □

Problem 3. Find the MGF of $X \sim Bern(p)$ and $Y \sim Bin(n, p)$.

Solution: For $X \sim Bern(p)$, $M(t) = pe^{-t} + 1 - p$.
For $Y \sim Bin(n, p)$, $M(t) = (1 - p + pe^{-t})^n$ □

Problem 4. Consider a setting where a Poisson approximation should work well: let $A_1, ..., A_n$ be independent, rare events, with n large and $p_j = P(A_j)$ small for all j. Let $X = I(A_1) + \cdots + I(A_n)$ count how many of the rare events occur, and Let $\lambda = E(X)$.

(a). Find the MGF of X.

(b). If the approximation $1 + x \approx e^x$ (this is a good approximation when x is very close to 0 but terrible when x is not close to 0) is used to write each factor in the MGF of X as e to a power.What happens to the MGF? (Hint: if $Y \sim Pois(\lambda)$, then $M_Y = e^{(e^{-t}-1)\lambda}$)

Solution:

(a).

$$\sum p_j = E(x) = \lambda$$
$$M_X(t) = \prod M_{I_j}(t)$$

而 $M_{I_j}(t) = 1 - p_j + p_j e^{-t}$ 因此

$$M_X(t) = \prod_{j=1}^{n} \left(1 - p_j + p_j e^{-t}\right)$$

(b).

$$M_X(t) = \prod_{j=1}^{n}\left(1 + p_j(e^{-t} - 1)\right) \approx \prod_{j=1}^{n} e^{(e^{-t}-1)p_j} = e^{(e^{-t}-1)\lambda}$$

□

Problem 5. Suppose $X \sim Bin(n, p)$. By using Binomial PGF, find the expectation $E(X)$ and variance $Var(X)$.

Solution: 二项分布的 PGF 为 $M(t) = (pe^{-t} + 1 - p)^n$, 因此

$$M'(t) = -np\left(pe^{-t} + 1 - p\right)^{n-1} e^{-t}$$

$$E(X) = -M'(0) = np$$

$$M''(t) = (n-1)np^2 e^{-2t}\left(pe^{-t} - p + 1\right)^{n-2} + npe^{-t}\left(pe^{-t} - p + 1\right)^{n-1}$$

$$E(X^2) = M''(0) = np + (n-1)np^2$$

$$Var(X) = E(X^2) - (E(X))^2 = np(1-p)$$

□

Problem 6. If a random variable X has the following moment-generating function:

$$M(t) = \frac{1}{10}e^{-t} + \frac{2}{10}e^{-2t} + \frac{3}{10}e^{-3t} + \frac{4}{10}e^{-4t}$$

for all t, then what is thePMF of X?

Solution: 根据 MGF 的定义

$$M_X(t) = \sum_{k=0}^{\infty} e^{-tk} p(X = k)$$

可知

$$\begin{cases} P(X=1) = \frac{1}{10} \\ P(X=2) = \frac{2}{10} \\ P(X=3) = \frac{3}{10} \\ P(X=4) = \frac{4}{10} \end{cases}$$

□

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Problem 7. Suppose that Y has the following moment-generating function:

$$M_Y(t) = \frac{e^{-t}}{4 - 3e^{-t}}$$

I).Find E(Y)

II).Find Var(Y)

Solution: I)

$$M_Y'(t) = \frac{-4e^t}{(4e^t - 3)^2}$$

$$E(Y) = -M_Y'(0) = 4$$

II)

$$M_Y''(t) = \frac{4e^t\,(4e^t + 3)}{(4e^t - 3)^3}$$

$$E(Y^2) = M_Y''(0) = 28$$

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = 12$$

□

Problem 8. If a random variable X has $E[X^k] = 0.2$ k=1,2,3....,then what is the PMF of X?

Solution: 题述条件说明 $E[x^k] = (-1)^n M^{(k)}(0) = 0.2$. 即

$$M(t) = 1 - 0.2t + \frac{0.2}{2}t^2 + \cdots = 0.2e^{-t} + 0.8$$

即

$$\begin{cases} P(X = 0) = 0.8 \\ P(X = 1) = 0.2 \end{cases}$$

□