SI 140
Probability and Statistics
Semester Spring 2019
Assignment 4

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Problem 1. Let $\lambda > 0$ and define f as follows:

$$f(u) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda u} & \text{if } u \geq 0; \\ \frac{1}{2}\lambda e^{+\lambda u} & \text{if } u < 0 \end{cases} \tag{1}$$

This f is called bilateral exponential. If X has density f, find the density of $|X|$.

Solution: density of $|X|$ = density of $X, -X$. so

$$f'(u) = f(u) + f(-u) = \lambda e^{-\lambda u}$$

□

Problem 2. If X is a positive random variable with density f, find the density of $+\sqrt{X}$. Apply this to the distribution of the side length of a square when its area is uniformly distributed in $[a, b]$.

Solution:

$$\int_0^{\sqrt{x_1}} f_{\sqrt{x}}(x)\,\mathrm{d}x = \int_0^{x_1} f(x)\,\mathrm{d}x \implies f_{\sqrt{2}}(x) = F'\left(x^2\right) = 2xf\left(x^2\right)$$

Now X is area of square, $X \in [a, b]$. $f(x) = \frac{1}{b-a}$,

$$\int_0^{\sqrt{x_1}} f_{\sqrt{x}}(x)\,\mathrm{d}x = \int_0^{x_1} \frac{1}{b-a}\mathrm{d}x = \frac{x_1}{b-a}$$

$$\implies F_{\sqrt{x}}(x) = \frac{x^2}{b-a} \implies f_{\sqrt{x}}(x) = \frac{2x}{b-a}$$

□

Problem 3. If X has density f, find the density of (i)$aX + b$ where a and b are constants; (ii) X^2.

Solution:

(i)

$$\int_{-\infty}^{ax+b} f_1(x)\,\mathrm{d}x = \int_{-\infty} s^x f(x)\,\mathrm{d}x$$

$$f_1(x) = \left|F'\left(\frac{x-b}{a}\right)\right| = \frac{1}{|a|} f\left(\frac{x-b}{a}\right)$$

.

(ii) For the same reason,

$$f_2(x) = \left|F'\left(\sqrt{x}\right)\right| + \left|F'\left(-\sqrt{x}\right)\right| = \frac{1}{2\sqrt{x}} f\left(\sqrt{x}\right) + \frac{1}{2\sqrt{x}} f\left(-\sqrt{x}\right)$$

□

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Problem 4. If f and g are two density functions, show that $\lambda f + \mu g$ is also a density function, where $\lambda + \mu = 1, \lambda \geq 0, \mu \geq 0$.

Solution:

(1)

$$\int_{-\infty}^{+\infty} \lambda f(x) + \mu g(x) \,\mathrm{d}x = \lambda + \mu = 1$$

(2)

$$\lambda, f, \mu, g > 0 \implies \lambda f + \mu g > 0$$

□

Problem 5. Let

$$f(u) = ue^{-u}, \quad u \geq 0$$

Show that f is a density function. Find $\int_0^\infty uf(u)\,du$.

Solution:

(1)

$$\int_0^{+\infty} ue^{-u}\mathrm{d}u = 1$$

(2)

$$\text{For any } u \geq 0, f(u) = ue^{-u} \geq 0$$

So that $f(u)$ is a density function.

$$E(u) = \int_0^{\infty} uf(u)\,\mathrm{d}u = \int_0^{+\infty} u^2e^{-u}\mathrm{d}u = 2$$

□

Problem 6. A number of μ is called the median of the random variable X iff $P(X \geq \mu) \geq 1/2$ and $P(X \leq \mu) \geq 1/2$. Show that such a number always exists but need not be unique. Here is a practical example. After n examination papers have been graded, they are arranged in descending order. There is one in the middle if n is odd, two if n is even, corresponding to the median(s). Explain the probability model used.

Solution:

Let x be the order of paper, and $P(x) = 1/n$.

(1) When n is odd.

Let $(n+1)/2_{th}$ paper be μ. There are $(n+1)/2$ pieces of paper before and after μ(include μ itself). Following the $P(x)$ define above, $P(X \leq \mu) = P(X \geq \mu) = \frac{1}{n}\frac{n+1}{2} = \frac{n+1}{2n} > \frac{1}{2}$.

(2) When n is even.

Let $n/2_{th}$ paper be μ_1, and $n/2_{th}+1$ paper be μ_2.

For μ_1, there are $n/2$ pieces of paper before μ_1 (include μ_1 itself) and $n/2+1$ pieces of paper after μ_1 (include μ_1 itself). Following the $P(x)$ define above, $P(X \le \mu_1) = \frac{1}{n}\frac{n}{2} = \frac{1}{2}$, and $P(X \ge \mu_1) = \frac{1}{n}\left(\frac{n}{2}+1\right) > \frac{1}{2}$

For μ_2. There are $n/2+1$ pieces of paper before μ_2 (include μ_2 itself) and $n/2$ pieces of paper after μ_2 (include μ_2 itself). Following the $P(x)$ define above, $P(X \le \mu_2) = \frac{1}{n}\left(\frac{n}{2}+1\right) > \frac{1}{2}$, and $P(X \ge \mu_2) = \frac{1}{n} \cdot \frac{n}{2} = \frac{1}{2}$

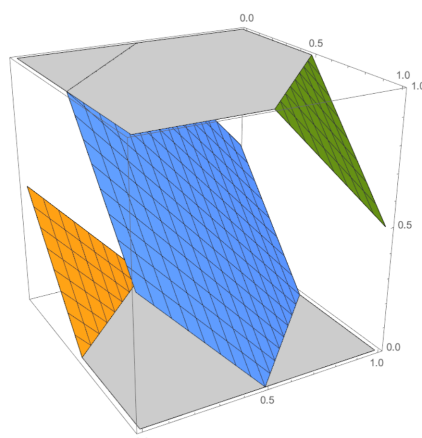
□

Problem 7. Suppose X_1, X_2, X_3 are independent identically distributed (i.i.d.) Unif$(0,1)$ random variables and let $Y = X_1 + X_2 + X_3$. (i). Find PDF of Y; (ii). Find $E(Y)$.

Solution:

(i)

$$F(X_1) = x_1, F(X_2) = x_2, F(X_3) = x_3$$



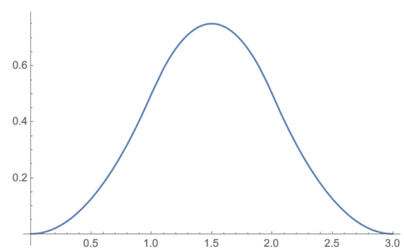
As is shown, plot a space of $x_1, x_2, x_3 \in (0,1)$, and the surface of $x_1+x_2+x_3=y$ is the probability density. Let the area of this surface be A.

$$A = \begin{cases} \frac{\sqrt{3}}{2}y^2, & 0 < y < 1 \\ \frac{\sqrt{3}}{2}y^2 - \frac{3\sqrt{3}}{2}(y-1)^2, & 1 \le y < 2 \\ \frac{\sqrt{3}}{2}(3-y)^2, & 2 \le y < 3 \end{cases}$$

If we denote vector $\boldsymbol{n} = (1,1,1)$, a thin slice $A\mathrm{d}\boldsymbol{n}$ is the probability of $P(y < Y \le y + \mathrm{d}y)$, which means $f(y)\,\mathrm{d}y = P(y < Y \le y + \mathrm{d}y) = A(y)\,\mathrm{d}\boldsymbol{n}$, $\mathrm{d}\boldsymbol{n} = \sqrt{3}\mathrm{d}x = \frac{1}{\sqrt{3}}\mathrm{d}y$. So that

$$f(y) = \begin{cases} \frac{1}{2}y^2, & 0 < y < 1 \\ \frac{1}{2}y^2 - \frac{3}{2}(y-1)^2, & 1 \le y < 2 \\ \frac{1}{2}(3-y)^2, & 2 \le y < 3 \end{cases}$$

Plot as:



(ii)

$$E(y) = \int_0^3 y f(y)\,\mathrm{d}y = \frac{3}{2}$$

□

Problem 8. There are 40 people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that peoples birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday?

Solution: The probability of everyone has unique birthday is

$$P_0 = \frac{365}{365}\cdot\frac{364}{365}\cdots\frac{365-40+1}{365} = \frac{365!}{365^{40}\times 325!} = 0.11$$

So the answer is

$$P = 1 - P_0 \approx 0.89$$

□

Problem 9. Let X_1,...,X_n be independent, with $X_j \sim \text{Expo}(\lambda_j)$. Let $L = \min\{X_1, ..., X_n\}$. Show that $L \sim \text{Expo}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ and find $E(L)$.

Solution:

$$P_j(t) = \lambda_j e^{-\lambda_j t}, \quad t \ge 0$$

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The probability X_j does not happen before t_0 is

$$1 - F_j(t) = \int_{t_0}^{\infty} P_j(t)\,\mathrm{d}t = e^{-\lambda_j t_0}$$

So that the probability that every X_j does not happen before t_0 is:

$$\prod_j \int_{t_0}^{\infty} P_j(t)\,\mathrm{d}t = \prod_j e^{-\lambda_j t_0} = \exp\left(-\sum_j \lambda_j t_0\right)$$

Which equals to $1 - F_L(t_0) \implies F_L(t) = 1 - \exp\left(-\sum_j \lambda_j t_0\right)$. So

$$P_L(t) = \frac{\mathrm{d}F_L(t)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(-\sum_j \lambda_j t\right) = \left(\sum_j \lambda_j\right)\exp\left(-\sum_j \lambda_j t\right)$$

This tells that

$$L \sim \mathrm{Expo}\left(\sum_j \lambda_j\right) = \mathrm{Expo}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

□

Problem 10. (Expectation via Survival Function) Let X be a nonnegative random variable. Let F be the CDF of X, and $G(x) = 1 - F(x) = P(X > x)$. The function G is called the survival function of X. Show that

(i). The expectation of a nonnegative integer-valued discrete random variable X is

$$E(X) = \sum_{n=0}^{\infty} G(n)$$

(ii). The expectation of a nonnegative continuous random variable X is

$$E(X) = \int_0^{\infty} G(x)\,dx$$

Solution:

(1)

$$\sum_{n=0}^{\infty} G(n) = \sum_{n=0}^{\infty}\left[1 - \sum_{i=0}^{n} p(i)\right] = \sum_{n=0}^{\infty}\sum_{i=n+1}^{\infty} p(i) = \sum_{0}^{\infty} np(n) = E(X)$$

(2)

$$E(y) = \int_0^{\infty} yF(y)\,dy = yF(y)\,\Big|_0^{\infty} - \int_0^{\infty}\int_0^{y} f(u)\,\mathrm{d}u\mathrm{d}y$$
$$= \int_0^{\infty}\left(y - \int_0^{y} f(u)\,\mathrm{d}u\right)\mathrm{d}y = \int_0^{\infty} G(y)\,\mathrm{d}y$$

□