SI 140
Probability and Statistics
Semester Spring 2019
Assignment 3

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Problem 1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $P(\omega_1) = P(\omega_2) = P(\omega_3) = 1/3$, and define X, Y and Z as follows:

$$X(\omega_1) = 1, X(\omega_2) = 2, X(\omega_3) = 3;$$

$$Y(\omega_1) = 2, Y(\omega_2) = 3, Y(\omega_3) = 1;$$

$$Z(\omega_1) = 3, Z(\omega_2) = 1, Z(\omega_3) = 2.$$

Show that these three random variables have the same probability distribution. Find the probability distributions of $X + Y$, $Y + Z$, and $Z + X$.

Solution:

$$\begin{cases} F_X(1) = & P(X(\omega_1)) = 1/3 \\ F_X(2) = & P(X(\omega_1)) + P(X(\omega_2)) = 2/3 \\ F_X(3) = & P(X(\omega_1)) + P(X(\omega_2)) + P(X(\omega_3)) = 1 \end{cases}$$

$$\begin{cases} F_Y(1) = & P(Y(\omega_3)) = 1/3 \\ F_Y(2) = & P(Y(\omega_3)) + P(Y(\omega_1)) = 2/3 \\ F_Y(3) = & P(Y(\omega_3)) + P(Y(\omega_1)) + P(Y(\omega_2)) = 1 \end{cases}$$

$$\begin{cases} F_Z(1) = & P(Z(\omega_2)) = 1/3 \\ F_Z(2) = & P(Z(\omega_2)) + P(Z(\omega_3)) = 2/3 \\ F_Z(3) = & P(Z(\omega_2)) + P(Z(\omega_3)) + P(Z(\omega_1)) = 1 \end{cases}$$

So X, Y, Z have the same probability distribution.

$$\begin{cases} F_{X+Y}(3) = & P((X+Y)(\omega_1)) = 1/3 \\ F_{X+Y}(4) = & P((X+Y)(\omega_1)) + P((X+Y)(\omega_3)) = 2/3 \\ F_{X+Y}(5) = & P((X+Y)(\omega_1)) + P((X+Y)(\omega_3)) + P((X+Y)(\omega_2)) = 1 \end{cases}$$

$$\begin{cases} F_{Y+Z}(3) = & (Y+Z)(\omega_3) = 1/3 \\ F_{Y+Z}(4) = & (Y+Z)(\omega_3) + (Y+Z)(\omega_2) = 2/3 \\ F_{Y+Z}(5) = & (Y+Z)(\omega_3) + (Y+Z)(\omega_2) + (Y+Z)(\omega_1) = 1 \end{cases}$$

$$\begin{cases} F_{Z+X}(3) = & (Z+X)(\omega_2) = 1/3 \\ F_{Z+X}(4) = & (Z+X)(\omega_2) + (Z+X)(\omega_1) = 2/3 \\ F_{Z+X}(5) = & (Z+X)(\omega_2) + (Z+X)(\omega_1) + (Z+X)(\omega_3) = 1 \end{cases}$$

$\square$

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Problem 2. In No.1 find the probability distribution of

$$X+Y-Z, \sqrt{(X^2+Y^2)Z}, \frac{Z}{|X-Y|}$$

Solution:

$$\begin{cases} (X+Y-Z)(\omega_1) = & 0 \\ (X+Y-Z)(\omega_2) = & 4 \\ (X+Y-Z)(\omega_3) = & 2 \end{cases}$$

$$\Longrightarrow \begin{cases} F_{X+Y-Z}(0) = & P((X+Y-Z)(\omega_1)) = 1/3 \\ F_{X+Y-Z}(2) = & P((X+Y-Z)(\omega_1)) + P((X+Y-Z)(\omega_3)) = 2/3 \\ F_{X+Y-Z}(4) = & P((X+Y-Z)(\omega_1)) + P((X+Y-Z)(\omega_3)) + P((X+Y-Z)(\omega_2)) = 1 \end{cases}$$

$$\begin{cases} (\sqrt{(X^2+Y^2)Z})(\omega_1) = & \sqrt{15} \\ (\sqrt{(X^2+Y^2)Z})(\omega_2) = & \sqrt{13} \\ (\sqrt{(X^2+Y^2)Z})(\omega_3) = & \sqrt{20} \end{cases}$$

$$\Longrightarrow \begin{cases} F_{\sqrt{(X^2+Y^2)Z}}(\sqrt{13}) = & P((\sqrt{(X^2+Y^2)Z})(\omega_2)) = 1/3 \\ F_{\sqrt{(X^2+Y^2)Z}}(\sqrt{15}) = & P((\sqrt{(X^2+Y^2)Z})(\omega_2)) + P((\sqrt{(X^2+Y^2)Z})(\omega_1)) = 2/3 \\ F_{\sqrt{(X^2+Y^2)Z}}(\sqrt{20}) = & P((\sqrt{(X^2+Y^2)Z})(\omega_2)) + P((\sqrt{(X^2+Y^2)Z})(\omega_1)) \\ & +P((\sqrt{(X^2+Y^2)Z})(\omega_3)) = 1 \end{cases}$$

$$\begin{cases} (\frac{Z}{|X-Y|})(\omega_1) = & 3 \\ (\frac{Z}{|X-Y|})(\omega_2) = & 1 \\ (\frac{Z}{|X-Y|})(\omega_3) = & 1 \end{cases}$$

$$\Longrightarrow \begin{cases} F_{\frac{Z}{|X-Y|}}(1) = & P((\frac{Z}{|X-Y|})(\omega_2)) + P((\frac{Z}{|X-Y|})(\omega_3)) = 2/3 \\ F_{\frac{Z}{|X-Y|}}(3) = & P((\frac{Z}{|X-Y|})(\omega_2)) + P((\frac{Z}{|X-Y|})(\omega_3)) + P((\frac{Z}{|X-Y|})(\omega_1)) = 1 \end{cases}$$

□

Problem 3. Let X be integer-valued and let F be its distribution function. Show that for every x and $a<b$

$$P(X=x) = \lim_{\varepsilon\downarrow 0}[F(x+\varepsilon) - F(x-\varepsilon)]$$

$$P(a<X<b) = \lim_{\varepsilon\downarrow 0}[F(b-\varepsilon) - F(a+\varepsilon)]$$

[The results are true for any random variable but require more advanced proofs even when Ω is countable.]

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Solution:

1.

$$F(x+\varepsilon)-F(x-\varepsilon)=\sum[P(X<x+\varepsilon)-P(X<x-\varepsilon)]=\sum P(x-\varepsilon\le X<x+\varepsilon)$$

And bacause X is integer-valued, $P(X)$ is not equal to zero around x only when $X=x$.

$$\lim_{\varepsilon\downarrow 0}\sum P(x-\varepsilon\le X<x+\varepsilon)=P(X=x)$$

So we have

$$P(X=x)=\lim_{\varepsilon\downarrow 0}[F(x+\varepsilon)-F(x-\varepsilon)]$$

2.

$$\lim_{\varepsilon\downarrow 0}[F(b-\varepsilon)-F(a+\varepsilon)]=P(a+\varepsilon\le X<b-\varepsilon)$$
$$=P(a<X<b)-P(a<X<a+\varepsilon)-P(b-\varepsilon\le X<b)$$

When a is an integer, let $\varepsilon<1$, $P(a<X<a+\varepsilon)=0$.
When a is not an integer, and a_{int} is the integer next to a, let $\varepsilon<a_{int}-a$, $P(a<X<a+\varepsilon)\le P(a<X<a_{int})=0$.
The same to b, Thus $P(a<X<a+\varepsilon)=P(b-\varepsilon\le X<b)=0$,

$$P(a\le X<b)=\lim_{\varepsilon\downarrow 0}[F(b-\varepsilon)-F(a+\varepsilon)]$$

□

Problem 4. (a) Is there a discrete distribution with support 1,2,3,..., such that the value of the PMF at n is proportional to $1/n$?

(b) Is there a discrete distribution with support 1,2,3,..., such a that the value of the PMF at n is proportional to $1/n^2$?

Solution:

(1) Suppose $P(x)=\dfrac{k}{n}$, Then $\sum P(x)=k\sum_{1}^{+\infty}\dfrac{1}{n}=+\infty$, so it's impossible.

(2) Suppose $P(x)=\dfrac{k}{n^2}$, Then $\sum P(x)=k\sum_{1}^{+\infty}\dfrac{1}{n^2}=\dfrac{k\pi^2}{6}\implies k=\dfrac{6}{\pi^2}$, so $P(x)=\dfrac{6}{\pi^2 n^2}$.

□

Problem 5. Let X have PMF

$$P(X = k) = cp^k/k \text{ for } k = 1, 2, ...$$

where p is a parameter with $0 < p < 1$ and c is a normalizing constant. We have $c = -1/\log(1-p)$, as seen from the Taylor series

$$-\log(1-p) = p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots .$$

This distribution is called the *Logarithmic* distribution (because of the log in the above Taylor series), and has often been used in ecology. Find the mean of X.

Solution:

$$E(X) = \sum_{1}^{+\infty} kP(k) = c\sum_{1}^{+\infty} p^k = \frac{cp}{1-p} = -\frac{p}{(1-p)\log(1-p)}$$

□

Problem 6. Suppose F is some cumulative distribution function. Then for any real number y, the function F_y defined by $F_y(x) = F(x\text{-}y)$ is also a cumulative distribution function. In fact, F_y is just a “shifted” version of F

Solution:

(1)

$$\lim_{x\to+\infty} F_y(x) = \lim_{x-y\to+\infty} F(x-y) = 1$$
$$\lim_{x\to-\infty} F_y(x) = \lim_{x-y\to-\infty} F(x-y) = 0$$

(2) If $x_1 < x_2$, then $x_1 - y < x_2 - y$, so $F_y(x_1) < F_y(x_2)$.

(3)

$$\lim_{x\to x_0^+} F_y(x) = \lim_{x-y\to x_0^+-y} F(x-y) = F(x_0 - y) = F_y(x_0)$$

□

Problem 7. Let X be a random variable, with cumulative distribution function F_X . Prove that $P(X = a) = 0$ if and only if the function F_X is continuous at a.

Solution:

$F(a^+) - F(a^-) = \sum_{a^-}^{a^+} P(x)$, if the function F_X is continuous at a, iff $F(a^+) - F(a^-) = 0$. Thus $F(a^+) - F(a^-) = \sum_{a^-}^{a^+} P(x) = 0 \iff P(a) = 0$. □

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Problem 8. Suppose that

$$p_n = cq^{n-1}p, 0 \leq n \leq m$$

where c is a constant and m is a positive integer; cf. (4.4.8). Determine c so that $\sum_{n=1}^{m} p_n = 1$. (This scheme corresponds to the waiting time for a success when it is supposed to occur within m trials.)

Solution:

$$\sum_{n=1}^{m} p_n = cp \sum_{n=1}^{m} q^{n-1} = cp \frac{1-q^m}{1-q} = 1$$

So that

$$c = \frac{1-q}{p\left(1-q^m\right)}.$$

□

Problem 9. A perfect coin is tossed n times. Let Y_n denote the number of heads obtained minus the number of tails. Find the probability distribution of Y_n and its mean.[Hint: there is a simple relation between Y_n and the S_n in Example 9 of 4.4]

Solution: Let H be times of heads, and T be times of tails.

$$Y_n = H - T = H - (n - H) = 2H - n.$$

$H = \frac{n+Y_n}{2}$.

$$P_Y\left(Y_n = x\right) = \begin{pmatrix} H \\ n \end{pmatrix} \bigg/ \left(2^n\right) = \begin{pmatrix} \frac{n+x}{2} \\ n \end{pmatrix} \bigg/ \left(2^n\right)$$

$$E_Y\left(x\right) = \sum_{x=-n}^{n} \frac{\begin{pmatrix} \frac{n+x}{2} \\ n \end{pmatrix} x}{2^n} = 0$$

□

Problem 10. Let

$$P\left(X = n\right) \text{ - } p_n = \frac{1}{n\left(n+1\right)}, n \geq 1$$

Show that it is a probability distribution for X? Find $P\left(X \geq m\right)$ for any m and $E\left(X\right)$.

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Solution:

1. For any $n > 0$, $p_n > 0$.
2. $\sum_{1}^{+\infty} \frac{1}{n(n+1)} = \sum_{1}^{+\infty} \frac{1}{n} - \frac{1}{n+1} = 1$

For reason 1,2, p_n is a probability distribution for X.

$$P(X \geq m) = \sum_{m}^{+\infty} \frac{1}{n} - \frac{1}{n+1} = \frac{1}{m}$$

$$E(x) = \sum_{1}^{+\infty} n \left(\frac{1}{n(n+1)} \right) = \sum_{1}^{+\infty} \frac{1}{n+1} = +\infty$$

$\square$