CheatSheet

肖涵薄 31360164

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1 傅里叶 & 拉普拉斯

$f(t)$	$\mathcal{F}[f(t)] = F(\omega)$	$\mathcal{L}[f(t)] = F(s)$
1	$2\pi\delta(\omega)$	$\dfrac{1}{s}$
$f(t-t_0)$	$F(\omega)e^{-i\omega t_0}$	
$e^{at}f(t)$	$F(\omega-\omega_0), i\omega_0 = a$	$F(s-a)$
$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F(\frac{\omega}{\alpha})$	
$u(t-a)$	$\frac{1}{i\omega}+\pi\delta(\omega), a=0$	$\dfrac{e^{-as}}{s}$
$f(t-a)u(t-a)$		$e^{-as}F(s)$
$\delta(t)$	1	1
$\delta(t-t_0)$	$e^{-i\omega t_0}$	e^{-st_0}
$t^n f(t)$	$i^n\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}F(\omega)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
$f'(t)$	$i\omega F(\omega)$	$sF(s)-f(0)$
$f^{(n)}(t)$	$(i\omega)^nF(\omega)$	$s^nF(s)-s^{(n-1)}f(0)$
		$-\cdots-f^{(n-1)}(0)$
$\int_0^t f(t)\,\mathrm{d}t$		$\frac{F(s)}{s}$
$\frac{1}{t}f(t)$		$\int_s^{+\infty}F(s)\,\mathrm{d}s$
$\int_{-\infty}^t f(\tau)\,\mathrm{d}\tau$	$\frac{F(\omega)}{i\omega}+\pi F(0)\delta(\omega)$	
t^n	$2\pi i\delta'(\omega), n=1$	$\dfrac{n!}{s^{n+1}}$
$\sin kt$	$-i\pi[\delta(\omega-k)-\delta(\omega+k)]$	$\dfrac{k}{s^2+k^2}$
$\cos kt$	$\pi[\delta(\omega-k)+\delta(\omega+k)]$	$\dfrac{s}{s^2+k^2}$

e^{at}	$2\pi\delta(\omega-\omega_0), i\omega_0 = a$	$\dfrac{1}{s-a}$
$f_1(t) * f_2(t)$	$F_1(\omega)F_2(\omega)$	same
$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega) * F_2(\omega)$	

2 一阶线性常微分方程的解

1. 通解法

$$\frac{dy}{dx} + P(x)y = Q(x) \implies y = Ce^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \int Q(x)e^{\int P(x)dx}dx$$

2. 特征线法:
   对于方程

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y)$$

   称

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}$$

   为特征方程, 其积分曲线称为特征线。设积分曲线为 ξ, $\eta = y$. 将 u 表示为 ξ, η 的函数, 可将原方程化简.

3 二阶线性常微分方程的解

二阶偏微分方程的标准形式为

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

特征方程为

$$A\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2B\frac{\mathrm{d}y}{\mathrm{d}x} + C = 0$$

令特征方程的解为 ξ, η. 并做变量代换 $u = u(\xi, \eta)$, 可如下化简

$$a\frac{\partial^2 u}{\partial \xi^2} + 2b\frac{\partial^2 u}{\partial \xi \partial \eta} + c\frac{\partial^2 u}{\partial \eta^2} + d\frac{\partial u}{\partial \xi} + e\frac{\partial u}{\partial \eta} + fu = g$$

$$
\begin{aligned}
&a = A\left(\frac{\partial\xi}{\partial x}\right)^2 + 2B\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + C\left(\frac{\partial\xi}{\partial y}\right)^2\\
&b = A\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + B\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + C\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\\
&c = A\left(\frac{\partial\eta}{\partial x}\right)^2 + 2B\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + C\left(\frac{\partial\eta}{\partial y}\right)^2\\
&d = A\frac{\partial^2\xi}{\partial x^2} + 2B\frac{\partial^2\xi}{\partial x\partial y} + C\frac{\partial^2\xi}{\partial y^2} + D\frac{\partial\xi}{\partial x} + E\frac{\partial\xi}{\partial y}\\
&e = A\frac{\partial^2\eta}{\partial x^2} + 2B\frac{\partial^2\eta}{\partial x\partial y} + C\frac{\partial^2\boldsymbol{\xi}}{\partial y^2} + D\frac{\partial\eta}{\partial x} + E\frac{\partial\eta}{\partial y}\\
&f = F\\
&g = G
\end{aligned}
$$

4 波动方程

1. 齐次波动方程 + 无界弦 (Cauchy 问题)

$$
\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < +\infty), \quad u|_{t=0} = \phi(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x)
$$

的达朗贝尔解为

$$
u(x,t) = f_1(x+at) + f_2(x-at) = \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(X)\mathrm{d}X
$$

物理意义: 假设最开始在位置 D 看到的波形为 $f_2(D)$, 观察者在移动 t 时间后，到达位置 $x = D + at$，看到的波形仍为 $f_2(D)$. 因此 $f_2(x-at)$ 表示以速度 a 沿 x 轴正向运动的行波. 在初始速度为零情况下特解是 (波形相同的) 正行波和反行波的叠加。但在初始速度不为零的情况下，特解包含正、反行波及“干涉项” $\int_{x-at}^{x+at}\psi(X)\mathrm{d}X$，后者的出现能使波形发生畸变（甚至变成单个的行波）。

2. 齐次波动方程 + 端点固定半无界弦: 设初始条件反对称于坐标原点

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, t > 0\\
u(x,0) = \varphi(x), \frac{\partial u(x,0)}{\partial t} = \psi(x), 0 < x < \infty\\
u(0,t) = 0 \qquad t > 0
\end{cases}
$$

的通解为

$$
u(x,t) = \frac{1}{2}[\varphi(x+at) - \varphi(at-x)] + \frac{1}{2a}\int_{at-x}^{x+at}\psi(\xi)\mathrm{d}\xi
$$

3. 齐次波动方程 + 端点自由半无界弦: 设初始条件对称于坐标原点 (方程同上) 的通解为

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(at-x)]+\frac{1}{2a}\left[\int_0^{x+at}\psi(\xi)\mathrm{d}\xi+\int_0^{at-x}\psi(\xi)\mathrm{d}\xi\right]$$

4. 同时具有非齐次初始条件和边界条件: 设 $u(x,t)=v(x,t)+w(x,t)$，分别满足非齐次初始条件和边界条件。其中 v 为端点固定半无界问题的解. w 满足齐次初始条件

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2}, & 0<x<\infty, t>0 \\ w(x,0)=0, w_t(x,0)=0, & 0<x<\infty \\ w(0,t)=\mu(t) & t>0 \end{cases}$$

其解为

$$w(x,t)=\mu\left(-\frac{x-at}{a}\right) \quad x-at<0$$

5. 非齐次波动方程 + 非齐次初始条件 = (齐次方程 + 非齐次初始条件)(B)+(非齐次方程 + 齐次初始条件)(C)

问题 (B) 可用达朗贝尔公式求解，下面用齐次化原理来求问题 (C):

设 $\tau\geq 0$ 为参数，如果函数 $w(x,t;\tau)$ 是如下初值问题

$$\begin{cases} \frac{\partial^2 W}{\partial t^2}=a^2\frac{\partial^2 W}{\partial x^2}, & t>\tau>0 \\ W|_{t-\tau=0}=0, W_t|_{t-\tau=0}=f(x,\tau) \end{cases}$$

的解，则函数 $u(x,t)=\int_0^t w(x,t;\tau)\mathrm{d}\tau$ 是非齐次方程初值问题 (C) 的解。

6. 三维波动方程的行波解法

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=a^2\Delta u, \\ u|_{t=0}=f(x,y,z), u_t|_{t=0}=g(x,y,z) \end{cases}$$

其解为泊松公式的累次积分形式:

$$u(x,y,z,t)=\frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}f(\xi,\eta,\zeta)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\right]+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}g(\xi,\eta,\zeta)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi$$

其中

$$\xi=x+at\sin\theta\cos\varphi, \eta=y+at\sin\theta\sin\varphi, \zeta=z+at\cos\theta$$

7. 高维泊松公式的物理意义

   三维: 当初始扰动限制在空间局部范围内时，空间中任意一点 M 受到的扰动总有清晰的“前锋”和“阵尾”，称为惠更斯原理或无后效现象。

   二维: 像这种当初始扰动限制在二维平面局部范围内时，二维平面中任意一点 M 受到的扰动只有清晰的“前锋”而无“阵尾”，称为波的弥散或有后效现象。

8. 二维齐次波动方程

$$u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{at}\frac{f(x+\rho\cos\theta,y+\rho\sin\theta)}{\sqrt{(at)^2-\rho^2}}\rho\mathrm{d}\rho\mathrm{d}\theta+\frac{1}{2\pi a}\int_0^{2\pi}\int_0^{at}\frac{g(x+\rho\cos\theta,y+\rho\sin\theta)}{\sqrt{(at)^2-\rho^2}}\rho\mathrm{d}\rho\mathrm{d}\theta$$

   其中 $\xi-x=\rho\cos\theta,\eta-y=\rho\sin\theta$

9.

4.1 影响区域

解在点 (x,t) 的值只与区间 $[x-at,x+at]$ 的初始条件有关，该区域称为点 (x,t) 的依赖区间。影响区域和决定区域: 在影响区域内任意一点的位移值都要受该区间上初始条件的影响，影响区域内包含一个决定区域，该区域内任意一点的位移值都由 $[x_1,x_2]$ 上的初始条件决定。

5 泊松方程

$$\nabla^2 u=-f(x,y,z)$$

Green 函数法求解:

1. 第一类边界条件 (狄利克雷条件)

$$\begin{cases}\nabla^2 u(\boldsymbol{r})=-f(\boldsymbol{r})\\ u(\boldsymbol{r})\big|_{\partial\Omega}=\varphi(\boldsymbol{r})\end{cases}$$

   其解为

$$u(\boldsymbol{r})=\iiint_\Omega G(\boldsymbol{r},\boldsymbol{r}_0)\,f(\boldsymbol{r}_0)\,\mathrm{d}V_0-\oint_{\partial\Omega}\varphi(\boldsymbol{r}_0)\,\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial n}\mathrm{d}S_0$$

   第一项物理意义为源点 r_0 处所有电荷在 r 处产生电势的累加；第二项代表边界处产生的感应电荷在 r 产生电势的累加。

2. 第二类边值问题（纽曼边值问题）

$$\begin{cases} \nabla^2 u = -f(\boldsymbol{r}) \\ \frac{\partial u}{\partial n}\big|_{\partial\Omega} = \varphi(\boldsymbol{r}) \end{cases}$$

其解为

$$u(\boldsymbol{r}) = \frac{1}{S}\iiint_\Omega u\left(\boldsymbol{r}_0\right) \mathrm{d}V_0 + \iiint_\Omega G\left(\boldsymbol{r}, \boldsymbol{r}_0\right) f\left(\boldsymbol{r}_0\right) \mathrm{d}V_0 + \oint_\Omega \varphi\left(\boldsymbol{r}_0\right) G\left(\boldsymbol{r}, \boldsymbol{r}_0\right) \mathrm{d}S_0$$

3. 第三类边值问题

$$\begin{cases} \nabla^2 u = -f(r) \\ \alpha u + \beta\ \frac{\partial u}{\partial n}\big|_{\partial\Omega} = \varphi(r) \end{cases}$$

其解为

$$u(\boldsymbol{r}) = \iiint_\Omega G\left(\boldsymbol{r}, \boldsymbol{r}_0\right) f\left(\boldsymbol{r}_0\right) \mathrm{d}V_0 + \frac{1}{\beta}\oint_{\partial\Omega} \varphi(\boldsymbol{r}) G\left(\boldsymbol{r}, \boldsymbol{r}_0\right) \mathrm{d}S_0$$

或

$$u(\boldsymbol{r}) = \iiint_\Omega G\left(\boldsymbol{r}, \boldsymbol{r}_0\right) f\left(\boldsymbol{r}_0\right) \mathrm{d}V_0 - \frac{1}{\alpha}\oint_{\partial\Omega} \varphi(\boldsymbol{r}) \frac{\partial G\left(\boldsymbol{r}, \boldsymbol{r}_0\right)}{\partial n} \mathrm{d}S_0$$

6 热传导方程

$$u_t = k\Delta u$$

分类变量法解:

$$T'(t) = -\lambda k T(t)$$

$$X''(x) = -\lambda X(x)$$

可解得

$$u(t,x) = \sum_{n=1}^{+\infty} D_n \left(\sin\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 kt}{L^2}}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

其中

$$D_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

7 分离变量法本征函数

边界条件	本征值 λ_n	本征函数 $X_n(x)$	n
$u\vert_{x=0}=0,\ u\vert_{x=L}=0$	$\left(\frac{n\pi}{L}\right)^2$	$B_n\sin\frac{n\pi}{L}x$	1,2..
$u\vert_{x=0}=0,\ \frac{\partial u}{\partial x}\Big\vert_{x=L}=0$	$\left[\frac{(2n+1)\pi}{2L}\right]^2$	$B_n\sin\frac{(2n+1)\pi}{2L}x$	0,1..
$\frac{\partial u}{\partial x}\Big\vert_{x=0}=0,\ u\vert_{x=L}=0$	$\left[\frac{(2n+1)\pi}{2L}\right]^2$	$A_n\cos\frac{(2n+1)\pi}{2L}x$	0,1..
$\frac{\partial u}{\partial x}\Big\vert_{x=0}=0,\ \frac{\partial u}{\partial x}\Big\vert_{x=L}=0$	$\left(\frac{n\pi}{L}\right)^2$	$A_n\cos\frac{n\pi}{L}x$	0,1..

8 Gamma 函数

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt \quad (x>0)$$

$$\Gamma(x+1)=x\Gamma(x)$$

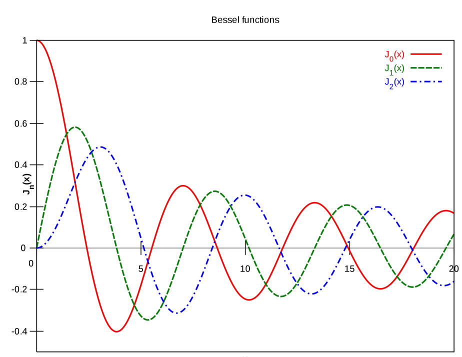
$$\Gamma(n+1)=n!$$

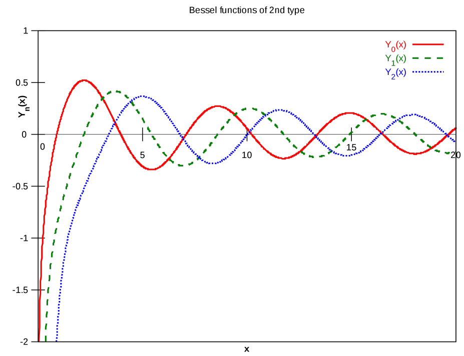
$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

斯特林公式

$$n!=\sqrt{2\pi n}n^n\mathrm{e}^{-n}$$

9 贝塞尔函数





9.1 第一类贝塞尔函数

贝塞尔方程

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left[x^2-v^2\right]y=0\quad(x>0)$$

的通解为:

$$y=AJ_v(x)+BJ_{-v}(x),v\text{不为整数}$$

定义为

$$J_v(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+1+v)}\left(\frac{x}{2}\right)^{2m+v}$$

$$J_\alpha(x)=\frac{1}{2\pi}\int_0^{2\pi}\cos(\alpha\tau-x\sin\tau)d\tau$$

$$J_\alpha(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(\alpha\tau-x\sin\tau)}d\tau$$

整数阶 $J_n(x)$ 和 $J_{-n}(x)$ 线性相关.

母函数

$$e^{(x/2)(t-1/t)}=\sum_{n=-\infty}^{\infty}J_n(x)t^n$$

递推公式 $(Y(x),H(x)$ 也满足)

$$\frac{d}{dx}\left[J_0(x)\right]=-J_1(x)$$

$$\begin{cases}\frac{d}{dx}\left[x^vJ_v(x)\right]=x^vJ_{v-1}(x)\\ \frac{d}{dx}\left[x^{-v}J_v(x)\right]=-x^{-v}J_{v+1}(x)\end{cases}$$

$$J_\nu'(x)=\tfrac{1}{2}\left[J_{\nu-1}(x)-J_{\nu+1}(x)\right]$$

$$J_{\nu-1}(x)+J_{\nu+1}(x)=\tfrac{2\nu}{x}J_\nu(x)$$

渐近公式

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

贝塞尔函数的正交完备性

$$\begin{aligned}&\textstyle\int_0^a rJ_n\left(\frac{\mu_m^{(n)}}{a}r\right) J_n\left(\frac{\mu_k^{(n)}}{a}r\right) dr = \\ &\begin{cases} 0 & m \neq k \\ \frac{a^2}{2} J_{n-1}^2\left(\mu_m^{(n)}\right) = \frac{a^2}{2} J_{n+1}^2\left(\mu_m^{(n)}\right) & m = k \end{cases}\end{aligned}$$

三角函数变换:

$$\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x), \quad \sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$$

9.2 第二类贝塞尔函数（诺依曼函数）

定义为

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

可以使得贝塞尔方程的解为

$$y = AJ_v(x) + BY_v(x), v\text{为整数}$$

渐近形式

$$Y_\alpha(x) \to \begin{cases} \frac{2}{\pi}\left[\ln(x/2) + \gamma\right] & \text{if } \alpha = 0 \\ -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{x}\right)^\alpha & \text{if } \alpha > 0 \end{cases}$$

9.3 第三类贝塞尔函数（汉克尔函数）

定义为

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x)$$

$$H_\alpha^{(2)}(x) = J_\alpha(x) - iY_\alpha(x)$$

$$H_\alpha^{(1)}(x) = \tfrac{J_{-\alpha}(x) - e^{-\alpha\pi i}J_\alpha(x)}{i\sin(\alpha\pi)}$$

$$H_\alpha^{(2)}(x) = \tfrac{J_{-\alpha}(x) - e^{\alpha\pi i}J_\alpha(x)}{-i\sin(\alpha\pi)}$$

$$H_{-\alpha}^{(1)}(x) = e^{\alpha\pi i}H_{\alpha}^{(1)}(x)$$
$$H_{-\alpha}^{(2)}(x) = e^{-\alpha\pi i}H_{\alpha}^{(2)}(x)$$

它们描述了二维波动方程的内行柱面波解和外行柱面波解.

9.4 球贝塞尔函数

球贝塞尔方程

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2r^2 - \omega^2\right)R = 0$$

的解为

$$y_{nm}(x) = j_n\left(\lambda_{nm}x\right)$$

9.5 例题

对于方程

$$\begin{cases} \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\mathrm{P}}{d\rho}\right) + \lambda^2\mathrm{P} = 0 \\ \mathrm{P}(0)\text{有界},\ P'(a) = 0 \end{cases}, \quad T' + \lambda^2\kappa T = 0$$

解本征值问题得 $\lambda_0 = 0, \quad \lambda_i = \frac{\mu_i'}{a}$,

$$\mathrm{P}_0(\rho) = A_0, \quad \mathrm{P}_i(\rho) = J_0\left(\frac{\mu_i'}{a}\rho\right), \quad T(t) = A_i\exp\left[-\kappa\left(\frac{\mu_i}{a}\right)^2 t\right]$$

对于本征值问题

$$\begin{cases} Z'' + k^2Z = 0 \\ Z(0) = 0, Z(h) = 0 \end{cases}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\mathrm{P}}{d\rho}\right) - k^2\mathrm{P} = 0$$

解得本征函数 $Z_n(z) = \sin\frac{n\pi}{h}z$,$\mathrm{P}_n(\rho) = I_0\left(\frac{n\pi}{h}\rho\right)$.

$$u(\rho, z) = \sum_{n=1}^{\infty} A_nI_0\left(\frac{n\pi}{h}\rho\right)\sin\frac{n\pi}{h}z$$

10 勒让德多项式

连带勒让德方程

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+\left(\omega^2-\frac{m^2}{1-x^2}\right)y=0$$

勒让德方程 ($m=0$):

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+\omega^2 y=0$$

勒让德方程的通解为：$y(x)=AP_l(x)+BQ_l(x)$

l 阶勒让德多项式 (罗德里格公式)

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}[\left(x^2-1\right)^l]$$

特点:l 为偶数时以 y 轴对称 (Pl(x) 为偶函数),l 为奇数时以坐标原点对称 (Pl(x) 为奇函数)$P_l(-1)=(-1)^l$

$$|\mathrm{P}_l(x)|\leq 1,\quad (-1\leq x\leq 1)$$

递推公式

$$\begin{aligned}
&(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)\\
&P_n(x)=P'_{n+1}(x)-2xP'_n(x)+P'_{n-1}(x)\\
&xP'_n(x)-P'_{n-1}(x)=nP_n(x)\\
&P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x)\\
&nP'_{n+1}(x)+(n+1)P'_{n-1}(x)=(2n+1)xP'_n(x)\\
&(n\geq 0)\quad P'_{n+1}(x)=(n+1)P_n(x)+xP'_n(x)
\end{aligned}$$

正交性

$$\int_{-1}^{1}P_l(x)P_m(x)\mathrm{d}x=0\quad (l\neq m)$$

模值

$$I_l\equiv\int_{-1}^{1}P_l^2(x)\mathrm{d}x=\frac{2}{2l+1}$$

函数展成勒让德多项式的级数

设展开式为

$$f(x)=\sum_{l=0}^{\infty}C_lP_l(x)$$

则展开系数为

$$C_l=\frac{2l+1}{2}\int_{-1}^{1}P_l(x)|x|\mathrm{d}x$$

11 Green 函数

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x}') = -\frac{1}{\varepsilon_0}\delta(\boldsymbol{x}-\boldsymbol{x}')$$

无界三维泊松方程对应的 Green 函数

$$G(x,x') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|r-r'|}$$

无界二维泊松方程对应的 Green 函数

$$u(r) = \frac{1}{2\pi\varepsilon}\ln\frac{1}{r}$$

无界三维波动方程的对应的 Green 函数

$$G(\boldsymbol{r},\boldsymbol{r}_0,t) = \frac{1}{4\pi a}\frac{\delta(|\boldsymbol{r}-\boldsymbol{r}_0|-at)}{|\boldsymbol{r}-\boldsymbol{r}_0|}$$

上半空间的 Green 函数

$$G(x,x') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$$

球外空间的 Green 函数

x' 为点电荷的位置，镜象点电荷的位置在 $\frac{R_0^2}{R'}\frac{\boldsymbol{x}'}{R'}$ 带电量为 $-\frac{R_0}{R'}$.

$$G(x,x') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{R_0}{R'r'}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+R'^2-2RR'\cos\alpha}} - \frac{1}{\sqrt{\left(\frac{RR'}{R_0}\right)^2+R_0^2-2RR'\cos\alpha}}\right]$$

12 坐标变换

12.1 柱坐标系 (ρ,ϕ,z)

$$\nabla\varphi = \widehat{e}_1\frac{\partial\varphi}{\partial\rho} + \widehat{e}_2\frac{1}{\rho}\frac{\partial\varphi}{\partial\phi} + \widehat{e}_3\frac{\partial\varphi}{\partial z}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_1)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\boldsymbol{A} = \widehat{e}_1(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}) + \widehat{e}_2(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}) + \widehat{e}_3\frac{1}{\rho}(\frac{\partial(\rho A_2)}{\partial\rho} - \frac{\partial A_1}{\partial\phi})$$

$$\nabla^2\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\varphi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2}$$

12.2 球坐标系 (r, θ, φ)

$$
\begin{gathered}
\nabla\varphi = \widehat{e}_1 \frac{\partial \varphi}{\partial r} + \widehat{e}_2 \frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \widehat{e}_3 \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi} q \\
\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial r^2 A_1}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial \phi} \\
\nabla \times \boldsymbol{A} = \widehat{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial \phi}\right] + \widehat{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \widehat{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial \theta}\right] \\
\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\frac{\partial \varphi}{\partial r}) + \frac{\partial}{\partial \theta}(\sin\theta\frac{\partial \varphi}{\partial \theta}) + \frac{1}{\sin\theta}\frac{\partial^2 \varphi}{\partial \phi^2}\right]
\end{gathered}
$$