The 12th HW of Electrodynamics

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Solve the equation:

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\varphi = -\frac{q}{\epsilon_0}\delta(\vec{x})$$

solutions:

两边 fourier 变换后:

$$-(k^2 + \frac{1}{\lambda^2})\widetilde{\varphi} = -\frac{q}{\varepsilon_0}$$

因此

$$\widetilde{\varphi} = \frac{\frac{q}{\varepsilon_0}}{k^2 + \frac{1}{\lambda^2}}$$

求逆变换:

$$\varphi = \frac{q}{(2\pi)^3\varepsilon_0}\iiint \frac{e^{ikr\cos\theta}}{k^2 + \frac{1}{\lambda^2}}\,\mathrm{d}\boldsymbol{k}$$

化简:

$$\begin{aligned}\varphi =&\frac{q}{(2\pi)^3\varepsilon_0}\iiint \frac{e^{ikr\cos\theta}}{k^2+\frac{1}{\lambda^2}}\,\mathrm{d}\boldsymbol{k}\\ =&\frac{q}{(2\pi)^3\varepsilon_0}\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\int_0^\infty k^2\sin\theta\frac{e^{ikr\cos\theta}}{k^2+\frac{1}{\lambda^2}}\,\mathrm{d}k\\ =&\frac{q}{(2\pi)^2\varepsilon_0}\int_0^\pi \mathrm{d}\theta\int_0^\infty k^2\sin\theta\frac{e^{ikr\cos\theta}}{k^2+\frac{1}{\lambda^2}}\,\mathrm{d}k\\ =&\frac{q}{i(2\pi)^2 r\varepsilon_0}\int_0^\infty \frac{2k\sin kr}{k^2+\frac{1}{\lambda^2}}\,\mathrm{d}k\\ =&\frac{q}{i(2\pi)^2 r\varepsilon_0}\int_{-\infty}^\infty \frac{ke^{ikr}}{k^2+\frac{1}{\lambda^2}}\,\mathrm{d}k\\ =&\frac{q}{i(2\pi)^2 r\varepsilon_0}\int_{-\infty}^\infty \frac{ke^{ikr}}{k^2+\frac{1}{\lambda^2}}\,\mathrm{d}k\\ =&\frac{q}{i(2\pi)^2 r\varepsilon_0}2\pi i\,\mathrm{Res}\left[\frac{ze^{izr}}{z^2+\frac{1}{\lambda^2}},z_0\right]\\ &\text{在}z_0=\frac{i}{\lambda}\text{取一阶极点}\\ =&\frac{q}{4\pi r\varepsilon_0}e^{-\frac{r}{\lambda}}\end{aligned}$$

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$$\begin{aligned}\varphi(\vec{r},t)&=\frac{1}{4\pi\epsilon_0}\iiint \mathrm{d}^3\vec{r'}\frac{1}{\left|\vec{r}-\vec{r'}\right|}\rho\left(\vec{r'},t-\frac{1}{c}\left|\vec{r}-\vec{r'}\right|\right)\\ &=\frac{q}{4\pi\varepsilon_0}\iiint\frac{\delta\left(\boldsymbol{r'}-\boldsymbol{r}_e\left(t'\right)\right)}{R}\,\mathrm{d}^3r'\end{aligned}$$

做变量替换: $\boldsymbol{r''}=\boldsymbol{r'}-\boldsymbol{r}_e(t')$. $J=\det\left(\nabla'\boldsymbol{r''}\right)=\det\left[\nabla'\boldsymbol{r'}-\nabla'\boldsymbol{r}_e\left(t'\right)\right]=1-\frac{\boldsymbol{v}\cdot\boldsymbol{R}}{cR}$

$$\begin{aligned}\varphi(\boldsymbol{r},t)=\frac{q}{4\pi\varepsilon_0}\iiint\frac{\delta\left(\boldsymbol{r''}\right)}{R}|J|^{-1}\mathrm{d}V''&=\frac{q}{4\pi\varepsilon_0}\iiint\frac{\delta\left(\boldsymbol{r''}\right)}{R\left(1-\frac{\boldsymbol{R}\cdot\boldsymbol{v}}{cR}\right)}\mathrm{d}V''\\ =\frac{q}{4\pi\varepsilon_0\left[R\left(1-\frac{\boldsymbol{v}\cdot\boldsymbol{R}}{cR}\right)\right]_{r''=0}}&=\frac{q}{4\pi\varepsilon_0\left[\left|\vec{r}-\vec{r'}\right|\left(1-\frac{v\cos\theta}{c}\right)\right]}\end{aligned}$$

同理

$$\begin{aligned}A(r,t)=\frac{\mu_0}{4\pi}\iiint\frac{\boldsymbol{J}\left(r',t-\frac{R}{c}\right)}{R}\mathrm{d}V'&=\frac{\mu_0 q}{4\pi}\iiint\frac{v\left(t'\right)\delta\left(\boldsymbol{r'}-\boldsymbol{r}_e\left(t'\right)\right)}{R}\mathrm{d}V'\\ \Longrightarrow\ \vec{A}(\vec{r},t)&=\frac{\vec{v}\left(t'\right)}{c^2}\varphi(\vec{r},t)\end{aligned}$$