The 4th HW of Electrodynamics

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Q1

A hollow cube has conducting walls defined by six planes $x = 0,\ y = 0,\ z = 0$, and $x = a$, $y = a$, $z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are all at zero potential.

a) Let $\varphi(x, y, z) = X(x) Y(y) Z(z)$, Plugging into the Laplace' s equation, we get $\frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z}{dz^2} = 0$.
The solutions of $X(x), Y(y), Z(z)$ are

$$\varphi = \sum_{l,m,n} (C_{xl}\cos\alpha_l x + D_{xl}\sin\alpha_l x)(C_{ym}\cos\beta_m y + D_{ym}\sin\beta_m y)(C_{zn}\cosh\gamma_n z + D_{zn}\sinh\gamma_n z)$$

Since $\varphi = 0$ for $x, y = 0, a$,

$$C_{xl} = C_{ym} = 0, C_{zn} = C_{zn}\cosh\gamma_n a + D_{zn}\sinh\gamma_n a,\ \alpha_l = \frac{l\pi}{a}, \beta_m = \frac{m\pi}{a}, \gamma_{lm} = \frac{\pi}{a}\sqrt{l^2+m^2}$$

The solution is:

$$\varphi = \sum_{l,m=1}^{\infty} A_{lm}\sin(\alpha_l x)\sin(\beta_m y)\left[\cosh(\gamma_{lm}z) + \frac{1-\cosh\left(\sqrt{l^2+m^2}\pi\right)}{\sinh\left(\sqrt{l^2+m^2}\pi\right)}\sinh(\gamma_{lm}z)\right]$$

$$V = \sum_{l,m=1}^{\infty} A_{lm}\sin(\alpha_l x)\sin(\beta_m y)\left[\cosh\left(\sqrt{l^2+m^2}\pi\right) + \frac{1-\cosh\left(\sqrt{l^2+m^2}\pi\right)}{\sinh\left(\sqrt{l^2+m^2}\pi\right)}\sinh\left(\sqrt{l^2+m^2}\pi\right)\right]$$
$$= \sum_{l,m=1}^{\infty} A_{lm}\sin(\alpha_l x)\sin(\beta_m y)$$

$$A_{lm} = \frac{4V}{a^2}\int_0^a dx\int_0^a dy\sin\left(\frac{l\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right) = \frac{16V}{ml\pi^2},\ for\ m, n\ odd.$$

b)

$$\varphi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)=\sum_{m,n\ odd}^{\infty}\frac{16V}{ml\pi^2}\sin\left(\frac{l\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\cdot$$
$$\left[\cosh\left(\sqrt{l^2+m^2}\frac{\pi}{2}\right)+\frac{1-\cosh\left(\sqrt{l^2+m^2}\pi\right)}{\sinh\left(\sqrt{l^2+m^2}\pi\right)}\sinh\left(\sqrt{l^2+m^2}\frac{\pi}{2}\right)\right]$$

With $\sin\left(\frac{(2n+1)\pi}{2}\right)=(-1)^n$, Let $2i+1=m,\ 2j+1=l$.

$$\varphi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right)=\sum_{i,j=0}^{\infty}\frac{16V}{(2i+1)(2j+1)\pi^2}(-1)^{i+j}\cdot$$
$$\left[\cosh\left(\sqrt{l^2+m^2}\frac{\pi}{2}\right)+\frac{1-\cosh\left(\sqrt{l^2+m^2}\pi\right)}{\sinh\left(\sqrt{l^2+m^2}\pi\right)}\sinh\left(\sqrt{l^2+m^2}\frac{\pi}{2}\right)\right]$$

Let$V=1$. For $i=j=0$, $\varphi=0.347546$.

For $i=j=1$, $\varphi=0.332958$.

For $i=1,j=2,\ m=5,n=3,\ \Delta\varphi=-0.000023$.

Thus 4 term is needed to achieve 3 significant figures.

c)

$$\sigma=\varepsilon_0\frac{\partial\varphi}{\partial z}=\frac{16\varepsilon_0 V}{\pi a}\sum_{l,m\ \text{odd}}\frac{\sqrt{l^2+m^2}}{lm}\tanh\left(\sqrt{l^2+m^2}\pi/2\right)\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

Q2

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta=\alpha$.

a) $\phi_{in}=\sum a_n r^n,\ \phi_{out}=\sum\frac{b_n}{r^{n+1}}$. As $E_{r\text{ out}}|_{r=R}=.E_{r\text{ in}}|_{r=R}+\frac{1}{\varepsilon_0}\sigma$,

$$\Phi_{\text{in}}=\sum_{l=0}^{\infty}\alpha_l\left(\frac{r}{R}\right)^l P_l(\cos\theta)$$
$$\Phi_{\text{out}}=\sum_{l=0}^{\infty}\alpha_l\left(\frac{R}{r}\right)^{l+1}P_l(\cos\theta)$$

Thus

$$E_{r\text{ in}}=-\sum_{l=1}^{\infty}\frac{l\alpha_l}{R}\left(\frac{r}{R}\right)^{l-1}P_l(\cos\theta)$$
$$E_{\text{rout}}=\sum_{l=0}^{\infty}\frac{(l+1)\alpha_l}{R}\left(\frac{R}{r}\right)^{l+2}P_l(\cos\theta)$$

Substituting this,

$$\sigma(\cos\theta) = \varepsilon_0[E_{r\ \mathrm{out}} - E_{r\ \mathrm{in}}]_{r=R} = \sum_{l=0}^{\infty} \frac{(2l+1)\,\varepsilon_0\alpha_l}{R} P_l(\cos\theta)$$

by the relation

$$\frac{(2l+1)\,\varepsilon_0\alpha_l}{R} = \frac{2l+1}{2}\int_{-1}^{1} \sigma(\cos\theta)\,P_l(\cos\theta)\,d(\cos\theta)$$

gives

$$\alpha_l = \frac{Q}{8\pi\varepsilon_0 R}\int_{-1}^{\cos\alpha} P_l(\cos\theta)\,d(\cos\theta) = \frac{Q}{8\pi\varepsilon_0 R}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]$$

Hence

$$\Phi = \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)]\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

b) Noting that $E_{in} \approx r^{l-1}$, we see that only the $l = 1$ component survives at the origin. Thus

$$\begin{aligned} E_r(r=0,\theta=0) &= -\frac{\alpha_1}{R}P_1(1) \\ &= -\frac{Q}{8\pi\varepsilon_0 R^2}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right] \\ &= -\frac{Q}{16\pi\varepsilon_0 R^2}\left(\cos^2\alpha - 1\right) = \frac{Q\sin^2\alpha}{16\pi\varepsilon_0 R^2} \end{aligned}$$

Also

$$\vec{E} = \frac{Q\sin^2\alpha}{16\pi\varepsilon_0 R^2}\widehat{z}$$

c) Series expanion:

$$P_l(\cos\alpha) \approx P_l\left(1 - \frac{1}{2}\alpha^2\right) \approx P_l(1) - \frac{1}{2}\alpha^2 P_l'(1) = 1 - 2\delta_{l,-1} - \frac{1}{2}\alpha^2 P_l'(1)$$

$$\implies P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) \approx 2\delta_{l,0} - \frac{1}{2}\alpha^2[P_{l+1}'(1) - P_{l-1}'(1)]$$

Using solution in b),

$$\Phi \approx \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>} - \frac{Q\alpha^2}{16\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

Where $r_< = \min(r,R)$, $\quad r_> = \max(r,R)$. Recalling the Green's function expansion,

$$\Phi \approx \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>} - \frac{Q\alpha^2/4}{4\pi\varepsilon_0}\frac{1}{|\vec{r} - R\widehat{z}|}$$

By linear superposition, the very small cap can be thought of electively as an oppositely charged particle located at R^z with charge given by

$$q = -\sigma dA = -\frac{Q}{4\pi R^2}\left(R^2 d\Omega\right) = -\frac{Q}{4\pi}\left(\pi\alpha^2\right) = -\frac{Q\alpha^2}{4}$$

Hence $\vec{E}(0) \approx \frac{Q\alpha^2/4}{4\pi\varepsilon_0}\frac{\hat{z}}{R^2}$ for $\alpha \approx 0$.

And then we consider the case $a \to \pi$.

$$P_l(\cos\alpha) = P_l(\cos(\pi-\beta)) = P_l(-\cos\beta) \approx P_l\left(-1+\frac{1}{2}\beta^2\right) \approx (-1)^l + \frac{1}{2}\beta^2 P_l'(-1)$$

$$\begin{aligned} P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) &\approx \frac{1}{2}\beta^2[P_{l+1}'(-1) - P_{l-1}'(-1)] \\ &= \frac{2l+1}{2}\beta^2 P_l(-1) = \frac{2l+1}{2}\beta^2(-1)^l \end{aligned}$$

Using solution in b),

$$\begin{aligned} \Phi &\approx \frac{Q\beta^2}{16\pi\varepsilon_0}\sum_{l=0}^{\infty}(-1)^l\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) = \frac{Q\beta^2}{16\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(-\cos\theta) \\ &= \frac{Q\beta^2/4}{4\pi\varepsilon_0}\frac{1}{|\vec{r}+R\hat{z}|} \end{aligned}$$

Finally, subsitute $\alpha = \pi - \beta$ in it,

$$\vec{E}(0) \approx \frac{Q\beta^2/4}{4\pi\varepsilon_0}\hat{z}R^2$$

Q3

A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a and dielectric constant ε_r.

a) Let q located at $(d,0,0)$. let φ_1 be the potential made by point charge, let φ_2 be the potential made by dielectric There is no free charge anywhere in the space, $\nabla^2\varphi = 0$. And we know

$$\Phi_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-d\hat{\mathbf{z}}|} + \frac{q'}{|\mathbf{r}-(a^2/d)\,\hat{\mathbf{z}}|}\right)$$

$$\Longrightarrow \begin{array}{ll} \Phi_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}P_l(\cos\theta)\frac{1}{r^{l+1}}\left(qd^l + q'\frac{a^{2l}}{d^l}\right) & \text{if } r > d \\ \Phi_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(q\frac{r^l}{d^{l+1}} + q'\frac{a^{2l}}{d^l r^{l+1}}\right) & \text{if } r < d \end{array}$$

There is no charge inside the sphere, so all we need is an image charge q'' outside the sphere at $z = d$ to simulate the effects of the dielectric material.

$$\Phi_{\text{in}} = \frac{1}{4\pi\varepsilon}\left(\sum_{l=0}^{\infty} q'' \frac{r^l}{d^{l+1}} P_l(\cos\theta)\right)$$

Apply the boundary condition. When $r = a$, $\varepsilon_0 E_{r-} = \varepsilon_r E_{r+}$, $\varepsilon_0 \frac{\partial \varphi_{r-}}{\partial r} = \varepsilon_r \frac{\partial \varphi_{r+}}{\partial r}$.

$$ql + q'(-l-1)\frac{d}{a} = q''l, \quad q'' = \frac{\varepsilon}{\varepsilon_0}q + \frac{\varepsilon}{\varepsilon_0}q'\frac{d}{a}$$

The final solution is:

$$\Phi_{\text{in}} = \frac{q}{4\pi\varepsilon_0 d}\left(1 + \sum_{l=1}^{\infty} \frac{2l+1}{(l+1) + l\varepsilon/\varepsilon_0}\left(\frac{r}{d}\right)^l P_l(\cos\theta)\right)$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\frac{d^{l+1}}{r^{l+1}}\left(1 + \frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right)l}{\left(\frac{\varepsilon_0}{\varepsilon}(l+1) + l\right)}\left(\frac{a}{d}\right)^{2l+1}\right), \text{ if } r > d.$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\left(\frac{r}{d}\right)^l + \frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right)l}{\left(\frac{\varepsilon_0}{\varepsilon}(l+1) + l\right)}\left(\frac{a}{d}\right)^l\left(\frac{a}{r}\right)^{l+1}\right) \text{ if } r < d.$$

b) near thecenter of the sphere $r << d$. the higher order terms become negligible.

$$\Phi_{\text{ in}} = \tfrac{q}{4\pi\varepsilon_0 d}[1 + \tfrac{3}{1+2\varepsilon_0/\varepsilon}\tfrac{r}{d}\cos\theta]$$

$$\Phi_{\text{ in}} = \tfrac{q}{4\pi\varepsilon_0 d}[1 + \tfrac{3}{2+\varepsilon/\varepsilon_0}\tfrac{z}{d}]$$

$$E = -\nabla\Phi$$

$$E = -\tfrac{q}{4\pi\varepsilon_0 d^2}[\tfrac{3}{2+\varepsilon/\varepsilon_0}]\hat{\mathbf{z}}$$

c)

$$\Phi_{\text{in}} = \frac{q}{4\pi\varepsilon_0 d}\left(1 + \sum_{l=1}^{\infty} \frac{2l+1}{(l+1) + l\varepsilon/\varepsilon_0}\left(\frac{r}{d}\right)^l P_l(\cos\theta)\right) \approx \frac{q}{4\pi\varepsilon_0 d}$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\frac{d^{l+1}}{r^{l+1}}\left(1 + \frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right)l}{\left(\frac{\varepsilon_0}{\varepsilon}(l+1) + l\right)}\left(\frac{a}{d}\right)^{2l+1}\right), \text{ if } r > d.$$

$$\approx \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\frac{d^{l+1}}{r^{l+1}}\left(1 - \left(\frac{a}{d}\right)^{2l+1}\right), \text{ if } r > d.$$

$$\Phi_{\text{out}} = \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\left(\frac{r}{d}\right)^l + \frac{\left(\frac{\varepsilon_0}{\varepsilon} - 1\right)l}{\left(\frac{\varepsilon_0}{\varepsilon}(l+1) + l\right)}\left(\frac{a}{d}\right)^l\left(\frac{a}{r}\right)^{l+1}\right), \text{ if } r < d.$$

$$\approx \frac{q}{4\pi\varepsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\left(\frac{r}{d}\right)^l - \left(\frac{a}{d}\right)^l\left(\frac{a}{r}\right)^{l+1}\right), \text{ if } r < d.$$

Q4

Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ε_r), as shown in the figure.

a)

$$2\pi r^2 (\varepsilon_0 + \varepsilon_r) E = Q \implies E = \frac{Q}{2\pi r^2 (\varepsilon_0 + \varepsilon_r)} \widehat{\boldsymbol{e}}_r$$

b) In the vacuum: σ_{free1}, and in the dielectric: σ_{free2}.

$$\sigma_{free1} = \varepsilon_0 E = \frac{\varepsilon_0 Q}{2\pi a^2 (\varepsilon_0 + \varepsilon_r)}, \quad \sigma_{free2} = \varepsilon_r E = \frac{\varepsilon_r Q}{2\pi a^2 (\varepsilon_0 + \varepsilon_r)}$$

c)

$$\sigma_{polar} = -D + \varepsilon_0 E_2 = \varepsilon_r E = \frac{(\varepsilon_0 - \varepsilon_r) Q}{2\pi a^2 (\varepsilon_0 + \varepsilon_r)}$$