The 2nd HW of Electrodynamics

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2019 年 3 月 1 日

Q1

1. Prove that $\nabla r = -\nabla' r = \frac{\boldsymbol{r}}{r}$:

for x direction:

$$(\nabla r)_x = \frac{\partial r}{\partial x} = \frac{1}{2r} 2\left(x - x'\right) = \frac{\boldsymbol{r}_x}{r}$$

$$(\nabla' r)_x = \frac{\partial r}{\partial x'} = \frac{1}{2r} 2\left(x - x'\right)(-1) = -\frac{\boldsymbol{r}_x}{r}$$

the same for y and z direcion, therefore $\nabla r = -\nabla' r = \frac{\boldsymbol{r}}{r}$

2. Prove that $\nabla \frac{1}{r} = -\nabla' \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$:

for x direction:

$$\left(\nabla \frac{1}{r}\right)_x = \frac{\partial 1/r}{\partial x} = -\frac{1}{2}\frac{1}{r^3} 2\left(x - x'\right) = -\frac{\boldsymbol{r}_x}{r^3}$$

$$\left(\nabla' \frac{1}{r}\right)_x = \frac{\partial 1/r}{\partial x'} = -\frac{1}{2}\frac{1}{r^3} 2\left(x - x'\right)(-1) = \frac{\boldsymbol{r}_x}{r^3}$$

the same for y and z direcion, therefore $\nabla \frac{1}{r} = -\nabla' \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$

3. Prove that $\nabla \times \frac{\boldsymbol{r}}{r^3} = 0$:

from the question above we know $\nabla \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$, and $\nabla \times (\nabla A) = 0$ for any scalar A:

$$\nabla \times \frac{\boldsymbol{r}}{r^3} = -\nabla \times \left(\nabla \frac{1}{r}\right) = 0$$

4. Prove that $\nabla \cdot \frac{\boldsymbol{r}}{r^3} = -\nabla \cdot \frac{\boldsymbol{r}}{r^3} = 0$ for any $r \neq 0$:

for x direction:

$$\left(\nabla \cdot \frac{\boldsymbol{r}}{r^3}\right)_x = \frac{\partial \boldsymbol{r}_x / r^3}{\partial x} = r^{-3} - \frac{3}{2} 2\boldsymbol{r}_x^2 \frac{1}{r^5} = \frac{1}{r^3} - 3\frac{\boldsymbol{r}_x^2}{r^5}$$

$$\left(\nabla' \cdot \frac{\boldsymbol{r}}{r^3}\right)_x = \frac{\partial \boldsymbol{r}_x / r^3}{\partial x'} = -r^{-3} + \frac{3}{2} 2\boldsymbol{r}_x^2 \frac{1}{r^5} = -\frac{1}{r^3} + 3\frac{\boldsymbol{r}_x^2}{r^5}$$

so for all directin:

$$\nabla \cdot \frac{\boldsymbol{r}}{r^3} = \frac{3}{r^3} - \frac{3}{r^5}\left(\boldsymbol{r}_x^2 + \boldsymbol{r}_y^2 + \boldsymbol{r}_z^2\right) = 0$$
$$\nabla' \cdot \frac{\boldsymbol{r}}{r^3} = -\frac{3}{r^3} + \frac{3}{r^5}\left(\boldsymbol{r}_x^2 + \boldsymbol{r}_y^2 + \boldsymbol{r}_z^2\right) = 0$$

Q2

Show that the interaction between two fixed current loops obeys Newton's third law:

$$F_{12} = \int_{C_1} I_1 \mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{B} = \frac{\mu_0 I_1 I_2}{4\pi} \iint_{C_1, C_2} \frac{\mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{r_{12}})}{r_{12}^3}$$

$$\frac{\mathrm{d}\boldsymbol{l}_1 \times (\mathrm{d}\boldsymbol{l}_2 \times \boldsymbol{r_{12}})}{r_{12}^3} = \frac{(\mathrm{d}\boldsymbol{l}_2 \cdot \boldsymbol{r}_2)\,\mathrm{d}\boldsymbol{l}_1}{r_{12}^3} - \frac{(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)\,\boldsymbol{r}_{12}}{r_{12}^3}$$
$$\int_{C_2} \frac{\mathrm{d}\boldsymbol{l}_2 \cdot \boldsymbol{r}_2}{r_{12}^3} = \iint_{S_2} \left(\nabla \times \frac{\boldsymbol{r}_2}{r_{12}^3}\right) \mathrm{d}\boldsymbol{S} = 0$$

$$\implies F_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \iint_{C_1, C_2} \frac{(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)\,\boldsymbol{r}_{12}}{r_{12}^3}$$

And $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$

$$F_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \iint_{C_1, C_2} \frac{\mathrm{d}\boldsymbol{l}_2 \times (\mathrm{d}\boldsymbol{l}_1 \times \boldsymbol{r_{21}})}{r_{21}^3} = \frac{\mu_0 I_1 I_2}{4\pi} \iint_{C_1, C_2} \frac{(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)\,\boldsymbol{r}_{12}}{r_{12}^3} = -F_{12}$$

Q3

Use the equation below to find the related equation for the conduction current $\boldsymbol{J} = n_f e \boldsymbol{v}$. Solve this equation for $\boldsymbol{E}(t) = \boldsymbol{E}_0 \delta(t)$ if $\boldsymbol{J}(t < 0) = 0$. What is $\boldsymbol{J}$ immediately after $t = 0$? Connect this with the sum rule.

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\gamma \boldsymbol{v} + \frac{e}{m}\boldsymbol{E}$$

After $t = 0, x = \int v \mathrm{d}t = 0$

$$v = -\gamma \int v \mathrm{d}t + \frac{e}{m} \int E \mathrm{d}t = \frac{e}{m} E_0$$
$$J = \frac{n_f e^2 E_0}{m}$$

As $\mathcal{F}(\delta(t)) = 1$, $\boldsymbol{E}(t) = \boldsymbol{E}_0 \delta(t)$ has all frequencies for the same strength. So that we can use it to determine n_f experimentally using sum rule,