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Problem 1. Let $U_1, U_2, ..., U_{60}$ be i.i.d. Unif(0,1) and $X = U_1 + U_2 + \cdots + U_{60}$

1. which important distribution is the distribution of X very close to? Specify what the parameters are, and state which theorem justifies your choice.
2. Give a simple but accurate approximation for $P(X > 17)$. Justify briefly.

Solution:

1. $M_i(t) = E(e^{tU}) = \int_0^1 e^{-tx}\,\mathrm{d}x = \frac{1}{t}(1-e^{-t}) = (1 - \frac{t}{2} + \frac{t^2}{6} + o(t^2))$
   $M(t) = \prod M_i(t) = (1 - \frac{t}{2} + \frac{t^2}{6} + o(t^2))^{60}$
   这近似是一个正态分布 $X \sim \mathcal{N}(30, 5)$. 遵从中心极限定理.
2. $\mu - 5.8\sigma \approx 17$, 因此 $P(X > 17) = 1 - P(X < \mu - 5.8\sigma) = 3.3 \times 10^{-9}$

$\square$

Problem 2. Let X and Y be Pois(λ) r.v.s. and $T = X + Y$. Suppose that X and Y are not independent, and in fact $X = Y$. Prove or disprove the claim that $T \sim Pois(2\lambda)$ in this scenario.

Solution: 根据矩的定义

$$\mu_{Xn} = \int_{-\infty}^{\infty} (x - E(x))^n f(x)\,\mathrm{d}x,$$

当 $T = X + Y = 2X$, $\mu_{Tn} = 2^n \mu_{Xn}$. 那么当 $M_X(t) = e^{\lambda(e^{-t}-1)}$, 需要 $M_T(t) = e^{2\lambda(e^{-t}-1)}$ 方能满足条件. 这个矩生成函数对应着

$$T \sim Pois(2\lambda)$$

$\square$

Problem 3. Let X, Y, Z be r.v.s such that $X \sim \mathcal{N}(0,1)$ and conditional on $X = x$, Y and Z are i.i.d. $\mathcal{N}(x, 1)$.

1. Find the joint PDF of X, Y, Z.
2. By definition, Y and Z are conditionally independent given X. Discuss intuitively whether or not Y and Z are also unconditianally independent.
3. Find the joint PDF of Y and Z. You can leave your answer as an integral, though the integral can be done with some algebra (such as completing the square) and facts about the Normal distribution.

Solution:

1.
$$\begin{aligned} f(x,y,z) =& f(x)f(y|x)f(z|x) \\ =& \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-x)^2}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z-x)^2}{2}\right) \\ =& (2\pi)^{-3/2}\exp\left(-\frac{1}{2}(3x^2+y^2+z^2-2xy-2xz)\right) \end{aligned}$$

2. Y 和 Z 是互相独立的.

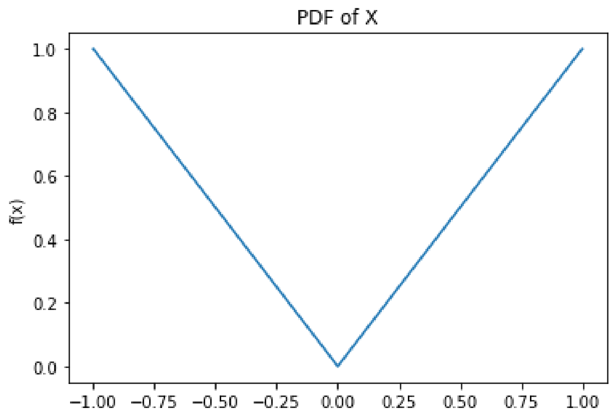
3.
$$\begin{aligned} f(y,z) =& \int_{-\infty}^{\infty} f(x,y,z)\,\mathrm{d}x \\ =& (2\pi)^{-3/2}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(3x^2+y^2+z^2-2xy-2xz)\right)\mathrm{d}x \\ =& \sqrt{\frac{2\pi}{3}}\exp\left(\frac{1}{3}\left(-y^2+yz-z^2\right)\right) \end{aligned}$$

□

Problem 4. A large freight elevator can transport a maximum of 9800 pounds. Suppose a load of cargo containing 49 boxes must be transported via the elevator. Experience has shown that the weight of boxes of this type of cargo follows a distribution with mean $\mu =$ 205 pounds and standard deviation $\sigma = 15$ pounds. Based on this information, what is the probability that all 49 boxes can be safely loaded onto the freight elevator and transported?

Solution: 每个箱子的重量为 $X_i \sim \mathcal{N}(205, 15^2)$, 则 49 个箱子的重量为 $X = \sum X_i = \mathcal{N}(10045, 105^2)$.
$9800 \approx \mu - 2.333\sigma$. 因此 $P(X \leq 9800) = P(X \leq \mu - 2.333\sigma) = 0.0098$ □

Problem 5. In this experiment we will sample 500 samples from probability distribute function $f(x)$. Using the 500 outcomes we will compute the sample mean . We will repeat until we obtain 1000 values of $\overline{X}$.
1).Plot the histogram of X.
2).Plot the histogram of $\overline{X}$.



Solution:

1) using Mathematica:

f = ProbabilityDistribution[Abs[x], {x, -1, 1}];

Show[Histogram[RandomVariate[f, 500], 40,“ProbabilityDensity”], Plot[PDF[f, x], {x, -1, 1}, PlotStyle -> Thick]]

2)

Show[Histogram[Table[Total[RandomVariate[f, 500]], {i, 1000}], 40,“ProbabilityDensity”]]

□

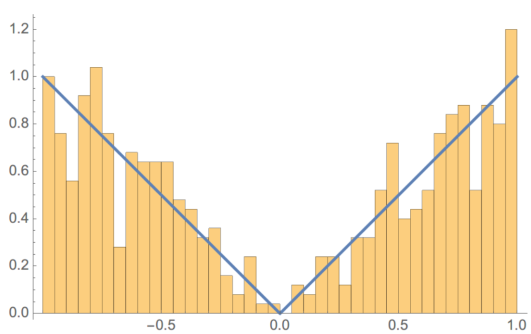


图 1: X

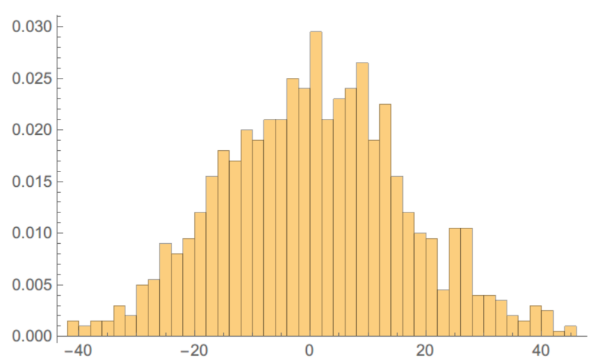


图 2: $\overline{X}$