数值分析第七次作业

肖涵溥 31360164

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1

(1)

$$\begin{aligned} A_{-1}+A_0+A_1= &\quad 4h \\ -hA_{-1}+hA_1= &\quad 0 \\ h^2A_{-1}+h^2A_1= \tfrac{16h^3}{3} \end{aligned}$$

可得

$$A_{-1}=\frac{8}{3}h, A_0=-\frac{4}{3}h, A_1=\frac{8}{3}h$$

取 $f(x)=x^n, h=1$, 那么 $A_{-1}f(-h)+A_0f(0)+A_1f(h)=0$(n 为奇数)$\frac{16}{3}$(n 为偶数). 当 n<4 时成立, 因此代数精度为 3.

(2)

$$\begin{aligned} A+B= &\quad \tfrac{1}{2} \\ B+C+D= &\quad \tfrac{1}{3} \\ B+2D= &\quad \tfrac{1}{4} \\ B+3D= &\quad \tfrac{1}{5} \end{aligned}$$

$$A=\frac{7}{20}, B=\frac{3}{20}, C=\frac{2}{15}, D=\frac{1}{20}$$

当 $f(x)=x^4$.

$$\int_0^1 xf(x)\,dx=\frac{1}{6}$$

$$Af(0)+Bf(1)+Cf'(0)+Df'(1)=\frac{7}{20}$$

不相等, 因此代数精度为 3.

2

(1)

利用带拉格朗日余项的泰勒公式在 a 处展开:

$$\left.\int_a^x f(x)\,\mathrm{d}x\right|_{x=b}=\int_a^a f(x)\,\mathrm{d}x+f(a)(b-a)+\frac{f'(\eta)}{2}(b-a)^2=f(a)(b-a)+\frac{f'(\eta)}{2}(b-a)^2$$

(2)

利用带拉格朗日余项的泰勒公式在 b 处展开:

$$\left.\int_x^b f(x)\,\mathrm{d}x\right|_{x=a}=\int_b^b f(x)\,\mathrm{d}x+f(b)(b-a)-\frac{f'(\eta)}{2}(b-a)^2=f(a)(b-a)-\frac{f'(\eta)}{2}(b-a)^2$$

(3)

$$\int_a^b f(x)\,\mathrm{d}x=\int_a^{\frac{a+b}{2}} f(x)\,\mathrm{d}x+\int_{\frac{a+b}{2}}^b f(x)\,\mathrm{d}x$$

利用带拉格朗日余项的泰勒公式在 $\frac{a+b}{2}$ 处展开:

$$\left.\int_x^{\frac{a+b}{2}} f(x)\,\mathrm{d}x\right|_{x=a}+\left.\int_{\frac{a+b}{2}}^x f(x)\,\mathrm{d}x\right|_{x=b}=-f\left(\frac{a+b}{2}\right)\left(a-\frac{a+b}{2}\right)-\frac{1}{2}f'\left(\frac{a+b}{2}\right)\left(a-\frac{a+b}{2}\right)^2-\frac{1}{6}f''(\eta)\left(a-\frac{a+b}{2}\right)^3+f\left(\frac{a+b}{2}\right)\left(b-\frac{a+b}{2}\right)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)\left(b-\frac{a+b}{2}\right)^2+\frac{1}{6}f''(\eta)\left(b-\frac{a+b}{2}\right)^3$$

$$=-f\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)-\frac{1}{2}f'\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)^2-\frac{1}{6}f''(\eta)\left(\frac{a-b}{2}\right)^3+f\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)^2+\frac{1}{6}f''(\eta)\left(\frac{b-a}{2}\right)^3$$

$$=f\left(\frac{a+b}{2}\right)(b-a)+\frac{1}{3}f''(\eta)\left(\frac{b-a}{2}\right)^3=f\left(\frac{a+b}{2}\right)(b-a)+\frac{f''(\eta)}{24}(b-a)^3$$

3

(1)

$$\int_0^1 f(x)\,\mathrm{d}x=A_0f(x_0)+A_1f(x_1)+A_2f(x_2)$$

满足

$$\begin{aligned}A_0+A_1+A_2 &= 1\\ \tfrac{A_0}{4}+\tfrac{A_1}{2}+\tfrac{3A_2}{4} &= \tfrac{1}{2}\\ \tfrac{A_0}{16}+\tfrac{A_1}{4}+\tfrac{9A_2}{16} &= \tfrac{1}{3}\end{aligned}$$

$$A_0=-\frac{1}{6},A_1=1,A_2=\frac{1}{6}$$

$$I\approx-\frac{1}{6}f\left(x_0\right)+f\left(x_1\right)+\frac{1}{6}f\left(x_2\right)$$

(2)

由于有三项, 设 $f\left(x\right)=x^3$.

$$\int_0^1 f\left(x\right)\mathrm{d}x=\frac{1}{4}$$

$$A_0f\left(x_0\right)+A_1f\left(x_1\right)+A_2f\left(x_2\right)=\frac{37}{192}$$

不相等, 因此代数精度为 2.

4

(1)

$$\begin{aligned}A_0+A_1+A_2 &= 3\\ A_1+2A_2 &= \tfrac{9}{2}\\ A_1+4A_2 &= 9\end{aligned}$$

$$A_0=\frac{3}{4},A_1=0,A_2=\frac{9}{4}$$

$$I\approx\frac{3}{4}f\left(0\right)+\frac{9}{4}f\left(2\right)$$

设 $f\left(x\right)=x^3$

$$I\left(x\right)=\frac{3^4}{4}=\frac{81}{4}$$

$$\frac{3}{4}f\left(0\right)+\frac{9}{4}f\left(2\right)=18$$

不相等, 因此代数精度为 2.

(2)

$$R_n\left(f\right)=\int_0^3 f\left(x\right)dx-\sum_{i=0}^{2}A_if\left(x_i\right)=\int_0^3\frac{f^{(3)}\left(\xi\right)}{6}x\left(x-1\right)\left(x-2\right)dx=\frac{3}{8}f^{(3)}\left(\xi\right)$$

5

取 $f(x)=1,x,x^2$, 使 $I=I_n$ 即可. 也就是

$$\begin{aligned} A_1+A_2= & \ 2 \\ A_1+2A_2+A_3= & \ 2 \\ A_1+4A_2+2A_3= & \ \tfrac{8}{3} \end{aligned}$$

$$A_1=\frac{4}{3},A_2=\frac{2}{3},A_3=-\frac{2}{3}$$

(2)

$$R[f]=\int_0^2\frac{f^{(3)}(\xi)}{6}(x-1)^2(x-2)\,dx=-\frac{f^{(3)}(\xi)}{9}$$

6

(1)

$$T_n(x)=\sum_{i=0}^{n-1}\frac{h}{2}[f(x_k)+f(x_{k+1})]$$

(2)

$$\begin{aligned} I(f)= & \sum_{i=0}^{n-1}\int_{x_k}^{x_{k+1}}f(x)\,\mathrm{d}x \\ = & \sum_{i=0}^{n-1}\frac{h}{2}[f(x_k)+f(x_{k+1})]-\sum_{i=0}^{n-1}\frac{h^3}{12}f''(\xi_k), \quad (\xi_k\in(x_k,x_{k+1})) \\ = & \ T_n(f)-\sum_{i=0}^{n-1}\frac{h^3}{12}f''(\xi_k) \\ \frac{I-T_n}{h^2}= & -\sum_{i=0}^{n-1}\frac{h}{12}f''(\xi_k) \end{aligned}$$

做近似

$$f''(\xi_k)=(f'(x_{k+1})-f'(x_k))/h$$

则

$$
\begin{aligned}
\frac{I-T_n}{h^2} &= -\sum_{i=0}^{n-1}\frac{h}{12}f''\left(\xi_k\right) \\
&= -\frac{h}{12}\frac{f'\left(b\right)-f'\left(a\right)}{h} \\
&= \frac{1}{12}[f'\left(a\right)-f'\left(b\right)]
\end{aligned}
$$