The 15th HW of Electrodynamics

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1

Prove that the above invariants are indeed invariant under Lorentz transformation.

$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu} = B^2 - \frac{1}{c^2}E^2$$

$$\frac{i}{8}\epsilon_{\mu\nu\lambda\tau}F_{\mu\nu}F_{\lambda\tau} = \frac{1}{c}\vec{B}\cdot\vec{E}$$

Proof:

1

对于式

$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu} = B^2 - \frac{1}{c^2}E^2$$

定义

$$a = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$\vec{B'} = \gamma(\vec{B} - \frac{\vec{v}}{c^2}\times\vec{E})$$

$$\vec{E'} = \gamma(\vec{E} - \vec{v}\times\vec{B})$$

则有

$$F'_{\mu\nu}=a_{\mu\lambda}a_{v\tau}F_{\lambda\tau}=aF\widetilde{a}=\begin{pmatrix} 0 & \gamma(B_3-\frac{v}{c^2}E_2) & -\gamma(B_2+\frac{v}{c^2}E_3) & -\frac{i}{c}E_1 \\ \gamma(-B_3+\frac{v}{c^2}E_2) & 0 & B_1 & -\frac{i}{c}\gamma(E_2-vB_3) \\ \gamma(B_2+\frac{v}{c^2}E_3) & -B_1 & 0 & -\frac{i}{c}\gamma(E_3+vB_2) \\ \frac{i}{c}E_1 & \frac{i}{c}\gamma(E_2-vB_3) & \frac{i}{c}\gamma(E_3+vB_2) & 0 \end{pmatrix}$$

$F'_{\mu\nu}F'_{\mu\nu}=2*B1^2-(2*E1^2)/c^2+2*B2^2*g^2+2*B3^2*g^2-(2*E2^2*g^2)/c^2-(2*E3^2*g^2)/c^2-(2*B2^2*g^2*v^2)/c^2-(2*B3^2*g^2*v^2)/c^2+(2*E2^2*g^2*v^2)/c^4+(2*E3^2*g^2*v^2)/c^4+(8*B2*E3*g^2*v)/c^2$

$B'^2-\frac{1}{c^2}E'^2=B1^2-E1^2/c^2+B2^2*g^2+B3^2*g^2-(E2^2*g^2)/c^2-(E3^2*g^2)/c^2-(B2^2*g^2*v^2)/c^2-(B3^2*g^2*v^2)/c^2+(E2^2*g^2*v^2)/c^4+(E3^2*g^2*v^2)/c^4$

可以发现

$$\frac{1}{2}(F'_{\mu\nu}F'_{\mu\nu})=B'^2-\frac{1}{c^2}E'^2$$

2

对于式

$$\frac{i}{8}\epsilon_{\mu\nu\lambda\tau}F_{\mu\nu}F_{\lambda\tau}=\frac{1}{c}\vec{B}\cdot\vec{E}$$

洛伦兹变换后:

$$\sum_{\mu=1}^{4}\sum_{\nu=1}^{4}\sum_{\lambda=1}^{4}\sum_{\tau=1}^{4}(\epsilon_{\mu\nu\lambda\tau}F'_{\mu\nu}F'_{\lambda\tau})=-\frac{8}{ic}\vec{B}\cdot\vec{E}$$

由于 $\vec{B}\cdot\vec{E}$ 是一个洛伦兹不变量, $\vec{B'}\cdot\vec{E'}=\vec{B}\cdot\vec{E}$ 因此

$$\sum_{\mu=1}^{4}\sum_{\nu=1}^{4}\sum_{\lambda=1}^{4}\sum_{\tau=1}^{4}(\epsilon_{\mu\nu\lambda\tau}F'_{\mu\nu}F'_{\lambda\tau})=-\frac{8}{ic}\vec{B'}\cdot\vec{E'}$$

2

(1) Lab 系中系统的动量为

$$p=\frac{\sqrt{E_1^2-m_1^2c^4}}{c}$$

洛伦兹变换得

$$p_1=\frac{p'_1+\frac{\beta_c}{c^2}E'_1}{\sqrt{1-\beta_c^2/c^2}};E_1=\frac{E'_1+\beta_c p'_1}{\sqrt{1-\beta_c^2/c^2}}$$
$$p_2=\frac{p'_2+\frac{\beta_c}{c^2}E'_2}{\sqrt{1-\beta_c^2/c^2}};\quad E_2=\frac{E'_2+\beta_c p'_2}{\sqrt{1-\beta_c^2/c^2}}$$

即

$$p_1 = \gamma \tfrac{\beta_c}{c^2} \left(E_1' + E_2'\right)$$

$$E_1 + E_2 = \gamma \left(E_1' + E_2'\right)$$

因此

$$\beta_c = \frac{p_1 c^2}{E_1 + E_2} = \frac{\sqrt{E_1^2 + m_1^2 c^4}}{E_1 + m_2 c^2} c$$

(2)

$$|\vec{p}_1'| = \frac{m_2 \sqrt{E_1^2 - m_1^2 c^4}}{Mc}, \quad |\vec{p}_2'| = |\vec{p}_1'|$$

$$E_1' = \sqrt{p_1'^2 c^2 + m_1^2 c^4} = \frac{m_1^2 c^2 + m_2 E_1}{M}, \quad E_2' = \sqrt{p_2'^2 c^2 + m_2^2 c^4} = \frac{m_2 E_1 + m_2^2 c^2}{M}$$

因此总能量

$$E' = E_1' + E_2' = \frac{\left(m_1^2 + m_2^2\right) c^2 + 2 m_2 E_1}{M}$$

其中

$$M^2 c^4 = m_1^2 c^4 + m_2^2 c^4 + 2 m_2 E_1 c^2$$

(3) 实验室系中

$$p_\mu = \left[\vec{p}_1 + \vec{p}_2, \frac{i}{c} \left(E_1 + E_2\right)\right] = \left[]\vec{p}, \frac{i}{c} \left(E_1 + E_2\right)\right]$$

质心系中

$$p_v' = \left[\vec{p}_1' + \vec{p}_2', \frac{i}{c} \left(E_1' + E_2'\right)\right] = \left[0, \frac{i}{c} 2E_1'\right]$$

可得

$$-2 m_e E_1 = -\frac{1}{c^2} 4 E_1'^2$$

即

$$E_1 = \frac{2 E_1'^2}{m_e c^2} = 1.9 \times 10^4 GeV$$