The 4th Homework of Theoretical Mechanics

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Q1

(i)

绕点 O 定轴旋转的刚体受惯性力为：

$$ma = -m\dot{\omega} \times r + m\omega^2 r$$

由达朗贝尔原理：

$$(F - ma)\delta r = (F + m\dot{\omega} \times r - m\omega^2 r)\delta r = 0$$

左乘 r 外积：

$$\implies (r \times F + mr \times (\dot{\omega} \times r) - mr \times (\omega^2 r))\delta r = 0$$

$$\implies r \times F - m\dot{\omega} r^2 = 0$$

$$\implies M = I\dot{\omega}$$

(ii)

绕点 O 定点旋转的刚体受惯性力为：

$$ma = -m\dot{\omega} \times r + m\omega^2 r - 2m\omega \times v$$

由达朗贝尔原理：

$$(F - ma)\delta r = (F + m\dot{\omega} \times r - m\omega^2 r + 2m\omega \times v)\delta r = 0$$

左乘 r 外积：

$$\implies (r \times F + mr \times (\dot{\omega} \times r) - mr \times (\omega^2 r) + 2mr \times (\omega \times v))\delta r = 0$$

$$\implies r \times F - m\dot{\boldsymbol{\omega}} r^2 - 2m(r \cdot \dot{r})\boldsymbol{\omega} = 0$$
$$\implies r \times F = \frac{\mathrm{d}(mr^2\omega)}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}t}$$

Q2

取 F 作用点，约束方程为：

$$f = x^2 + y^2 - l_1^2 = 0$$
$$\implies F_x + \lambda \frac{\partial f}{\partial x} = 0$$
$$m_2 g + \lambda \frac{\partial f}{\partial y} = 0$$
$$\implies \lambda = \frac{\sqrt{F^2 + m_2^2 g^2}}{2l_1}$$
$$x = -\frac{F}{2\lambda},\ y = -\frac{m_2 g}{2\lambda}$$
$$R = \lambda \nabla f = (-F, -m_2 g)$$

Q3

(1)

圆柱体质心：

$$x^2 + y^2 = (R - r)^2$$

(2)

自由度为 1, 广义坐标：θ

(3)

$$T = \frac{1}{2}m(\dot{\theta}(R-r))^2 + \frac{1}{2}I(R\dot{\theta}/r)^2$$

$$I = \frac{1}{2}mr^2$$

$$V = -mg(R-r)\cos\theta$$

$$\implies L = T - V = \frac{1}{2}m(\dot{\theta}(R-r))^2 + \frac{1}{4}m(R\dot{\theta})^2 + mg(R-r)\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mg(R-r)\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\dot{\theta}(R-r)^2 + \frac{1}{2}mR^2\dot{\theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = m\ddot{\theta}(R-r)^2 + \frac{1}{2}mR^2\ddot{\theta}$$

$$\implies \ddot{\theta}(R-r)^2 + \frac{1}{2}R^2\ddot{\theta} = -g(R-r)\cos\theta$$

(4)

展开 $V = -mg(R-r)\cos\theta \approx -mg(R-r)(1-\frac{\theta^2}{2})$

$$\frac{\partial L}{\partial \theta} = -mg(R-r)\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = m\ddot{\theta}(R-r)^2 + \frac{1}{2}mR^2\ddot{\theta}$$

$$\implies \left((R-r)^2 + \frac{1}{2}mR^2\right)\ddot{\theta} = -mg(R-r)\theta$$

设 $\omega^2 = \frac{mg(R-r)}{(R-r)^2+\frac{1}{2}mR^2}$，则有

$$\theta = A\sin(\omega t + \varphi)$$

其中 A, φ 取决于初速度和初始位置。

Q4

由几何关系 $\angle AOB = \pi/2$

$$I_1 = \frac{m_1 r^2}{2} + \frac{m_1 r^2}{6} = \frac{2}{3} m_1 r^2$$

$$I_2 = m_2 r^2$$

$$T = \frac{1}{2} I_1 \omega^2 + \frac{1}{2} I_2 \omega^2 + \frac{1}{2}(m_1 + m_2)\omega^2 r^2$$

$$V = -m_1 g \frac{r}{\sqrt{2}} \cos\theta$$

$$\implies L = \frac{5}{6} m_1 \omega^2 r^2 + m_2 \omega^2 r^2 + m_1 g \frac{r}{\sqrt{2}} \cos\theta$$

$$\frac{\partial L}{\partial \theta} = -m_1 g \frac{r}{\sqrt{2}} \sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \omega} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{5}{3} m_1 \omega r^2 + 2 m_2 \omega r^2) = \frac{5}{3} m_1 \ddot{\theta} r^2 + 2 m_2 \ddot{\theta} r^2$$

$$\implies -m_1 g \frac{\sin\theta}{\sqrt{2}} = \frac{5}{3} m_1 \ddot{\theta} r + 2 m_2 \ddot{\theta} r$$

Q5

AB 质心速度

$$v_{c2} = (\dot{\theta}_1 l \cos\theta_1 + \frac{1}{2}\dot{\theta}_2 l \cos\theta_2)^2 + (\dot{\theta}_1 l \sin\theta_1 + \dot{\theta}_2 l \sin\theta_2)^2$$

$$T = \frac{1}{6} m l^2 \dot{\theta}_1^2 + \frac{1}{24} m l^2 \dot{\theta}_2^2 + \frac{1}{2} m v_{c2}^2$$

碰撞瞬间 $\theta_1 = \theta_2 = 0$:

$$T = \frac{1}{6} m l^2 \dot{\theta}_1^2 + \frac{1}{24} m l^2 \dot{\theta}_2^2 + \frac{1}{2} m(\dot{\theta}_1 l + \frac{1}{2}\dot{\theta}_2 l)^2$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = \frac{4}{3} m l^2 \dot{\theta}_1 + \frac{1}{2} m \dot{\theta}_2 l^2 = I_1 = I \frac{\partial x}{\partial \theta_1} = I l \cos\theta_1 \approx I l$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = \frac{1}{2} m l^2 \dot{\theta}_1 + \frac{1}{3} m \dot{\theta}_2 l^2 = I_2 = I \frac{\partial x}{\partial \theta_2} = I l \cos\theta_2 \approx I l$$

$$\implies \dot{\theta}_1 = -\frac{6I}{7ml}, \quad \dot{\theta}_2 = \frac{30I}{7ml}$$