The 11th HW of Electrodynamics

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$$E_z = E_0 J_m\left(k_{mn} r\right) \cos m\phi \cos \tfrac{p\pi z}{l} e^{-i\omega t}$$
$$E_r = -\tfrac{p\pi}{l} \tfrac{1}{k_{mn}} E_0 J_m'\left(k_{mn} r\right) \cos m\phi \sin \tfrac{p\pi z}{l} e^{-i\omega t}$$
$$E_\phi = \tfrac{p\pi}{l} \tfrac{m}{r k_{mn}^2} E_0 J_m\left(k_{mn} r\right) \sin m\phi \sin \tfrac{p\pi z}{l} e^{-i\omega t}$$

代入 $mnp = 010$,

$$E_z = E_0 J_0\left(k_{01} r\right) \phi e^{-i\omega t}$$
$$E_r = 0$$
$$E_\phi = 0$$

代入 $k_{01} = \frac{2.405}{a}$

$$E_z = E_0 J_0\left(\frac{2.405}{a} r\right) \cos(\omega t)$$

总能量为电场能两倍:

$$w = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 J_0^2\left(\frac{2.405}{a} r\right) \cos^2(\omega t)$$

$$U = 4\pi\varepsilon_0 E_0^2 \cos^2 \omega t l \int_0^a J_0^2 r \, \mathrm{d}r$$

电流密度为:

$$J = \sigma E = E_0 J_0\left(2.405\right) \cos(\omega t)$$

$$I = (l + 2a) * \delta * J = \delta l E_0 J_0\left(2.405\right) \cos \omega t$$

代入 $\omega = \frac{2.405}{\sqrt{\mu_0 \varepsilon_0} a}$ 因此

$$Q = \frac{\omega U}{P} = \frac{2.405\sqrt{\mu_0/\epsilon_0}}{2R_S}\frac{1}{1+a/l}$$

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For an electromagnetic wave traveling in an waveguide...

由 Maxwell 方程,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = -i\omega\varepsilon_0 E_x$$
$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0 E_y$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon_0 E_z = 0$$

代入 $H = H(x,y)e^{i(k_z z - \omega t)}$

$$\frac{\partial H_z}{\partial y} - ik_z H_y = -i\omega\varepsilon_0 E_x$$
$$ik_z H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0 E_y$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\omega\varepsilon_0 E_z = 0$$

E 同理, 则可以得到:

$$\boxed{\begin{aligned} E_x &= \tfrac{i}{k^2-k_z^2}\left(\omega\mu_0 \tfrac{\partial H_z}{\partial y} + k_z \tfrac{\partial E_z}{\partial x}\right) \\ E_y &= \tfrac{i}{k^2-k_z^2}\left(-\omega\mu_0 \tfrac{\partial H_z}{\partial x} + k_z \tfrac{\partial E_z}{\partial y}\right) \\ H_x &= \tfrac{i}{k^2-k_z^2}\left(-\omega\varepsilon_0 \tfrac{\partial E_z}{\partial y} + k_z \tfrac{\partial H_z}{\partial x}\right) \\ H_y &= \tfrac{i}{k^2-k_z^2}\left(\omega\varepsilon_0 \tfrac{\partial E_z}{\partial x} + k_z \tfrac{\partial H_z}{\partial y}\right) \end{aligned}}$$

如果将

$$E_z = E_0 J_m\left(k_{mn} r\right) \cos m\phi \cos\frac{p\pi z}{l} e^{-i\omega t}$$

代入上式, 再由于:

$$\frac{\partial E_z}{\partial x} = E_0 J_m'\left(k_{mn} r\right)\frac{x}{r}(-m)\sin m\phi \frac{y}{r^2}\cos\frac{p\pi z}{l}e^{-i\omega t}$$
$$\frac{\partial E_z}{\partial y} = E_0 J_m'\left(k_{mn} r\right)\frac{y}{r}(-m)\sin m\phi \frac{x}{r^2}\cos\frac{p\pi z}{l}e^{-i\omega t}$$
$$H_z = 0$$

便有

$$E_x = \frac{-mik_z xy}{k_{mn}^2 r^3} E_0 J_m' (k_{mn} r) \sin m\phi \cos \frac{p\pi z}{l} e^{-i\omega t}$$

代入 $k_z = \frac{p\pi}{l_3}$,

$$E_x = \frac{-imp\pi xy}{l_3 k_{mn}^2 r^3} E_0 J_m' (k_{mn} r) \sin m\phi \cos \frac{p\pi z}{l} e^{-i\omega t}$$

对于 y 方向:

$$E_y = \frac{-imp\pi xy}{l_3 k_{mn}^2 r^3} E_0 J_m' (k_{mn} r) \sin m\phi \cos \frac{p\pi z}{l} e^{-i\omega t}$$

变换直角坐标为柱坐标可得:

$$\boxed{\begin{aligned} E_r &= -\tfrac{p\pi}{l} \tfrac{1}{k_{mn}} E_0 J_m' (k_{mn} r) \cos m\phi \sin \tfrac{p\pi z}{l} e^{-i\omega t} \\ F_\phi &= \tfrac{p\pi}{l} \tfrac{m}{r k_{mn}^2} E_0 J_m (k_{mn} r) \sin m\phi \sin \tfrac{p\pi z}{l} e^{-i\omega t} \end{aligned}}$$

同理, 如果将

$$E_z = E_0 J_m (k_{mn} r) \cos m\phi \cos \frac{p\pi z}{l} e^{-i\omega t}$$

代入

$$H_x = \tfrac{i}{k^2 - k_z^2} \left(-\omega \varepsilon_0 \tfrac{\partial E_z}{\partial y} + k_z \tfrac{\partial H_z}{\partial x} \right)$$
$$H_y = \tfrac{i}{k^2 - k_z^2} \left(\omega \varepsilon_0 \tfrac{\partial E_z}{\partial x} + k_z \tfrac{\partial H_z}{\partial y} \right)$$

再由于:

$$\frac{\partial E_z}{\partial x} = E_0 J_m' (k_{mn} r) \frac{x}{r} (-m) \sin m\phi \frac{y}{r^2} \cos \frac{p\pi z}{l} e^{-i\omega t}$$
$$\frac{\partial E_z}{\partial y} = E_0 J_m' (k_{mn} r) \frac{y}{r} (-m) \sin m\phi \frac{x}{r^2} \cos \frac{p\pi z}{l} e^{-i\omega t}$$
$$H_z = 0$$

便有

$$\boxed{\begin{aligned} B_r &= i\omega \tfrac{m}{r k_{mnc^2}^2 c^2} E_0 J_m (k_{mn} r) \sin m\phi \cos \tfrac{p\pi z}{l} e^{-i\omega t} \\ B_\phi &= i\omega \tfrac{1}{k_{mn} c^2} E_0 J_m' (k_{mn} r) \cos m\phi \cos \tfrac{p\pi z}{l} e^{-i\omega t} \end{aligned}}$$

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电场能量为

$$\begin{aligned} \int U_E \, \mathrm{d}V &= \frac{1}{2} \varepsilon E_0^2 \int_0^a dx \int_0^a \sin^2 k_y y dy \int_0^a \sin^2 k_z z dz \cos^2(\omega t) \\ &= \frac{1}{8} \varepsilon E_0^2 a^3 \cos^2(\omega t) \end{aligned}$$

磁场能量为

$$\begin{aligned}\int U_H dV =& \frac{1}{2}\frac{k_z^2}{\mu\omega^2}E_0^2\int_0^a dx\int_0^a \sin^2(ky)dy\int_0^a \cos^2(kz)dz\sin^2(\omega t)\\&+\frac{1}{2}\frac{k_y^2}{\mu\omega^2}E_0^2\int_0^a dx\int_0^a \cos^2(ky)dy\int_0^a \sin^2(kz)dz\sin^2(\omega t)\\=&\frac{1}{8}\frac{\left(k_y^2+k_z^2\right)}{\mu\omega^2}E_0^2a^3\sin^2(\omega t)\end{aligned}$$

代入 $k_y^2+k_z^2=\frac{w_2}{c^2}$ 则磁场能量可被化为

$$\int U_H dV=\frac{1}{8}\frac{1}{\mu c^2}E_0^2a^3\sin^2(\omega t)=\frac{1}{8}\varepsilon E_0^2a^3\sin^2(\omega t)$$

由于 $\sin^2\omega t$ 与 $\cos^2\omega t$ 的周期平均是相同的, 因此二者能量相等, 且和为

$$\mathcal{E}_E+\mathcal{E}_H=\frac{1}{8}\varepsilon E_0^2a^3$$

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将三角形波导看为矩形波导 + 对角线边界条件, 对于 TE 波, 方波导中的磁场为

$$B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)e^{ikz-i\omega t}$$

再令其满足边界条件 $\frac{\partial B}{\partial n}\Big|_{z=y}=0$, 即 $\frac{1}{\sqrt{2}}\left[\frac{\partial B_Z}{\partial x}-\frac{\partial B_Z}{\partial y}\right]_{y=x}=0$ 即得三角波导的 B

$$B_z=B_0\left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)+\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\right]e^{ikz-i\omega t}$$

对于 TM 波, 将 $[E_z]_{x=y}=0$ 代入方波表达式

$$E_z=E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)e^{ikz-i\omega t}$$

可得到

$$E_z=E_0\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)-\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right]e^{ikz-i\omega t}$$

这个计算方法显然可看出截止频率与方波相同, 为

$$\omega_{mn}=\frac{c\pi}{a}\sqrt{m^2+n^2}$$