数理方法 II 第三次作业

肖涵薄 31360164

2019 年 4 月 21 日

1

(1)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+(2+\lambda/x)y=0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(x\frac{\mathrm{d}y}{\mathrm{d}x})-(-2)y+\frac{\lambda}{x}y=0$$

即

$$k(x)=x,q(x)=-2,\rho(x)=\frac{1}{x}$$

(2)

原式化为 $y'' + \frac{a-bx}{x-x^2}y' - \frac{\lambda}{x-x^2}y = 0$

$\exp(\displaystyle\int \frac{a-bx}{x-x^2}\,\mathrm{d}x) = \exp(a\displaystyle\int \frac{1}{x}\,\mathrm{d}x + (a-b)\displaystyle\int \frac{1}{1-x}\,\mathrm{d}x) = \exp(a\ln x - (a-b)\ln(1-x)) = \dfrac{x^a}{(1-x)^{a-b}}$

则最后可化为标准形式:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\frac{x^a}{(1-x)^{a-b}}y') + \lambda(\frac{x^{a-1}}{(1-x)^{a-b+1}})y = 0$$

2

设 $y_m, y_n, n \neq m$ 是函数不同本征值的两个解.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}(py_m') + (\lambda_m \rho - q)y_m = 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(py_n') + (\lambda_n \rho - q)y_n = 0 \end{cases}$$

两式分别乘以 y_n, y_m，相减, $y_n \frac{\mathrm{d}}{\mathrm{d}x}(py_m') - y_m \frac{\mathrm{d}}{\mathrm{d}x}(py_n') + (\lambda_m - \lambda_n)y_m y_n = 0$. 求区间 $[a, b]$ 积分,

$$\int_a^b \left[y_n \frac{d}{dx}(py_m') - y_m \frac{d}{dx}(py_n') \right] dx + \int_a^b (py_m' \frac{d}{dx} y_n - py_n' \frac{d}{dx} y_m) dx + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx$$

$$= \int_a^b \frac{d}{dx}(py_n y_m' - py_m y_n') dx + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx$$

$$= (py_n y_m' - py_m y_n')|_{x=b} - (py_n y_m' - py_m y_n')|_{x=a} + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx$$

由于 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, 有 $y_n(b)y_m'(b) = y_m(b)y_n'(b)$, $y_n(a)y_m'(a) = y_m(a)y_n'(a)$

$$= (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx = 0$$

当 $\lambda_m \neq \lambda_n$, $\int_a^b \rho y_m y_n dx = 0$.

3

(1)

根据定义, $\int \delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}r^3=1$, 在球坐标下即为

$$\int_0^\infty \mathrm{d}r \int_0^{2\pi} r\,\mathrm{d}\varphi \int_0^\pi r\sin\varphi\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}\theta=1$$

$$\Longrightarrow \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\cos\varphi \int_0^\pi r^2\delta(\boldsymbol{r}-\boldsymbol{r}_0)\,\mathrm{d}\theta=1$$

根据直角坐标下形式, 可知

$$\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)=r^2\delta(\boldsymbol{r}-\boldsymbol{r}_0)$$

移项即得

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=\frac{1}{r^2}\delta(r-r_0)\delta(\cos\theta-\cos\theta_0)\delta(\varphi-\varphi_0)$$

(2)

$$\nabla^2 \frac{1}{|r - r_0|} = -\nabla \cdot \frac{1}{(r - r_0)^2}$$

由高斯定理可知

$$-\int \nabla \cdot \frac{1}{(r - r_0)^2} \, \mathrm{d}V = -\int_\Omega \frac{1}{(r - r_0)^2} \, \mathrm{d}S$$

取积分面为 $r - r_0 = a$ 的球壳, a 为任意常数.

$$-\int_\Omega \frac{1}{(r - r_0)^2} \, \mathrm{d}S = -4\pi a^2 \frac{1}{a^2} = -4\pi$$

即

$$\int \nabla^2 \frac{1}{|r - r_0|} \, \mathrm{d}r^3 = -4\pi$$

根据定义,

$$\nabla^2 \frac{1}{|r - r_0|} = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}_0)$$

4

(1) 先计算 $\mathcal{F}(e^{-a|t|})$

$$\mathcal{F}(e^{-a|t|}) = \int_{-\infty}^{0} e^{at-iwt}\,\mathrm{d}t + \int_{0}^{\infty} e^{-at-iwt}\,\mathrm{d}t$$

$$= \int_{0}^{\infty} (e^{-(a+iw)t} + e^{-(a-iw)t})\,\mathrm{d}t = \frac{2a}{a^2+w^2}$$

则 $\mathcal{F}^{-1}(\frac{2a}{a^2+w^2}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+w^2} e^{iwt}\,\mathrm{d}w = e^{-a|t|}$, 两边求实部:

$$\int_{-\infty}^{\infty} \frac{1}{a^2+w^2}\cos wt\,\mathrm{d}w = \frac{\pi}{a}e^{-a|t|}$$

(2)

(i)

$$\int \frac{1}{r}e^{ikr\cos\theta}\,\mathrm{d}r = \int_0^\infty \frac{1}{r}r^2\,\mathrm{d}r \int_1^{-1} e^{ikr\cos\theta}\,\mathrm{d}\cos\theta \int_0^{2\pi}\mathrm{d}\varphi$$

$$= 2\pi\int_0^\infty \frac{1}{r}r^2\frac{1}{ikr}(e^{ikr}-e^{-ikr})\,\mathrm{d}r = \frac{2\pi}{ik}\int_0^\infty (e^{ikr}-e^{-ikr})\,\mathrm{d}r$$

$$= \frac{2\pi}{ik}(2i\int_0^\infty \sin kr\,\mathrm{d}r) = \frac{4\pi}{k}\int_0^\infty \sin kr\,\mathrm{d}r$$

$$\begin{aligned}\int_0^\infty \sin kr\,\mathrm{d}r = & \lim_{\varepsilon\to 0^+}\mathrm{Im}(\int_0^\infty e^{\varepsilon r}e^{ikr}\,\mathrm{d}r) \\ = & \lim_{\varepsilon\to 0^+}\mathrm{Im}(\int_0^\infty e^{(\varepsilon+ik)r}\,\mathrm{d}r) \\ = & \lim_{\varepsilon\to 0^+}\mathrm{Im}(\frac{e^{(\varepsilon+ik)\infty}-1}{\varepsilon+ik}) \\ = & \frac{1}{k}\end{aligned}$$

则

$$\mathcal{F}(\frac{1}{r}) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{r}e^{ikr\cos\theta}\,\mathrm{d}r = \frac{\sqrt{2}}{\sqrt{\pi}k}\int_0^\infty \sin kr\,\mathrm{d}r = \frac{\sqrt{2}}{\sqrt{\pi}k^2}$$

(ii)

$$\int \frac{\delta(r-a)}{r} e^{-ikr\cos\theta}\,\mathrm{d}r = \int_0^\infty \frac{\delta(r-a)}{r} r^2\,\mathrm{d}r \int_1^{-1} e^{-ikr\cos\theta}\,\mathrm{d}\cos\theta \int_0^{2\pi}\mathrm{d}\varphi$$
$$= -2\pi \int_0^\infty \frac{\delta(r-a)}{ik}(e^{ikr}-e^{-ikr})\,\mathrm{d}r = -\frac{2\pi}{ik}(e^{ika}-e^{-ika}) = \frac{\pi}{k}\sin ka$$

则可得到

$$\mathcal{F}^{-1}(\frac{\sin ak}{k}) = \sqrt{\frac{\pi}{2}}\frac{\delta(r-a)}{r}$$

5

根据周期性有:

$$\mathcal{L}(f(t-a)) = \mathcal{L}(f(t)u(t-a))$$

又由于

$$\mathcal{L}(f(t-a)) = \int_0^{\infty} \frac{f(t-a)}{e^{ap}} e^{-pt+ap}\,\mathrm{d}t = e^{-ap}F(p)$$

$$\mathcal{L}(f(t)u(t-a)) = F(p) - \int_0^{a} f(t)e^{-pt}\,\mathrm{d}t$$

上面两式相等即可得到

$$(1-e^{-ap})F(p) = \int_0^{a} f(t)e^{-pt}\,\mathrm{d}t \implies F(p) = \frac{1}{1-e^{-ap}}\int_0^{a} f(t)e^{-pt}dt$$

6

(1)

$$pU(x,p)-u(x,0)=a^2\frac{\partial^2 U}{\partial x^2}+F(x,p)$$

$$\Longrightarrow\ pU(x,p)-\varphi(x)=a^2\frac{\partial^2 U}{\partial x^2}+F(x,p)$$

有通解

$$\begin{aligned}U(x,p)= & -\frac{a}{2\sqrt{p}}\int_{-\infty}^{\infty}\exp(-\frac{\sqrt{p}}{a}|x-x'|)(-\frac{F(x')+\varphi(x')}{a^2})\,\mathrm{d}x'\\ = & \frac{1}{2a}\int_{-\infty}^{\infty}\exp(-\frac{\sqrt{p}}{a}|x-x'|)(\frac{F(x')+\varphi(x')}{\sqrt{p}})\,\mathrm{d}x'\\ = & \frac{1}{2a}\int_{-\infty}^{\infty}\frac{1}{\sqrt{p}}\exp(-\frac{\sqrt{p}}{a}|x-x'|)(F(x')+\varphi(x'))\,\mathrm{d}x'\end{aligned}$$

记 $m(x,x',p)=\frac{1}{p}e^{-\frac{|x-x'|}{a}p}$, 则 $\mathcal{L}^{-1}(m)=\eta(t-\frac{|x-x'|}{a})$. 那么

$$m(x,x',p)=\int_0^{\infty}\eta(t-\frac{|x-x'|}{a})e^{-pt}\,\mathrm{d}t$$

或写为

$$m(x,x',\sqrt{p})=\int_0^{\infty}\eta(t-\frac{|x-x'|}{a})e^{-\sqrt{p}t}\,\mathrm{d}t=\int_0^{\infty}\eta(t-\frac{|x-x'|}{a})\frac{e^{-\sqrt{p}t}}{e^{-pt}}e^{-pt}\,\mathrm{d}t$$

因此 $\mathcal{L}^{-1}(m(x,x',\sqrt{p}))=\eta(t-\frac{|x-x'|}{a})\frac{e^{-\sqrt{p}t}}{e^{-pt}}$

$$u(x,t)=\frac{1}{2a}\int_{-\infty}^{\infty}\eta(t-\frac{|x-x'|}{a})\frac{e^{-\sqrt{p}t}}{e^{-pt}}(F(x')+\varphi(x'))\,\mathrm{d}x'$$

$$u(x,t)=\int_{-\infty}^{\infty}\phi(\xi)K(x-\xi,t)\mathrm{d}\xi+\int_0^t\mathrm{d}\tau\int_{-\infty}^{\infty}f(\xi,\tau)K(x-\xi,t-\tau)\mathrm{d}\xi$$

(2)

$$\frac{\partial U}{\partial t}(w,t) = -a^2w^2U(w,t) + F(w,t)$$

$$U|_{t=0} = \Phi(w)$$

则

$$U(w,t) = e^{-a^2w^2t}\Phi(w) + \int_0^t F(w,\tau)e^{-a^2w^2(t-\tau)}\,\mathrm{d}\tau$$

做逆变换:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(w)\mathrm{e}^{-a^2w^2t}\mathrm{e}^{\mathrm{i}wx}\mathrm{d}w + \mathcal{F}^{-1}\left\{\int_0^t F(w,\tau)\mathrm{e}^{-a^2w^2(t-\tau)}\mathrm{d}\tau\right\}$$

$$u(x,t) = \int_{-\infty}^{\infty}\phi(\xi)K(x-\xi,t)\mathrm{d}\xi + \int_0^t \mathrm{d}\tau\int_{-\infty}^{\infty}f(\xi,\tau)K(x-\xi,t-\tau)\mathrm{d}\xi$$

7

先对 t 求拉普拉斯, $\mathcal{L}(u(x,t)) = U(x,p)$:

$$p^2U(x,p) - pu(x,0) - \frac{\partial u}{\partial t}(x,0) = c^2\frac{\partial^2 U}{\partial x^2}(x,p)$$

$$\Longrightarrow\ p^2U(x,p) - p\varphi(x) - \psi(x) = c^2\frac{\partial^2 U}{\partial x^2}(x,p)$$

再对 x 求傅里叶, $\mathcal{F}(U(x,p)) = U^*(w,p)$:

$$p^2U^*(w,p) - p\Phi(w) - \Psi(w) = -c^2w^2U^*(w,p)$$

$$\Longrightarrow\ U^*(w,p) = \frac{p\Phi(w)+\Psi(w)}{p^2+c^2w^2}$$

拉普拉斯逆变换得到

$$U(w,t) = \Phi(w)\cos(cwt) + \frac{\Psi(w)}{cw}\sin(cwt)$$

$$\mathcal{F}^{-1}[\Phi(w)e^{icwt}] = \phi(x+ct)$$

$$\mathcal{F}^{-1}[\Phi(w)e^{-icwt}] = \phi(x-ct)$$

$$\mathcal{F}^{-1}[\frac{1}{cw}\Psi(w)e^{icwt}] = \frac{i}{c}\int_{-\infty}^{x+ct}\psi(x+ct)\,\mathrm{d}x$$

$$\mathcal{F}^{-1}[\frac{1}{cw}\Psi(w)e^{-icwt}] = \frac{i}{c}\int_{-\infty}^{x-ct}\psi(x-ct)\,\mathrm{d}x$$

$$\Longrightarrow\ u(x,t) = \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)+\frac{1}{c}\int_{-\infty}^{x+ct}\psi(x+ct)\,\mathrm{d}x - \frac{1}{c}\int_{-\infty}^{x-ct}\psi(x-ct)\,\mathrm{d}x\right]$$

$$\Longrightarrow\ u(x,t) = \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x+ct)\,\mathrm{d}x$$