The 4th Homework of Theoretical Mechanics

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2019 年 1 月 1 日

Q1

a

设 M 坐标 $(x, 0)$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(b^2\dot{\theta}^2 + 2b\dot{\theta}\dot{x} + \dot{x}^2\right) - \frac{mgb\theta^2}{2}$$

b

取坐标 $b\theta, x$

$$a_{11} = m \qquad a_{12} = m \quad a_{22} = M + m$$

$$b_{11} = mg/b \quad b_{12} = 0 \qquad b_{22} = 0$$

$$\Longrightarrow \alpha = \frac{M+m}{m}, \quad \beta = 0$$

$$\Longrightarrow q_1 = b\theta + \frac{M+m}{m}x, \quad q_2 = b\theta$$

c

$$x = (q_1 - q_2)\frac{m}{m+M}$$

将简正坐标代入 L：

$$L = \frac{m^2}{2(M+m)}\dot{q}_1^2 + \left(\frac{m}{2} - \frac{m^2}{2(M+m)}\right)\dot{q}_2^2 - \frac{mg}{2b}q_2^2$$

代入拉格朗日方程

$$\dot{q}_1 = const$$
$$q_2 = A\sin\left(\sqrt{\frac{b}{g} - \frac{bm}{(M+m)\,g}}t + \varphi\right), \quad A, \varphi = any\ constant$$

Q2

$$L = \frac{1}{2}me^{\alpha t}\left(\dot{x}^2 - \omega^2 x^2\right)$$

1

代入拉格朗日方程

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x}e^{\alpha t} + m\dot{x}\alpha e^{\alpha t} + \omega^2 m e^{\alpha t}x$$
$$\implies \ddot{x} = -\omega^2 x - \alpha\dot{x}$$

2

$$p = \frac{\partial L}{\partial \dot{x}} = me^{\alpha t}\dot{x}$$
$$H = p\dot{x} - L = \frac{p^2}{me^{\alpha t}} - \frac{p}{2} + \frac{1}{2}me^{\alpha t}\omega^2 x^2$$
$$\implies \dot{x} = \frac{p}{me^{\alpha t}}, \dot{p} = -me^{\alpha t}\omega^2 x$$
$$\implies \ddot{x} = \frac{m\dot{p}e^{\alpha t} - m\alpha p e^{\alpha t}}{m^2 e^{2\alpha t}} = -\omega^2 x - \alpha\dot{x}$$

Q3

$$H = \frac{1}{2}M\left(R\dot{\theta}\right)^2 + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}m\left(\left(R\dot{\theta} - R\dot{\theta}\cos\theta\right)^2 + \left(R\dot{\theta}\sin\theta\right)^2\right) - mgR\cos\theta$$
$$= \frac{1}{2}M\left(R\dot{\theta}\right)^2 + \frac{1}{4}MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2\left(1 - \cos\theta\right) - mgR\cos\theta$$

$$\frac{\partial H}{\partial \dot{\theta}} = p = \left(\frac{3}{2}M + 2\left(1-\cos\theta\right)m\right)R^2\dot{\theta}$$

$$\Longrightarrow\ H = \frac{p^2}{\left(3M + 4m\left(1-\cos\theta\right)\right)R^2} - mgR\cos\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{2p}{\left(3M + 4m\left(1-\cos\theta\right)\right)R^2}$$

$$\dot{p} = -\frac{\partial H}{\partial \theta} = \frac{4mp^4R^2\sin\theta}{\left(3M + 4m\left(1-\cos\theta\right)\right)^2R^4} - mgR\sin\theta$$

Q4

1

沿 OC 方向建 x 轴，小环受向外的离心力 $m\omega^2 r$ 和垂直于圆圈的科里奥利力，科氏力被支持力抵消。

$$a_\theta = R\ddot{\theta} = -\omega^2 r\sin\theta$$

2

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + 2mR^2\dot{\theta}\omega\cos\left(\theta/2\right) + 2m\omega^2R^2\cos^2\left(\theta/2\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} + 2mR^2\omega\cos\left(\theta/2\right)$$

$$\Longrightarrow\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta} - mR^2\omega\sin\left(\theta/2\right)\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mR^2\dot{\theta}\omega\sin\left(\theta/2\right) - 2m\omega^2R^2\cos\left(\theta/2\right)\sin\left(\theta/2\right) = -mR^2\dot{\theta}\omega\sin\left(\theta/2\right) - m\omega^2R^2\sin\theta$$

$$\Longrightarrow\ \ddot{\theta} = -\omega^2\sin\theta$$

3

$$p = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} + 2mR^2\omega\cos\left(\theta/2\right)$$

$$\Longrightarrow\ H = p\dot{\theta} - L = p\left(\frac{p}{mR^2} - 2\omega\cos\left(\theta/2\right)\right) - \frac{p^2}{2mR^2}$$

$$\dot{\theta} = \frac{\partial H}{\partial p} = \frac{p}{mR^2} - \omega \cos(\theta/2)$$
$$\dot{p} = -\frac{\partial H}{\partial \theta} = -p\omega \sin(\theta/2)$$

Q5

受到离心力和科氏力，

$$L = \frac{1}{2}mv'^2 + mv'(\omega_0 \times r') + \frac{1}{2}m(\omega_0 \times r')^2 - V$$
$$p = \frac{\partial L}{\partial v'} = mv' + m(\omega_0 \times r')$$
$$\Longrightarrow L = \frac{1}{2}m\left(\frac{p}{m} - (\omega_0 \times r')\right)^2 + m\left(\frac{p}{m} - (\omega_0 \times r')\right)(\omega_0 \times r') + \frac{1}{2}m(\omega_0 \times r')^2 - V = \frac{p^2}{2m} - V$$
$$\Longrightarrow H = pv' - L = \frac{p^2}{2m} - p(\omega_0 \times r') + V$$
$$\dot{p} = -\frac{\partial H}{\partial r'} = p\omega_0 - \frac{\partial V}{\partial r'}$$
$$v' = \frac{\partial H}{\partial p} = p/m - \omega_0 \times r'$$