ElectroDynamics

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1 方程

真空麦克斯韦方程

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\epsilon_0} \qquad \oint_S \boldsymbol{E}\,\mathrm{d}\boldsymbol{s}=\frac{Q}{\epsilon_0} \qquad \text{高斯定律}$$

$$\nabla\cdot\boldsymbol{B}=0 \qquad \oint_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{s}=0 \qquad \text{高斯磁定律}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \qquad \oint_L \boldsymbol{E}\,\mathrm{d}\boldsymbol{l}=-\frac{\mathrm{d}\varphi_B}{\mathrm{d}t} \qquad \text{法拉第电磁感应定律}$$

$$\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{J}+\mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \qquad \oint_L \boldsymbol{B}\,\mathrm{d}\boldsymbol{l}=\mu_0 I+\mu_0\epsilon_0\frac{\mathrm{d}\varphi_E}{\mathrm{d}t} \qquad \text{安培定律}$$

物质内麦克斯韦方程

$$\nabla\cdot\boldsymbol{D}=\rho_f \qquad \oint_S \boldsymbol{D}\,\mathrm{d}\boldsymbol{s}=Q_f \qquad \text{高斯定律}$$

$$\nabla\cdot\boldsymbol{B}=0 \qquad \oint_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{s}=0 \qquad \text{高斯磁定律}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \qquad \oint_L \boldsymbol{E}\,\mathrm{d}\boldsymbol{l}=-\frac{\mathrm{d}\varphi_B}{\mathrm{d}t} \qquad \text{法拉第电磁感应定律}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f+\frac{\partial\boldsymbol{D}}{\partial t} \qquad \oint_L \boldsymbol{H}\,\mathrm{d}\boldsymbol{l}=I_f+\frac{\mathrm{d}\varphi_D}{\mathrm{d}t} \qquad \text{安培定律}$$

镜像法

距半径为 R_0 的球的球心距离为 a 处有一点电荷 q, 则镜像电荷 $-\frac{R_0}{a}q$ 距球心 $\frac{R_0^2}{a}$ 远, 且在靠近 q 方向.

球谐函数解拉普拉斯方程

$$\varphi(r,\theta)=\sum_{n=0}^{\infty}(a_n r^n+\frac{b_n}{r^{n+1}})P_n(\cos\theta)$$

$$\begin{cases} P_0(\cos\theta) = 1 \\ P_1(cos\theta) = \cos\theta \\ P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \end{cases}$$

$$\frac{1}{|\boldsymbol{R}-\boldsymbol{a}'|} = \frac{1}{\sqrt{R^2+a'^2-2Ra\cos\theta}} = \begin{cases} \frac{1}{a}\sum_{n=0}^{\infty}(\frac{R}{a})^n P_n(\cos\theta), & (R<a) \\ \frac{1}{R}\sum_{n=0}^{\infty}(\frac{a}{R})^n P_n(\cos\theta), & (R>a) \end{cases}$$

格林函数法

$$\nabla^2 G(x,x') = -\frac{1}{\varepsilon}\delta^3(x-x')$$

(1) 无界空间中

$$G(x,x') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|r-r'|}$$

(2) 上半平面中

$$G(x,x') = \frac{1}{4\pi\varepsilon_0}(\frac{1}{|r-r'|} - \frac{1}{|r+r'|})$$

(3) 球外空间 (R' 为电荷位置, α 为场点与电荷位置夹角,R_0 为球半径).

$$G(x,x') = \frac{1}{4\pi\varepsilon_0}(\frac{1}{R^2+R'^2-2RR'\cos\alpha} - \frac{1}{(\frac{RR'}{R_0})^2+R_0^2-2RR'\cos\alpha})$$

给定 $\rho(x')$, 第一类边值问题的解为 (G 交换了 x,x'):

$$\varphi(x) = \int_V G(x',x)\rho(x')\,\mathrm{d}V' + \varepsilon_0\oint_S (G(x',x)\frac{\partial\varphi}{\partial n'} - \varphi(x')\frac{\partial G(x',x)}{\partial n'}\,\mathrm{d}S')$$

若给定边界 φ, 则应使 G 在边界为 0, 若给定边界 $\frac{\partial\varphi}{\partial n}$, 则应使 $\frac{\partial G}{\partial n}$ 在边界为 0.

泊松方程

$$\nabla^2\Phi = \nabla\cdot\boldsymbol{E} = -\frac{\rho}{\epsilon_0}$$

电多极矩

$$\varphi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}(\frac{q}{R} + \frac{\boldsymbol{p}\cdot\boldsymbol{R}}{R^3} + \frac{1}{6}\sum_{i,j}\mathfrak{D}_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{R} + \cdots)$$

$$\boldsymbol{E} = -\frac{1}{4\pi\varepsilon_0}(\frac{3(\boldsymbol{p}\cdot\boldsymbol{R})\boldsymbol{R}}{R^5} - \frac{\boldsymbol{p}}{R^3})$$

$$\boldsymbol{p} = \iiint_V \rho(x')x'\,\mathrm{d}^3x'$$

$$\mathfrak{D} = \iiint_V 3\boldsymbol{x}'\boldsymbol{x}'\rho(\boldsymbol{x}')d^3x'$$

磁偶极矩

$$\varphi = \frac{\boldsymbol{m}\cdot\boldsymbol{R}}{4\pi R^3}$$

$$\boldsymbol{m} = \frac{1}{2}\iiint_V \boldsymbol{x}'\times\boldsymbol{J}(\boldsymbol{x'})\,\mathrm{d}^3x'$$

保角变换 (z_1 为原来的点, a 为夹角出现的位置的横坐标, α 为边界夹角).

$$\frac{\mathrm{d}z_1}{\mathrm{d}z_2} = C_1\prod_{i=1}^{n}(z_2-a_i)^{\frac{\alpha_i}{\pi}-1}$$

电荷

面电荷	电场	磁场
$\sigma_{polar} = P$	$\rho_p = -\nabla\cdot\boldsymbol{P}$	$\rho_M = -\mu_0\nabla\cdot\boldsymbol{M}$
$\sigma_{free} = D$	$\rho_f = \nabla\cdot\boldsymbol{D}$	$\rho_f = 0$
$\sigma_{total} = \epsilon_0 E$	$\rho_{tot} = \varepsilon_0\nabla\cdot\boldsymbol{E}$	$\rho_{tot} = \rho_p = \mu_0\nabla\cdot\boldsymbol{H}$
	$\varepsilon_0 E = D - P$	$B = \mu_0(H+M)$

边界条件

磁场	电场
$\varphi_1 = \varphi_2$	$\varphi_1 = \varphi_2$
$\mu_1\frac{\partial\varphi_1}{\partial n} = \mu_2\frac{\partial\varphi_2}{\partial n}$	$\varepsilon_1\frac{\partial\varphi_1}{\partial n} + \sigma_f = \varepsilon_2\frac{\partial\varphi_2}{\partial n}$
	$\sigma_{\text{电导}1}\frac{\partial\varphi_1}{\partial n} = \sigma_{\text{电导}2}\frac{\partial\varphi_2}{\partial n}$
$H_{1\parallel} + \boldsymbol{\alpha}_f = H_{2\parallel}$	$E_{1\parallel} = E_{2\parallel}$
$B_{1\perp} = B_{2\perp}$	$D_{1\perp} + \sigma_f = D_{2\perp}$
$M_{1\perp} + \alpha_M = M_{2\perp}$	
$B_{2\perp} - B_{1\perp} = \mu_0(\alpha_f + \alpha_M)$	$E_{2\perp} - E_{1\perp} = (\sigma_f + \sigma_p)/\varepsilon_0$
$B_\perp = 0$(超导球)	$D_\perp = \sigma_f$(导体)

洛伦兹力:

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}$$

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}$$

磁偶极子:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\widehat{R})\widehat{R} - \vec{m}}{R^3}$$

$$\varphi = \frac{\boldsymbol{m} \cdot \boldsymbol{R}}{4\pi R^3}$$

电磁场：

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

$$w = \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) = \frac{1}{2}(\rho\varphi + \boldsymbol{J}_f \cdot \boldsymbol{A})$$

电流：

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$J = \sigma E$$

毕奥——萨伐尔定律 $\boldsymbol{B} = \frac{\mu_0}{4\pi} \int \frac{I\,\mathrm{d}l \times \boldsymbol{e}_r}{r^2}$，若 I 为直线，$B = \frac{\mu_0 Il}{4\pi r^2}$.

磁矢势：

库仑规范：

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$$

$$\nabla^2 \varphi = -\frac{\varphi}{\epsilon_0}$$

$$\boldsymbol{\Box} \boldsymbol{A} = -\mu_0 \boldsymbol{J} + \frac{1}{c^2} \nabla \frac{\partial \varphi}{\partial t}$$

洛伦兹规范：

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$$

达朗贝尔方程:

$$\Box \varphi = -\frac{\rho}{\epsilon_0}$$

$$\boldsymbol{\Box} \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

1.1 超导体

临界磁场: 超过 $H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$ 时, 超导电性会被破坏.

迈斯纳效应: 超导体内部 $B = 0$.

伦敦第一方程

$$\frac{\partial J_s}{\partial t} = \alpha E, \alpha = \frac{n_s e^2}{m}$$

伦敦第二方程

$$\nabla \times \boldsymbol{J_s} = -\alpha B$$

2 电磁波的传播

2.1 电磁波

$$
\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} &=0 \qquad & \boldsymbol{\nabla^2}\boldsymbol{E}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}=\Box\boldsymbol{E} &=0\\
\boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t} &=0 \qquad & \boldsymbol{\nabla^2}\boldsymbol{B}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}=\Box\boldsymbol{B} &=0
\end{aligned}
$$

$$
\left|\frac{E}{B}\right|=\frac{1}{\sqrt{\mu\varepsilon}}=v
$$

$$
S=\sqrt{\frac{\varepsilon}{\mu}}E^2\boldsymbol{e}_k=vw\boldsymbol{e}_k
$$

群速与相速关系

$$
v_g=\frac{\mathrm{d}\omega}{\mathrm{d}k}=v_p+k\frac{\mathrm{d}v_p}{\mathrm{d}k}=\frac{c}{n+\omega(\mathrm{d}n/\mathrm{d}\omega)}
$$

导体内波矢量 $k=\beta+i\alpha, v\omega/\beta$. 垂直入射时 $\alpha\approx\beta\approx\sqrt{\omega\mu\sigma/2}, B\approx\sqrt{\mu\sigma/\omega}e^{i\frac{\pi}{4}}e_n\times E$. B 的相位比 E 滞后 1/4. 金属内部主要是磁场能. 电磁波穿透深度为 $\delta=\frac{1}{\alpha}=\sqrt{2/\omega\mu\sigma}$, 此为趋肤效应.

2.2 反射

介质界面上的边界条件为

$$
\begin{aligned}
e_n\times(E_2-E_2) &=0\\
e_n\times(H_2-H_1) &=\alpha
\end{aligned}
$$

入射波, 反射波和折射波分别为 E, E', E''. 从介质 1 射向介质 2. 有边界条件 $\boldsymbol{k}\cdot\boldsymbol{x}=\boldsymbol{k'}\cdot\boldsymbol{x}=\boldsymbol{k''}\cdot\boldsymbol{x}$. 菲涅耳公式:

当 $E\perp$ 入射面 (s 光):

$$
\frac{E'}{E}=\frac{\sqrt{\varepsilon_1}\cos\theta-\sqrt{\varepsilon_2}\cos\theta''}{\sqrt{\varepsilon_1}\cos\theta+\sqrt{\varepsilon_2}\cos\theta''}=-\frac{\sin(\theta-\theta'')}{\sin(\theta+\theta'')} \tag{1}
$$

$$
\frac{E''}{E}=\frac{2\sqrt{\varepsilon_1}\cos\theta}{\sqrt{\varepsilon_1}\cos\theta+\sqrt{\varepsilon_2}\cos\theta''}=\frac{2\cos\theta\sin\theta''}{\sin(\theta+\theta'')} \tag{2}
$$

当 $E\parallel$ 入射面 (p 光):

$$
\begin{aligned}
\frac{E'}{E} &=\frac{\tan(\theta-\theta'')}{\tan(\theta+\theta'')}\\
\frac{E''}{E} &=\frac{2\cos\theta\sin\theta''}{\sin(\theta+\theta'')\cos(\theta-\theta'')}
\end{aligned}
$$

布儒斯特角: 当 $\theta+\theta''=90°$, $E'_{\parallel}$ 消失. $\tan\theta_B=n_{21}$
半波损失: 前一种情况反射波与入射波反相.
反射系数和透射系数

$$R=\frac{E_0'^2}{E_0^2},T=\frac{n_2\cos\theta''}{n_1\cos\theta}\frac{E_0''^2}{E_0^2}$$

$$R_s=(1)^2,R_p=(2)^2,T_s=\frac{\sin 2\theta\sin 2\theta''}{\sin^2(\theta+\theta'')},T_p=\frac{4\sin 2\theta\sin 2\theta''}{(\sin 2\theta+\sin 2\theta'')^2}$$

当 $\theta=0$, $R_s=R_p=\frac{(n_2-n_1)^2}{(n_2+n_1)^2}$.
全反射:

$$k_z''=i\kappa,\ \kappa=k\sqrt{\sin^2\theta-n_{21}^2}$$

全反射能量守恒为:

$$R+\frac{\cos\theta''}{\cos\theta}T=1$$

垂直入射到良导体 ($\frac{\sigma}{\omega\varepsilon}>>1$) 表面:

$$R=1-2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}}=\frac{(n-n_1)^2+\kappa}{(n+n_1)+\kappa^2}$$

$$\varepsilon'=\varepsilon+i\frac{\sigma}{\omega}$$

折射波场沿 x 轴传播, 场强沿 z 轴指数衰减:

$$\boldsymbol{E}''=\boldsymbol{E}_0''e^{-\kappa z}e^{i(k_x''x-\omega t)}$$

其厚度 $\sim\kappa^{-1}$.

$$\kappa^{-1}=\frac{1}{\lambda_1}2\pi\sqrt{\sin^2\theta-n_{21}^2}$$

2.3 矩形谐振腔

1 代表导体, 2 代表真空. 法线由导体指向介质. 满足亥姆霍兹方程 $\nabla^2u+k^2u=0$. 边界条件为 $E_{\parallel}=H_{\perp}=\frac{\partial E_n}{\partial n}=0.k=\omega\sqrt{\mu\varepsilon}$.
满足 $k_xA_1+k_yA_2+k_zA_3=0$. 本征频率 $\omega=\frac{\pi}{\sqrt{\mu\varepsilon}}\sqrt{(m/l_1)^2+\cdots}$.

$$\begin{cases}E_x=A_1\cos k_xx\sin k_yy\sin k_zze^{-i\omega t}\\E_y=A_2\sin k_xx\cos k_yy\sin k_zze^{-i\omega t}\\E_z=A_3\sin k_xx\sin k_yy\cos k_zze^{-i\omega t}\end{cases},\ k_x=\frac{m\pi}{l_1},k_y=\frac{n\pi}{l_2},k_z=\frac{p\pi}{l_3}$$

2.4 矩形波导

z 方向无穷长的解为 $k_x^2+k_y^2+k_z^2=k^2.k_xA_1+k_yA_2-ik_zA_3=0.$

$$\begin{cases} E_x = & A_1\cos k_xx\sin k_yye^{-k_zz} \\ E_y = & A_2\sin k_xx\cos k_yye^{-k_zz} \\ E_z = & A_3\sin k_xx\sin k_yye^{-k_zz} \end{cases},\ k_x=\frac{m\pi}{a},k_y=\frac{n\pi}{b}$$

$$H=-\frac{i}{\omega\mu}\nabla\times E$$

由上式, $E_z=0$ 则 $H_z\neq 0$. 因此波导中的波不能是 TEM. 由于 $k_z=\sqrt{(\omega/c)^2-(k_x^2+k_y^2)}$ 为实数, 截止频率为 $\omega=\pi c\sqrt{(m/a)^2+(n/b)^2}$. 相速度 <c, 群速度 >c.

2.5 等离子体

振荡频率 $\omega_p=\sqrt{n_0e^2/m\varepsilon_0}$. $m\ddot{r}=-eE=eE_0e^{i(kx-\omega t)}$. $J(\omega)=-n_0ev=\sigma(\omega)E,\sigma=i\frac{n_0e^2}{m\omega}$. 稀薄等离子折射率为 $n=\sqrt{1-\omega_p^2/\omega^2}$. 当 $\omega>\omega_p$,$v_p>c$ 全反射, 可传播电磁波.

2.6 推迟势

以 R 表示原点 x' 到场点 x 的距离. $r\approx R-e_R\cdot x'$.

$$\varphi(x,t)=\int_V\frac{\rho\left(x',t-\frac{r}{c}\right)}{4\pi\varepsilon_0r}\,\mathrm{d}V'$$

$$A(x,t)=\frac{\mu_0}{4\pi}\int_V\frac{\boldsymbol{J}\left(x',t-\frac{r}{c}\right)}{r}\,\mathrm{d}V'=\frac{\mu_0}{4\pi}\int_V\frac{\boldsymbol{J}(x')e^{ik(R-e_R\cdot x')}}{R-e_R\cdot x'}\,\mathrm{d}V'$$

展开第一项为:

$$A(x)=\frac{\mu_0e^{ikR}}{4\pi R}\int_V J(x')\,\mathrm{d}V'=\frac{\mu_0e^{ikR}}{4\pi R}\dot{\boldsymbol{p}}$$

可得电偶极辐射:

$$\begin{cases} \boldsymbol{B} = & \frac{1}{4\pi\varepsilon_0c^3R}\ddot{p}e^{ikR}\sin\theta\boldsymbol{e}_\varphi \\ \boldsymbol{E} = & \frac{1}{4\pi\varepsilon_0c^3R}\ddot{p}e^{ikR}\sin\theta\boldsymbol{e}_\theta \\ \overline{\boldsymbol{S}} = & \frac{|\ddot{p}|^2}{32\pi^2\varepsilon_0c^3R^2}\sin^2\theta\boldsymbol{e}_R \\ P = \oint|\overline{\boldsymbol{S}}|R^2\,\mathrm{d}\Omega=\frac{1}{4\pi\varepsilon_0}\frac{|\ddot{p}|^2}{3c^3} \end{cases}$$

磁偶极辐射和电四极辐射展开第二项:($\mathcal{D} = \sum 3qx_i'x_j' - r'^2\delta_{ij}$)

$$A(x) = \frac{-ik\mu_0 e^{ikR}}{4\pi R}\int_V \boldsymbol{J}(x')(\boldsymbol{e_R}\cdot\boldsymbol{x'}\,\mathrm{d}V') = \frac{-ik\mu_0 e^{ikR}}{4\pi R}\left(-\boldsymbol{e_R}\times\boldsymbol{m} + \frac{1}{6}\boldsymbol{e_R}\cdot\dot{\mathcal{D}}\right)$$

先计算磁偶极辐射 $A = \frac{ik\mu_0 e^{ikR}}{4\pi R}\boldsymbol{e_R}\times\boldsymbol{m}$.

$$\begin{cases} \boldsymbol{B} = \dfrac{\mu_0 e^{ikR}}{4\pi c^2 R}(\ddot{m}\times e_R)e_R \\ \boldsymbol{E} = -\dfrac{\mu_0 e^{ikR}}{4\pi c R}(\ddot{m}\times e_R) \\ \overline{\boldsymbol{S}} = \dfrac{\mu_0\omega^4|\boldsymbol{m}|^2}{32\pi^2c^3R^2}\sin^2\theta e_R \\ P = \dfrac{\mu_0\omega^4|\boldsymbol{m}|^2}{12\pi c^3} \end{cases}$$

再计算电四极辐射 $A = \frac{-ik\mu_0 e^{ikR}}{24\pi R}\dot{\mathcal{D}}$. 定义 $\boldsymbol{D} = e_R\cdot\mathcal{D}$.

$$\begin{cases} A(x) = \dfrac{e^{ikR}}{24\pi\varepsilon_0 c^4 R}\dddot{\boldsymbol{D}}\times e_R \\ B = ik\boldsymbol{e_R}\times\boldsymbol{A} \\ E = c\boldsymbol{B}\times\boldsymbol{e_R} \\ S = \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{288\pi c^5R^2}(\dddot{\boldsymbol{D}}\times e_R)^2 e_R \end{cases}$$

2.7 衍射

基尔霍夫公式:$\psi(x) = -\frac{1}{4\pi}\oint_S \frac{e^{ikr}}{r}\boldsymbol{e}_n\cdot\left[\nabla'\psi + \left(ik - \frac{1}{r}\right)\frac{\boldsymbol{r}}{r}\psi\right]\mathrm{d}S'$.
夫琅禾费衍射: x' 为小孔上一点, x 为空间远处一点, R 为小孔中心到远处距离. k_1 沿入射方向, k_2 沿 R 方向. θ_1, θ_2 为入射出射角.

$$\phi(x) = -\frac{i\psi_0 e^{ikR}}{4\pi R}\int_S e^{i(k_1-k_2)\cdot x'}(\cos\theta_1 + \cos\theta_2)\,\mathrm{d}S'$$

长宽为 α, β 的矩形孔夫琅禾费衍射为:

$$I = I_0\left(\frac{1+\cos\theta_2}{2}\right)^2\left(\frac{\sin ka\alpha}{ka\alpha}\right)^2\left(\frac{\sin kb\beta}{kb\beta}\right)^2$$

电磁场动量:
动量密度 $\boldsymbol{g} = \varepsilon_0\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{S} = \frac{w}{c}\boldsymbol{e}_k$.

3 狭义相对论

$$a=\begin{pmatrix}\gamma & 0 & 0 & i\beta\gamma\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -i\beta\gamma & 0 & 0 & \gamma\end{pmatrix},\quad \beta=\frac{v}{c},\quad \gamma=\frac{1}{\sqrt{1-\beta^2}}$$

定义固有时 $d\tau=\frac{1}{c}ds$ 和 4-速度:$U_\mu=\frac{dx_\mu}{d\tau}=\gamma_u\left(u_1,u_2,u_3,ic\right)$.
相对论多普勒效应

$$\omega\approx\frac{\omega_0}{1-\frac{v}{c}\cos\theta}$$

定义场强张量

$$F_{\mu\nu}=\begin{bmatrix}0 & B_3 & -B_2 & -\frac{1}{c}E_1\\ -B_3 & 0 & B_1 & -\frac{\mathbf{i}}{c}E_2\\ B_2 & -B_1 & 0 & -\frac{\mathbf{i}}{c}E_3\\ \frac{\mathbf{i}}{c}E_1 & \frac{\mathbf{i}}{\boldsymbol{c}}\boldsymbol{E}_2 & \frac{\mathbf{i}}{c}\boldsymbol{E}_3 & 0\end{bmatrix}$$

Maxwell 方程变为

$$\frac{\partial F_{\mu v}}{\partial x_v}=\mu_0 J_\mu$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{v\lambda}}{\partial x_\mu}+\frac{\partial F_{\lambda\mu}}{\partial x_v}=0$$

且满足

$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu}=B^2-\frac{1}{c^2}E^2$$

$$\frac{i}{8}\epsilon_{\mu\nu\lambda\tau}F_{\mu\nu}F_{\lambda\tau}=\frac{1}{c}\vec{B}\cdot\vec{E}$$

能动量洛伦兹变换

$$p_1=\frac{p_1'+\frac{\beta_c}{c^2}E_1'}{\sqrt{1-\beta_c^2/c^2}};E_1=\frac{E_1'+\beta_c p_1'}{\sqrt{1-\beta_c^2/c^2}}$$

$$p_2=\frac{p_2'+\frac{\beta_c}{c^2}E_2'}{\sqrt{1-\beta_c^2/c^2}};E_2=\frac{E_2'+\beta_c p_2'}{\sqrt{1-\beta_c^2/c^2}}$$

4 数学

4.1 柱坐标系 (ρ, ϕ, z)

$$\nabla\varphi = \widehat{e}_1\frac{\partial\varphi}{\partial\rho} + \widehat{e}_2\frac{1}{\rho}\frac{\partial\varphi}{\partial\phi} + \widehat{e}_3\frac{\partial\varphi}{\partial z}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial(\rho A_1)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\boldsymbol{A} = \widehat{e}_1(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}) + \widehat{e}_2(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}) + \widehat{e}_3\frac{1}{\rho}(\frac{\partial(\rho A_2)}{\partial\rho} - \frac{\partial A_1}{\partial\phi})$$

$$\nabla^2\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\varphi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2}$$

4.2 球坐标系 (r, θ, φ)

$$\nabla\varphi = \widehat{e}_1\frac{\partial\varphi}{\partial r} + \widehat{e}_2\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \widehat{e}_3\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}q$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial r^2 A_1}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\boldsymbol{A} = \widehat{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \widehat{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \widehat{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\varphi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\frac{\partial\varphi}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\varphi}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2\varphi}{\partial\phi^2}\right]$$

4.3 矢量变换

$$\nabla\boldsymbol{r} = -\nabla'\boldsymbol{r} = \boldsymbol{e}_r$$

$$\nabla\frac{1}{\boldsymbol{r}} = -\nabla'\frac{1}{\boldsymbol{r}} = -\frac{1}{r^2}\boldsymbol{e}_r$$

$$\nabla\times\frac{1}{\boldsymbol{r}^2} = \nabla\cdot\frac{1}{\boldsymbol{r}} = 0$$

$$\nabla\cdot\varphi\boldsymbol{A} = \varphi\nabla\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\nabla\varphi$$

$$\nabla\times\varphi\boldsymbol{A} = \varphi\nabla\times\boldsymbol{A} + \nabla\varphi\times\boldsymbol{A}$$

$$\nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A})$$

$$\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{G} - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \times (\boldsymbol{C} \times \boldsymbol{D}) = [\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{D})]\boldsymbol{C} - [\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})]\boldsymbol{D}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = \begin{vmatrix} \boldsymbol{A} \cdot \boldsymbol{C} & \boldsymbol{A} \cdot \boldsymbol{D} \\ \boldsymbol{B} \cdot \boldsymbol{C} & \boldsymbol{B} \cdot \boldsymbol{D} \end{vmatrix}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = 0$$