SI 140
Probability and Statistics
Semester Spring 2019
Assignment 2

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Problem 1. Show that if P and Q are two probability measures defined on the same (countable) sample space, then $aP + bQ$ is also a probability measure for any two nonnegative numbers a and b satisfying $a + b = 1$. Give a concrete illustration of such a mixture.

Solution:
(i) for all $H(A) = aP(A) + bQ(A)$, $P(A), Q(A) \geq 0$. Thus $H(A) \geq 0$.
(ii)

$$\begin{aligned} H(A_1 + A_2) &= aP(A_1 + A_2) + bQ(A_1 + A_2) \\ &= [aP(A_1) + bQ(A_1)] + [aP(A_2) + bQ(A_2)] \\ &= H(A_1) + H(A_2) \end{aligned}$$

(iii) $H(\Omega) = aP(\Omega) + bQ(\Omega) = a + b = 1$

$\square$

Problem 2. If P is a probability measure, show that the function $P/2$ satisfies Axioms (i) and (ii) but not (iii). The function P^2 satisfies (i) and (iii) but not necessarily (ii); give a conterexample to (ii).

Solution:
$P/2$:
(1) $P/2(A) = \frac{P(A)}{2} \geq 0$.
(2) $P/2(A+B) = \frac{P(A+B)}{2} = \frac{P(A)}{2} + \frac{P(B)}{2} = P/2(A) + P/2(B)$.
(3) $P/2(\Omega) = \frac{P(\Omega)}{2} = \frac{1}{2} \neq 1$.
P^2:
(1) $P^2(A) = P(A)^2 \geq 0$.
(2) $P^2(A+B) = [P(A) + P(B)]^2 = P(A)^2 + P(B)^2 + 2P(A)P(B) \neq P^2(A) + P^2(B)$.
(3) $P^2(\Omega) = 1^2 = 1$.
conterexample to (ii): Suppose $P(A) = P(B) = 0.1$, $P^2(A+B) = 0.2^2 = 0.04$, however $P^2(A) + P^2(B) = 0.1^2 + 0.1^2 = 0.02$. $\square$

Problem 3. Show that if the two events (A, B) are independent, then so are (A, B^c), (A^c, B) and (A^c, B^c). Generalize this result to three independent events.

Solution:
The independence follows that $P(A)P(B) = P(AB)$.
$P(A)P\left(B^C\right) = P(A)(1 - P(B)) = P(A) - P(AB) = P\left(AB^C\right)$.
$P\left(A^C B\right) = (1 - P(A))P(B) = P(B) - P(AB) = P\left(A^C B\right)$.

$P\left(A^C\right)P\left(B^C\right) = (1-P(A))(1-P(B)) = 1-P(A)-P(B)+P(AB) = P\left(A^CB^C\right)$.
$P\left(A^C\right)P(B)P(C) = (1-P(A))P(BC) = P(BC)-P(ABC) = P\left(A^CBC\right)$.
$P\left(A^C\right)P\left(B^C\right)P(C) = P\left(A^CB^C\right)P(C) = P\left(A^CB^CC\right)$.
$P\left(A^C\right)P\left(B^C\right)P\left(C^C\right) = P\left(A^CB^C\right)P\left(C^C\right) = P\left(A^CB^CC^C\right)$.

□

Problem 4. Show that if A, B, C are independent events, then A and $B \cup C$ are independent, and $A \setminus B$ and C are independent.

Solution:
$P(A)P(B\cup C) = P(A)(P(B)+P(C)-P(BC)) = P(AB)+P(AC)-P(ABC) = P[(A\cap B)\cup(A\cap C)] = P[A\cap(B\cup C)]$.
$P(A\backslash B)P(C) = (P(A)-P(AB))P(C) = P(AC)-P(ABC) = P\left(AB^CC\right) = P[(A\backslash B)C]$.

□

Problem 5. Let Ω be a set and $\mathscr{F} \subset 2^{\Omega}$ be a σ-algebra. A function $P\colon \mathscr{F} \to \mathbb{R}\cup\{+\infty, -\infty\}$ is called a probability measure if it satisfies the following three properties:

1. For all $A \in \mathscr{F}$, $P(A) \geq 0$
2. $P(\Omega) = 1$
3. For all countable collections disjoint $A_1, A_2, \ldots$ in $\mathscr{F}$,
$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

Given a nested increasing sequence of events $A_1 \subset A_2 \subset A_3 \ldots \subset A_n \subset \ldots$ such that $\cup_{i=1}^{\infty} A_i$ is also an event, prove that
$$\lim_{n\to\infty} P(A_n) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$$

Solution:
For any $i > j$, $A_i \cup A_j = A_i$, so that $\bigcup_{i=1}^{n} A_i = A_n \implies P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_n)$.
Using probability axiom: $P\left(\lim_{n\to\infty}\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$.
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\lim_{n\to\infty}\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P(A_n)$$

□

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Problem 6. Find an example where

$$P(AB) < P(A)P(B)$$

Solution:
Throw a coin, A = front face. B = back face. Then $P(A) = P(B) = 1/2$, and $P(AB) = 0$.
□

Problem 7. What can you say about the event A if it is independent of itself? If the events A and B are disjoint and independent, what can you say of them?

Solution:

If the event A is independent of itself, $P(A)^2 = P(A) = 1$.
If the events A and B are disjoint and independent, $P(AB) = 0 = P(A)P(B)$. So that $P(A) = 0$ or $P(B) = 0$. □

Problem 8. Prove that

$$P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

when A, B, C are independent by considering $P(A^c B^c C^c)$

Solution:

$$\begin{aligned} P(A \cup B \cup C) &= 1 - P\left(A^C B^C C^C\right) \\ &= 1 - (1 - P(A))(1 - P(B))(1 - P(C)) \\ &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) + P(ABC) \end{aligned}$$

□

Problem 9. Let $S = (-\infty, +\infty)$, the real line. Then $\mathscr{F}$ is chosen to contain all sets of the form

$$(a, b], [a, b], [a, b), (a, b)$$

for all real numbers a and b.Show that $\mathscr{F}$ is a Borel field.

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Solution:

1. For every element $A = (a,b], [a,b], [a,b), (a,b)$ in $\mathscr{F}$,
$A^C = (-\infty,a] \cup (b,+\infty)\,,\ (-\infty,a) \cup (b,+\infty)\,,\ (-\infty,a) \cup [b,+\infty)\,,\ (-\infty,a] \cup [b,+\infty)$ is also in $\mathscr{F}$.
2. If $A_i = [/\,(a_i,b_i]/)\,,\ A = \bigcup A_i$. Then

$$A = \bigcup_{i=1}^{\infty} A_i = [/\,(min\,(a_i)\,, max\,(b_i)]/) \in \mathscr{F}$$

As $\mathscr{F}$ satisfy 1,2, it's a Borel field. □

Problem 10. Suppose that the land of a square kingdom is divided into three strips A, B, C of equal area and suppose the value per unit is in the ratio of 1:3:2. For any piece of (measurable) land S in this kingdom, the relative value with respect to that of the kingdom is then given by the formula:

$$V\,(S) = \frac{P\,(SA) + 3P\,(SB) + 2P\,(SC)}{2}$$

where P is as in Example 2 of 2.1. Show that V is a probability measure.

Solution:

1. Since $P\,(S) = \frac{|A|}{|\Omega|} \geq 0$, $V\,(S) \geq 0$.
2. $SA = SB = SC = \frac{1}{3}|\Omega|$, $P\,(SA) = P\,(SB) = P\,(SC) = \frac{\frac{1}{3}|\Omega|}{|\Omega|} = \frac{1}{3}$, Thus $V\,(|\Omega|) = \frac{\frac{1}{3}+3\frac{1}{3}+2\frac{1}{3}}{2} = 1$.
3. For any A, B, because A, B are disjoint, $|A + B| = |A| + |B|$. Thus $P[S\,(A+B)] = \frac{|A+B|}{|\Omega|} = \frac{|A|+|B|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = P\,(SA) + P\,(SB)$.

□