The 4th Homework of Theoretical Mechanics

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Q1

取广义坐标：C 点坐标 x, y, 圆盘位置及角度 θ, φ，以 C 为动系，

$$L = \frac{1}{2}m(b\dot{\theta})^2 + \frac{1}{4}m(r\dot{\varphi})^2 + \frac{1}{4}M(R\dot{\theta})^2 - (M+m)\boldsymbol{r_{OC}}\frac{\mathrm{d}\boldsymbol{v_C}}{\mathrm{d}t}$$

$$= \frac{1}{2}m(b\dot{\theta})^2 + \frac{1}{4}m(r\dot{\varphi})^2 + \frac{1}{4}M(R\dot{\theta})^2 - (M+m)(x\ddot{x} + y\ddot{y})$$

由能量守恒：

$$E = \frac{1}{2}M\sqrt{x^2+y^2} + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}m\sqrt{(x+b\cos\theta)^2 + (y+b\sin\theta)^2} + \frac{1}{4}mr^2\dot{\varphi}^2 = const$$

由角动量守恒:

$$J = \frac{1}{2}MR^2\dot{\theta} + \frac{1}{2}mr^2\dot{\varphi} + mb^2\dot{\theta} = const$$

由动量守恒：

$$P_x = (M+m)\dot{x} + mb\theta\cos\varphi = const$$

$$P_y = (M+m)\dot{y} + mb\theta\sin\varphi = const$$

Q2

$$T = \frac{1}{2}m(l\dot{\theta})^2 + \frac{1}{4}mr^2\left(\frac{r+l}{r}\dot{\theta}\right)^2 + \frac{1}{2}ml^2\dot{\theta}^2$$

$$= \left(\frac{1}{4}r^2 + \frac{1}{2}rl + \frac{5}{4}l^2\right)m\dot{\theta}^2$$

$$V = \frac{1}{2}k\left(l\sin\theta\right)^2 \approx \frac{1}{2}kl^2\theta^2$$

$$L = \left(\frac{1}{4}r^2 + \frac{1}{2}rl + \frac{5}{4}l^2\right)m\dot{\theta}^2 - \frac{1}{2}kl^2\theta^2$$

$$\Longrightarrow \frac{\partial L}{\partial \theta} = -kl^2\theta = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = \left(\frac{1}{2}r^2 + rl + \frac{5}{2}l^2\right)m\ddot{\theta}$$

设

$$\omega^2 = \frac{m(\frac{1}{2}r^2 + rl + \frac{5}{2}l^2)}{kl^2}$$

则运动为：

$$\theta = A\sin(\omega t + \varphi)$$

其中 A, φ 由初始条件决定，周期

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{kl^2}{m(\frac{1}{2}r^2 + rl + \frac{5}{2}l^2)}}$$

Q3

在 $\theta_1 = \theta_2 = 0$ 附近：

$$L = MR^2\dot{\theta}_1^2 + \frac{1}{2}mR^2(\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) + \frac{1}{2}MgR\theta_1^2 + \frac{1}{2}mgR\theta_1^2 + \frac{1}{2}mgR\theta_2^2$$

式中：

$$\begin{aligned} a_{11} &= \tfrac{(2M+m)R^2}{2} & a_{12} &= \tfrac{mR^2}{2} & a_{22} &= \tfrac{1}{2}mR^2 \\ b_{11} &= \tfrac{1}{2}(M+m)gR & b_{12} &= 0 & b_{22} &= \tfrac{1}{2}mgR \end{aligned}$$

求解久期方程：

$$\left(\frac{M+m}{2}gR - \frac{2M+m}{2}R^2\omega^2\right)\left(\frac{1}{2}mgR - \frac{1}{2}mR^2\omega^2\right) = \left(-\frac{mR^2}{2}\omega^2\right)^2$$

$$\Longrightarrow \omega_1^2 = \frac{g}{2R}, \qquad \omega_2^2 = \frac{m+M}{M}\frac{g}{R}$$

Q4

设杆角速度为 $\omega, A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)$，其中 $\omega = \dot{\theta}$ 为常数，有 $x_C = x_B - \frac{1}{2}l\cos\omega t$

$$\frac{\dot{y}_C}{\dot{x}_C} = \tan\omega t = \frac{\dot{y}_A + \dot{y}_B}{\dot{x}_A + \dot{x}_B}$$

代入 $\dot{x}_B = \dot{x}_A - l\sin\omega t$，$\dot{y}_B = \dot{y}_A + l\cos\omega t$

$$\implies \frac{2\dot{y}_A + l\cos\omega t}{2\dot{x}_A - l\sin\omega t} = \tan\omega t$$

回代：

$$\implies \frac{2\dot{y}_B - l\cos\omega t}{2\dot{x}_B + l\cos\omega t} = \tan\omega t$$

Q5

设质点距 y 轴水平距离为 r，抛物线为 $y = ar^2$

$$T = \frac{1}{2}mv^2$$

$$v^2 = v_r^2 + v_y^2$$

$$v_r = r\omega,\ v_y = \frac{\mathrm{d}}{\mathrm{d}t}(ar^2) = 2arv_r$$

$$\implies T = \frac{1}{2}mr^2\omega^2(1 + 4a^2r^2)$$

$$V = mgy$$

$$E = T + V = \frac{1}{2}m\omega^2(r^2 + 4a^2r^4) + mgy = const$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \implies \omega^2 + 8a^2r^2\omega^2 + 2ga = 0$$

$$\implies r = -\frac{2ga + \omega^2}{8a^2\omega^2}$$