The 6th HW of Electrodynamics

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Q1

$n = 1,\ a = 0,\ \alpha = \pi/3$

$$\frac{\mathrm{d}z_1}{\mathrm{d}z_2} = C_1 \prod_{i=1}^{n} (z_2 - a_i)^{\alpha_i/\pi - 1} = C_1 z_2^{-2/3}$$

$$\Longrightarrow\ z_1 = 3C_1 z_2^{1/3} + C_2$$

$$W = i|\vec{E}|z_2 = iC_3|\vec{E}|z_1^3 = \left(y^3 - 3x^2y\right) + i\left(x^3 - 3xy^2\right)$$

$$\Longrightarrow\ \phi = \left(y^3 - 3x^2y\right),\ \psi = x^3 - 3xy^2$$

Q2

(a)

该分布满足 Laplace 方程:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

其分离变量解为:

$$\Phi(\rho,\phi) = \left(a_0 + b_0\ln\rho\right)\left(A_0 + B_0\phi\right) + \sum_{\nu\neq 0}\left(a_\nu\rho^\nu + b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}\right)$$

代入边界条件 $\Phi(\phi = 0) = 0$,

$$0 = \left(a_0 + b_0\ln\rho\right)\left(A_0\right) + \sum_{\nu\neq 0}\left(a_\nu\rho^\nu + b_\nu\rho^{-\nu}\right)\left(A_\nu + B_\nu\right)$$

要使上式对任意 ρ 均成立, 必然有 $A_0 = 0,\ A_\nu = -B_\nu$. 则解的形式可以简化为:

$$\Phi(\rho,\phi) = (a_0 + b_0 \ln\rho)(B_0\phi) + \sum_{\nu\neq 0}\left(a_\nu\rho^\nu + b_\nu\rho^{-\nu}\right)A_\nu\sin(\nu\phi)$$

代入边界条件 $\Phi(\phi=\beta)=0$,

$$0 = (a_0 + b_0 \ln\rho)(B_0\beta) + \sum_{\nu\neq 0}\left(a_\nu\rho^\nu + b_\nu\rho^{-\nu}\right)A_\nu\sin(\nu\beta)$$

要使上式对任意 ρ 均成立, 必然有 $B_0 = 0,\ \sin(\nu\beta) = 0 \implies \nu\beta = n\pi$. 则解的形式可以简化为:

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty}\left(a_n\rho^{n\pi/\beta} + b_n\rho^{-n\pi/\beta}\right)A_n\sin\left(\frac{n\pi\phi}{\beta}\right)$$

代入边界条件 $\Phi(\rho=a)=0$,

$$0 = \sum_{n=1}^{\infty}\left(a_n a^{n\pi/\beta} + b_n a^{-n\pi/\beta}\right)A_n\sin\left(\frac{n\pi\phi}{\beta}\right)$$

要使上式对任意 ϕ 均成立, 必然有 $b_n = -a_n a^{2n\pi/\beta}$. 则解的形式可以简化为:

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty}A_n\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)$$

(b)

$$E_\rho = -\frac{\partial\Phi}{\partial\rho} = -\frac{\partial}{\partial\rho}\sum_{n=1}^{\infty}A_n\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$\implies E_\rho = -\sum_{n=1}^{\infty}A_n\frac{n\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\sin\left(\frac{n\pi\Phi}{\beta}\right)$$

最低阶项为:

$$E_\rho = -A_1\frac{\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\sin\left(\frac{\pi\phi}{\beta}\right)$$

同理对于 E_ϕ,

$$E_\phi = -\sum_{n=1}^{\infty}A_n\frac{n\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\cos\left(\frac{n\pi\phi}{\beta}\right)$$

最低阶项为:

$$E_\phi = -A_1\frac{\pi}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)\cos\left(\frac{\pi\phi}{\beta}\right)$$

$$\sigma(\rho,0) = \left[\epsilon_0 E_\phi\right]_{\Phi=0}$$

$$\sigma(\rho, 0) = -A_1 \frac{\pi\epsilon_0}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)$$

$$\sigma(\rho, \beta) = [-\epsilon_0 E_\phi]_{\phi=\beta}$$

$$\sigma(\rho, \beta) = -A_1 \frac{\pi\epsilon_0}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)$$

$$\sigma(a, \phi) = [\epsilon_0 E_\rho]_{\rho=a}$$

$$\sigma(a, \phi) = -A_1 \frac{2\pi\epsilon_0}{a\beta} \sin\left(\frac{\pi\phi}{\beta}\right)$$

(c)

由 (b) 中计算,

$$\mathbf{E} = \sum_{n=1}^{\infty} A_n \frac{n\pi}{a\beta}\left[-\widehat{\boldsymbol{\rho}}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} + \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\sin\left(\frac{n\pi\phi}{\beta}\right) - \widehat{\Phi}\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta-1}\right)\cos\left(\frac{n\pi\phi}{\beta}\right)\right]$$

当 $\beta = \pi$,

$$\mathbf{E} = \sum_{n=1}^{\infty} A_n \frac{n}{a}\left[-\widehat{\boldsymbol{\rho}}\left(\left(\frac{\rho}{a}\right)^{n-1} + \left(\frac{\rho}{a}\right)^{-n-1}\right)\sin(n\Phi) - \widehat{\Phi}\left(\left(\frac{\rho}{a}\right)^{n-1} - \left(\frac{\rho}{a}\right)^{-n-1}\right)\cos(n\Phi)\right]$$

其最低阶项为:

$$\mathbf{E} = A_1 \frac{1}{a}\left[-\widehat{\boldsymbol{\rho}}\left(1 + \left(\frac{a}{\rho}\right)^2\right)\sin(\Phi) - \widehat{\Phi}\left(1 - \left(\frac{a}{\rho}\right)^2\right)\cos(\phi)\right]$$

当 $\rho >> a$:

$$\mathbf{E} = -A_1 \frac{1}{a}[\widehat{\boldsymbol{\rho}}\sin(\phi) + \widehat{\phi}\cos(\phi)] = -\frac{A_1}{a}\widehat{\mathbf{j}}$$

半球面上的面电荷为:

$$\sigma(a, \phi) = \sigma_0 \sin(\phi),\ \ \sigma_0 = -A_1 \frac{2\epsilon_0}{a}$$

在边缘处:

$$\sigma(\rho, 0) = \sigma(\rho, \beta) = \frac{\sigma_0}{2}\left(1 - \left(\frac{\rho}{a}\right)^{-2}\right)$$

半球面上的总电荷为:

$$Q_{\text{ half-cyl }} = -A_1 \frac{2\epsilon_0}{a}\int_0^\pi \sin(\phi) a \, \mathrm{d}\phi = -4A_1\varepsilon_0$$

边缘处平均面电荷为:

$$\sigma_{\text{side}} = \varepsilon_0 E = -\frac{\varepsilon_0 A_1}{a}$$

长度为 $2a$ 的一段的总电荷为:

$$Q_{\text{side}} = 2a\sigma_{\text{side}} = -2\varepsilon_0 A_1$$

对比可以发现

$$2Q_{\text{side}} = Q_{\text{ half-cyl}}$$

包含了半球的总电荷为:

$$\begin{aligned}
Q_1 &= 2\int_a^l (-A_1)\frac{\epsilon_0}{a}\left(1-\left(\frac{\rho}{a}\right)^{-2}\right)d\rho + Q_{\text{ half-cyl}} \\
Q_1 &= -2(A_1)\frac{\epsilon_0}{a}(l-a) - 2(A_1)\,a\epsilon_0(1/l-1/a) + Q_{\text{ half-cyl}} \\
Q_1 &= 2A_1\epsilon_0\left[\frac{-l}{a}-\frac{a}{l}\right]
\end{aligned}$$

当 $l >> a$:

$$Q_1 = \frac{-2l\epsilon_0 A_1}{a}$$

当没有半球面时, 总电荷为

$$\begin{aligned}
Q_2 &= \sigma 2l \\
Q_2 &= [\epsilon_0 E_y]_{y=0}\,\sigma 2l \\
Q_2 &= (-\epsilon_0 A_1/a)\,2l \\
Q_2 &= \frac{-2l\epsilon_0 A_1}{a}
\end{aligned}$$

二式相同.