Group Theory
Homework Assignment 02
Spring, 2019

1. Show that the intersection S of two invariant subgroups S_1 and S_2 of a group G is an invariant subgroup.
   $S = S_1 \cap S_2$, for $T \in S$, $X \in G$, $S \in S_1, S_2$, thus $XTX^{-1} \in S_1, S_2 \implies XTX^{-1} \in S$.

2. The multiplication table of a finite group G is given by

	E	A	B	C	D	F	I	J	K	L	M	N
E	E	A	B	C	D	F	I	J	K	L	M	N
A	A	E	F	I	J	B	C	D	M	N	K	L
B	B	F	A	K	L	E	M	N	I	J	C	D
C	C	I	L	A	K	N	E	M	J	F	D	B
D	D	J	K	L	A	M	N	E	F	I	B	C
F	F	B	E	M	N	A	K	L	C	D	I	J
I	I	C	N	E	M	L	A	K	D	B	J	F
J	J	D	M	N	E	K	L	A	B	C	F	I
K	K	M	J	F	I	D	B	C	N	E	L	A
L	L	N	I	J	F	C	D	B	E	M	A	K
M	M	K	D	B	C	J	F	I	L	A	N	E
N	N	L	C	D	B	I	J	F	A	K	E	M

   (a) Find the inverse of each element of G.

$$
\begin{aligned}
E^{-1} &= E \\
A^{-1} &= A \\
B^{-1} &= F \\
C^{-1} &= I \\
D^{-1} &= J \\
F^{-1} &= B \\
I^{-1} &= C \\
J^{-1} &= D \\
K^{-1} &= L \\
L^{-1} &= K \\
M^{-1} &= N \\
N^{-1} &= M
\end{aligned}
$$

   (b) Find the elements in each class of G.

$$\{E\},\ \{A\},\ \{B, C, D\},\ \{F, I, J\},\ \{K, L, M\},\ \{N\}.$$

   (c) Find all invariant subgroups of G.

$$\{E\},\ \{E, A\},\ \{E, A, K, L, M, N\}\ \{E, A, B, C, D, F, I, J, K, L, M, N\}$$

3. Consider the group D_3.

   (a) List all the classes of D_3.

$$\{E\},\ \{D, F\},\ \{A, B, C\}$$

   (b) Find the right and left cosets of the subgroup $S = \{E, A\}$ of D_3.

Right cosets:

$$\begin{aligned} SE = SA = & \; \{E, A\} \\ SD = SC = & \; \{D, C\} \\ SF = SB = & \; \{F, B\} \end{aligned}$$

Left cosets:

$$\begin{aligned} ES = AS = & \; \{E, A\} \\ DS = BS = & \; \{D, B\} \\ FS = CS = & \; \{F, C\} \end{aligned}$$

4. For the group D_3 and its invariant subgroup $S = \{E, D, F\}$, find the factor group D_3/S. Construct the multiplication table for the factor group.
   Right cosets:

$$SE = SD = SF = S$$
$$SA = SB = SC = \{A, B, C\}$$

   Multiplication table:

	SE	SA
SE	SE	SA
SA	SA	SE

5. Consider $C_6 = \{E,\, a,\, a^2,\, a^3,\, a^4,\, a^5\}$ and its two subgroups $S_1 = \{E,\, a^3\}$ and $S_2 = \{E,\, a^2,\, a^4\}$. Show that $C_6 = S_1 \otimes S_2$.

$$\begin{aligned} EE = & \; E \\ Ea^2 = & \; a^2 \\ Ea^4 = & \; a^4 \\ a^3E = & \; a^3 \\ a^3a^2 = & \; a^5 \\ a^3a^4 = & \; a \end{aligned}$$

$$\Longrightarrow C_6 = S_1 \otimes S_2.$$