数值分析第五次作业

肖涵薄 31360164

2019 年 4 月 2 日

3

其差分为:

阶数	0	1	2	3
1	-1			
1.5	0.5	3		
2	2.5	4	1	
2.5	5	5	1	0
3	8	6	1	0
3.5	11.5	7	1	0

则 Newton 插值多项式为:

$$N\left(x\right) = -1 + 3\left(x - 1\right) + \left(x - 1.5\right)$$

4

设存在一点 $x_{n+1}, f\left(x_{n+1}\right)$. 对这 $n+1$ 个点进行插值,

阶数	0	1	2	$\cdots$	n
x_1	0				
x_2	0	0			
$\vdots$	$\vdots$	$\vdots$	$\ddots$		
x_n	0	0	0	$\cdots$	
x_{n+1}	$f\left(x_{n+1}\right)$	$\frac{f\left(x_{n+1}\right)}{x_{n+1}-x_n}$	$\frac{f\left(x_{n+1}\right)}{\left(x_{n+1}-x_n\right)\left(x_{n+1}-x_{n-1}\right)}$	$\cdots$	$\frac{f\left(x_{n+1}\right)}{\prod_1^n\left(x_{n+1}-x_i\right)}$

$n+1$ 个点插值 n 次多项式可得到精确解, 因此

$$f(x)=N(x)=\frac{f\left(x_{n+1}\right)}{\prod_1^n\left(x_{n+1}-x_i\right)} \prod_{i=1}^n\left(x-x_i\right)$$

求导:

$$f^{\prime}\left(x_j\right)=\frac{f\left(x_{n+1}\right)}{\prod_1^n\left(x_{n+1}-x_i\right)} \prod_{i=1, i \neq j}^n\left(x_j-x_i\right)$$

因此

$$\frac{x_j^{n-1}}{f^{\prime}\left(x_j\right)}=\frac{\prod_1^n\left(x_{n+1}-x_i\right)}{f\left(x_{n+1}\right)} \cdot \frac{1}{\prod_{i=1, i \neq j}^n\left(1-\frac{x_i}{x_j}\right)}$$

通过 $f(x)$ 表达式可以发现 $\frac{\prod_1^n\left(x_{n+1}-x_i\right)}{f\left(x_{n+1}\right)}$ 正是 x 最高次项系数的倒数, 即 a_n^{-1}. 即

$$\frac{x_j^{n-1}}{f^{\prime}\left(x_j\right)}=a_n^{-1} \cdot \frac{1}{\prod_{i=1, i \neq j}^n\left(1-\frac{x_i}{x_j}\right)}$$

又由于

$$\left\{\begin{array}{r}
\sum_{j=1}^n \frac{1}{\prod_{i=1, i \neq j}^n\left(1-\frac{x_i}{x_j}\right)}=\frac{\prod_{i, j=1}^n\left(1-\frac{x_i}{x_j}\right)}{\prod_{i, j=1}^n\left(1-\frac{x_i}{x_j}\right)}=1 \\
\sum_{j=1}^n \frac{1}{x_j \prod_{i=1, i \neq j}^n\left(1-\frac{x_i}{x_j}\right)}=0
\end{array}\right.$$

即

$$\left\{\begin{array}{l}
\sum_{j=1}^n \frac{x_j^{n-1}}{f^{\prime}\left(x_j\right)}=a_n^{-1} \\
\sum_{j=1}^n \frac{x_j^k}{f^{\prime}\left(x_j\right)}=0, \quad(k<n-1)
\end{array}\right.$$

5

(1)

$$P(x)=\alpha_0(x)y_0+\alpha_1(x)y_1+\alpha_2(x)y_2+\beta_2(x)y_0'=\alpha_0(x)-\alpha_2(x)$$

根据 $\alpha(x_0)=1,\alpha(x_{1,2})=0$, 设 $\alpha_0=(Ax+B)l_0$, 满足

$$Ax_0+B=1$$
$$\left((Ax+B)l_0\ (x)\right)'\Big|_{x_0}=0$$
$$A=\frac{3}{2},B=\frac{5}{2}$$
$$\implies \alpha_0=\frac{1}{4}x(3x+5)(x-1)$$

同理, 设 $\alpha_2=\frac{(x-x_0)^2(x-x_1)}{(x_2-x_0)^2(x_2-x_1)}=\frac{1}{4}x(x+1)^2$. 则

$$P(x)=\alpha_0(x)-\alpha_2(x)=\frac{1}{4}x(3x+5)(x-1)-\frac{1}{4}x(x+1)^2=-\frac{3}{2}x+\frac{x^3}{2}$$

(2)

$$P(0.5)=-\frac{11}{16},f(0)-P(0)=f(1)-P(1)=0$$
$$R_4(0.5)=\left|\frac{f^{(4)}(x)}{4!}(x-0)(x-1)(x+1)^2\right|_{x=0.5}<\frac{1}{4!}\frac{1}{2}\frac{1}{2}\left(\frac{3}{2}\right)^2=\frac{3}{128}$$

6

(1) 设

$$\begin{aligned}p(x)=&f(x_0)+f[x_0,x_1](x-x_0)\\&+f[x_0,x_1,x_2](x-x_0)(x-x_1)\\&+A(x-x_0)(x-x_1)(x-x_2)\end{aligned}$$

则 $P(x)=-\frac{3}{4}+(x-x_0)(x-x_1)+A(x-x_0)(x-x_1)(x-x_2)$.
代入 $P'(x_1)=f'(x_1)$:

$$A=\frac{f'(x_1)-(x_1-x_0)-(x_1-x_1)}{(x_1-x_0)(x_1-x_2)}=0$$

即

$$P(x)=-\frac{3}{4}+\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)=x^2-1$$

(2)

$$R\left(1\right)=\left|f\left(1\right)-P\left(1\right)\right|=\left|\frac{f^{(4)}\left(x\right)}{4!}\left(x-2\right)\left(x-1/2\right)^{2}\left(x+1/2\right)\right|_{x=1}<\frac{1}{64}$$

又由于 $P\left(1\right)=0$ 即 $\left|f\left(1\right)\right|<\frac{1}{64}$.

7

(1) 给定节点 $a\leq x_0<x_1<\cdots<x_n\leq b$, 若函数 $S\left(x\right)\in C^3[a,b]$, 且每个小区间 $[x_j,x_{j+1}]$ 上是 4 次多项式, 则称 $S\left(x\right)$ 是节点 $a\leq x_0<x_1<\cdots<x_n\leq b$ 上的 4 次样条函数. 若在节点上给定函数数值 $f\left(x_j\right)=y_j$, 且满足

$$S\left(x_j\right)=y_j,\quad j=0,1,\cdots,n.$$

则称 $S\left(x\right)$ 为 4 次样条插值函数.

(2)

$$S\left(x\right)=\begin{cases}x^4+2x+1 & 0\leq x\leq 1\\ \left(x-1\right)^4+a\left(x-1\right)^3+b\left(x-1\right)^2+c\left(x-1\right)+d & 1\leq x\leq 3\end{cases}$$

$$S\left(1_-\right)=S\left(1_+\right)\implies 3=d$$

$$S'\left(1_-\right)=S'\left(1_+\right)\implies 6=c$$

$$S''\left(1_-\right)=S''\left(1_+\right)\implies 12=2b\implies b=6$$

$$S'''\left(1_-\right)=S'''\left(1_+\right)\implies 24=6a\implies a=4$$