

数理方法 II 第三次作业

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Q1

(1)

$$\begin{aligned}x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + (2 + \lambda/x) y &= 0 \\ \frac{d}{dx} \left(x \frac{dy}{dx} \right) - (-2) y + \frac{\lambda}{x} y &= 0\end{aligned}$$

即

$$k(x) = x, q(x) = -2, \rho(x) = \frac{1}{x}$$

(2)

原式化为 $y'' + \frac{a-bx}{x-x^2} y' - \frac{\lambda}{x-x^2} y = 0$

$$\exp \left(\int \frac{a-bx}{x-x^2} dx \right) = \exp \left(a \int \frac{1}{x} dx + (a-b) \int \frac{1}{1-x} dx \right) = \exp(a \ln x - (a-b) \ln(1-x)) =$$
$$\frac{x^a}{(1-x)^{a-b}}$$

则最后可化为标准形式:

$$\frac{d}{dx} \left(\frac{x^a}{(1-x)^{a-b}} y' \right) + \lambda \left(\frac{1}{x(x-1)} \frac{x^a}{(1-x)^{a-b}} \right) y = 0$$

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设 $y_m, y_n, n \neq m$ 是函数不同本征值的两个解.

$$\begin{cases} \frac{d}{dx} (py'_m) + (\lambda_m \rho - q) y_m = 0 \\ \frac{d}{dx} (py'_n) + (\lambda_n \rho - q) y_n = 0 \end{cases}$$

两式分别乘以 y_n, y_m , 相减, $y_n \frac{d}{dx} (py'_m) - y_m \frac{d}{dx} (py'_n) + (\lambda_m - \lambda_n) y_m y_n = 0$. 求区间 $[a, b]$ 积分,

$$\begin{aligned} & \int_a^b \left[y_n \frac{d}{dx} (py'_m) - y_m \frac{d}{dx} (py'_n) \right] dx + \int_a^b \left(py'_m \frac{d}{dx} y_n - py'_n \frac{d}{dx} y_m \right) dx + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx \\ &= \int_a^b \frac{d}{dx} (py_n y'_m - py_m y'_n) dx + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx \\ &= (py_n y'_m - py_m y'_n)|_{x=b} - (py_n y'_m - py_m y'_n)|_{x=a} + (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx \end{aligned}$$

由于 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, 有 $y_n(b) y'_m(b) = y_m(b) y'_n(b)$, $y_n(a) y'_m(a) = y_m(a) y'_n(a)$

$$= (\lambda_m - \lambda_n) \int_a^b \rho y_m y_n dx = 0$$

当 $\lambda_m \neq \lambda_n$, $\int_a^b \rho y_m y_n dx = 0$.

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(1)

根据定义, $\int \delta(\mathbf{r} - \mathbf{r}_0) d\mathbf{r}^3 = 1$, 在球坐标下即为

$$\begin{aligned} & \int_0^\infty dr \int_0^{2\pi} r d\varphi \int_0^\pi r \sin \varphi \delta(\mathbf{r} - \mathbf{r}_0) d\theta = 1 \\ \implies & \int_0^\infty dr \int_0^{2\pi} d\cos \varphi \int_0^\pi r^2 \delta(\mathbf{r} - \mathbf{r}_0) d\theta = 1 \end{aligned}$$

根据直角坐标下形式, 可知

$$\delta(r - r_0) \delta(\cos \theta - \cos \theta_0) \delta(\varphi - \varphi_0) = r^2 \delta(\mathbf{r} - \mathbf{r}_0)$$

移项即得

$$\delta(\mathbf{r} - \mathbf{r}_0) = \frac{1}{r^2} \delta(r - r_0) \delta(\cos \theta - \cos \theta_0) \delta(\varphi - \varphi_0)$$

(2)

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}_0|} = -\nabla \cdot \frac{1}{(\mathbf{r} - \mathbf{r}_0)^2}$$

由高斯定理可知

$$-\int \nabla \cdot \frac{1}{(r-r_0)^2} dV = -\int_{\Omega} \frac{1}{(r-r_0)^2} dS$$

取积分面为 $r-r_0=a$ 的球壳, a 为任意常数.

$$-\int_{\Omega} \frac{1}{(r-r_0)^2} dS = -4\pi a^2 \frac{1}{a^2} = -4\pi$$

即

$$\int \nabla^2 \frac{1}{|r-r_0|} dr^3 = -4\pi$$

根据定义,

$$\nabla^2 \frac{1}{|r-r_0|} = -4\pi \delta(r-r_0)$$

4

(1) 先计算 $\mathcal{F}(e^{-a|t|})$

$$\begin{aligned} \mathcal{F}(e^{-a|t|}) &= \int_{-\infty}^0 e^{at-iwt} dt + \int_0^{\infty} e^{-at-iwt} dt \\ &= \int_0^{\infty} (e^{-(a+iw)t} + e^{-(a-iw)t}) dt = \frac{2a}{a^2+w^2} \end{aligned}$$

则 $\mathcal{F}^{-1}\left(\frac{2a}{a^2+w^2}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2+w^2} e^{iwt} dw = e^{-a|t|}$, 两边求实部:

$$\int_{-\infty}^{\infty} \frac{1}{a^2+w^2} \cos wt dw = \frac{\pi}{a} e^{-a|t|}$$

(2)

$$\begin{aligned} \int \frac{1}{r} e^{ikr \cos \theta} dr &= \int_0^{\infty} \frac{1}{r} r^2 dr \int_1^{-1} e^{ikr \cos \theta} d \cos \theta \int_0^{2\pi} d\varphi \\ &= 2\pi \int_0^{\infty} \frac{1}{r} r^2 \frac{1}{ikr} (e^{ikr} - e^{-ikr}) dr = \frac{2\pi}{ik} \int_0^{\infty} (e^{ikr} - e^{-ikr}) dr \\ &= \frac{2\pi}{ik} \left(2i \int_0^{\infty} \sin kr dr \right) = \frac{4\pi}{k} \int_0^{\infty} \sin kr dr \\ \int_0^{\infty} \sin kr dr &= \lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} \left(\int_0^{\infty} e^{\varepsilon r} e^{ikr} dr \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} \left(\int_0^{\infty} e^{(\varepsilon+ik)r} dr \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} \left(\frac{e^{(\varepsilon+ik)\infty} - 1}{\varepsilon+ik} \right) \\ &= \frac{1}{k} \end{aligned}$$

则

$$\mathcal{F}\left(\frac{1}{r}\right) = \frac{1}{(2\pi)^{3/2}} \int \frac{1}{r} e^{ikr \cos \theta} dr = \frac{\sqrt{2}}{\sqrt{\pi}k} \int_0^\infty \sin kr dr = \frac{\sqrt{2}}{\sqrt{\pi}k^2}$$

(ii)

$$\begin{aligned} \int \frac{\delta(r-a)}{r} e^{-ikr \cos \theta} dr &= \int_0^\infty \frac{\delta(r-a)}{r} r^2 dr \int_1^{-1} e^{-ikr \cos \theta} d \cos \theta \int_0^{2\pi} d\varphi \\ &= -2\pi \int_0^\infty \frac{\delta(r-a)}{ik} (e^{ikr} - e^{-ikr}) dr = -\frac{2\pi}{ik} (e^{ika} - e^{-ika}) = \frac{\pi}{k} \sin ka \end{aligned}$$

则可得到

$$\mathcal{F}^{-1}\left(\frac{\sin ak}{k}\right) = \sqrt{\frac{\pi}{2}} \frac{\delta(r-a)}{r}$$

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根据周期性有:

$$\mathcal{L}(f(t-a)) = \mathcal{L}(f(t)u(t-a))$$

又由于

$$\mathcal{L}(f(t-a)) = \int_0^\infty \frac{f(t-a)}{e^{ap}} e^{-pt+ap} dt = e^{-ap} F(p)$$

$$\mathcal{L}(f(t)u(t-a)) = F(p) - \int_0^a f(t) e^{-pt} dt$$

上面两式相等即可得到

$$(1 - e^{-ap}) F(p) = \int_0^a f(t) e^{-pt} dt \implies F(p) = \frac{1}{1 - e^{-ap}} \int_0^a f(t) e^{-pt} dt$$

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(1)

$$pU(x, p) - u(x, 0) = a^2 \frac{\partial^2 U}{\partial x^2} + F(x, p)$$

$$\implies pU(x, p) - \varphi(x) = a^2 \frac{\partial^2 U}{\partial x^2} + F(x, p)$$

设 $U = \sum a_n \sinh nx$, 则

$$p \sum a_n \sinh nx = a^2 \sum n^2 a_n \sinh nx + F + \varphi$$

(2)

$$\frac{\partial U}{\partial t}(w, t) = -a^2 w^2 U(w, t) + F(w, t)$$

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先对 t 求拉普拉斯, $\mathcal{L}(u(x, t)) = U(x, p)$:

$$p^2 U(x, p) - pu(x, 0) - \frac{\partial u}{\partial t}(x, 0) = c^2 \frac{\partial^2 U}{\partial x^2}(x, p)$$

$$\implies p^2 U(x, p) - p\varphi(x) - \psi(x) = c^2 \frac{\partial^2 U}{\partial x^2}(x, p)$$

再对 x 求傅里叶, $\mathcal{F}(U(x, p)) = U^*(w, p)$:

$$p^2 U^*(w, p) - p\varphi^*(w) - \psi^*(w) = -c^2 w^2 U^*(w, p)$$

$$\implies U^*(w, p) = \frac{p\varphi^*(w) + \psi^*(w)}{p^2 + c^2 w^2}$$