数理方法 II 第二次作业

肖涵薄 31360164

2019 年 3 月 31 日

Q1

1. 取齐次方程,

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, (-\infty < x < +\infty, t > 0) \\
u|_{t=0} = e^{-2x^2}, (-\infty < x < +\infty) \\
\frac{\partial u}{\partial t}\Big|_{t=0} = \sin(x), (-\infty < x < +\infty)
\end{cases}
$$

由达朗贝尔公式:

$$u(x,t) = \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(x)\,\mathrm{d}x$$

$$u = \frac{1}{2}\left[e^{-2(x+at)^2} + e^{-2(x-at)^2}\right] + \frac{1}{2a}(\cos(x-at) - \cos(x+at))$$

$$u = \frac{1}{2}\left[e^{-2(x+at)^2} + e^{-2(x-at)^2}\right] + \frac{1}{a}\sin x \sin at$$

2. 取齐次初始条件,

$$
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + \cos(\omega t)\cos(x), (-\infty < x < +\infty, t > 0) \\
u|_{t=0} = 0, (-\infty < x < +\infty) \\
\frac{\partial u}{\partial t}\Big|_{t=0} = 0, (-\infty < x < +\infty)
\end{cases}
$$

$$
\begin{aligned}
w(x,t;\tau) = & \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} \cos(\omega\tau)\cos(\xi)\,\mathrm{d}\xi \\
= & \frac{1}{2a}\cos(\omega\tau)[\sin(x+a(t-\tau)) - \sin(x-a(t-\tau))]
\end{aligned}
$$

$$\begin{aligned} u(x,t) = & \int_0^t w(x,t;\tau)\,\mathrm{d}\tau \\ = & \tfrac{1}{2a}\cos x\left[\tfrac{\cos(at-a\tau+\omega\tau)}{a-\omega}+\tfrac{\cos(at-a\tau-\omega\tau)}{a+\omega}\right]\Bigg|_{\tau=0}^{\tau=t} \\ = & \cos x\tfrac{\cos(\omega t)-\cos(at)}{a^2-\omega^2} \end{aligned}$$

叠加 1,2 的 u:

$$u=\frac{1}{2}\left[e^{-2(x+at)^2}+e^{-2(x-at)^2}\right]+\frac{1}{a}\sin x\sin at+\cos x\frac{\cos(\omega t)-\cos(at)}{a^2-\omega^2}$$

Q2

设 $u = \sum X(x)T(t) + C_0$.

$$\begin{cases} XT'' = a^2X''T \\ X'(0)T = X'(l)T = 0 \\ XT(0) = e^{-x^2} \\ XT'(0) = 2axe^{-x^2} \end{cases}$$

代入边界条件,

$$\Longrightarrow \begin{matrix} X = X_0 \cos\left(\frac{n\pi x}{l}\right) \\ T = \left[T_1 \cos\left(\frac{an\pi}{l}t\right) + T_2 \sin\left(\frac{an\pi}{l}t\right)\right] \end{matrix}$$

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{l}\right)\left[C_n \cos\left(\frac{an\pi}{l}t\right) + D_n \sin\left(\frac{an\pi}{l}t\right)\right]$$

代入初始条件 1:

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{l}\right) = e^{-x^2}$$

$$C_0 = \frac{1}{l}\int_0^l e^{-x^2}dx = \frac{\sqrt{\pi}\,\mathrm{erf}(l)}{2l}$$

$$C_n = \frac{2}{l}\int_0^l e^{-x^2}\cos\left(\frac{n\pi}{l}x\right)dx = \frac{\sqrt{\pi}e^{-\frac{\pi^2n^2}{4l^2}}\left(\mathrm{erf}\left(l+\frac{i\pi n}{2l}\right) + \mathrm{erf}\left(l-\frac{i\pi n}{2l}\right)\right)}{2l}$$

代入初始条件 2:

$$\frac{\partial u}{\partial t}(x,0) = \sum_{0}^{\infty} \frac{an\pi D_n}{l}\cos\left(\frac{n\pi x}{l}\right) = 2axe^{-x^2}$$

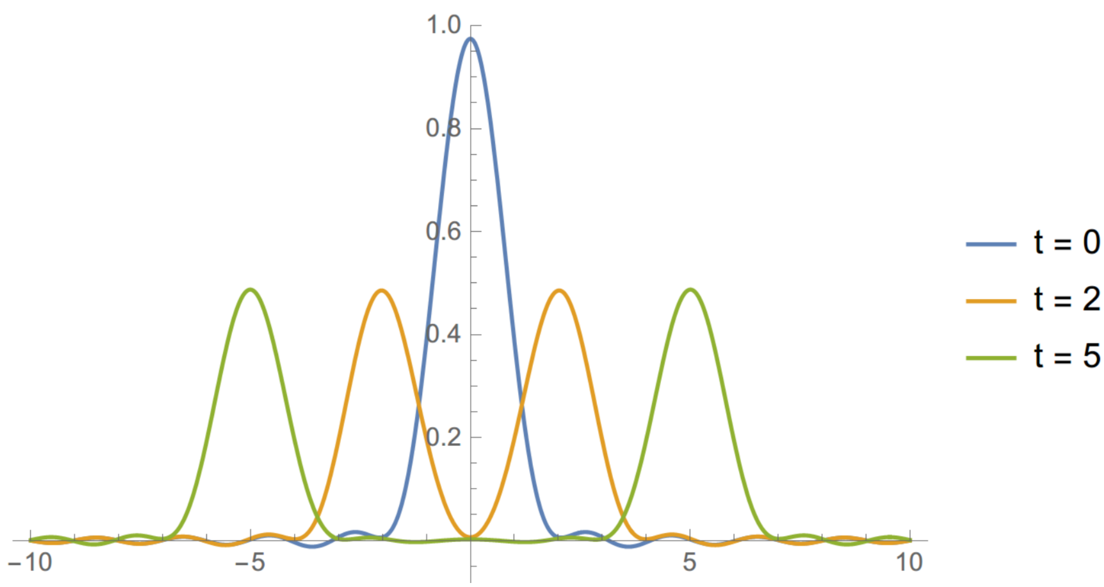
$$D_n = \frac{4}{n\pi}\int_0^l xe^{-x^2}\cos\frac{n\pi}{L}x\,\mathrm{d}x$$

$$\begin{aligned} D_n = \quad & \frac{e^{-l^2-i\pi n}}{2\pi ln}\Bigg\{i\pi^{3/2}ne^{\frac{\left(2l^2+i\pi n\right)^2}{4l^2}}\left(-\,\mathrm{erf}\left(l+\frac{i\pi n}{2l}\right) + \mathrm{erf}\left(l-\frac{i\pi n}{2l}\right) + 2i\,\mathrm{erfi}\left(\frac{\pi n}{2l}\right)\right) \\ & -2l\left(-2e^{l^2+i\pi n} + e^{2i\pi n} + 1\right)\Bigg\} \end{aligned}$$

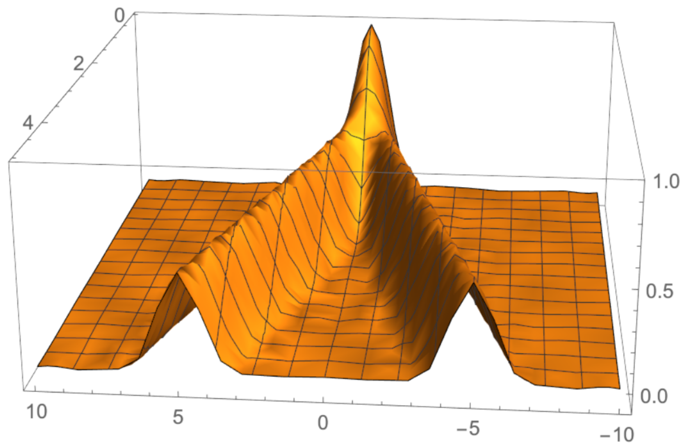
最后可以得到 $u(x,t)$:

$$\boxed{\begin{aligned} u(x,t) = \;& \frac{\sqrt{\pi}\operatorname{erf}(l)}{2l} + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{l}\right)\Bigg\{ \\ & \frac{\sqrt{\pi}e^{-\frac{\pi^2 n^2}{4l^2}}\left(\operatorname{erf}\left(l+\frac{i\pi n}{2l}\right)+\operatorname{erf}\left(l-\frac{i\pi n}{2l}\right)\right)}{2l}\cos\left(\frac{an\pi}{l}t\right)\frac{e^{-l^2-i\pi n}}{2\pi l n}\Bigg[\\ & +i\pi^{3/2}ne^{\frac{\left(2l^2+i\pi n\right)^2}{4l^2}}\left(-\operatorname{erf}\left(l+\frac{i\pi n}{2l}\right)+\operatorname{erf}\left(l-\frac{i\pi n}{2l}\right)+2i\operatorname{erfi}\left(\frac{\pi n}{2l}\right)\right) \\ & -2l\left(-2e^{l^2+i\pi n}+e^{2i\pi n}+1\right)\Bigg]\sin\left(\frac{an\pi}{l}t\right)\Bigg\} \\ & \text{其中包含误差函数:} \\ \operatorname{erf}(x) = \;& \frac{1}{\sqrt{\pi}}\int_{-x}^{x}e^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^2}\,\mathrm{d}t \\ \operatorname{erfi}(x) = \;& -i\operatorname{erf}(ix). \end{aligned}}$$

数值模拟, 代入 $a=1, l=100, n=1\to 100$, 当 $t=0,2,5$, 可以看到波向两端传播的过程:



如图为时间连续变化波的传递状况:


0
2
4
1.0
0.5
0.0
10
5
0
−5
−10

Q3

代入球坐标系,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

并做代换

$$v\left(r,t\right)=ru\left(r,t\right)$$

该函数满足一维波动方程解, $m\left(r\right)$ 为单位阶跃函数.

$$\begin{cases}\dfrac{\partial^2 v}{\partial t^2}=a^2\dfrac{\partial^2 v}{\partial r^2}, \quad r,t>0\\ v\left(r,0\right)=ru\left(0\right)m\left(R-r\right)\\ \dfrac{\partial v}{\partial t}\left(r,0\right)=0\end{cases}$$

$$v\left(r,t\right)=\frac{1}{2}u\left(0\right)\left[\left(r+at\right)m\left(R-r-at\right)+\left(r-at\right)m\left(R-r+at\right)\right]$$

回代 u, 通解为 ($m\left(r\right)$ 为单位阶跃函数.)

$$u\left(r,t\right)=\frac{u\left(0\right)}{2r}\left[\left(r+at\right)m\left(R-r-at\right)+\left(r-at\right)m\left(R-r+at\right)\right], \quad r=\sqrt{x^2+y^2+z^2}$$

Q4

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = bx(l-x)/l^2 \end{cases}$$

设 $u = X(x)T(t)$. $X = X_0 \sin\left(\frac{n\pi x}{l}\right)$, $T = T_0 \exp[-\left(\frac{na\pi}{l}\right)^2 t]$, 则

$$u(x,t) = \sum_1^\infty C_n \sin\left(\frac{n\pi x}{l}\right) \exp[-\left(\frac{na\pi}{l}\right)^2 t]$$

初始条件为:

$$u(x,0) = \sum_1^\infty C_n \sin\left(\frac{n\pi x}{l}\right) = \frac{b}{l^2} x(l-x)$$

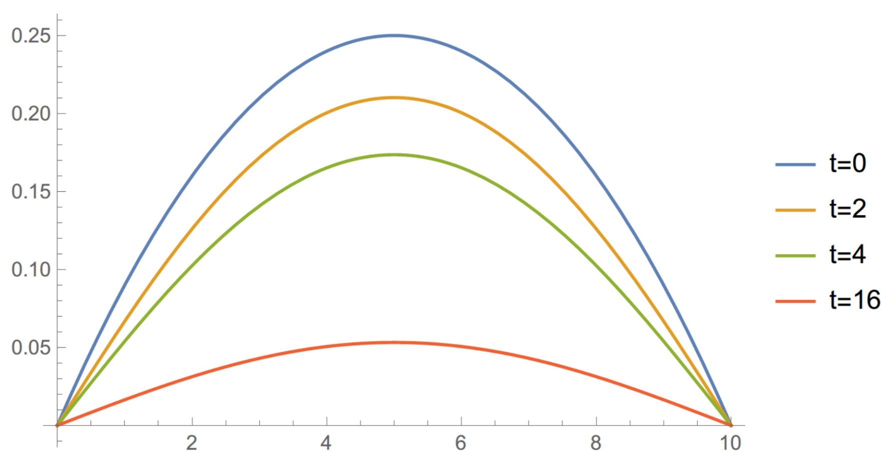
展开初始条件:

$$C_n = \frac{2b}{l^3} \int_0^l x(l-x) \sin\frac{n\pi x}{l} dx = -\frac{4b((-1)^n - 1)}{\pi^3 n^3}$$

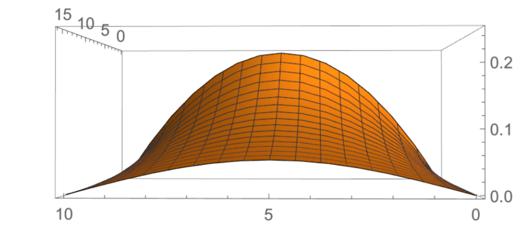
最后可得:

$$u(x,t) = \sum_1^\infty -\frac{4b((-1)^n - 1)}{\pi^3 n^3} \sin\left(\frac{n\pi x}{l}\right) \exp[-\left(\frac{na\pi}{l}\right)^2 t]$$

取 $a = b = 1, l = 10$, 各时间点图像为:



从 $t = 0$ 到 $t = 16$ 连续演化为:


15
10
5
0
0.2
0.1
0.0
10
5
0

Q5

(1)

$$a_1(x)\frac{\partial^2 u}{\partial x^2}+b_1(y)\frac{\partial^2 u}{\partial y^2}+a_2(x)\frac{\partial u}{\partial x}+b_2(y)\frac{\partial u}{\partial y}=0$$

设 $u=X(x)Y(y)$, 则上式变为:

$$a_1X''/X+b_1Y''/Y+a_2X'/X+b_2Y'/Y=0$$

由函数间无关性 (c 为任意常数):

$$a_1X''/X+a_2X'/X=c$$

$$\implies \boxed{a_1(x)X''(x)+a_2(x)X'(x)+cX(x)=0}$$

同理可得 y 的方程

$$b_1Y''/Y+b_2Y'/Y=-c$$

$$\implies \boxed{b_1(y)Y''(y)+b_2(y)Y'(y)-cY(y)=0}$$

(2)

原式 $=\Delta u(\rho,\varphi)=0$, 设 $u=R(\rho)\Psi(\varphi)$, 原式可化为

$$\frac{R''\rho^2}{R}+\frac{R'\rho}{R}+\frac{\Psi''}{\Psi}=0$$

设常数 c, 即

$$\frac{R''\rho^2}{R}+\frac{R'\rho}{R}=\boxed{c\implies \rho^2R''+\rho R'=cR}$$

$$\frac{\Psi''}{\Psi}=-c\implies \boxed{\Psi''+c\Psi=0}$$

(3)

设 $u(r,\theta)=R(r)\Theta(\theta)$. 原式可化为

$$\frac{2R'r}{R}+\frac{R''r^2}{R}+\frac{\Theta'}{\Theta\tan\theta}+\frac{\Theta''}{\Theta}=0$$

设常数 c, 即

$$\frac{2R'r}{R}+\frac{R''r^2}{R}=c \implies \boxed{r^2R''(r)+2rR'(r)=cR(r)}$$

$$\frac{\Theta'}{\Theta\tan\theta}+\frac{\Theta''}{\Theta}=-c \implies \boxed{\Theta'(\theta)+\tan\theta\Theta''(\theta)+c\tan\theta\Theta(\theta)=0}$$

Q6

当 $\lambda \leq 0$, 仅有 $X=0$ 的平庸解. 由泛定方程, $X_i = A_i \cos\left(\sqrt{\lambda_i}x\right) + B_i \sin\left(\sqrt{\lambda_i}x\right)$. 由 0 处边界条件,

$$\alpha_1 A_i + \beta_1 \sqrt{\lambda_i} B_i = 0 \implies A_i = -\frac{\beta_1 \sqrt{\lambda_i} B_i}{\alpha_1}$$

再代入 l 处边界条件,

$$\begin{gathered} -\frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\alpha_2 \cos\left(\sqrt{\lambda_i}l\right) + \alpha_2 \sin\left(\sqrt{\lambda_i}l\right) \\ +\frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\beta_2\sqrt{\lambda_i} \sin\left(\sqrt{\lambda_i}l\right) + \beta_2\sqrt{\lambda_i} \cos\left(\sqrt{\lambda_i}l\right) = 0 \\ \implies \left(\alpha_2 + \frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\beta_2\sqrt{\lambda_i}\right) \sin\left(\sqrt{\lambda_i}l\right) + \left(\beta_2 - \frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\alpha_2\right) \cos\left(\sqrt{\lambda_i}l\right) = 0 \end{gathered}$$

可用辅助角公式化为:

$$\sin\left(\sqrt{\lambda_i}l + \phi\right) = 0, \quad \phi = \arctan \frac{\beta_2 - \frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\alpha_2}{\alpha_2 + \frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\beta_2\sqrt{\lambda_i}}$$

即解为:

$$\boxed{\sqrt{\lambda_i}l + \arctan \frac{\beta_2 - \frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\alpha_2}{\alpha_2 + \frac{\beta_1\sqrt{\lambda_i}}{\alpha_1}\beta_2\sqrt{\lambda_i}} = i\pi, \quad i = 1, 2, 3...}$$

取 $\beta_1 = \beta_2 = 0, l = 1, i = 1$, 上式可化简为:

$$\sqrt{\lambda_1} = \pi$$

$$\begin{cases} \lambda_1 = & \pi^2 \\ A_1 = & 0 \end{cases}$$

$$\implies X_1 = B_1 \sin(\pi x)$$

取 $\beta_1 = \beta_2 = 0, l = 1, i = 2$, 上式可化简为:

$$\sqrt{\lambda_2} = 2\pi$$

$$\begin{cases} \lambda_2 = & 4\pi^2 \\ A_2 = & 0 \end{cases}$$

$$\implies X_2 = B_2 \sin(2\pi x)$$

归一化:

$$\int_0^1 X_1^2 \,\mathrm{d}x = B_1^2 \int_0^1 \sin^2(\pi x) \,\mathrm{d}x = \frac{B_1^2}{2} = 1 \implies B_1 = \sqrt{2}$$

$$\int_0^1 X_2^2 \,\mathrm{d}x = B_2^2 \int_0^1 \sin^2(2\pi x) \,\mathrm{d}x = \frac{B_2^2}{2} = 1 \implies B_2 = \sqrt{2}$$

正交性:

$$\boxed{\int_0^1 X_1 X_2 \,\mathrm{d}x = 2 \int_0^1 \sin(\pi x) \sin(2\pi x) \,\mathrm{d}x = 0}$$

Q7

换用极坐标系,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2}=6\rho^2+6\rho^2\sin 2\theta$$

$$u\left(a,\theta\right)=1,\quad \frac{\partial u}{\partial\rho}\left(b,\theta\right)=0$$

设

$$u\left(\rho,\theta\right)=\sum_{n=0}^{\infty}\left\{A_n\left(\rho\right)\cos n\theta+B_n\left(\rho\right)\sin n\theta\right\}$$

泛定方程可以化为:

$$\Delta u\left(\rho,\theta\right)=\frac{1}{\rho}\sum[A_n'\cos n\theta+B_n'\sin n\theta]+\sum[A_n''\cos n\theta+B_n''\sin n\theta]-\frac{1}{\rho^2}\sum[n^2A_n\cos n\theta+n^2B_n\sin n\theta]$$

$$=\sum\left\{\left[\frac{A_n'}{\rho}+A_n''-\frac{n^2A_n}{\rho^2}\right]\cos n\theta+\left[\frac{B_n'}{\rho}+B_n''-\frac{n^2B_n}{\rho^2}\right]\sin n\theta\right\}=6\rho^2\left(1+\sin 2\theta\right)$$

第 0,2 阶为:

$$\begin{aligned}\tfrac{A_0'}{\rho}+A_0''= &\quad 6\rho^2 \implies A_0=\frac{3\rho^4}{8}+c_0\ln\rho+d_0\\ \tfrac{B_2'}{\rho}+B_2''-\tfrac{4B_2}{\rho^2}= &\quad 6\rho^2 \implies B_2=m\rho^2+n\rho^{-2}+\frac{1}{2}\rho^4\end{aligned}$$

其余项均为齐次方程, 其通解为:

$$\begin{aligned}B_0\left(\rho\right)&=c_0'+d_0'\ln\rho &&(n=0)\\ A_n\left(\rho\right)&=c_n\rho^n+d_n\rho^{-n} &&(n\neq 0)\\ B_n\left(\rho\right)&=c_n'\rho^n+d_n'\rho^{-n} &&(n\neq 2)\end{aligned}$$

代入边界条件:

$$A_n\left(\rho\right)=0\implies c_n=d_n=0,\ (n\neq 0)$$

$$B_n\left(\rho\right)=0\implies c_n'=d_n'=0,\ (n\neq 2)$$

由于

$$u\left(a,\theta\right)=\sum_{n=0}^{\infty}\left\{A_n\left(a\right)\cos n\theta+B_n\left(a\right)\sin n\theta\right\}=1$$

代入 A_0, B_2 表达式:

$$A_0\left(a\right)=\frac{3a^4}{8}+c_0\ln a+d_0=1$$

$$B_2\left(a\right)=ma^2+na^{-2}+\frac{1}{2}a^4=0$$

又由于:

$$\frac{\partial u}{\partial \rho}(b,\theta)=\sum_{n=0}^{\infty}\{A_n'(b)\cos n\theta+B_n'(b)\sin n\theta\}=0$$

代入 A_0', B_2' 表达式

$$\begin{aligned}A_0' &= \frac{3\rho^3}{2}+\frac{c_0}{\rho}\\ B_2' &= 2m\rho-2n\rho^{-3}+2\rho^3\end{aligned}$$

即得:

$$\begin{aligned}A_0'(b) &= \frac{3b^3}{2}+\frac{c_0}{b}=0\\ B_2'(a) &= 2mb-2nb^{-3}+2b^3=0\end{aligned}$$

$$c_0=-\frac{3}{2}b^4, \qquad d_0=1-\frac{3a^4}{8}+\frac{3}{2}b^4\ln a$$

$$m=-\frac{a^6+2b^6}{2\left(a^4+b^4\right)}, \quad n=-\frac{a^6b^4-2a^4b^6}{2\left(a^4+b^4\right)}$$

$$\begin{aligned}u(\rho,\theta)= &\left(\frac{3\rho^4}{8}-\frac{3}{2}b^4\ln\rho+1-\frac{3a^4}{8}+\frac{3}{2}b^4\ln a\right)\\ &+\left(-\frac{a^6+2b^6}{2\left(a^4+b^4\right)}\rho^2-\frac{a^6b^4-2a^4b^6}{2\left(a^4+b^4\right)}\rho^{-2}+\frac{1}{2}\rho^4\right)\sin 2\theta\end{aligned}$$