数理方法 II 第四次作业

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两边 fourier 变换后:

$$-(k_0^2+k^2)\widetilde{G}=-\frac{1}{\varepsilon_0}$$

因此

$$\widetilde{G}=\frac{\frac{1}{\varepsilon_0}}{k_0^2+k^2}$$

求逆变换:

$$G=\frac{1}{(2\pi)^3\varepsilon_0}\iiint\frac{e^{ikr\cos\theta}}{k_0^2+k^2}\,\mathrm{d}\boldsymbol{k_0}$$

化简:

$$
\begin{aligned}
G =&\frac{1}{(2\pi)^3\varepsilon_0}\iiint \frac{e^{ikr\cos\theta}}{k_0^2+k^2}\,\mathrm{d}\boldsymbol{k_0} \\
=&\frac{1}{(2\pi)^3\varepsilon_0}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}k_0^2\sin\theta\frac{e^{ikr\cos\theta}}{k_0^2+k^2}\,\mathrm{d}k_0 \\
=&\frac{1}{(2\pi)^2\varepsilon_0}\int_0^{\pi}\mathrm{d}\theta\int_0^{\infty}k_0^2\sin\theta\frac{e^{ikr\cos\theta}}{k_0^2+k^2}\,\mathrm{d}k_0 \\
=&\frac{1}{i(2\pi)^2r\varepsilon_0}\int_0^{\infty}\frac{2k\sin kr}{k_0^2+k^2}\,\mathrm{d}k_0 \\
=&\frac{1}{i(2\pi)^2r\varepsilon_0}\int_{-\infty}^{\infty}\frac{ke^{ikr}}{k_0^2+k^2}\,\mathrm{d}k_0 \\
=&\frac{1}{i(2\pi)^2r\varepsilon_0}\int_{-\infty}^{\infty}\frac{ke^{ikr}}{k_0^2+k^2}\,\mathrm{d}k_0 \\
=&\frac{1}{i(2\pi)^2r\varepsilon_0}2\pi i\,\mathrm{Res}\left[\frac{ze^{izr}}{z^2+k^2},z_0\right] \\
&\text{在}z_0=ik_0\text{取一阶极点} \\
=&\frac{1}{4\pi r\varepsilon_0}e^{-kr}
\end{aligned}
$$

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根据定义

$$
K_\alpha(x)=\frac{\pi}{2}\frac{I_{-\alpha}(x)-I_\alpha(x)}{\sin(\alpha\pi)} \tag{1}
$$

且

$$
I_\alpha(x)=i^{-\alpha}J_\alpha(ix) \tag{2}
$$

将 (2) 代入 (1) 式,

$$
K_\alpha(x)=\frac{\pi}{2}\frac{i^{\alpha}J_{-\alpha}(ix)-i^{-\alpha}J_\alpha(ix)}{\sin(\alpha\pi)}
$$

对比 $H_\alpha^{(1)}(x)$ 的定义

$$
H_\alpha^{(1)}(x)=\frac{J_{-\alpha}(x)-e^{-\alpha\pi i}J_\alpha(x)}{i\sin(\alpha\pi)} \tag{3}
$$

可以发现

$$
K_\alpha(x)=\frac{\pi}{2}i^{\alpha+1}H_\alpha^{(1)}(ix)
$$

现将

$$
N_\alpha(x)=\frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}
$$

与 (3) 式对比, 可以发现 $H_\alpha^{(1)}(ix) = J_\alpha(ix) + iN_\alpha(ix)$. 因此有

$$K_\alpha(x) = \frac{\pi}{2} i^{\alpha+1} H_\alpha^{(1)}(ix) = J_\alpha(ix) + iN_\alpha(ix)$$

$\square$

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(1) $\int xJ_2(x)\,\mathrm{d}x$

由递推式得

$$J_2(x) = -J_0(x) + \frac{2}{x} J_1(x)$$

又由于

$$\int xJ_0(x)\,\mathrm{d}x = xJ_1(x), \quad \int J_1(x)\,\mathrm{d}x = -J_0(x)$$

从而

$$\int xJ_2(x)\,\mathrm{d}x = -xJ_1(x) - 2J_0(x)$$

(2)$\int x^4 J_1(x)\,\mathrm{d}x$

$$\begin{aligned}\int x^4 J_1(x)\,\mathrm{d}x =& x^4 J_2(x) - 2\int x^3 J_2(x)\,\mathrm{d}x \\ =& x^4 J_2(x) - 2\int \mathrm{d}(x^3 J_3(x)) \\ =& x^4 J_2(x) - 2x^3 J_3(x)\end{aligned}$$

(3)$\int_0^R J_0(x)\cos x\,\mathrm{d}x$

$$\begin{aligned}\int_0^R J_0(x)\cos x\,\mathrm{d}x =& xJ_0(x)\cos x\Big|_0^R - \int_0^R x(-J_1\cos x - J_0\sin x)\,\mathrm{d}x \\ =& xJ_0(x)\cos x\Big|_0^R + \int_0^R xJ_1\cos x\,\mathrm{d}x + \int_0^R \sin x\,\mathrm{d}(xJ_1) \\ =& xJ_0(x)\cos x\Big|_0^R + \int_0^R xJ_1\cos x\,\mathrm{d}x + \left(xJ_1\sin x\Big|_0^R - \int_0^R xJ_1\cos x\,\mathrm{d}x\right) \\ =& xJ_0(x)\cos x\Big|_0^R + xJ_1\sin x\Big|_0^R \\ =& RJ_0(R)\cos R + RJ_1(R)\sin R\end{aligned}$$

(4)$3J_0'(x)+4J_0'''(x)$

$$J_0'=-J_1$$

$$J_0'''=-J_1''=-\frac{1}{2}\left(J_0'-J_2'\right)=\frac{1}{2}J_1+\frac{1}{2}J_2'=\frac{1}{2}J_1+\frac{1}{4}(J_1-J_3)=\frac{3}{4}J_1(x)-\frac{1}{4}J_3(x)$$

因此

$$3J_0'(x)+4J_0'''(x)=-3J_1(x)+3J_1(x)-J_3(x)=J_3(x)$$

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$$\begin{aligned} W(J_v,J_{-v})=&\begin{vmatrix} J_v & J_{-v} \\ J_v' & J_{-v}' \end{vmatrix} \\ =&\begin{vmatrix} J_v & J_{-v} \\ \frac{1}{2}\left[J_{\nu-1}(x)-J_{\nu+1}(x)\right] & \frac{1}{2}\left[J_{-\nu-1}(x)-J_{-\nu+1}(x)\right] \end{vmatrix} \\ =&J_vJ_{-v-1}-J_vJ_{-v+1}-J_{-v}J_{v-1}+J_{-v}J_{v+1} \end{aligned}$$

由 Bessel 方程,

$$\frac{d}{dx}\left[x\frac{dJ_v(x)}{dx}\right]+x\left(1-\frac{v^2}{x^2}\right)J_v(x)=0$$

$$\frac{d}{dx}\left[x\frac{dJ_{-v}(x)}{dx}\right]+x\left(1-\frac{v^2}{x^2}\right)J_{-v}(x)=0$$

可得

$$J_{-v}(x)\frac{d}{dx}\left[x\frac{dJ_v(x)}{dx}\right]-J_v(x)\frac{d}{dx}\left[x\frac{dJ_{-v}(x)}{dx}\right]=0$$

即

$$\frac{d}{dx}\left\{x\left[J_{-v}(x)J_v'(x)-J_v(x)J_{-v}'(x)\right]\right\}=0\implies x\left[J_{-v}(x)J_v'(x)-J_v(x)J_{-v}'(x)\right]=C$$

由 Bessel 函数表达式可确认 C:

$$C=\frac{1}{\Gamma(-v+1)}\frac{v}{\Gamma(v+1)}-\frac{1}{\Gamma(v+1)}\frac{-v}{\Gamma(-v+1)}=\frac{2v}{\Gamma(v+1)\Gamma(-v+1)}=\frac{2}{\Gamma(v)\Gamma(1-v)}=\frac{2\sin\pi v}{\pi}$$

因此有

$$W\left(J_v,J_{-v}\right)=J_v(x)J_{-v}'(x)-J_{-v}(x)J_v'(x)=-\frac{C}{x}=-\frac{2\sin\pi v}{\pi x}$$

$$W\left(J_v, Y_v\right)=\left|\begin{array}{cc} J_v & Y_v \\ J_v^{\prime} & Y_v^{\prime} \end{array}\right|=\cot \pi v\left|\begin{array}{cc} J_v & J_v \\ J_v^{\prime} & J_v^{\prime} \end{array}\right|-\frac{1}{\sin \pi \nu}\left|\begin{array}{cc} J_v & J_{-v} \\ J_v^{\prime} & J_{-\nu}^{\prime} \end{array}\right|=\frac{2}{\pi x}$$

(1)

将

$$W\left(J_v, J_{-v}\right)=J_v(x) J_{-v}^{\prime}(x)-J_{-v}(x) J_v^{\prime}(x)=-\frac{C}{x}=-\frac{2 \sin \pi v}{\pi x}$$

两边同时乘 $-\frac{\pi}{2 \sin \pi \nu} \frac{1}{J_v^2(x)}$,

$$\frac{1}{x J_v^2(x)}=-\frac{\pi}{2 \sin \pi v} \frac{J_v(x) J_{-\nu}^{\prime}(x)-J_{-v}(x) J_v^{\prime}(x)}{J_v^2(x)}=-\frac{\pi}{2 \sin \pi v} \frac{d}{d x} \frac{d_{-v}(x)}{J_v(x)}$$

因此

$$\int \frac{d x}{x J_v^2(x)}=-\frac{\pi}{2 \sin \pi v} \int d \frac{J_{-v}(x)}{J_v(x)}=-\frac{\pi}{2 \sin \pi v} \frac{J_{-v}(x)}{J_v(x)}+C=\frac{\pi}{2} \frac{Y_v(x)}{J_v(x)}+C^{\prime}$$

(2)

由 $W\left(J_v, Y_v\right)$ 可得

$$\frac{1}{x Y_v^2(x)}=-\frac{\pi}{2} \frac{d}{d x} \frac{J_v(x)}{Y_v(x)}$$

因此

$$\int \frac{d x}{x Y_v^2(x)}=-\frac{\pi}{2} \frac{J_v(x)}{Y_v(x)}+C$$

(3)

同理

$$\frac{1}{x J_v(x) Y_v(x)}=\frac{\pi}{2}\left[\frac{Y_v^{\prime}(x)}{Y_v(x)}-\frac{J_v^{\prime}(x)}{J_v(x)}\right]$$

积分得

$$\int \frac{d x}{x J_v(x) Y_v(x)}=\frac{\pi}{2} \int\left[\frac{Y_v^{\prime}(x)}{Y_v(x)}-\frac{J_v^{\prime}(x)}{J_v(x)}\right] d x=\frac{\pi}{2} \ln \frac{Y_v(x)}{J_v(x)}+C$$

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令 $x=\beta z^\gamma, u=z^\alpha y$,

$$\frac{d u}{d z}=\alpha z^{\alpha-1} y+z^\alpha \frac{d y}{d z}=\alpha z^{\alpha-1} y+\beta \gamma z^{\alpha+\gamma-1} \frac{d y}{d x}=\alpha z^{\alpha-1} y+\gamma x z^{\alpha-1} \frac{d y}{d x}$$

$$\begin{aligned} \frac{d^2 u}{d z^2} & =\alpha(\alpha-1) z^{\alpha-2} y+\alpha z^{\alpha-1} \frac{d y}{d z}+\beta \gamma(\alpha+\gamma-1) z^{\alpha+\gamma-2} \frac{d y}{d x}+\beta \gamma z^{\alpha+\gamma-1} \frac{d}{d z} \frac{d y}{d x} \\ & =\alpha(\alpha-1) z^{\alpha-2} y+\gamma(2 \alpha+\gamma-1) x z^{\alpha-2} \frac{d y}{d x}+\gamma^2 x^2 z^{\alpha-2} \frac{d^2 y}{d x^2} \end{aligned}$$

代入方程可化简为

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-v^2\right)y=0$$

这是 v 阶 Bessel 方程, 通解为

$$y=c_1J_v(x)+c_2Y_v(x)$$

因此

$$u=z^\alpha y=c_1z^\alpha J_v(x)+c_2z^\alpha Y_v(x)=c_1z^\alpha J_v\left(\beta z^\gamma\right)+c_2z^\alpha Y_v\left(\beta z^\gamma\right)$$

1\.

令 $\alpha=\frac{1}{2},\quad \beta=\frac{2\sqrt{a}}{b+2},\gamma=\frac{b}{2}+1,\quad v=\frac{1}{b+2}$, 变为上述通解, 因此其解为

$$u=c_1\sqrt{z}J_{\frac{1}{b+2}}\left(\frac{2\sqrt{a}}{b+2}z^{\frac{b}{2}+1}\right)+c_2\sqrt{z}Y_{\frac{1}{b+2}}\left(\frac{2\sqrt{a}}{b+2}z^{\frac{b}{2}+1}\right)$$

2\.

令 $\alpha=2,\quad \beta=1,\quad \gamma=1,\quad v=2$, 解为

$$u=c_1zJ_{1/2}\left(z^2\right)+c_2zY_{1/2}\left(z^2\right)$$

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问题为

$$\left\{\begin{array}{c}\frac{\partial u}{\partial t}-\kappa\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)=0\\ u\big|_{\rho=0}\text{有界},\ \frac{\partial u}{\partial\rho}\Big|_{\rho=a}=0\\ u\big|_{t=0}=u_0\left(1-\frac{\rho^2}{a^2}\right)\end{array}\right.$$

分离变量得

$$\left\{\begin{array}{l}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\mathrm{P}}{d\rho}\right)+\lambda^2\mathrm{P}=0\\ \mathrm{P}(0)\text{有界},\ P'(a)=0\end{array}\right.,\quad T'+\lambda^2\kappa T=0$$

解本征值问题得 $\lambda_0=0,\quad \lambda_i=\frac{\mu_i'}{a}$,

$$\mathrm{P}_0(\rho)=A_0,\quad \mathrm{P}_i(\rho)=J_0\left(\frac{\mu_i'}{a}\rho\right)$$

$$T(t)=A_i\exp\left[-\kappa\left(\frac{\mu_i}{a}\right)^2t\right]$$

所以

$$u(\rho,t)=A_0+\sum_{i=1}^{\infty}A_iJ_0\left(\frac{\mu_i'}{a}\rho\right)\exp\left[-\kappa\left(\frac{\mu_i}{a}\right)^2t\right]$$

$$u|_{t=0}=A_0+\sum_{i=1}^{\infty}A_iJ_0\left(\frac{\mu_i'}{a}\rho\right)=u_0\left(1-\frac{\rho^2}{a^2}\right)$$

$$A_0=\frac{2u_0}{a^2}\int_0^a\left(1-\frac{\rho^2}{a^2}\right)\rho d\rho=\frac{u_0}{2}$$

$$A_i=-\frac{4u_0}{\mu'^2J_0\left(\mu_i'\right)}$$

最后得到

$$u(\rho,t)=\frac{u_0}{2}-4u_0\sum_{i=1}^{\infty}\frac{1}{\mu'^2J_0\left(\mu_i'\right)}J_0\left(\frac{\mu_i'}{a}\rho\right)\exp\left[-\kappa\left(\frac{\mu_i}{a}\right)^2t\right]$$

稳态为

$$u\to\frac{u_0}{2}$$

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分离变量得本征值问题

$$\begin{cases}Z''+k^2Z=0\\Z(0)=0,Z(h)=0\end{cases}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\mathrm{P}}{d\rho}\right)-k^2\mathrm{P}=0$$

解得本征函数 $Z_n(z)=\sin\frac{n\pi}{h}z$,$\mathrm{P}_n(\rho)=I_0\left(\frac{n\pi}{h}\rho\right)$.

$$u(\rho,z)=\sum_{n=1}^{\infty}A_nI_0\left(\frac{n\pi}{h}\rho\right)\sin\frac{n\pi}{h}z$$

根据边界条件得

$$A_2=\frac{u_0}{I_0\left(\frac{2\pi a}{h}\right)},A_{n\neq2}=0$$

即

$$u(\rho,z)=\frac{u_0}{I_0\left(\frac{2\pi a}{h}\right)}I_0\left(\frac{2\pi}{h}\rho\right)\sin\frac{2\pi}{h}z$$

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$$
\begin{aligned}
&\int_{-1}^{1}(1+x)^k P_l(x)dx = \frac{1}{2^l l!}\int_{-1}^{1}(1+x)^k \frac{d^l}{dx^l}\left(x^2-1\right)^l dx \\
&= \frac{1}{2^l l!}(1+x)^k \frac{d^{l-1}}{dx^{l-1}}\left(x^2-1\right)^l\Big|^1 - \frac{1}{2^l l!}\int_{-1}^{1}\frac{d(1+x)^k}{dx}\frac{d^{l-1}}{dx^{l-1}}\left(x^2-1\right)^l dx \\
&= -\frac{1}{2^l l!}\int_{-1}^{1}\frac{d(1+x)^k}{dx}\frac{d^{l-1}}{dx^{l-1}}\left(x^2-1\right)^l dx = \cdots = \frac{1}{2^l l!}\int_{-1}^{1}\left(1-x^2\right)^l\frac{d^l(1+x)^k}{dx^l}dx \qquad (4) \\
&= \frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}\left(1-x^2\right)^l(1+x)^{k-l}dx = \frac{k!}{2^l l!(k-l)!}\int_{-1}^{1}(1-x)^l(1+x)^k dx
\end{aligned}
$$

作代换 $1+x=2t$

$$
\begin{aligned}
&= \frac{2^{k+1}k!}{l!(k-l)!}\int_0^1(1-t)^l t^k dt = \frac{2^{k+1}k!}{l!(k-l)!}\mathrm{B}(l+1,k+1) \\
&= \frac{2^{k+1}k!}{l!(k-l)!}\frac{\Gamma(l+1)\Gamma(k+1)}{\Gamma(l+k+2)} = \frac{2^{k+1}(k!)^2}{(k-l)!(l+k+1)!}
\end{aligned}
$$

$k<l$ 时根据 (4) 式, 积分为 0.

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$$u=\sum_{n=0}^{\infty}(a_n r^n+\frac{b_n}{r^{n+1}})P_n(\cos\theta)$$

$$u|_{r=a}=\sum_{l=0}^{\infty}\left(A_i a^l+\frac{B_l}{a^{l+1}}\right)P_l(\cos\theta)=u_0P_0(\cos\theta)$$

$$u|_{r=b}=\sum_{l=0}^{\infty}\left(A_l b^l+\frac{B_l}{b^{l+1}}\right)P_l(\cos\theta)=u_0\cos^2\theta=\frac{1}{3}u_0P_0(\cos\theta)+\frac{2}{3}u_0P_2(\cos\theta$$

比较系数得

$$A_0=\frac{b-3a}{3(b-a)}u_0, B_0=\frac{2ab}{3(b-a)}u_0, A_2=\frac{2b^3}{3\left(b^5-a^5\right)}u_0, B_2=\frac{2a^5b^3}{3\left(a^5-b^5\right)}u_0$$

因此

$$u(r,\theta)=\frac{b-3a}{3(b-a)}u_0+\frac{2b}{3(b-a)}\frac{a}{r}u_0+\frac{2b^3a^2u_0}{3\left(b^5-a^5\right)}\left[\left(\frac{r}{a}\right)^2-\left(\frac{a}{r}\right)^3\right]P_2(\cos\theta)$$

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(1)

$$(\sin\theta - 2\cos^2\theta)\cos^2\varphi = \frac{1}{2}(\sqrt{1-\cos^2\theta} - 2\cos^2\theta) + \frac{1}{2}(\sqrt{1-\cos^2\theta} - 2\cos^2\theta)\cos 2\varphi$$

上式右端第一项需展开为 P_l^0, 第二项需展开为 P_l^2

$$\begin{aligned}\sqrt{1-\cos^2\theta} &= \sum_0^\infty A_l P_l^0(\cos\theta), \quad A_l = \frac{2l+1}{2}\int_{-1}^1 \sqrt{1-x^2}P_l^0\,\mathrm{d}x \\ &= \sum_0^\infty B_l P_l^2(\cos\theta), \quad B_l = \frac{(2l+1)(l+2)!}{2(l-2)!}\int_{-1}^1 \sqrt{1-x^2}P_l^2\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\cos^2\theta =& \frac{2}{3}P_2^0(\cos\theta) + \frac{1}{3}P_0^0(\cos\theta) \\ =& \sum_0^\infty C_l P_l^2(\cos\theta), \quad C_l = \frac{(2l+1)(l+2)!}{2(l-2)!}\int_{-1}^1 x^2 P_l^2\,\mathrm{d}x\end{aligned}$$

于是有

$$\begin{gathered}(\sin\theta - 2\cos^2\theta)\cos^2\varphi = \\ \frac{1}{2}(\sum_0^\infty A_l P_l^0(\cos\theta) - 2(\frac{2}{3}P_2^0(\cos\theta) + \frac{1}{3}P_0^0(\cos\theta)))\Phi^0 \\ +\frac{1}{2}(\sum_0^\infty B_l P_l^2(\cos\theta) - 2\sum_0^\infty C_l P_l^2(\cos\theta))\Phi^2 \\ = \frac{1}{2}\sum_0^\infty A_l Y_l^0 - \frac{2}{3}Y_2^0 - \frac{1}{3}Y_0^0 + \frac{1}{2}\sum_0^\infty B_l Y_l^2 - \sum_0^\infty C_l Y_l^2\end{gathered}$$

*A, B, C 的定义在前面几个式子中.

(2)

$$\begin{gathered}(1-2\sin\theta)\cos\theta\cos\varphi = (\cos\theta - 2\sin\theta\cos\theta)\cos\varphi \\ \Longrightarrow = \int_1^\infty A_l Y_l^1 - \frac{1}{2}3Y_2^1, \quad A_l = \frac{(2l+1)(l+2)!}{2(l-2)!}\int_{-1}^1 xP_l^1\,\mathrm{d}x\end{gathered}$$

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令

$$u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}P_l^m(\cos\theta)\left[R_{l,m}(r)\sin m\varphi+S_{l,m}(r)\cos m\varphi\right]$$

代入方程可化简为

$$=A+Br^2\sin 2\theta\cos\varphi=AP_0^0(\cos\theta)-\frac{2}{3}Br^2P_2^1(\cos\theta)\cos\varphi$$

因此

$$\frac{d}{dr}\left(r^2\frac{dR_{l,m}}{dr}\right)-l(l+1)R_{l,m}=0,\frac{d}{dr}\left(r^2\frac{dS_{0,0}}{dr}\right)=Ar^2,\frac{d}{dr}\left(r^2\frac{dS_{2,1}}{dr}\right)-6S_{2,1}=-\frac{2}{3}Br^2$$

由边界条件可得

$$R_{l,m}(r)=0,\quad S_{0,0}(r)=\frac{A}{6}\left(r^2-a^2\right),\quad S_{2,1}(r)=\frac{1}{21}Br^2\left(a^2-r^2\right)$$

代入即为

$$\begin{aligned}u(r,\theta,\varphi)&=\frac{A}{6}\left(r^2-a^2\right)+\frac{B}{21}r^2\left(a^2-r^2\right)P_2^1(\cos\theta)\cos\varphi\\&=\frac{A}{6}\left(r^2-a^2\right)+\frac{B}{14}r^2\left(r^2-a^2\right)\sin 2\theta\cos\varphi\end{aligned}$$