ElectroDynamics

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1 方程

真空麦克斯韦方程

$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \boldsymbol{E}\,\mathrm{d}\boldsymbol{s} = \frac{Q}{\epsilon_0}$	高斯定律
$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{s} = 0$	高斯磁定律
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\oint_L \boldsymbol{E}\,\mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\varphi_B}{\mathrm{d}t}$	法拉第电磁感应定律
$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$	$\oint_L \boldsymbol{B}\,\mathrm{d}\boldsymbol{l} = \mu_0 I + \mu_0\epsilon_0 \frac{\mathrm{d}\varphi_E}{\mathrm{d}t}$	安培定律

物质内麦克斯韦方程

$\nabla \cdot \boldsymbol{D} = \rho_f$	$\oint_S \boldsymbol{D}\,\mathrm{d}\boldsymbol{s} = Q_f$	高斯定律
$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{s} = 0$	高斯磁定律
$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\oint_L \boldsymbol{E}\,\mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\varphi_B}{\mathrm{d}t}$	法拉第电磁感应定律
$\nabla \times \boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{\partial t}$	$\oint_L \boldsymbol{H}\,\mathrm{d}\boldsymbol{l} = I_f + \frac{\mathrm{d}\varphi_D}{\mathrm{d}t}$	安培定律

镜像法

距半径为 R_0 的球的球心距离为 a 处有一点电荷 q, 则镜像电荷 $-\frac{R_0}{a}q$ 距球心 $\frac{R_0^2}{a}$ 远, 且在靠近 q 方向.

球谐函数解拉普拉斯方程

$$\varphi\left(r,\theta\right) = \sum_{n=0}^{\infty}\left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n\left(\cos\theta\right)$$

$$\begin{cases} P_0\left(\cos\theta\right) = 1 \\ P_1\left(cos\theta\right) = \cos\theta \\ P_2\left(\cos\theta\right) = \frac{1}{2}\left(3\cos^2\theta - 1\right) \end{cases}$$

$$\frac{1}{|\boldsymbol{R} - \boldsymbol{a}'|} = \frac{1}{\sqrt{R^2 + a'^2 - 2Ra\cos\theta}} = \begin{cases} \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{R}{a}\right)^n P_n\left(\cos\theta\right), & (R < a) \\ \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^n P_n\left(\cos\theta\right), & (R > a) \end{cases}$$

格林函数法

$$\nabla^2 G\left(x, x'\right) = -\frac{1}{\varepsilon}\delta^3\left(x - x'\right)$$

(1) 无界空间中

$$G\left(x, x'\right) = \frac{1}{4\pi\varepsilon_0}\frac{1}{|r - r'|}$$

(2) 上半平面中

$$G\left(x, x'\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{|r - r'|} - \frac{1}{|r + r'|}\right)$$

(3) 球外空间 (R' 为电荷位置, α 为场点与电荷位置夹角,R_0 为球半径).

$$G\left(x, x'\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{R^2 + R'^2 - 2RR'\cos\alpha} - \frac{1}{\left(\frac{RR'}{R_0}\right)^2 + R_0^2 - 2RR'\cos\alpha}\right)$$

给定 $\rho\left(x'\right)$, 第一类边值问题的解为 (G 交换了 x, x'):

$$\varphi\left(x\right) = \int_V G\left(x', x\right)\rho\left(x'\right)\mathrm{d}V' + \varepsilon_0\oint_S\left(G\left(x', x\right)\frac{\partial\varphi}{\partial n'} - \varphi\left(x'\right)\frac{\partial G\left(x', x\right)}{\partial n'}\mathrm{d}S'\right)$$

若给定边界 φ, 则应使 G 在边界为 0, 若给定边界 $\frac{\partial\varphi}{\partial n}$, 则应使 $\frac{\partial G}{\partial n}$ 在边界为 0.

泊松方程

$$\nabla^2\Phi = \nabla\cdot\boldsymbol{E} = -\frac{\rho}{\epsilon_0}$$

电多极矩

$$\varphi\left(\boldsymbol{x}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R} + \frac{\boldsymbol{p}\cdot\boldsymbol{R}}{R^3} + \frac{1}{6}\sum_{i,j}\mathfrak{D}_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{R} + \cdots\right)$$

$$\boldsymbol{E} = -\frac{1}{4\pi\varepsilon_0}\left(\frac{3\left(\boldsymbol{p}\cdot\boldsymbol{R}\right)\boldsymbol{R}}{R^5} - \frac{\boldsymbol{p}}{R^3}\right)$$

$$\boldsymbol{p} = \iiint_V \rho\left(x'\right)x'\,\mathrm{d}^3x'$$

$$\mathfrak{D} = \iiint_V 3\boldsymbol{x}'\boldsymbol{x}'\rho\left(\boldsymbol{x}'\right) d^3x'$$

磁偶极矩

$$\varphi = \frac{\boldsymbol{m}\cdot\boldsymbol{R}}{4\pi R^3}$$

$$\boldsymbol{m} = \frac{1}{2}\iiint_V \boldsymbol{x'}\times\boldsymbol{J}\left(\boldsymbol{x'}\right)\mathrm{d}^3x'$$

保角变换 (z_1 为原来的点, a 为夹角出现的位置的横坐标, α 为边界夹角).

$$\frac{\mathrm{d}z_1}{\mathrm{d}z_2} = C_1\prod_{i=1}^{n}\left(z_2 - a_i\right)^{\frac{\alpha_i}{\pi}-1}$$

电荷

面电荷	电场	磁场
$\sigma_{polar} = P$	$\rho_p = -\nabla\cdot\boldsymbol{P}$	$\rho_M = -\mu_0\nabla\cdot\boldsymbol{M}$
$\sigma_{free} = D$	$\rho_f = \nabla\cdot\boldsymbol{D}$	$\rho_f = 0$
$\sigma_{total} = \epsilon_0 E$	$\rho_{tot} = \varepsilon_0\nabla\cdot\boldsymbol{E}$	$\rho_{tot} = \rho_p = \mu_0\nabla\cdot\boldsymbol{H}$
	$\varepsilon_0 E = D - P$	$B = \mu_0\left(H + M\right)$

边界条件

磁场	电场
$\varphi_1 = \varphi_2$	$\varphi_1 = \varphi_2$
$\mu_1\frac{\partial\varphi_1}{\partial n} = \mu_2\frac{\partial\varphi_2}{\partial n}$	$\varepsilon_1\frac{\partial\varphi_1}{\partial n} + \sigma_f = \varepsilon_2\frac{\partial\varphi_2}{\partial n}$
	$\sigma_{电导1}\frac{\partial\varphi_1}{\partial n} = \sigma_{电导2}\frac{\partial\varphi_2}{\partial n}$
$H_{1\parallel} + \boldsymbol{\alpha}_f = H_{2\parallel}$	$E_{1\parallel} = E_{2\parallel}$
$B_{1\perp} = B_{2\perp}$	$D_{1\perp} + \sigma_f = D_{2\perp}$
$M_{1\perp} + \alpha_M = M_{2\perp}$	
$B_{2\perp} - B_{1\perp} = \mu_0\left(\alpha_f + \alpha_M\right)$	$E_{2\perp} - E_{1\perp} = \left(\sigma_f + \sigma_p\right)/\varepsilon_0$
$B_\perp = 0$ (超导球)	$D_\perp = \sigma_f$ (导体)

洛伦兹力：

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B}$$

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}$$

磁偶极子:

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{3\left(\vec{m}\cdot\widehat{R}\right)\widehat{R}-\vec{m}}{R^3}$$

$$\varphi=\frac{\boldsymbol{m}\cdot\boldsymbol{R}}{4\pi R^3}$$

电磁场：

$$\boldsymbol{S}=\boldsymbol{E}\times\boldsymbol{H}$$

$$w=\frac{1}{2}\left(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B}\right)=\frac{1}{2}\left(\rho\varphi+\boldsymbol{J}_f\cdot\boldsymbol{A}\right)$$

电流：

$$\nabla\cdot J=-\frac{\partial\rho}{\partial t}$$

$$J=\sigma E$$

毕奥——萨伐尔定律 $\boldsymbol{B}=\frac{\mu_0}{4\pi}\int\frac{I\,\mathrm{d}l\times\boldsymbol{e}_r}{r^2}$，若 I 为直线，$B=\frac{\mu_0 Il}{4\pi r^2}$.

电磁波：

$$\begin{array}{r|r}\boldsymbol{\nabla}\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}=0 & \boldsymbol{\nabla}^2\boldsymbol{E}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}=\Box\boldsymbol{E}=0\\ \boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0 & \boldsymbol{\nabla}^2\boldsymbol{B}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}=\Box\boldsymbol{B}=0\end{array}$$

磁矢势：

库仑规范：

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$$

$$\nabla^2\varphi=-\frac{\varphi}{\epsilon_0}$$

$$\Box\boldsymbol{A}=-\mu_0\boldsymbol{J}+\frac{1}{c^2}\nabla\frac{\partial\varphi}{\partial t}$$

洛伦兹规范：

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0$$

$$\Box\varphi=-\frac{\rho}{\epsilon_0}$$

$$\Box\boldsymbol{A}=-\mu_0\boldsymbol{J}$$

2 数学

2.1 柱坐标系 (ρ,ϕ,z)

$$\nabla\varphi=\widehat{e}_1\frac{\partial\varphi}{\partial\rho}+\widehat{e}_2\frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}+\widehat{e}_3\frac{\partial\varphi}{\partial z}$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{\rho}\frac{\partial\left(\rho A_1\right)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_2}{\partial\phi}+\frac{\partial A_3}{\partial z}$$

$$\nabla\times\boldsymbol{A}=\widehat{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi}-\frac{\partial A_2}{\partial z}\right)+\widehat{e}_2\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial\rho}\right)+\widehat{e}_3\frac{1}{\rho}\left(\frac{\partial\left(\rho A_2\right)}{\partial\rho}-\frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\varphi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}$$

2.2 球坐标系 (r,θ,φ)

$$\nabla\varphi=\widehat{e}_1\frac{\partial\varphi}{\partial r}+\widehat{e}_2\frac{1}{r}\frac{\partial\varphi}{\partial\theta}+\widehat{e}_3\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}q$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial r^2A_1}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_2\right)+\frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\boldsymbol{A}=\widehat{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_3\right)-\frac{\partial A_2}{\partial\phi}\right]+\widehat{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}\left(rA_3\right)\right]+\widehat{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_2\right)-\frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\varphi=\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\varphi}{\partial\phi^2}\right]$$

2.3 矢量变换

$$\nabla\boldsymbol{r}=-\nabla'\boldsymbol{r}=\boldsymbol{e}_r$$

$$\nabla\frac{1}{\boldsymbol{r}}=-\nabla'\frac{1}{\boldsymbol{r}}=-\frac{1}{r^2}\boldsymbol{e}_r$$

$$\nabla\times\frac{1}{\boldsymbol{r}^2}=\nabla\cdot\frac{1}{\boldsymbol{r}}=0$$

$$\nabla\cdot\varphi\boldsymbol{A}=\varphi\nabla\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\nabla\varphi$$

$$\nabla\times\varphi\boldsymbol{A}=\varphi\nabla\times\boldsymbol{A}+\nabla\varphi\times\boldsymbol{A}$$

$$\nabla\cdot(\boldsymbol{A}\cdot\boldsymbol{B})=(\boldsymbol{A}\cdot\nabla)\,\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{A}+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})$$

$$\nabla\cdot(\boldsymbol{F}\times\boldsymbol{G})=(\nabla\times\boldsymbol{F})\cdot\boldsymbol{G}-\boldsymbol{F}\cdot(\nabla\times\boldsymbol{G})$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\,\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\,\boldsymbol{C}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})=[\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{D})]\boldsymbol{C}-[\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})]\boldsymbol{D}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=\begin{vmatrix}\boldsymbol{A}\cdot\boldsymbol{C}&\boldsymbol{A}\cdot\boldsymbol{D}\\\boldsymbol{B}\cdot\boldsymbol{C}&\boldsymbol{B}\cdot\boldsymbol{D}\end{vmatrix}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})+\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A})+\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B})=0$$