The 1st HW of Electrodynamics

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1. Prove that $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$:

$$
\begin{aligned}
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{A}\times[(B_yC_z-B_zC_y)\,\boldsymbol{i}+(B_zC_x-B_xC_z)\,\boldsymbol{j}+(B_xC_y-B_yC_x)\,\boldsymbol{k}] \\
&= [A_y(B_xC_y-B_yC_x)-A_z(B_zC_x-B_xC_z)]\boldsymbol{i} \\
&+[A_z(B_yC_z-B_zC_y)-A_x(B_xC_y-B_yC_x)]\boldsymbol{j} \\
&+[A_x(B_zC_x-B_xC_z)-A_y(B_yC_z-B_zC_y)]\boldsymbol{k}
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} &= (A_xC_x+A_yC_y+A_zC_z)\boldsymbol{B}-(A_xB_x+A_yB_y+A_zB_z)\boldsymbol{C} \\
&= [(A_xC_x+A_yC_y+A_zC_z)B_x-(A_xB_x+A_yB_y+A_zB_z)C_x]\boldsymbol{i} \\
&+[(A_xC_x+A_yC_y+A_zC_z)B_y-(A_xB_x+A_yB_y+A_zB_z)C_y]\boldsymbol{j} \\
&+[(A_xC_x+A_yC_y+A_zC_z)B_z-(A_xB_x+A_yB_y+A_zB_z)C_z]\boldsymbol{k} \\
&= [(A_yC_y+A_zC_z)B_x-(A_yB_y+A_zB_z)C_x]\boldsymbol{i} \\
&+[(A_xC_x+A_zC_z)B_y-(A_xB_x+A_zB_z)C_y]\boldsymbol{j} \\
&+[(A_xC_x+A_yC_y)B_z-(A_xB_x+A_yB_y)C_z]\boldsymbol{k}
\end{aligned}
$$

Therefore we have $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$,

and for the same reason, $\nabla\times(\nabla\times\boldsymbol{A})=\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$

$$
\begin{aligned}
\nabla\times(\nabla\times\boldsymbol{A}) &= \nabla\times[\left(\tfrac{\partial}{\partial y}A_z-\tfrac{\partial}{\partial z}A_y\right)\boldsymbol{i}+\left(\tfrac{\partial}{\partial z}A_x-\tfrac{\partial}{\partial x}A_z\right)\boldsymbol{j}+\left(\tfrac{\partial}{\partial x}A_y-\tfrac{\partial}{\partial y}A_x\right)\boldsymbol{k}] \\
&= [\tfrac{\partial}{\partial y}\left(\tfrac{\partial}{\partial x}A_y-\tfrac{\partial}{\partial y}A_x\right)-\tfrac{\partial}{\partial z}\left(\tfrac{\partial}{\partial z}A_x-\tfrac{\partial}{\partial x}A_z\right)]\boldsymbol{i} \\
&+[\tfrac{\partial}{\partial z}\left(\tfrac{\partial}{\partial y}A_z-\tfrac{\partial}{\partial z}A_y\right)-\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial x}A_y-\tfrac{\partial}{\partial y}A_x\right)]\boldsymbol{j} \\
&+[\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial z}A_x-\tfrac{\partial}{\partial x}A_z\right)-\tfrac{\partial}{\partial y}\left(\tfrac{\partial}{\partial y}A_z-\tfrac{\partial}{\partial z}A_y\right)]\boldsymbol{k}
\end{aligned}
$$

$$
\begin{aligned}
\nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A} &= \nabla\left(\tfrac{\partial}{\partial x}A_x+\tfrac{\partial}{\partial y}A_y+\tfrac{\partial}{\partial z}A_z\right)-\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}+\tfrac{\partial^2}{\partial z^2}\right)\boldsymbol{A}\\
&= [\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial x}A_x+\tfrac{\partial}{\partial y}A_y+\tfrac{\partial}{\partial z}A_z\right)-\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}+\tfrac{\partial^2}{\partial z^2}\right)A_x]\boldsymbol{i}\\
&+[\tfrac{\partial}{\partial y}\left(\tfrac{\partial}{\partial x}A_x+\tfrac{\partial}{\partial y}A_y+\tfrac{\partial}{\partial z}A_z\right)-\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}+\tfrac{\partial^2}{\partial z^2}\right)A_y]\boldsymbol{j}\\
&+[\tfrac{\partial}{\partial z}\left(\tfrac{\partial}{\partial x}A_x+\tfrac{\partial}{\partial y}A_y+\tfrac{\partial}{\partial z}A_z\right)-\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}+\tfrac{\partial^2}{\partial z^2}\right)A_z]\boldsymbol{k}\\
&= [\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial y}A_y+\tfrac{\partial}{\partial z}A_z\right)-\left(\tfrac{\partial^2}{\partial y^2}+\tfrac{\partial^2}{\partial z^2}\right)A_x]\boldsymbol{i}\\
&+[\tfrac{\partial}{\partial y}\left(\tfrac{\partial}{\partial x}A_x+\tfrac{\partial}{\partial z}A_z\right)-\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial z^2}\right)A_y]\boldsymbol{j}\\
&+[\tfrac{\partial}{\partial z}\left(\tfrac{\partial}{\partial x}A_x+\tfrac{\partial}{\partial y}A_y\right)-\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\right)A_z]\boldsymbol{k}\\
&= [\tfrac{\partial}{\partial y}\left(\tfrac{\partial}{\partial x}A_y-\tfrac{\partial}{\partial y}A_x\right)-\tfrac{\partial}{\partial z}\left(\tfrac{\partial}{\partial z}A_x-\tfrac{\partial}{\partial x}A_z\right)]\boldsymbol{i}\\
&+[\tfrac{\partial}{\partial z}\left(\tfrac{\partial}{\partial y}A_z-\tfrac{\partial}{\partial z}A_y\right)-\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial x}A_y-\tfrac{\partial}{\partial y}A_x\right)]\boldsymbol{j}\\
&+[\tfrac{\partial}{\partial x}\left(\tfrac{\partial}{\partial z}A_x-\tfrac{\partial}{\partial x}A_z\right)-\tfrac{\partial}{\partial y}\left(\tfrac{\partial}{\partial y}A_z-\tfrac{\partial}{\partial z}A_y\right)]\boldsymbol{k}
\end{aligned}
$$

Q.E.D.

2. Prove that $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \begin{vmatrix}\tfrac{\partial}{\partial x} & \tfrac{\partial}{\partial y} & \tfrac{\partial}{\partial z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\\
&= \tfrac{\partial}{\partial x}(A_yB_z-A_zB_y)+\tfrac{\partial}{\partial y}(A_zB_x-A_xB_z)+\tfrac{\partial}{\partial z}(A_xB_y-A_yB_x)
\end{aligned}
$$

and for the same reason:

$$
\begin{aligned}
\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) &= \quad B_x\left(\tfrac{\partial}{\partial y}A_z-\tfrac{\partial}{\partial z}A_y\right)+B_y\left(\tfrac{\partial}{\partial z}A_x-\tfrac{\partial}{\partial x}A_z\right)+B_z\left(\tfrac{\partial}{\partial x}A_y-\tfrac{\partial}{\partial y}A_x\right)\\
-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) &= -A_x\left(\tfrac{\partial}{\partial y}B_z-\tfrac{\partial}{\partial z}B_y\right)-A_y\left(\tfrac{\partial}{\partial z}B_x-\tfrac{\partial}{\partial x}B_z\right)-A_z\left(\tfrac{\partial}{\partial x}B_y-\tfrac{\partial}{\partial y}B_x\right)
\end{aligned}
$$

They are all equal. Q.E.D.

3. Prove $\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{A}(\nabla\cdot\boldsymbol{B})-\boldsymbol{B}(\nabla\cdot\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$

$$
\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-(\nabla\cdot\boldsymbol{A})\boldsymbol{B}
$$

$$
\begin{aligned}
\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) &= \nabla \times [(A_y B_z - A_z B_y)\,\boldsymbol{i} + (A_z B_x - A_x B_z)\,\boldsymbol{j} + (A_x B_y - A_y B_x)\,\boldsymbol{k}] \\
&= [\tfrac{\partial}{\partial y}\,(A_x B_y - A_y B_x) - \tfrac{\partial}{\partial z}\,(A_z B_x - A_x B_z)]\boldsymbol{i} \\
&+ [\tfrac{\partial}{\partial z}\,(A_y B_z - A_z B_y) - \tfrac{\partial}{\partial x}\,(A_x B_y - A_y B_x)]\boldsymbol{j} \\
&+ [\tfrac{\partial}{\partial x}\,(A_z B_x - A_x B_z) - \tfrac{\partial}{\partial y}\,(A_y B_z - A_z B_y)]\boldsymbol{k}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{A}\,(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}\,(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\,\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\,\boldsymbol{B} &= \left(\tfrac{\partial B_x}{\partial x} + \tfrac{\partial B_y}{\partial y} + \tfrac{\partial B_z}{\partial z} + B_x \tfrac{\partial}{\partial x} + B_y \tfrac{\partial}{\partial y} + B_z \tfrac{\partial}{\partial z}\right)\boldsymbol{A} \\
&- \left(\tfrac{\partial A_x}{\partial x} + \tfrac{\partial A_y}{\partial y} + \tfrac{\partial A_z}{\partial z} + A_x \tfrac{\partial}{\partial x} + A_y \tfrac{\partial}{\partial y} + A_z \tfrac{\partial}{\partial z}\right)\boldsymbol{B} \\
&= [\tfrac{\partial}{\partial y}\,(A_x B_y - A_y B_x) - \tfrac{\partial}{\partial z}\,(A_z B_x - A_x B_z)]\boldsymbol{i} \\
&+ [\tfrac{\partial}{\partial z}\,(A_y B_z - A_z B_y) - \tfrac{\partial}{\partial x}\,(A_x B_y - A_y B_x)]\boldsymbol{j} \\
&+ [\tfrac{\partial}{\partial x}\,(A_z B_x - A_x B_z) - \tfrac{\partial}{\partial y}\,(A_y B_z - A_z B_y)]\boldsymbol{k}
\end{aligned}
$$

Q.E.D.