数值分析第三次作业

肖涵薄 31360164

2019 年 3 月 19 日

1

(1)

$$D^{-1}=\begin{pmatrix}1/\alpha & 0 & 0\\ 0 & 1/\alpha & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$\implies J=I-D^{-1}A=-\begin{pmatrix}0 & 2/\alpha & 1/\alpha\\ 2/\alpha & 0 & -1/\alpha\\ 2 & 2 & 0\end{pmatrix}$$

(2) $\lambda=0,\frac{2}{\alpha},-\frac{2}{\alpha} \implies rho=\frac{2}{\alpha}$. 当 $rho<1 \implies 2>\alpha$ 时收敛.

2

$$(D-L)^{-1}=\begin{pmatrix}2 & & \\ 1 & 1 & \\ 1 & 1 & -2\end{pmatrix}^{-1}=-\frac{1}{2}\begin{pmatrix}-1 & & \\ 1 & -2 & \\ 0 & -1 & 1\end{pmatrix}$$

$$G=(D-L)^{-1}U=\frac{1}{2}\begin{pmatrix}0 & 1 & -1\\ 0 & -1 & -1\\ 0 & 0 & -1\end{pmatrix},\ f=(D-L)^{-1}b=\frac{1}{2}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

由于 $|G|=0<1$, 收敛.

3

(1)

$$(D-L)^{-1}=\begin{pmatrix} 2 & & \\ 2 & 2 & \\ 0 & -1 & 2 \end{pmatrix}^{-1}=\begin{pmatrix} 1/2 & & \\ -1/2 & -1/2 & \\ -1/4 & 1/4 & 1/2 \end{pmatrix}$$

$$G=\begin{pmatrix} 0 & -a/2 & -1/2 \\ -1 & a/2-1 & (1-a)/2 \\ -1/2 & a/4 & -(a-3)/4 \end{pmatrix}$$

$$\begin{aligned} \lambda_1 = & \quad -1 \\ \lambda_2 = & \quad a/8-((a+1)(a+25))^{1/2}/8-3/8 \\ \lambda_3 = & \quad a/8+((a+1)(a+25))^{1/2}/8-3/8 \end{aligned}$$

令 $\lambda_3=1$, 则 $a=2$.

(2) 令 $\lambda_3=0$, 则 $a=-\frac{1}{2}$.

4

迭代公式等价于

$$x=D^{-1}\left(b+(L+U)\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}\right)$$

则收敛的充要条件为: $\rho\left(D^{-1}(L+U)\right)<1$.

$$D^{-1}(L+U)=\begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 \end{pmatrix}$$

$$\lambda=\pm\frac{a_{12}a_{21}}{a_{11}a_{22}}$$

则收敛条件为:

$$\rho = |\lambda| = \left|\frac{a_{12}a_{21}}{a_{11}a_{22}}\right| < 1$$

5

设 $I - \omega A$ 的特征值为 λ', 收敛的充要条件为: $\rho\left(I - \omega A\right) = \|I - \omega A\|_2 < 1 \iff |\lambda'_{max}| < 1$. 满足 $|\left(1 - \lambda'\right) I - \omega A| = 0$, 可得 $\lambda' = \omega\lambda - 1$.
当 $0 < \omega < \frac{2}{\beta}$, $-1 < \lambda' = \omega\lambda - 1 < \omega\beta - 1 < 1$. 即 $|\lambda'| = \rho\left(I - \omega A\right) < 1$, 收敛.

6

(1) 上式等价于

$$x^{(k+1)} = \frac{2I - \omega A}{2I + \omega A} x^{(k)} - \frac{b\omega}{I + \frac{\omega A}{2}}$$

则收敛条件为 $\rho\left(\frac{2I-\omega A}{2I+\omega A}\right) < 1$. 设 A 的特征值为 λ, $\frac{2I-\omega A}{2I+\omega A}$ 的特征值为 λ'.

$$\lambda' = \frac{2 - \lambda\omega}{2 + \lambda\omega} = 1 - \frac{2\lambda\omega}{2 + \lambda\omega}$$

$$\lambda' = \frac{2 - \lambda\omega}{2 + \lambda\omega} = -1 + \frac{4}{2 + \lambda\omega}$$

由于 $\lambda, \omega > 0$, $-1 < \lambda' < 1$. 因此 $\rho\left(\frac{2I-\omega A}{2I+\omega A}\right) = |\lambda'_{max}| < 1$, 收敛.

(2) 矩阵 A 的特征值为 $1, 3$. 则其谱半径为 $\rho = \frac{1}{2}$. 渐进迭代收敛速度 $R\left(A\right) = -\ln\,\rho = \ln\,2$.

7

设 B 的特征值为 λ, $\left(1 - \omega\right) E + \omega B$ 的特征值为 λ'. $\lambda' = 1 + \left(\lambda - 1\right)\omega$. 由题设, $-1 < \lambda < 1$, 因此:

$$-1 < 1 + \left(-1 - 1\right)\omega < \lambda' < 1 + \left(1 - 1\right)\omega = 1$$

即 $-1<\lambda'<1$, $\rho=|\lambda'|_{max}<1$, 该迭代收敛. 将 $x^{(k+1)}=Bx^{(k)}+f$ 代入该迭代,

$$\begin{aligned} x^{(k+1)} = & \ (1-\omega)Ex^{(k)}+\omega Bx^{(k)}+\omega f \\ = & \ (1-\omega)Ex^{(k)}+\omega x^{(k+1)} \\ (1-\omega)x^{(k+1)} = & \ (1-\omega)Ex^{(k)} \\ x^{(k+1)} = & \ x^{(k)} \end{aligned}$$

说明该迭代是 $x^{(k+1)}=Bx^{(k)}+f$ 的解, 也是 $Ax=b$ 的解.

8

(a)

$$Az_1^{(m+1)}=b_1-B\left(A^{-1}b_2-A^{-1}Bz_1^{(m-1)}\right)$$
$$\Longrightarrow z^{(m+1)}=A^{-1}b_1-A^{-1}BA^{-1}b_2+A^{-1}BA^{-1}Bz_1^{(m-1)}$$

同理,

$$z_2^{(m+1)}=A^{-1}b_2-A^{-1}BA^{-1}b_1+A^{-1}BA^{-1}Bz_2^{(m-1)}$$

收敛的充要条件为

$$\sqrt{\rho\left(A^{-1}BA^{-1}B\right)}=\left|\frac{\lambda_B}{\lambda_A}\right|_{max}<1$$

(b)

$$Az_1^{(m+1)}=b_1-B\left(A^{-1}b_2-A^{-1}Bz_1^{(m)}\right)\Longrightarrow z_1^{(m+1)}=A^{-1}b_1-A^{-1}BA^{-1}b_2-A^{-1}BA^{-1}Bz_1^{(m)}$$

同理,

$$z_2^{(m+1)}=A^{-1}b_2-A^{-1}BA^{-1}b_1-A^{-1}BA^{-1}Bz_1^{(m)}$$

收敛的充要条件为

$$\rho\left(A^{-1}BA^{-1}B\right)=\left(\frac{\lambda_B^2}{\lambda_A^2}\right)_{max}<1$$

(3)

(a) 中收敛速度为 $R_1=-\ln\,\rho_1=|\lambda_A|-|\lambda_B|$, $R_2=-\ln\,\rho_2=2\left(|\lambda_A|-|\lambda_B|\right)=2R_1$, (b) 的收敛速度是 (a) 的 2 倍.