The 5th HW of Electrodynamics

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Q1

Following the Example 2 above, i.e., a charge q is located at...

a)

we guess that there is an image charge inside the ball. Let us assume that the charge is $-q$ and the location is $(b, 0, 0)$. The potential outside of the ball is

$$\varphi = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} - \frac{q'}{r'}\right)$$

where

$$r = \sqrt{(x-d)^2 + y^2 + z^2}$$
$$r' = \sqrt{(x-b)^2 + y^2 + z^2}$$

From the boundary condition on the surface of the ball, we have

$$\frac{q}{\sqrt{R^2 + d^2 - 2Rd\cos\theta}} - \frac{q'}{\sqrt{R^2 + b^2 - 2Rb\cos\theta}} = 0$$
$$q\sqrt{R^2 + b^2 - 2Rb\cos\theta} = q'\sqrt{R^2 + d^2 - 2Rd\cos\theta}$$

Solving the two equations, we obtain

$$b = \frac{R^2}{d}, q' = \frac{R}{d}q$$

$$F_x = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(b+d)^2} = \frac{1}{4\pi\epsilon_0}\frac{Rq^2}{d}\frac{1}{\left(\frac{R^2}{d}+d\right)^2} = \frac{1}{4\pi\epsilon_0}\frac{Rdq^2}{R^4 + 2R^2d^2 + d^4}$$

$$W = -\frac{Rq^2}{4\pi\epsilon_0}\int_d^\infty \frac{x}{(R^2+x^2)^2}\mathrm{d}x = -\frac{Rq^2}{8\pi\epsilon_0 (R^2 + d^2)}$$

b)

Put a charge q'' at the center of the ball. It must have $q'' = Q - q' = Q - \frac{R}{d}q$. Now

$$W' = -\frac{1}{4\pi\epsilon_0}\int_d^\infty \frac{\left(Q - \frac{R}{d}q\right)q}{x^2}\mathrm{d}x = -\frac{\left(Q - \frac{R}{d}q\right)q}{4\pi\epsilon_0}$$

$$W_{Total} = W + W' = -\left(\frac{Rq^2}{8\pi\epsilon_0\left(R^2+d^2\right)} + \frac{\left(Q - \frac{R}{d}q\right)q}{4\pi\epsilon_0}\right)$$

Q2

Solve the example above using the image charge method.

将点电荷看做平行于另外两个板的无穷大平面, 并保持总电荷不变, 则感应电荷也不变. 感应电荷之和为 Q: $Q_1 + Q_2 = -Q$.

且二者电势 $U = \frac{Qd}{A\varepsilon}$ 相等, 即 $Q_1 d_1 = Q_2 d_2$

$$\Longrightarrow \begin{aligned} Q_1 &= -\tfrac{d_2 q}{d_1+d_2} \\ Q_2 &= -\tfrac{d_1 q}{d_1+d_2} \end{aligned}$$

Q3

Show that the Green function ...

a)

$$\frac{\partial^2 G}{\partial x^2} = -2\sum n^2\pi^2 g_n\left(y,y'\right)\sin\left(n\pi x\right)\sin\left(n\pi x'\right)$$

$$\frac{\partial^2 G}{\partial y^2} = 2\sum \frac{\partial^2 g_n\left(y,y'\right)}{\partial y^2}\sin\left(n\pi x\right)\sin\left(n\pi x'\right)$$

则有:

$$\nabla^2 G = 2\sum\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right)g_n\left(y,y'\right)\sin\left(n\pi x\right)\sin\left(n\pi x'\right) = -8\pi\delta\left(y-y'\right)\sum\left(\sin\left(n\pi x\right)\sin\left(n\pi x'\right)\right)$$

$$\sum\left(\sin\left(n\pi x\right)\sin\left(n\pi x'\right)\right) = \frac{1}{2}\delta\left(x-x'\right)$$

代入上式得

$$\nabla^2 G = -4\pi\delta\left(y-y'\right)\delta\left(x-x'\right)$$

b) 由题意, g 可以写作

$$g_n(y,y') = \begin{cases} g_< \equiv a_< \sinh(n\pi y') + b_< \cosh(n\pi y') & y' < y \\ g_> \equiv a_> \sinh(n\pi y') + b_> \cosh(n\pi y') & y' > y \end{cases}$$

边界条件为: $y = y'$ 时 $g_> = g_<, \quad \partial_{y'} g_> = \partial_{y'} g_< - 4\pi$. 由题意 $g_n(y,0) = g_n(y,1) = 0$.

$$\implies b_< = 0, a_> \sinh(n\pi) + b_> \cosh(n\pi) = 0$$

$$\implies g_n(y,y') = \begin{cases} a_< \sinh(n\pi y') & y' < y \\ a_> [\sinh(n\pi y') - \tanh(n\pi) \cosh(n\pi y') & y' > y \end{cases}$$

可以解得:

$$g_n(y,y') = \frac{4}{n \sinh(n\pi)} \times \begin{cases} \sinh(n\pi y') [\sinh(n\pi) \cosh(n\pi y) - \cosh(n\pi) \sinh(n\pi y)] & y' < y \\ \sinh(n\pi y) [\sinh(n\pi) \cosh(n\pi y') - \cosh(n\pi) \sinh(n\pi y')] & y' > y \end{cases}$$

化简可得:

$$g_n(y,y') = \frac{4}{n \sinh(n\pi)} \sinh(n\pi y_<) \sinh[n\pi(1-y_>)]$$

代入 G 表达式即为

$$G(x,y;x',y') = \sum_n \frac{8}{n \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh(n\pi y_<) \sinh[n\pi(1-y_>)]$$

Q4

A two-dimensional potential exists on a unit square area...

$$\begin{aligned} \Phi(x,y) =& \frac{1}{4\pi\epsilon_0} \int_0^1 dx' \int_0^1 dy' G(x,y;x'y') \rho(x',y') \\ =& \frac{2}{\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{\sin n\pi}{n \sinh n\pi} \int_0^1 \sin n\pi x' dx' \Big[\sinh n\pi(1-y) \int_0^y \sinh n\pi y' dy' \\ & \sinh n\pi y \int_y^1 \sinh n\pi(1-y') dy'] \rho(x',y') \end{aligned}$$

当 ρ 为常数,

$$\int_0^1 \sin n\pi x' dx' = \frac{2}{n\pi} \text{ for odd } n \text{ and } 0 \text{ for even } n$$

$$\int_0^1 \sinh n\pi y' dy' = \frac{1}{n\pi} [\cosh n\pi y - 1]$$

$$\int_y^1 \sinh n\pi(1-y') dy' = \frac{1}{n\pi} [\cosh n\pi(1-y) - 1]$$

$$
\begin{aligned}
&(\cosh n\pi y-1)\sinh n\pi(1-y)+(\cosh n\pi(1-y)-1)\sinh n\pi y\\
&=\sinh n\pi-\sinh n\pi y-\sinh n\pi(1-y)\\
&=\sinh n\pi\left[1-\frac{2\sinh\frac{n\pi}{2}\cosh n\pi\left(y-\frac{1}{2}\right)}{\sinh n\pi}\right]\\
&=\sinh n\pi\left[1-\frac{\cosh n\pi\left(y-\frac{1}{2}\right)}{\cosh\frac{n\pi}{2}}\right]
\end{aligned}
$$

代入并取 $n=m+1$ 可得:

$$
\Phi(x,y)=\frac{4}{\pi^3\epsilon_0}\sum_{m=0}^{\infty}\frac{\sin(2m+1)\pi x}{(2m+1)^3}\left[1-\frac{\cosh(2m+1)\pi\left(y-\frac{1}{2}\right)}{\cosh\frac{(2m+1)\pi}{2}}\right]
$$

Q5

Find the electric quadrupole moment of the problem above. One more problem is problem 5 on page 71 of the textbook.

$\mathcal{D}=\int 3x_ix_j\rho\left(x\right)\mathrm{d}V$. p65 题中的电荷分布为 $\rho=Q\delta\left(0,0,\frac{l}{2}\right)-Q\delta\left(0,0,-\frac{l}{2}\right)$. 则

$$
\mathcal{D}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}
$$

problem 5 on page 71:

导体球内电荷分布不影响外界电场因此对于 $R>R_2$:

$$
\varphi=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}
$$

$$
\sigma_2=\frac{Q}{4\pi R_2^2}
$$

导体表面 $\varphi=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R_2}$, 于是导体内部

$$
\varphi=\left(a_nr^n+\frac{b_n}{r^{(n+1)}}\right)P_n\left(r\right)+\frac{1}{4\pi\varepsilon_0}\frac{Q}{R_2}
$$

$r=0$ 处 φ 有限, 于是 $b_n=0$.

$$
\varphi_1=\frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}\cdot\vec{r}}{r^3}+\frac{Q}{R_2}-\frac{\vec{P}\cdot\vec{r}}{R_1^3}\right]
$$

$$\sigma_1 = -\varepsilon_0 \frac{\partial \varphi_1}{\partial r} = -\frac{3p\cos\theta}{4\pi R_1^3}$$