统计力学第七次作业

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8.3

弱简并费米 (玻色) 气体的内能为

$$U = \frac{3}{2}NkT\left[1 \pm \frac{1}{2^{5/2}}\frac{1}{g}\frac{N}{V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\right]$$

由于 $p = \frac{2}{3}U/V$, 设 $n = N/V$, 压强为:

$$nkT\left[1 \pm \frac{1}{2^{5/2}}\frac{1}{g}n\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\right]$$

定容热容为

$$C_v = \frac{3}{2}Nk\left[1 \mp \frac{1}{2^{5/2}}\frac{1}{g}n\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\right]$$

可计算熵为:

$$\begin{aligned} S &= \frac{3}{2}Nk\ln T \pm Nk\frac{1}{2^{7/2}}\frac{1}{g}n\left(\frac{h^2}{2\pi mkT}\right)^{3/2} + S_0 \\ &= Nk\left\{\ln[\frac{1}{g}n\left(\frac{h^2}{2\pi mkT}\right)^{3/2}] + \frac{5}{2} \pm \frac{1}{2^{7/2}}\frac{1}{g}n\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\right\} \end{aligned}$$

8.4

二维粒子的状态密度为

$$D\,\mathrm{d}\varepsilon = \frac{2\pi L^2}{h^2}m\,\mathrm{d}\varepsilon$$

代入 T_c 表达式可得

$$\frac{2\pi L^2}{h^2}m\int_0^\infty \frac{1}{e^{\frac{\varepsilon}{kT_c}} - 1} = n$$

该积分发散, 因此不存在玻色凝聚.

8.9

$$\lambda T = \frac{hv}{4.9651k}$$

代入各参数为

$$T = 6000K$$

8.17

压强为

$$p = \frac{2}{5}n\mu(0) = \frac{2}{5}\frac{\hbar^2}{2m}(3\pi^2)^{2/3}\left(N/V\right)^{5/3}$$

0K 下

$$\kappa_T = -\frac{1}{V}(\frac{\partial V}{\partial p})_T = \frac{3}{2}\frac{V}{\frac{\hbar^2}{2m}(3\pi^2)^{2/3}\left(N/V\right)^{5/3}} = \frac{3}{2}\frac{1}{n\mu(0)}$$

由于 0K 下 T=0 与 S=0 等温线重合,

$$\kappa_s = \frac{3}{2}\frac{1}{n\mu(0)}$$

8.19

面积 A 内自由电子量子态数为:

$$D\,\mathrm{d}\varepsilon = \frac{4\pi A}{h^2}m\,\mathrm{d}\varepsilon$$

费米能量由下式确定: $N = \frac{4\pi A}{h^2}m\int_0^{\mu(0)}\mathrm{d}\varepsilon = \frac{4\pi A}{h^2}m\mu(0)$ 即

$$\mu(0) = \frac{h^2}{4\pi m}\frac{N}{A} = \frac{h^2}{4\pi m}n$$

0K 时内能为

$$U = \frac{4\pi A}{h^2}m\int_0^{\mu(0)}\varepsilon\mathrm{d}\varepsilon = \frac{4\pi A}{h^2}\frac{m}{2}\mu^2(0) = \frac{N}{2}\mu(0)$$

代入 $p = U/A$

$$p = \frac{1}{2}n\mu(0)$$

9.1

将 $\rho_s = \frac{1}{\Omega}$ 代入 $S = -k\sum \rho_s \ln \rho_s$. 得

$$S = -k\sum_s \frac{1}{\Omega}\ln\frac{1}{\Omega}$$

再代入 $\sum \frac{1}{\Omega} = 1$,

$$S = -k\ln\frac{1}{\Omega} = k\ln\Omega$$

这正是玻耳兹曼关系

9.2

$$\rho_s = \frac{1}{Z}\mathrm{e}^{-\beta E_s}$$
$$Z = \sum_s \mathrm{e}^{-\beta E_s}$$

由于

$$\sum_s \rho_s = 1$$

$$\begin{aligned}S &= k\left(\ln Z - \beta\frac{\partial}{\partial\beta}\ln Z\right)\\ &= k(\ln Z + \beta U)\\ &= k\sum \rho_s\left(\ln Z + \beta E_s\right)\end{aligned}$$

因此

$$\ln\rho_s = -\left(\ln Z + \beta E_s\right)$$

所以

$$S = -k\sum \rho_s \ln\rho_s$$

9.3

其能量为

$$E = \sum_{i=1}^{3N}\frac{p_i^2}{2m}$$

$$
\begin{aligned}
Z &= \frac{V^N}{N!h^{3N}} \prod_{i=1}^{3N} \int \mathrm{e}^{-\beta_{2i}^2} \mathrm{d}p_i \\
&= \frac{V^N}{N!} \left(\frac{2\pi m}{\beta h^2} \right)^{\frac{3N}{2}}
\end{aligned}
$$

气体压强为

$$
p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z = \frac{N}{\beta} \frac{\partial}{\partial V} \ln V = \frac{NkT}{V}
$$

因此物态方程为

$$
pV = NkT
$$

$$
U = -\frac{\partial}{\partial \beta} \ln Z = -\frac{3N}{2} \frac{\partial}{\partial \beta} \ln \frac{1}{\beta} = \frac{3N}{2} kT
$$

$$
\begin{aligned}
S &= k \left(\ln Z - \beta \frac{\partial}{\partial \beta} \ln Z \right) \\
&= k(\ln Z + \beta U) \\
&= \frac{3}{2} Nk \ln T + Nk \ln \frac{V}{N} + Nk \left[\ln \left(\frac{2\pi mk}{h^2} \right)^{\frac{3}{2}} + \frac{5}{2} \right]
\end{aligned}
$$

$$
\begin{aligned}
\mu &= \left(\frac{\partial F}{\partial N} \right)_{r,v} = -kT \frac{\partial}{\partial N} \ln Z \\
&= -kT \left[N \ln V - N(\ln N - 1) + \frac{3N}{2} \ln \left(\frac{2\pi m}{\beta h^2} \right) \right] \\
&= kT \ln \frac{N}{V} \left(\frac{h^2}{2\pi mkT} \right)^{3/2}
\end{aligned}
$$

9.5

$$
E = \sum_{i=1}^{3N_A} \frac{p_{Ai}^2}{2m_A} + \sum_{j=1}^{3N_B} \frac{p_{B_j}^2}{2m_B}
$$

$$
Z = \frac{1}{N_A! h^{3N_A}} \int \mathrm{e}^{-\beta E_A} \mathrm{d}\Omega_\mathrm{A} \cdot \frac{1}{N_\mathrm{B}! h^{3N_\mathrm{B}}} \int \mathrm{e}^{-\beta E_\mathrm{B}} \mathrm{d}\Omega_\mathrm{B} = Z_A Z_B
$$

$$
p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln Z = (N_A + N_B) \frac{kT}{V}
$$

因此物态方程为

$$
pV = (N_\mathrm{A} + N_\mathrm{B}) kT
$$

$$
U = -\frac{\partial}{\partial \beta} \ln Z = \frac{3}{2} (N_A + N_\mathrm{B}) kT
$$

$$
\begin{aligned}
S &= k\left(\ln Z - \beta\frac{\partial}{\partial\beta}\ln Z\right)\\
&= k(\ln Z + \beta U)\\
&= N_{\mathrm{A}} k \ln\left[\frac{V}{N_{\mathrm{A}}}\left(\frac{2\pi m_A kT}{h^2}\right)^{\frac{3}{2}}\right] + \frac{5}{2}Nk +\\
&\quad N_{\mathrm{B}} k \ln\left[\frac{V}{N_{\mathrm{A}}}\left(\frac{2\pi m_{\mathrm{B}} kT}{h^2}\right)^{\frac{3}{2}}\right] + \frac{5}{2}Nk
\end{aligned}
$$

9.5

$$
E = \sum_{i=1}^{N} \varepsilon_i = \sum_{i=1}^{N} c p_i
$$

$$
Z = \frac{1}{N!}\left(Z_1\right)
$$

其中

$$
Z_1 = \frac{4\pi V}{h^3}\int_0^\infty \mathrm{e}^{-\beta\varepsilon p} p^2 \mathrm{d}p = 8\pi V\left(\frac{kT}{hc}\right)^3
$$

所以

$$
Z = \frac{1}{N!}\left[8\pi V\left(\frac{kT}{hc}\right)^3\right]^N
$$

$$
\begin{aligned}
F &= -kT\ln Z\\
&= -NkT\ln\left[\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right] - NkT
\end{aligned}
$$

$$
p = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = \frac{NkT}{V}
$$

物态方程为

$$
pV = NkT
$$

$$
S = -\left(\frac{\partial F}{\partial T}\right)_{v,N} = Nk\ln\left[\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right] + 4Nk
$$

$$
U = F + TS = 3NkT
$$

$$
\mu = \left(\frac{\partial F}{\partial N}\right)_{r,v} = -kT\ln\left[\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right]
$$

9.8

$$E=\sum_{i=1}^{2N}\frac{1}{2m}p_i^2+\sum_{i<j}\phi\left(r_{ij}\right)$$

$$Z=\frac{1}{N!h^{2N}}\int\cdots\int \mathrm{e}^{-\beta E}\mathrm{d}\boldsymbol{r}_1\cdots\mathrm{d}\boldsymbol{r}_N\mathrm{d}\boldsymbol{p}_1\cdots\mathrm{d}\boldsymbol{p}_N=\frac{1}{N!h^{2N}}\left(\frac{2\pi m}{\beta h^2}\right)^N Q$$

$$Q=\int\cdots\int \mathrm{e}^{-\beta\sum_{i<j}\phi(r_{ij})}\mathrm{d}\boldsymbol{r}_1\cdots\mathrm{d}\boldsymbol{r}_N$$

近似为

$$\begin{aligned}
Q&=S^N+\frac{N^2}{2}\int\cdots\int f_{12}\mathrm{d}\boldsymbol{r}_1\cdots\mathrm{d}\boldsymbol{r}_N\\
&=S^N\left(1+\frac{N^2}{2}S^{N-2}\right] f_{12}\mathrm{d}\boldsymbol{r}_1\mathrm{d}\boldsymbol{r}_2)\\
&=A^N\left[1+\frac{N^2}{2S}\int_0^{+\infty}\left(\mathrm{e}^{-\beta\phi(r)}-1\right)2\pi r\mathrm{d}r\right]\\
&=S^N\left(1-\frac{N^2}{N_AS}B\right)
\end{aligned}$$

其中

$$B=-\frac{N_{\mathrm{S}}}{2}\int\left(\mathrm{e}^{-\beta\phi}-1\right)2\pi r\mathrm{d}r$$

因此

$$Z=\frac{1}{N!}\left(\frac{2\pi m}{\beta h^2}\right)^N S^N\left(1-\frac{N^2}{N_AS}B\right)$$

$$p=\frac{1}{\beta}\frac{\partial}{\partial S}\ln Z=\frac{NkT}{S}\left(1+\frac{N}{N_A}\frac{B}{S}\right)$$

□

$$Z=\mathrm{e}^{-\beta\phi_0}\prod_i^{3N}\frac{\mathrm{e}^{-\beta\frac{\hbar\omega_i}{2}}}{1-\mathrm{e}^{-\beta\hbar\omega_i}}$$

$$\ln Z=-\beta U_0-\sum_{i=1}^{3N}\ln\left(1-\mathrm{e}^{-\beta\hbar\omega_i}\right)$$

其中

$$U_0=\phi_0+\sum_{i=1}^{3N}\frac{\hbar\omega_i}{2}$$

德拜频谱为

$$D(\omega)\mathrm{d}\omega = \begin{cases} \frac{9N}{\omega_{\mathrm{D}}^3}\omega^2\mathrm{d}\omega, & \omega \le \omega_{\mathrm{D}} \\ 0, & \omega > \omega_{\mathrm{D}} \end{cases}$$

于是

$$\ln Z = -\beta U_0 - \frac{9N}{\omega_{\mathrm{D}}^3}\int_0^{\omega_0} \omega^2 \ln\left(1 - \mathrm{e}^{-\beta\hbar\omega}\right)\mathrm{d}\omega$$

设

$$y = \tfrac{\hbar\omega}{kT}$$

$$x = \tfrac{\hbar\omega_0}{kT} = \tfrac{\theta_0}{T}$$

高温 (x«1) 下有近似

$$\begin{aligned} \ln Z &= -\beta U_0 - 3N\ln x + N \\ &= -\beta U_0 - 3N\ln\left(\beta\hbar\omega_{\mathrm{D}}\right) + N \end{aligned}$$

$$U = -\frac{\partial}{\partial\beta}\ln Z = U_0 + 3NkT$$

$$S = k(\ln Z + \beta U) = 3Nk\ln\frac{T}{\theta_{\mathrm{D}}} + 4Nh$$

$$S = k(\ln Z + \beta U) = 3Nk\ln\frac{T}{\theta_{\mathrm{D}}} + 4Nk$$

低温下有近似

$$\ln Z = -\beta U_0 + \frac{N\pi^4}{5}\frac{1}{x^3} = -\beta U_0 + \frac{N\pi^4}{5}\left(\frac{1}{\beta\hbar\omega_{\mathrm{D}}}\right)^3$$

$$U = U_0 + 3Nk\frac{\pi^4}{5}\frac{T^4}{\theta_{\mathrm{D}}^3}$$

$$S = k(\ln Z + \beta U) = \frac{4\pi^4}{5}Nk\left(\frac{T}{\theta_{\mathrm{D}}}\right)^3$$