数理方法 II 第一次作业

肖涵薄 31360164

2019 年 3 月 17 日

1

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} &= \frac{\mathrm{d}r\boldsymbol{e_r}}{\mathrm{d}t} \\
&= r\frac{\mathrm{d}\boldsymbol{e_r}}{\mathrm{d}t} + \dot{r}\boldsymbol{e_r} \\
&= r\frac{\mathrm{d}\theta}{\mathrm{d}t}\boldsymbol{e_\theta} + \dot{r}\boldsymbol{e_r} \\
&= r\dot{\theta}\boldsymbol{e_\theta} + \dot{r}\boldsymbol{e_r}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(r\dot{\theta}\boldsymbol{e_\theta} + \dot{r}\boldsymbol{e_r}\right) \\
&= \dot{r}\dot{\theta}\boldsymbol{e_\theta} + r\ddot{\theta}\boldsymbol{e_\theta} - r\theta^2\boldsymbol{e_r} + \ddot{r}\boldsymbol{e_r} + r\frac{\mathrm{d}\boldsymbol{e_r}}{\mathrm{d}t} \\
&= \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\boldsymbol{e_\theta} + \left(\ddot{r} - r\theta^2\right)\boldsymbol{e_\theta}
\end{aligned}
$$

2

$$r = |\boldsymbol{r}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

(1) x 方向:

$$\left(\nabla\frac{1}{r}\right)_x = \frac{\partial 1/r}{\partial x} = -\frac{1}{2}\frac{1}{r^3}2\left(x-x'\right) = -\frac{\boldsymbol{r}_x}{r^3}$$

y, z 方向同理, 则有:

$$\nabla\frac{1}{r} = -\nabla'\frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$$

(2) 由于 $\nabla\frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$, 且对任何标量函数 ϕ 有: $\nabla\times(\nabla A) = 0$:

$$\nabla\times\frac{\boldsymbol{r}}{r^3} = -\nabla\times\left(\nabla\frac{1}{r}\right) = 0$$

(3)

$$\nabla\times\boldsymbol{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-x' & y-y' & z-z' \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\boldsymbol{e}_x + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\boldsymbol{e}_y + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\boldsymbol{e}_z$$

3

设存在等距的三个点 a, b, c, 其间距为 $\mathrm{d}x$. 在 b 点左侧受力:

$$f_L \mathrm{d}x = Y\left(u\left(b,t\right) - u\left(a,t\right)\right) \implies f_L = Y\frac{\partial u}{\partial x}\left(b,t\right)$$

总应力为左右两侧之差:

$$f = f_R - f_L = Y\left(\frac{\partial u}{\partial x}\left(b,t\right) - \frac{\partial u}{\partial x}\left(a,t\right)\right)$$

单位长度牛顿第二定律:

$$\rho S\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = f_0 + \frac{f}{\mathrm{d}x} = f_0 + Y\frac{\partial^2 u}{\partial x^2}$$

边界条件 1, $x = 0$ 处固定:

$$u\left(0,t\right) = 0$$

边界条件 2, $x = 0$ 处受力:

$$G\left(t\right) = Yu\left(0,t\right)$$

4

$\alpha = \frac{\partial u}{\partial \rho}$, $\beta = \frac{\partial u}{\partial \phi}$, 设 ϕ 方向的两个方向的方向矢量为 $\boldsymbol{n}_1 = \left(-\cos\frac{\mathrm{d}\phi}{2}, -\sin\frac{\mathrm{d}\phi}{2}, 0\right), \boldsymbol{n}_2 = \left(\cos\frac{\mathrm{d}\phi}{2}, -\sin\frac{\mathrm{d}\phi}{2}, 0\right)$, 分别沿 x, y 轴旋转小角度 α, β:

$$\boldsymbol{n}' = \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ -\beta & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \alpha\beta y \\ y \\ -\beta x + \alpha y \end{pmatrix}$$

$\mathrm{d}\alpha = \frac{\partial u(\rho+\mathrm{d}\rho)}{\partial \rho} - \frac{\partial u(\rho+\mathrm{d}\rho)}{\partial \rho} = \frac{\partial^2 u}{\partial \rho^2}\mathrm{d}\rho$, 则这两个方向在 Z 分量的力为:

$$F_{12} = T\mathrm{d}\rho\mathrm{d}\phi\left(\frac{1}{\rho}\frac{\partial^2 u}{\partial \phi^2} - \frac{\partial u}{\partial \rho}\right)$$

再结合 ρ 方向的两个力:

$$m\frac{\partial^2 u}{\partial t^2} = T\left(\rho + \mathrm{d}\rho\right)\mathrm{d}\phi\frac{\partial u}{\partial \rho}\left(\rho + \mathrm{d}\rho\right) - T\rho\mathrm{d}\phi\frac{\partial u}{\partial \rho}\left(\rho\right) + F_{12}$$

$$\Longrightarrow \rho_m\frac{\partial^2 u}{\partial t^2} = T\mathrm{d}\rho\left(\rho\frac{\partial^2 u}{\partial \rho^2}\right)\mathrm{d}\phi + T\frac{1}{\rho}\frac{\partial^2 u}{\partial \phi^2}\mathrm{d}\rho\mathrm{d}\phi = T\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) + T\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = T\nabla^2 u$$

化为直角坐标:

$$\rho_m\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

5

(1)

$$\Delta = 4 - 5 < 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \pm \sqrt{4-5} = 2 \pm i \implies (2 \pm i)\, x - y = \gamma$$

$$\implies \xi = 2x + y,\ \eta = x$$

$$a = 4 - 4 \times 2 + 5 = 1, \quad d = 2 - 2 = 0, \quad e = 1$$

方程变为:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = -\frac{\partial u}{\partial \eta}$$

(2)

$$\Delta = -y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm (-y)^{1/2} \implies x \pm 2\, (-y)^{1/2} = \gamma$$

当 $y > 0, \Delta < 0$:

$$\xi = x,\ \eta = 2\sqrt{y}$$

$$a = 1, \quad d = 0, \quad e = \frac{y^{-3/2}}{2} + \frac{1}{2\sqrt{y}}$$

方程变为:

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = -\frac{y^{-3/2}}{2}\frac{\partial u}{\partial \eta} = -\frac{4}{\eta^3}\frac{\partial u}{\partial \eta}$$

当 $y < 0, \Delta > 0$:

$$\xi = x + 2\sqrt{-y},\ \eta = x - 2\sqrt{-y}$$

$$\mu = x,\ \nu = 2\sqrt{-y}$$

$$b = 2, \quad d = -\frac{1}{\sqrt{-y}}, \quad e = \frac{1}{\sqrt{-y}}$$

方程变为:

$$\frac{\partial^2 u}{\partial \mu^2} - \frac{\partial^2 u}{\partial \nu^2} = -\frac{1}{2}\left(-2\frac{1}{\sqrt{-y}}\frac{\partial u}{\partial \nu}\right) = \frac{1}{\sqrt{-y}}\frac{\partial u}{\partial \nu} = \frac{2}{\nu}\frac{\partial u}{\partial \nu}$$

当 $y = \Delta = 0$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\frac{\partial u}{\partial y} = 0$$

(3)

$$\Delta = 4 > 0$$

5

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\cos x\pm 2 \implies \sin x+y\pm 2x=\gamma$$

$$\xi=\sin x+2x+y,\ \eta=\sin x-2x+y,\quad \mu=\sin x+y,\ \nu=2x$$

$$b=4\left(\cos x-1\right),\quad d=-\sin x-1,\quad e=-\sin x-1$$

方程变为:

$$\frac{\partial^2 u}{\partial\mu^2}-\frac{\partial^2 u}{\partial\nu^2}=-\frac{1}{4\left(\cos x-1\right)}\left[-2\left(\sin x+1\right)\frac{\partial u}{\partial\mu}-2y\right]=\frac{1}{2\left(\cos\frac{\nu}{2}-1\right)}\left[\left(\sin\frac{\nu}{2}+1\right)\frac{\partial u}{\partial\mu}+\left(\mu-\sin\frac{\nu}{2}\right)\right]$$