Pauli Vector Construction

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1 Derivation

We wish to construct a vector such that,

$$\hat{n} \cdot \vec{\sigma} |\pm_n\rangle = \pm |\pm_n\rangle$$

We can construct this by rotating $|\pm_z\rangle$

$$\begin{aligned} \hat{n} \cdot \vec{\sigma} |\pm_n\rangle &= \pm |\pm_n\rangle \\ \hat{n} \cdot \vec{\sigma} e^{i\theta \hat{m} \cdot \vec{\sigma}} |\pm_z\rangle &= \pm e^{i\theta \hat{m} \cdot \vec{\sigma}} |\pm_z\rangle \\ e^{-i\theta \hat{m} \cdot \vec{\sigma}} \hat{n} \cdot \vec{\sigma} e^{i\theta \hat{m} \cdot \vec{\sigma}} &= \sigma_z \end{aligned}$$

By geometrical ansatz, let $\hat{m} = \hat{z} \times \hat{n} / \sin\theta$. Thus,

$$\begin{aligned} \sigma_z &= e^{-i\theta \hat{m} \cdot \vec{\sigma}} \hat{n} \cdot \vec{\sigma} e^{i\theta \hat{m} \cdot \vec{\sigma}} \\ &= \left(\cos\frac{\theta}{2} - i\hat{m} \cdot \vec{\sigma} \sin\frac{\theta}{2} \right) \hat{n} \cdot \vec{\sigma} \left(\cos\frac{\theta}{2} + i\hat{m} \cdot \vec{\sigma} \sin\frac{\theta}{2} \right) \\ &= \hat{n} \cdot \vec{\sigma} \cos^2\frac{\theta}{2} + i \left[\hat{m} \cdot \vec{\sigma}, \hat{n} \cdot \vec{\sigma} \right] \sin\frac{\theta}{2} \cos\frac{\theta}{2} + (\hat{m} \cdot \vec{\sigma}) \hat{n} \cdot \vec{\sigma} (\hat{m} \cdot \vec{\sigma}) \sin^2\frac{\theta}{2} \\ &= \hat{n} \cdot \vec{\sigma} + i/2 \left[\hat{m} \cdot \vec{\sigma}, \hat{n} \cdot \vec{\sigma} \right] \sin\theta + \hat{m} \cdot \vec{\sigma} \left[\hat{n} \cdot \vec{\sigma}, \hat{m} \cdot \vec{\sigma} \right] \sin^2\frac{\theta}{2} \\ &= \hat{n} \cdot \vec{\sigma} - (\hat{m} \times \hat{n}) \cdot \vec{\sigma} \sin\theta + i2(\hat{m} \cdot \vec{\sigma})(\hat{n} \times \hat{m}) \cdot \vec{\sigma} \sin^2\frac{\theta}{2} \\ &= \hat{n} \cdot \vec{\sigma} + \hat{n} \times (\hat{z} \times \hat{n}) \cdot \vec{\sigma} + 2(\hat{m} \times (\hat{m} \times \hat{n})) \cdot \vec{\sigma} \sin^2\frac{\theta}{2} \\ &= \hat{n} \cdot \vec{\sigma} + (\hat{z} - \hat{n}\cos\theta) \cdot \vec{\sigma} - 2\hat{n} \cdot \vec{\sigma} \sin^2\frac{\theta}{2} \\ &= \hat{n} \cdot \vec{\sigma} + (\hat{z} - \hat{n}\cos\theta) \cdot \vec{\sigma} - \hat{n} \cdot \vec{\sigma}(1 - \cos\theta) \\ &= \hat{z} \cdot \vec{\sigma} \end{aligned}$$