Equations of Motion for Schwarzschild Solution

See "General Theory of Relativity" by P.A.M. Dirac for derivation.

The solution in question gives a space time metric as the following,

In[1]:=
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ds = dt Sqrt[(1 - 2 m / r[t]) -
    r'[t]^2 (1 - 2 m / r[t])^-1 - r[t]^2 θ'[t]^2 - r[t]^2 Sin[θ[t]] ϕ'[t]^2]
```

Out[1]=
$$dt \sqrt{\left(1-\frac{2\,m}{r[t]}-\frac{r'[t]^2}{1-\frac{2\,m}{r[t]}}-r[t]^2\,\theta'[t]^2-r[t]^2\,\mathrm{Sin}[\theta[t]]\,\phi'[t]^2\right)}$$

The Geodesic assumption states that the $\delta\int ds = 0$ for the physical path of the particle. By explicit time independance of the metric, we can see that this reproduces the action principle of classical mechanics where the Euler-Lagrange equations hold. Otherwise, we go through the calculus of variations to find the equations of motion. Thus,

In[2]:=
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L = ds / dt
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Out[2]=
$$\sqrt{1-\frac{2\,m}{r[t]}-\frac{r'[t]^2}{1-\frac{2\,m}{r[t]}}-r[t]^2\,\theta'[t]^2-r[t]^2\,\mathrm{Sin}[\theta[t]]\,\phi'[t]^2}$$

By spherical symmetry, we can move only along the θ[t] = π/2 plane (ie. equatorial)

In[3]:=
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L = L /. {θ'[t] → 0, θ[t] → π / 2}
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Out[3]=
$$\sqrt{1-\frac{2\,m}{r[t]}-\frac{r'[t]^2}{1-\frac{2\,m}{r[t]}}-r[t]^2\,\phi'[t]^2}$$

There is a conservation of angular momentum,

In[5]:=
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D[L, ϕ[t]]
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Out[5]= 0

Thus,

In[7]:=
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pϕ = D[L, ϕ'[t]]
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Out[7]=
$$-\frac{r[t]^2\,\phi'[t]}{\sqrt{1-\frac{2\,m}{r[t]}-\frac{r'[t]^2}{1-\frac{2\,m}{r[t]}}-r[t]^2\,\phi'[t]^2}}$$

Working out the relations for the r[t] coordinate, In[10]:= **F_r = D[L, r[t]]**

Out[10]=
$$\frac{\frac{2\,m}{r[t]^2}+\frac{2\,m\,r'[t]^2}{\left(1-\frac{2\,m}{r[t]}\right)^2 r[t]^2}-2\,r[t]\,\phi'[t]^2}{2\sqrt{1-\frac{2\,m}{r[t]}-\frac{r'[t]^2}{1-\frac{2\,m}{r[t]}}-r[t]^2\,\phi'[t]^2}}$$

In[11]:= **p_r = D[L, r'[t]]**

Out[11]=
$$-\frac{r'[t]}{\left(1-\frac{2\,m}{r[t]}\right)\sqrt{1-\frac{2\,m}{r[t]}-\frac{r'[t]^2}{1-\frac{2\,m}{r[t]}}-r[t]^2\,\phi'[t]^2}}$$

We can use the conservation of angular momentum to simplify both equations before continuing with the differentiation.

In[19]:= **D[p_r / p_ϕ, t] == F_r / p_ϕ**

Out[19]=
$$-\frac{2\,m\,r'[t]^2}{\left(1-\frac{2\,m}{r[t]}\right)^2 r[t]^4\,\phi'[t]}-\frac{2\,r'[t]^2}{\left(1-\frac{2\,m}{r[t]}\right) r[t]^3\,\phi'[t]}+\frac{r''[t]}{\left(1-\frac{2\,m}{r[t]}\right) r[t]^2\,\phi'[t]}-\frac{r'[t]\,\phi''[t]}{\left(1-\frac{2\,m}{r[t]}\right) r[t]^2\,\phi'[t]^2}==-\frac{\frac{2\,m}{r[t]^2}+\frac{2\,m\,r'[t]^2}{\left(1-\frac{2\,m}{r[t]}\right)^2 r[t]^2}-2\,r[t]\,\phi'[t]^2}{2\,r[t]^2\,\phi'[t]}$$

These equations are miserable. So, perhaps an approximation on the Lagrangian would be better. We will expand to a few orders and get our effective potential and kinetic terms,

In[26]:= **T = Series[L, {r'[t], 0, 3}]**

Out[26]=
$$\sqrt{1-\frac{2\,m}{r[t]}-r[t]^2\,\phi'[t]^2}-\frac{r'[t]^2}{2\left(\left(1-\frac{2\,m}{r[t]}\right)\sqrt{1-\frac{2\,m}{r[t]}-r[t]^2\,\phi'[t]^2}\right)}+O[r'[t]]^4$$

In[27]:= **U = Series[L, {r[t], 0, 3}]**

Out[27]=
$$\frac{\sqrt{2}\,\sqrt{-m}}{\sqrt{r[t]}}-\frac{\sqrt{-m}\,\sqrt{r[t]}}{2\left(\sqrt{2}\,m\right)}-\frac{\left(\sqrt{-m}\,\left(1+4\,r'[t]^2\right)\right) r[t]^{3/2}}{16\left(\sqrt{2}\,m^2\right)}+\frac{\sqrt{-m}\,\left(-1-12\,r'[t]^2+32\,m^2\,\phi'[t]^2\right) r[t]^{5/2}}{64\sqrt{2}\,m^3}+O[r[t]]^{7/2}$$

Still horrible!