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Classical field theory

Field theory is one of the cornerstones of classical physics. The most notable examples of classical fields are the force fields that one encounters in the description of gravitational and electromagnetic phenomena. These fields are caused by the presence of masses and electric charges, respectively. In this chapter we present a framework for classical field theory, which is known as the Lagrangian formulation. In this formulation the dynamics of a system is described by a single function, the Lagrangian. Via a variational principle the Lagrangian yields the equations of motion which govern the time evolution of that system, so it is a useful mnemonic for summarizing a theory in a concise form. The use of a variational principle to express the equations of classical physics is very old. Fermat's principle in optics (1657) and Maupertuis' principle in mechanics (1744) are famous examples.

Apart from its conciseness and its mathematical elegance we mention two important reasons why the Lagrangian formulation is so convenient in field theory. The first one is that the Lagrangian, or rather the integral of the Lagrangian density over space and time, should be invariant under all symmetries of the theory in question. This aspect of the Lagrangian formulation makes it rather attractive for relativistic theories, because it allows one to treat space and time on an equal footing from the very beginning, in contrast to other approaches where one aims directly for a description of the time evolution. The second advantage of the Lagrangian formulation, emphasized by Dirac and elaborated on by Feynman, is that it plays a natural role in the path integral formulation of quantum mechanics. It turns out that the evolution operator for a quantum-mechanical wave function can be expressed as a sum over all paths with fixed endpoints in space-time weighted by a phase factor that depends only on the Lagrangian integrated along the path.

1.1. The Lagrangian and Hamilton's principle

In order to introduce the basic concepts of the Lagrangian formulation it is convenient to start from classical mechanics. Consider, for example, a system of N particles of equal mass. Its classical motion is described in terms of coordinates $q_i(t)$ $(i = 1, \ldots, 3N)$ which are functions of time. These $3N$ coordinates describe a trajectory in space-time; one determines the velocities by $\dot{q}_i(t) = \mathrm{d}q_i(t)/\mathrm{d}t$ and the accelerations by $\ddot{q}_i(t) = \mathrm{d}^2 q_i(t)/\mathrm{d}t^2$. The equation

of motion in classical mechanics, which determines the acceleration in terms of the force applied to these particles, or acting between these particles, is Newton's law,

$$m\ddot{q}_i(t) = F_i(t)\,. \tag{1.1}$$

We will restrict ourselves to the case in which the forces are derivable from a potential $V(q_i)$ according to

$$F_i = -\frac{\mathrm{d}}{\mathrm{d}q_i}V\,. \tag{1.2}$$

The forces are then called "conservative".

Given the value of the coordinates q_i and the velocities $\dot{q}_i$ at a given time, Newton's law allows us to construct the full trajectory in terms of the coordinate functions $q_i(t)$. Alternatively, one can uniquely determine a trajectory by specifying the value of the coordinates $q_i(t)$ at two different times, say t_l and t_2, i.e.

$$q_{i1} = q_i(t_1)\,, \qquad q_{i2} = q_i(t_2)\,. \tag{1.3}$$

Then, among the infinite variety of ways in which this physical system can move from q_{i1} to q_{i2}, Newton's equation uniquely picks out one particular trajectory. Now suppose that one assigns a real number to each of the possible trajectories between q_{i1} and q_{i2}. An object that assigns a *number* to a *function* is called a "functional", and we denote it here by $S[q_i(t)]$. It turns out that it is possible to define a functional, called the *action*, such that the number assigned to the physical path between q_{i1} and q_{i2} that is prescribed by Newton's law corresponds to a stationary value (usually a minimum) of this functional. There are thus two alternative approaches to the problem at hand which lead to equivalent results. One is to simply solve Newton's equation (1.1); the other one is to determine the trajectory for which the action functional that we are about to define acquires a minimum (see fig. 1.1). The latter formulation thus amounts to solving a variational principle.

In order to present the variational principle we first define the action functional $S[q_i(t)]$. It is written as the time integral of the *Lagrangian* $L(q_i, \dot{q}_i)$, which is usually a function of coordinates and velocities, i.e.

$$S[q_i(t)] = \int_{t_1}^{t_2} \mathrm{d}t\; L(q_i(t), \dot{q}_i(t))\,. \tag{1.4}$$

Note that (1.4) is indeed a functional because it assigns a number to every given trajectory described by $q_i(t)$. The variational principle now states that the trajectory that is followed by the system is the one for which $S[q_i(t)]$ has an extremum. Hence

$$\delta S[q_i(t)] = 0\,, \tag{1.5}$$

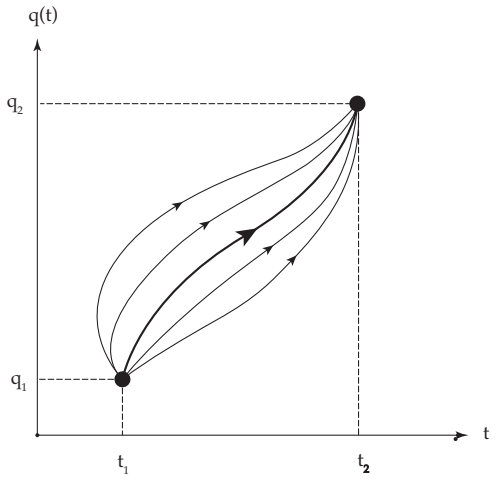

Figure 1.1: Various paths from q_1 to q_2. The thick line indicates the path for which the action is stationary. This path satisfies the equation of motion.

for all trajectories $q_i(t)$ that have the required endpoints at $t = t_1$ and $t = t_2$ (cf. 1.3). This variational principle is known as Hamilton's principle.

Irrespective of the precise form of the Lagrangian, it can be shown that solutions of Hamilton's principle satisfy the so-called Euler-Lagrange equations, also called the equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0\,. \tag{1.6}$$

Usually (1.6) is a second-order differential equation which can be solved for the coordinates $q_i(t)$. Its solutions are unique provided we impose certain boundary conditions, such as the specification of all the coordinates and velocities at one instant in time, or of all the coordinates at two different times t_1 and t_2 as we did in (1.3).

Before establishing the equivalence of (1.5) and (1.6) we first elucidate the meaning of (1.6). Consider the motion of a single particle in a potential. The Lagrangian is the difference between the kinetic energy $T = \frac{1}{2}m\dot{q}^2$ and the potential energy $V(\mathbf{q})$, i.e.

$$L = \tfrac{1}{2}m\dot{\mathbf{q}}^2 - V(\mathbf{q})\,, \tag{1.7}$$

so the Euler-Lagrange equation yields

$$m\ddot{\mathbf{q}} + \frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} = m\ddot{\mathbf{q}} - F = 0\,,$$

which is just Newton's law (1.1). For a free particle ($V(\mathbf{q}) = 0$) this equation can easily be solved. If the particle is located at $\mathbf{q}_1$ at $t = t_1$ and at $\mathbf{q}_2$ at $t = t_2$, it moves according to

$$\mathbf{q}(t) = \mathbf{q}_1 \frac{t - t_2}{t_1 - t_2} - \mathbf{q}_2 \frac{t - t_1}{t_1 - t_2} \,. \tag{1.8}$$

This result can be substituted into the action corresponding to the Lagrangian $L = \frac{1}{2} m \, \dot{\mathbf{q}}^2$.

One then finds

$$S[\mathbf{q}(t)] = \frac{m}{2(t_2 - t_1)} (\mathbf{q}_2 - \mathbf{q}_1)^2 \,, \tag{1.9}$$

which depends only on the boundary values of the trajectory. According to Hamilton's principle all other trajectories from $\mathbf{q}_1$ to $\mathbf{q}_2$ must have an action whose value is larger than (1.9); the action thus exhibits a stationary point for the path (1.8) (Hamilton's principle in its weak form requires only that the action has an extremum; in this example the extremum is a minimum).

Let us now show that the variational principle (1.5) is equivalent to the Euler-Lagrange equations (1.6). To see this consider infinitesimal changes of a given trajectory, and examine the corresponding variation of the action. A well-defined way of doing this is to construct a continuous set of curves parametrized by a parameter α. All curves, denoted by $q_i(t;\alpha)$, go from q_{i1} to q_{i2}, so

$$q_i(t_1;\alpha) = q_{i1} \,, \qquad q_i(t_2;\alpha) = q_{i2} \,.$$

The action corresponding to these curves is now a function of α, and the variational principle requires that we find an extremum, i.e.

$$\frac{\partial S}{\partial \alpha} = 0 \,. \tag{1.10}$$

However, this parametric procedure is rather cumbersome, and because $q_i(t;\alpha)$ cannot parametrize all possible trajectories the condition (1.10) is not necessarily equivalent to (1.5). Fortunately it is also possible to consider arbitrary infinitesimal changes of the trajectory

$$q_i(t) \to q_i(t) + \delta q_i(t) \,, \qquad \dot{q}_i(t) \to \dot{q}_i(t) + \frac{\mathrm{d}}{\mathrm{d}t} \delta q_i(t) \,. \tag{1.11}$$

The corresponding change in the action is

$$\begin{aligned} \delta S[q_i(t)] &= S[q_i(t) + \delta q_i(t)] - S[q_i(t)] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \Big\{ \frac{\partial L}{\partial q_i(t)} \delta q_i(t) + \frac{\partial L}{\partial \dot{q}_i(t)} \frac{\mathrm{d}}{\mathrm{d}t} \delta q_i(t) \Big\} \\ &= \int_{t_1}^{t_2} \mathrm{d}t \Big\{ \frac{\partial L}{\partial q_i(t)} - \frac{\mathrm{d}}{\mathrm{d}t} p_i(t) \Big\} \delta q_i(t) + p_i(t) \, \delta q_i(t) \Big|_{t_1}^{t_2} \,, \end{aligned} \tag{1.12}$$

where we have used the definition

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}\,. \tag{1.13}$$

The quantities p_i are called generalized momenta. Imposing the boundary condition (1.3) so that $\delta q_i(t_1) = \delta q_i(t_2) = 0$, the last term in (1.12) vanishes. Thus the requirement that the action be stationary with respect to variations of the coordinate functions yields the Euler-Lagrange equations (1.6). The classical motion of a system of particles therefore follows from Hamilton's principle, and conversely any solution of the Euler-Lagrange equations corresponds to a stationary trajectory of the action functional.

Since we have assumed so far that the Lagrangian depends only on coordinates and velocities the boundary conditions $\delta q_i(t_1) = \delta q_i(t_2) = 0$ are sufficient for deriving the Euler-Lagrange equations from the variational principle. The Euler-Lagrange equations are second-order differential equations in this case, which require the same number of conditions for uniquely specifying a solution. If the Lagrangian depends on higher derivatives of $q_i(t)$ one needs to impose more conditions on the derivatives of $q_i(t)$ at the boundary. Likewise the corresponding differential equation involves higher-order time derivatives, and requires additional initial conditions for specifying its solution. We may bypass this question of boundary conditions by observing that the change of the action under *arbitrary* variations $\delta q_i(t)$ can always be written as the sum of two terms. One term involves an integral over $\delta q_i(t)$ multiplied by an expression that *by definition* is the Euler-Lagrange equation; the other term depends entirely on the values of $\delta q_i(t)$ (and possibly its derivatives) taken at the boundary of the integration domain. This can easily be shown by performing a series of integrations by parts. The Euler-Lagrange equations thus govern the dynamics inside the integration domain. This is not influenced by adding the time derivative of some function of the coordinates and their derivatives to the Lagrangian, because such a term will only depend on the boundary values at $t = t_1$ or t_2. Therefore, the equation of motion will not be affected by this modification, and the boundary conditions can be imposed separately on its solutions.

Once the Lagrangian is known, one may attempt to construct constants of the motion, i.e. quantities that remain constant in time. For instance, when the Lagrangian does not depend explicitly on q_i, the generalized momenta p_i, defined in (1.13), are constant according to (1.6). These constants are then called the momentum integrals of the equations of motion. Another example is the energy integral, which is found by multiplying (1.6) by $\dot{q}_i$, so that

$$\dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \dot{q}_i \frac{\partial L}{\partial q_i} = 0\,.$$

We rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\Big) - \ddot{q}_i\frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i\frac{\partial L}{\partial q_i} = 0\,.$$

Assuming that L does not depend explicitly on t, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}L(q_i,\dot{q}_i) = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\,.$$

Combining the last two equations yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L\Big) = 0\,, \tag{1.14}$$

or

$$\dot{q}_i\,p_i - L = E\,, \tag{1.15}$$

where the constant E is the energy of the system. We return to the question of conserved quantities in section 1.5.

The advantage of the Lagrangian formulation is that it is based on a single scalar function, the Lagrangian, which can be defined on the basis of any set of generalized coordinates $q_i(t)$. If the theory in question exhibits a certain symmetry, then the action functional should be invariant under that symmetry, modulo possible effects from the boundary of the time integral (1.4). In that case one can show that the corresponding Euler-Lagrange equations are invariant in the sense that the symmetry transformation applied to a given solution of these equations will lead to another solution. The invariance property is sometimes all one needs in order to deduce the action for a given system. For example, consider a free relativistic particle. If such a particle travels from one point to another, the obvious relativistic invariant is the proper time, i.e. the time that it takes measured in the rest frame of the particle. During an infinitesimal amount of time $\mathrm{d}t$ a particle with velocity $\dot{\mathbf{q}}$ is displaced over a distance $\mathrm{d}\mathbf{q} = \dot{\mathbf{q}}\,\mathrm{d}t$. As is well-known, the corresponding time interval is shorter in the particle rest frame, and equal to

$$\mathrm{d}\tau = \sqrt{1-(\dot{\mathbf{q}}/c)^2}\mathrm{d}t\,,$$

where c is the velocity of light and m is the mass of the particle. We obtain a relativistic invariant when integrating the proper time during the particle motion. Hence we assume a Lagrangian

$$L = -mc^2\sqrt{1-(\dot{\mathbf{q}}/c)^2}\,, \tag{1.16}$$

which leads to an invariant action. Observe that the proportionality constant mc^2 is fixed by dimensional arguments. The generalized momentum is equal to

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{m\dot{\mathbf{q}}}{\sqrt{1-(\dot{\mathbf{q}}/c)^2}}\,, \tag{1.17}$$

and the energy is equal to

$$E = \dot{\mathbf{q}} \cdot \mathbf{p} - L = \frac{m\dot{\mathbf{q}}^2}{\sqrt{1-(\dot{\mathbf{q}}/c)^2}} + mc^2\sqrt{1-(\dot{\mathbf{q}}/c)^2}\,,$$

or

$$E = \frac{mc^2}{\sqrt{1-(\dot{\mathbf{q}}/c)^2}}\,. \tag{1.18}$$

These expressions for energy and momentum are subject to the relativistic dispersion relation, $E^2 = \mathbf{p}^2c^2 + m^2c^4$, as expected.

1.2. The Lagrangian for continuous systems

Until now we have discussed a system with a finite number of degrees of freedom. The transition to an infinite number of degrees of freedom is necessary for the treatment of continuous systems, such as a vibrating solid, since their motion is described by specifying the position coordinates of all points. The continuum case can be approached by taking the appropriate limit of a system with a finite number of discrete coordinates. To illustrate this procedure consider an elastic rod of fixed length l, undergoing small longitudinal vibrations. The continuous rod can be approximated by a set of discrete coordinates representing a long chain of n equal mass particles spaced a distance a apart and connected by uniform massless springs having force constants k. The total length of the system is $l = (n+1)a$ as depicted in fig. 1.2

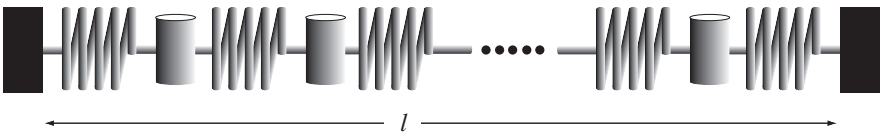

Figure 1.2: Equal masses connected by springs between fixed endpoints separated by a distance l.

If the displacement of the i-th particle from its equilibrium position is measured by the quantity ϕ_i, then the kinetic energy of this one-dimensional chain of particles equals

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m\,\dot{\phi}_i^2\,, \tag{1.19}$$

where m is the mass of each particle. The potential energy is the sum of $n+1$ potential energies of each spring as the result of being stretched or compressed from its equilibrium distance,

$$V = \tfrac{1}{2}\sum_{i=0}^{n} k(\phi_{i+1} - \phi_i)^2 \,, \tag{1.20}$$

where k is the force constant. The force on the i-th particle follows from the potential via $F_i = -\partial V/\partial\phi_i$, so that we obtain

$$F_i = k(\phi_{i+1} - \phi_i) - k(\phi_i - \phi_{i-1}) = k(\phi_{i+1} + \phi_{i-1} - 2\phi_i) \,. \tag{1.21}$$

The force thus decomposes into two parts; the force exerted by the spring on the right of the i-th particle, equal to $k(\phi_{i+1} - \phi_i)$, and the force exerted by the spring on the left, equal to $k(\phi_i - \phi_{i-1})$. Combining (1.19) and (1.20) gives the Lagrangian

$$L = T - V = \tfrac{1}{2}\sum_{i=1}^{n} m\dot{\phi}_i^2 - \tfrac{1}{2}\sum_{i=0}^{n} k(\phi_{i+1} - \phi_i)^2 \,. \tag{1.22}$$

The corresponding Euler-Lagrange equations yield Newton's law $m\,\ddot{\phi}_i = F_i$.

In order to describe the elastic rod we must take the continuum limit of the system discussed above. Hence we increase the number of particles to infinity ($n \to \infty$), keeping the total length $l = (n+1)a$ and the mass per unit length, $\mu = m/a$, fixed. Furthermore $Y = ka$ must be kept fixed as well. This follows from Hooke's law, which tells us that the extension of the rod per unit length is directly proportional to the force exerted on the rod, with Young's modulus being the constant of proportionality. In the discrete case the force between two particles equals $F = k(\phi_{i+1} - \phi_i)$, and the extension of the interparticle spacing per unit length equals $(\phi_{i+1} - \phi_i)/a$. Hence we identify $Y = ka$ as Young's modulus which should be kept constant in the continuum limit.

Rewriting the Lagrangian (1.22) as

$$L = \tfrac{1}{2}\sum_{i=1}^{n} a\Big(\frac{m}{a}\dot{\phi}_i^2\Big) - \tfrac{1}{2}\sum_{i=0}^{n} a(ka)\Big(\frac{\phi_{i+1} - \phi_i}{a}\Big)^2 \,, \tag{1.23}$$

it is straightforward to take the limit $a \to 0$, $n \to \infty$ with $l = (n+1)a$, $\mu = m/a$ and $Y = ka$ kept fixed. The continuous position coordinate x now replaces the label i, and ϕ_i becomes a function of x, i.e. $\phi_i \to \phi(x)$. Hence the Lagrangian becomes an integral over the length of the rod,

$$L = \tfrac{1}{2}\int_0^l \mathrm{d}x\, \Big[\mu\,\dot{\phi}^2 - Y(\partial_x\phi)^2\Big] \,, \tag{1.24}$$

where we have used

$$\lim_{a\to 0}\frac{\phi_{i+1} - \phi_i}{a} = \lim_{a\to 0}\frac{\phi(x+a) - \phi(x)}{a} = \frac{\partial\phi}{\partial x} \equiv \partial_x\phi \,.$$

Also the equation of motion for the coordinate ϕ_i can be obtained by this limiting procedure. Starting from

$$\frac{m}{a}\ddot{\phi}_i - ka\frac{\phi_{i+1}+\phi_{i-1}-2\phi_i}{a^2} = 0\,, \tag{1.25}$$

and using

$$\lim_{a\to 0}\frac{\phi_{i+1}+\phi_{i-1}-2\phi_i}{a^2} = \frac{\partial^2\phi}{\partial x^2} \equiv \partial_{xx}\phi\,,$$

the equation of motion becomes

$$\mu\,\ddot{\phi} - Y\,\partial_{xx}\phi = 0\,. \tag{1.26}$$

We see from this example that x is a continuous variable replacing the discrete label i. Just as there is a generalized coordinate ϕ_i for each i, there is a generalized coordinate $\phi(x)$ for each x, i.e. the finite number of coordinates ϕ_i has been replaced by a function of x. In fact ϕ depends also on time, so we are dealing with a function of two variables. This function $\phi(x,t)$ is called the *displacement field*, and $\dot{\phi} = \partial_t\phi$ and $\partial_x\phi$ are its partial derivatives with respect to time and position.

The Lagrangian (1.24) appears as an integral over x of

$$\mathcal{L} = \tfrac{1}{2}\mu\,\dot{\phi}^2 - \tfrac{1}{2}Y\,(\partial_x\phi)^2\,, \tag{1.27}$$

which is called the Lagrangian density. In this case it is a function of $\phi(x,t)$ and its first-order derivatives $\partial_t\phi(x,t)$ and $\partial_x\phi(x,t)$, but one can easily envisage further generalizations. It has become common practice in field theory to simply call the Lagrangian density the Lagrangian. What is relevant is the action, which can now be written as an integral over both space and time, i.e.

$$S[\phi(x,t)] = \int_{t_1}^{t_2}\mathrm{d}t\int_0^l \mathrm{d}x\;\mathcal{L}(\phi(x,t),\dot{\phi}(x,t),\partial_x\phi(x,t))\,. \tag{1.28}$$

It is a functional of $\phi(x,t)$, i.e. it assigns a number to any function $\phi(x,t)$ of space and time. Observe that the Lagrangian equals the difference of the kinetic and the potential energy, as usual. The latter is proportional to the integral over $(\partial_x\phi)^2$.

It is possible to obtain the equations of motion for $\phi(x,t)$ directly from Hamilton's principle by following the same arguments as in the previous section. One then simply investigates the change in the action under an infinitesimal change in the field,

$$\begin{aligned}
\phi(x,t) &\to \phi(x,t) + \delta\phi(x,t)\,,\\
\partial_t\phi(x,t) &\to \partial_t\phi(x,t) + \frac{\partial}{\partial t}\delta\phi(x,t)\,,\\
\partial_x\phi(x,t) &\to \partial_x\phi(x,t) + \frac{\partial}{\partial x}\delta\phi(x,t)\,,
\end{aligned} \tag{1.29}$$

leading to

$$\begin{aligned}\delta S[\phi(x,t)] &= S[\phi(x,t)+\delta\phi(x,t)] - S[\phi(x,t)] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \int_0^l \mathrm{d}x \\ &\quad \times \Big\{ \frac{\partial \mathcal{L}}{\partial \phi(x,t)} \delta\phi(x,t) + \frac{\partial \mathcal{L}}{\partial(\partial_t \phi(x,t))} \frac{\partial}{\partial t} \delta\phi(x,t) \\ &\qquad + \frac{\partial \mathcal{L}}{\partial(\partial_x \phi(x,t))} \frac{\partial}{\partial x} \delta\phi(x,t) \Big\} . \end{aligned} \tag{1.30}$$

Integrating the second and third term by parts,

$$\begin{aligned} \int_{t_1}^{t_2} \mathrm{d}t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \frac{\partial \delta\phi}{\partial t} &= \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta\phi \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t \frac{\partial}{\partial t} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \Big) \delta\phi , \\ \int_0^l \mathrm{d}x \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \frac{\partial \delta\phi}{\partial x} &= \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta\phi \Big|_{t_1}^{t_2} - \int_0^l \mathrm{d}x \frac{\partial}{\partial x} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \Big) \delta\phi , \end{aligned} \tag{1.31}$$

leads to

$$\begin{aligned}\delta S[\phi(x,t)] &= \int_{t_1}^{t_2} \mathrm{d}t \int_0^1 \mathrm{d}x \, \delta\phi \Big\{ \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \Big) - \frac{\partial}{\partial x} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \Big) \Big\} \\ &\quad + \int_0^l \mathrm{d}x \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta\phi \Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \mathrm{d}t \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta\phi \Big|_{x=0}^{x=l} . \end{aligned} \tag{1.32}$$

Hamilton's principle requires that the action be stationary with respect to infinitesimal variations of the fields that leave the field values at the initial and final time unaffected, i.e. $\phi(x,t_1) = \phi_1(x)$ and $\phi(x,t_2) = \phi_2(x)$. Therefore we have $\delta\phi(t_1,x) = \delta\phi(t_2,x) = 0$. On the other hand, because the rod is clamped, the displacement at the endpoints must be zero, i.e. $\delta\phi(x,t) = 0$ for $x = 0$ and $x = l$. Under these circumstances the last two terms in (1.32) vanish, and Hamilton's principle gives

$$\frac{\partial}{\partial t} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \Big) + \frac{\partial}{\partial x} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \Big) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 . \tag{1.33}$$

This is the Euler-Lagrange equation for a continuous system.

As a check one can insert the Lagrangian (1.27) into (1.33) to derive the equation of motion, which indeed gives (1.26). Note that with a suitable choice of units we can write the Lagrangian (1.27) as

$$\mathcal{L} = \tfrac{1}{2} (\partial_t \phi)^2 - \tfrac{1}{2} (\partial_x \phi)^2 . \tag{1.34}$$

The generalization to continuous systems in more space dimensions is now straightforward, and one can simply extend the definitions of the Lagrangian and the Euler-Lagrange equations.

For example, in two dimensions one may start with a two-dimensional system of springs. The displacement of the particle at the site labelled by (i, j) is measured by the quantity $\boldsymbol{\phi}_{ij}(t)$, which is labelled by a two-dimensional vector that characterizes the displacement in the two-dimensional plane. In the limit when we go to a continuous system this becomes the two-dimensional displacement field $\boldsymbol{\phi}_{ij}(x, y, t)$, of a membrane subjected to small vibrations in the (x, y) plane. In three dimensions one may consider a crystal whose sites are labelled by (i, j, k). The displacement of a particle at a site is measured by the vector field $\boldsymbol{\phi}_{ijk}(t)$. The continuous limit yields a three-dimensional displacement field $\boldsymbol{\phi}(x, y, z, t)$ of a continuous solid subjected to small vibrations in the x, y, z directions. In such a case the equations of motion would involve a partial differential equation of the type

$$\begin{aligned} &\ddot{\boldsymbol{\phi}} - c_1\,\partial_{xx}\boldsymbol{\phi} - c_2\,\partial_{yy}\boldsymbol{\phi} - c_3\,\partial_{zz}\boldsymbol{\phi} \\ &- c_4\,\partial_{xy}\boldsymbol{\phi} - c_5\,\partial_{yz}\boldsymbol{\phi} - c_6\,\partial_{xz}\boldsymbol{\phi} = 0\,, \end{aligned} \tag{1.35}$$

where the coefficient functions $c_1(x, y, z), \ldots, c_6(x, y, z)$ depend on the properties of the solid.

The solutions of the classical wave equations for both the discrete and the continuous systems yield the so-called normal modes of oscillation. These are waves with wave number and angular frequency related by some dispersion relation. In the simple case of a continuous rod with fixed endpoints the eigenfunctions for each normal mode are proportional to $\sin kx\,\cos(\omega t + \alpha)$, where α is a constant. Because $\phi(0, t) = \phi(l, t) = 0$ the wave numbers are given by $k_n = \pi n/l$, with $n = 0, 1, 2, \ldots$. The wave equation (1.26) restricts the eigenfrequencies to $\omega_n = k_n c$, where $c = \sqrt{Y/\mu}$ is the propagation velocity of the vibrations through the rod. The most general solution is a superposition of normal modes:

$$\phi(x, t) = \sum_n \sin k_n x\,\left[A_n \cos\omega_n t + B_n \sin\omega_n t\right]\,, \tag{1.36}$$

where the constants are fixed by the initial conditions.

1.3. Relativistic classical field theory

In the previous section we have introduced fields to measure the displacement from an equilibrium position. From now on we will consider arbitrary types of fields, without precisely specifying the degrees of freedom that they describe. These fields will be functions of the four-vector of space-time $x^\mu = (x^0, x^1, x^2, x^3) = (c\,t, \mathbf{x})$, where t denotes the time and c the velocity of light in vacuum. In this book we will ususally choose units in which the velocity of light in vacuum is taken equal to unity. The relativistic invariant

associated with an infinitesimal displacement in space and time is equal to

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - c^2\mathrm{d}t^2\,, \end{aligned} \tag{1.37}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is a diagonal metric invariant under Lorentz transformations (note that unless stated otherwise we use the convention where a summation is implied over repeated indices). The invariant (1.37) is equal to $-c^2$ times the square of the change in the so-called *proper time*. The latter, denoted by $\mathrm{d}\tau$, measures the time span for a particle moving over an infinitesimal distance $\mathrm{d}\boldsymbol{x}$ in a time interval $\mathrm{d}t$, as measured in its own rest-frame. For a light signal, (1.37) will vanish, because the infinitesimal change in the spatial coordinates is related to the infinitesimal time interval $\mathrm{d}t$ through the velocity of light c.

The metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ can be used to raise and lower indices, so that we have

$$\mathrm{d}x_\mu \equiv \eta_{\mu\nu}\mathrm{d}x^\nu\,, \qquad \mathrm{d}x^\mu = \eta^{\mu\nu}\mathrm{d}x_\nu\,. \tag{1.38}$$

Observe that this implies $\mathrm{d}x_\mu = (-c\,\mathrm{d}t, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$. The notation with upper and lower indices is very common in the literature and will be used throughout the book. Obviously there are different ways to choose $\eta_{\mu\nu}$, and we relegate a discussion of the various conventions to appendices A and B.

The relativistic invariance of the theory will be guaranteed if the Lagrangian that we consider is invariant under Lorentz transformations, i.e. if it is a Lorentz scalar. To establish this fact requires information on the way in which the fields transform under Lorentz transformations. We return to this aspect in section 1.4. If the Lagrangian depends on the fields and their first-order derivatives the Euler-Lagrange equations are a generalization of (1.33),

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0\,, \tag{1.39}$$

where the four-gradient is defined by

$$\partial_\mu \equiv \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = (c^{-1}\partial_t, \boldsymbol{\nabla})\,. \tag{1.40}$$

and the summation convention is implied over the index $\mu = 0,1,2,3$ which labels the coordinates x^μ. Observe that the derivative with respect to x^μ carries lower indices, so that $\partial x^\mu/\partial x^\nu = \delta^\mu{}_\nu$. Contractions between four-vectors with upper and with lower indices are always Lorentz invariant, in view of $X_\mu Y^\mu = \eta_{\mu\nu}X^\mu\,Y^\nu$, for any two four-vectors X^μ and Y^μ. This is further discussed in section 1.4 (see also appendix A). In relativistic quantum field theory it is convenient to choose the velocity of light in vacuum and Planck's

constant to be dimensionless and equal to unity ($c = \hbar = 1$), so we will usually suppress these factors. With this convention there is only one dimensional unit; for example one may choose length, in which case mass parameters have dimension $[\text{length}]^{-1}$, or one may adopt mass as the basic unit so that length and time have dimension $[\text{mass}]^{-1}$. The action is then dimensionless, which implies that the Lagrangian must have dimension $[\text{mass}]^4$. However, in this subsection we will retain the light velocity c for easy comparison with the literature.

The simplest example of a relativistic field theory is a generalization of (1.34) with Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2c^2\,\phi^2\,. \tag{1.41}$$

Note that the square of the four-vector $\partial_\mu\phi$ is always defined by contraction with the Lorentz invariant inverse metric $\eta^{\mu\nu}$. Hence we have $(\partial_\mu\phi)^2 \equiv \eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi = -c^{-2}(\partial_t\phi)^2 + (\boldsymbol{\nabla}\phi)^2$. Here $\phi(x)$ is a scalar field, which means that, in two different Lorentz frames with coordinates x^μ and x'^μ related by a Lorentz transformation, the scalar fields are related by

$$\phi'(x') = \phi(x)\,. \tag{1.42}$$

The Euler-Lagrange equation corresponding to (1.41) is

$$\Box\phi - m^2c^2\phi = 0\,, \tag{1.43}$$

where $\Box = \partial_\mu^2 = \boldsymbol{\nabla}^2 - c^{-2}\partial_t^2$. This equation is the so-called Klein-Gordon equation which was originally postulated as a generalization of the non-relativistic Schrödinger equation. Solutions of this equation consist of superpositions of plane waves

$$\phi(x) \propto \mathrm{e}^{\mathrm{i}p\cdot x}\,, \tag{1.44}$$

where we have introduced the four-vector $p^\mu = (p^0, \mathbf{p})$ so that $p\cdot x = \eta_{\mu\nu}p^\mu x^\nu = \mathbf{p}\cdot\mathbf{x} - p^0x^0$. Equation (1.43) implies that $p^2 = \mathbf{p}^2 - (p^0)^2 = -(m\,c)^2$, which can be solved for p^0 by

$$p^0 = \pm\sqrt{\mathbf{p}^2 + m^2c^2}\,. \tag{1.45}$$

Clearly p^0 can be identified with the energy divided by c, so that (1.45) takes the form of the dispersion law for a free relativistic particle (cf. 1.18). This suggests that the degrees of freedom described by the Lagrangian (1.41) will correspond to those of a free relativistic particle with mass m. The correspondence with physical particles will be further pursued from a somewhat different perspective in chapter 2.

One can introduce interaction terms into the theory by adding a more complicated function of the fields to (1.41). For instance

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2c^2\,\phi^2 - g\,\phi^4\,, \tag{1.46}$$

where g is a coupling constant. The corresponding equation of motion reads

$$\Box\phi - m^2c^2\,\phi - 4g\phi^3 = 0\,. \tag{1.47}$$

Field theories based on fields that have a more complicated behaviour under Lorentz transformations can be investigated by similar methods. A well-known example is Maxwell's theory of electromagnetism, which we will now briefly describe. Since we only consider the electromagnetic field in vacuum we use the so-called rationalized c.g.s units where $\varepsilon_0 = 1$ and $\mu_0 = 1$. This has the advantage that there are no factors of 4π in the Maxwell equations. In this system of units the fine structure constant is $\alpha = e^2/4\pi\hbar c$.

Suppose we want to derive the classical Maxwell equations from a Lagrangian formulation. These equations involve six field components, three for the electric field $\mathbf{E}(\mathbf{x},t)$ and three for the magnetic field $\mathbf{B}(\mathbf{x},t)$. They are usually written as two homogeneous and two inhomogeneous equations. The homogeneous equations are

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,, \qquad \boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\,\frac{\partial\mathbf{B}}{\partial t} = 0\,. \tag{1.48}$$

Some of their physical consequences become transparent when we cast them in an integral form by making use of the well-known integral theorems of Gauss and Stokes. The result is

$$\oint_{\mathrm{S}} \mathbf{B}\cdot\mathrm{d}\mathbf{O} = 0\,, \qquad \frac{1}{c}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{S}} \mathbf{B}\cdot\mathrm{d}\mathbf{O} = -\oint_{\mathrm{C}} \mathbf{E}\cdot\mathrm{d}\mathbf{l}\,. \tag{1.49}$$

The first equation implies that the magnetic flux through a closed surface S must vanish, thus excluding magnetic monopoles (unless, following Dirac, one attaches a "string" to the monopole to transport the flux back through the surface S; in the limit that the thickness of the string tends to zero the magnetic field on S develops a singularity at the point where the string passes through it, but one can show that the string itself has no observable effect if the monopole magnetic charge is quantized). The second equation expresses Faraday's law of induction, which relates the time derivative of the magnetic flux through an open (fixed) surface S to the electric field integrated along the closed circuit C that forms the boundary of S (this line integral is called the electromotive force).

The inhomogeneous Maxwell equations are Coulomb's law and a generalization of Ampère's law

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\,\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{c}\,\mathbf{J}\,, \tag{1.50}$$

where $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ are the electric charge and current densities. From (1.50) one can derive that electric charge must be locally conserved,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0\,. \tag{1.51}$$

The four-vector current equals $J^\mu = (c\,\rho, \boldsymbol{J})$, so that the above equation reads $\partial_\mu J^\mu = 0$. To prove that charge is locally conserved, one integrates (1.51) over a given volume V. The time derivative of the total charge contained in V is then equal to the charge that leaves or enters through the boundary surface S of that volume per unit of time, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = -c\oint_S \mathbf{J}\cdot\mathrm{d}\mathbf{O}\,. \tag{1.52}$$

Similarly we can integrate the first equation (1.50), which leads to

$$\int_V \rho\,\mathrm{d}V = \oint_{\mathrm{S}} \mathbf{E}\cdot\mathrm{d}\mathbf{O}\,. \tag{1.53}$$

This is Gauss' law which relates the total charge in a volume to the flux of $\mathbf{E}$ through the boundary surface S.

The homogeneous equations (1.48) can be solved by expressing the electric and magnetic fields in terms of a scalar and a vector potential, $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$, respectively. Indeed, the homogeneous equations are satisfied for

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}\,. \tag{1.54}$$

Again we can bring these relations into integral form. The second one then relates the magnetic flux through a surface to the line integral of $\mathbf{A}$ along the boundary of that surface, which follows upon using Stokes' theorem. Explicitly one finds

$$\int_{\mathbf{S}} \mathbf{B}\cdot\mathrm{d}\mathbf{O} = \oint_{\mathbf{C}} \mathbf{A}\cdot\mathrm{d}\mathbf{l}\,, \tag{1.55}$$

where C forms the boundary of S.

To construct a Lagrangian for electrodynamics we use the potentials $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ as the generalized coordinates of the electromagnetic field. We start from the Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - \rho\,\phi + \frac{1}{c}\,\mathbf{J}\cdot\mathbf{A}\,, \tag{1.56}$$

where $\mathbf{E}$ and $\mathbf{B}$ are now *defined* by (1.54), and we will show that the Euler-Lagrange equations corresponding to (1.56) yield precisely the two inhomogeneous Maxwell equations (1.50). The coupling of the current J^μ in (1.56) indicates that the potentials ϕ and $\boldsymbol{A}$ constitute a four-vector $A_\mu = (-\phi, \boldsymbol{A})$, so that $J^\mu A_\mu = -c\,\rho\,\phi + \boldsymbol{J}\cdot\boldsymbol{A}$.

The equation for ϕ involves

$$\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_i)} = E_j\frac{\partial E_j}{\partial(\partial\phi/\partial x_i)} = -E_i \qquad (i,j=1,2,3)\,,$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\rho \,. \tag{1.57}$$

The Lagrangian (1.56) does not contain $\partial\phi/\partial x_0$. Hence (1.39) reduces to

$$\frac{\partial E_j}{\partial x_j} - \rho = 0 \qquad \text{or} \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = \rho \,, \tag{1.58}$$

which is Coulomb's law. The equation for $\mathbf{A}$ can be derived component by component. Then

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial A_1} = c^{-1} J_1 \,, \\
&\frac{\partial \mathcal{L}}{\partial(\partial A_1/\partial t)} = E_1 \frac{\partial E_1}{\partial(\partial A_1/\partial t)} = -E_1 \,, \\
&\frac{\partial \mathcal{L}}{\partial(\partial A_1/\partial x_2)} = -B_3 \frac{\partial B_3}{\partial(\partial A_1/\partial x_2)} = B_3 \,, \\
&\frac{\partial \mathcal{L}}{\partial(\partial A_1/\partial x_3)} = -B_2 \,.
\end{aligned} \tag{1.59}$$

Since $\partial A_1/\partial x_1$ is absent in the definitions of $\mathbf{E}$ and $\mathbf{B}$ the Euler-Lagrange gives

$$\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{\partial E_1}{\partial t} - c^{-1} J_1 = 0 \,, \tag{1.60}$$

which is the first component of the equation (1.50) for the vector curl of $\mathbf{B}$. By similar methods the other components can be checked. Hence the two inhomogenous Maxwell equations follow from the Lagrangian (1.56), while the homogeneous equations (1.48) have been implemented from the start by virtue of the definitions (1.54).

The Maxwell equations (1.48) and (1.50) can be written more concisely when introducing an antisymmetric field-strength tensor $F_{\mu\nu}(x) \equiv F_{\mu\nu}(\mathbf{x}, t)$, defined by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} . \tag{1.61}$$

The inverse relations, defining B_i and E_i in terms of $F_{\mu\nu}$ are

$$B_i = \tfrac{1}{2} \varepsilon_{ijk} F^{jk} \,, \qquad (i, j, k = 1, 2, 3) \,, \tag{1.62}$$

where ε_{ijk} is a totally antisymmetric tensor normalized by $\varepsilon_{123} = 1$, and

$$E_i = F_{0i}\,. \tag{1.63}$$

Using the potential $A_\mu = (-\phi, \boldsymbol{A})$, and the four-vector current J^μ, defined previously, the relation (1.54) becomes

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1.64}$$

and the Maxwell equations read

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0\,, \tag{1.65}$$

$$\partial_\nu F^{\mu\nu} = c^{-1} J^\mu\,. \tag{1.66}$$

The advantage of writing Maxwell's equations in this form is that they are manifestly Lorentz invariant once we assume that A_μ and J_μ transform as four-vectors under Lorentz transformations.

In terms of $F_{\mu\nu}$ the Lagrangian (1.56) reads

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + c^{-1} J^\mu A_\mu\,, \tag{1.67}$$

which indeed transforms as a Lorentz scalar. The relativistic formulation facilitates the derivation of the equation of motion. One easily verifies that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}\,, \qquad \frac{\partial \mathcal{L}}{\partial A_\mu} = c^{-1} J^\mu\,, \tag{1.68}$$

so that the Euler-Lagrange equations lead directly to (1.66). Written in terms of A_μ this equation reads

$$\Box A_\mu - \partial_\mu(\partial^\nu A_\nu) = -c^{-1} J_\mu\,. \tag{1.69}$$

There is an ambiguity involved in the relation (1.64) between $F_{\mu\nu}$ and A_μ because the vector potential can be changed to

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \xi(x), \tag{1.70}$$

without changing $F_{\mu\nu}$. The parameter $\xi(x)$ in this so-called *gauge transformation* is some arbitrary function of $\mathbf{x}$ and t. We can avoid this ambiguity by imposing a gauge condition. For instance, one may impose the Lorentz condition by choosing a function ξ such that

$$\partial^\mu A'_\mu(x) = \partial^\mu A_\mu + \Box \xi(x) = 0\,.$$

In terms of A_μ the equations of motion reduce to

$$\Box A'_\mu = -c^{-1} J_\mu\,. \tag{1.71}$$

Observe that (1.70) is only consistent provided that $\partial^\mu J_\mu = 0$ [the fact that the current is conserved follows already from the Maxwell equations, as was previously shown in (1.51)]. However, even in the gauge $\partial^\mu A'_\mu = 0$ the potential $A'_\mu(x)$ is not yet uniquely given because $A'_\mu(x)$ can be changed into $A'_\mu(x) + \partial_\mu \xi(x)$ provided that $\xi(x)$ is a solution of the homogeneous equation

$$\Box \xi(x) = 0\,.$$

$A_\mu(x)$ is generally called a vector *gauge field*, and Maxwell's theory of electromagnetism is just the simplest example of a gauge theory.

The possibility of performing gauge transformations on the vector potential indicates that there is more information contained in A_μ than is of physical significance. Indeed, in classical physics the measurable fields are the electric and magnetic fields induced by current and charge distributions, and the introduction of A_μ is done for reasons of convenience (in order to implement the homogeneous Maxwell equations from the start). For any given value of A_μ it is straightforward to determine $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$. There are, however, certain subtleties connected with gauge transformations on $A_\mu(x)$ which should be mentioned here. Suppose we have a finite region in space where there are no electric or magnetic fields. Using suitable gauge transformations, we can then consistently define $A_\mu(x) = 0$ in this region. Consequently the line integral of the vector potential around any closed loop in this region vanishes, which is consistent with (1.55). Let us now introduce a confined flux of a static magnetic field through a long straight tube in the middle of this region, so that the electromagnetic fields vanish outside the tube. According to (1.55) the line integral of the vector potential around the tube is then nonvanishing and proportional to the amount of magnetic flux through the tube. Therefore, even though the magnetic field $\mathbf{B}$ is zero outside the tube, we cannot define $\mathbf{A} = 0$ everywhere by means of a suitable gauge transformation. Indeed, the line integral of the vector potential along a curve from $\mathbf{r}_1$ to $\mathbf{r}_2$ changes under a gauge transformation according to

$$\int_{r_1}^{r_2} \mathbf{A} \cdot d\mathbf{l} \to \int_{r_1}^{r_2} \mathbf{A} \cdot d\mathbf{l} + \xi(\mathbf{r}_2, t) - \xi(\mathbf{r}_1, t)\,, \tag{1.72}$$

so that the integral along a closed loop (i.e. $\mathbf{r}_1 = \mathbf{r}_2$) is invariant under all gauge transformations that are single valued. Although one can thus bring the vector potential to zero by means of a gauge transformation at every given point, it is not possible to define a gauge transformation globally so that the vector potential will vanish everywhere around the flux tube.

To measure this effect one can envisage a situation where a charged particle is forced to follow a trajectory around the flux tube. A more practical set-up is shown in fig. 1.3 where a coherent beam of electrons is separated by a double slit, such that the two parts pass at different sides of the flux tube. Behind the

tube the beams are recombined. In this way both beams pass through regions where the magnetic field is zero, but the vector potential is not. Even though the latter can be made to vanish locally by a gauge transformation, it is not possible to ensure that both beams pass through regions where $\mathbf{A}$ vanishes, because of the arguments given above. In quantum mechanics the vector potential appears in the Schrödinger equation, where it affects the phase of the wavefunction. Therefore the nonzero vector potential will induce a difference in the phase of the wave functions corresponding to the two beams, which becomes visible as an interference effect when the two beams are brought together again. This effect, noted by Ehrenberg and Siday and by Aharonov and Bohm, has been confirmed by experiment. Thus we conclude that $F_{\mu\nu}(x)$ is not always sufficient to describe electromagnetic effects in quantum theory. Knowledge of the (gauge invariant) phase-factors $\exp(\mathrm{i}e \oint A_\mu \mathrm{d}x^\mu)$ around any unshrinkable loop may also be necessary.

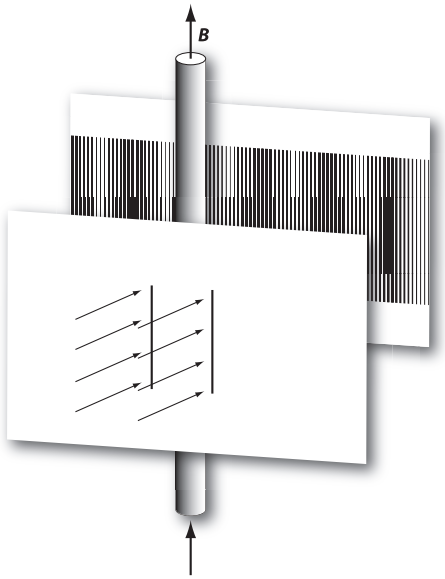

Figure 1.3: A schematic drawing of an experimental apparatus to measure the Aharonov-Bohm effect. When $\mathbf{B} = 0$, the electron intensity on the screen shows a diffraction pattern with maxima and minima. When $\mathbf{B} \neq 0$, this diffraction pattern shifts, rellecting the additional phase difference $e \oint \mathbf{A} \cdot d\mathbf{l}$. This effect is *not* seen when both slits are on the same side of the solenoid.

1.4. Symmetries

Symmetries will play a central role throughout this book, so we had better take a first look at their definition and consequences. A transformation, or a set of transformations, defines a symmetry of a theory if it leaves the equations of motion invariant. Such transformations may be continuous or discrete. A typical example of a discrete symmetry is parity reversal, under which the four-vector of space and time transforms as

$$(t, \mathbf{x}) \xrightarrow{\mathrm{P}} (t, -\mathbf{x})\,. \tag{1.73}$$

This transformation induces a change in a field $\phi(\mathbf{x}, t)$ given by

$$\phi(\mathbf{x}, t) \xrightarrow{\mathrm{P}} \phi^{\mathrm{p}}(\mathbf{x}, t) = \eta_{\mathrm{p}}\, \phi(-\mathbf{x}, t)\,, \tag{1.74}$$

where $\eta_{\mathrm{p}} = \pm 1$ is a phase factor (called "intrinsic" parity). For $\eta_{\mathrm{p}} = 1$ we have a scalar, and for $\eta_{\mathrm{p}} = -1$ a pseudoscalar field. Its space-time derivative transforms according to

$$\partial_\mu \phi(\mathbf{x}, t) \xrightarrow{\mathrm{P}} \partial_\mu \phi^{\mathrm{p}}(\mathbf{x}, t) = -(1 - 2\delta_{\mu 0})\eta_{\mathrm{p}}\, \partial_\mu \phi(-\mathbf{x}, t)\,, \tag{1.75}$$

and thus transforms as a vector ($\eta_{\mathrm{p}} = 1$) or an axial vector ($\eta_{\mathrm{p}} = -1$). The electromagnetic vector potential transforms as a vector under parity reversal,

$$A_\mu(\mathbf{x}, t) \xrightarrow{\mathrm{P}} A^{\mathrm{p}}_\mu(\mathbf{x}, t) = -(1 - 2\delta_{\mu 0}) A_\mu(-\mathbf{x}, t)\,, \tag{1.76}$$

so that the electric field transforms as a three-dimensional vector, and the magnetic field as a three-dimensional pseudovector. Of course, the Lagrangian is strictly speaking not invariant under parity reversal, but the action is as the space-time integration domain also changes in accordance with (1.73). From this one may infer that the equations of motion are invariant under parity reversal, as can be verified by explicit computation.

Before giving more examples let us emphasize another property of symmetry transformations. Two successive transformations define a new transformation under which the action must again be invariant. In the case of parity reversal this is trivial, since two parity transformations lead back to the original fields; hence P^2 is simply the identity transformation. Similarly one can argue that the inverse transformation must also leave the action invariant, so that the full set of transformations, consisting of P and the identity, satisfies the defining properties of what mathematicians call a *group*. Since any discussion of symmetry properties will thus be based on group theory, we have collected some useful definitions and results on group theory in appendix C.

Let us now consider an example of a field theory which is invariant under continuous transformations. We first generalize the Lagrangian (1.46) for a

real scalar field to the case of a complex-scalar field. The latter is equivalent to two real scalar fields, according to

$$\phi = \frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}}\,, \qquad \phi^* = \frac{\phi_1 - \mathrm{i}\phi_2}{\sqrt{2}}\,, \tag{1.77}$$

and the Lagrangian can be written in terms of either (ϕ_1, ϕ_2) or (ϕ, ϕ^*). Choosing the second option we consider

$$\begin{aligned} \mathcal{L} &= -\partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\phi^* - g(\phi\phi^*)^2 \\ &= -|\partial_\mu\phi|^2 - m^2|\phi|^2 - g|\phi|^4\,. \end{aligned} \tag{1.78}$$

This Lagrangian is invariant under phase transformations,

$$\phi \,\to\, \phi' = \mathrm{e}^{\mathrm{i}\xi}\phi\,, \qquad \phi^* \,\to\, (\phi^*)' = \mathrm{e}^{-\mathrm{i}\xi}\phi^*\,, \tag{1.79}$$

which depend on a continuous parameter ξ, and obviously form a group; this group is called U(1), the group of 1×1 unitary matrices. Incidentally, the Lagrangian is also invariant under the interchange of ϕ and ϕ^*. This discrete transformation is often called "charge" conjugation.

The phase transformations (1.79) simply correspond to rotations of the real and imaginary parts of the fields ϕ. Hence, when expressed in terms of (ϕ_1, ϕ_2) the theory will be invariant under two-dimensional real rotations; the corresponding group is called SO(2), and it is locally equivalent to U(1). Indeed, the invariance under SO(2) is manifest in

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi_1)^2 - \tfrac{1}{2}(\partial_\mu\phi_2)^2 - \tfrac{1}{2}m^2(\phi_1^2 + \phi_2^2) - \tfrac{1}{4}g(\phi_1^2 + \phi_2^2)^2\,, \tag{1.80}$$

which follows from (1.78) after substitution of (1.77). The theory under consideration thus describes two real fields, which behave in a similar fashion owing to the invariance under (1.79). As we will discuss in the next section there exists a conserved quantity associated with the behaviour under phase transformations. Since ϕ and ϕ^* have opposite phase factors in (1.79), we say that ϕ and ϕ^* have opposite "charges", to indicate that their contributions to this conserved quantity will have opposite sign. Therefore, the degrees of freedom associated with ϕ and ϕ^* are often called "particle" and "antiparticle". The only property that distinguishes them is that they carry opposite "charge".

As another example of a continuous symmetry let us briefly discuss Lorentz transformations. They are important because all field theories that we intend to discuss here will be Lorentz invariant. Under Lorentz transformations four-vectors X^μ, such as the four-vector of space and time, transform according to

$$X^\mu \to X'^\mu = L^\mu{}_\nu\, X^\nu\,, \tag{1.81}$$

where $L^\mu{}_\nu$ is a 4×4 matrix. The defining condition for the transformation (1.81) is that inner products of two four-vectors X^μ and Y^μ, defined by $X \cdot Y = \eta_{\mu\nu} X^\mu Y^\nu$, must be invariant under Lorentz transformations. This implies that

$$L^\mu{}_\rho \, L^\nu{}_\sigma \, \eta_{\mu\nu} = \eta_{\rho\sigma} \,,$$

which means that the metric $\eta_{\mu\nu}$ is a Lorentz invariant tensor. It also follows that the inverse of the matrix $L^\mu{}_\nu$ is equal to

$$(L^{-1})^\mu{}_\nu = L_\nu{}^\mu \equiv \eta_{\mu\rho} \, L^\rho{}_\sigma \, \eta^{\sigma\nu} \,. \tag{1.82}$$

If the metric $\eta_{\mu\nu}$ were proportional to the identity, then this would be the defining condition for four-dimensional orthogonal transformations where the inverse matrices are equal to their transpose. These matrices constitute the group O(4). This group contains the group SO(4) of real rotations in four dimensions as a subgroup. However, the metric $\eta_{\mu\nu}$ has eigenvalues $(-1, 1, 1, 1)$ and therefore we are dealing with a "non-compact" version of SO(4), which is called $\mathrm{SO}(3,1)$ (for the definition of a compact group, see appendix C). The prefix S in this terminology indicates that the Lorentz transformations are restricted to have unit determinant,

$$\det L = 1 \,, \tag{1.83}$$

so that reflections of an odd number of coordinates x^μ are excluded.

Upon lowering the index of a four-vector, one derives that four-vectors X_μ are subject to the following transformation rule under Lorentz transformations,

$$X_\mu \;\to\; X'_\mu = L_\mu{}^\nu X_\nu \,. \tag{1.84}$$

Here the matrix $L_\mu{}^\nu$ was defined in (1.82)

The Lorentz transformations belonging to $\mathrm{SO}(3,1)$ depend on six continuous parameters. Three of them are the angles that specify all possible spatial rotations, i.e. the three-dimensional rotations of $\mathbf{x}$ that leave the time unaffected. The other three parameters specify the Lorentz boosts that relate two inertial frames moving with some relative velocity $\mathbf{v}$. For example, the Lorentz boost that relates two frames moving with a relative velocity v in the direction of the positive z-axis is given by (see also appendix A),

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{1.85}$$

where we use the standard symbols

$$\beta = \frac{v}{c}\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,. \tag{1.86}$$

Another parametrization that is sometimes used, expresses β and γ according to

$$\gamma = \cosh\alpha\,, \qquad \beta\gamma = \sinh\alpha\,. \tag{1.87}$$

In this parametrization (1.85) resembles an ordinary rotation. Note that the hyperbolic "angle" α ranges from $-\infty$ to $+\infty$, which is a characteristic feature of a non-compact group.

All fields that we consider transform under Lorentz transformations in some characteristic way. First of all, their space-time arguments change according to (1.81). For a *scalar* field, this gives the complete result, i.e.

$$\phi'(x') = \phi(x)\,, \qquad \text{or} \qquad \phi'(x) = \phi(L^{-1}x)\,. \tag{1.88}$$

For a *vector* field, such as the vector potential in electromagnetism, the Lorentz transformations also act on the indices attached to the fields; we have

$$A'_\mu(x') = L_\mu{}^\nu A_\nu(x)\,, \quad \text{or} \quad A'_\mu(x) = L_\mu{}^\nu A_\nu(L^{-1}x)\,. \tag{1.89}$$

It is straightforward to show that also $\partial_\mu\phi$ transforms as a vector (cf. appendix A). The transformation (1.89) is easily generalized to *tensor* fields, such as the Maxwell field strength tensor $F_{\mu\nu}$, defined in (1.64), i.e.

$$F'_{\mu\nu}(x') = L_\mu{}^\rho L_\nu{}^\sigma F_{\rho\sigma}(x)\,, \quad \text{or} \quad F'_{\mu\nu}(x) = L_\mu{}^\rho L_\nu{}^\sigma F_{\rho\sigma}(L^{-1}x)\,. \tag{1.90}$$

The fact that ϕ, A_μ and $F_{\mu\nu}$ transform differently under Lorentz transformations is expressed by saying that they transform according to different *representations* of the Lorentz group. Not all representations are so easy to construct. For instance, the Lorentz group in four space-time dimensions has also another representation in terms of 4×4 matrices, which is inequivalent to the matrices $L^\mu{}_\nu$. This is the *spinor* representation of SO(3, 1), which is relevant for fermion fields as will be discussed in chapter 5. The fact that the spinor dimension is also four is only a coincidence; in general the dimensionality of the spinor representation is $2^{D/2}$, where D is the number of space-time dimensions (for odd dimensions one has $2^{(D-1)/2}$). The spinor representation is familiar from the rotation group in three dimensions, where it is two-dimensional. As one knows from quantum mechanics, this is a crucial ingredient in the description of spin-$\frac{1}{2}$ particles, and we are essentially dealing here with an extension of this situation from SO(3) to SO(3,1) motivated by our quest for relativistically invariant field theories.

1.5. Currents and conservation laws

We now restrict ourselves to continuous symmetries and exhibit their relation with conservation laws. For this purpose it suffices to consider only infinitesimal symmetry transformations. Hence the fields change according to $\phi \to \phi' = \phi + \delta\phi$, where $\delta\phi$ is linear in the infinitesimal transformation parameters ξ. For instance, the infinitesimal transformation corresponding to (1.79) is

$$\delta_\xi \phi = \mathrm{i}\xi\phi\,, \qquad \delta_\xi \phi^* = -\mathrm{i}\xi\phi^*\,, \tag{1.91}$$

whereas the infinitesimal transformation corresponding to (1.85) is expressed by ($c = 1$)

$$\delta_\xi x = \delta_\xi y = 0\,, \qquad \delta_\xi z = -\xi\, t\,, \qquad \delta_\xi t = -\xi\, z\,,$$

which induces a change of a scalar field equal to (cf. 1.88)

$$\delta_\xi \phi = \xi(z\partial_t\phi + t\partial_z\phi)\,. \tag{1.92}$$

In the latter case ξ is an infinitesimally small velocity measured in units of the velocity of light. Yet another example are the infinitesimal translations which are given in terms of an infinitesimal four-vector, i.e. $\delta x^\mu = \xi^\mu$.

If the Lagrangian is invariant under infinitesimal transformations then the equations of motion are invariant as well. The converse is not necessarily true. In order that the field equations be invariant it is sufficient that the Lagrangian changes by a total divergence, i.e.

$$\delta_\xi \mathcal{L} = \partial_\mu K^\mu(\phi, \xi)\,, \tag{1.93}$$

where ϕ generically denotes the fields on which the Lagrangian depends, and K^μ depends linearly on the infinitesimal parameter ξ. This follows from the observation that the action changes by the integral of (1.93), which by Gauss' theorem is equal to a surface integral over the boundary of the corresponding space-time domain. However, as we have already emphasized in section 1.1, the equations of motion describe the dynamics inside the integration domain, and the boundary conditions are separately imposed as independent conditions on the possible solutions.

This last observation reminds us that, under arbitrary infinitesimal variations $\delta\phi$, the Lagrangian must change into the field variations times their corresponding field equations and a total derivative term associated with the boundary contribution. Hence, we may write,

$$\delta\mathcal{L} = \delta\phi\, E_\phi(\phi) + \partial_\mu N^\mu(\phi, \delta\phi)\,, \tag{1.94}$$

where $E_\phi(\phi)$ denotes the equation of motion for the field ϕ, and $\partial_\mu N^\mu$ corresponds to the boundary term. Obviously, N^μ depends linearly on $\delta\phi$. Combining (1.93) and (1.94), where in the latter we choose $\delta\phi = \delta_\xi\phi$, we deduce that the divergence of $K^\mu(\phi,\xi) - N^\mu(\phi,\delta_\xi\phi)$ must be proportional to the field equation $E_\phi(\phi)$. Hence we have discovered that, for any continuous symmetry, there exists a corresponding conserved current J^μ. This result is known as Noether's theorem. In the formulation adopted above, the corresponding *Noether current* $J^\mu(\phi,\xi)$ thus takes the form,

$$J^\mu(\phi,\xi) = N^\mu(\phi,\delta_\xi\phi) - K^\mu(\phi,\xi)\,, \tag{1.95}$$

which, according to the above arguments satisfies

$$\partial_\mu J^\mu(\phi,\xi) = 0\,, \tag{1.96}$$

upon imposing the equations of motion.

Obviously, the Noether current is only defined modulo terms that vanish upon contraction by ∂_μ. This means that one is free to add so-called "improvement terms", which are conserved simply because of their form. Such terms take the form of a derivative contracted with an anti-symmetric tensor. Hence one may always modify a current by

$$J^\mu \;\to\; J^\mu_{\rm imp} = J^\mu + \partial_\nu A^{\mu\nu}\,, \tag{1.97}$$

where $A^{\mu\nu}$ is an arbitrary antisymmetric tensor ($A^{\mu\nu} = -A^{\nu\mu}$).

Before investigating further consequences of Noether's theorem, let us consider some examples. The first one concerns the Lagrangian for a complex scalar field (1.78), which is invariant under phase transformations (1.79) whose infinitesimal form was given in (1.91). Since the Lagrangian is invariant we have $K^\mu = 0$. It therefore remains to determine N^μ for which we simply consider (1.94),

$$\begin{aligned}\delta\mathcal{L} =& \,\delta\phi(\Box\phi^* - m^2\phi^* - 2g\phi^{*2}\phi) + \delta\phi^*(\Box\phi - m^2\phi - 2g\phi^2\phi^*)\\ &- \partial_\mu(\delta\phi\,\partial^\mu\phi^* + \delta\phi^*\,\partial^\mu\phi)\,.\end{aligned} \tag{1.98}$$

This shows that

$$N^\mu = -\delta\phi\,\partial^\mu\phi^* - \delta\phi^*\,\partial^\mu\phi\,. \tag{1.99}$$

Substituting the symmetry variations (1.91) into this result leads to the corresponding Noether current,

$$\begin{aligned} J^\mu &= \mathrm{i}\phi^*(\partial^\mu\phi) - \mathrm{i}(\partial^\mu\phi^*)\phi\,,\\ &\equiv \mathrm{i}\phi^*\overset{\leftrightarrow}{\partial}{}^\mu\phi\,. \end{aligned} \tag{1.100}$$

Indeed this current is conserved when the fields satisfy their field equations. To see this, derive the Euler-Lagrange equations corresponding to (1.78), which are manifest from (1.98),

$$\Box\phi - m^2\phi - 2g\phi^2\phi^* = 0\,, \qquad \Box\phi^* - m^2\phi^* - 2g(\phi^*)^2\phi = 0\,, \tag{1.101}$$

which imply straightforwardly that

$$\partial_\mu J^\mu = \mathrm{i}\phi^*(\Box\phi) - \mathrm{i}(\Box\phi^*)\phi$$

vanishes.

Since the field equations corresponding to the Lagrangian (1.78) are also invariant under translations, let us construct the Noether current associated with translations. The effect of an infinitesimal translation follows from

$$\phi(x) \to \phi'(x) = \phi(x-\xi) = \phi(x) - \xi^\mu\,\partial_\mu\phi(x) + \cdots\,. \tag{1.102}$$

This time the Lagrangian (1.78) is not invariant, and we find

$$\delta\mathcal{L} = -\xi^\nu\,\partial_\nu\mathcal{L}\,, \tag{1.103}$$

so that $K^\mu(\phi,\phi^*,\xi) = \xi^\mu\mathcal{L}$. Furthermore, using (1.99) and (1.102), one derives,

$$N^\mu(\phi,\phi^*,\delta_\xi\phi,\delta_\xi\phi^*) = \xi^\nu(\partial_\nu\phi\,\partial^\mu\phi^* + \partial_\nu\phi^*\,\partial^\mu\phi)\,. \tag{1.104}$$

In this way we find the Noether currents associated with translations

$$T^{\mu\nu} = \partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi + \eta^{\mu\nu}\mathcal{L}\,, \tag{1.105}$$

where we extracted the parameter ξ^ν. Note that although $T^{\mu\nu}$ is symmetric in μ and ν, the two indices have different origins. One of them corresponds to the standard four-vector index of the current, whereas the other one labels the four currents associated with the four independent translations in space and time. The currents (1.105) constitute the so-called energy-momentum tensor of this theory, for reasons that we will explain shortly.

According to Noether's theorem the energy momentum tensor must be conserved, and again this can be verified directly by using the field equations (1.101). Sometimes one prefers to include an extra term to (1.105), namely

$$T^{\mu\nu}_{\rm imp} = T^{\mu\nu} + \tfrac{1}{3}(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)(\phi^*\phi)\,. \tag{1.106}$$

The extra term is an example of an improvement term, as its divergence vanishes without having to rely on the field equations. The reason why one adds this term is that the trace of $T^{\mu\nu}_{\rm imp}$ takes a particularly simple form if one uses the equations of motion (1.101), namely

$$\eta_{\mu\nu}\,T^{\mu\nu}_{\rm imp} = -2m^2|\phi|^2\,, \tag{1.107}$$

To see what the consequences are of a conserved Noether current we decompose J^μ into spatial and time-like components according to $J^\mu = (J^0, \mathbf{J})$. Equation (1.96) then reads

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial}{\partial t} J^0 = 0 \,. \tag{1.108}$$

This is the same continuity equation as for the electric charge and current densities in electrodynamics (cf. 1.51). In that case we argued that this equation implies that electric charge is locally conserved (cf. 1.52). The interpretation is the same here, and we conclude that every Noether current leads to the analogue of a conserved "charge" by integrating the "charge density" J_0 over a spatial volume, i.e.,

$$Q = \int_V \mathrm{d}^3 x J^0 \,. \tag{1.109}$$

If no "charge" associated with this conservation law flows through the boundary surface of this volume (i.e. if the component of $\mathbf{J}$ normal to the boundary surface vanishes) then Q must be time independent by virtue of the continuity equation (1.108),

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = \int_V \mathrm{d}^3 x \frac{\partial}{\partial t} J^0(\mathbf{x}, t) = - \int_V \mathrm{d}^3 x \, \boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{x}, t) = 0 \,. \tag{1.110}$$

Again we have made use of Gauss' theorem here.

One may wonder what is the conserved quantity associated with the currents (1.100) and (1.105). The first current can be viewed as the electromagnetic current associated with the fields ϕ; to see this one must couple J^μ to the vector potential of electrodynamics. This will be discussed systematically in later parts of this book. The corresponding conserved charge is the electric charge. The conserved quantities associated with the energy-momentum tensor are energy and momentum, since invariance under time translations (i.e. the absence of time-dependent forces) implies that the energy must be constant, whereas the invariance under spatial translations implies that the three-momentum must be constant. Energy and momentum associated with a field configuration are constructed according to the prescription (1.109), and one finds

$$\begin{aligned} E &= \int_V \mathrm{d}^3 x \, T^{00}(\mathbf{x}, t) \,, \\ P^i &= \int_V \mathrm{d}^3 x \, T^{i0}(\mathbf{x}, t) \,. \end{aligned} \tag{1.111}$$

Before closing this chapter let us present one more example that will allow us to make contact with standard expressions in electrodynamics. In the absence

of external sources the Maxwell equations are obviously invariant under space time translations. Hence we may construct the energy momentum tensor from the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \,. \tag{1.112}$$

Following the same steps as before, we find that $\mathcal{L}$ changes under a space-time translation $A_\mu \to A_\mu - \xi^\nu \, \partial_\nu A_\mu$ according to,

$$\delta\mathcal{L} = -\xi^\mu \partial_\mu \mathcal{L} \,, \tag{1.113}$$

so that $K^\mu = -\xi^\mu \mathcal{L}$. Furthermore, the reader may verify that

$$N^\mu(A, \delta A) = -\delta A_\nu \, F^{\mu\nu} \,. \tag{1.114}$$

Combining the two contributions according to (1.95), the energy-momentum tensor takes the form

$$T^{\mu\nu} = F^{\mu\rho} \, \partial^\nu A_\rho - \tfrac{1}{4}\eta^{\mu\nu} F_{\rho\sigma}^2 \,. \tag{1.115}$$

which is indeed conserved by virtue of the source-free Maxwell equations ($\partial^\mu F_{\mu\nu} = 0$), i.e.,

$$\partial_\mu T^{\mu\nu} = 0 \,. \tag{1.116}$$

However, this time the energy-momentum tensor is not symmetric in (μ, ν). Furthermore, the expression (1.115) has the troublesome feature that it depends explicitly on the vector potential, rather than exclusively on the electric and magnetic fields. Therefore $T^{\mu\nu}$ will change under electromagnetic gauge transformations, and does not seem directly related with physical quantities. However, it is possible to obtain a conserved and gauge invariant energy-momentum tensor by adding an improvement term,

$$\begin{aligned} T^{\mu\nu}_{\rm imp} &= T^{\mu\nu} - \partial_\rho (F^{\mu\rho} \, A^\nu) \\ &= F^{\mu\rho} \, F^\nu{}_\rho - \tfrac{1}{4}\eta^{\mu\nu} F_{\rho\sigma}^2 + (\partial_\rho F^{\rho\mu}) \, A^\nu \,. \end{aligned} \tag{1.117}$$

If we drop the last term because it is just the source-free Maxwell equation, we are left with the canonical expression for the energy-momentum tensor in electrodynamics

$$T^{\mu\nu}_{\rm imp} = F^{\mu\rho} \, F^\nu{}_\rho - \tfrac{1}{4}\eta^{\mu\nu} F_{\rho\sigma}^2 \,, \tag{1.118}$$

which is gauge invariant, symmetric and traceless (see problem 1.7). The energy density thus equals

$$T^{00}_{\rm imp} = \tfrac{1}{2}F_{i0}^2 + \tfrac{1}{4}F_{ij}^2 = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \,, \tag{1.119}$$

and the momentum density equals

$$T^{i0}_{\rm imp} = F^i{}_j\, F^{0j} = (\mathbf{E}\times\mathbf{B})^i\,. \tag{1.120}$$

The latter is the familiar Poynting vector of electrodynamics.

Problems

1.1. The Lagrangian for the harmonic oscillator is

$$L(q,\dot q) = \tfrac{1}{2}m\,\dot q^2 - \tfrac{1}{2}m\,\omega^2 q^2\,.$$

Derive the corresponding equation of motion and write down its solution satisfying $q(t_1) = q_1$, $q(t_2) = q_2$. Show that the action for this solution is

$$S[q(t)] = \frac{m\,\omega}{2\,\sin\omega(t_2-t_1)}\left\{(q_1^2+q_2^2)\cos\omega(t_2-t_1) - 2\,q_1q_2\right\}\,.$$

1.2. The length of a curve between two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in a three-dimensional Euclidean space is given by:

$$I = \int_{P_1}^{P_2} {\rm d}s\,, \tag{1}$$

where ${\rm d}s = \sqrt{{\rm d}x^2 + {\rm d}y^2 + {\rm d}z^2}$. The curve can be parametrized by expressing x and y as functions of z. Then (1) becomes

$$I = \int_{z_1}^{z_2} {\rm d}z\sqrt{x'^2 + y'^2 + 1}\,, \tag{2}$$

where $x' = {\rm d}x/{\rm d}z$, $x(z_1) = x_1$, $x(z_2) = x_2$ and similarly for y. The *geodesic* or shortest path between P_1 and P_2 is given by the minimum of (1) and (2).
Geometric optics is based on a generalization of (1) and (2). The *optical length* is defined as the time it takes a ray of light to follow a path in a medium, multiplied by c, the velocity of light in vacuum, i,e.

$$I = c\int {\rm d}t = \int {\rm d}s\, n = \int_{z_1}^{z_2} {\rm z}\, n(x,y,z)\sqrt{x'^2 + y'^2 + 1}\,. \tag{3}$$

In (3) the refractive index n is defined by the ratio of c to v, the velocity of light in the medium. Fermat's variational principle states that a ray of light follows the path with smallest optical length. Compare the similarities and differences between Fermat's variational principle in optics with Hamilton's variational principle in mechanics. Derive the Euler-Lagrange equation for $x(z)$ and $y(z)$. Find that the solution of these equations for a homogeneous medium where $n(x,y,z)$ = constant, implies that light rays follow straight lines. Use the variational principle to derive Snell's

law for refraction and reflection at the boundary of two homogeneous media with different refractive indices.

1.3. The action for a free relativistic particle follows from (1.16). Consider the Poincaré transformations consisting of

$$\text{translations in space}: \quad \mathbf{q} \to \mathbf{q} + \mathbf{a}\,, \tag{1}$$

$$\text{translations in time}: \quad t \to t + b\,, \tag{2}$$

$$\text{rotations in space (infinitesimally)}: \quad \mathbf{q} \to \mathbf{q} + \boldsymbol{\Lambda} \times \mathbf{q}\,, \tag{3}$$

$$\text{Lorentz boosts (infinitesimally)}: \quad t \to t + c^{-1}\boldsymbol{\xi}\cdot\mathbf{q}\,, \qquad \mathbf{q} \to \mathbf{q} + c\,t\,\boldsymbol{\xi}\,. \tag{4}$$

Show that these transformations can be described by:

$$\begin{aligned}
&\delta\mathbf{q}(t) = a\,,\\
&\delta\mathbf{q}(t) = -b\,\dot{\mathbf{q}}(t)\,,\\
&\delta\mathbf{q}(t) = \boldsymbol{\Lambda}\times\mathbf{q}(t),\\
&\delta\mathbf{q}(t) = c\,t\,\boldsymbol{\xi} - c^{-1}(\boldsymbol{\xi}\cdot\mathbf{q}(t))\,\dot{\mathbf{q}}(t)\,.
\end{aligned} \tag{5}$$

Show that the Lagrangian transforms into a time-derivative, $\delta\mathcal{L} = \mathrm{d}K^0/\mathrm{d}t$, and that $N^0(\mathbf{q},\delta\mathbf{q})$ is equal to

$$N^0(\mathbf{q},\delta\mathbf{q}) = -\frac{m\,\dot{\mathbf{q}}\cdot\delta\mathbf{q}}{\sqrt{1-(\dot{\mathbf{q}}/c)^2}}\,.$$

Use Noether's theorem to obtain the corresponding conserved quantities, and show that, when expressed in terms of $\mathbf{q}$, t, $\mathbf{p}$, and E, they are proportional to

$$\mathbf{p}\,, E\,, \mathbf{q}\times\mathbf{p} \quad \text{and} \quad \mathbf{q}E - c^2\mathbf{p}t,$$

respectively.

1.4. To make the Poincare invariance of the action corresponding to (1.16) more manifest we parametrize the world line not in terms of the time t but in terms of an arbitrary parameter s. Hence $x^\mu(s)$ specifies the position $\mathbf{x}(s)$ at time $c^{-1}x^0(s)$ (we replace $\mathbf{q}$ by $\mathbf{x}$). Show that the action now takes the form

$$S[x^\mu(s)] = -mc\int \mathrm{d}s\,\sqrt{-\left(\dot{x}^\mu(s)\right)^2}\,,$$

with

$$(\dot{x}^\mu)^2 = \left(\frac{\mathrm{d}\mathbf{x}}{ds}\right)^2 - \left(\frac{\mathrm{d}x^0}{\mathrm{d}s}\right)^2.$$

Prove that this action preserves its form under reparametrization by replacing s by $s(s')$, where s' is a new parameter, and writing derivatives with respect to s'.
Find the conjugate momentum $p^\mu(s)$ corresponding to $x^\mu(s)$ and show that $p^2 = -m^2c^2$. Show that $(\dot{x}^\mu)^2$ is invariant under Poincaré transformations. Find the corresponding conserved (i.e., s-independent) quantities and express them in terms of $p^\mu(s)$ and $x^\mu(s)$.

1.5. Show that, with suitable boundary conditions, an improvement term in the Noether current does not contribute to the associated conserved charge.

1.6. Examine a model in four space-time dimensions with the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - g\phi^4\,.$$

Consider infinitesimal Lorentz transformations

$$\delta x^\mu = \theta^{\mu\nu}x_\nu \quad \text{with} \quad \theta^{\nu\mu} = -\theta^{\mu\nu}\,,$$

and infinitesimal dilatations (scale transformations)

$$\delta x^\mu = \alpha\, x^\mu\,, \qquad \delta\phi = -\alpha\phi\,.$$

The combined effect of these transformations is to induce a change of the field

$$\delta\phi = -\theta^{\mu\nu}x_\nu\,\partial_\mu\phi - \alpha\,(\phi + x^\mu\partial_\mu\phi)\,.$$

Substitute $\delta\phi$ into the variation of the action and show that it changes into a total derivative both under infinitesimal Lorentz transformations and, for $m = 0$, infinitesimal dilatations. Use Noether's theorem to define the corresponding conserved currents $(M^{\rho\sigma})^\mu$ and D^μ, respectively.
Calculate the energy-momentum tensor for the above Lagrangian and add an improvement term such that $\eta_{\mu\nu}T_{\rm imp}{}^{\mu\nu} = -m^2\phi^2$. Define

$$(M^{\rho\sigma}_{\rm imp})^\mu = -\tfrac{1}{2}[x^\rho T^{\sigma\mu}_{\rm imp} - x^\sigma T^{\rho\mu}_{\rm imp}]\,, \tag{1}$$

and

$$D^\mu_{\rm imp} = x_\nu T^{\mu\nu}_{\rm imp}\,. \tag{2}$$

Show that the current (1) is conserved by virtue of $T^{\mu\nu}_{\rm imp} = T^{\nu\mu}_{\rm imp}$, and that the current (2) is conserved when $m = 0$, by virtue of $\eta_{\mu\nu}T^{\mu\nu}_{\rm imp} = 0$. Display the difference between the currents (1) and (2) and the currents $(M^{\rho\sigma})^\mu$ and D^μ, and show that they take the form of improvement terms.

1.7. A more straightforward derivation of the energy momentum tensor (1.118) follows directly from the infinitesimal transformation,

$$\delta_\xi A_\mu = \xi^\nu F_{\mu\nu}\,. \tag{1}$$

Show that the field strength transforms according to $\delta_\xi F_{\mu\nu} = \xi^\rho(\partial_\mu F_{\nu\rho} - \partial_\nu F_{\mu\rho}) = -\xi^\rho\,\partial_\rho F_{\mu\nu}$, where we used the homogeneous Maxwell equation (1.65). Show that the Lagrangian (1.112) transforms as,

$$\delta\mathcal{L} = \tfrac{1}{4}\xi^\rho\,\partial_\rho(F_{\mu\nu})^2\,, \tag{2}$$

and thus transforms into a total derivative. Using (1.114), show that the Noether current is indeed given by (1.118).

To clarify the nature of the transformation (1) we first write it in the form

$$\delta A_\mu = -\partial_\mu \xi^\nu A_\nu - \xi^\nu \, \partial_\nu A_\mu + \partial_\mu (\xi^\nu A_\nu) \, . \tag{3}$$

Now observe that the last term is just a (field-dependent) gauge transformation under which the Maxwell Lagrangian is separately invariant. For constant ξ^ν the first term vanishes and the second term is precisely a translation. If ξ^ν is not a constant then the first two terms define an infinitesimal general coordinate transformation (with A_μ transforming as a covariant vector).
Consider the variation (1), but nnow for a vector ξ^μ that depends on the space-time coordinates. Derive again the variation of the field strength $F_{\mu\nu}$ and show that the Lagrangian (1.112) changes according to

$$\delta \mathcal{L} = -\partial_\rho \left(\xi^\rho \, \mathcal{L} \right) + \tfrac{1}{2} \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \tfrac{1}{2} \eta_{\mu\nu} \, \partial_\sigma \xi^\sigma \right) \, F^{\mu\rho} F^\nu{}_\rho \, . \tag{4}$$

Let us now search for special choices of ξ^μ that the transformation leaves the Lagrangian invariant modulo a total derivative. There exist only fifteen independent vectors of this type. Ten of them are generated by a vector a^μ and $\theta^{\mu\nu} x_\nu$ where α^μ and $\theta^{\mu\nu}$ are constant, and and $\theta^{\mu\nu} = -\theta^{\nu\mu}$. Do you know to which space-time symmetries these vectors correspond? The Noether current corrsponding to a^μ is the energy-momentum tensor (1.118). Determine also the Noether current corresponding to $\theta^{\mu\nu}$ and express it in terms of the energy-momentum tensor.
As it turns out there are five other independent vectors ξ^μ, parametrized in terms of a constant α and in terms of a constant vector c^μ. Demonstrate that they are given by $\alpha \, x^\mu$ and $2 \, x^\mu (x \cdot c) - x^2 c^\mu$. These vecors generate scale transformations and so-called conformal boosts, respectively. All these transformations leave the light-cone, defined by $x^2 = 0$, invariant. Find also the Noether currents associated with scale transformations and with conformal boosts and express the in terms of the energy-momentum tensor.

References

General reading material relevant for chapter 1

M. Born and E. Wolf, Principles of Optics, 6th Ed. (Pergamon Press, Oxford, 1980).

H. Goldstein. Classical Mechanics (Addison-Wesley, Reading, MA, 1950).

L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields, 4th Ed. (Pergamon Press, Oxford, 1975).

D.E. Soper, Classical Field Theory (Wiley, New York, 1976).

J.D. Jackson, Classical Electrodynamics, 2nd Ed. (Wiley, New York, 1975).

For the path integral formulation in quantum mechanics

P.A.M. Dirac, Phys. Z. der Sowjetunion 3 (1933) 64.

R.P. Feynman, Rev. Mod. Phys. 20 (1948) 267; Phys. Rev. 76 (1949) 769;

the above papers are reprinted in: Selected Papers on Quantum Electrodynamics, ed. J. Schwinger (Dover Publications, New York, 1958).

R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965).

L.S. Schulman, Techniques and Applications of Path Integration (Wiley, New York, 1981).

For the quantum interterence effects discussed in section 1.3

W. Ehrenberg and R.E. Siday, Proc. Roy. Soc. (London) B62 (1949) 8.
Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485.
R.G. Chambers, Phys. Rev. Lett. 5 (1960) 3.
M. Peshkin, Phys. Rep. 80 (1981) 375.
G. Möllenstedt, H. Schmid and H. Lichte, in: Proc. Int. Congr. on Electron Microscopy, Hamburg 1982, Vol. I (Deutsche Gesellschaft für Elektronenmikroskopie, Frankfurt, 1982) p. 433.
S. Olariu and I. Iovitzu Popescu, Rev. Mod. Phys. 57 (1985) 339.
C.N. Yang, in: Proc. Int. Symp. "Foundations of Quantum Mechanics in the Light of New Technology", ed. S. Kamefuchi (Physical Society of Japan, 1983) p. 5.

For the relation between currents and conservation laws Some early references are

E. Noether, Nachr. Ges. Wiss. Göttingen (1918) 37, 235.
E.L. Hill, Rev. Mod. Phys. 23 (1951) 253.