A Double Generalization of the Spiral of Theodorus

Aidan Lorenzo

December 2024

1 Introduction

In this work we will develop a double generalization of Theodorus Spiral (see Figure 4 next page). On one side we will extend the formation angle of new triangles: in the aforementioned Spiral, the angle is fixed and equal to $\frac{\pi}{2}$, in order that the distance from the center to each new point of the Spiral is equal to the function $\sqrt{n}$ for n a positive integer.

On the other side, we will generalize this construction to non-triangular formation blocks, and we will prove some Theorems describing the relationship among the behaviors of both iterative geometric constructions. Furthermore, we will connect notions of Number Theory by introducing Euler's Totient Function and proving some symmetries regarding this function and which will serve as Lemma.

2 First Construction

2.1 Definition of the construction

Definition 1.1. Let $x \in (0, \pi) = X$ be a *formation angle* in its *domain of generation.*

Definition 1.2. Let the *fundamental block* be a triangle of sides a_i, b_i y c_i, such that

$$\forall i \in \mathbb{Z}^+, a_i = 1 \text{ and } b_1 = 1$$

and O the *center* of the figure, which is the intersection between b_i and c_i. x is the angle that separates a_i and b_i.

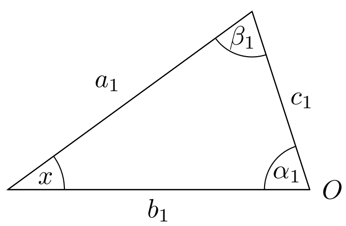


Figure 1: First Triangle.

Definition 1.3. Let T_x be the figure generated by angle x juxtaposing fundamental blocks according to the following generation rule:

$$b_i = c_{i-1}$$

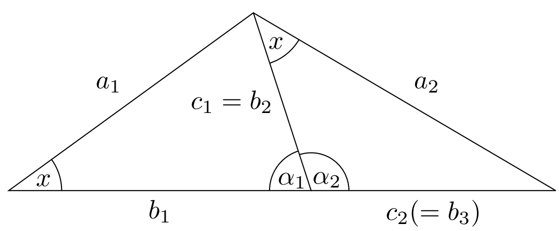


Figure 2: First two Blocks of T_{36°.

Definition 1.4. Let $c_i(x)$ be the length of segment c_i of T_x.

2.2 Convergence study

Proposition 1.1.

$$c_i(x) = \begin{cases} 1 & \text{si } i = 0 \\ \sqrt{1 + c_{i-1}^2 - 2c_{i-1}\cos x} & \text{si } i > 0 \end{cases}$$

Proof. *By the Cosine Theorem and the previous definitions:*

$$c_i^2(x) = a_i^2 + b_i^2 - 2a_i b_i \cos x \iff c_i(x) = \sqrt{1 + c_{i-1}^2 - 2c_{i-1}\cos x}$$

The function is defined recursively, so we need an initial value of definition:

$$b_i = c_{i-1} \wedge b_1 = 1 \implies c_0 = 1$$

Proposition 1.2.

$$i) \lim_{i\to\infty} c_i(x) = \infty \iff \cos(x) \leq 0$$

$$ii) \lim_{i\to\infty} (c_i(x) - c_{i-1}(x)) = |\cos(x)| \iff \cos(x) < 0$$

$$iii) \lim_{i\to\infty} c_i(x) = \frac{1}{2\cos(x)} \iff \cos(x) > 0$$

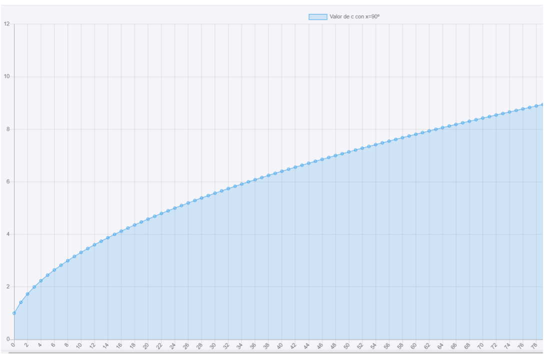


Figure 3: $c_i(\frac{\pi}{2})$

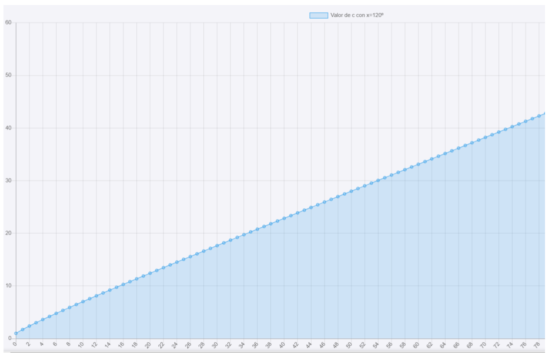


Figure 4: $c_i(\frac{2}{3}\pi)$

Proof. *We shall note that, by construction,* $c_i(x) > 0$*, for it refers to the length of a segment, which must be positive.*

i). Let's see that the functions associated to x *angles with negative or null cosine are monotonically increasing.*

$$\forall i \in \mathbb{Z}^+ \forall x \in X, c_i(x) > 0 \wedge \cos(x) \leq 0 \implies c_i(x) > c_{i-1}(x)$$

$$\sqrt{1 + c_{i-1}^2 + 2c_{i-1}|\cos(x)|} > c_{i-1} \iff 2c_{i-1}|\cos(x)| > -1$$

which will always be true. Also, the series does not have an upper boundary; therefore it diverges. As seen in the Introduction, $c_i(90°) = \sqrt{i}$*. . Let's study how the function behaves in different intervals of* $(0, \frac{\pi}{2})$ *by visualizing the corresponding graphs. Note that* $c_i(\frac{\pi}{3}) = 1, \forall i \in \mathbb{Z}^+$*. With this, we only have an analytical proof of convergence for the trivial case where the series is constant:*

$$c_1(\frac{\pi}{3}) = \sqrt{2 - 2\cos(\frac{\pi}{3})} = \sqrt{2-1} = \sqrt{1} = 1$$

Hence we see that every evaluation of the function is equal. This implies that the blocks will be equilateral triangles.

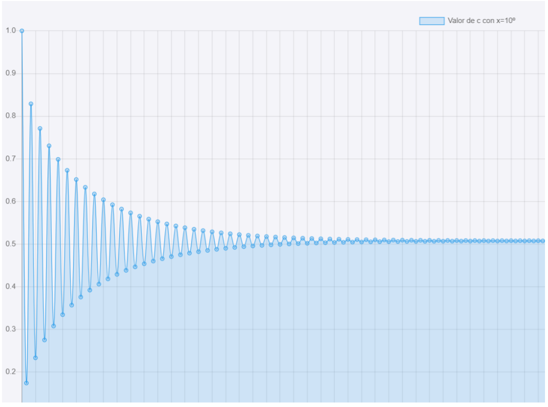


Figure 5: $c_i(x)$ para $x \in (0, \frac{\pi}{4})$

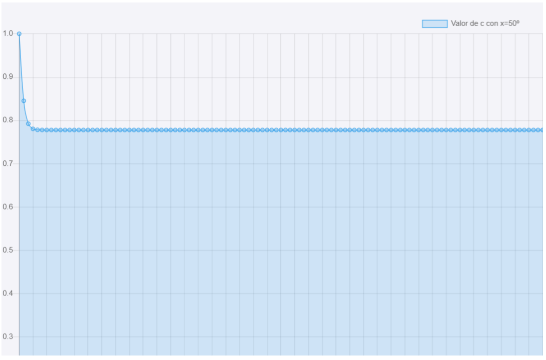


Figure 6: $c_i(x)$ para $x \in [\frac{\pi}{4}, \frac{\pi}{3})$

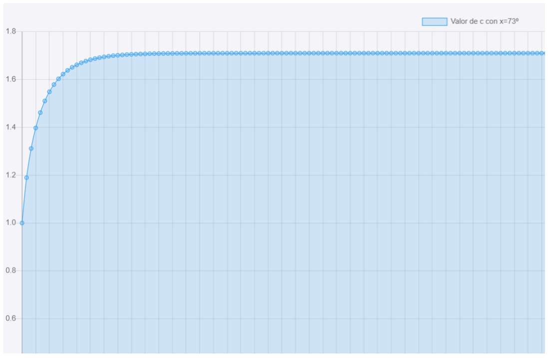


Figure 7: $c_i(x)$ para $x \in (\frac{\pi}{3}, \frac{\pi}{2})$

Definition 2.1. Let the *domain of convergence* be the interval $X_1 = (0, \frac{\pi}{2})$, for which

$$x \in (0, \frac{\pi}{2}) \iff cos(x) > 0 \iff c_i(x) \text{ converges}$$

Definition 2.2. Let the *domain of non-convergence* be the interval $X_2 = [\frac{\pi}{2}, \pi)$, for whom

$$x \in [\frac{\pi}{2}, \pi) \iff \cos(x) \leq 0 \iff \lim_{i \to \infty} c_i(x) = \infty$$

Definition 2.3 Let $r(x) = \frac{1}{2\cos(x)}$ be the *radius* function, $\forall x \in X_1$

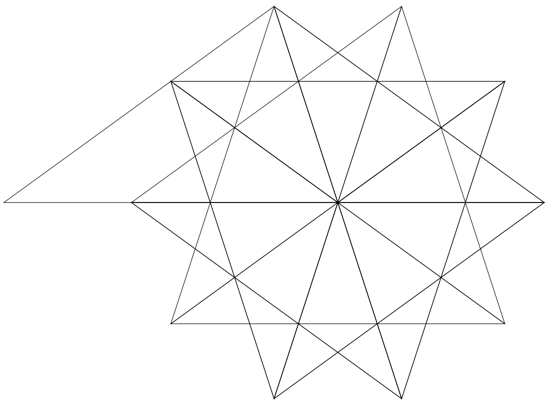


Figure 8: T_{36°

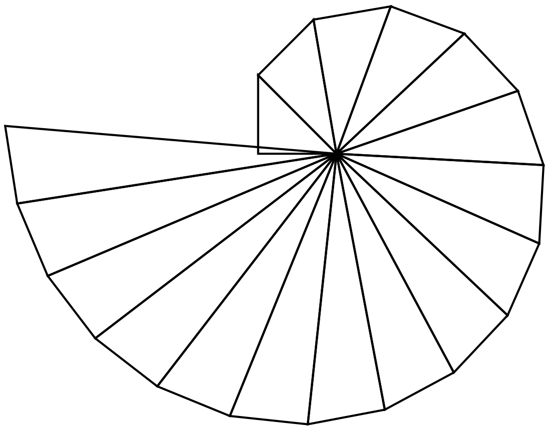


Figure 9: T_{90° (Spiral of Theodorus, V. BC)

2.3 Study of a function: $p(x)$

Definition 3.1.
Let $\varphi_n = |\{\frac{m}{n}\pi : m \in \mathbb{Z}^+{}_{<n} \wedge \gcd(m,n) = 1\}|$ and $\varphi'_n = |\{m \in \mathbb{Z}^+{}_{<n} : \gcd(m,n) = 1\}|$

Definition 3.2.
Let $\gamma_n = |\{\frac{m}{n}\pi : 2m < n \wedge m \in \mathbb{Z}^+ \wedge \gcd(m,n) = 1\}|$ and $\gamma'_n = |\{m \in \mathbb{Z}^+ : 2m < n \wedge \gcd(m,n) = 1\}|$

Definition 3.3.
$\chi_n = |\{\frac{m}{n}\pi : m \in \mathbb{Z}^+{}_{<n} \wedge \gcd(m,n) = 1 \wedge 2 \mid m\}|$, $\chi'_n = |\{m \in \mathbb{Z}^+{}_{<n} : \gcd(m,n) = 1 \wedge 2 \mid m\}|$

Observation. Trivially, $|\varphi_n| = |\varphi'_n|, |\gamma_n| = |\gamma'_n|$ and $|\chi_n| = |\chi'_n|$.

Definition 4.1. Let $p : X_1 \to \mathbb{Z}^+ \cup \{\infty\}$ be the function $p(x) = |\{P \in F_x : |\overline{OP}| = r(x)\}|$.

Theorem 1.1. Value of $p(x)$

$$\forall n \in \mathbb{Z}^+ : x \in \gamma_n \iff \begin{cases} 2 \nmid n \iff p(x) = 2n \\ 4 \mid n \iff p(x) = n \\ 2 \mid n \wedge 4 \nmid n \implies p(x) = \frac{1}{2}n \end{cases}$$

Proof. Let's observe that, if $c_i(x)$ converges, so does $c_{i-1}(x)$, which by Definition 1.3. is equal to b_i. This implies that the blocks will be isosceles triangles. O being the vertex which connects all these segments, and $\alpha_i = \hat{O}$, we see that α_i converges to $\alpha = \pi - 2x$. Then, we'll see that there exists a finite amount of blocks whose sum of internal constant angles α is $v \cdot 2\pi$ for some $v \in \mathbb{Z}^+$.

$$\exists n \in \mathbb{Z}^+ : x \in \gamma_n \iff x = \frac{m}{n}\pi \iff \frac{2\pi}{\alpha} = \frac{2n}{n-2m} \iff \exists k, v \in \mathbb{Z}^+ : \gcd(k,v) = 1 \wedge k \cdot \alpha = v \cdot 2\pi$$

This means that, after k constant isosceles triangles, the new blocks will be drawn over preexistent ones, for α is constant. Now, to elucidate the value of k as a function of x, let's do the following:

$$k \cdot \alpha = v \cdot 2\pi \iff \frac{k}{v} = \frac{2n}{n-2m}$$

In order to equate k to the numerator of the expression we must study what values of m and n generate an irreducible fraction.

$$\begin{cases} 2 \nmid n \wedge 1 = \gcd(n,m) = \gcd(n, n-m) \implies \gcd(n, n-2m) = \gcd(2n, n-2m) = 1 \\ 4 \mid n \wedge 1 = \gcd(n,m) = \gcd(n, n-m) \implies \gcd(n, n-2m) = \gcd(2n, n-2m) = 2 \\ 2 \mid n \wedge 4 \nmid n \wedge 1 = \gcd(n,m) = \gcd(n, n-m) \implies \gcd(n, n-2m) = 2 \implies \gcd(2n, n-2m) = 4 \end{cases}$$

Therefore we have that

$$\begin{cases} 2 \nmid n \implies \gcd(2n, n-2m) = 1 \implies k(x) = 2n \\ 4 \mid n \implies \gcd(2n, n-2m) = 2 \implies \gcd\left(n, \frac{n-2m}{2}\right) = 1 \implies k(x) = n \\ 2 \mid n \wedge 4 \nmid n \implies \gcd(2n, n-2m) = 4 \implies \gcd\left(\frac{n}{2}, \frac{n-2m}{4}\right) = 1 \implies k(x) = \frac{n}{2} \end{cases}$$

We still have to see that $k(x)$, whose expression we've just found, coincides with $p(x)$, object of this Theorem.

i	1	2	3	4	5	6	$\dots$	$k-1$	k
$\sum_{j=1}^{i} p_j$	2	3	4	5	6	7	$\dots$	k	k

Having noticed that the blocks are isosceles triangles, where $c_i = b_i = r(x)$, we see that the first block will draw the first 2 points of the figure; after this one, each new triangle will have 1 external vertex, for it will share one with the previous block; finally, the last block does not add more points to the figure, since it connects two existing ones.

Corollary 1.1.

$$\nexists n \in \mathbb{Z}^+ : x \in \gamma_n \iff \frac{m}{n} \notin \mathbb{Q} \iff p(\frac{m}{n}\pi) = \infty$$

If x cannot be expressed as a rational fraction of π, $\nexists k, v \in \mathbb{Z}^+ : k \cdot \alpha = v \cdot 2\pi$, which translates into the construction never closing after a finite number of blocks.

Corollary 1.2.

$$\forall n \in \mathbb{Z}^+ : x \in \gamma_n \iff \begin{cases} 2 \nmid n \iff v(x) = n - 2m \\ 4 \mid n \iff v(x) = \frac{n-2m}{2} \\ 2 \mid n \wedge 4 \nmid n \implies v(x) = \frac{n-2m}{4} \end{cases}$$

Definition 4.2. Let $p^{-1}(p) = \{x \in X_1 : p(x) = p\}, \forall p \in \mathbb{Z}_{\geq 3}$ be the anti-image set of the application explained in Definition 3.1, and $f(p) := |p^{-1}(p)|$ the cardinal of this set.

Definition 5.1. Let $\varphi(n) = |\varphi_n|$ be Euler's Totient Function.

Lemma. Symmetries in φ_n

$$i)\ \forall n \in \mathbb{Z}_{>2},\ |\varphi_n| = 2|\chi_n|$$
$$ii)\ \forall n \in \mathbb{Z}_{>2} : 2 \nmid n \iff |\varphi_n| = 2|\gamma_n|$$
$$iii)\ \forall n \in \mathbb{Z}_{>2} : 2 \nmid n \iff \varphi(2n) = \varphi(n)$$

The proof may be found in the Appendix of this Paper.

Theorem 1.2. Inverse of $p(x)$ and Totient Function

$$\forall p \in \mathbb{Z}_{\geq 3},\ f(p) = \frac{\varphi(p)}{2}$$

Proof. First, let's see what denominator n will cause the figure to have p points, following Theorem 1.1:

$$\begin{cases} 2 \nmid p \iff n = 2p \\ 4 \mid n \iff n = p \\ 2 \mid p \wedge 4 \nmid p \iff n = \frac{1}{2}p \end{cases}$$

Given an independent value of p, we know what unique denominator gives rise to figures that satisfy $p(x) = p$.

Now we expand $p^{-1}(p) = \gamma_n = \begin{cases} \gamma_{2p} \iff 2 \nmid p \\ \gamma_n \iff 4 \mid p \\ \gamma_{\frac{p}{2}} \iff 2 \mid p \wedge 4 \nmid p \end{cases}$

The set of all $x \in X$ which satisfy $p(x) = p$ are the angles $x \in X_1$ with denominator a positive integer n, and numerator also a positive integer and relatively prime to n is what γ_n, introduced in Definition 3.2. represents by construction.

Following the Lemma on Symmetries and Definition 5.1:

$$p^{-1}(p) = \gamma_n \iff |p^{-1}(p)| = f(p) = |\gamma_n| = \frac{\varphi(n)}{2} = \begin{cases} \frac{1}{2}\varphi(2p) \iff 2 \nmid p \\ \frac{1}{2}\varphi(p) \iff 4 \mid p \\ \frac{1}{2}\varphi(\frac{p}{2}) \iff 2 \mid p \wedge 4 \nmid p \end{cases} = \frac{\varphi(p)}{2}$$

2.4 Extension of $p(x)$

Definition 6.1. Let $k : X \times \mathbb{R}^+ \to \mathbb{R}^+$ be the function

$$k(x, y) = k \iff \sum_{i=1}^{\lfloor k \rfloor} \alpha_i(x) + (k - \lfloor k \rfloor)\alpha_{\lfloor k \rfloor + 1}(x) = y$$

Proposition 2.1.

$$\exists n \in \mathbb{Z}^+ : x \in \gamma_n \iff p(x) = k(x, 2\pi v(x))$$

Proof. By Theorem 1.1. and Definition 6.1. we note the following:

$$p(x) = k \iff \exists k, v \in \mathbb{Z}^+ : k \cdot \alpha = v \cdot 2\pi \wedge \gcd(k, v) = 1$$

$$k(x, y) = k \iff \sum_{i=1}^{\lfloor k \rfloor} \alpha_i(x) + (k - \lfloor k \rfloor)\alpha_{\lfloor k \rfloor + 1}(x) = y$$

In the domain of convergence, for $k \in \mathbb{Z}^+$:

$$k(x, 2\pi v(x)) = k \iff \sum_{i=1}^{k} \alpha(x) = y \iff k \cdot \alpha = 2\pi v(x)$$

Theorem 2.1. Unique Spirals

$$\forall k \in \mathbb{Z}_{\geq 3}, \exists! x \in X : k(x, 2\pi) = k$$

Let's prove that $k(x, 2\pi)$ is an injective function:

$$1.\ \forall x_1, x_2 \in X : x_1 < x_2 \implies c_i(x_1) < c_i(x_2)$$

At $i = 0$, $c_0(x_1) = c_0(x_2) = 1$. Then:

$$x_1 < x_2 \implies c_1(x_1) = \sqrt{2(1 - \cos(x_1))} < \sqrt{2(1 - \cos(x_2))} = c_1(x_2) \iff \cos(x_1) > \cos(x_2)$$

Which is true if and only if $x_1, x_2 \in (0, \pi) = X$. For the remaining values of i we proceed as follows:

$$\forall a, b \in \mathbb{R}^+ : a < b \implies \sqrt{1 + a^2 - 2a\cos x_1} < \sqrt{1 + b^2 - 2b\cos(x_2)} \iff a(a - 2\cos(x_1)) < b(b - 2\cos(x_2))$$

By hypothesis we have that $a < b$, so we need to prove $a - 2\cos(x_1) < b - 2\cos(x_2)$. With the same logic, $a < b$, so we need to see that $2\cos(x_1) > 2\cos(x_2)$, which we just did in the case $i = 0$:

$$2.\ \forall x_1, x_2 \in X : c_i(x_1) < c_i(x_2) \implies \alpha_i(x_1) > \alpha_i(x_2)$$

In the first iteration of some two constructions:

$$\alpha_1(x_1) > \alpha_1(x_2) \iff \arccos(\frac{1^2 + c_1^2(x_1) - 1}{2 \cdot 1 \cdot c_1(x_1)}) > \arccos(\frac{1^2 + c_1^2(x_2) - 1}{2 \cdot 1 \cdot c_1(x_2)}) \iff \frac{c_1(x_1)}{2} < \frac{c_1(x_2)}{2}$$

$$\arccos(\frac{c_i(x_1)^2 + c_{i-1}(x_1)^2 - 1}{2c_i(x_1)c_{i-1}(x_1)}) > \arccos(\frac{c_i(x_2)^2 + c_{i-1}(x_2)^2 - 1}{2c_i(x_2)c_{i-1}(x_2)}) \iff$$

$$\frac{c_i(x_1)^2 + c_{i-1}(x_1)^2 - 1}{c_i(x_1)c_{i-1}(x_1)} < \frac{c_i(x_2)^2 + c_{i-1}(x_2)^2 - 1}{c_i(x_2)c_{i-1}(x_2)} \iff$$

$$\frac{(\sqrt{1+c_{i-1}^2-2c_{i-1}\cos(x_1)})^2+c_{i-1}^2-1}{c_i(x_1)c_{i-1}(x_1)} < \frac{(\sqrt{1+c_{i-1}^2-2c_{i-1}\cos(x_2)})^2+c_{i-1}^2-1}{c_i(x_2)c_{i-1}(x_2)} \iff$$

$$\frac{c_{i-1}(x_1)-\cos(x_1)}{c_i(x_1)} < \frac{c_{i-1}(x_2)-\cos(x_2)}{c_i(x_2)}$$

3. $\forall k \in \mathbb{R}^+ \forall x_1, x_2 \in X : x_1 < x_2 \implies \sum_{i=1}^{\lfloor k \rfloor} \alpha_i(x_1) + (k - \lfloor k \rfloor)\alpha_{\lfloor k \rfloor}(x_1) > \sum_{i=1}^{\lfloor k \rfloor} \alpha_i(x_2) + (k - \lfloor k \rfloor)\alpha_{\lfloor k \rfloor}(x_2)$

4. $\forall x_1, x_2 \in X\ \forall y \in \mathbb{R}^+ : x_1 < x_2 \implies k(x_1, y) < k(x_2, y)$

Let's consider $k(x_2, y) = k_2 \iff \sum_{i=1}^{\lfloor k_2 \rfloor} \alpha_i(x_2) + (k_2 - \lfloor k_2 \rfloor)\alpha_{\lfloor k_2 \rfloor}(x_2) = y$*. Then:*

$$\sum_{i=1}^{\lfloor k_2 \rfloor} \alpha_i(x_1) + (k_2 - \lfloor k \rfloor)\alpha_{\lfloor k_2 \rfloor}(x_1) > y$$

Therefore, necessarily, $k(x_1, y) = k_1 < k_2 = k(x_2, y)$*. According to this principle:*

5. $\forall x_1, x_2 \in X\ \forall y \in \mathbb{R}^+, \exists k \in \mathbb{Z}^+ : k(x_1, y) < k < k(x_2, y) \implies \exists! x_m \in X : x_1 < x_m < x_2 \wedge k(x_m, y) = k$

If we could compute $k(x, y)$*, we could use its monotonic growth to find those unique angles* x_m *which satisfy* $k(x_m, y) \in \mathbb{Z}^+$*.*

In Figure 4 one can observe 17 *blocks of the infinite figure* T_{90°*. It is patent that the sum of its internal angles* $\alpha_i(90^\circ)$ *is greater than* 2π*, and that the sum of the first* 16 *is less that* 2π*. In consequence, we know that the integer part of* $k(90^\circ, 360^\circ) = k \in \mathbb{R}^+$ *is* 16*. In general:*

$$\lfloor k(x, y) \rfloor = n \in \mathbb{Z}^+ \iff \sum_{i=1}^{n} \alpha_i(x) < y < \sum_{i=1}^{n+1} \alpha_i(x)$$

$$k(x, y) = k \iff \sum_{i=1}^{\lfloor k \rfloor} \alpha_i(x) + (k - \lfloor k \rfloor)\alpha_{\lfloor k \rfloor + 1}(x) = y \iff k = \lfloor k \rfloor + \frac{y - \sum_{i=1}^{\lfloor k \rfloor} \alpha_i(x)}{\alpha_{\lfloor k \rfloor + 1}(x)}$$

Now we can easily compute the value of $k(x, y)$ *for all* $(x, y) \in X \times \mathbb{R}^+$*. In particular,* $k(90^\circ, 360^\circ) \approx 16.649128$*, y* $k(91^\circ, 360^\circ) \approx 17.445935$*. Now we apply principle* 5*:*

$$k(90^\circ, 360^\circ) < 17 < k(91^\circ, 360^\circ) \implies \exists! x_m \in (90^\circ, 91^\circ) : k(x_m, 360^\circ) = 17$$

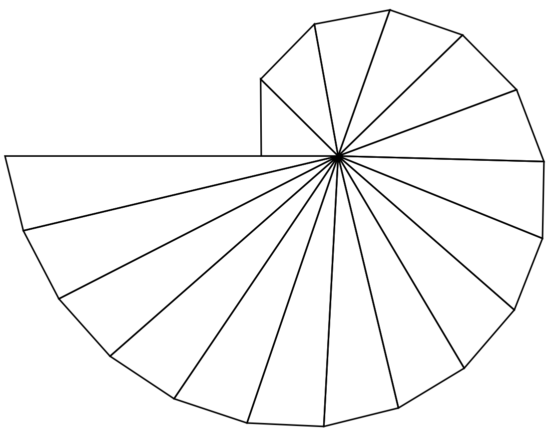


Figure 10: 17 blocks of $T_{x=90.45332215^\circ}$

$k \in \mathbb{Z}^+$	x°
3	19.019079333
6	60
9	74.15292718
12	82.2524055
16	89.13404388
17	90.45332215
18	91.65964018
19	92.76863062
20	93.79304858
21	94.74341811
60	110.8178438
450	128.8014110

Figure 11: $\{(k, x) : k(x, 2\pi) = k\}$

Corollary 2.

$$i) \ |\{x_1 \in X_1 : k(x_1, 2\pi) \in \mathbb{Z}^+\}| = 14$$

$$ii) \ |\{x_2 \in X_2 : k(x_2, 2\pi) \in \mathbb{Z}^+\}| = \infty$$

There is a finite number of $x_1 \in X_1$ that manifest this injective behavior with respect to the application $k(x, 2\pi)$, while there are infinite $x_2 \in X_2$ which satisfy $k(x_2, 2\pi) \in \mathbb{Z}^+$

Conjecture 1.

$$\forall x \in X \setminus \{\frac{\pi}{3}\} : k(x, 2\pi) \in \mathbb{Z}^+ \implies \nexists n : x \in \gamma_n$$

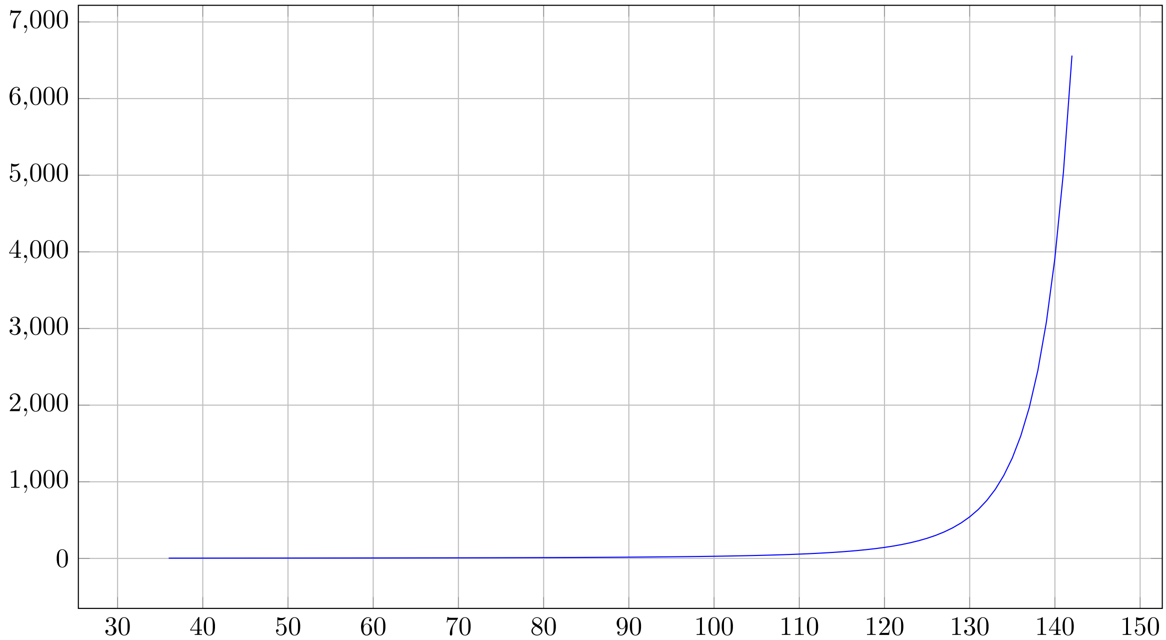


Figure 12: Graph of $k(x, 2\pi)$.

Thus we conclude that $k(x, y)$ is a function that grows exponentially.

3 Second Construction

3.1 Definition of the construction

Definition 7.1. In this case, the *fundamental block* will be the equilateral parallelogram with sides a_i, b_i, c_i y d_i such that

$$a_i = b_i = c_i = d_i = \lambda \in \mathbb{R}^+$$

and x the angle which forms between a_i and b_i (as well as between c_i and d_i)

Definition 7.2. Let P_x be the figure generated juxtaposing blocks according to the same pattern as T_x, and O the center of the figure:

$$b_i = c_{i-1}$$

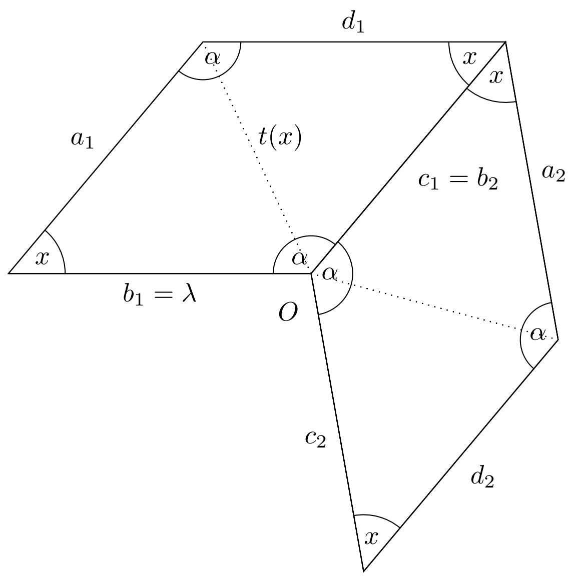


Figure 13: Two first blocks of P_{50°.

Definition 7.3. Let $r'(x) \coloneqq \max(\lambda, t(x)), \forall x \in X$, where $t(x)$ is the distance between the opposed vertices whose angles are α.

3.2 Homologue function to $p(x) : q(x)$

Proposition 3.1.

$$r'(x) = \begin{cases} \lambda & \text{si } 0 < x \leq \frac{\pi}{3} \\ \lambda\sqrt{2(1-\cos x)} & \text{si } \frac{\pi}{3} < x < \pi \end{cases}$$

Proof. By the Cosine Theorem, we have that:

$$t^2(x) = a^2 + b^2 - 2ab\cos x \iff t(x) = \sqrt{2\lambda^2 - 2\lambda^2\cos(x)} = \sqrt{2\lambda^2(1-\cos(x))} = \lambda\sqrt{2(1-\cos(x)}$$

Then:

$$\begin{cases} r'(x) = \lambda \iff \max(\lambda, \lambda\sqrt{2(1-\cos x)}) = \lambda \iff \lambda \geq \lambda\sqrt{2(1-\cos(x)} \iff \cos x \geq \frac{1}{2} \iff x \in (0, \frac{\pi}{3}] \\ \\ r'(x) = \lambda\sqrt{2(1-\cos x)} \iff \lambda < \lambda\sqrt{2(1-\cos(x)} \iff \cos x < \frac{1}{2} \iff x \in (\frac{\pi}{3}, \pi) \end{cases}$$

We hence observe that all figures generated by this second construction will have a real radius.

Definition 8.1. Let $q : X \to \mathbb{Z}^+ \cup \{\infty\}$ be the function $q(x) = |\{S \in P_x : |\overline{OS}| = r'(x)\}|$

Theorem 3.1. Value of $q(x)$

$$i)\ \forall n \in \mathbb{Z}^+ \setminus \{3\}, x \in \varphi_n \iff \begin{cases} q(x) = 2n \iff 2 \mid n \vee 2 \mid m \\ \\ q(x) = n \iff 2 \nmid n \wedge 2 \nmid m \end{cases} \qquad ii)\ q(\frac{1}{3}\pi) = 2n = 6$$

Proof. $2\alpha + 2x = 2\pi \iff \alpha = \pi - x$. *Let's proceed as in Theorem 1.1:*

$$\exists n \in \mathbb{Z}^+ : x \in \varphi_n \iff x = \frac{m}{n}\pi \wedge m < n \iff \frac{2\pi}{\alpha} = \frac{2n}{n-m} \iff$$
$$\iff \exists k, v \in \mathbb{Z}^+ : \gcd(k, v) = 1 \wedge k \cdot \alpha = v \cdot 2\pi \iff \frac{k}{v} = \frac{2n}{n-m}$$

$$\begin{cases} (2 \mid n \vee 2 \mid m) \wedge 1 = \gcd(n, m) = \gcd(n, n-m) \implies \gcd(2n, n-m) = 1 \implies k(x) = 2n \\ \\ 2 \nmid n \wedge 2 \nmid m \wedge 1 = \gcd(n, m) = \gcd(n, n-m) \implies \gcd(2n, n-m) = 2 \implies k(x) = n \end{cases}$$

Now we observe the concordance of $q(x)$ *with* $k(x)$*:*

$r'(x) = \lambda \iff x \in (0, \frac{\pi}{3}) \iff$

i	1	2	3	4	5	6	...	$k-1$	k
$\sum_{j=1}^{i} q_j$	2	3	4	5	6	7	...	k	k

$r'(x) = \lambda\sqrt{2(1-\cos(x))} \iff$

i	1	2	3	4	5	6	...	$k-1$	k
$\sum_{j=1}^{i} q_j$	1	2	3	4	5	6	...	$k-1$	k

$x = \frac{\pi}{3} \iff$

i	1	2	3
$\sum_{j=1}^{i} q_j$	3	5	6

When $r'(x) = \lambda$*, the first block adds two points, for it has two segments with that length:* $b_1 = \lambda = c_1$*. Conversely, for* $r'(x) = t(x)$*, the first triangle contains only one segment from* O *with length* $t(x) = \lambda\sqrt{2(1-\cos(x))}$*. In the exceptional case* $x = 60°$ *the blocks are equilateral triangles.*

Corollary 3.1.

$$\forall n \in \mathbb{Z}^+, x \in \varphi_n \iff \begin{cases} (2 \mid n \vee 2 \mid m) \iff v'(x) = n - m \\ \\ 2 \nmid n \wedge 2 \nmid m2 \iff v'(x) = \frac{n-m}{2} \end{cases}$$

We will study this function in Relations between the Constructions.

Corollary 3.2.

$$\nexists n \in \mathbb{Z}^+ : x \in \varphi_n \iff \frac{m}{n} \notin \mathbb{Q} \iff q(\frac{m}{n}\pi) = \infty$$

4 Relations between Constructions

4.1 Equivalence Theorems

Definition 8.2. Let $q^{-1}(q) = \{x \in X : q(x) = q\}$ be the anti-image set of the function $q(x)$, and $g(q) \coloneqq |q^{-1}(q)|$

Theorem 4.1. First Equivalence Theorem

$$i)\ \forall p, q \in \mathbb{Z}_{\geq 3} \setminus \{6\},\ p = q \implies f(p) = g(q)$$
$$ii)\ \forall p \in \mathbb{Z}_{\geq 3} \setminus \{6\},\ f(p) + g(p) = \varphi(p)$$
$$iii)\ f(6) + g(6) = 3 = \varphi(6) + 1$$

Proof. By Theorem 1.2. we have that $f(p) = \frac{\varphi(p)}{2}$*, so we'll see*

$$f(p) + g(p) = \varphi(p) \iff g(p) = \frac{\varphi(p)}{2} \iff f(p) = g(p)$$

proving $g(p) = \frac{\varphi(p)}{2}$*. Let's study* q *to know what numerators and denominators* m *and* n *satisfy* $q(\frac{m}{n}\pi) = q$*:*

$$\begin{cases} 2 \nmid n \wedge 2 \mid m \iff n = \frac{q}{2} \wedge 2 \mid m \iff 2 \mid q \wedge 4 \nmid q \\ 2 \mid n \wedge 2 \nmid m \iff n = \frac{q}{2} \wedge 2 \nmid m \iff 4 \mid q \\ 2 \nmid n \wedge 2 \nmid m \iff n = q \wedge 2 \nmid m \iff 2 \nmid q \end{cases} \implies q^{-1}(q) = \begin{cases} \chi_{\frac{q}{2}} \iff 2 \mid q \wedge 4 \nmid q \\ \varphi_{\frac{q}{2}} \iff 4 \mid q \\ \varphi_q \setminus \chi_q \iff 2 \nmid q \end{cases}$$

Then, following the Lemma on Symmetries:

$$|q^{-1}(q)| = g(q) = \begin{cases} \varphi(q) - |\chi_q| = \frac{1}{2}\varphi(q) \iff 2 \nmid q \\ \varphi(\frac{q}{2}) \iff 4 \mid q \\ |\chi_{\frac{q}{2}}| = \frac{1}{2}\varphi(\frac{q}{2}) \iff 2 \mid q \wedge 4 \nmid q \end{cases} = \frac{\varphi(q)}{2}$$

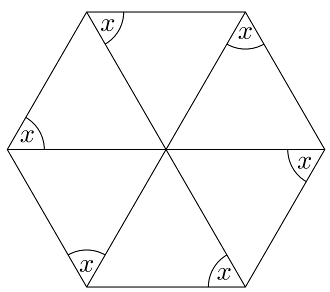


Figure 14: T_{60°

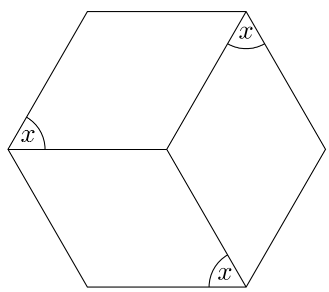


Figure 15: P_{60°

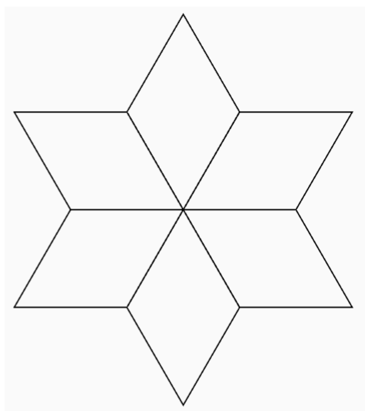


Figure 16: P_{120°

Definition 9.1. $T_x \sim P_x \iff p(x) = q(x) \wedge v(x) = v'(x)$

Definition 9.2. $T_{x_1} \sim T_{x_2} \iff p(x_1) = p(x_2) \wedge v(x_1) = v(x_2)$

Definition 9.3. $P_{x_1} \sim P_{x_2} \iff q(x_1) = q(x_2) \wedge v'(x_1) = v'(x_2)$

Theorem 4.2. Second Equivalence Theorem

$$
\begin{aligned}
i)\ \forall x \in X,\ T_x \sim P_x \iff x = \frac{\pi}{3}\\
ii)\ \forall x_t, x_p \in X,\ T_{x_t} \sim P_{x_p} \iff 2x_t = x_p\\
iii)\ \forall x_1, x_2 \in X_1,\ T_{x_1} \sim T_{x_2} \iff x_1 = x_2\\
iv)\ \forall x_1, x_2 \in X,\ P_{x_1} \sim P_{x_2} \iff x_1 = x_2 \lor x_1 = \frac{\pi}{3} = \frac{x_2}{2}
\end{aligned}
$$

Proof. i) Let's evaluate separately what values of $x \in X$ satisfy $p(x) = q(x)$ y $v(x) = v'(x)$.

$$
\begin{cases}
1. \quad p(x) = q(x) = 2n \iff 2 \nmid n \land (2 \mid n \lor 2 \mid m) \iff 2 \mid m \land 2 \nmid n \iff x \in \chi_n \\
2. \quad p(x) = q(x) = n \iff 4 \mid n \land (2 \nmid n \land 2 \nmid m) \quad \text{is a contradiction} \\
3. \quad p\left(\frac{\pi}{3}\right) = q\left(\frac{\pi}{3}\right) = 6
\end{cases}
$$

Therefore, $p(x) = q(x) \iff 2 \mid m \land 2 \nmid n$. Then:

$$
\begin{cases}
4. \quad v(x) = v'(x) \land 2 \nmid n \land 2 \nmid m \iff \frac{n-m}{2} = n - 2m \iff n - m = 2n - 4m \iff 3m = n \iff \frac{m}{n} = \frac{1}{3} \\
5. \quad v(x) = v'(x) \land 2 \nmid n \land 2 \mid m \iff n - 2m = n - m \iff m = 0 \iff x = 0 \notin X \\
6. \quad v(x) = v'(x) \land 4 \mid n \land 2 \nmid m \iff \frac{n-2m}{2} = n - m \iff n - 2m = 2n - 2m \iff n = 0 \notin \mathbb{Z}_{\geq 2} \\
7. \quad v(x) = v'(x) \land 2 \mid n \land 4 \nmid n \land 2 \nmid m \iff \frac{n-2m}{4} = n - m \iff 3n = 2m \iff \frac{m}{n} = \frac{3}{2} \iff x \notin X
\end{cases}
$$

Hence we know $v(x) = v'(x) \iff x = \frac{\pi}{3}$. Also, as a special case, $p(\frac{\pi}{3}) = q(\frac{\pi}{3}) = 6$.

$$
ii)\ T_{x_t} \sim P_{x_p} \iff \frac{p}{v} = \frac{q}{v'} \iff \frac{2n_t}{n_t - 2m_t} = \frac{2n_p}{n_p - m_p} \iff \frac{2m_t}{n_t} = \frac{m_p}{n_p} \iff 2x_t = x_p
$$

for $\gcd(p, v) = \gcd(q, v') = 1 \iff p = q \land v = v'$. *Remember that, by i), $x_t = x_p = 60^\circ$ is the exception.*

$$
iii)\ T_{x_1} \sim T_{x_2} \iff \frac{p_1}{v_1} = \frac{p_2}{v_2} \iff \frac{2n_1}{n_1 - 2m_1} = \frac{2n_2}{n_2 - 2m_2} \iff \frac{m_1}{n_1} = \frac{m_2}{n_2} \iff x_1 = x_2
$$

$$
iv)\ P_{x_1} \sim P_{x_1} \iff \frac{q_1}{v_1'} = \frac{q_2}{v_2'} \iff \frac{2n_1}{n_1 - m_1} = \frac{2n_2}{n_2 - m_2} \iff \frac{m_1}{n_1} = \frac{m_2}{n_2} \iff x_1 = x_2
$$

Definition 10. Let $x_t : \mathbb{Z}_{\geq 3} \times \gamma_p' \to X_1$ and $x_p : \mathbb{Z}_{\geq 3} \times \gamma_q' \to X$ be the functions

$$
\begin{aligned}
x_t(p, v) = x_t \iff p(x_t) = p \land v(x_t) = v\\
x_p(q, v') = x_p \iff q(x_p) = q \land v'(x_p) = v'
\end{aligned}
$$

Proposition 4.

$$i)\ \forall (p,v) \in \mathbb{Z}_{\geq 3} \times \gamma'_p,\ x_t(p,v) = \frac{p-2v}{2p}\pi \in X_1$$

$$ii)\ \forall (q,v') \in \mathbb{Z}_{\geq 3} \times \gamma'_q,\ x_p(q,v') = \frac{q-2v'}{q}\pi \in X$$

Proof. We'll prove i) first:

$$\frac{p}{v} = \frac{2n}{n-2m} \iff p(n-2m) = 2nv \iff pn - 2pm = 2nv \iff n(p-2v) = 2pm \iff \frac{m}{n} = \frac{p-2v}{2p}$$

$$1.\ 2m < n \iff p - 2v < p \iff v > 0$$
$$2.\ p - 2v > 0 \iff 2v < p$$
$$3.\ p, v \in \mathbb{Z}^+ \wedge \gcd(p,v) = 1$$

so, in conclusion, $v \in \gamma'_p$. This reaffirms Theorem 1.2, since

$$\forall p \in \mathbb{Z}_{\geq 3} : v \in \gamma'_p \wedge |\gamma'_p| = \frac{\varphi(p)}{2}$$

$$ii)\ \frac{q}{v'} = \frac{2n}{n-m} \iff q(n-m) = 2nv' \iff qn - qm = 2nv' \iff n(q-2v') = qm \iff \frac{m}{n} = \frac{q-2v'}{q}$$

$$1.\ m < n \iff q - 2v' < q \iff v' > 0$$
$$2.\ q - 2v' > 0 \iff 2v' < p$$
$$3.\ q, v' \in \mathbb{Z}^+ \wedge \gcd(q,v') = 1$$

Analogously, $v' \in \gamma'_q$. We have found a way to prove the First Equivalence Theorem which does not require any information about χ_n.

Corollary 4.

$$\forall (p,v) \in \mathbb{Z}_{\geq 3} \times \gamma_p : 2x_t(p,v) = x_p(p,v)$$

This isn't but a re-expression of part *ii)* of the Second Equivalence Theorem, and it follows immediately from Proposition 4. $2x_t = x_p$ implies that the internal angles will be equal for both constructions. In consequence, $p(x_t) = q(x_p)$ and $v(x_t) = v'(x_p)$.

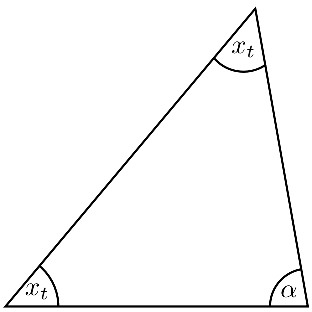


Figure 17: $x_t = 50°$

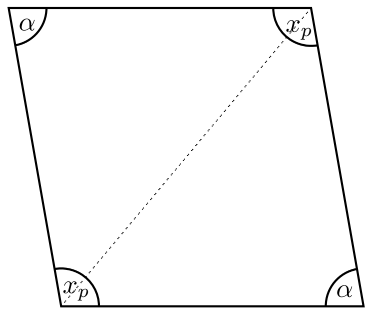


Figure 18: $x_p = 2x_t = 100°$

5 Appendix

5.1 Proof of the Lemma on Symmetries

$$i)\ \forall n \in \mathbb{Z}_{>2},\ 2 \nmid n \iff |\varphi_n| = 2|\chi_n|$$

Proof.

$$1.\ \forall n \in \mathbb{Z}^+ : \exists f_n : [1, \frac{n-1}{2}] \cap \mathbb{Z}^+ \to [\frac{n+1}{2}, n-1] \cap \mathbb{Z}^+ \textit{ bijective}$$

It is trivial that the domain and co-domain of this function are finite and numerable sets with cardinal $\frac{n-1}{2} \in \mathbb{Z}^+ \iff 2 \nmid n$*:*

$$\forall a, b \in \mathbb{Z}^+ : a \leq b \implies |[a, b] \cap \mathbb{Z}^+| = b - a + 1$$

$$n - 1 - \frac{n+1}{2} + 1 = \frac{2(n-1) - (n+1) + 2}{2} = \frac{n-1}{2}$$

Also:

$$2 \nmid n \iff [1, \frac{n-1}{2}] \cap \mathbb{Z}^+ \ \cup\ [\frac{n+1}{2}, n-1] \cap \mathbb{Z}^+ = [1, n-1] \cap \mathbb{Z}^+ = \{m \in \mathbb{Z}^+ : m < n\}$$

Specifically, we are going to consider the following rule of association

$$f_n(m) = n - m \iff f_n(m) + m = n$$

m	1	2	3	4	5	6	7
$f_{15}(m)$	14	13	12	11	10	9	8

$$2.\ \forall n \in \mathbb{Z}^+ : 2 \nmid n \iff \forall m \in [1, \frac{n-1}{2}] \cap \mathbb{Z}^+, \begin{cases} 2 \mid m \iff 2 \nmid f_n(m) \\ 2 \nmid m \iff 2 \mid f_n(m) \end{cases}$$

This occurs because of the rule of association of the function. We see that the sum of the argument m *and its image* $f_n(m)$ *must be equal to* n*, an odd number; and only the sum of an odd and an even number give an odd result.*

$$3.\ \forall m, n \in \mathbb{Z}^+ : \gcd(m, n) = 1 \iff \gcd(n, n-m) = 1 \iff (m \in \gamma'_n \iff f_n(m) \in \varphi'_n \setminus \gamma'_n)$$

m	1	2	$\cancel{3}$	4	5	$\cancel{6}$	7	8	$\cancel{9}$	10	11	$\cancel{12}$	13
$f_{27}(m)$	26	25	$\cancel{24}$	23	22	$\cancel{21}$	20	19	$\cancel{18}$	17	16	$\cancel{15}$	14

With this information we can see that there will exist a bijection among the odd and even elements of φ'_n*:*

χ_{27}	2	4	8	10	14	16	20	22	26
$\varphi_{27} \setminus \chi_{27}$	25	23	19	17	13	11	7	5	1

$$\forall a \in \chi'_n, \exists! b \in \varphi'_n \setminus \chi'_n : a + b = n$$

$$\forall b \in \varphi'_n \setminus \chi'_n, \exists! a \in \varphi'_n : a + b = n$$

So we would have already proven Symmetry ii) for $2 \nmid n$*. Let's do it in general:*

$$ii)\ \forall n \in \mathbb{Z}_{>2}, |\varphi_n| = 2|\gamma_n|$$

m	1	$\cancel{2}$	$\cancel{3}$	$\cancel{4}$	$\cancel{5}$	$\cancel{6}$	7	$\cancel{8}$	$\cancel{9}$	$\cancel{10}$	11	$\cancel{12}$	13	$\cancel{14}$
$f_{30}(m)$	29	$\cancel{28}$	$\cancel{27}$	$\cancel{26}$	$\cancel{25}$	$\cancel{24}$	23	$\cancel{22}$	$\cancel{21}$	$\cancel{20}$	19	$\cancel{18}$	17	$\cancel{16}$

Where we are considering the same rule of association for $f_n(m)$. Let's note that, for $2 \mid n$, the union of the domain and co-domain isn't equal to all positive integers less than n, but it excludes $\frac{n}{2}$ when $2 \mid n$, which will not affect the cardinal of neither φ_n nor γ_n, for $n > 2 \implies \frac{n}{2} > 1 \implies mcd(n, \frac{n}{2}) > 1 \implies \frac{n}{2} \notin \varphi_n$

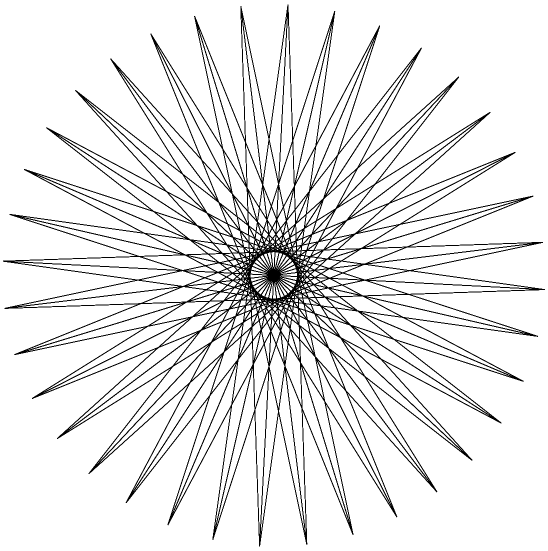
$$\forall m \in [1, \frac{n}{2}) \cap \mathbb{Z}^+, (2m < n \wedge \frac{n}{2} < f_n(m) < n) \wedge (m \in \gamma_n \iff f_n(m) \in \varphi_n \setminus \gamma_n)$$

$$|\gamma_n| + |\varphi_n \setminus \gamma_n| = |\gamma_n| + (|\varphi_n| - |\gamma_n|) = |\varphi_n| \wedge |\gamma_n| = |\varphi_n \setminus \gamma_n| \implies |\varphi_n| = 2|\gamma_n|$$

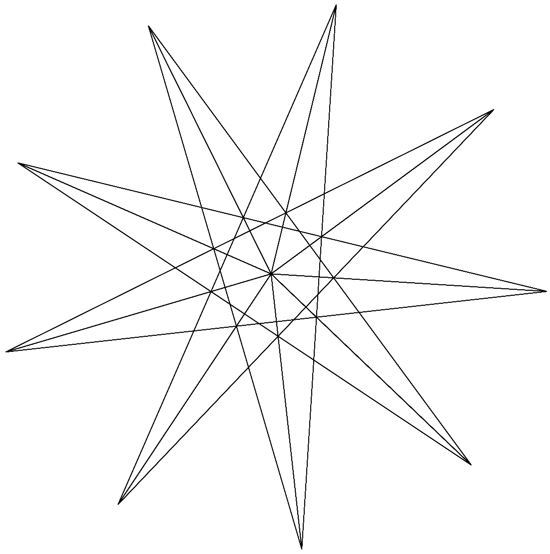
5.2 Software

`https://aidanLorenzo.github.io`
`https://github.com/aidanLorenzo/aidanLorenzo.github.io`

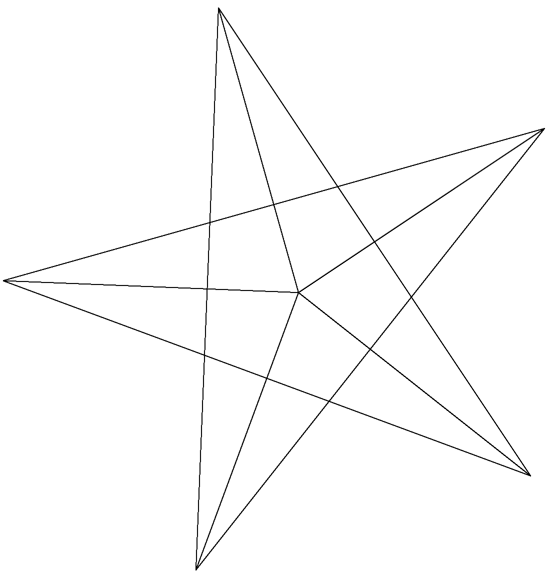
5.3 Figures



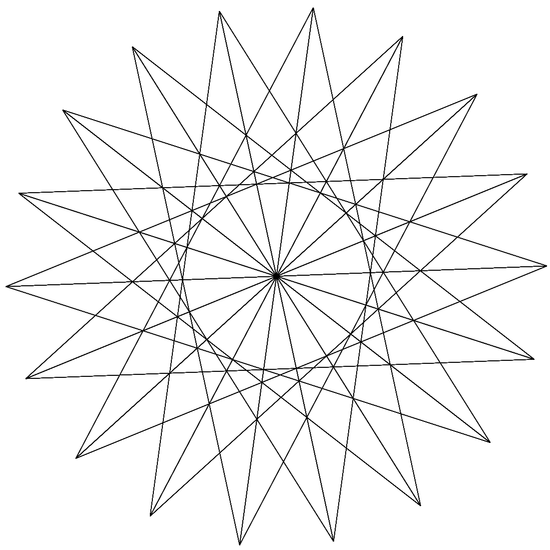
T_{5°



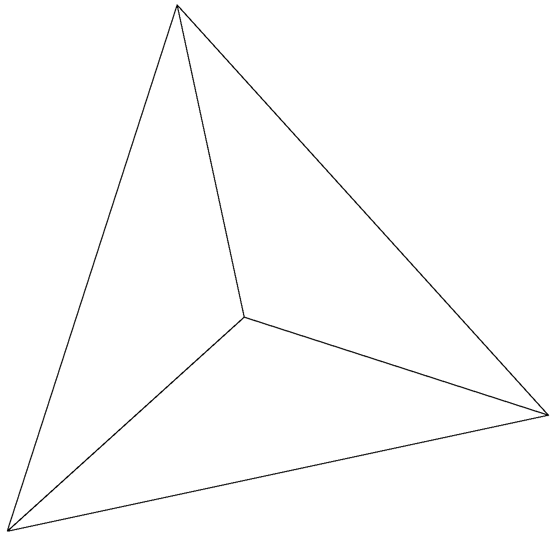
T_{10°



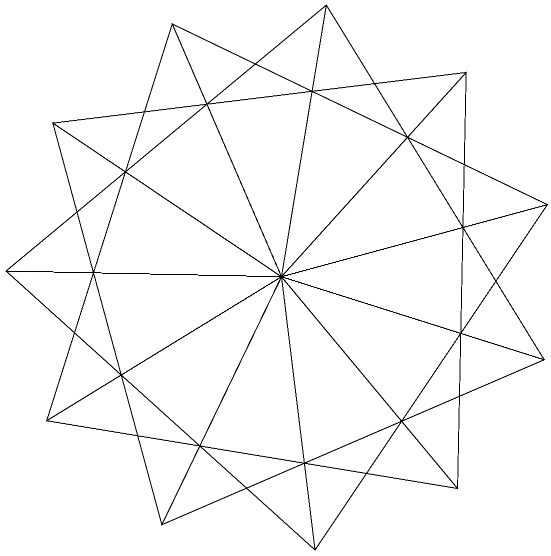
T_{18°



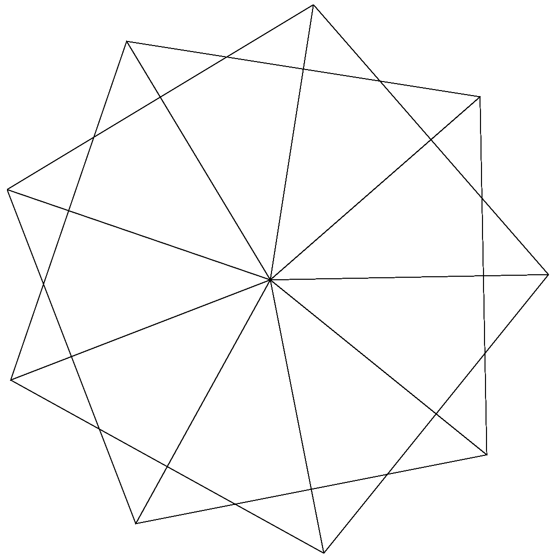
T_{20°



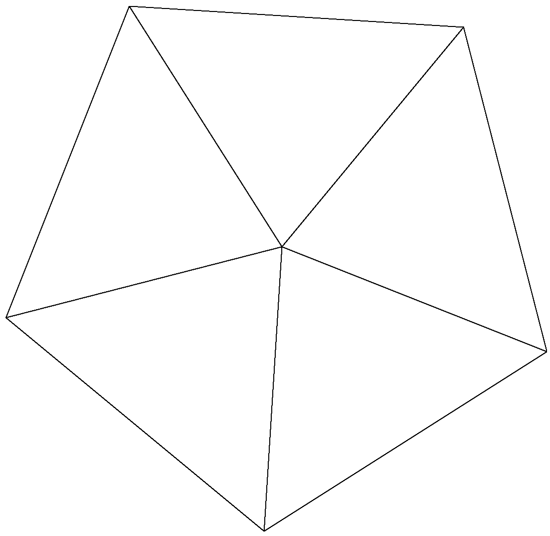
T_{30°



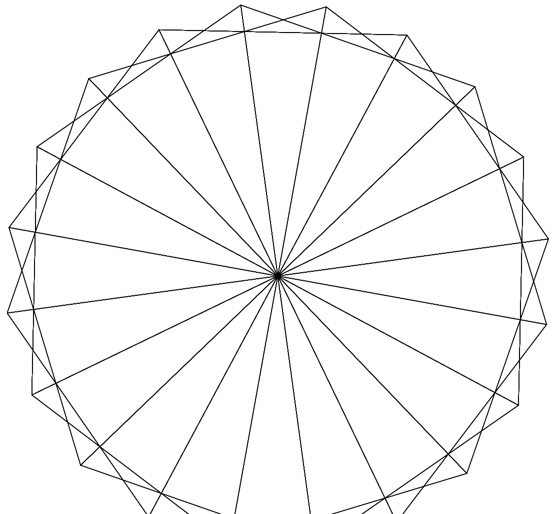
$T_{40,\overline{90}^\circ}$



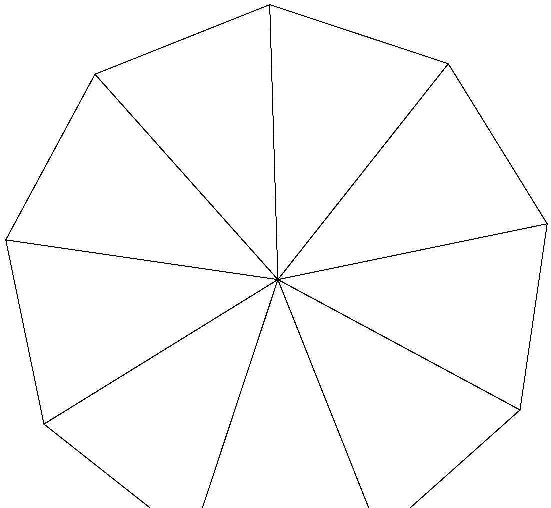
T_{50°



T_{54°



T_{63°



T_{70°