Advanced Electromagnetism I

PHYS 435

Fall 2025

Contents

1. LECTURE 2: GAUSS'S LAW

August 27, 2025

1.1 Gauss's Law

The electric field from a collection of charges is the vector sum of the fields from each charge.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}_i|^2}\hat{\vec{r}}_i,\ \vec{r}_i = \vec{r} - \vec{r'}_i$$

1.2 Gauss's Law in Differential Form

The divergence of the electric field is proportional to the charge density.

$$\oint_C v(\vec{r})\cdot d\vec{a} = \int_v (\nabla \cdot \vec{v}(\vec{r}))d\tau$$

$$\vec{v}(\vec{r}) = \text{any differentiable vector field}$$

$$\nabla \cdot \vec{v} = \text{divergence} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Apply to Gauss's Law :}$$

$$\oint \vec{E}\cdot d\vec{a} = \int_v (\nabla \cdot \vec{E}(\vec{r}))d\tau = \frac{Q_{enc}}{\epsilon_0},\ Q_{enc} = \int_v \rho(\vec{r})d\tau$$

$$\int_v (\nabla \cdot \vec{E})d\tau = \int_v (\rho(\vec{r})/\epsilon_0)d\tau$$

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\text{Divergence Identity:}$$

$$\nabla_r \cdot (\frac{\vec{r}}{r^2}) = 4\pi\delta^3(\vec{r})$$

2. LECTURE 3: THE CURL OF $\vec{E}(\nabla \times \vec{E})$

August 29, 2025

Integral:

$$\oint_C \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

Differential:

$$\nabla \times \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

2.1 Problem 2.18 Griffiths

Approach: Use superposition to add contributions from each sphere.

2.2 Practice using the differential form of Gauss's Law

Example 2.2.1. *For the differential form, it is important to remember we are considering a specified point in space.*

$$\text{Consider two identical charged plates and the electric field between them is constant.}$$
$$\text{The charge density is } 0 \text{ between the plates.} \nabla \cdot \vec{E} = 0, \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \text{Charge density is 0}$$
$$\nabla \cdot \vec{E} \neq 0 \Rightarrow \text{charge density}$$

Example 2.2.2. *Now consider* $\nabla \times \vec{E}$ *(curl)*

$$\text{By Stokes's theorem: } \int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_C \vec{v} \cdot d\vec{l}$$

The curl of any electric field due to a fixed charge distribution is zero:

$$\nabla \times \vec{E}(\vec{r}) = 0$$

For any closed loop,

$$\oint \vec{E} \cdot d\vec{l} = 0$$

3. LECTURE 4: ELECTRIC POTENTIAL

September 3, 2025

3.1 Electric Potential

For static charge distributions

$$\nabla \times \vec{E} = 0 \rightarrow \text{implies that 3 components of } \vec{E} \text{ are related to each other.}$$

Stokes's Theorem states that:

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = \oint_P \vec{E} \cdot d\vec{l}$$

$$\oint_P \vec{E} \cdot d\vec{l} = 0$$

$$\oint_P \vec{E} \cdot d\vec{l} \text{ is independent of path}$$

$$\int_a^b \vec{E} \cdot d\vec{l} + \int_b^a \vec{E} \cdot d\vec{l} = 0$$

Only the endpoints a and b matter

We can define a scalar function $V(\vec{r})$ such that:

$$\int_a^b \vec{E} \cdot d\vec{l} = V(\vec{a}) - V(\vec{b})$$

$$V(\vec{r}) = -\int_a^r \vec{E} \cdot d\vec{l}$$

$$V(b) - V(a) = -\int_0^b \vec{E} \cdot d\vec{l} + \int_0^a \vec{E} \cdot d\vec{l} = -\int_a^b \vec{E} \cdot d\vec{l}$$

Fundamental thereom of gradients: $V(b) - V(a) = \int_a^b \nabla V \cdot d\vec{l}$

$\rightarrow \vec{E} = -\nabla V(\vec{r})$

3.2 Potential

1. V is the (electric) potential.

$E = \frac{N}{C}, \quad V = \frac{N \cdot m}{C} = \frac{J}{C} = \text{Volt}$

2. The potential at one point has no physical significance.

Potential differences matter. We always define some reference point O. Typically choose O such that $V(O) = 0$. We can always change the reference point if we wish $V(\vec{r}) \to V(\vec{r}) + C$.

$\vec{E} \to \vec{E}$

3. Superposition also works with the potential

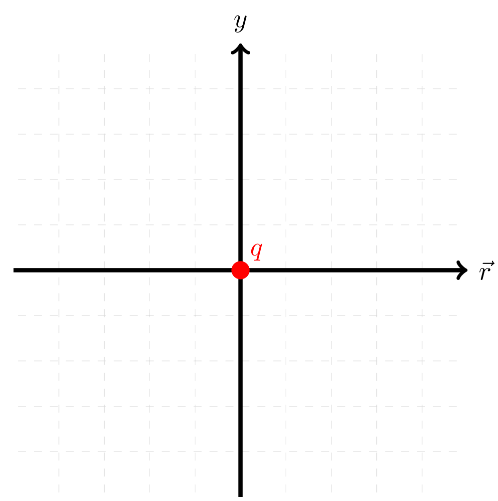
$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N$

$\vec{E} = -\nabla V$ or $V(\vec{r}) = -\int_0^r \vec{E} \cdot d\vec{l}$

$V_{total} = -\int_0^r \vec{E}_1 \cdot d\vec{l} + -\int_0^r \vec{E}_2 \cdot d\vec{l} + \cdots + -\int_0^r \vec{E}_N \cdot d\vec{l}$

$= v_1(\vec{r}) + v_2(\vec{r}) + \cdots + v_N(\vec{r})$

Example 3.2.1 (Potential due to a point charge).



$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \vec{r}$$
$$V(\vec{r}) = -\int_0^{\vec{r}} \vec{E} \cdot d\vec{l}$$
$$= -\int_\infty^r \frac{q}{4\pi\epsilon_0} \frac{1}{r'^2} dr'$$
$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r}$$

3.3 Poisson's Equation

$$\vec{E} = -\nabla V, \textit{Gauss's law} \nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \to \nabla^2 V = \frac{\rho(\vec{r})}{\epsilon_0} + \textit{Boundary conditions}$$

4. LECTURE 5: ELECTROSTATIC BOUNDARY CONDITIONS

September 5, 2025

Recall: $\vec{E}(\vec{r}) = -\nabla V(\vec{r})$, *and also* $\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$.

Invert: $\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0} \to V(\vec{r}) = f(p)$.

For a point charge, $V(\vec{r}) = \frac{q}{4\pi\epsilon_0 \nabla}$, *where* $\vec{\nabla} = \vec{r} - \vec{r}'$.

If we add another charge, $V(\vec{r}) = \frac{q}{4\pi\epsilon_0 \nabla} + \frac{q'}{4\pi\epsilon_0 \nabla'}$ *(just superpose).*

For N *charges:* $V(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0 \nabla_i}$.

For a continuous distribution: $V(\vec{r}) = \int \frac{dq}{4\pi\epsilon_0 \nabla} = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_0 \nabla} d\tau'$.

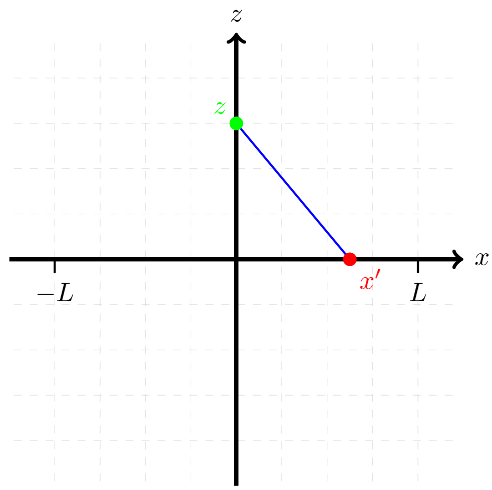
For a line charge: $V(\vec{r}) = \int \frac{\lambda(\vec{r}')}{4\pi\epsilon_0 \nabla} dl'$.

For a surface charge: $V(\vec{r}) = \int \frac{\sigma(\vec{r}')}{4\pi\epsilon_0 \nabla} da'$.

For a volume charge: $V(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_0 \nabla} d\tau'$.

4.1 Line Charge

Example 4.1.1 (Find e-field from line charge).



$$\int dV = \int \frac{dq}{4\pi\epsilon_0 \nabla} = \int \frac{\lambda dx'}{4\pi\epsilon_0 \sqrt{x'^2 + z^2}}$$

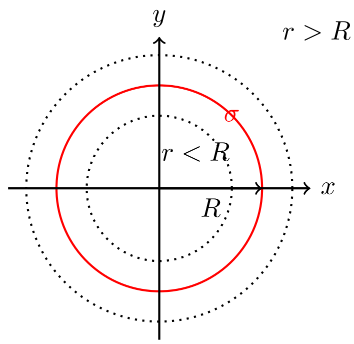
$$V = \frac{\lambda}{4\pi\epsilon_0} \int_{-L}^{L} \frac{dx'}{\sqrt{x'^2 + z^2}} = \frac{\lambda}{4\pi\epsilon_0} \left[\ln(x' + \sqrt{x'^2 + z^2}) \right]_{-L}^{L}$$

$$V(z) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{L + \sqrt{L^2 + z^2}}{-L + \sqrt{L^2 + z^2}} \right)$$

$$\vec{E} = -\nabla V = -(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}) = -\frac{\partial V}{\partial z}\hat{z} = \frac{2L\lambda}{4\pi\epsilon_0} \frac{1}{z\sqrt{L^2 + z^2}}\hat{z}$$

4.2 Spherical Shell

Example 4.2.1 (Spherical Shell)**.**



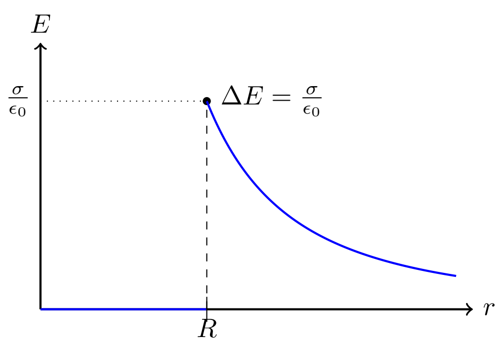
Spherical shell with radius R and surface charge density σ

For $r < R$: $\quad Q_{enc} = 0, \quad \vec{E} = 0$

For $r > R$: $\quad \oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}, \quad \vec{E} = \frac{\sigma}{\epsilon_0} \frac{R^2}{r^2} \hat{r}$ *where* $Q_{enc} = 4\pi R^2 \sigma$

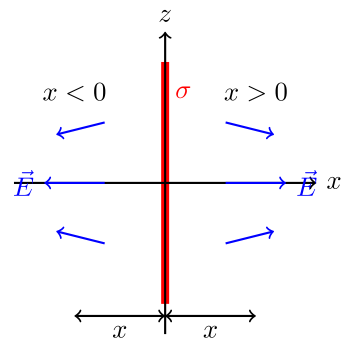
At $r = R$: $\quad E_{out} - E_{in} = \frac{\sigma}{\epsilon_0}$

Since $E_{in} = 0$: $\quad E_{out} = \frac{\sigma}{\epsilon_0}$



4.3 Sheet of Charge

Example 4.3.1 (Imagine sheet of charge)**.**



Infinite sheet of charge with surface density σ

Using Gauss's law with cylindrical Gaussian surface:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$E \cdot 2A = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

For $x > 0: \quad \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{x}$

For $x < 0: \quad \vec{E} = -\frac{\sigma}{2\epsilon_0}\hat{x}$

Magnitude: $|E| = \frac{\sigma}{2\epsilon_0}$ *(constant)*

4.3.1 Loop

$\oint \vec{E} \cdot d\vec{l} = 0$

$= E''_{above} l - E''_{below} l = 0 \rightarrow E''_{above} = E''_{below}$

4.3.2 General Statement

$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$, $\hat{n}$ *is a unit vector defining the surface normal..*

5. LECTURE 6: ELECTROSTATIC ENERGY

September 8, 2025

5.1 Recap: Boundary Conditions

From the previous lecture, we found the boundary conditions at a sheet of charge:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n} \tag{1}$$

$$\nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\epsilon_0}\hat{n} \tag{2}$$

$$\left.\frac{\partial V}{\partial n}\right|_{above} - \left.\frac{\partial V}{\partial n}\right|_{below} = -\frac{\sigma}{\epsilon_0} \tag{3}$$

where $\hat{n}$ is an outward pointing normal and $\frac{\partial V}{\partial n} = \nabla V \cdot \hat{n}$.

5.2 Work and Potential Energy

How much energy does it take to move a charge from point a to point b?

For a charge q in an electric field:

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}) \tag{4}$$

$$\vec{F}_{ext} = -q\vec{E}(\vec{r}) \quad \textit{(external force needed)} \tag{5}$$

The work done by the external force:

$$W = \int_a^b \vec{F}_{ext} \cdot d\vec{l} = -q\int_a^b \vec{E} \cdot d\vec{l} \tag{6}$$

$$= q[V(b) - V(a)] \quad \textit{(conservative force)} \tag{7}$$

Key insight: *The potential $V(\vec{r})$ has units of $\frac{energy}{charge}$. If we set the reference point at infinity where $V(\infty) = 0$, then:*

$$W(\vec{r}) = qV(\vec{r}) \tag{8}$$

5.3 Energy in Discrete Charge Arrangements

Imagine we are in a vacuum with no initial fields.

Step 1: *Bring in charge q_1 at location $\vec{r}_1$*

$$W_1 = 0 \quad \textit{(no electric field to work against)} \tag{9}$$

Step 2: *Bring in charge q_2 at location $\vec{r}_2$*

$$W_{12} = q_2 V_1(\vec{r}_2) = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|} \tag{10}$$

Step 3: *Bring in charge q_3 at location $\vec{r}_3$*

$$W_{123} = q_3 V_1(\vec{r}_3) + q_3 V_2(\vec{r}_3) \tag{11}$$

$$= \frac{q_1 q_3}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_1|} + \frac{q_2 q_3}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_2|} \tag{12}$$

Total energy:

$$
\begin{aligned}
W_{total} &= W_{12} + W_{13} + W_{23} && (13)\\
&= \frac{q_1 q_2}{4\pi\epsilon_0|\vec{r}_2 - \vec{r}_1|} + \frac{q_1 q_3}{4\pi\epsilon_0|\vec{r}_3 - \vec{r}_1|} + \frac{q_2 q_3}{4\pi\epsilon_0|\vec{r}_3 - \vec{r}_2|} && (14)
\end{aligned}
$$

General expression for N charges:

$$
\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j>i}^{N}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} && (15)\\
&= \frac{1}{2}\sum_{i=1}^{N} q_i V_i(\vec{r}_i) && (16)
\end{aligned}
$$

where $V_i(\vec{r}_i)$ is the potential at $\vec{r}_i$ due to all other charges.

5.4 Energy in Continuous Charge Distributions

For continuous charge distributions:

$$
W = \frac{1}{2}\int_{all\ space} \rho(\vec{r})V(\vec{r})\,d\tau \tag{17}
$$

Using Gauss's law: $\rho(\vec{r}) = \epsilon_0 \nabla \cdot \vec{E}(\vec{r})$

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\int (\nabla \cdot \vec{E})V\,d\tau && (18)\\
&= \frac{\epsilon_0}{2}\left[-\int \vec{E}\cdot\nabla V\,d\tau + \oint \vec{E}\cdot d\vec{a}\right] && (19)\\
&= \frac{\epsilon_0}{2}\int E^2\,d\tau && (20)
\end{aligned}
$$

Final result:

$$
W = \frac{\epsilon_0}{2}\int E^2(\vec{r})\,d\tau \tag{21}
$$

The quantity $\frac{\epsilon_0 E^2}{2}$ is the ***energy density*** *with units of $\frac{energy}{volume}$.*

Example 5.4.1 (Point charge energy)**.** *For a point charge q:*

$$
\begin{aligned}
W_{pt\ charge} &= \frac{\epsilon_0}{2}\int \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 d\tau && (22)\\
&= \frac{\epsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{q^2}{(4\pi\epsilon_0)^2 r^4} r^2\sin\theta\,dr\,d\theta\,d\phi && (23)\\
&= \frac{q^2}{32\pi^2\epsilon_0}\int_0^{\infty}\frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0}\left[-\frac{1}{r}\right]_0^{\infty} = \infty && (24)
\end{aligned}
$$

Note: *The self-energy of a point charge is infinite, indicating the classical model breaks down at small scales.*

6. LECTURE 7: PERFECT CONDUCTORS

September 10, 2025

6.1 Recap: Energy Superposition

Energy is stored in the electric field:

$$W = \frac{\epsilon_0}{2} \int E^2(\vec{r})\, d\tau$$

For superposition, energy is not simply additive:

$$\begin{aligned} W_{total} &= \frac{\epsilon_0}{2} \int [E_1(\vec{r}) + E_2(\vec{r})]^2 \, d\tau \\ &= W_1 + W_2 + \frac{\epsilon_0}{2} \int 2\vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r})\, d\tau \end{aligned}$$

The cross term represents interaction energy between the fields.

6.2 Material Classification

Materials can be characterized by how free charges are to move within them:

- ***Insulators**: Charges are tightly bound to atoms (ceramics, rubber, Teflon)*
- ***Conductors**: Charges are free to move, weakly bound (gold, platinum, aluminum, salt water)*

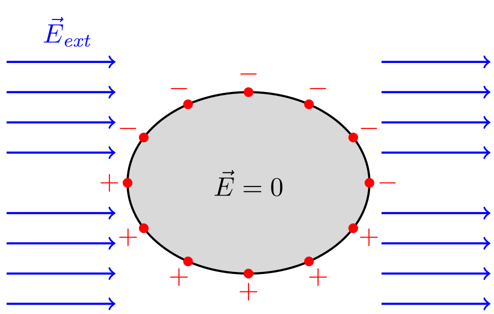
6.3 Properties of Perfect Conductors

A perfect conductor in electrostatic equilibrium has seven fundamental properties:

6.3.1 Property 1: Zero Internal Electric Field

$$\vec{E} = 0 \text{ inside conductor at equilibrium}$$

If an electric field were present, charges would experience force $\vec{F} = q\vec{E}$ and move until the field is cancelled. The charges separate to create an internal field that exactly cancels the external field.



Conductor in External Field

6.3.2 Property 2: Zero Volume Charge Density

$$\rho(\vec{r}) = 0 \text{ inside conductor}$$

*From Gauss's law: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$. Since $\vec{E} = 0$ inside, we have $\rho = 0$. This is called **screening**.*

6.3.3 Property 3: Surface Charge Distribution

If the conductor has net charge, it must reside entirely on the surface. Since $\vec{E} = 0$ inside, there is no mechanism to support volume charge density.

6.3.4 Property 4: Equipotential Surface

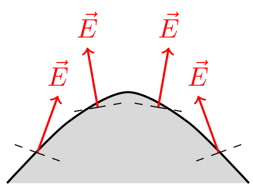
$$V = \text{constant throughout conductor}$$

Since $\vec{E} = -\nabla V = 0$, the potential cannot vary within the conductor:

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = 0$$

6.3.5 Property 5: Perpendicular Surface Field

The electric field just outside the conductor surface is perpendicular to the surface. Any tangential component would cause surface charges to move, violating equilibrium.



E-field perpendicular to conductor surface

6.3.6 Property 6: Faraday Cage Effect

A conductor with an internal cavity shields the cavity from external fields. The field inside the cavity is zero regardless of external conditions.

6.3.7 Property 7: Loss of Spatial Information

If a charge is placed inside a cavity within a conductor, observers outside the conductor cannot determine the position of the internal charge - only its magnitude affects the external field.

6.4 Surface Charge and Boundary Conditions

From our previous boundary condition analysis:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

For a conductor where $\vec{E}_{below} = 0$:

The electric field just outside a conductor surface is:

$$\vec{E}_{surface} = \frac{\sigma}{\epsilon_0}\hat{n}$$

where σ is the local surface charge density and $\hat{n}$ is the outward normal.

6.5 Electrostatic Pressure and Forces

The electric field exerts a force on the surface charges of a conductor, creating an electrostatic pressure.

Example 6.5.1 (Force per unit area on conductor surface)**.** *Consider the force on a small patch of conductor surface with area dA and charge $dq = \sigma\, dA$.*

The field acting on this charge is the average of the field just inside (zero) and just outside (σ/ϵ_0):

$$\vec{E}_{avg} = \frac{1}{2}(0 + \frac{\sigma}{\epsilon_0}\hat{n}) = \frac{\sigma}{2\epsilon_0}\hat{n}$$

The force per unit area (electrostatic pressure) is:

$$\vec{P} = \sigma\vec{E}_{avg} = \frac{\sigma^2}{2\epsilon_0}\hat{n} = \frac{\epsilon_0 E^2}{2}\hat{n}$$

This pressure always acts outward, trying to expand the conductor.

6.6 Introduction to Capacitance

When we have two conductors at different potentials, we can define capacitance.

For two conductors with charges $+Q$ and $-Q$ at potential difference V:

$$C = \frac{Q}{V}$$

Units: $[Farad] = \frac{Coulomb}{Volt}$

Capacitance depends only on geometry and material properties, not on Q or V.

The relationship $V \propto Q$ arises because:

- *Laplace's equation: $\nabla^2 V = 0$ in regions with $\rho = 0$*
- *Boundary conditions: $V =$ constant on conductor surfaces*
- *Linearity of Laplace's equation ensures $V \propto Q$*

Example 6.6.1 (Parallel plate capacitor preview)**.** *For two parallel plates with surface charge density $\pm\sigma$:*

$$\begin{aligned} E &= \frac{\sigma}{\epsilon_0} \\ V &= Ed = \frac{\sigma d}{\epsilon_0} \\ Q &= \sigma A \\ C &= \frac{Q}{V} = \frac{\sigma A}{\sigma d/\epsilon_0} = \frac{\epsilon_0 A}{d} \end{aligned}$$

7. LECTURE 8: CAPACITANCE AND LAPLACE'S EQUATION

September 12, 2025

7.1 Recap: Conductor Properties

From our study of conductors, we established:

$$V = V_+ - V_- \quad \textit{(potential difference)}$$
$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \textit{(Poisson's equation)}$$
$$V \propto Q \quad \textit{(linear relationship)}$$

7.2 Capacitance

For two conductors with charges $\pm Q$ at potential difference V:

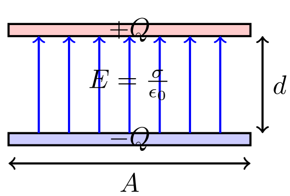
$$C = \frac{Q}{V}$$

Units: $[Farad] = \frac{Coulomb}{Volt}$

Capacitance depends only on geometry and material properties, not on Q or V.

7.3 Parallel Plate Capacitor

Example 7.3.1 (Parallel plate capacitor)**.** *Consider two parallel plates with area A, separation d, carrying charges $\pm Q$.*



Using Gauss's law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$
$$E \cdot A = \frac{\sigma A}{\epsilon_0}$$
$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Potential difference:

$$V = -\int \vec{E} \cdot d\vec{l} = Ed = \frac{Qd}{\epsilon_0 A}$$

Capacitance:

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

7.4 Energy Stored in Capacitor

The energy stored in a capacitor can be calculated as:

$$W = \int V\,dq = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C}$$
$$= \frac{1}{2}CV^2 = \frac{1}{2}QV$$

7.5 General Potential Calculations

For arbitrary charge configurations, we have two approaches:

7.5.1 High Symmetry Cases

When symmetry allows direct integration:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'$$

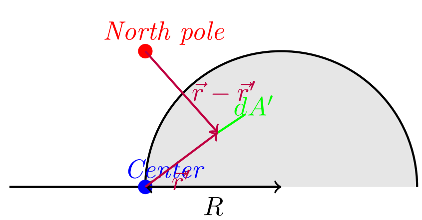
7.5.2 General Cases

When direct integration is difficult, solve Poisson's equation:

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

7.6 Example: Hemispherical Shell (Griffiths 2.48)

Example 7.6.1 (Hemispherical shell potential difference)**.** *An inverted hemispherical bowl of radius R carries uniform surface charge density σ. Find the potential difference between the "north pole" and the center.*



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dA'$$

At center: $|\vec{r}-\vec{r}'| = R$

$$V_{center} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{R}\,dA'$$
$$= \frac{\sigma R}{2\epsilon_0}$$

At north pole: $|\vec{r}-\vec{r}'| = \sqrt{2}R\sqrt{1-\cos\theta}$

$$V_{pole} = \frac{\sigma R}{\sqrt{2}\epsilon_0}$$

$$\Delta V = V_{pole} - V_{center}$$
$$= \frac{\sigma R}{\epsilon_0}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$$
$$= \frac{\sigma R}{2\epsilon_0}(\sqrt{2}-1)$$

7.7 Laplace's Equation

When $\rho = 0$ in a region, we have Laplace's equation:

$$\nabla^2 V = 0$$

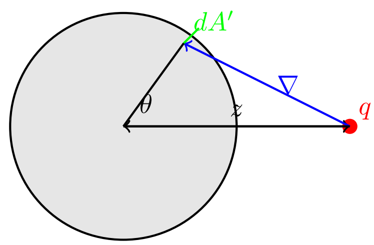
In Cartesian coordinates:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

7.8 Mean Value Theorem

Theorem 7.8.1 (Mean Value Theorem for Potential)**.** *The potential at the center of a sphere is equal to the average potential over the surface of the sphere, provided there are no charges inside the sphere.*

Example 7.8.2 (Verification of mean value theorem)**.** *For a point charge q outside a sphere of radius R:*



$$\begin{aligned}
\nabla &= \sqrt{z^2 + R^2 - 2zR\cos\theta} \\
V_{center} &= \frac{q}{4\pi\epsilon_0 z} \\
V_{ave} &= \frac{1}{4\pi R^2}\int \frac{q}{4\pi\epsilon_0 \nabla}\, dA' \\
&= \frac{q}{4\pi\epsilon_0 z} = V_{center}
\end{aligned}$$

This confirms the mean value theorem.

7.9 Uniqueness Theorem

Theorem 7.9.1 (Uniqueness of Solutions to Laplace's Equation)**.** *If a function V satisfies Laplace's equation in a region and satisfies the boundary conditions on the surface of that region, then V is unique.*

This theorem is crucial because it means that if we find any solution to Laplace's equation that satisfies the boundary conditions, we have found the only solution.

8. LECTURE 9: THE UNIQUENESS THEOREM

September 15, 2025

8.1 Recap: Properties of Laplace's Equation

From our previous study of Laplace's equation for finding potentials $\nabla^2 V = 0$, we established two key properties:

Properties of solutions to Laplace's equation:

1. ***Mean Value Theorem**: The potential at any point equals the average potential over any sphere centered at that point:*

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint V(\vec{r}')\, da'$$

2. ***No Local Extrema**: $V(\vec{r})$ has no local maxima or minima in the interior - extrema occur only at boundaries.*

To solve electrostatic problems, we need both Laplace's equation and appropriate boundary conditions.

8.2 The Uniqueness Theorem

Theorem 8.2.1 (Uniqueness Theorem for Electrostatics)**.** *Given boundary conditions on a closed surface, there exists one and only one solution to Laplace's equation (or Poisson's equation) in the volume enclosed by that surface.*

8.2.1 Proof for Laplace's Equation

***Setup**: Consider a volume V bounded by surface S, where we specify $V(\vec{r}) = f(\vec{r})$ on the boundary.*

***Proof by Contradiction**: Assume two different solutions V_1 and V_2 exist that both satisfy:*

$$\begin{aligned} &\nabla^2 V_1 = 0 \quad \textit{in } V \\ &\nabla^2 V_2 = 0 \quad \textit{in } V \\ &V_1(\vec{r}) = V_2(\vec{r}) = f(\vec{r}) \quad \textit{on } S \end{aligned}$$

Define the difference: $V_3 = V_1 - V_2$

Then:

$$\begin{aligned} &\nabla^2 V_3 = \nabla^2 (V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0 - 0 = 0 \\ &V_3(\vec{r}) = V_1(\vec{r}) - V_2(\vec{r}) = f(\vec{r}) - f(\vec{r}) = 0 \quad \textit{on } S \end{aligned}$$

Since V_3 satisfies Laplace's equation and equals zero on the boundary, and since Laplace's equation has no local extrema in the interior, V_3 must be zero everywhere in V.

Therefore: $V_3 = 0 \Rightarrow V_1 = V_2$

8.2.2 Extension to Poisson's Equation

Theorem 8.2.2 (Uniqueness for Poisson's Equation)**.** *The uniqueness theorem also applies to Poisson's equation: $\nabla^2 V = -\frac{\rho(\vec{r})}{\epsilon_0}$*

Proof: *If V_1 and V_2 are solutions to Poisson's equation with the same boundary conditions:*

$$\nabla^2 V_3 = \nabla^2(V_1 - V_2) = -\frac{\rho}{\epsilon_0} - \left(-\frac{\rho}{\epsilon_0}\right) = 0$$

The rest follows identically to the Laplace case.

8.3 Alternative Boundary Conditions

The uniqueness theorem applies to different types of boundary conditions:

- ***Dirichlet**: Specify $V(\vec{r})$ on the boundary surface S*
- ***Neumann**: Specify $\frac{\partial V}{\partial n}$ on the boundary surface S*
- ***Conductor charges**: Specify total charges Q_i on conductor surfaces*

8.3.1 Uniqueness with Conductor Boundary Conditions

For conductors, we can specify total charges instead of potentials. Consider two electric fields $\vec{E}_1$ and $\vec{E}_2$ that satisfy:

$$\nabla \cdot \vec{E}_1 = \nabla \cdot \vec{E}_2 = \frac{\rho(\vec{r})}{\epsilon_0}$$
$$\oint_{S_i} \vec{E}_1 \cdot d\vec{a} = \oint_{S_i} \vec{E}_2 \cdot d\vec{a} = \frac{Q_i}{\epsilon_0} \quad \text{for each conductor } i$$

Define $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$. Then:

$$\nabla \cdot \vec{E}_3 = 0$$
$$\oint_{S_i} \vec{E}_3 \cdot d\vec{a} = 0 \quad \text{for each conductor}$$

Since conductors are equipotentials, $V_3 = V_1 - V_2$ is constant on each conductor surface.
Using the vector identity and divergence theorem:

$$\begin{aligned}\nabla \cdot (V_3 \vec{E}_3) &= V_3(\nabla \cdot \vec{E}_3) + \vec{E}_3 \cdot (\nabla V_3) \\ &= 0 - \vec{E}_3 \cdot \vec{E}_3 = -E_3^2\end{aligned}$$

Integrating over the volume:

$$\int_V \nabla \cdot (V_3 \vec{E}_3)\, d\tau = \oint_S V_3 \vec{E}_3 \cdot d\vec{a} = -\int_V E_3^2\, d\tau$$

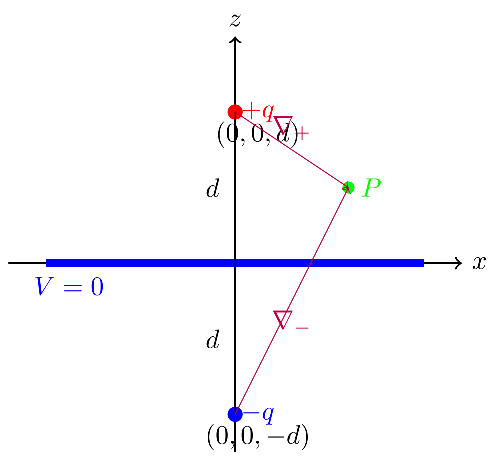
Since V_3 is constant on each conductor surface and $\oint_{S_i} \vec{E}_3 \cdot d\vec{a} = 0$, the left side equals zero.
Therefore: $E_3^2 = 0 \Rightarrow \vec{E}_3 = 0 \Rightarrow \vec{E}_1 = \vec{E}_2$

8.4 Method of Images

The uniqueness theorem enables powerful solution techniques like the method of images.

Example 8.4.1 (Point charge above grounded conducting plane)**.** *Consider a point charge q at position $(0, 0, d)$ above a grounded conducting plane at $z = 0$.*
Boundary conditions:

- $V(x, y, 0) = 0$ *(grounded plane)*
- $V(\vec{r}) \to 0$ *as* $|\vec{r}| \to \infty$



Solution Strategy:
Place an image charge $-q$ *at* $(0, 0, -d)$ *(outside the region of interest).*
Potential:

$$V = \frac{q}{4\pi\epsilon_0 \nabla_+} - \frac{q}{4\pi\epsilon_0 \nabla_-}$$

where:

$$\nabla_+ = \sqrt{x^2 + y^2 + (z-d)^2}$$
$$\nabla_- = \sqrt{x^2 + y^2 + (z+d)^2}$$

Verification:

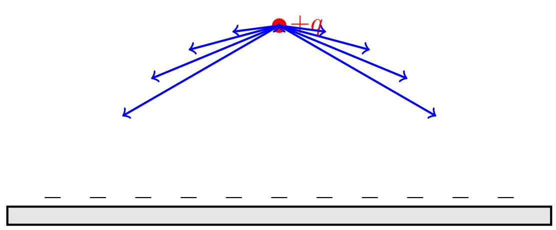
$$V|_{z=0} = \frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+d^2}} - \frac{q}{4\pi\epsilon_0\sqrt{x^2+y^2+d^2}} = 0\checkmark$$

Final Solution:

$$\boxed{V(x, y, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]}$$

The image charge $-q$ *at* $(0, 0, -d)$ *is called the* ***image charge****. It can only be placed outside the region where we're solving for the potential. The conductor acts like a mirror, creating the image charge to satisfy boundary conditions.*

8.5 Physical Interpretation



Conducting Plane

Induced negative charges screen the field inside

The method of images works because:

- *The conductor redistributes surface charge to maintain* $V = 0$
- *The image charge mimics this redistribution effect*
- *By uniqueness, this is the only possible solution*

9. COMMON MATH

9.1 Vector Calculus

9.1.1 Vector Operations

Example 9.1.1 (Vector Dot Product). *The dot product of two vectors:*

$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= |\vec{A}||\vec{B}| \cos\theta\end{aligned}$$

Example 9.1.2 (Vector Cross Product). *The cross product in component form:*

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}\end{aligned}$$

9.1.2 Differential Operators

Example 9.1.3 (Gradient). *The gradient of a scalar function:*

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f\end{aligned}$$

Example 9.1.4 (Divergence). *The divergence of a vector field:*

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Example 9.1.5 (Curl). *The curl of a vector field:*

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

9.2 Line Integrals and Path Integrals

Example 9.2.1 (Line Integral of a Vector Field). *Work done by a force along a path:*

$$\begin{aligned}W &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt\end{aligned}$$

where $\vec{r}(t)$ parametrizes the curve C from $t = a$ to $t = b$.

Example 9.2.2 (Line Integral of a Scalar Field). *Integral of a scalar function along a curve:*

$$\int_C f(x, y, z)\, ds = \int_a^b f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

Example 9.2.3 (Closed Path Integral). *Circulation around a closed loop:*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

This is Stokes' theorem.

9.3 Surface and Volume Integrals

Example 9.3.1 (Surface Integral). *Flux through a surface:*

$$\begin{aligned}\Phi &= \iint_S \vec{F} \cdot \hat{n}\, dS \\ &= \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv\end{aligned}$$

where $\vec{r}(u,v)$ parametrizes the surface S.

Example 9.3.2 (Volume Integral). *Integral over a volume:*

$$\iiint_V f(x,y,z)\, dV = \iiint_V f(x,y,z)\, dx\, dy\, dz$$

9.4 Differential Equations

Example 9.4.1 (First-Order Linear ODE).

$$\begin{aligned}&\frac{dy}{dx} + P(x)y = Q(x) \\ &\textit{Solution: } y = e^{-\int P(x)dx} \left[\int Q(x) e^{\int P(x)dx} dx + C \right]\end{aligned}$$

Example 9.4.2 (Second-Order Linear ODE with Constant Coefficients).

$$\begin{aligned}\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by &= f(x) \\ \textit{Characteristic equation: } r^2 + ar + b &= 0\end{aligned}$$

Example 9.4.3 (Wave Equation).

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2 \nabla^2 u \\ \frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)\end{aligned}$$

9.5 Complex Numbers and Phasors

Example 9.5.1 (Complex Exponential). *Euler's formula and complex representation:*

$$\begin{aligned}e^{i\theta} &= \cos\theta + i\sin\theta \\ z &= re^{i\theta} = r(\cos\theta + i\sin\theta)\end{aligned}$$

Example 9.5.2 (Phasor Notation). *AC voltage representation:*

$$\begin{aligned}V(t) &= V_0 \cos(\omega t + \phi) \\ \tilde{V} &= V_0 e^{i\phi} \quad \textit{(phasor)}\end{aligned}$$

9.6 Series and Summations

Example 9.6.1 (Taylor Series).

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Example 9.6.2 (Fourier Series).

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

9.7 Coordinate Systems

Example 9.7.1 (Spherical Coordinates).

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \\ dV &= r^2 \sin\theta \, dr \, d\theta \, d\phi \end{aligned}$$

Example 9.7.2 (Cylindrical Coordinates).

$$\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \\ z &= z \\ dV &= \rho \, d\rho \, d\phi \, dz \end{aligned}$$

9.8 Special Functions

Example 9.8.1 (Dirac Delta Function).

$$\delta(x) = \begin{cases} \infty & \textit{if } x = 0 \\ 0 & \textit{if } x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a) \, dx = f(a)$$

Example 9.8.2 (Heaviside Step Function).

$$H(x) = \begin{cases} 1 & \textit{if } x > 0 \\ 0 & \textit{if } x < 0 \end{cases}$$

$$\frac{d}{dx}H(x) = \delta(x)$$

9.9 Matrix Operations

Example 9.9.1 (Eigenvalue Problem)**.**

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ \det(A - \lambda I) &= 0 \end{aligned}$$

Example 9.9.2 (Matrix Exponential)**.**

$$\begin{aligned} e^A &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \\ &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \end{aligned}$$

9.10 Statistical Physics

Example 9.10.1 (Boltzmann Distribution)**.**

$$\begin{aligned} P(E) &= \frac{1}{Z} e^{-\beta E} \\ Z &= \sum_i e^{-\beta E_i} \quad \textit{(partition function)} \\ \beta &= \frac{1}{k_B T} \end{aligned}$$

Example 9.10.2 (Maxwell-Boltzmann Distribution)**.**

$$f(v) = 4\pi v^2 \left(\frac{m}{2\pi k_B T} \right)^{3/2} e^{-mv^2/(2k_B T)}$$