Advanced Electromagnetism I

PHYS 435

Fall 2025

Contents

1. LECTURE 2: GAUSS'S LAW

August 27, 2025

1.1 Gauss's Law

The electric field from a collection of charges is the vector sum of the fields from each charge.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}_i|^2}\hat{\vec{r}}_i,\ \vec{r}_i = \vec{r} - \vec{r'}_i$$

1.2 Gauss's Law in Differential Form

The divergence of the electric field is proportional to the charge density.

$$\oint_C v(\vec{r}) \cdot d\vec{a} = \int_v (\nabla \cdot \vec{v}(\vec{r}))d\tau$$

$$\vec{v}(\vec{r}) = \text{any differentiable vector field}$$

$$\nabla \cdot \vec{v} = \text{divergence} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Apply to Gauss's Law}:$$

$$\oint \vec{E} \cdot d\vec{a} = \int_v (\nabla \cdot \vec{E}(\vec{r}))d\tau = \frac{Q_{enc}}{\epsilon_0},\ Q_{enc} = \int_v \rho(\vec{r})d\tau$$

$$\int_v (\nabla \cdot \vec{E})d\tau = \int_v (\rho(\vec{r})/\epsilon_0)d\tau$$

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\text{Divergence Identity:}$$

$$\nabla_r \cdot (\frac{\vec{r}}{r^2}) = 4\pi\delta^3(\vec{r})$$

2. LECTURE 3: THE CURL OF $\vec{E}(\nabla \times \vec{E})$

August 29, 2025

Integral:

$$\oint_C \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

Differential:

$$\nabla \times \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

2.1 Problem 2.18 Griffiths

Approach: Use superposition to add contributions from each sphere.

2.2 Practice using the differential form of Gauss's Law

Example 2.2.1. *For the differential form, it is important to remember we are considering a specified point in space.*

$$\textit{Consider two identical charged plates and the electric field between them is constant.}$$
$$\textit{The charge density is } 0 \textit{ between the plates.} \nabla \cdot \vec{E} = 0, \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \textit{Charge density is 0}$$
$$\nabla \cdot \vec{E} \neq 0 \Rightarrow \textit{charge density}$$

Example 2.2.2. *Now consider $\nabla \times \vec{E}$ (curl)*

$$\textit{By Stokes's theorem: } \int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_C \vec{v} \cdot d\vec{l}$$

The curl of any electric field due to a fixed charge distribution is zero:

$$\nabla \times \vec{E}(\vec{r}) = 0$$

For any closed loop,

$$\oint \vec{E} \cdot d\vec{l} = 0$$

3. LECTURE 4: ELECTRIC POTENTIAL

September 3, 2025

3.1 Electric Potential

For static charge distributions

$$\nabla \times \vec{E} = 0 \rightarrow \text{implies that 3 components of } \vec{E} \text{ are related to each other.}$$

Stokes's Theorem states that:

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = \oint_P \vec{E} \cdot d\vec{l}$$

$$\oint_P \vec{E} \cdot d\vec{l} = 0$$

$$\oint_P \vec{E} \cdot d\vec{l} \text{ is independent of path}$$

$$\int_a^b \vec{E} \cdot d\vec{l} + \int_b^a \vec{E} \cdot d\vec{l} = 0$$

Only the endpoints a and b matter

We can define a scalar function $V(\vec{r})$ such that:

$$\int_a^b \vec{E} \cdot d\vec{l} = V(\vec{a}) - V(\vec{b})$$

$$V(\vec{r}) = -\int_a^r \vec{E} \cdot d\vec{l}$$

$$V(b) - V(a) = -\int_0^b \vec{E} \cdot d\vec{l} + \int_0^a \vec{E} \cdot d\vec{l} = -\int_a^b \vec{E} \cdot d\vec{l}$$

Fundamental thereom of gradients: $V(b) - V(a) = \int_a^b \nabla V \cdot d\vec{l}$

$\rightarrow \vec{E} = -\nabla V(\vec{r})$

3.2 Potential

1. V is the (electric) potential.

$E = \frac{N}{C}, \quad V = \frac{N \cdot m}{C} = \frac{J}{C} = \text{Volt}$

2. The potential at one point has no physical significance.

Potential differences matter. We always define some reference point O. Typically choose O such that $V(O) = 0$. We can always change the reference point if we wish $V(\vec{r}) \to V(\vec{r}) + C$.

$\vec{E} \to \vec{E}$

3. Superposition also works with the potential

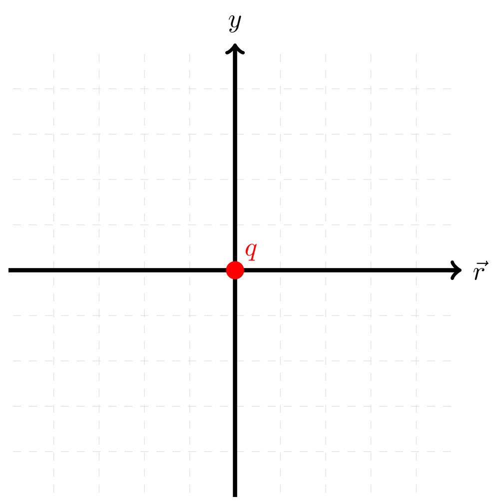
$\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N$

$\vec{E} = -\nabla V$ or $V(\vec{r}) = -\int_0^r \vec{E} \cdot d\vec{l}$

$V_{total} = -\int_0^r \vec{E}_1 \cdot d\vec{l} + -\int_0^r \vec{E}_2 \cdot d\vec{l} + \cdots + -\int_0^r \vec{E}_N \cdot d\vec{l}$

$= v_1(\vec{r}) + v_2(\vec{r}) + \cdots + v_N(\vec{r})$

Example 3.2.1 (Potential due to a point charge).



$$
\begin{aligned}
\vec{E} &= \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \vec{r} \\
V(\vec{r}) &= -\int_0^{\vec{r}} \vec{E} \cdot d\vec{l} \\
&= -\int_\infty^r \frac{q}{4\pi\epsilon_0} \frac{1}{r'^2} dr' \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{r}
\end{aligned}
$$

3.3 Poisson's Equation

$$\vec{E} = -\nabla V, \textit{Gauss's law} \nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \to \nabla^2 V = \frac{\rho(\vec{r})}{\epsilon_0} + \textit{Boundary conditions}$$

4. LECTURE 5: ELECTROSTATIC BOUNDARY CONDITIONS

September 5, 2025

Recall: $\vec{E}(\vec{r}) = -\nabla V(\vec{r})$, *and also* $\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$.

Invert: $\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0} \to V(\vec{r}) = f(p)$.

For a point charge, $V(\vec{r}) = \frac{q}{4\pi\epsilon_0 \nabla}$, *where* $\vec{\nabla} = \vec{r} - \vec{r}'$.

If we add another charge, $V(\vec{r}) = \frac{q}{4\pi\epsilon_0 \nabla} + \frac{q'}{4\pi\epsilon_0 \nabla'}$ *(just superpose).*

For N *charges:* $V(\vec{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0 \nabla_i}$.

For a continuous distribution: $V(\vec{r}) = \int \frac{dq}{4\pi\epsilon_0 \nabla} = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_0 \nabla} d\tau'$.

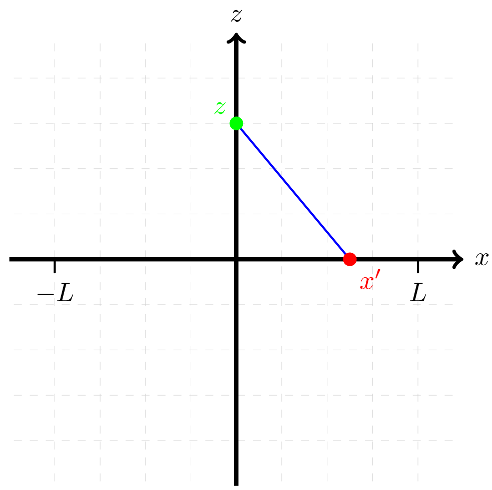
For a line charge: $V(\vec{r}) = \int \frac{\lambda(\vec{r}')}{4\pi\epsilon_0 \nabla} dl'$.

For a surface charge: $V(\vec{r}) = \int \frac{\sigma(\vec{r}')}{4\pi\epsilon_0 \nabla} da'$.

For a volume charge: $V(\vec{r}) = \int \frac{\rho(\vec{r}')}{4\pi\epsilon_0 \nabla} d\tau'$.

4.1 Line Charge

Example 4.1.1 (Find e-field from line charge).



$$\int dV = \int \frac{dq}{4\pi\epsilon_0 \nabla} = \int \frac{\lambda dx'}{4\pi\epsilon_0\sqrt{x'^2+z^2}}$$

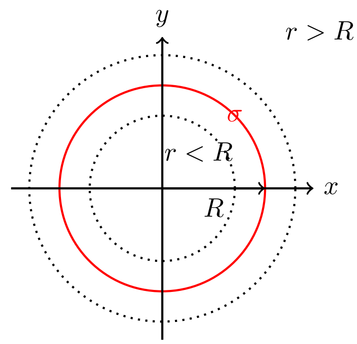
$$V = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dx'}{\sqrt{x'^2+z^2}} = \frac{\lambda}{4\pi\epsilon_0}\left[\ln(x'+\sqrt{x'^2+z^2})\right]_{-L}^{L}$$

$$V(z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{L+\sqrt{L^2+z^2}}{-L+\sqrt{L^2+z^2}}\right)$$

$$\vec{E} = -\nabla V = -(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}) = -\frac{\partial V}{\partial z}\hat{z} = \frac{2L\lambda}{4\pi\epsilon_0}\frac{1}{z\sqrt{L^2+z^2}}\hat{z}$$

4.2 Spherical Shell

Example 4.2.1 (Spherical Shell).



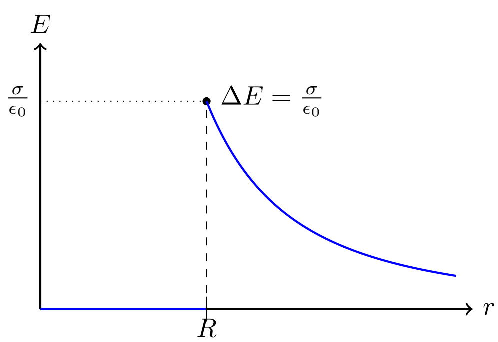
Spherical shell with radius R and surface charge density σ

For $r < R$: $\quad Q_{enc} = 0, \quad \vec{E} = 0$

For $r > R$: $\quad \oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}, \quad \vec{E} = \frac{\sigma}{\epsilon_0}\frac{R^2}{r^2}\hat{r}$ *where* $Q_{enc} = 4\pi R^2 \sigma$

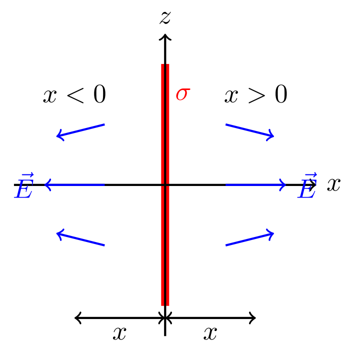
At $r = R$: $\quad E_{out} - E_{in} = \frac{\sigma}{\epsilon_0}$

Since $E_{in} = 0$: $\quad E_{out} = \frac{\sigma}{\epsilon_0}$



4.3 Sheet of Charge

Example 4.3.1 (Imagine sheet of charge).



Infinite sheet of charge with surface density σ
Using Gauss's law with cylindrical Gaussian surface:

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$E\cdot 2A = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

For $x>0: \quad \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{x}$

For $x<0: \quad \vec{E} = -\frac{\sigma}{2\epsilon_0}\hat{x}$

Magnitude: $|E| = \frac{\sigma}{2\epsilon_0}$ *(constant)*

4.3.1 Loop

$\oint \vec{E}\cdot d\vec{l} = 0$
$= E''_{above} l - E''_{below} l = 0 \to E''_{above} = E''_{below}$

4.3.2 General Statement

$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$, $\hat{n}$ *is a unit vector defining the surface normal..*

5. LECTURE 6: ELECTROSTATIC ENERGY

September 8, 2025

5.1 Recap: Boundary Conditions

From the previous lecture, we found the boundary conditions at a sheet of charge:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n} \tag{1}$$

$$\nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\epsilon_0}\hat{n} \tag{2}$$

$$\left.\frac{\partial V}{\partial n}\right|_{above} - \left.\frac{\partial V}{\partial n}\right|_{below} = -\frac{\sigma}{\epsilon_0} \tag{3}$$

where $\hat{n}$ is an outward pointing normal and $\frac{\partial V}{\partial n} = \nabla V \cdot \hat{n}$.

5.2 Work and Potential Energy

How much energy does it take to move a charge from point a to point b?

For a charge q in an electric field:

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}) \tag{4}$$

$$\vec{F}_{ext} = -q\vec{E}(\vec{r}) \quad \textit{(external force needed)} \tag{5}$$

The work done by the external force:

$$W = \int_a^b \vec{F}_{ext} \cdot d\vec{l} = -q\int_a^b \vec{E} \cdot d\vec{l} \tag{6}$$

$$= q[V(b) - V(a)] \quad \textit{(conservative force)} \tag{7}$$

Key insight: *The potential $V(\vec{r})$ has units of $\frac{energy}{charge}$. If we set the reference point at infinity where $V(\infty) = 0$, then:*

$$W(\vec{r}) = qV(\vec{r}) \tag{8}$$

5.3 Energy in Discrete Charge Arrangements

Imagine we are in a vacuum with no initial fields.

Step 1: *Bring in charge q_1 at location $\vec{r}_1$*

$$W_1 = 0 \quad \textit{(no electric field to work against)} \tag{9}$$

Step 2: *Bring in charge q_2 at location $\vec{r}_2$*

$$W_{12} = q_2V_1(\vec{r}_2) = \frac{q_1q_2}{4\pi\epsilon_0|\vec{r}_2 - \vec{r}_1|} \tag{10}$$

Step 3: *Bring in charge q_3 at location $\vec{r}_3$*

$$W_{123} = q_3V_1(\vec{r}_3) + q_3V_2(\vec{r}_3) \tag{11}$$

$$= \frac{q_1q_3}{4\pi\epsilon_0|\vec{r}_3 - \vec{r}_1|} + \frac{q_2q_3}{4\pi\epsilon_0|\vec{r}_3 - \vec{r}_2|} \tag{12}$$

Total energy:

$$
\begin{aligned}
W_{total} &= W_{12} + W_{13} + W_{23} && (13)\\
&= \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|} + \frac{q_1 q_3}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_1|} + \frac{q_2 q_3}{4\pi\epsilon_0 |\vec{r}_3 - \vec{r}_2|} && (14)
\end{aligned}
$$

General expression for N charges:

$$
\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} && (15)\\
&= \frac{1}{2} \sum_{i=1}^{N} q_i V_i(\vec{r}_i) && (16)
\end{aligned}
$$

where $V_i(\vec{r}_i)$ is the potential at $\vec{r}_i$ due to all other charges.

5.4 Energy in Continuous Charge Distributions

For continuous charge distributions:

$$
W = \frac{1}{2} \int_{all\ space} \rho(\vec{r}) V(\vec{r})\, d\tau \qquad (17)
$$

Using Gauss's law: $\rho(\vec{r}) = \epsilon_0 \nabla \cdot \vec{E}(\vec{r})$

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E}) V\, d\tau && (18)\\
&= \frac{\epsilon_0}{2} \left[-\int \vec{E} \cdot \nabla V\, d\tau + \oint \vec{E} \cdot d\vec{a} \right] && (19)\\
&= \frac{\epsilon_0}{2} \int E^2\, d\tau && (20)
\end{aligned}
$$

Final result:

$$
W = \frac{\epsilon_0}{2} \int E^2(\vec{r})\, d\tau \qquad (21)
$$

The quantity $\frac{\epsilon_0 E^2}{2}$ is the ***energy density*** *with units of $\frac{energy}{volume}$.*

Example 5.4.1 (Point charge energy)**.** *For a point charge q:*

$$
\begin{aligned}
W_{pt\ charge} &= \frac{\epsilon_0}{2} \int \left(\frac{q}{4\pi\epsilon_0 r^2} \right)^2 d\tau && (22)\\
&= \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{q^2}{(4\pi\epsilon_0)^2 r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi && (23)\\
&= \frac{q^2}{32\pi^2\epsilon_0} \int_0^{\infty} \frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0} \left[-\frac{1}{r} \right]_0^{\infty} = \infty && (24)
\end{aligned}
$$

Note: *The self-energy of a point charge is infinite, indicating the classical model breaks down at small scales.*

6. LECTURE 7: PERFECT CONDUCTORS

September 10, 2025

Recap: We defined the energy as being stored in the electric field $W = \frac{\epsilon_0}{2} \int E^2(\vec{r})\, d\tau$
Note Superposition is not just adding the energy of each field $W_{total} = \frac{\epsilon_0}{2} \int E_1^2(\vec{r}) + E_2^2(\vec{r})\, d\tau = W_1 + W_2 + \frac{\epsilon_0}{2} \int E_1(\vec{r}) \cdot E_2(\vec{r})\, d\tau$
Materials can be characterized by how free charges are to move in that material.
Insulator: charges are tightly bound to the atoms that form the material (ceramics, rubbery teflon).
Conductor: charges are free to move, very weakly bound (gold, platinum, aluminum, salt water).
Properties of conductors:
1. $\vec{E} = 0$ *inside conductor at equilibrium. If electric field is present, the charges will move (*$\vec{F} = q\vec{E}$*) until the field is cancelled out. The net effect is that charges separate.*
2. $\rho(\vec{r}) = 0$ $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = 0 \rightarrow \rho = 0$*. This is called screening.*
3. if the conductor has net charge, it must lie at the surface. if $E = 0$ *inside the conductor, there is no place for the charge to be except for on the surface.*
4. A conductor is an equipotential. $V = -\int_a^b \vec{E} \cdot d\vec{l} = 0 \rightarrow V(b) = V(a)$ *potential everywhere is the same*
5. The electric field just outside conductor is perpendicular to the surface.
6. If a conductor with an internal cavity is placed in an electric field, the field in the cavity is zero.
7. If a charge is placed in the cavity, all spatial information is lost to those outside the conductor.
Surface Charge: The boundary conditions we found tell us the perpendicular field outside of the conductor
For a sheet charge $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$
For a conductor $\vec{E}_{below} = 0 \rightarrow \vec{E}_{above} = \frac{\sigma}{\epsilon_0}\hat{n} \rightarrow \frac{\partial V}{\partial n} = \nabla \cdot \hat{n}$
Force on the conductor:
Capacitance:
$V = V_+ - V_-$
$\nabla^2 V = \frac{\rho}{\epsilon_0}$
$V \propto Q$

7. COMMON MATH

7.1 Vector Calculus

7.1.1 Vector Operations

Example 7.1.1 (Vector Dot Product). *The dot product of two vectors:*

$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= |\vec{A}||\vec{B}| \cos\theta\end{aligned}$$

Example 7.1.2 (Vector Cross Product). *The cross product in component form:*

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}\end{aligned}$$

7.1.2 Differential Operators

Example 7.1.3 (Gradient). *The gradient of a scalar function:*

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f\end{aligned}$$

Example 7.1.4 (Divergence). *The divergence of a vector field:*

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Example 7.1.5 (Curl). *The curl of a vector field:*

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

7.2 Line Integrals and Path Integrals

Example 7.2.1 (Line Integral of a Vector Field). *Work done by a force along a path:*

$$\begin{aligned}W &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt\end{aligned}$$

where $\vec{r}(t)$ parametrizes the curve C from $t = a$ to $t = b$.

Example 7.2.2 (Line Integral of a Scalar Field). *Integral of a scalar function along a curve:*

$$\int_C f(x, y, z)\, ds = \int_a^b f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

Example 7.2.3 (Closed Path Integral). *Circulation around a closed loop:*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

This is Stokes' theorem.

7.3 Surface and Volume Integrals

Example 7.3.1 (Surface Integral). *Flux through a surface:*

$$\begin{aligned}\Phi &= \iint_S \vec{F} \cdot \hat{n}\, dS \\ &= \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv\end{aligned}$$

where $\vec{r}(u, v)$ parametrizes the surface S.

Example 7.3.2 (Volume Integral). *Integral over a volume:*

$$\iiint_V f(x, y, z)\, dV = \iiint_V f(x, y, z)\, dx\, dy\, dz$$

7.4 Differential Equations

Example 7.4.1 (First-Order Linear ODE).

$$\begin{aligned}&\frac{dy}{dx} + P(x)y = Q(x) \\ &\text{Solution: } y = e^{-\int P(x)dx} \left[\int Q(x) e^{\int P(x)dx} dx + C \right]\end{aligned}$$

Example 7.4.2 (Second-Order Linear ODE with Constant Coefficients).

$$\begin{aligned}\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \\ \text{Characteristic equation: } r^2 + ar + b = 0\end{aligned}$$

Example 7.4.3 (Wave Equation).

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2 \nabla^2 u \\ \frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)\end{aligned}$$

7.5 Complex Numbers and Phasors

Example 7.5.1 (Complex Exponential). *Euler's formula and complex representation:*

$$\begin{aligned}e^{i\theta} &= \cos\theta + i\sin\theta \\ z &= re^{i\theta} = r(\cos\theta + i\sin\theta)\end{aligned}$$

Example 7.5.2 (Phasor Notation). *AC voltage representation:*

$$\begin{aligned}V(t) &= V_0 \cos(\omega t + \phi) \\ \tilde{V} &= V_0 e^{i\phi} \quad \text{(phasor)}\end{aligned}$$

7.6 Series and Summations

Example 7.6.1 (Taylor Series).

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Example 7.6.2 (Fourier Series).

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

7.7 Coordinate Systems

Example 7.7.1 (Spherical Coordinates).

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ dV &= r^2\sin\theta\, dr\, d\theta\, d\phi \end{aligned}$$

Example 7.7.2 (Cylindrical Coordinates).

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \\ dV &= \rho\, d\rho\, d\phi\, dz \end{aligned}$$

7.8 Special Functions

Example 7.8.1 (Dirac Delta Function).

$$\begin{aligned} \delta(x) = \begin{cases} \infty & \textit{if } x = 0 \\ 0 & \textit{if } x \neq 0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(x)\, dx = 1 \\ \int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a) \end{aligned}$$

Example 7.8.2 (Heaviside Step Function).

$$\begin{aligned} H(x) = \begin{cases} 1 & \textit{if } x > 0 \\ 0 & \textit{if } x < 0 \end{cases} \\ \frac{d}{dx}H(x) = \delta(x) \end{aligned}$$

7.9 Matrix Operations

Example 7.9.1 (Eigenvalue Problem)**.**

$$
\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
\det(A - \lambda I) &= 0
\end{aligned}
$$

Example 7.9.2 (Matrix Exponential)**.**

$$
\begin{aligned}
e^A &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \\
&= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots
\end{aligned}
$$

7.10 Statistical Physics

Example 7.10.1 (Boltzmann Distribution)**.**

$$
\begin{aligned}
P(E) &= \frac{1}{Z} e^{-\beta E} \\
Z &= \sum_i e^{-\beta E_i} \quad \textit{(partition function)} \\
\beta &= \frac{1}{k_B T}
\end{aligned}
$$

Example 7.10.2 (Maxwell-Boltzmann Distribution)**.**

$$
f(v) = 4\pi v^2 \left(\frac{m}{2\pi k_B T} \right)^{3/2} e^{-mv^2/(2k_B T)}
$$