Quantum Information Theory

ECE 404

Fall 2025

Contents

1. LECTURE 2: QUANTUM INFORMATION THEORY

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1.1 Axiom I: The State Space Axiom and Density Operators

- The starting point in formulating quantum mechanics is the state space axiom.
It consists of two parts:

- Every quantum system is represented by a complex Hilbert space $\mathcal{H}$ called state space.
- States of the system are represented by trace-one positive semi-definite operators acting on $\mathcal{H}$ called density operators. The set of all density operators is denoted by $\mathcal{D}(\mathcal{H})$.

1.2 Gauss's Law in Differential Form

The divergence of the electric field is proportional to the charge density.

$$\oint_C v(\vec{r}) \cdot d\vec{a} = \int_v (\nabla \cdot \vec{v}(\vec{r}))d\tau$$

$$\vec{v}(\vec{r}) = \text{any differentiable vector field}$$

$$\nabla \cdot \vec{v} = \text{divergence} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Apply to Gauss's Law :}$$

$$\oint \vec{E} \cdot d\vec{a} = \int_v (\nabla \cdot \vec{E}(\vec{r}))d\tau = \frac{Q_{enc}}{\epsilon_0},\ Q_{enc} = \int_v \rho(\vec{r})d\tau$$

$$\int_v (\nabla \cdot \vec{E})d\tau = \int_v (\rho(\vec{r})/\epsilon_0)d\tau$$

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\text{Divergence Identity:}$$

$$\nabla_r \cdot (\frac{\vec{r}}{r^2}) = 4\pi\delta^3(\vec{r})$$

2. PHYSICS MATH EXAMPLES REFERENCE

Common Mathematical Notation in Physics

2.1 Vector Calculus

2.1.1 Vector Operations

Example 2.1.1 (Vector Dot Product). *The dot product of two vectors:*

$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= |\vec{A}||\vec{B}| \cos\theta\end{aligned}$$

Example 2.1.2 (Vector Cross Product). *The cross product in component form:*

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}\end{aligned}$$

2.1.2 Differential Operators

Example 2.1.3 (Gradient). *The gradient of a scalar function:*

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f\end{aligned}$$

Example 2.1.4 (Divergence). *The divergence of a vector field:*

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Example 2.1.5 (Curl). *The curl of a vector field:*

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

2.2 Line Integrals and Path Integrals

Example 2.2.1 (Line Integral of a Vector Field). *Work done by a force along a path:*

$$\begin{aligned}W &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt\end{aligned}$$

where $\vec{r}(t)$ parametrizes the curve C from $t = a$ to $t = b$.

Example 2.2.2 (Line Integral of a Scalar Field). *Integral of a scalar function along a curve:*

$$\int_C f(x, y, z)\, ds = \int_a^b f(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

Example 2.2.3 (Closed Path Integral). *Circulation around a closed loop:*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

This is Stokes' theorem.

2.3 Surface and Volume Integrals

Example 2.3.1 (Surface Integral). *Flux through a surface:*

$$\begin{aligned} \Phi &= \iint_S \vec{F} \cdot \hat{n}\, dS \\ &= \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, du\, dv \end{aligned}$$

where $\vec{r}(u,v)$ parametrizes the surface S.

Example 2.3.2 (Volume Integral). *Integral over a volume:*

$$\iiint_V f(x,y,z)\, dV = \iiint_V f(x,y,z)\, dx\, dy\, dz$$

2.4 Differential Equations

Example 2.4.1 (First-Order Linear ODE).

$$\begin{aligned} &\frac{dy}{dx} + P(x)y = Q(x) \\ &\textit{Solution: } y = e^{-\int P(x)dx} \left[\int Q(x) e^{\int P(x)dx} dx + C \right] \end{aligned}$$

Example 2.4.2 (Second-Order Linear ODE with Constant Coefficients).

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$$
$$\textit{Characteristic equation: } r^2 + ar + b = 0$$

Example 2.4.3 (Wave Equation).

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \nabla^2 u \\ \frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \end{aligned}$$

2.5 Complex Numbers and Phasors

Example 2.5.1 (Complex Exponential). *Euler's formula and complex representation:*

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ z &= re^{i\theta} = r(\cos\theta + i\sin\theta) \end{aligned}$$

Example 2.5.2 (Phasor Notation). *AC voltage representation:*

$$\begin{aligned} V(t) &= V_0 \cos(\omega t + \phi) \\ \tilde{V} &= V_0 e^{i\phi} \quad \textit{(phasor)} \end{aligned}$$

2.6 Series and Summations

Example 2.6.1 (Taylor Series).

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Example 2.6.2 (Fourier Series).

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

2.7 Coordinate Systems

Example 2.7.1 (Spherical Coordinates).

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \\ dV &= r^2 \sin\theta \, dr \, d\theta \, d\phi \end{aligned}$$

Example 2.7.2 (Cylindrical Coordinates).

$$\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \\ z &= z \\ dV &= \rho \, d\rho \, d\phi \, dz \end{aligned}$$

2.8 Special Functions

Example 2.8.1 (Dirac Delta Function).

$$\delta(x) = \begin{cases} \infty & \textit{if } x = 0 \\ 0 & \textit{if } x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x) \, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a) \, dx = f(a)$$

Example 2.8.2 (Heaviside Step Function).

$$H(x) = \begin{cases} 1 & \textit{if } x > 0 \\ 0 & \textit{if } x < 0 \end{cases}$$

$$\frac{d}{dx} H(x) = \delta(x)$$

2.9 Matrix Operations

Example 2.9.1 (Eigenvalue Problem)**.**

$$
\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
\det(A - \lambda I) &= 0
\end{aligned}
$$

Example 2.9.2 (Matrix Exponential)**.**

$$
\begin{aligned}
e^A &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \\
&= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots
\end{aligned}
$$

2.10 Statistical Physics

Example 2.10.1 (Boltzmann Distribution)**.**

$$
\begin{aligned}
P(E) &= \frac{1}{Z} e^{-\beta E} \\
Z &= \sum_i e^{-\beta E_i} \quad \textit{(partition function)} \\
\beta &= \frac{1}{k_B T}
\end{aligned}
$$

Example 2.10.2 (Maxwell-Boltzmann Distribution)**.**

$$
f(v) = 4\pi v^2 \left(\frac{m}{2\pi k_B T} \right)^{3/2} e^{-mv^2/(2k_B T)}
$$