Quantum Computing Fundamentals

PHYS 370

Spring 2025

Contents

1. LECTURE 3: QUANTUM GATES AND CIRCUITS

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1.1 Matrix Representation of Quantum Gates

In quantum computing, the state of an n-qubit system is represented by a 2^n-dimensional vector. Quantum gates are represented by $2^n \times 2^n$ unitary matrices. A six-qubit quantum gate must be represented by a 64×64 unitary matrix. The unitary condition:

$$U^\dagger U = UU^\dagger = I,$$

where $U^\dagger$ is the Hermitian conjugate (conjugate transpose) of U.

1.2 Pauli Matrices as Quantum Gates

Definition 1.2.1 (Pauli Matrices)**.** *The Pauli matrices (X, Y, Z) are fundamental single-qubit gates:*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \textit{(Bit-flip gate)}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \textit{(Bit \& Phase-flip)}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textit{(Phase-flip gate)}$$

These matrices are unitary ($U^\dagger = U^{-1}$) and Hermitian ($U = U^\dagger$), ensuring valid quantum operations.

1.3 Controlled-NOT (CNOT) Gate

Definition 1.3.1 (CNOT Gate)**.** *The CNOT gate acts on two qubits, with one acting as a control and the other as a target. It performs a NOT operation on the target if and only if the control qubit is $|1\rangle$.*

Example 1.3.2 (CNOT Matrix Representation)**.**

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem 1.3.3 (CNOT Action)**.** *Using Dirac notation, its action is:*

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |11\rangle \quad \textit{(flips target qubit)} \\ |11\rangle &\to |10\rangle \end{aligned}$$

1.4 Hadamard Gate and Bell State Generation

Definition 1.4.1 (Hadamard Gate)**.** *The Hadamard gate (H) puts a qubit into an equal superposition:*

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example 1.4.2 (Bell State). *A fundamental Bell state:*

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The circuit construction consists of:

1. Applying H to the first qubit.
2. Applying CNOT, with the first qubit as the control.
3. H turns the $|0\rangle$ into $|+\rangle$ and $|1\rangle$ into $|-\rangle$

1.5 Quantum Circuit Representation

- Wires represent qubits and their evolution through time, not physical movement.
- Single-qubit gates (X, H, Z, etc.) act on one qubit.
- Multi-qubit gates (CNOT, Toffoli) involve entanglement.
- Measurement collapses the quantum state and is usually represented by an encircled M in diagrams.

1.6 Controlled Gates and Universal Quantum Gates

Definition 1.6.1 (Controlled Gates). *Controlled gates allow conditional operations:*

- Controlled-Z (CZ): Applies a Z gate to the target when control is $|1\rangle$.
- Controlled-Hadamard (CH): Applies H to the target when control is $|1\rangle$.
- Toffoli Gate (CCNOT): Uses two control qubits and one target, flipping the target only when both controls are $|1\rangle$.
- Fredkin Gate (CSWAP): Swaps two qubits based on a control qubit.

1.7 Gate Decomposition

Complex multi-qubit gates can be decomposed into single-qubit gates and CNOTs. Example: Controlled-U decomposition replaces a controlled-U gate with a sequence of CNOTs and single-qubit rotations, used for efficient quantum circuit optimization.

1.8 No-Cloning Theorem and Quantum Cloning

Theorem 1.8.1 (No-Cloning). *The CNOT gate cannot clone an arbitrary quantum state due to the no-cloning theorem. Example: Trying to copy $\alpha|0\rangle + \beta|1\rangle$ results in entanglement, not an identical copy. Some specific states, like $|0\rangle$ or $|1\rangle$, can be cloned, but arbitrary superpositions cannot.*

1.9 Multi Qubit or Composite States

Definition 1.9.1 (Multi-Qubit States).

- *An n-qubit diagram will have n wires/lines*
- *Tensor product state*
- *entangled states (Bell) $\alpha|0\rangle + \beta|1\rangle$ and $|1\rangle$* ***bold***

1.9.1 Tensor Product State

Definition 1.9.2 (Tensor Product)**.** *A tensor product state is a product of single-qubit states. An example of this is 2* $|0\rangle$ *qubits in tensor product state:* $|0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle$

Example 1.9.3 (Matrix Representations)**.**

- $|00\rangle = \begin{bmatrix} 1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ *and* $|11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ *and* $|01\rangle = \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
- *Keep piling on from the right for more qubits.*

1.9.2 Bell States

Bell states are quintessential entangled 2-qubit states. Pauli exclusion principle states that no two fermions can occupy the same quantum state. So if you have one electron spin up and one spin down, the sign is minus. Also, a cheat sheet for signs is any state with a 1 first is minus, and 0 first is plus.

Bell State 1 $|\beta_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$

- In this state you know the bit on the right by measuring the bit on the left.
- If the left bit is $|0\rangle$ then the right bit is $|0\rangle$
- If the left bit is $|1\rangle$ then the right bit is $|1\rangle$
- This is a Bell state because it is an entangled state.

Bell State 2 $|\beta_{01}\rangle = \frac{|01\rangle+|10\rangle}{\sqrt{2}}$

- In this state you know the bit on the left by measuring the bit on the right.
- If the right bit is $|0\rangle$ then the left bit is $|1\rangle$
- If the right bit is $|1\rangle$ then the left bit is $|0\rangle$
- This is a Bell state because it is an entangled state.

Bell State 3 $|\beta_{10}\rangle = \frac{|00\rangle-|11\rangle}{\sqrt{2}}$

- In this state you know the bit on the right by measuring the bit on the left.
- If the left bit is $|0\rangle$ then the right bit is $|1\rangle$
- If the left bit is $|1\rangle$ then the right bit is $|0\rangle$
- This is a Bell state because it is an entangled state.

Bell State 4 $|\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

- In this state you know the bit on the left by measuring the bit on the right.
- If the right bit is $|0\rangle$ then the left bit is $|1\rangle$
- If the right bit is $|1\rangle$ then the left bit is $|0\rangle$
- This is a Bell state because it is an entangled state.

1.9.3 CNOT 2-qubit Operator

Definition 1.9.4 (CNOT 2-qubit Operator)**.** *To make Bell states we need the CNOT 2-qubit operator. Given by:*

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- If the control qubit is $|0\rangle$ then the target qubit is unchanged.
- If the control qubit is $|1\rangle$ then the target qubit is flipped.

1.10 Key Takeaways for Class Questions

- What size matrix represents an n-qubit gate? $2^n \times 2^n$ unitary matrix.
- Why are Pauli matrices quantum gates? They are unitary and Hermitian.
- What is the role of CNOT? Flips the target qubit if the control is $|1\rangle$.
- How does Hadamard create superposition? Equal probability of $|0\rangle$ and $|1\rangle$.
- What do wires in a circuit mean? Time evolution of a quantum state.
- Can quantum states be cloned? No, due to the no-cloning theorem.

2. LECTURE 4: ADVANCED QUANTUM OPERATIONS

February 3, 2025

2.1 Checkpoint: Quiz Questions and Answers

Key Questions and Answers:

1. **Q: What is the problem with trying to make copies of quantum data?**
   A: There is no unitary operator that exists which could be used to clone arbitrary quantum states, and so it simply isn't possible.

2. **Q: In the teleportation example with Alice and Bob, why is the overall interaction remarkable?**
   A: Because a measurement on Alice's end seemingly 'caused' a change on Bob's end across space and time through no classical communication, only quantum.

3. **Q: Why is it necessary for Alice to make a measurement during the teleportation process?**
   A: Because the very act of measurement will 'collapse' the unknown state into one of the other four possible states, depending on what Alice measures.

2.2 No-Cloning Theorem

Definition 2.2.1 (No-Cloning Theorem)**.** *It is impossible to create an identical copy of an arbitrary unknown quantum state.*

Theorem 2.2.2 (Mathematical Explanation)**.** *Suppose a unitary operation U could clone states:*

$$U(|\psi\rangle \otimes |\chi\rangle) = |\psi\rangle \otimes |\psi\rangle$$

$$U(|\phi\rangle \otimes |\chi\rangle) = |\phi\rangle \otimes |\phi\rangle$$

By taking the inner product and using the properties of unitarity, we reach a contradiction. This means cloning only works for orthogonal states but not for arbitrary quantum states.

2.3 Quantum Teleportation

Definition 2.3.1 (Quantum Teleportation)**.** *Quantum teleportation allows the transfer of a quantum state from Alice to Bob using an entangled pair and classical communication.*

- **Key Steps:**
  1. Alice and Bob share an **entangled EPR pair**
  2. Alice **applies a CNOT gate** and **Hadamard gate** to her qubit
  3. Alice **measures her qubits**, collapsing the system
  4. Alice sends **classical information** (two classical bits) to Bob
  5. Bob applies appropriate **quantum gate operations** (X, Z, or both)

Important Note: Teleportation **does not allow faster-than-light communication** because classical communication is required.

2.4 Superdense Coding

Definition 2.4.1 (Superdense Coding)**.** *A technique for sending two classical bits of information using only one qubit, leveraging entanglement.*

- **Process:**
  1. Alice and Bob share an entangled qubit pair
  2. Alice applies specific operations to encode two classical bits:
     - Identity (I) → **00**
     - X gate → **01**
     - Z gate → **10**
     - ZX gate → **11**
  3. Alice sends the qubit to Bob
  4. Bob measures in the Bell basis to extract information

2.5 Tools of Quantum Information Theory

2.5.1 Fidelity and Distance Measures

- **Trace Distance:** Measures how distinguishable two quantum states are
- **Fidelity:** Measures how similar two quantum states are
- **Entanglement Measures:** Concurrence and Entanglement of Formation quantify entanglement

Key Summary Points:

- Quantum teleportation and superdense coding demonstrate practical applications of entanglement
- The No-Cloning Theorem ensures security in quantum communication
- Quantum operations (Hadamard, CNOT) are fundamental for state manipulation
- Classical communication remains necessary despite quantum advantages