1 Matrix Representation of Quantum Gates

In quantum computing, the state of an n-qubit system is represented by a 2^n-dimensional vector. Quantum gates are represented by $2^n \times 2^n$ unitary matrices. A six-qubit quantum gate must be represented by a 64×64 unitary matrix. The unitary condition:

$$U^\dagger U = UU^\dagger = I,$$

where $U^\dagger$ is the Hermitian conjugate (conjugate transpose) of U.

2 Pauli Matrices as Quantum Gates

The Pauli matrices (X, Y, Z) are fundamental single-qubit gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(Bit-flip gate)}$$
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{(Bit \& Phase-flip)}$$
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(Phase-flip gate)}$$

‘’ These matrices are unitary ($U^\dagger = U^{-1}$) and Hermitian ($U = U^\dagger$), ensuring valid quantum operations.

3 Controlled-NOT (CNOT) Gate

The CNOT gate acts on two qubits, with one acting as a control and the other as a target. It performs a NOT operation on the target if and only if the control qubit is $|1\rangle$. The matrix representation:

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Using Dirac notation, its action is:

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |11\rangle \quad \text{(flips target qubit)} \\ |11\rangle &\to |10\rangle \end{aligned}$$

The CNOT gate can be used to create Bell states when combined with the Hadamard gate.

4 Hadamard Gate and Bell State Generation

The Hadamard gate (H) puts a qubit into an equal superposition:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Used to create Bell states in combination with CNOT. Example Bell state:

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The circuit consists of:

- Applying H to the first qubit.
- Applying CNOT, with the first qubit as the control.

5 Quantum Circuit Representation

- Wires represent qubits and their evolution through time, not physical movement.
- Single-qubit gates (X, H, Z, etc.) act on one qubit.
- Multi-qubit gates (CNOT, Toffoli) involve entanglement.
- Measurement collapses the quantum state and is usually represented by an encircled M in diagrams.

6 Controlled Gates and Universal Quantum Gates

Controlled gates allow conditional operations:

- Controlled-Z (CZ): Applies a Z gate to the target when control is $|1\rangle$.
- Controlled-Hadamard (CH): Applies H to the target when control is $|1\rangle$.
- Toffoli Gate (CCNOT): Uses two control qubits and one target, flipping the target only when both controls are $|1\rangle$.
- Fredkin Gate (CSWAP): Swaps two qubits based on a control qubit.

7 Gate Decomposition

Complex multi-qubit gates can be decomposed into single-qubit gates and CNOTs. Example: Controlled-U decomposition replaces a controlled-U gate with a sequence of CNOTs and single-qubit rotations, used for efficient quantum circuit optimization.

8 No-Cloning Theorem and Quantum Cloning

The CNOT gate cannot clone an arbitrary quantum state due to the no-cloning theorem. Example: Trying to copy $\alpha|0\rangle + \beta|1\rangle$ results in entanglement, not an identical copy. Some specific states, like $|0\rangle$ or $|1\rangle$, can be cloned, but arbitrary superpositions cannot.

9 Key Takeaways for Class Questions

- What size matrix represents an n-qubit gate? $2^n \times 2^n$ unitary matrix.
- Why are Pauli matrices quantum gates? They are unitary and Hermitian.
- What is the role of CNOT? Flips the target qubit if the control is $|1\rangle$.
- How does Hadamard create superposition? Equal probability of $|0\rangle$ and $|1\rangle$.
- What do wires in a circuit mean? Time evolution of a quantum state.
- Can quantum states be cloned? No, due to the no-cloning theorem.