Quantum Computing Fundamentals

PHYS 370

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Contents

Chapter 1: Quantum Gates and Circuits

Lecture 3: Matrix Representations and Basic Gates (January 29, 2025)

1 Matrix Representation of Quantum Gates

In quantum computing, the state of an n-qubit system is represented by a 2^n-dimensional vector. Quantum gates are represented by $2^n \times 2^n$ unitary matrices. A six-qubit quantum gate must be represented by a 64×64 unitary matrix. The unitary condition:

$$U^\dagger U = UU^\dagger = I,$$

where $U^\dagger$ is the Hermitian conjugate (conjugate transpose) of U.

2 Pauli Matrices as Quantum Gates

The Pauli matrices (X, Y, Z) are fundamental single-qubit gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(Bit-flip gate)}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{(Bit \& Phase-flip)}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(Phase-flip gate)}$$

'' These matrices are unitary ($U^\dagger = U^{-1}$) and Hermitian ($U = U^\dagger$), ensuring valid quantum operations.

3 Controlled-NOT (CNOT) Gate

The CNOT gate acts on two qubits, with one acting as a control and the other as a target. It performs a NOT operation on the target if and only if the control qubit is $|1\rangle$. The matrix representation:

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Using Dirac notation, its action is:

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |11\rangle \quad \text{(flips target qubit)} \\ |11\rangle &\to |10\rangle \end{aligned}$$

The CNOT gate can be used to create Bell states when combined with the Hadamard gate.

4 Hadamard Gate and Bell State Generation

The Hadamard gate (H) puts a qubit into an equal superposition:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Used to create Bell states in combination with CNOT. Example Bell state:

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

The circuit consists of:

- Applying H to the first qubit.
- Applying CNOT, with the first qubit as the control.
- H turns the $|0\rangle$ into $|+\rangle$ and $|1\rangle$ into $|-\rangle$

5 Quantum Circuit Representation

- Wires represent qubits and their evolution through time, not physical movement.
- Single-qubit gates (X, H, Z, etc.) act on one qubit.
- Multi-qubit gates (CNOT, Toffoli) involve entanglement.
- Measurement collapses the quantum state and is usually represented by an encircled M in diagrams.

6 Controlled Gates and Universal Quantum Gates

Controlled gates allow conditional operations:

- Controlled-Z (CZ): Applies a Z gate to the target when control is $|1\rangle$.
- Controlled-Hadamard (CH): Applies H to the target when control is $|1\rangle$.
- Toffoli Gate (CCNOT): Uses two control qubits and one target, flipping the target only when both controls are $|1\rangle$.
- Fredkin Gate (CSWAP): Swaps two qubits based on a control qubit.

7 Gate Decomposition

Complex multi-qubit gates can be decomposed into single-qubit gates and CNOTs. Example: Controlled-U decomposition replaces a controlled-U gate with a sequence of CNOTs and single-qubit rotations, used for efficient quantum circuit optimization.

8 No-Cloning Theorem and Quantum Cloning

The CNOT gate cannot clone an arbitrary quantum state due to the no-cloning theorem. Example: Trying to copy $\alpha|0\rangle + \beta|1\rangle$ results in entanglement, not an identical copy. Some specific states, like $|0\rangle$ or $|1\rangle$, can be cloned, but arbitrary superpositions cannot.

9 Multi Qubit or Composite States

- An n-qubit diagram will have n wires/lines
- Tensor product state
- entangled states (Bell) $\alpha|0\rangle + \beta|1\rangle$ and $|1\rangle$ **bold**

9.1 Tensor Product State:

A tensor product state is a product of single-qubit states. An example of this is 2 $|0\rangle$ qubits in tensor product state: $|0\rangle \otimes |0\rangle = |0\rangle|0\rangle = |00\rangle$

- $|00\rangle = \begin{bmatrix} 1 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $|11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $|01\rangle = \begin{bmatrix} 1 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
- Keep piling on from the right for more qubits.

9.2 Bell States:

Bell states are quintessential entangled 2-qubit states. Pauli exclusion principle states that no two fermions can occupy the same quantum state. So if you have one electron spin up and one spin down, the sign is minus. Also, a cheat sheet for signs is any state with a 1 first is minus, and 0 first is plus.

9.2.1 Bell State 1 $|\beta_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$:

- In this state you know the bit on the right by measuring the bit on the left.

- If the left bit is $|0\rangle$ then the right bit is $|0\rangle$
- If the left bit is $|1\rangle$ then the right bit is $|1\rangle$
- This is a Bell state because it is an entangled state.

9.2.2 Bell State 2 $|\beta_{01}\rangle = \frac{|01\rangle+|10\rangle}{\sqrt{2}}$:

- In this state you know the bit on the left by measuring the bit on the right.
- If the right bit is $|0\rangle$ then the left bit is $|1\rangle$
- If the right bit is $|1\rangle$ then the left bit is $|0\rangle$
- This is a Bell state because it is an entangled state.

9.2.3 Bell State 3 $|\beta_{10}\rangle = \frac{|00\rangle-|11\rangle}{\sqrt{2}}$:

- In this state you know the bit on the right by measuring the bit on the left.
- If the left bit is $|0\rangle$ then the right bit is $|1\rangle$
- If the left bit is $|1\rangle$ then the right bit is $|0\rangle$
- This is a Bell state because it is an entangled state.

9.2.4 Bell State 4 $|\beta_{11}\rangle = \frac{|01\rangle-|10\rangle}{\sqrt{2}}$:

- In this state you know the bit on the left by measuring the bit on the right.
- If the right bit is $|0\rangle$ then the left bit is $|1\rangle$
- If the right bit is $|1\rangle$ then the left bit is $|0\rangle$
- This is a Bell state because it is an entangled state.

9.3 CNOT 2-qubit Operator:

To make Bell states we need the CNOT 2-qubit operator. Given by:

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- If the control qubit is $|0\rangle$ then the target qubit is unchanged.
- If the control qubit is $|1\rangle$ then the target qubit is flipped.

10 Key Takeaways for Class Questions

- What size matrix represents an n-qubit gate? $2^n \times 2^n$ unitary matrix.
- Why are Pauli matrices quantum gates? They are unitary and Hermitian.
- What is the role of CNOT? Flips the target qubit if the control is $|1\rangle$.
- How does Hadamard create superposition? Equal probability of $|0\rangle$ and $|1\rangle$.
- What do wires in a circuit mean? Time evolution of a quantum state.
- Can quantum states be cloned? No, due to the no-cloning theorem.

Chapter 2: Advanced Quantum Operations

Lecture 4: Coming Soon (February 3, 2025)

Content for this lecture will be added in the next class.