Problem Set - 22 Jan 2024

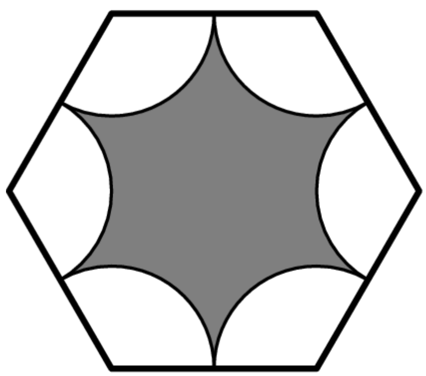
PROBLEM 1 (2017 AMC 10B #11)

At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

$\textbf{(A)}\ 10\% \qquad \textbf{(B)}\ 12\% \qquad \textbf{(C)}\ 20\% \qquad \textbf{(D)}\ 25\% \qquad \textbf{(E)}\ 33\frac{1}{3}\%$

PROBLEM 2 (2014 AMC 10A #12)

A regular hexagon has side length 6. Congruent arcs with radius 3 are drawn with the center at each of the vertices, creating circular sectors as shown. The region inside the hexagon but outside the sectors is shaded as shown What is the area of the shaded region?



(A) $27\sqrt{3}-9\pi$ (B) $27\sqrt{3}-6\pi$ (C) $54\sqrt{3}-18\pi$ (D) $54\sqrt{3}-12\pi$ (E) $108\sqrt{3}-9\pi$

PROBLEM 3 (2014 AMC 10A #15)

David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

$\textbf{(A)}\ 140 \qquad \textbf{(B)}\ 175 \qquad \textbf{(C)}\ 210 \qquad \textbf{(D)}\ 245 \qquad \textbf{(E)}\ 280$

PROBLEM 4 (2018 AMC 10A #18)

How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

$\textbf{(A)}\ 512 \qquad \textbf{(B)}\ 729 \qquad \textbf{(C)}\ 1094 \qquad \textbf{(D)}\ 3281 \qquad \textbf{(E)}\ 59,048$

PROBLEM 5 (2022 AMC 10B #17)

One of the following numbers is not divisible by any prime number less than 10. Which is it?

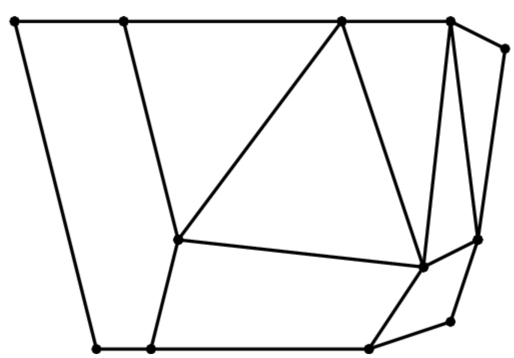
$\textbf{(A)}\ 2^{606} - 1 \qquad \textbf{(B)}\ 2^{606} + 1 \qquad \textbf{(C)}\ 2^{607} - 1 \qquad \textbf{(D)}\ 2^{607} + 1 \qquad \textbf{(E)}\ 2^{607} + 3^{607}$

PROBLEM 6 (2017 AMC 10B #19)

Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

$\textbf{(A)}\ 9:1 \qquad \textbf{(B)}\ 16:1 \qquad \textbf{(C)}\ 25:1 \qquad \textbf{(D)}\ 36:1 \qquad \textbf{(E)}\ 37:1$

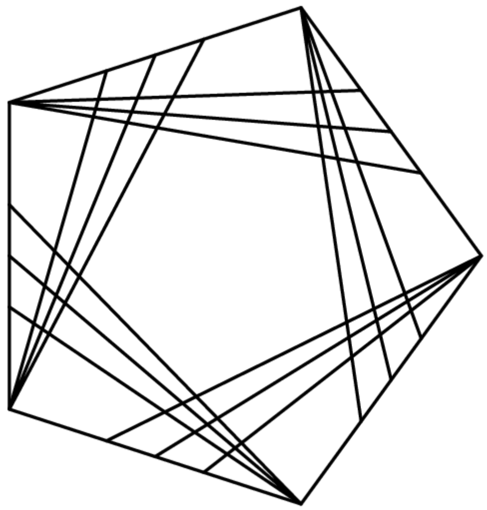
PROBLEM 7 (2015 UNCO MATH CONTEST II #5)



A termite nest has the shape of an irregular polyhedron. The bottom face is a quadrilateral. The top face is another polygon. The sides comprise 9 triangles, 6 quadrilaterals, and 1 pentagon. The nest has 10 vertices on its sides and bottom, not counting the several around the top face. How many edges does the top face have?

You may use Euler's polyhedral identity, which says that on a convex polyhedron the number of faces plus the number of vertices is two more than the number of edges. (A vertex is a corner point and an edge is a line segment along which two faces meet.)

PROBLEM 8 (2017 UNCO MATH CONTEST II #6)



The Spider's Divider

On a regular pentagon, a spider forms segments that connect one endpoint of each side to n different non-vertex points on the side adjacent to the other endpoint of that side, going around clockwise, as shown. Into how many non-overlapping regions do the segments divide the pentagon? Your answer should be a formula involving n. (In the diagram, n = 3 and the pentagon is divided into 61 regions.)

PROBLEM 9 (2021 AIME I #7)

Find the number of pairs (m, n) of positive integers with $1 \le m < n \le 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

PROBLEM 10 (2020 AMC 12A #22)

Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \ge 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}\,?$$

$\textbf{(A) } \frac{3}{8} \qquad \textbf{(B) } \frac{7}{16} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } \frac{9}{16} \qquad \textbf{(E) } \frac{4}{7}$