# SIR models

S- Susceptible

I: Infected and Exposed combined

R: removed/recovered — Individuals who cannot contract the disease again

Assume total population fixed at N

$$S(t) + I(t) + R(t) = N \qquad (*)$$

$$\boxed{S} \xrightarrow{\beta} \boxed{I} \xrightarrow{\alpha} \boxed{R}$$

$\beta$ is the proportionality constant in $\frac{dS}{dt} = -\beta IS$

$IS$: proportional to the probability of S meeting I

__Assumption__

\# of S decreases proportional to contacts with infected $\qquad \frac{dS}{dt} = -\beta IS$

\# of infected decreases proportional to itself $\qquad \frac{dI}{dt} = \beta IS - \alpha I$

\# of removed/recovered is everybody who leaves infected $\qquad \frac{dR}{dt} = \alpha I$

Note that $\frac{d}{dt}(S+I+R) = (-\beta IS) + (\beta IS - \alpha I) + \alpha I = 0$

$\Rightarrow (S+I+R)(t)$ is constant, which we want

Progress Report due today

Template at Canvas > Pages > Course Materials

Not much to report this time

Dimension (or units) of quantities in SIR model

$[S] = \#$ of susceptible

$[I] = \#$ of infected

$[R] = \#$ of recovered

"#" means a count of people

$$\left[\frac{dS}{dt}\right] = \frac{[S]}{[t]} = \frac{\#\text{ susc.}}{\text{time}} \Rightarrow \frac{dS}{dt} = -\beta I S \Rightarrow \frac{[S]}{[t]} = [\beta][I][S]$$

$$\Rightarrow [\beta] = \frac{1}{[t][I]}$$

Similarly $[\alpha] = \dfrac{[R]}{[t][I]}$ from $\dfrac{dR}{dt} = \alpha I$

Analyzing

$I(t), S(t), R(t) \geq 0$

$\alpha, \beta > 0$

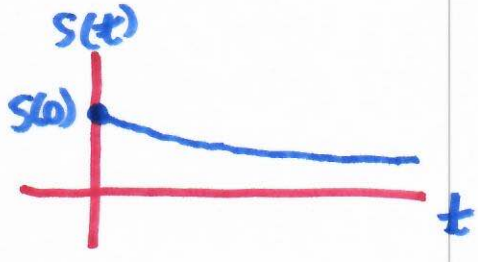
$\dfrac{dS}{dt} = -\beta I S \Rightarrow S(t)$ is decreasing

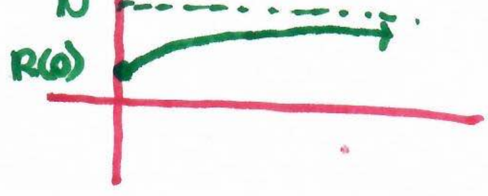
$\therefore \underbrace{\lim_{t\to\infty} S(t)}_{\text{call this } S_\infty \geq 0}$ exists

$\dfrac{dI}{dt} = \beta I S - \alpha I$

$\dfrac{dR}{dt} = \alpha I \Rightarrow R(t)$ is increasing and $\lim_{t\to\infty} R(t) =: R_\infty$ exists

Also $R(t) \leq N$ for all $t$



Is $S_\infty > 0$ or $S_\infty = 0$

$S = S(t)$
$R = R(t)$

Look at $$\frac{dS}{dR} \underset{\text{Chain Rule}}{=} \frac{dS/dt}{dR/dt} = \frac{-\beta I S}{\alpha I} = -\frac{\beta}{\alpha} S$$

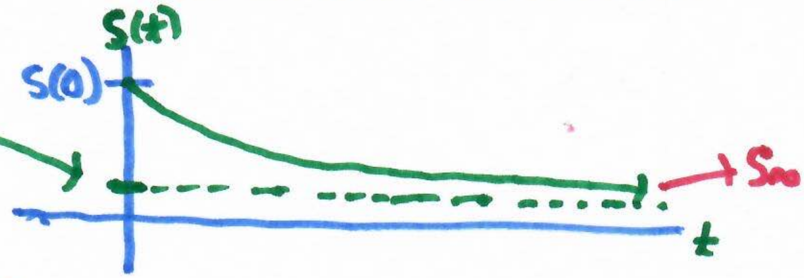
$\therefore \frac{dS}{dR} = -\frac{\beta}{\alpha} S$ This is solveable!

$$\Rightarrow S(R) = \underset{\text{this is } R=0}{S(0)}\, e^{-\frac{\beta}{\alpha} R}$$

Note that $R \leq N \Rightarrow -R \geq -N \Rightarrow \frac{\beta}{\alpha}(-R) > \frac{\beta}{\alpha}(-N)$

$$S(R) = S(0)e^{\left(-\frac{\beta}{\alpha} R\right)} \geq \underbrace{S(0)e^{-\frac{\beta}{\alpha} N}}_{\text{independent of time}} > 0$$

This shows that $S(t) = S(R(t))$ is bounded below by $S(0)e^{-\frac{\beta}{\alpha} N}$




$\therefore$ $S_\infty > 0$, so in the long term, so individuals do **not** get infected

What about $I(t)$?

$$\frac{dS}{dt} = -\beta S(t) I(t)$$

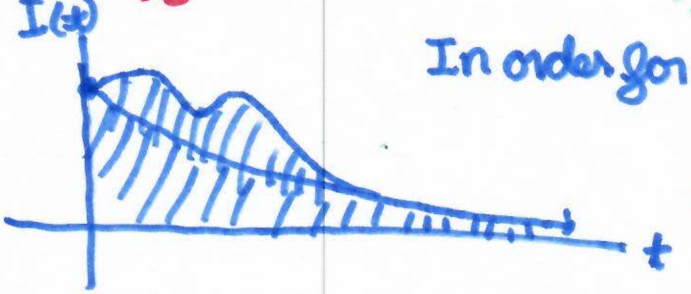
$$\int_0^\infty \frac{dS}{dt}(t)\,dt = -\beta \int_0^\infty S(t) I(t)\,dt$$

$$S(t)\Big|_0^\infty = -\beta \int_0^\infty \underbrace{S(t)}_{\geq S_\infty} I(t)\,dt \geq -\beta \int_0^\infty S_\infty I(t)\,dt$$

$$\underbrace{S(\infty)}_{S_\infty} - S(0) \geq -\beta S_\infty \int_0^\infty I(t)\,dt$$

or $\displaystyle\int_0^\infty I(t)\,dt \leq \frac{S_0 - S_\infty}{\beta S_\infty} < \infty$ (since $S_\infty \neq 0$)

so $\displaystyle\int_0^\infty I(t)\,dt < \infty$



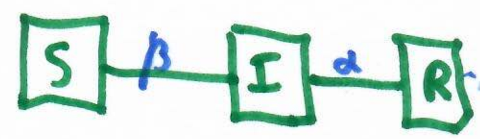

In order for the area to be finite,

$$\lim_{t\to\infty} I(t) = 0$$

From the SIR equations, we get

$\lim_{t\to\infty} S(t) = S_\infty > 0$ (no everybody gets the disease)

$\lim_{t\to\infty} I(t) = 0$




$\lim_{t\to\infty} R(t) = R_\infty \leq N$

also $S_\infty + 0 + R_\infty = N$, so $R_\infty < N$.

$$R_\infty = N - S_\infty$$