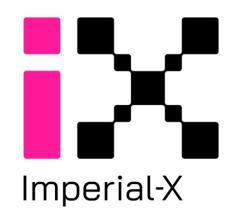
Differentiable simulators: a bridge between machine learning and scientific computing

Dr Aidan Crilly

Eric and Wendy Schmidt AI in Science postdoctoral fellow Centre for Inertial Fusion Studies

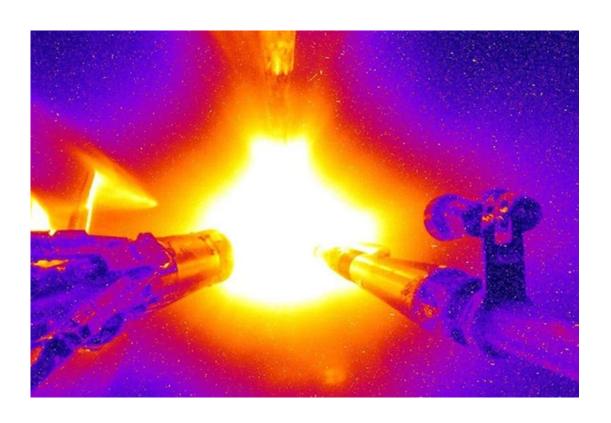


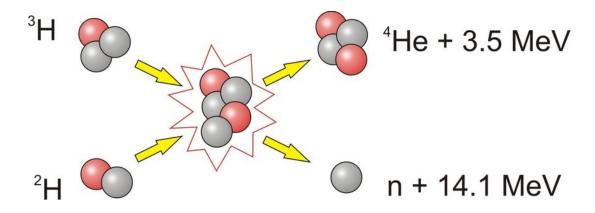


Outline

- Context
- Differential equations
 - Adjoint state problem
- Differentiable programming
 - For differential equations
- Neural differential equations
- Exercises and Assessment

Nuclear fusion – "inertial confinement"





Charting the First Year of Ignition











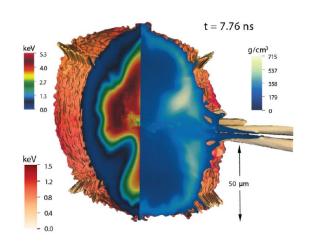


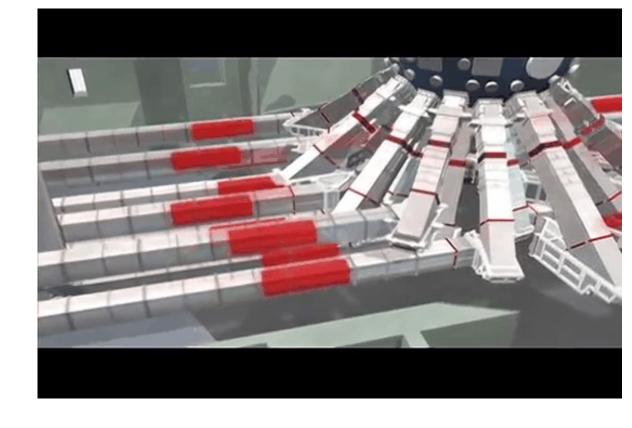




Nuclear fusion – "inertial confinement"

 Predictive modelling of plasma behaviour and observable signals





Charting the First Year of Ignition











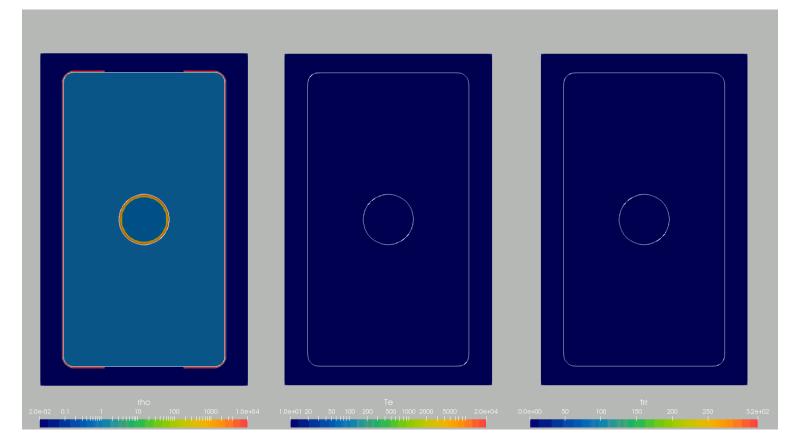








- Multi-physics code in 3D:
 - Magneto-hydrodynamics, radiation transport, thermal conduction, split electron-ion energy equations, laser ray trace, alpha particle transport, non-ideal equation of state, extended Ohm's law, material strength...





- Multi-physics code in 3D:
 - Magneto-hydrodynamics, radiation transport, thermal conduction, split electron-ion energy equations, laser ray trace, alpha particle transport, non-ideal equation of state, extended Ohm's law, material strength...

$$\partial_{t}\rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\partial_{t} (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \vec{u}) = -\nabla P + J \times B$$

$$\partial_{t} (\rho e_{tot}) + \nabla \cdot (\rho e_{tot} \vec{u}) = -\nabla \cdot (P \vec{u}) + \text{Sources}$$

$$\frac{\partial B}{\partial t} = \nabla \times \underline{v}_{B} \times \underline{B} - \nabla \times \eta_{\parallel} \nabla \times \underline{B} + \nabla \times \frac{\nabla P_{e}}{e n_{e}}$$

$$\left[\frac{\partial}{\partial t} + \kappa_{\nu} c\right] E_{\nu} = 4\pi j_{\nu} - \nabla \cdot \vec{F}_{\nu} ,$$

$$\left[\frac{\partial}{\partial t} + \kappa_{\nu} c\right] \vec{F}_{\nu} = -c^{2} \nabla \cdot \mathbf{P}_{\nu} ,$$

$$C_{e} \frac{\partial T_{e}}{\partial t} = \kappa_{e} \nabla^{2} T_{e} + S_{e} + \omega_{ie} (T_{i} - T_{e})$$

$$C_{i} \frac{\partial T_{i}}{\partial t} = \kappa_{i} \nabla^{2} T_{i} + S_{i} + \omega_{ie} (T_{e} - T_{i})$$

$$\begin{split} \frac{\partial \underline{v}_{ray}}{\partial t} &= \nabla \left(\frac{-c^2}{2} \frac{n_e}{n_c} \right) \\ \left[\frac{1}{v} \frac{\partial}{\partial t} + \hat{\Omega} \cdot \nabla + n(\vec{r}, t) \sigma(E) \right] \psi(\vec{r}, \hat{\Omega}, E, t) \\ &= S_{ex}(\vec{r}, \hat{\Omega}, E, t) \\ &+ \int_0^\infty dE' \int d\hat{\Omega}' n(\vec{r}, t) \sigma_s(\hat{\Omega}' \cdot \hat{\Omega}, E' \to E) \psi(\vec{r}, \hat{\Omega}', E', t), \\ \frac{\partial f_\alpha(v_\alpha, t)}{\partial t} &= C(f_\alpha(v_\alpha, t)) + \frac{S_0 \delta(v_\alpha - v_b)}{4\pi v_\alpha^2} \\ &= \frac{1}{\tau_S v_\alpha^2} \frac{\partial}{\partial v_\alpha} \left\{ \frac{v_\alpha^3 T_e + v_c^3 T_i \Phi(x_i)}{m_\alpha v_\alpha} \frac{\partial f_\alpha}{\partial v_\alpha} \right. \\ &+ \left[v_\alpha^3 + v_c^3 \Phi(x_i) \right] f_\alpha \right\} + \frac{S_0 \delta(v_\alpha - v_b)}{4\pi v_\alpha^2}, \quad \text{And the list goes on...} \end{split}$$

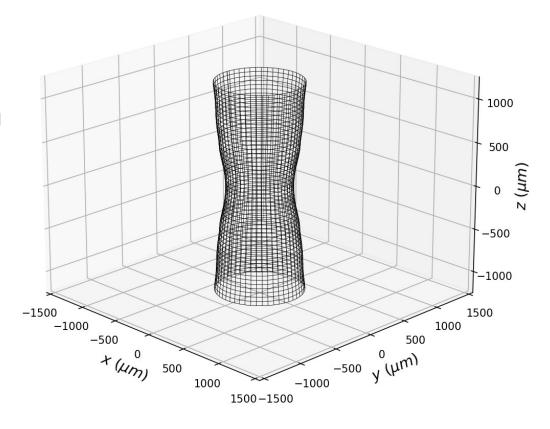
Nature and differential equations

 Differential equations have "unreasonable effectiveness" in describing nature*

- Newton noted that the circular orbit of the moon and parabolic trajectory of a thrown rock were special cases of an ellipse
 - 2nd order derivative
- Can we use ML/Al to augment differential equation models?

- Ordinary differential equations (ODEs) dependent on only a single independent variable
 - For the most part, this variable is time
- Example: the path of a laser in a plasma (geometric optics)

$$\frac{d\underline{v}_{ray}}{dt} = \nabla \left(\frac{-c^2}{2} \frac{n_e}{n_c} \right)$$



• Numerical solutions to ODEs use finite steps (h or dt) to approximate derivatives

$$\frac{dy}{dt} = f(t, y)$$

We use finite differencing to approximate derivatives

Forward differencing:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y^{n+1} - y^n}{x^{n+1} - x^n}$$

Backward differencing:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y^n - y^{n-1}}{x^n - x^{n-1}}$$

First order in accuracy

Centred differencing:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y^{n+1} - y^{n-1}}{x^{n+1} - x^{n-1}} \longrightarrow$$

Second order in accuracy

• Numerical solutions to ODEs use finite steps (h or dt) to approximate derivatives

$$\frac{dy}{dt} = f(t, y)$$

• Simplest = Forward Euler:

$$y_{n+1} = y_n + h f(t_n, y_n)$$

• Numerical solutions to ODEs use finite steps (h or dt) to approximate derivatives

$$\frac{dy}{dt} = f(t, y)$$

• Simplest = Forward Euler:

$$y_{n+1} = y_n + h f(t_n, y_n)$$

Very common = 4th order Runge-Kutta (RK4) with adaptive stepping

$$y_{n+1} = y_n + rac{h}{6} \left(k_1 + 2k_2 + 2k_3 + k_4
ight), \qquad egin{aligned} k_1 &= f(t_n, y_n), \ k_2 &= f igg(t_n + rac{h}{2}, y_n + h rac{k_1}{2} igg), \ k_3 &= f igg(t_n + rac{h}{2}, y_n + h rac{k_2}{2} igg), \ k_4 &= f(t_n + h, y_n + h k_3). \end{aligned}$$

Forward vs Inverse problems

• Solutions to differential equations are often concerned with the forward problem

Differential equation: Initial condition:

$$h(x, p, t) = 0$$
$$g(x(0), p) = 0$$

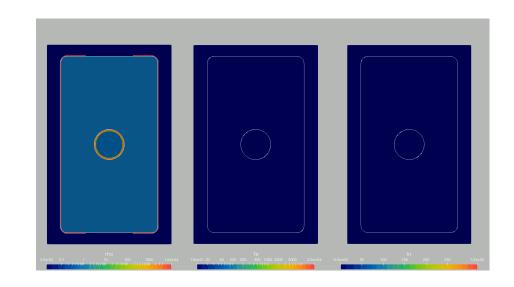
x = state variablesp = parameters

t = time

*Initial conditions:*Densities, temperatures, etc.

Differential equations: Hydrodynamics ++

Other parameters: Laser power vs time



Forward vs Inverse problems

Solutions to differential equations are often concerned with the forward problem

```
Differential equation: h(x, p, t) = 0 p = parameters Initial condition: g(x(0), p) = 0 t = time
```

• However, if we want to minimize some other scalar function at the same time = *inverse problem*

Figure of merit: Minimise w. r. t. p: f(x, p)

Forward vs Inverse problems

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Differential equation: h(x, p, t) = 0 p = parameters
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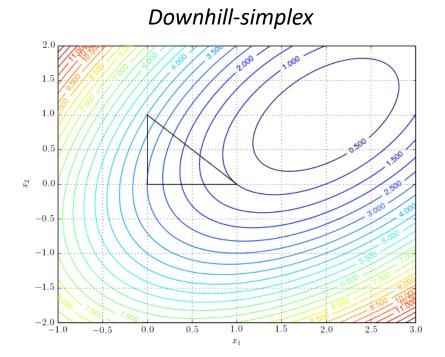
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Figure of merit: Minimise w. r. t. p: f(x, p)

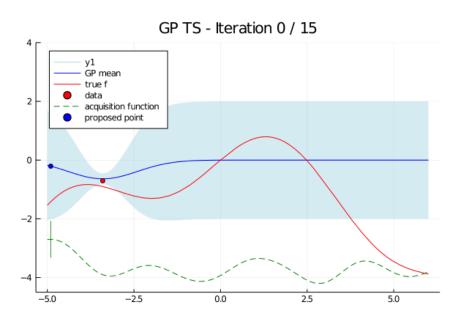
Optimisation

Optimisation

- Minimisation = optimisation = root-finding of gradient
- Gradient-based vs gradient-free optimisation
- Gradient-free:

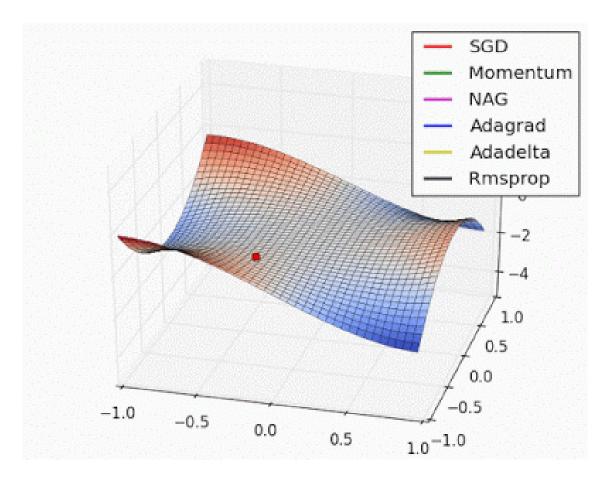


Bayesian optimisation



Optimisation

- Minimisation = optimisation = root-finding of gradient
- Gradient-based:
 - Simple idea, roll down-hill (to minimize)



The adjoint state problem

• How do we solve the inverse problem in differential equations efficiently?

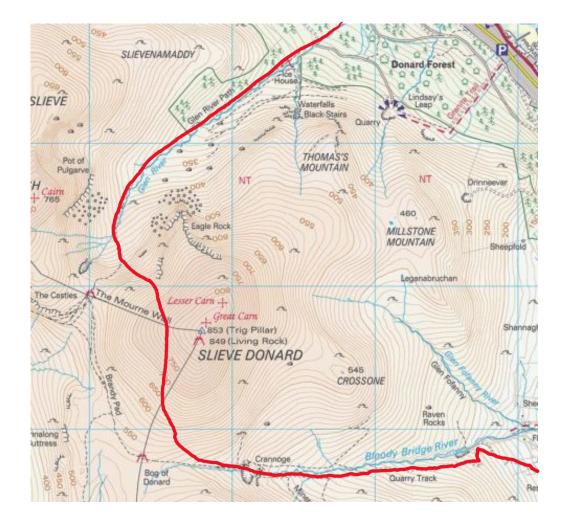
 We could run additional forward solutions to get finite difference gradients – very inefficient

• Instead of predicting how a *single* design change influences *every* aspect of the flow, the adjoint method predicts how *every* design change influences a *single* aspect of the flow.

Constrained Optimisation

• Inverse problem in simplified form: Maximise w. r. t. p: f(p)g(p) = 0

- Intuitive example: Reach highest point (f) while sticking to path (g)
 - When path falls below you both in front and behind you
 - Or the path is tangent with contour
 - Or the path of steepest ascent (gradient) is 90 degree to path



In this example:

- f(p) gives the height given your position, p, i.e. x and y coordinates.
- g(p) parameterises the path you take, such that if you are on the path g(p) = 0

Constrained Optimisation – the maths

 Combine target and constraint using "Lagrange multipliers"



$$L(p,\lambda) = f(p) + \lambda g(p)$$

Find optima of this function:

$$\partial_p L = 0 \rightarrow \partial_p f = -\lambda \partial_p g$$

 $\partial_\lambda L = g(p) = 0$

• In other words, at the optima the gradients are parallel

Let's consider p = [x, y] and: $f(x,y) = e^{-x^2 - y^2}$ $g(x,y) = y - (x-1)^2 = 0$ 2.0 1.5 1.0 -0.5 -0.0 -0.5 --1.0 --1.5-2.0-1.5 -1.0 -0.50.5 0.0

The adjoint state problem

• Inverse problem in simplified form:

Minimise w. r. t.
$$p$$
: $f(x,p)$ $g(x,p) = 0$

Constrained optimisation → Lagrange multipliers:

$$L(x, p, \lambda) = f(x, p) + \lambda^{T} g(x, p)$$

• Gives adjoint equation and gradient w.r.t. parameters, p:

$$\frac{\partial f}{\partial x} + \lambda^T \frac{\partial g}{\partial x} = 0, \qquad \frac{df}{dp} = \frac{dL}{dp} = \frac{\partial f}{\partial p} + \lambda^T \frac{\partial g}{\partial p}$$

The adjoint state problem – differentiable programming

 For differential equations, the adjoint equation is itself another differential equation

- Backpropagation of gradients through our differential equation would implicitly solve adjoint equations
 - We will show this later...

 How do we do get gradients of numerical solutions to differential equations?

Automatic differentiation introduction

• Numerical differentiation, finite difference methods:

$$\frac{dy}{dt} = \lim_{\varepsilon \to 0} \frac{y(t+\varepsilon) - y(t)}{\varepsilon} = f(t, y)$$

• Numerical error dependent on the step size used:

$$\frac{y(t+\varepsilon)-y(t)}{\varepsilon} = \dot{y}(t) + \frac{1}{2}\varepsilon \ddot{y}(t) + \dots = f(t,y) + \text{Error}(\varepsilon)$$

• Enter automatic differentiation...

Differentiable Programming?



Yann LeCun:

"Deep Learning est mort. Vive Differentiable Programming!

...

An increasingly large number of people are defining the networks procedurally in a data-dependent way (with loops and conditionals), allowing them to change dynamically as a function of the input data fed to them. It's really very much like a regular program, except it's parameterized, automatically differentiated, and trainable/optimizable. Dynamic networks have become increasingly popular..."

Automatic-differentiation/differentiableprogramming frameworks

Python: PyTorch – autograd

Python: Tensorflow – GradientTape

Python: JAX

• C++: autodiff

Check out autodiff.org for other languages/libraries

Automatic-differentiation/differentiableprogramming frameworks

Python: PyTorch – autograd

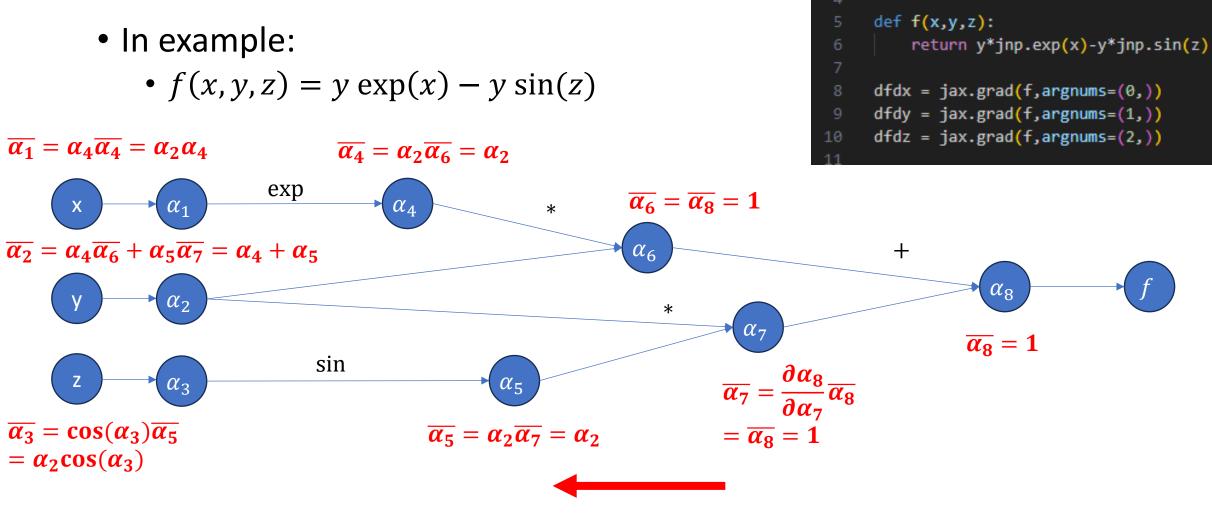
Python: Tensorflow – GradientTape

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Check out autodiff.org for other languages/libraries

PythonJAX simple example



import jax

import jax.numpy as jnp

import matplotlib.pyplot as plt

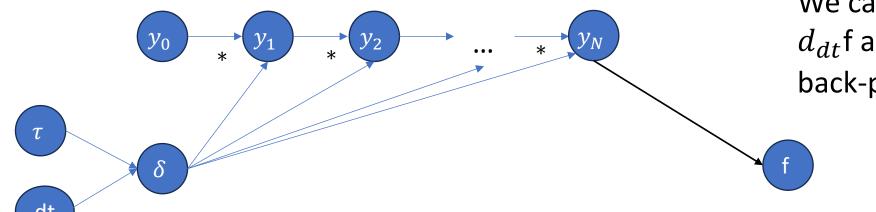
A single 'back-propagation' gives all gradients!

Back to differential equations...

A simple example ODE, solved using Euler's method

$$\frac{dy}{dt} = -\frac{1}{\tau}y$$
, $y_{n+1} = \left(1 - \frac{dt}{\tau}\right)y_n = \delta y_n$

- Define some loss, f, which uses values of y
- It's just a graph



We can compute d_{τ} f, d_{dt} f and d_{y_0} f using back-propagation

• A simple example ODE, with parameters y_0 and au

$$\frac{dy}{dt} = -\frac{1}{\tau}y, \qquad f = L(y(T))$$

- What are the formal adjoint equations?
 - There will be differential equations for
 - 1. Adjoint process (Lagrange multiplier)
 - 2. Parameter gradients (df/dp)

• A simple example ODE, with parameters y_0 and τ

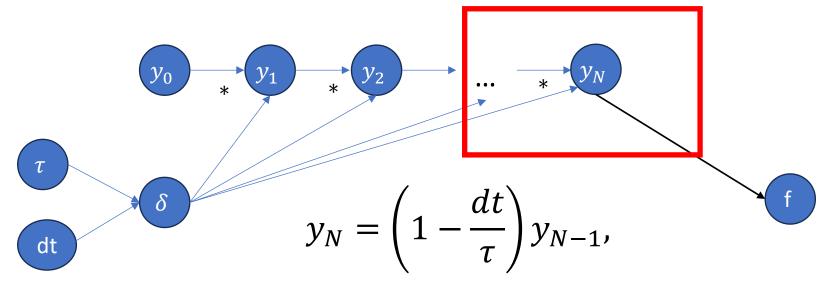
$$\frac{dy}{dt} = -\frac{1}{\tau}y, \qquad f = L(y(T))$$

• What are the formal adjoint equations?

Adjoint process:
$$\frac{d\alpha}{dt} = \frac{1}{\tau}\alpha$$
, $\alpha(T) = \frac{dL}{dy(T)}$ Exponential growth

Parameter gradients:
$$\frac{d\beta}{dt} = -\frac{1}{\tau^2}\alpha y$$
, $\beta(T) = 0$

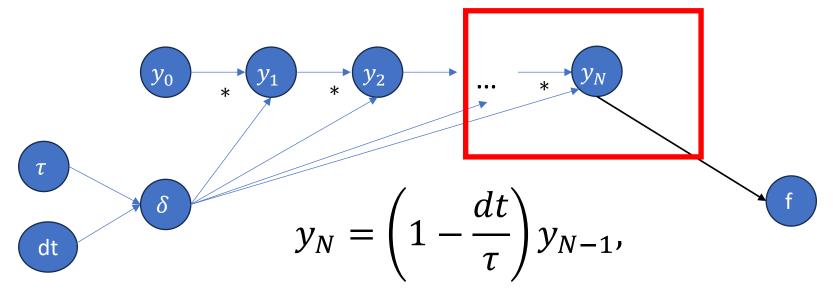
• Solved backwards in time, $\beta(0) = d_{\tau}f = \left(\frac{\mathrm{T}}{\tau^2}\right)\left(\frac{dL}{dy(T)}\right)y_0\exp\left[-\frac{\mathrm{T}}{\tau}\right]$



Backpropagation a single step:

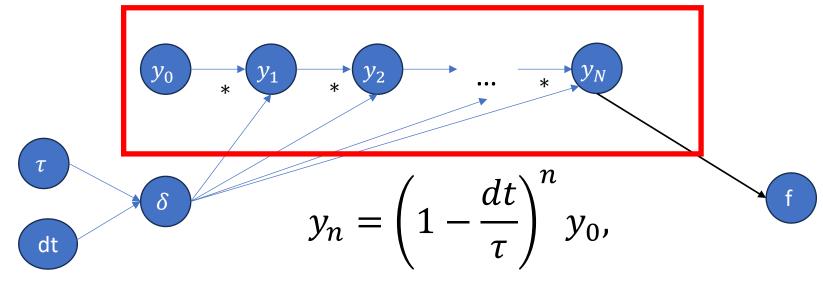
$$\frac{df}{d\tau} = \frac{dL}{dy_N} \cdot \frac{dy_N}{d\tau}$$

$$\frac{dy_N}{d\tau} = \frac{d}{d\tau} \left[\left(1 - \frac{dt}{\tau} \right) y_{N-1} \right]$$



• Backpropagation a single step:

$$\frac{df}{d\tau} = \frac{dL}{dy_N} \cdot \frac{dy_N}{d\tau} = \frac{dL}{dy_N} \cdot \left(\frac{dt}{\tau^2} y_{N-1} + \left(1 - \frac{dt}{\tau}\right) \frac{dy_{N-1}}{d\tau}\right)$$



Backpropagation all the way:

AD + ODE:
$$\frac{df}{d\tau} = \frac{T}{\tau^2} \frac{dL}{dy_N} y_0 \left(1 - \frac{T}{N\tau} \right)^{N-1}, \qquad T = \text{Ndt}$$

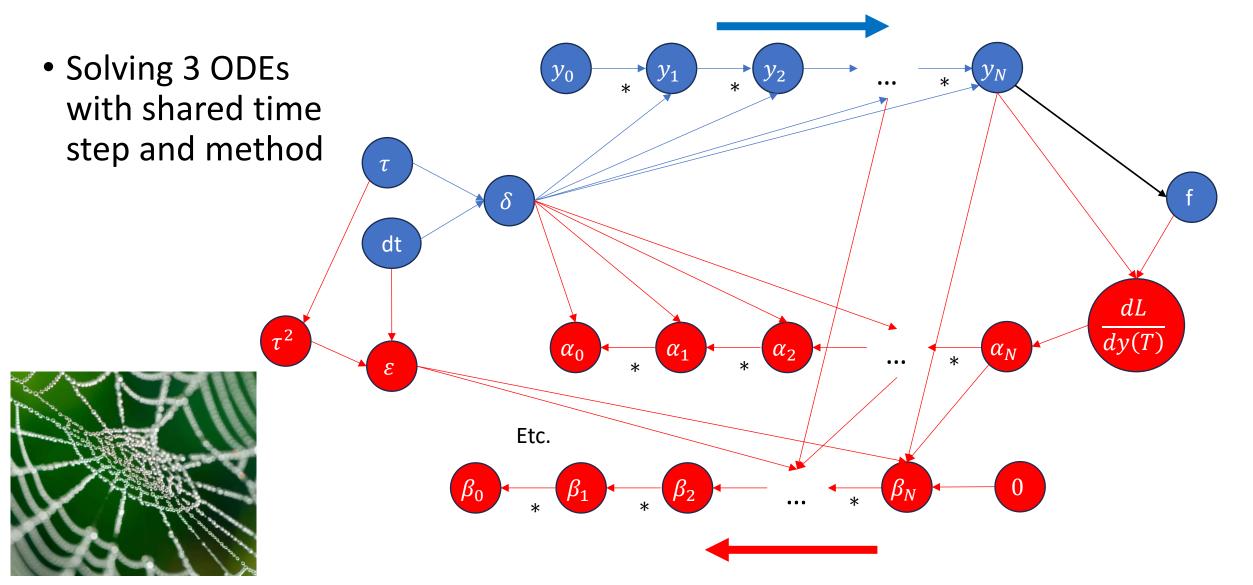
Analytic adjoint state method:

$$\beta(0) = \frac{T}{\tau^2} \frac{dL}{dy(T)} y_0 \exp\left[-\frac{T}{\tau}\right]$$

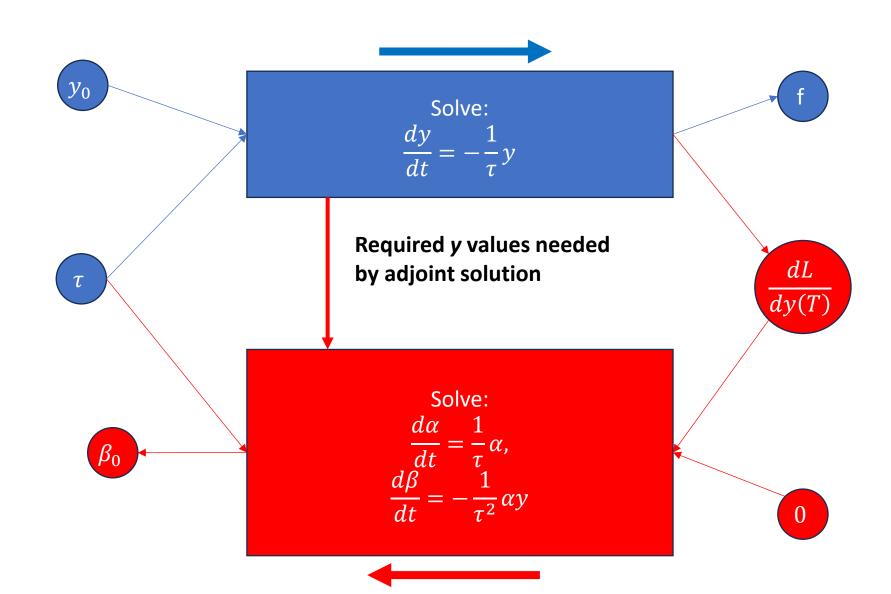
Backprop of ODE solution numerically solves adjoint state method

= "discretise-then-optimise"

Extending the computational graph



An abstraction



Numerical modelling — PDEs

- Partial differential equations (PDEs) dependent on only many independent variables
 - For the most part, these variables are space and time

Total derivative vs. partial derivative

$$\frac{d}{dt}[f(x,t)] = \frac{\partial f}{\partial t} + \frac{\partial x}{\partial t} \cdot \frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} + v_x \cdot \frac{\partial f}{\partial x}$$

• Example: Hydrodynamics
$$\begin{cases} \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g} \\ \frac{\partial e}{\partial t} + \mathbf{u} \cdot \nabla e + \frac{p}{\rho} \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Numerical modelling – PDEs

- Method of lines for time dependent PDEs
 - Discretising all but the time dimension turns PDE → system of ODEs
- For example, the heat equation:

$$\frac{\partial T(x,t)}{\partial t} = k \frac{\partial^2 T(x,t)}{\partial x^2}$$

Becomes (centred-space differencing):

$$\frac{dT(x_i,t)}{dt} = k \left(\frac{T(x_{i+1},t) - 2T(x_i,t) - T(x_{i-1},t)}{\Delta x^2} \right)$$

Numerical modelling – PDEs

- Method of lines for time dependent PDEs
 - Discretising all but the time dimension turns PDE → system of ODEs
- For example, the heat equation:

$$\frac{\partial \dot{T}(x,t)}{\partial t} = k \frac{\partial^2 T(x,t)}{\partial x^2}$$

Becomes:

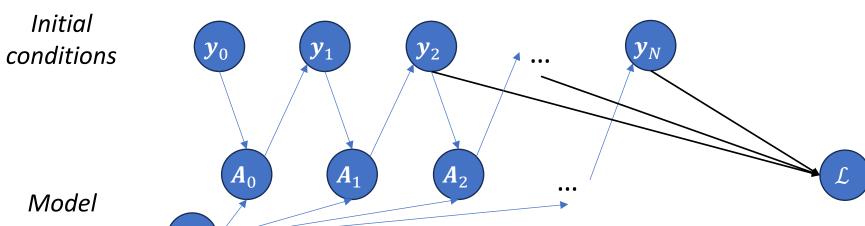
$$\frac{d\mathbf{T}(t)}{dt} = k\underline{D}\mathbf{T}(t)$$

• Where T is a vector of $T(x_i, t)$ at discrete points x_i

What if our function is a solution to a PDE?

- Spatial discretisation makes time-space PDEs into time ODEs
- Explicit time-stepping easiest:

$$\frac{d\mathbf{y}}{dt} = L(\mathbf{y}, \boldsymbol{\theta}) \xrightarrow{\text{discretise}} \mathbf{y}_{n+1} = (\underline{1} - \underline{L}(\mathbf{y}_n, \boldsymbol{\theta})dt)\mathbf{y}_n = \underline{A}_n\mathbf{y}_n$$



We can modify our initial conditions or our model parameters to minimise the loss

parameters

Recipe for differentiable solver

1. Specify your ODE/PDE

2. Choose spatial (and temporal) differencing scheme

3. Define loss term for optimisation purposes

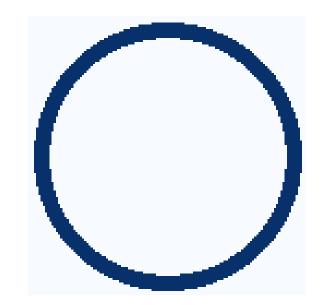
4. Compute adjoint system via 'discretise-then-optimise' or 'optimise-then-discretise' strategies

Backpropagating through a fluid simulation

 Modify the initial conditions (velocities) to match some input image after a set simulation time

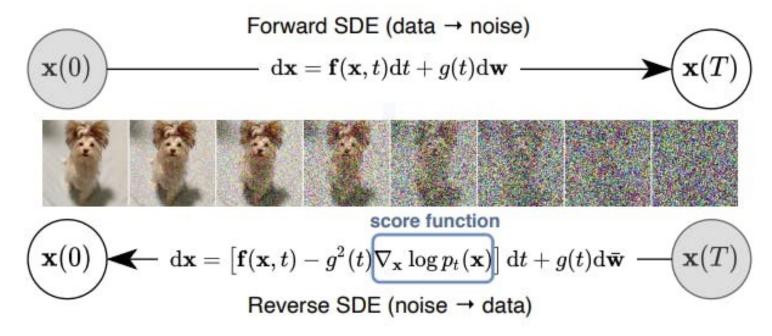
Steps:

- 1. Write a fluid simulator in a differentiable programming framework in this case AutoGrad
- 2. Initialise some densities and x,y velocities on nx by ny grid
- 3. Find distance to target image (loss) after N simulation iterations
- 4. Use Jacobian of loss to update initial velocities (2x nx x ny values)



Differential equations in pure ML

Diffusion models are based on stochastic differential equations (SDEs)



 SDEs solve for a single trajectory, subject to deterministic and white noise forces

Fokker-Planck equation

One can instead solve for the PDF of an ensemble of trajectories
 SDE → Fokker-Planck equation

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t \qquad \qquad rac{\partial}{\partial t}p(x,t) = -rac{\partial}{\partial x}\left[\mu(x,t)p(x,t)
ight] + rac{\partial^2}{\partial x^2}\left[D(x,t)p(x,t)
ight].$$

- Fokker-Planck describes the probability flow in diffusion models
- Fokker-Planck is used to describe the motion of particles in a plasma
- In exercise 2, you will encounter the various simplified forms of the Fokker-Planck equation

Colab Exercise 1

- Computational graphs and AD
- Euler's method for simple ODEs
- The adjoint method for ODEs

ODE differentiable programming libraries

 Diffrax – JAX-based library providing numerical differential equation solvers

 SciMLSensitivity.jl – Julia-based library, AD and adjoints for differential equations solvers (and more...)

ODE differentiable programming libraries

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Diffrax example

• Diffrax has 4 key features (for our purposes):

- 1. Terms
- 2. Solvers
- 3. Adaptive stepping
- 4. Adjoints

```
def dydt(t,y,args):
  return -y/args['tau']
def diffrax solve(dydt,t0,t1,Nt,rtol=1e-5,atol=1e-5):
  Here we wrap the diffrax diffeqsolve function such that we can run with
  different y0s and taus over the same time interval easily
  # We convert our python function to a diffrax ODETerm
  term = diffrax.ODETerm(dydt)
  # We chose a solver (time-stepping) method from within diffrax library
  # Heun's method (https://en.wikipedia.org/wiki/Heun%27s method)
  solver = diffrax.Heun()
  # At what time points you want to save the solution
  saveat = diffrax.SaveAt(ts=jnp.linspace(t0,t1,Nt))
  # Diffrax uses adaptive time stepping to gain accuracy within certain tolerances
  stepsize controller = diffrax.PIDController(rtol=rtol, atol=atol)
  return lambda y0,tau : diffrax.diffeqsolve(term, solver,
                         y0=y0, args = {'tau' : tau},
                         t0=t0, t1=t1, dt0=(t1-t0)/Nt,
                         saveat=saveat, stepsize_controller=stepsize_controller)
t0 = 0.0
t1 = 1.0
Nt = 100
ODE solve = diffrax solve(dydt,t0,t1,Nt)
# Solve for specific y0 and tau
y0 = 1.0
tau = 0.5
sol = ODE solve(y0,tau)
```

Diffrax example

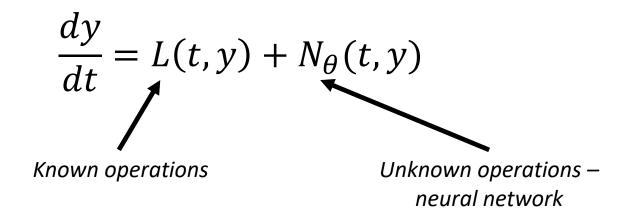
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                         t0=t0, t1=t1, dt0=(t1-t0)/Nt,
                         saveat=saveat, stepsize controller=stepsize controller)
t0 = 0.0
t1 = 1.0
Nt = 100
ODE_solve = diffrax_solve(dydt,t0,t1,Nt)
def loss(inputs):
  y0 = inputs['y0']
  tau = inputs['tau']
  sol = ODE solve(y0,tau)
  return sol.ys[-1]
inputs = {'y0' : y0, 'tau' : tau}
# Returns gradient of loss with respect to all inputs, i.e. dLdtau and dLdy0
jax.grad(loss)(inputs)
```

Neural Differential Equations

 What if what to describe a dynamical system but don't know the form of (some of) the terms?



Neural network parameters learnt using adjoint state method

Colab Exercise 2

- Diffrax library for solving ODEs
- Numerical solution to PDEs
- Differentiable PDE solvers
- Neural Differential Equations

Feedback



Assessment

- Quiz questions
- Refer to slides and previous exercises
- Coding exercise
- Training data on the github: https://github.com/aidancrilly/ AIMSLecture/data

Feedback

