Math 450 Portfolio

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December 20, 2024

1 Sets and Topology

Problem 1. Give a vector proof of the following geometry proposition.

Theorem. *Let A, B, C, and D be any four points in the plane and form a quadrilateral by joining them in that order; then, the midpoints of the four sides are the vertices of a parallelogram.*

Proof. Let P, Q, R, S be the midpoints of sides AB, BC, CD, AD, respectively. Thus,

$$P = \frac{A+B}{2}, Q = \frac{B+C}{2}, R = \frac{C+D}{2}, S = \frac{A+D}{2}$$

Intuitively, we find that

$$P + R = Q + S \Rightarrow \frac{P+R}{2} = \frac{Q+S}{2}$$

This common point of intersection is a midpoint of both diagonals, thus the diagonals bisect each other. This proves that PQRS is a parallelogram.

Problem 2. Prove that for any two vectors $p, q \in \mathbb{R}^n$ that

$$|p \cdot q| \leq |p||q|$$

with equality holding if and only if p and q are parallel.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathbb{R}^n$. Let $v = \alpha p - \beta q$, with α, β to be determined.

Thus,

$$0 \leq v \cdot v = [\alpha p - \beta q] \cdot [\alpha p - \beta q] = \alpha p[\alpha p - \beta q] - \beta q[\alpha p - \beta q]$$

Simplifying, we determine

$$0 \leq \alpha^2(p \cdot p) - 2\alpha\beta(p \cdot q) + \beta^2(q \cdot q)$$

So for any $\alpha, \beta \in \mathbb{R}$,

$$0 \leq \alpha^2|p|^2 - 2\alpha\beta(p \cdot q) + \beta^2|q|^2 \Rightarrow 2\alpha\beta[p \cdot q] \leq \alpha^2|p|^2 + \beta^2|q|^2$$

We set $\alpha = |q|, \beta = |p|$ so

$$2|q||p|[p \cdot q] \leq 2|p|^2|q|^2 \Rightarrow p \cdot q \leq |p||q|$$

Note:

$$-p \cdot q \leq |-p||q| \Rightarrow p \cdot q \geq -|p||q|$$

We then find that

$$-|p||q| \leq p \cdot q \leq |p||q| \Rightarrow -1 \leq \frac{p \cdot q}{|p||q|} \leq 1$$

Let θ be the angle between p and q. We know that if $\theta = 0, \pi$ that $cos(\theta) = 1$. Thus, the equality

$$\frac{p \cdot q}{|p||q|} = 1$$

must hold if and only if the vectors p and q are parallel.

Problem 3. Prove the triangle inequality. If $p, q \in \mathbb{R}^n$ then,

$$|p + q| \leq |p| + |q|.$$

Proof. Given $|p| + |q|$, we want to show $|p + q| \leq |p| + |q|$.

We will start by squaring $|p| + |q|$. That is,

$$(|p| + |q|)^2 = |p|^2 + |q|^2 + 2|p||q| \geq p^2 + q^2 + 2pq$$

Thus,

$$(|p| + |q|)^2 \geq (p + q)^2 (\because \forall x \in \mathbb{R}; x^2 = |x|^2)$$

We can simplify then by reducing both sides. We find

$$||p| + |q|| \geq |p + q| \Rightarrow |p| + |q| \geq |p + q|$$

Problem 4. Prove the "reverse triangle identity": If $p \in \mathbb{R}^n$, then

$$|p - q| \geq ||p| - |q||.$$

Proof.

$$p = q + |p - q| \Rightarrow |p| = |q + (p - q)| \leq |q| + |p - q|$$

$$q = p + |q - p| \Rightarrow |q| = |p + (q - p)| \leq |p| + |q - p|$$

Thus,

$$|q| - |p| \leq |q - p| \Rightarrow |p| - |q| \geq -|q - p|.$$

Finally,

$$||p| - |q|| \leq |p - q| \Rightarrow |p - q| \geq ||p| - |q||.$$

Definition. A metric or distance function on $\mathbb{R}^n$ is a function $d(p, q)$ of the variables $p, q \in \mathbb{R}^n$ that satisfy the three conditions:

1. $d(p, q) = d(q, p)$
2. $d(p, q) \geq 0$ with $d(p, q) = 0 \iff p = q$
3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in \mathbb{R}^n$. This is called the triangle inequality.

Problem 5. Prove that $d(p, q) = |p - q|$ is a metric.

Proof. We will start by proving the equality satisfies the first condition of a metric, then the following two:

1. $d(p, q) = d(q, p)$
$$d(p, q) = |p - q|$$
$$d(q, p) = |q - p| = |-(p - q)| = |p - q|$$
2. $d(p, q) \geq 0$ *with* $d(p, q) = 0 \iff p = q$.
If $p = q$, then $d(p, q) = d(p, p)$. We then find
$$d(p, p) = |p - p| = 0$$
Suppose $p \neq q$, then $d(p, q) = |p - q| > 0$. This must be the case since $d(p, q) = 0 \iff p = q$
3. $d(p, q) \leq d(p, r) + d(r, q)$ *for any* $r \in \mathbb{R}^n$.
$$p - q = p - r + r - q$$
$$\Rightarrow |p - q| = |(p - r) + (r - q)| \leq |p - r| + |r - q|$$

Thus we have proven that $d(p, q) = |p - q|$ is a metric.

Problem 7. Prove that arbitrary unions of open sets are open and finite intersections of open sets are open.

Suppose A, B are open sets. Let $p \in A \cup B$, WLOG $p \in A$. $\exists B(p, r) \subset A \subset A \cup B$.

$\bigcup_{\alpha=1}^{\infty} A_\alpha = U$ is open. $p \in U$. Suppose $p \in A_\alpha$. $\exists B(p,r) \subset A_\alpha \subset U$.

$p \in A \cap B \to p \in A, p \in B$. $B(p, r_a) \subset A, B(p, r_b) \subset B$. WLOG, $r_a < r_b, B(p, r_a) \subset B(p, r_b) \subset B$. $p \in \bigcap_{\alpha=1}^{n} A_\alpha = I.p \in A_k$ for each $k.B(p, r_k) \to B(p, min(r_1, r_2, ..., r_n)) \subset B(p, r_k)$ for each $k = 1...n$. So the finite intersection of open sets is open.

Problem 8. Prove that finite unions of closed sets are closed and that arbitrary intersections of closed sets are closed.

Let $F_1, F_2, ...$ be closed sets. $n \in \mathbb{N}$. Consider the compliment of the union of the closed sets up to F_n. $(F_1 \cup F_2 \cup ... \cup F_n)^c = F_1^c \cap F - 2^c \cap ... \cap F_n^c$ by De Morgan's Law. Since $F_k, k \in \mathbb{N}$ is a closed set, the compliments of F_k must be open. From 7. we know arbitrary unions of open sets are open. So the finite union of closed sets must be closed.

Let $A = \bigcap_{\alpha=1}^{\infty} F_\alpha$ where any $F_k, k \in \mathbb{N}$ is closed. $A^c = (\bigcap_{\alpha=1}^{\infty} F_\alpha)$. By De Morgan's law, $A^c = \bigcup_{\alpha=1}^{\infty} F_\alpha^C$ is open, since infinite union of open sets is open. Thus infinite intersection of closed sets is closed.

Problem 9. Prove that a set is closed if and only if it contains all its limit points.

Proof. Suppose F is any non-empty set and $F \neq R^n$. In order to prove this, we must show:

1. If F is closed $\to F$ contains all its limit points.

   We want to use the contrapositive. Suppose F does not contain all its limit points. (We want to show that F is not closed.)

   Suppose $\exists p \in F', p \notin F$. This means $p \in F^c$. Therefore, $\exists$ some $B(p,r)$ for any $r > 0, r \in \mathbb{R}$ such that $B(p,r) \cap F \neq \varnothing$

   This implies that F^c is not open, meaning that F is not closed.

2. If F contains all its limit points $\to F$ is closed.

   Suppose that F contains all its limit points. Let $p \in F^c$. Then p is not a limit point of F. There must be some open set U such that $p \in U$ and $U \cap F = \varnothing$. Since $p \notin F$, U does not contain any other point of F. This implies that $p \in U \subseteq F^c$, so F^c is open. Therefore, F must be closed.

Thus, we have shown that for any non-empty set F and $F \neq R^n$, F is closed if and only if F contains all its limit points.

Problem 10. Show that the closure of a set is equal to the union of the set and its limit points.

Proof.

a) $S \cup S' \subseteq \overline{S}$
$S \subseteq \overline{S}$ by definition. Is $S' \subseteq \overline{S}$? Suppose not, that is, suppose $x \in S', \overline{S}$ but $S' \nsubseteq \overline{S}$. Then there exists some neighborhood $V_x \subset S'$ such that $V_x \cap \overline{S} = \varnothing$. Then by definition $V_x \cap S = \varnothing$. This contradicts $x \in S'$, so $S \subseteq \overline{S}, S \cup S' \subseteq \overline{S}$.

b) $\overline{S} \subseteq S \cup S'$
Let $x \in \overline{S}$. If $x \in S$, we are done. If $x \notin S$, for every neighborhood V_x, it must follow $V_x \cup S = \varnothing$. Thus $x \in S', x \in S \cup S'$.

Problem 13. Suppose that an open set Ω may be written as a union of two disjoint nonempty open sets U and V. Prove that Ω is disconnected.

Proof. Let U, V be any open, disjoint, non-empty sets. $U \neq R^n$, $V \neq R^n$. Let $\Omega = U \cup V$. (We want to show Ω is disconnected.)

Suppose not. That is, suppose $U^c \cap V \neq \varnothing$. So $\exists p \in U^c \cap V$, $p \in U^c$, $p \in V$. Since V is open, $\exists r > 0$, $r \in \mathbb{R}$ such that $B(p,r) \subset V$. Thus, $B(p,r) \cap U = \varnothing$. Moreover, $p \notin U$, since $U \cap V = \varnothing$. But since $p \in U^c$ and U^c contains its limit points, we see $p \in U'$. This implies $B(p,r) \cap U \neq \varnothing$. This is a contradiction, so $U^c \cap V = \varnothing$. Thus, $\Omega = U \cup V$.

Claim. We claim that the monotonic sequence property implies the least upper bound property.

Proof. Suppose S is some bounded set, $S \neq \varnothing$. Without loss of generality, $S \subseteq (0, \infty)$.

Consider the greatest lower bound case first. Let L represent the set of lower bounds of S. (Note: $0 \in L$) We will use logical construction.

Let $n_0 =$ the greatest integer of L. Let $n_1 =$ the greatest integer but for when $\frac{n_1}{2} \in L$. (Note $n_0 = \frac{2n_0}{2}$, so $2n_0 \leq n_1$.) Let $n_2 =$ the greatest integer but for when $\frac{n_2}{4} \in L$.(Note $\frac{n_2}{2} = \frac{2n_1}{4}$, so $2n_1 \leq n_2 \Rightarrow \frac{n_1}{2} \leq \frac{n_2}{4}$.)

Thus, $n_0 \leq \frac{n_1}{2} \leq \frac{n_2}{4}...$, so we define the sequence $\frac{n_k}{2^k}$. By the monotonic sequence property:

$$\lim_{k \to \infty} \frac{n_k}{2^k} = L_0$$

We want to show L_0 is the greatest of the lower bounds. ($\frac{n_k}{2^k} \leq L_0$, $L_0 \in L$.) Suppose $x \in S$, $\frac{n_k}{2^k} \leq x$. Thus,

$$\lim_{k\to\infty} \frac{n_k}{2^k} \leq \lim_{k\to\infty} x = x \Rightarrow L_0 \leq x$$

Suppose a bigger lower bound for S We will refer to it as b. Then $L_0 < b$. Thus,

$$\frac{n_k}{2^k} \leq L_0 < b \Rightarrow n_k \leq 2^k L_0 < 2^k b$$

Then for some N,

$$2^k L_0 < N < 2^k b \Rightarrow L_0 < \frac{N}{2^k} < b$$

Problem 15. Prove an open connected set Ω is polygon connected.

Proof.

Let Ω be an open connected set. $x, y \in \Omega$. Let A be the set of all points $a \in \Omega$ that can be connected to x by a polygon in Ω. $x \in A$ by definition, given all points in A are open, A must be open. Assume B is the set of all points in Ω that cannot be polygon connected. $\Omega = A \cup B$. By definition no point in A can connect to a point in B, thus the sets are disjoint. This contradicts Ω being a connected set, so $A = \varnothing$ or $B = \varnothing$. If $\Omega = A \cup \varnothing, \Omega = A$. Since A is polygon connected, Ω is polygon connected.

Problem 17. Assume $\{p_n\}$ is a sequence in $\mathbb{R}^n$ and assume there exists a real number $0 < c < 1$ such that $|p_{n+1} - p_n| < Ac^n, A > 0$. Prove the limit of this sequence as $n \to \infty$ exists.

Proof.

Assume $n > m$. Then $p_n = (p_m - p_{m-1}) + (p_{m-1} - p_{m-2}) + ...$ In a similar way we can write: $|p_n - p_m| = |(p_n - p_{n-1}) + ... + (p_{m+1} - p_m)|$. By the triangle inequality, $|p_n - p_m| \leq |(p_n - p_{n-1})| + ... + |(p_{m+1} - p_m)| < Ac^{n-1} + ... + Ac^m$ $Ac^{n-1} + ... + Ac^m = A(c^{n-1} + ... + c^m) = Ac^m(c^{n-m-1} + ... + 1)$

$$Ac^m \frac{1 - c^{n-m}}{1 - c} \leq \frac{A^m}{1 - c}. \lim_{m\to\infty} \frac{A^m}{1 - c} = 0$$

Thus $\{p_n\}$ is a Cauchy Sequence in $\mathbb{R}^n$ and the sequence converges.

Problem 23. Show for any bounded sequences $\{a_n\}, \{b_n\}$ of real numbers,

$$\lim_{n\to\infty} inf(a_n) + \lim_{n\to\infty} inf(b_n) \leq \lim_{n\to\infty} inf(a_n + b_n)$$

$$\lim_{n\to\infty} sup(a_n) + \lim_{n\to\infty} sup(b_n) \leq \lim_{n\to\infty} sup(a_n + b_n)$$

Proof.

let $I_n = [a_n, b_n]$. Define $I_n \subset I_1 = [a_1, b_1]$. $a_n \leq a_1 \leq b_n \leq b_1$. $\{a_n\}, \{b_n\}$ are bounded monotonic sequences, By the monotonic sequence property,

$$\lim_{n\to\inf} a_n = A, \lim_{n\to\inf} b_n = B$$

$A \leq B$ by the transitive property. $[A, B] \subseteq \bigcup_{n=1}^{\infty} I_n$. If $A = B = C$, $a_n \leq A = B = C \leq b_n$. If $A < B$, $a_n \leq A < B \leq b_n$. $[A, B] \subset [a_n, b_n]$.
$[A, B] = \bigcup_{n=1}^{\infty} I_n = \varnothing$

Problem 25. Every bounded infinite set of real numbers has a limit point.

Proof.

Let $S \in \mathbb{R}$ be a bounded set with infinite elements. Then $S \subseteq I - 1 = [n, m]$. $I_1 \cap S = S \Rightarrow |I_1 \cap S| = \infty$. Let m be the midpoint of a, b. Thus $[a, b] = [a, m] \cup [m, b]$. $S \cap I_n = (S \cap [a, m]) \cup (S \cap [m, b])$. $|S \cap [a, m]| = \infty$ or $|S \cap [a, m]| = \infty$. WLOG, $I_{n+1} = [m, b]$ if $|S \cap I_{n+1} = \infty$. $\forall k \in \mathbb{N}, f(k) = y - x, I_k = [x, y]$. $I_{n+1} = f(n+1) = b - m = b - \frac{a+b}{2} = \frac{b-a}{2}$. $2f(n+1) = 2 * \frac{b-a}{2} = b - a = f(n)$ $I_n = I_{n+1} \cup [a, m] \Rightarrow I_{n+1} \subseteq I_n$. Since $a \in I_n, a \notin I_{n+1}, I_{n+1} \subsetneq I_n$. The nested intervals property shows $\exists L \in \mathbb{R}, \forall n \in \mathbb{N}, L \in I_n$. Let $u \in \mathbb{R}, u > 0, \exists N > n, f(N) < u$. $I_N = [g, h], c, L \in [g, h], g \leq c, L \leq h,$. $c - L \leq h - L$. $h - L \leq g - L$. $c - L \leq L - g$. By symmetry, $L - c \leq h - g$. $|c - L| \leq h - g \Rightarrow |c - L| \leq f(n) < u$. $c \in B(L, u)$. Thus $\forall c \in I_n, I_n \subseteq B(L, u) \Rightarrow |S \cap B(L, u)| = \infty$. Thus L is a limit point.

Challenge Problem 1. Given a_n an infinite convergent sequence with terms in $R^1, \lim_{n\to\inf} a_n = A, \sigma_n = \frac{a_1+...+a_n}{n}$, show $\lim_{n\to\infty} \sigma_n = A$

Proof.

Case $A = 0$. Show $\forall \epsilon > 0, \exists N, \forall n > N, |\sigma_n| < \epsilon$. $\exists i \forall j > i, |a_j| < \epsilon * 1/2$. $a_1, ..., a_i$ finite terms in $R^1, \exists a_b, |a_b| \geq |a_v| \forall v \leq i$. $B = |a_b|$.

$$\sigma_{i+D} = \frac{a_1 + ... + a_i + ... + a_{i+D}}{i + D} < \frac{i}{i + D} * B + \frac{D}{i + D} * 1/2\epsilon$$

If $\frac{D}{i+D} < 1, \sigma_{i+D} < \frac{i}{i+D} * B + 1/2\epsilon$.

$$D > \frac{k * i * b}{\epsilon} - i, 2 < k > \epsilon/b$$

$$\sigma_{i+D} < \frac{i}{1 + \frac{k*i*b}{\epsilon} - i} * B + 1/2\epsilon = 1/k * \epsilon + 1/2 * \epsilon < \epsilon$$

$N = i + d, \forall \epsilon \exists N \forall n > N, |\sigma_n| < \epsilon$. Thus the limit exists.

Case $A \neq 0$. $a'_n = a_1 - A, ..., a_n - A, ...$ $\lim_{n\to\infty} a'_n = \lim_{n\to\infty} a_n - A = A - A = 0$.

$$\lim_{n\to\infty} \sigma'_n = 0 \Rightarrow \lim_{n\to\infty} \sigma'_n + A = A$$

$$\sigma'_n + A = \frac{(a'_1 + A) + ... + (a'_n + A)}{n} = \sigma'_n + A = \frac{(a_1 - A + A) + ... + (a_n - A + A)}{n} = \sigma_n$$

$$\lim_{n\to\infty} (\sigma'_n + A) \geq \lim_{n\to\infty} \sigma_n = A$$

Challenge Problem 4. Cauchy criterion in R^1. A sequence p_n is convergent $\iff$ p_n is a Cauchy sequence

Proof.

($\Rightarrow$) Suppose p_n is convergent. Show it is Cauchy. Let $\lim_{n\to\infty} p_n = L, \forall \epsilon > 0, \exists N, n > N \rightarrow |p_n - L| < \epsilon$. $\exists N', n > N' \rightarrow |p_n - L| < \epsilon/2$. Suppose $m > N'$. Then $|p_m - L| < \epsilon/2$. Thus $p_n - p_m = (p_n + L) + (L - p_m)$. $|p_n - p_m| \leq |p_n + L| + |L - p_m| < \epsilon/2 + \epsilon/2 = \epsilon$
($\Leftarrow$) Suppose p_n is a Cauchy sequence. Show it is convergent. Corollary: Every bounded sequence p_n has a convergent subsequence p_{n_k}, where the limit of this subsequence as k approaches infinity is L. Let $\epsilon = 1$. Since p_n is Cauchy, $\exists N, \forall n, m > N, |p_n - p_m| < 1$. Consider $n = N+1$, then $\forall m > N, |p_{N+1} - p_m| < 1$. So p_m, p_n are bounded. p_n has a convergent subsequence. $\lim_{k\to\infty} p_{n_k} = L$. Let ϵ be chosen, consider $\epsilon/2$. There will be a N' such that $k > N'$. $|P_{N_k} - L| < \epsilon/2$. Since p_n is Cauchy, $\exists N'', n, n_k > N'' \rightarrow |p_n - p_{n_k}| < \epsilon/2$. $|P_N - L| \leq |p_n - p_{n_k}| + |p_{n_k} - L| < \epsilon/2 + \epsilon/2 = \epsilon$

Problem 32. Every bounded infinite set in R^n has a limit point.

Proof.

Let $S \subset R^n$ be a bounded infinite set. There exists some closed square such that $S \subset R_0$. $R_0 \supset R_1 \supset ... \supset R_n$ by the nested rectangle property. Let $p \in \bigcap_{n=0}^{\infty} R_n$. Want to show $p \in S'$. $diam(R_n) < \epsilon$.

$$\frac{diam(R_0)}{2^n} = diam(R_n) < \epsilon$$

$$\frac{diam(R_0)}{\epsilon} < 2^n$$

Claim: $R_n \subset B(p, \epsilon)$ Let $q \in R_n, |q - p| \leq diam(R_n) < \epsilon$. $q \in B(p, \epsilon)$, so $R_n \subset B(p, \epsilon)$. Thus p is a limit point in R^n

Problem 34. Prove all compact sets are closed

Proof.

Suppose $K \subset R^n$ is a compact set, but K is not closed. Then K does not contain all it's limit points. There exists some $p \in K^c \in R^n$. Look at the closed ball $B(p, r) = q \in R^n||p - q| \leq r$. Suppose a sequence of the converse $U_r = B(p, r)^c$ must be open. Let $r = 1/n \forall n \in \mathbb{N}$. Suppose $\lim_{n \to \infty} r = 0$, but $|p - q| \geq r$. For any $x \in R^n | x \neq p, U_{1/n}$ is an open cover of K. p is on the boundary of K. K is compact, so $U_{1/n}$ must have a finite subcover. Since p is a limit point, there exists no finite n such that $B(p, r)^c$ can be a superset of K. $U_{1/n}$ has no finite subcover. This is a contradiction. So K must be closed.

Challenge Problem 5. Prove the Heine-Borel Theorem in R^n.

Proof.

Suppose not. That is, suppose R_0 is not compact. By nested rectangles property, at least one quarter of R_0, call this R_1 does not admit a finite subcover of O. $R_0 \supset R_1 \supset \ldots$ Let $p \in \bigcap_{n=0}^{\infty} R_n \neq \varnothing$. $p \in R_0, R_1, R_2, \ldots\ \exists \alpha | p \in O_\alpha$ with O_α open. $\forall n > N, R_n \subset O_\alpha$. This is a contradiction, O_α is an open cover, so R_n is compact.

Problem 36. In R^n show that $K_1 \supset K_2 \supset \ldots$ is a nested sequence of nonempty compact sets.

Proof.

$\forall n \in \mathbb{N}, K_n$ is bounded and closed. Since K_1 is bounded, all K_{n+1} is bounded. Let some $x_n \in K_n \forall n \in \mathbb{N}, \{x_n\}$ is bounded. By #32, x_{n_k} converges to some limit point p. Since $K_{n_k} \subset \ldots \subset K_1$, each K_n is closed, the limit point $p \in K_n \forall n \in \mathbb{N}$. Thus the points $x \in K_n, x \in \bigcap_{n=1}^{\infty} K_n$. $p \in \bigcap_{n=1}^{\infty} K_n \neq \varnothing$.

Problem 37. Prove that if S is any bounded set in $R^n, S > 0$ is given, then it is possible to choose a finite set of points $p_i \in S, \forall p \in S$ is within a distance δ of at least one of the points.

Proof.

Claim 1: If S is bounded then $\overline{S}$ is bounded. Let S be bounded but suppose $\overline{S}$ is not bounded. Then for any positive number $m > 0 \exists q \in \overline{S}, |q| > m$. Choosing $m_n = n, n \in \mathbb{N}$ we find $q_n \in \overline{S}, |q_n| > n$. Since $q_n \in \overline{S}$, it is either in S or S'. If $q \in S$ it contradicts the boundedness of S. Thus assume that $q \in S'$. Then pick $q_1 \in S, |q_1| > R$. Then $q_1 \in B(q_1, \epsilon) \cap B(0, r)$. But the intersection is empty. So q_1 is not a limit point. This is a contradiction, so the closure of S is bounded.

Claim 2: Since $\overline{S}$ is closed by definition, bounded by claim 1, by Heine-Borel theorem it must be compact. Give $\delta > 0$ by property of compactness, the collection of open balls $\{B(p, \delta)\}$ forms an open cover of S. By the compactness of the closure of S we can extract a finite number of points p_i such that the balls $B(p_i, \delta)$ covers S where $p_i \in S$.
Thus we have shown it is possible to choose a finite set of points in S such that every point in S is withing δ of at least one of the points.

2 Functions and Continuity

Problem 42. $f(\bigcap A_\alpha) \subset \bigcap f(A_\alpha)$

Proof.

If $y \in f(\bigcap A_\alpha), \exists x \in \bigcap A_\alpha, f(x) = y$. Since $x \in A_\alpha, \forall \alpha, f(x) \in f(A_\alpha)$. $f : x \to y, A_\alpha \subset x$. Thus $f(x) \in \bigcap f(A_\alpha), y = f(x) \to y \in \bigcap f(A_\alpha)$

Problem 46. Sequential continuity implies continuity

Proof.

Let $f : D \subset R^n \to R^1$. Suppose f is sequentially continuous but not generally continuous. So $\exists p_0 \in D$, but f is not continuous at p_0. $\exists \epsilon > 0 \forall \delta > 0, f(D \cap B(p_0, \delta)) \not\subset B(f(p_0), \epsilon)$. Consider $\delta_n = 1/n. \forall n \exists p_n \in D \cap B(p_0, 1/n)$, but $f(p_n) \notin B(f(p_0), \epsilon) \to |f(p_n) - f(p_0)| \leq \epsilon > 0$.Then:

$$\lim_{n\to\infty} p_n = p_0$$

as f is sequentially continuous. Thus

$$\lim_{n\to\infty} f(p_n) = f(p_0)$$

then for $\epsilon > 0, \exists N, n > N \to |f(p_n) - f(p_0)| < \epsilon$. This is a contradiction, so f must be generally continuous.

Problem 48. A function f is said to be continuous at a point p_0 if for every $\epsilon > 0$ there is a neighborhood U_{p_0} such that for all $p \in U \cap D, |f(p) - f(p_0)| < \epsilon$.

Proof.

$\forall \epsilon > 0, \exists \delta, |p - p_0| < \delta, |f(p) - f(p_0)| < \epsilon$. $B(p_0, \delta = U$. $-\epsilon < f(p) - f(p_0) < \epsilon \Rightarrow f(p_0) - \epsilon < f(p). \epsilon = f(p_0)/2 \Rightarrow 2\epsilon - \epsilon < f(p) \Rightarrow \epsilon < f(p)$

Problem 49. Let $f : D \subset R^n \to R^1$ be continuous, and c be a real number. Then the set $f^{-1}((-\infty, c))$ and $f^{-1}((c, \infty))$ are relatively open in D. In other words, $f^{-1}((-\infty, c)) = U \cap D, f^{-1}((c, \infty)) = V \cap D$, where U, V are open sets in $\mathbb{R}^n$.

Proof.

For any point $p \in f^{-1}((-\infty, c))$ choose $\epsilon, 0 < \epsilon < c - f(p)$. Note $\exists B(f(p), \epsilon) \subset (-\infty, c)$. f is continuous, so $\exists \delta > 0, f(B(p, \delta) \cap D) \subset B(f(p), \epsilon) \subset (-\infty, c)$. Then $f^{-1}(f(B(p, \delta)) \cap D) \subset f^{-1}((-\infty, c)) \Rightarrow B(p, \delta) \cap D) \subset f^{-1}((-\infty, c))$. We define the open set $\mu = \bigcup_{p=0} B(p, \epsilon)$. Then $\mu \cap D = \bigcup_{p=0}(B(p, \delta) \cap D) \subset f^{-1}((-\infty, c)) \to \mu \cap D \subset f^{-1}((-\infty, c))$. If $p \in f^{-1}((-\infty, c))$, then $p \in B(p, \delta) \cap D \subset \mu \cap D$. Thus $f^{-1}((-\infty, c)) \subset \mu \cap D \Rightarrow f^{-1}((-\infty, c)) = \mu \cap D$

Problem 51. If $f : K \subset R^n \to R^1$ is continuous and K is compact then f is uniformly continuous on k.

Proof.

Suppose f is generally continuous on k but not uniformly continuous. $\exists \epsilon > 0, \forall \delta > 0, \exists p, q \in K, |p - q| < \delta, |f(p) - f(q)| \geq \epsilon$. Choose $\delta_n = 1/n, \forall \delta_n \exists p_n, q_n, |p_n - q_n| < \delta, |f(p_n) - f(q_n)| \geq \epsilon$. Since K is bounded, $\{p_n\}$ has a convergent subsequence. $p_{n_k} \to p* \in K$. Now $\{q_{n_k}\}$ had a convergent subsequence $\{q_{n_{k_l}}\} \to q* \in K$. $|p_{n_{k_l}} - q_{n_{k_l}}| < 1/n_{k_l}, |f(p_{n_{k_l}}) - f(q_{n_{k_l}})| \geq \epsilon$.
Claim: $p* = q*$. $|p * -q *| \leq |p * -p_{n_{k_l}}| + |q * -q_{n_{k_l}}| + |p_{n_{k_l}} - q_{n_{k_l}}| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Thus $p* = q*$. Since continuous implies sequentially continuous, $f(p_{n_{k_l}}) \to f(p*), f(q_{n_{k_l}}) \to f(q*)$. $f(p_{n_{k_l}}) - f(q_{n_{k_l}}) \to f(p*) - f(q*) = 0$. $|f(p_{n_{k_l}}) - f(q_{n_{k_l}})| < \epsilon$.

Problem 52. Prove the Extreme Value Theorem. That is, prove that a continuous function on a compact set K attains a maximum and a minimum.

Proof.

$f(p) = max(f(k)) = sup(f(k)), f(q) = min(f(k)) = inf(f(k))$. Define $F : K \subset R^n \to R^1$, K is compact $\to K' = F(K)$ is also compact.

max:
$$s = sup(K') \to \exists q_n \in K$$
$$s - 1/N < q_n \leq s$$
$$s \in K' \exists q \in K, f(q) = s$$

min:
$$i = inf(K') \to \exists p_n \in K$$
$$i \leq p_n < i + i/Ns$$
$$i \in K' \exists p \in K, f(p) = s$$

The convergent sequences p_n, q_n are proven in #16

Problem 57. Prove a continuous image of a compact set is compact. (Show if K compact in $R^n, F : K \subset R^n \to R^m$ is cont, $F(K)$ is compact)

Proof.

1. $F(K) is closed.$ Let any sequence $y_n \in F(K)$. $\exists (x_n) \in k, f(x_n) = y_n, \forall n \in \mathbb{N}$. Since K is compact, $\exists x_{n_k} \to x \in K$. By continuity of F: $y = \lim_{k\to\infty} x_{n_k} = f(x)$. Thus $y \in F(K)$, $F(K)$ is closed.
2. $F(K)$ is bounded. Suppose $w_n \in F, w_n \geq n, \forall n \in \mathbb{N}$. $\forall n \exists v_n + n, F(v_n) = w_n(v_n)$ is contained in the compact K, and thus admits a subsequence $v_{n_k} \to v \in K, \lim_{k\to\infty} n_k = \infty \leq \lim_{k\to\infty}(v_{n_k}) = f(v)$. This is a contradiction, thus $F(K)$ is bounded. By Heine-Borel then F is compact.

Problem 59. Prove that the continuous image of a connected set is connected. That is, if $f : R^n \to R^m$ is continuous and $S \subset R^n$ is connected. Show $f(S)$ is connected.

Proof.

Suppose $f(s) = A \cup B$, A, B are mutually separated. Since no point of A is arbitrarily close to B, A, B are relatively open to $F(S)$. Since F is continuous, $F^{-1}(A), F^{-1}(B)$ must be relatively open to S. This means $F^{-1}(A) \cup F^{-1}(B) = S$ where these sets are disjoint and mutually seperated. Since S was assumed connected, one set must be empty. WLOG, say $F^{-1}(A) = \varnothing \to A = \varnothing$. Thus $F(S)$ is not disconnected

3 Differentiation

Problem 71. Suppose f is defined on a neighborhood of x_0 and is differentiable at x_0. Show that f is continuous at x_0.

Proof.

f is differentiable at x_0 if $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = L$ by definition of differentiability. $\lim_{x\to x_0} f(x) - f(x_0) = 0$ by definition of continuity.

$$\lim_{x\to x_0} [f(x) - f(x_0)]\frac{x - x_0}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} * \lim_{x\to x_0} (x - x_0) = f'(x_0) * 0 = 0$$

Thus if f is differentiable at x_0, then it is differentiable at x_0.

Problem 72. If f is defined on a neighborhood of $x_0 \in \mathbb{R}$, and is differentiable at x_0, an extremum, then $f'(x_0) = 0$.

Proof.

$$f(x_0) \geq f(x_0+h) \Rightarrow 0 \geq f(x_0+h) - f(x_0) \Rightarrow 0 \geq \frac{f(x_0+h)-f(x_0)}{h}$$
$$0 \geq \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

cannot possibly be negative. If h is negative, we have

$$0 \leq \lim_{h\to 0^-} \frac{f(x+0)-f(x+0+h)}{h}$$

which cannot be positive. Since f is continuous, the limit of every point must agree, so

$$0 = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

Problem 73. If f is continuous function on $[a,b]$, and differentiable on (a,b), $f(a) = f(b) \exists c, f'(c) = 0$.

Proof.

Case 1: $f'(x) = f'(a) = f'(b) \forall x \in (a,b)$. In this case any value between a, b can serve as c. Thus f is constant on $[a,b]$. $f' = 0$ so $f'(c) = 0$
Case 2: $f' \neq 0$. By EVT, $\exists c, f(c)/geqf(x_0) \forall x_0 \in f$, or $f(c) \leq f(x_0)$. By Fermat's theorem, $F'(c) = 0$.

Problem 76. Prove $(D_{-\beta}f)(p_0) = -(D_\beta f)(p_0)$

Proof.

Since β is the direction derivative of f, if the direction is reversed, $(D_\beta f)(p_0)$ becomes $(D_{-\beta}f)(p_0)$ then:

$$\lim_{t\to 0} \frac{f(p_o+t\beta)-f(p_0)}{t} \to \lim_{t\to 0} \frac{f(p_o-t\beta)-f(p_0)}{t}$$

let $\lambda = -t$. Then:

$$\lim_{t\to 0} \frac{f(p_o-t\beta)-f(p_0)}{t} = \lim_{\lambda\to 0} \frac{f(p_o-\lambda\beta)+f(p_0)}{-\lambda} = -1 * (D_\beta f)(p_0)$$

Problem 77. $g : (a,b) \to R^1, g'(x) \neq 0 \to g(x)$ is strictly monotonic

Proof.

WLOG, $\exists x_0 \in (a,b), g'(x_0) > 0, x_1 \in (a,b), g'(x_1) < 0$.

$$g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}, g(x) - g(x_0) > 0$$

on $V_\delta(x_o)$.

$$g'(x_1) = \lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1}, g(x) - g(x_1) < 0$$

$x - x_1 < 0, g(x) > g(x_1)$. $[x_0, x_1] \subset (a,b)$. By Fermat's theorem, $\exists x_2 \in [x_0, x-1], g'(x_2) = 0$. This is contradiction, $g'(x) \neq 0$, $g(x)$ is strictly monotonic.

Problem 78. If all first derivatives of f exist and are continuous in open set $D \subset R^n$, then f itself is continuous on D

Proof.

$f : D \subset R^n \to R^1, \partial f/\partial x_1, ..., \partial f/\partial x_n \in D$ are continuous. $p_0 \in D$, since D is open, $\exists B(p_0, r) \subset D, r > 0$. Consider the closed ball $\overline{B}(p_0, r) \subset D$. By EVT, a continuous function on a compact set is bounded.

$$|\partial f|/|\partial x_1| \leq B_1 ... \partial f/\partial x_n \leq B_n$$

$\Delta p = p - p_0, p = (x_1, ..., x_n), p_0 = (a_1, ..., a_n)$. Then $\Delta p = (x_1 - a_1 = \Delta x_1, ..., \Delta x_n) \to |\Delta x_i| \leq |\Delta p|$. By MVL,

$$|f(p) - f(p_0)| \leq |\frac{\partial f}{\partial x_1}(p_1)| * |\Delta x_1| + ... + |\frac{\partial f}{\partial x_n}(p_n)| * |\Delta x_n| \leq B_1 * |\Delta x_1| + ... + B_n * |\Delta x_n|$$

$$\leq B_1 * |\Delta p| + ... + B_n * |\Delta p| = |\Delta p| \hat{B} \to |f(p) - f(p_0)| \leq |\Delta p| \hat{B}$$

For $\epsilon > 0, p - p_0 = \Delta p < \epsilon / \hat{B} = \delta$. $|f(p) - f(p_0)| < \hat{B} * \epsilon / \hat{B} = \epsilon$. Thus f is continuous on D.

Problem 79. Let f be differentiable at x_0, then the remainder function $R = R(\Delta x) = f(x) - f(x_0) - f'(x_0) * \Delta x$ approaches 0 faster than Δx, meaning $\lim_{x \to x_0} \frac{R(\Delta x)}{|\Delta x|} = 0$. Equivalently, for $\epsilon > 0 \exists V_\delta(x_0), |R(\Delta x)| < |\delta x| * \epsilon \forall x \in V_\delta(x_0)$

Proof.

$\lim_{x \to x_0} \frac{R(\Delta x)}{|\Delta x|} = \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0) * \Delta x}{|\Delta x|}$. Consider $x > x_0$:

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{\Delta x} - \frac{f'(x_0) * \Delta x}{\Delta x} = \lim_{x \to x_0} f'(x_0) - \lim_{x \to x_0} f'(x_0) = 0$$

$$\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{\Delta x}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{\Delta x}=f'(x)\to\lim_{x\to x_0}\frac{R(\Delta x)}{|\Delta x|}=0$$

Claim: For $\epsilon > 0 \exists V_\delta(x_0), |R(\Delta x)| < |\delta x| * \epsilon \forall x \in V_\delta(x_0)$. Consider

$$\lim_{x\to x_0}\frac{|R(\Delta x)|}{|\Delta x|}=\begin{cases}\lim_{\Delta x\to 0}\frac{-R(\Delta x)}{x}\\ \lim_{\Delta x\to 0}\frac{R(\Delta x)}{x}=0\end{cases}$$

$\lim_{\Delta x\to 0}\frac{-R(\Delta x)}{x}=-\lim_{\Delta x\to 0}\frac{R(\Delta x)}{x}\to -f'(x_0)+f'(x_0)=0\to$
$\to \lim_{x\to x_0}\frac{|R(\Delta x)|}{|\Delta x|}=0$. Given $\epsilon > 0, \exists\delta > 0, \frac{|R(\Delta x)|}{|\Delta x|} < \epsilon$. $B(x_0,\delta)$ defines an interval in one dimension where: $\frac{|R(\Delta x)|}{|\Delta x|} < \epsilon \Rightarrow |R(\Delta x)| < |\Delta x| * \epsilon$

Problem 81. Suppose f continuously differentiable in the open set $S \subset R^n$. Define the remainder function: $R = f(p_0) - f(p) - \sum_{j=1}^n f_j(p_0)\Delta x_j$ where $p, p_0 \in S, \Delta p = p - p_0$. Show R approaches - faster that ΔP. Show $\forall\epsilon > 0\exists V_\delta(p_0) = B(p_0,\delta), |R|/|\Delta p|$

Proof.

Recall the MVL, $\exists p_1, ..., p_n \in S, f(p) - f(p_0) = \sum_{j=1}^n f_j(p_0)\Delta x_j$. Rewrite $R = [\sum_{j=1}^n f_j(p_j) - f_j(p_0)] * \Delta p$. By triangle inequality:

$$|R| \le [\sum_{j=1}^n |f_j(p_j) - f_j(p_0)|] * |\Delta p| \Rightarrow \frac{|R|}{|\Delta p|} \le \sum_{j=1}^n |f_j(p_j) - f_j(p_0)|$$

By continuity of $f_j, \exists\delta_j > 0, |p_j - p_0| < \delta_j \to |f_j(p_j) - f_j(p_0)| < \epsilon/n$. Taking the minimum δ_j gives us our neighborhood $V_{\delta_j}(p_0) = B(p_0,\delta_j)$, ensuring the sum $< \epsilon$, thus $\frac{|R|}{|\Delta p|} < \epsilon$.

Problem 84. Let S be an open set, $f : S \subset R^n \to R^1$ be a function differentiable at $p_0 \in S$, and has a local max/min at p_0. Show $Df(p_0) = 0$.

Proof.

f has a max at p_0. Since S is open and p_0 is a max, $\exists K = B(p_0, r) \subset S, r > 0, \forall p \in K, f(p) < f(p_0)$. Let $\beta \in R^n, 0 < t < r/|\beta| \Rightarrow |t\beta| < r \Rightarrow |(p_0 + t\beta) - p_0| < r \Rightarrow (p_0 - t\beta) \in K$. Let $|\beta| = 1$.

$$f(p_0-t\beta)-f(p_0) < 0 \Rightarrow \frac{f(p_0 - t\beta) - f(p_0)}{t} < 0 \Rightarrow \lim_{t\to 0^+}\frac{f(p_0 - t\beta) - f(p_0)}{t} \le 0$$

$(D_\beta f)(p_0)$ exists so:

$$\lim_{t\to 0}\frac{f(p_0 - t\beta) - f(p_0)}{t} \le 0$$

so $\forall\beta \in R^n, (D_\beta f)(p_0) \le 0$. From #76: $-(D_\beta f)(p_0) \le 0 \Rightarrow (D_\beta f)(p_0) \ge 0 \Rightarrow (D_\beta f)(p_0) = 0 \forall\beta \in R^n$.

Problem 87. Prove the best linear approximation theorem.

Proof.

Let $F(t) = f(x, y), x = g(t), y = h(t), p_0 = (x_0, y_0)$
$F'(t) = f_1(p)g'(t) + f_2(p)h'(t), F'(t_0) = f_1(p_0)g'(t_0) + f_2(p_0)h'(t_0)$
$R = f(p_0 + \Delta p) - f(p_0) - D_f(p_0) * \Delta p = o(\Delta p), p \to 0$

$$\lim_{\Delta p \to 0} \frac{f(p_0 + \Delta p) - f(p_0) - D_f(p_0) * \Delta p}{|\Delta p|} = r$$

$R = f(p_0 + \Delta p) - f(p_0) - f_1(p_0)\Delta x - f_2(p_0)\Delta y$

$$\frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t} = \frac{f_1(p_0)\Delta x - f_2(p_0)\Delta y + R}{\Delta t}$$

$$F'(t_0) = \lim_{\Delta t_0 \to 0} \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t} = f_1(p_0)\Delta x/\Delta t + f_2(p_0)\Delta y/\Delta t = f_1(p_0)g'(t_0) + f_2(p_0)h'(t_0)$$

Problem 89. Prove the best linear approximation property.

Proof.

$f : S \subset R^n \to R^1$. S is an open set, $p_0 \in S$ The total derivative is linear transformation $L : R^n \to R^1$ satisfying $R = f(p_0 + \Delta p) - f(p_0) - L(\Delta p) = O(\Delta p), \Delta \to 0$. In other words:

$$\lim_{\Delta p \to 0} \frac{|f(p_0 + \Delta p) - f(p_0) - L(\Delta p)|}{|\Delta p|} = 0$$

L is the total derivative or "differential", write $Lp_0 = Df(p_0) = \triangledown f(p_0)$.

Problem 90. Let $f \in C^{m+1}$ on an open interval I about $x = c$ and let $P_c(x)$ be the Taylor polynomial of degree n at c. Then $f(x) = P_c(x) + R_n(x)$, for any $x \in I$ where R_n is given by

$R_n(x) = \dfrac{1}{n!} \displaystyle\int_c^x f^{(n+1)}(t)(x-1)^n dt$, the integral form of the Taylor remainder.

Proof.

$g(t) = P_t(x) = f(t) + \frac{f'(t)(x-t)^1}{1!} + ... + \frac{f^n(t)(x-t)^n}{n!}$
Note: $g(c) = P_c(x), g(x) = f(x)$.
$R_n(x) = f(x) - P_c(x) = g(x) - g(c) = \int_c^x g'(t)dt$

$$g'(t) = f'(t) + \frac{(f''(t)(x-t) - f'(t)}{1!} + ... + \frac{(f^{n+1}(t)(x-t)^n - nf^n(t)(x-t)^{n-1}}{n!}$$

$$g(x) - g(c) = \int_c^x \frac{f^{n+1}(t)(x-t)^n}{n!} dt = R_n(x)$$

Problem 94. Suppose Ω is a convex open set in R^n and $\phi \in C^1(\Omega)$. Prove that ϕ does not depend on x_i if and only if $\partial\phi/\partial x_i = 0$ on Ω.

Proof.

$\phi(t) = (1-t)p_1 + tp_2, 0 \le t \le 1, \phi : [0,1] \to S \subset R^n$. Define $F(t) = f(\phi(t)) = f((1-t)p_1 + tp_0)$. $\phi(0) = p_1, \phi(1) = p_2$. $F(1) - F(0) = F'(c)(1-0) = F'(c)$. $F'(t) = Df(\phi(t)) * \phi'(t) = (f_1, ...f_n)(p_2 - p_1) = Df(p^t)(p_2 - p_1)$
$F(1) - F(0) = F'(c) = Df(p^t)(p_2 - p_1)$

Challenge Problem 16. Evaluate the integral

$$\int_0^1 \frac{ln(1+x)}{1+x^2} dx$$

Proof.

$$\int_0^1 \frac{ln(1+x)}{1+x^2} dx = \int_0^1 \frac{ln(1+ax)}{1+x^2} dx = I(a)$$

$$I'(a) = \int_0^1 \frac{x}{(1+ax)(1+x^2)} dx = \int_0^1 \frac{1}{(1+a^2)} \left[\frac{x+a}{1+x^2} - \frac{a}{1+ax} \right] dx$$

$$= \frac{1}{1+a^2} \left(\frac{ln2}{2} + \frac{a\pi}{4} - ln(1+a) \right)$$

By integrating $I'(a) \to I(a)$ we find

$$I(a) = \frac{ln2}{2} + atan(a) + ln(1+a^2) * \pi/8 - \int \frac{ln(1+a)}{1+a^2} da$$

$$I(a) = \frac{ln2}{2} + atan(a) + ln(1+a^2) * \pi/8 - \int_0^a \frac{ln(1+x)}{1+x^2} dx$$

Let a=1.

$$I(1) = \frac{\pi ln2}{8}$$

4 Riemann Integration

Problem 124. Let $f(x,y)$ be a continuous function on $[a,b]x[c,d]$. $\partial f/\partial x$ exists and is continuous on $[a,b]x[c,d]$. $F(x) = \int_c^d f(x,y)dy$, show $F'(x)$ exists and

$$F'(x) = \frac{d}{dx}\int_c^d f(x,y)dy = \int_c^d \frac{\partial f}{\partial x}dy$$

Proof.

Claim: $F(x)$ is continuous. $x_0 \in [a,b]$.

$$F(x) - F(x_0) = \int_c^d f(x,y) - f(x_0,y)dy \to |F(x) - F(x_0)| \leq \int_c^d |f(x,y) - f(x_0,y)|dy$$

Note: $|(x,y) - (x_0,y)| = |(x-x_0,0)| = |x-x_0|$. Since $f(x,y)$ is uniformly continuous, $\forall \epsilon' > 0, \epsilon = \epsilon'(d-c) > 0, \exists \delta, |x-x_0| < \delta, |f(x,y) - f(x_0,y)| < \epsilon'$.
Recall:

$$|F(x) - F(x_0)| \leq \int_c^d |f(x,y) - f(x_0,y)|dy < \int_c^d \epsilon' dy = \epsilon'(d-c) = \epsilon$$

Thus $\forall \epsilon > 0, \exists \delta, |x-x_0| < \delta, |F(x) - F(x_0)| < \epsilon$. $F(x)$ is continuous.
Claim: $F'(x) = \int_c^d \frac{\partial f}{\partial x}dy$. Let $\phi(x) = \int_c^d \frac{\partial f}{\partial x}dy, \psi(x_0) = \int_a^{x_0} \phi(x)dx = \int_a^{x_0}\int_c^d \frac{\partial f}{\partial x}dydx$
By Fubini's Theorem:

$$\int_a^{x_0}\int_c^d \frac{\partial f}{\partial x}dydx = \int_c^d\int_a^{x_0} \frac{\partial f}{\partial x}dxdy \to f(x,y)|_a^{x_0} = f(x_0,y) - f(a,y)$$

$$\psi(x_0) = \int_c^d f(x_0,y) - f(a,y)dy = F(x_0) - F(a) \text{ by FTC}$$

Notice that $F(a)$ is a constant.

$$F(x_0) = \psi(x_0) + F(a) \to F'(x_0) = \psi'(x_0) = \phi(x_0) = \int_c^d \frac{\partial f}{\partial x}dydx$$

Problem 131. Prove that

$$\int_1^\infty \frac{sin(x)}{x}dx$$

is conditionally convergent but not absolutely convergent

Proof.

$$\int_1^\infty \frac{sin(x)}{x}dx = -\frac{cos(x)}{x}\bigg|_1^\infty - \int_1^\infty \frac{cos(x)}{x^2}dx = 1 - \int_1^\infty \frac{cos(x)}{x^2}dx$$

Compare this to

$$\int_1^\infty \frac{1}{x^2}dx = 1$$

$$\int_0^{2\pi N}\left|\frac{sin(x)}{x}\right|dx = \sum_{n=1}^{N-1}\int_{2\pi n}^{2\pi n+1}\left|\frac{sin(x)}{x}\right|dx \geq \sum_{n=1}^{N-1}\frac{1}{2\pi(n+1)}\int_{2\pi n}^{2\pi(n+1)}|sin(x)|dx$$

5 Sequences and Series

Problem 146. Prove this discrete analogue of integration by parts, called the Abel partial summation formula or summation by parts. Let $\{a_n\}, \{b_n\}$ be sequences. Then

$$\sum_{k=1}^{n} a_k b_k = a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k$$

where $B_k = \sum_{k=1}^{n} b_k$

Proof.

Suppose $\{a_n\}, \{b_n\}$ be finite sequences with $B_k = \sum_{k=1}^{n} b_k, n \in \mathbb{N}$. Consider $\sum_{k=1}^{n} a_k b_k$ and it's expansion. $\sum_{k=1}^{n} a_k b_k = a_1 b_1 + ... + a_n b_n$. By def of B_k, write $b_k = B_k - B_{k-1}$. $b_1 = B_1, b_n = B_n - B_{n-1}$ in general. Now rewrite $\sum_{k=1}^{n} a_k b_k$ in terms of B_k.

$$\sum_{k=1}^{n} a_k b_k = a_1 B_1 + a_2 (B_2 - B_1) + ... + a_n (B_n - B_{n-1}) = B_1 (a_1 - a_2) + ... + B_{n-1} (a_{n-1} - a_n) + a_n B_n$$

$$= \sum_{k=1}^{n-1} [B_k (a_k - a_k + 1)] + a_n B_n = -\sum_{k=1}^{n-1} [B_k (a_k + 1 - a_k)] + a_n B_n = a_n B_n - \sum_{k=1}^{n-1} [B_k (a_k + 1 - a_k)]$$