PHYS417HW5

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May 2025

1)

2)

If we're given some vector potential $\vec{A}$ we can show that the resulting magnetic field $\vec{B}$ is gauge invariant; $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}'$.

We are given the two potentials

$$\vec{A} = \frac{B_0}{2}\left(-y\vec{e}_x + x\vec{e}_y\right) \quad \text{and}$$

$$\vec{A}' = B_0 y\vec{e}_x$$

To find $\vec{B}$ we simply take the curl of $\vec{A}$ and $\vec{A}'$.

$$\vec{\nabla} \times \vec{A} = \frac{B_0}{2}\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix}$$

$$= \frac{B_0}{2}\left(\vec{e}_z + \vec{e}_z\right)$$
$$= B_0\vec{e}_z$$

and for $\vec{A}'$

$$\vec{\nabla} \times \vec{A} = B_0\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & x & 0 \end{vmatrix}$$

$$= B_0\left(\vec{e}_z\right)$$
$$= B_0\vec{e}_z$$

b)

More generally, we can say that $\vec{A'} = \vec{A} + \nabla\Lambda$ for some function $\Lambda(\vec{r})$. This will only be true if $\Lambda(\vec{r})$ is curless, or, in other words, $\vec{\nabla} \times \vec{\nabla}\Lambda = 0$.

$$\vec{\nabla} \times \vec{\nabla}\Lambda = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_y & \partial_z \\ \partial_x\Lambda & \partial_y\Lambda & \partial_z\Lambda \end{vmatrix}$$

$$= \vec{e}_x \left(\partial_y\partial_z\Lambda - \partial_z\partial_y\Lambda\right) - \vec{e}_y \left(\partial_x\partial_z\Lambda - \partial_z\partial_x\Lambda\right) + \vec{e}_z \left(\partial_x\partial_y\Lambda - \partial_y\partial_x\Lambda\right)$$

$$= 0$$

Thus we can pick essentially any $\Lambda(\vec{r})$ we want, so long as it is possible to take the gradient of it.

c)

The addition of a $\vec{\nabla}\Lambda(\vec{r})$ term corresponds to the addition of a $e^{i\phi(\vec{r})}$ term to $|\psi\rangle$. I'm going to make a dimensionality based argument for the relation. $\phi(\vec{r})$ must be a unitless quantity. $\vec{A}$ has units of $\frac{\text{momentum}}{\text{charge}}$. Consequently Λ has units of $\frac{\text{length}\times\text{momentum}}{\text{charge}}$. $p \times \ell = \frac{E}{s}$, which conveniently has the same units as $\hbar$. To that end the relationship between our choice of gauge and our state's phase is;

$$\phi = \frac{q}{\hbar}\Lambda$$

The book states that

$$g(\vec{r}) = \frac{q}{\hbar}\int_{\mathcal{O}}^{r} d\vec{r'} \cdot \vec{A}(\vec{r'})$$

Which, given that $\vec{A}(\vec{r}) = \vec{A'}(\vec{r}) + \nabla\Lambda(\vec{r})$, the vector potential term will drop out and we'l only be left with

$$g(\vec{r}) = \frac{q}{\hbar}\int_{\mathcal{O}}^{r} d\vec{r'} \cdot \nabla\Lambda(\vec{r'}) \quad \text{Which by the fundamental theorem for line integrals}$$

$$= \frac{q}{\hbar}\Lambda(\vec{r})$$

d)

The difference between the cannonical momentum $\langle\hat{p}\rangle = \langle ih\hat{\nabla}\rangle$ and the particle momentum $m\frac{d\langle r\rangle}{dt}$ will be that $\langle\hat{p}\rangle$ will pull down a derivative of $\Lambda(\vec{r})$ from the exponent, making it so that it *will* depend on the gauge, or more accurately, a derivative of the gauge. $\langle r \rangle$ on the other hand will not act on the phase term at all, allowing them to cancel out as we would expect $|e^{i\phi}|^2 = 1$. In this way the classical momentum will be gauge invariant.

2)

Preparing a state $|\Psi\rangle = \frac{1}{\sqrt{3}}[|A\rangle + |B\rangle + |C\rangle]$ to determine which box $(A, B, \text{ or}, C)$ a particle will end up in after the following measurements:

$$\hat{A} = |A\rangle\langle A| = |B\rangle\langle B| + |C\rangle\langle C| \quad \text{or}$$

$$\hat{B} = |B\rangle\langle B| = |A\rangle\langle A| + |C\rangle\langle C|$$

After this I perform someas yet undetermined measurement that projects us into the state

$$|\Phi\rangle = \frac{1}{\sqrt{3}}[|A\rangle + |B\rangle - |C\rangle]$$

The fact that we end up in this state allows us to figure out which measurement you did.

a)

There are 4 potential outcomes from your measurement, if you perform the first measurement, you could either end up in the state $|\psi\rangle = |A\rangle$ or $|\psi\rangle = \frac{1}{\sqrt{2}}[|B\rangle + |C\rangle]$. Similarly, if you performed the second measurement you could end up in the state $|\psi\rangle = |B\rangle$ or $|\psi\rangle = \frac{1}{\sqrt{2}}[|A\rangle + |C\rangle]$.

We then perform another measurement, which projects us into $|\Phi\rangle = \frac{1}{\sqrt{3}}[|A\rangle + |B\rangle - |C\rangle]$.

The overlap of our new state with the state that resulted from your measurement $|\psi\rangle_{prev}$ will be a coefficient that describes the probability that that specific transition happened.

Starting with the case where you measured box A

$$\langle\Phi|\psi_{prev}\rangle = \frac{1}{\sqrt{3}}\left(\langle A|A\rangle + \langle B|0\rangle - \langle C|0\rangle\right) = \frac{1}{\sqrt{6}} \quad \text{or}$$

$$= \frac{1}{\sqrt{6}}\left(\langle A|0\rangle + \langle B|B\rangle - \langle C|C\rangle\right) = 0$$

! So if you measured A and we were able to get into state $|\Phi\rangle$ then the result of your measurement must have been that the particle was in box A.

Similar logic holds if you measure $|B\rangle\langle B|$.

$$\langle\Phi|\psi_{prev}\rangle = \frac{1}{\sqrt{3}}\left(\langle A|0\rangle + \langle B|B\rangle - \langle C|0\rangle\right) = \frac{1}{\sqrt{6}} \quad \text{or}$$

$$= \frac{1}{\sqrt{6}}\left(\langle A|A\rangle + \langle B|0\rangle - \langle C|C\rangle\right) = 0$$

So then, we know what outcome was obtained if we know what measurement you did. If you measured $|A\rangle\langle A|$ you found the particle in that box, and if you measured $|B\rangle\langle B|$ you found the particle in *that* box.

b)

This is kind of a weird result. In a classical world the particle would simply *be* in one of the boxes, and we wouldn't be able to do this sort of manipulation. We're doing a measurement that tells us something about the outcome of a previous measurement. The idea that you had to have found the particle in whichever box you looked in has no real classical analogue.

3)

Weird stuff with a spin-$\frac{3}{2}$ particle.

a)

Measuring S_z^2 will result in a measurement of the "length" of spin in the z direction. We did something similar to this for the spin-$\frac{1}{2}$ case last term.

A measurement of S_z will return $\pm\frac{3}{2}, \pm\frac{1}{2}$ and so a measurement of S_z^2 will return $\frac{9}{4}, \frac{1}{4}$.

If we measured S_x^2 or S_y^2 we could return the same values, as those operators will have the same eigenvalues.

b)

Classically, the sum of all three would be $\frac{27}{4}, \frac{19}{4}, \frac{11}{4}$, or $\frac{3}{4}$, assuming that each direction returned one of its two possible eigenvalues.

c)

But of course, S^2 can only return $s(s+1)$, or, in our case $\frac{3}{2}(\frac{3}{2}+1) = \frac{15}{4}$, which is different than any of the values we found in the classical case.