# PHYS 417 HW 2

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## 1)

For the hyperfine line in Hydrogen, $A \approx 2.9 \times 10^{-15}$ s$^{-1}$.

### a)

This is not allowed via dipole transition it would need to obey the transition rules.

$$\Delta\ell \equiv \ell' - \ell = \pm 1$$

$$\Delta m \equiv m' - m = 0 \text{ or } \pm 1$$

The hyperfine line comes from a transition from $|n\ell m\rangle$ to $|n'\ell' m'\rangle$ Under these rules, the hyperfine splitting transition is not allowed because $|n'\ell' m'\rangle = |n\ell m\rangle$, it's just that the total spin of the configuration is different.

Additionally,

### b)

If we were considering a dipole transition with $|p| = ea_0$ we could note that $\lambda = 2.1 \times 10^{-1}$ m corresponds to $E = \frac{\hbar c}{\lambda} = \hbar\omega_0$ so $\omega_0 = \frac{c}{\lambda}$.

From Griffiths 11.63

$$A' = \frac{\omega_0^3 |p|^2}{3\pi\epsilon_0 \hbar c^3}$$

$$= \frac{|p|^2}{3\pi\epsilon_0 \hbar \lambda^3}$$

$$= \frac{e^2 a_0^2}{3\pi\epsilon_0 \hbar \lambda^3}$$

Which we know all the values of, so we can calculate $A'$.

$$A' = 1.4 \times 10^{-13} \times (2\pi)^3$$

$$= 2.18 \times 10^{-10}\ \mathrm{s}^{-1}$$

Which is several orders of magnitude faster than the actual transition rate.

### c)

We can now approximate the actual decay rate by noting that this will be due to a magnetic dipole, using the substitution $p = ea_0 = \frac{\mu_B}{c}$

The ratio $\frac{\mu_B}{cea_0}$ is equal to $\approx 0.037$. Which means that if we multiply by that ratio squared;

$A'' = A'(0.0036)^2 = 2.9 \times 10^{-15}\ \mathrm{s}^{-1}$ Which is exactly what we want.

## 2)

### a)

Starting with the typical 1D QHO Hamiltonian $\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$ and try to calculate the expectation value of a coherent state $|\alpha\rangle$ we calculate

$$E_\alpha = \langle\alpha|H|\alpha\rangle = \langle\alpha|\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|\alpha\rangle$$

We remember that the coherent states can be expanded in terms of the number states $|n\rangle$

$$|\alpha\rangle = \sum_n^\infty a_n|n\rangle$$

Which, when subbed in yields

$$E_\alpha = \hbar\omega\sum_n^\infty\langle a_n^\star n|\left(a^\dagger a + \frac{1}{2}\right)|a_n n\rangle$$

$$= \hbar\omega\sum_n^\infty |a_n|^2\left(\langle n|a^\dagger a|n\rangle + \frac{1}{2}\langle n|n\rangle\right)$$

$$= \hbar\omega\sum_n^\infty |a_n|^2\left(n + \frac{1}{2}\right)$$

## b)

The variance is defined as $\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2$. We already know $\langle H \rangle$, so $\langle H \rangle^2$ is trivial. We must now calculate $\langle H^2 \rangle$

$$\hat{H}^2 = \hbar^2 \omega^2 \left( a^\dagger a a^\dagger a + a^\dagger a + \frac{1}{4} \right)$$

So this means that $\langle \hat{H}^2 \rangle$ is

$$\langle \hat{H}^2 \rangle = \hbar^2 \omega^2 \sum_n^\infty |a_n|^2 \langle n| \left( a^\dagger a a^\dagger a + a^\dagger a + \frac{1}{4} \right) |n\rangle$$

$$= \hbar^2 \omega^2 \sum_n^\infty |a_n|^2 \left( \langle n|n^2|n\rangle + \langle n|n|n\rangle + \frac{1}{4} \langle n|n\rangle \right)$$

$$= \hbar^2 \omega^2 \sum_n^\infty |a_n|^2 \left( n + \frac{1}{2} \right)^2$$

And, quickly calculating $\langle H \rangle^2$

$$\langle H \rangle^2 = \hbar^2 \omega^2 \sum_n |a_n|^4 \left( n + \frac{1}{2} \right)^2$$

So $\sigma^2$ is

$$\sigma^2 = \hbar^2 \omega^2 \sum_n^\infty |a_n|^2 \left( n + \frac{1}{2} \right)^2 - \hbar^2 \omega^2 \sum_n |a_n|^4 \left( n + \frac{1}{2} \right)^2$$

$$= \sum_n \left( 1 - |a_n|^2 \right)$$

Which actually makes some amount of sense, if we are only allowed 1 state, there is no variance.

# 3)

If we try to construct a state $|\gamma\rangle$ such that

$$a^\dagger |\gamma\rangle = \gamma |\gamma\rangle$$

We can start out with the expansion

$$|\gamma\rangle = \sum_{n}^{\infty} c_n |n\rangle$$

and sub

$$a^{\dagger} \sum_{n}^{\infty} c_n |n\rangle = \gamma \sum_{n}^{\infty} c_n |n\rangle$$

$$\sum_{n}^{\infty} c_n a^{\dagger} |n\rangle = \gamma \sum_{n}^{\infty} c_n |n\rangle$$

$$\sum_{n}^{\infty} c_n \sqrt{n+1} |n+1\rangle = \gamma \sum_{n}^{\infty} c_n |n\rangle$$

Which will clearly never be equal

## 4)

We have a 50/50 beam splitter as set up in the lecture, two inputs, $1, 2$ and two outputs $3, 4$. We are interested in the ratio

$$G = \frac{N_{3,4}}{N_3 N_4}$$

With $N_k$ and $N_{3,4}$ being defined as

$$N_k = \sum_{n_k} P(n_k) n_k$$

$$N_{3,4} = \sum_{n_3, n_4} P(n_3, n_4) n_3 n_4$$

From class, we know that this is dependant on Poisonnian statistics, with $P(n_k) = \left(\frac{|\alpha|^2}{2n_k!}^{n_k}\right) \times \exp\left(\frac{-|\alpha|^2}{2}\right)$. $P(n_3, n_4)$ is $P(n_3)P(n_4)$.

### a)

If we send a coherent state $|\alpha\rangle$ into input 1 and $|0\rangle$ into input 2, $G$ will be a function of the amplitudes $|\alpha\rangle_3$ and $|\alpha\rangle_4$. These are;

$$\alpha_3 = \frac{\alpha_1 + i\alpha_2}{\sqrt{2}}$$

$$\alpha_4 = \frac{\alpha_2 + i\alpha_1}{\sqrt{2}}$$

So, with $\alpha_1 = \alpha$ and $\alpha_2 = 0$

$$|\alpha_3|^2 = |\alpha_4|^2 = \frac{\alpha}{2}$$

But aha, the Poissonian statistics as play here, we can point out $P(n_3, n_4) = P(n_3)P(n_4)$, and so $G$ will simply be 1.

$$\begin{aligned} G &= \frac{N_{3,4}}{N_3 N_4} \\ &= \frac{P(n_3, n_4)n_3 n_4}{P(n_3)n_3 P(n_4)n_4} \\ &= \frac{P(n_3)n_3 P(n_4)n_4}{P(n_3)n_3 P(n_4)n_4} \\ &= 1 \end{aligned}$$

## b)

If we send $|\alpha\rangle$ into input 1 and $|\beta\rangle$ into input 2, $G$ will still be 1. We can make the same argument.

$$\begin{aligned} G &= \frac{N_{3,4}}{N_3 N_4} \\ &= \frac{P(n_3, n_4)n_3 n_4}{P(n_3)n_3 P(n_4)n_4} \\ &= \frac{P(n_3)n_3 P(n_4)n_4}{P(n_3)n_3 P(n_4)n_4} \\ &= 1 \end{aligned}$$

Even though $\alpha_3$ and $\alpha_4$ have an additional term in them, everything will still cancel out.

## c)

We now send the *number* states $|n\rangle$ and $|0\rangle$ into inputs 1 and 2. Our argument is no longer true, and the input state will be ;

$$|\psi\rangle_{in} = \frac{(a_1^\dagger)^n}{\sqrt{n!}}|n\rangle_1|0\rangle_2$$

so $|\psi\rangle_{out}$ is

$$|\psi\rangle_{out} =$$