Cartpole Dynamics Research Using Lagrangian Mechanics

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September 2021

1 Simple Pendulum Scenario

1.1 Mathematical Derivation of Pendulum Motion Using the Lagrangian

Founded by and named after scientist Joseph-Louis Lagrange, Lagrangian Mechanics is a systematic method to analyze a system of motion in three dimensions through energy consideration using calculus, as opposed to the traditional study using Newton's Laws. One of the most common applications of Lagrangian Mechanics is on pendulum motion.

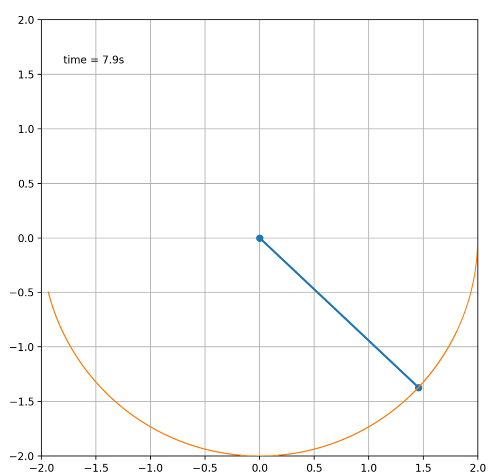


Figure 1

Consider a simple pendulum with mass, m, hanging from a string of length L at an angle θ away from the vertical hanging position of the pendulum. From this scenario, we can lay out the energy system of the pendulum, consisting of kinetic and potential energy, represented as T and U respectively:

$$T = \frac{1}{2}\mathrm{m}v^2$$
$$U = \text{-mgh}$$
(h represents perp. dist. between current and lowest position)

These equations can be further expanded through consideration of the motion of the pendulum in both the x and y directions:

$$x = \mathrm{Lcos}(\theta)\ y = \mathrm{Lsin}(\theta)$$
$$v_x = -Lsin(\theta)\dot{\theta}$$
$$v_y = Lcos(\theta)\dot{\theta}$$
$$v^2 = v_x^2 + v_y^2 = L^2 sin^2(\theta)\dot{\theta} + L^2 cos^2(\theta)\dot{\theta} = L^2\dot{\theta}^2$$
$$T = \frac{1}{2}\mathrm{m}L^2\dot{\theta}^2$$
$$U = \text{-mgLcos}(\theta)$$

Now that we have solved the energy equations, we can put these together to utilize the Lagrangian, L, characterized by the formula L = T - U. This is shown below:

$$L = T - U$$
$$L = \frac{1}{2}\mathrm{m}L^2\dot{\theta}^2 + \mathrm{mgLcos}(\theta)$$

From the Euler-Lagrange equation, shown below, we can then determine the equation we are looking for through this derivation:

$$\frac{d}{dt}\frac{dL}{d\dot{q}_i} - \frac{dL}{dq_i} = 0$$
$$\mathrm{m}L^2\ddot{\theta} = \text{-mgLsin}(\theta)$$
$$\mathrm{m}L^2\ddot{\theta} + mgLsin(\theta) = 0$$
$$\mathrm{L}\ddot{\theta} + \mathrm{gsin}(\theta) = 0$$
$$\ddot{\theta} = -\frac{g}{L}\ \sin(\theta)$$

From this, we learn that the angular acceleration of the pendulum is directly proportional to the sine of the theta, which is the angle away from the equilibrium point. And we are able to deduce the equation for the motion of a simple pendulum using the Lagrangian and the Euler-Lagrange equation to be: $\boldsymbol{\ddot{\theta} = -\frac{g}{L}\ \sin(\theta)}$

1.2 Locating Pendulum Equilibria

Assume a simple pendulum as in Figure 2 below with damping and taking position 0 as the upright (vertical) hanging position. Let's assume that the right and left are the positive and negative x-axes respectively and that the initial position of the pendulum starts on the positive end as shown below.

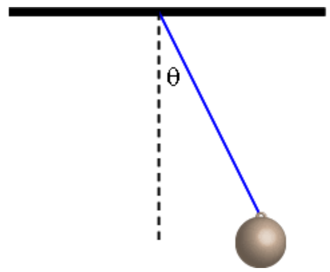
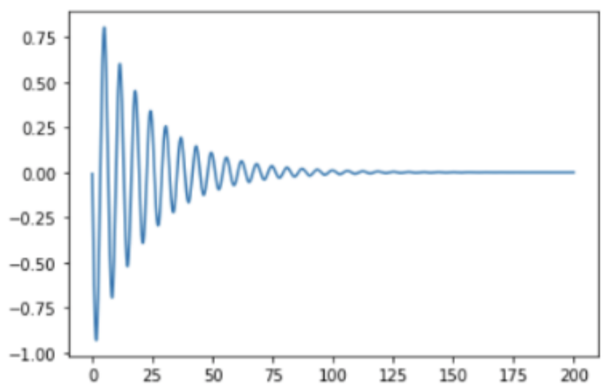


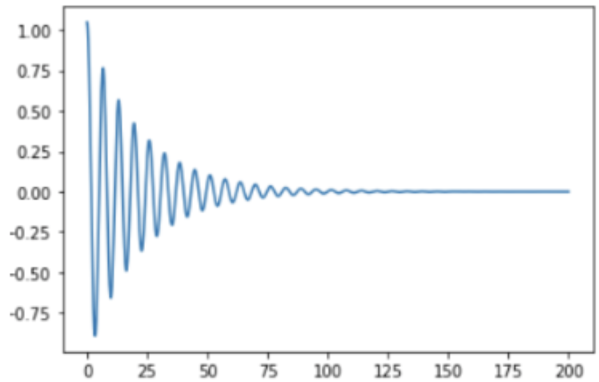
Figure 2

When a pendulum is in equilibrium at a position, the hanging mass is considered to have a velocity of 0 at that specific point. Taking a look at the velocity versus time graph of a simple pendulum below, we can see that there exist several x-intercept points which represent a velocity of 0.



Simulation Result: $\dot{\theta}$ vs Time of a Simple Pendulum with added Damping Proportional to $\dot{\theta}$

Each of these points represent times in which the pendulum is in equilibrium. Furthermore, if we look at the position versus time graph with damping, we see that the points of equilibrium in the velocity graph line up in time with the highest position in the position versus time graph with damping. This means that the positions of equilibrium occur at the highest point of the pendulum.



Simulation Result: θ vs Time of a Simple Pendulum with added Damping Proportional to $\dot{\theta}$

However, in addition to those points mentioned above, there exists another equilibrium point at position 0. This is because as time approaches infinity, the pendulum will reach the vertical position and a velocity of 0, due to the damping that is present in the scenario. Therefore, the **equilibrium positions of a simple pendulum are at the top position and the hanging vertical position, a total of 2 points.**

1.3 Pendulum Stability

Even though we determined points of equilibrium of a damped pendulum, we have yet to determine which equilibrium points are stable and unstable equilibrium points. If we revisit the velocity versus time graph of the damped simple pendulum, we can see that there are oscillations within the graph. An unstable equilibrium is different from a stable equilibrium point because of their distinctions in acceleration at the very instant that the pendulum mass is at the equilibrium position. A stable equilibrium position is where both the velocity and acceleration of the mass is 0, whereas an unstable equilibrium satisfies the former but not the latter. Therefore, **a simple pendulum, when at the top, represents an unstable equilibrium point, as shown in Figure 3.**

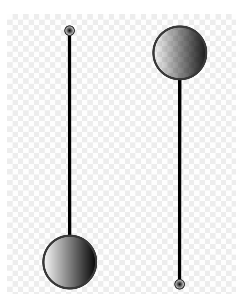


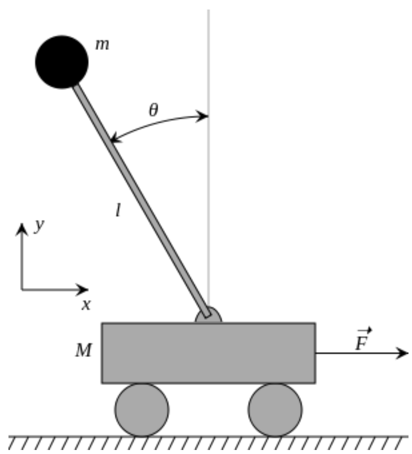
Figure 3

On the other hand, the equilibrium point at position 0 is a stable equilibrium point because it remains at the hanging position when time reaches infinity. This leads to its resting position at the equilibrium position, meaning that velocity and acceleration of the mass are both 0. For this reason, the **equilibrium at position 0 (hanging upright position) is a stable equilibrium point.**

2 Cartpole Scenario

2.1 Mathematical Derivation of Cartpole Motion Using the Lagrangian

Now that we have gone over the pendulum scenario, the cartpole scenario is relatively similar, allowing us to apply concepts from the previous scenario. To do this, let's assume a cartpole, consisting of an inverted pendulum attached to a cart, as shown below. We will take theta as the angle between the pendulum and its equilibrium position at the top, and will let L represent the length of the pendulum.



Cartpole

When taking into consideration both the kinetic and potential energy of the system, we have:

$$K = \frac{1}{2}(\mathrm{m+M})v^2 + \frac{1}{2}\mathrm{m}[(v + L\omega\cos(\theta))^2 + (\mathrm{L}\omega\sin(\theta))^2]$$
$$\mathrm{U = -mgh}$$

where m is the mass of the cart, v is the linear velocity of the cart, M is the mass of the pendulum mass, ω is the rotational velocity of the mass, and h is the height of the pendulum above the cart. However, the equations above which model energy in the system can be further simplified:

$$K = \frac{1}{2}(m+M)v^2 + \frac{1}{2}mL^2\omega^2 + mLv\omega\cos(\theta)$$
$$U = -mgL\cos(\theta)$$

From Lagrangian Mechanics, we know that the Lagrangian, L, is equal to the K - U, where K and U represent the kinetic and potential energy of the system respectively.

$$K - U = \frac{1}{2}(m+M)v^2 + \frac{1}{2}mL^2\omega^2 + mLv\omega\cos(\theta) + mgL\cos(\theta)$$

To derive the equations of motion for the cartpole, we can use the Euler-Lagrange formula.

$$\frac{d}{dt}\frac{dL}{d\dot{q}_i} - \frac{dL}{dq_i} = 0$$
$$(m+M)a + mL\alpha\cos(\theta) - mL\omega^2\sin(\theta) = F$$
and
$$mLa\cos(\theta) + mL^2\alpha + mgL\sin(\theta) = 0$$

After simplification and solving for angular and linear acceleration, we get:

$$a = [F + m\sin(\theta)(L\omega^2 + g\cos(\theta))]\frac{}{M+msin(\theta)*sin(\theta)}$$
$$\alpha = [-F\cos(\theta) - mL\omega^2\cos(\theta)sin(\theta) - (m+M)gsin(\theta)]\frac{}{L(M+msin(\theta)*sin(\theta))}$$

Now that we have derived the linear and rotational acceleration equations of the cartpole system in terms of θ, the equations can be implemented to simulate the dynamics of a cartpole.

2.2 Full Animation of a Cartpole

To see the full animation, access this link: https://github.com/aidanlee09/Cartpole-Simulation

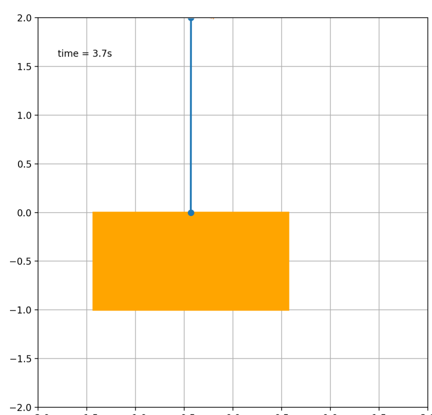


Image of Cartpole Simulation (”Cart” - Orange Rectangle; ”Pole” - Blue Line)