$\mathbb{P}(T=3) = [1 - \mathbb{P}(T=2)]\,\mathbb{P}(\text{all grandchildren die})$

$\mathbb{P}(\text{all grandchildren die}) = \sum \mathbb{P}(\text{all grandchildren die} \mid \text{children are set } S)\,\mathbb{P}(\text{children are set } S)$

I need to start over

---

We need a good notation for the tree structure induced by messages.

Let $M$ be the tree. It has levels $1, 2, 3, \ldots$ denoted $M_1, M_2, M_3, \ldots$

ex:

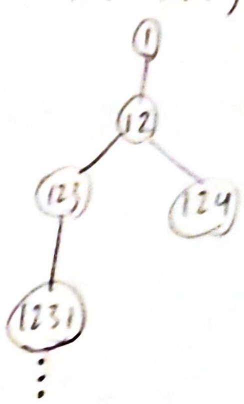
G: 

$L=0$

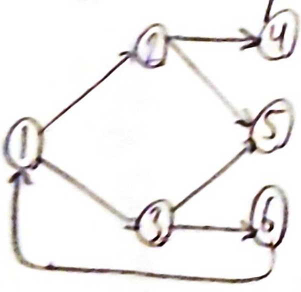
$M_1 = \{1\}$
$M_2 = \{2\}$
$M_3 = \{3, 4\}$
$M_4 = \{1\}$

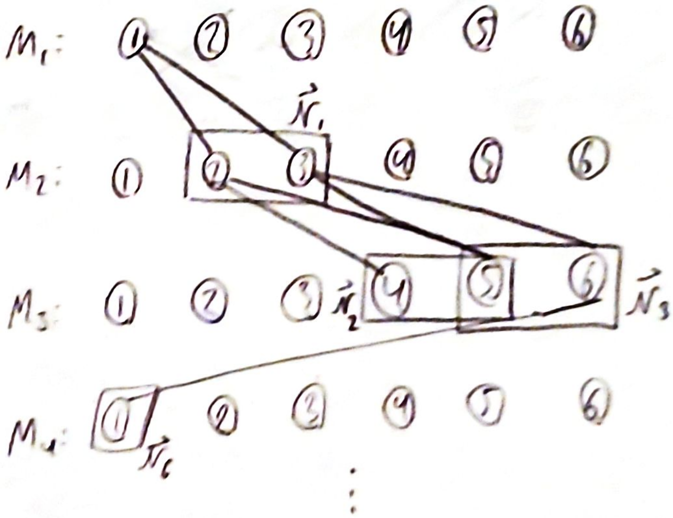
Simple labeling

Full labeling



Consider this example

G: 




$L=0$

$M_1 = \{1\}$

Observe that if a message occupies two nodes $i_1$ and $i_2$ that share a common downstream neighbor $j$, then $j$ will be OFF in the next state.

It may be helpful to have a notation for the downstream neighbors ~~with a unique~~ of $i$ whose only upstream neighbor is $i$: let

$$\boxed{P_i = \vec{N}_i - \bigcup_{\substack{j=1 \\ j \neq i}}^{n} \vec{N}_j}$$

denote this set of nodes.

---

Okay, let $M$ be a message injected at node $i$ (meaning $M_1 = \{i\}$), and let $T$ be the extinction time of $M$ (meaning the index of the deepest level of $M$).

~~$T=1$ iff an element of $\overleftarrow{N}_i$ was active in the previous~~

For $T=1$, since only up to 1 message can be injected into a node ~~~~ ~~~~ on each time step, it must be that an element of $\overleftarrow{N}_i$ is active in $X_t$ (where $t$ is some time of injection $\geq 0$). Thus,

$$\boxed{\begin{aligned} P(T=1) &= \mathbb{P}(\text{some node in } \overleftarrow{N}_i \text{ is active}) \\ &\approx \sum \{\pi_x : \textstyle\sum_{k \in \overleftarrow{N}_i} x_k \geq 1\} \end{aligned}}$$

For $T=2$, note that if $L=n$, then $\mathbb{P}(T=2 \mid T>1)=1$. If $L<n$, then it's possible that $T>2$.

$$\mathbb{P}(T=2) \overset{?}{=} [1-\mathbb{P}(T=1)]\,\mathbb{P}(\text{every element of } M_2 \text{ is in a collision})$$

$$\text{event}(T=2) = \underbrace{\text{event}(T>1)}_{\text{event }1} \cap \underbrace{\text{event}(\text{every element of } M_2 \text{ is in a collision})}_{\text{event }2}$$

So the question is if the two events on the right hand side are independent.

$$\mathbb{P}(\text{event }1 \;\&\; \text{event }2) \overset{?}{=} \mathbb{P}(\text{event }1)\,\mathbb{P}(\text{event }2)$$

~~$\mathbb{P}(T=2 \mid T>1)$~~

$\mathbb{P}(T=2 \mid T\geq 2)$ hazard function (also instantaneous hazard function because domain is discrete)

If $T$ is given to $k \geq 2$, then $T=2$ if every element of $M_2$ is in a collision, and vice versa. So

$$\mathbb{P}(T=2 \mid T\geq 2) = \mathbb{P}(\text{every element of } M_2 \text{ is in a collision}).$$

I think. Tired. Sleep.