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Lab 2: Period of a Pendulum Continued

Part 1: Method

Objective

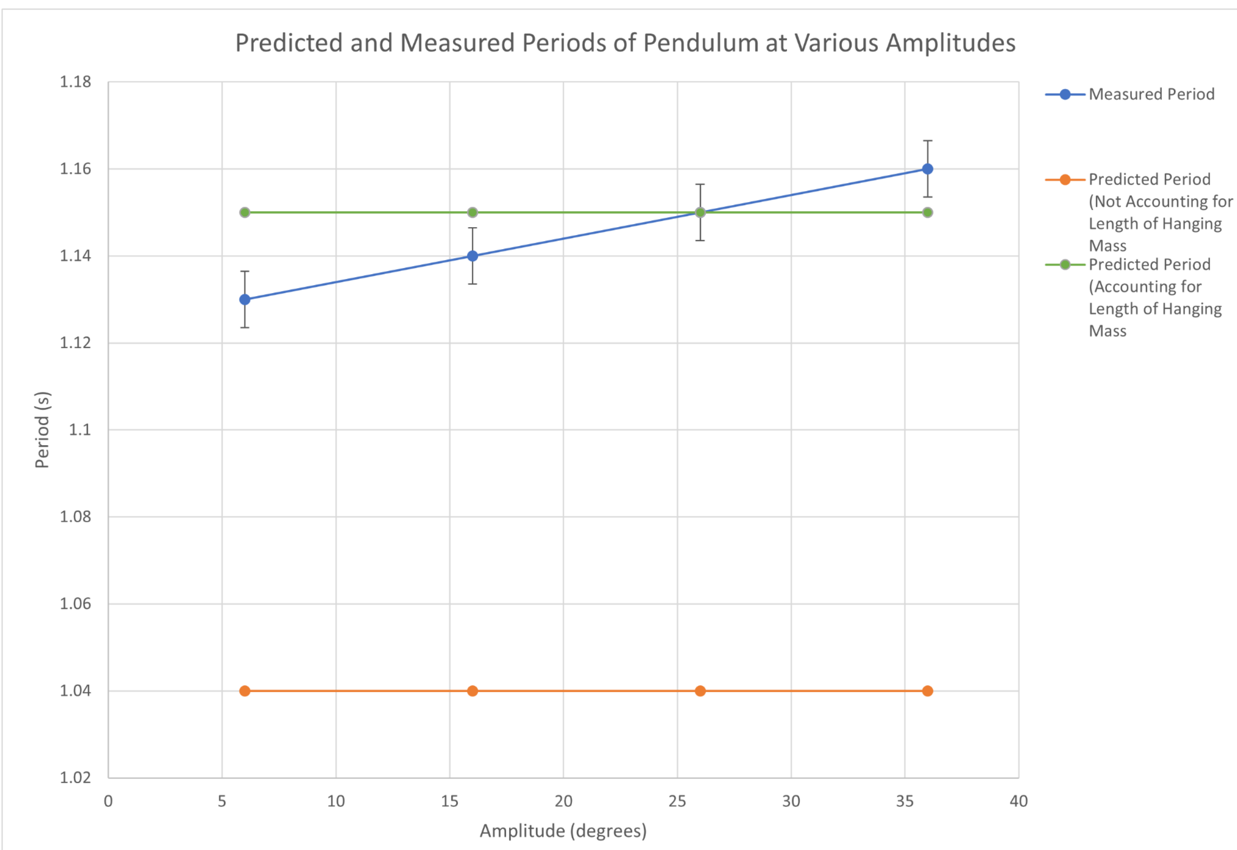
The objective of this experiment, as well as part 2, was to conduct further investigation of the model for the period of a pendulum introduced in lab 1:

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum in meters, and g is acceleration due to gravity in meters per second squared.

There is one result from lab 1 that we can correct at the outset. In that lab, we measured l as the length of the string alone, which came out to 27.10 +/- 0.05 cm. Upon further consideration, we realized that in the derivation of the model for T, it was likely assumed that the mass on the pendulum was a point mass; therefore, we took another measurement of the length, this time including the length of the mass hanging on the string. This length was 32.90 +/- 0.05 cm, and it gave a much better estimate for the period. See Figure 1.

Figure 1



Materials

The materials required for this lab were: one pendulum stand, one circular protractor, one string, one weight with a hook, one stopwatch, one PASCO photogate, one PASCO interface, and one computer to record the photogate output. We also used another computer for the analysis, although this could be carried out on the first computer.

The weight was stamped with a measurement of its mass, which was 100 g; no error was given with this measurement. The length of the pendulum (including the length of the mass and the string, see the commentary preceding Figure 1) was 32.90 +/- 0.05 cm.

Hypothesis

Many potential angles from which to investigate the model were suggested, one of which was to see if there was a set of amplitudes within which the model was accurate, and outside of which it was not. We chose to investigate this route.

We hypothesized that the model would be accurate for amplitudes less than 10 degrees, and that it would not be accurate for amplitudes greater than or equal to 10 degrees.

Method

This experiment required two people, one to ensure the correct amplitude was set, and another to position the mass such that the string was vertical.

Once the pendulum state was initialized, one person would hit record on the PASCO software, and the other person would release the pendulum. Six measurements of the period were taken, and then the pendulum was stopped. The first measurement was discarded, as we were instructed that it was most likely erroneous.

As in lab 1, the independent variable was the amplitude, the dependent variable was the period, and the control variables included: the length of the string, the mass of the hanging weight, and all of the equipment used.

Because the conclusions of parts 1 and 2 of lab 1 contradicted each other, we chose to keep the length of the string and the hanging mass the same, so that we could get a closer investigation of this exact situation.

Part 1: Data

Table 1: Data for Part 1

Amplitude (degrees)	Trial 1 (s)	Trial 2 (s)	Trial 3 (s)	Trial 4 (s)	Trial 5 (s)
5	1.12 +/- 0.01	0.69 +/- 0.01	0.51 +/- 0.01	1.05 +/- 0.01	1.12 +/- 0.01
8	1.12 +/- 0.01	0.63 +/- 0.01	1.08 +/- 0.01	0.54 +/- 0.01	1.12 +/- 0.01
10	1.12 +/- 0.01	1.13 +/- 0.01	1.12 +/- 0.01	1.12 +/- 0.01	1.12 +/- 0.01
20	1.13 +/- 0.01	1.13 +/- 0.01	1.13 +/- 0.01	1.13 +/- 0.01	1.13 +/- 0.01
30	1.14 +/- 0.01	1.14 +/- 0.01	1.14 +/- 0.01	1.14 +/- 0.01	1.14 +/- 0.01

Table 2: *t*-scores Between Measured Periods and Predicted Periods

Amplitude (degrees)	*t*-score	Interpretation
5	2.01	Inconclusive
8	0.36	Indistinguishable
10	2.80	Inconclusive
20	2.00	Inconclusive
30	1.00	Inconclusive*

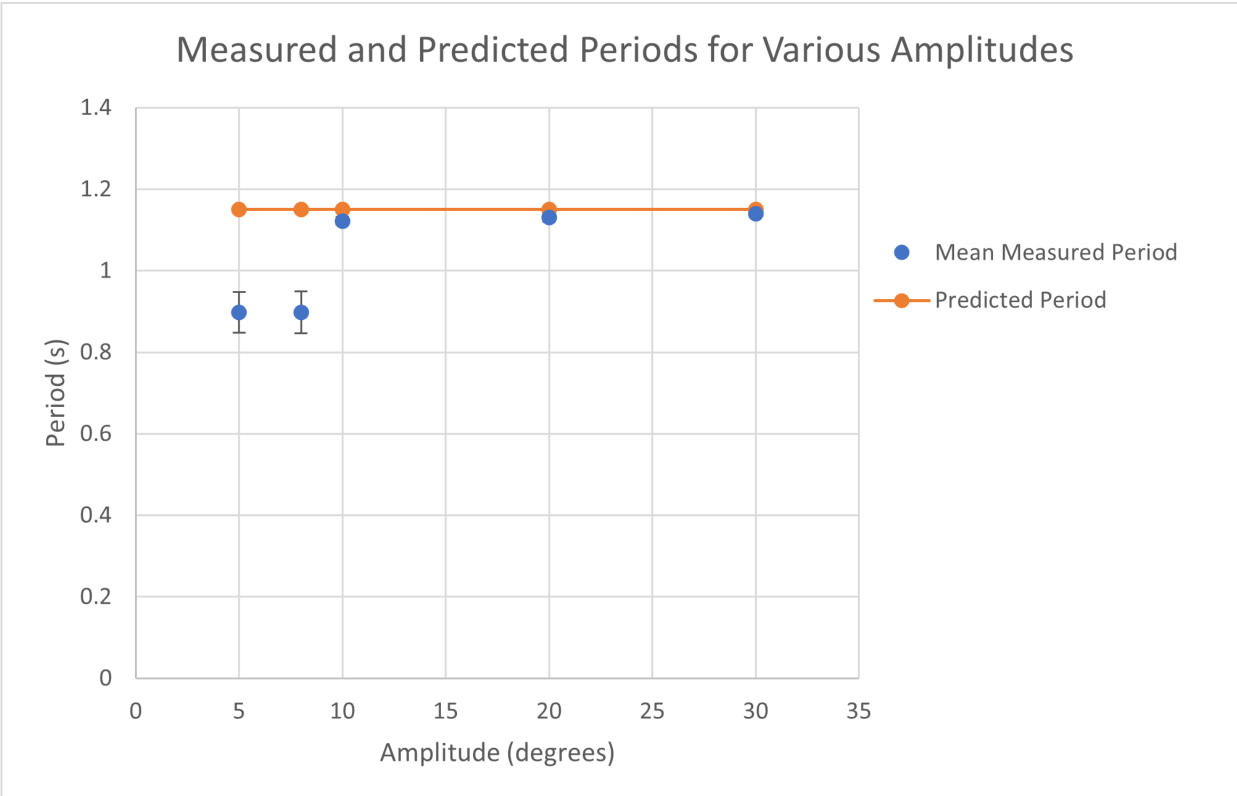
* A *t*-score of 1.00 is right between being inconclusive and indistinguishable. Since this is a border case, it is best to err on the side of caution and interpret this value as meaning the data was inconclusive.

Table 3: Mean Period for Each Amplitude (Part 1)

Amplitude (degrees)	Mean Period (s)
5	0.90 +/- 0.05
8	0.90 +/- 0.05
10	1.12 +/- 0.01
20	1.13 +/- 0.01
30	1.14 +/- 0.01

The errors in this table were calculated as the maximum of the systematic error (0.01 seconds) and the standard deviation of the mean. Calculations were carried out in Microsoft Excel.

Figure 2



<u>Analysis</u>

Because we used a program to calculate the *t*-scores in Table 2, **in lieu of a by-hand calculation of the *t*-score, we have just provided the code in Figure 2. The values plugged into this function were the trials in Table 1.** The if-else block in Figure 2 enforces that the model error is 0, the value we were instructed to use. Note that the model data was always passed as data2.

Figure 3: *t*-score Calculation Code

```
def stdev_of_mean(data):
    N = len(data)

    # Setting ddof = 1 ensures estimate for standard deviation is unbiased
    sigma = np.std(data, ddof=1)

    # Now we get the standard deviation of the mean
    delta = sigma/sqrt(N)

    return delta
```

```
def t_score(data1, data2, sys_error, model=0):
    mu1 = np.mean(data1)
    mu2 = np.mean(data2)
    delta1 = max(stdev_of_mean(data1), sys_error)
    if model == 0:
        delta2 = max(stdev_of_mean(data2), sys_error)
    else:
        delta2 = 0

    return abs(mu1 - mu2) / sqrt( delta1**2 + delta2**2 )
```

Now, based on the *t*-scores in Table 2, we can see that for an amplitude of 8 degrees, the model period was indistinguishable from the measured periods, while in all other cases, the data were inconclusive. The fact that we found indistinguishability between the data and the model at an amplitude of 8 degrees suggests that our hypothesis may be correct. Regarding the amplitude of 5 degrees, there was some error introduced in that the swinging mass did not always pass completely through the photogate. We attempted to rectify this by moving the photogate as close to the bottom of the mass's trajectory as possible, but it was still an issue, and it caused a lot of variation in the trials for the measurement of period at the amplitude of 5 degrees. The fact that we got inconclusive *t*-scores for amplitudes of at least 10 degrees is actually encouraging; while it isn't enough to suggest our hypothesis, continued experimentation at different lengths might yield distinguishable *t*-scores.

To answer the questions in the assignment page. (1) Where does the model fail and what might be the reason? While our results were inconclusive for amplitudes greater than 10 degrees, it seems that the model fails here, and this would be due to actual dependence on the amplitude. (2) What parameters would you change and why? It would be good to change the length of the string and the size of the hanging mass; the first change would allow for more exploration of the parameter space, while the second change would allow the photogate to get more accurate measurements at small amplitudes. (3) Limitations of your hypothesis. This hypothesis is only a qualitative statement; we prescribe a set of amplitudes where the model is accurate and one where it is not, but we don't make a claim as to how it depends on the amplitude in any quantitative. We address this in part 2. (4) Does your data and conclusion match with the data from last week? Why or why not? The data does not match exactly because we used different amplitudes from those used in lab 1. An interesting point on which the two data sets are different is that the small amplitude trials from this week have much more variation than those in lab 1, as can be seen from Table 3. The conclusion (see below) is not the same as in lab 1, because we were testing a different hypothesis in this experiment, so the conclusions wouldn't be the same. Please find the requested plot in Figure 2.

Part 1: Conclusion

Because the data were so inconclusive, we are forced into a weak conclusion that the data weakly supports our hypothesis. This is because of the agreement at 8 degrees, and the inconclusivity for higher amplitudes allowing us to interpret that our hypothesis *may* be correct in that region of amplitudes.

Along the way, we found a number of improvements that might give us more conclusive data. First, it would be better to use a smaller mass, that way the photogate would have more accuracy for smaller amplitudes. Second, it would be good to test periods for a variety of string lengths, as this would allow for a fuller exploration of the parameter space. Finally, for the small amplitudes, the random error was higher than the systematic error, so it would be wise to take more trials at the smaller amplitudes.

Part 2: Method

Objective

Based on our weak conclusion in part 1, our goal in part 2 was to determine if a linear model, i.e. one of the form

$$c_1 + c_2\theta$$

would give better predictions for the period.

Materials

All materials used in this part were the same as in part 1, except for that the data from part 1 should also be considered in the materials. This is because we were instructed to use the data from part 1 to test our hypothesis in part 2, so as to avoid collecting excessive data.

Hypothesis

The model

$$(0.8576847192 + 0.0053992651 \times \theta) \times 2\pi\sqrt{\frac{l}{g}}$$

would be more successful in predicting the period of the pendulum than the provided model. Here, l = 32.90 +/- 0.05 cm and g = 9.81 m/s/s. An important note is that the coefficients in the linear factor above were calculated for these specific values of l and g.

Before moving onto the method, we give the derivation of this model. The model was derived using least squares. Let

$$X = \begin{bmatrix} 1 & 5\text{ deg} \\ 1 & 8\text{ deg} \\ 1 & 10\text{ deg} \\ 1 & 20\text{ deg} \\ 1 & 30\text{ deg} \end{bmatrix}, T = \begin{bmatrix} 0.90\ s \\ 0.90\ s \\ 1.12\ s \\ 1.13\ s \\ 1.14\ s \end{bmatrix}, c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Also, let $a = 2\pi\sqrt{\frac{l}{g}}$. Now, the equation we wish to solve by least squares if

$$Xc = \frac{1}{a}T.$$

We will show the sequence of steps. Note that if *T* appears in the exponent, it denotes matrix transpose, whereas if *T* appears on the main line, it refers to the matrix *T* as defined above.

$$Xc = \frac{1}{a}T \Rightarrow X^T Xc = \frac{1}{a}X^T T \Rightarrow c = \frac{1}{a}(X^T X)^{-1} X^T T,$$

where the last step is justified because a matrix times its transpose is always invertible. We used a TI-84 to carry out the matrix product, and then by multiplying both sides by *a*, we were able to derive the model in the hypothesis.

Method

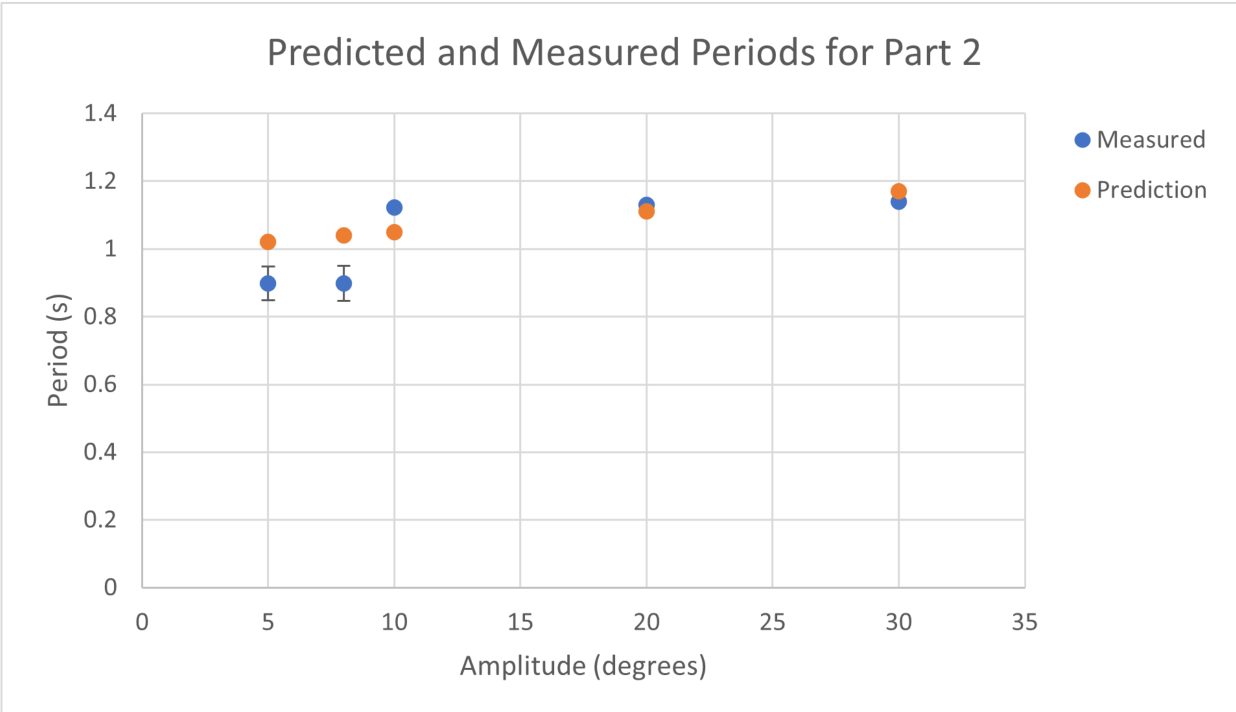
The method in this section was very straightforward. We would use the data from part 1 to fit the model in the hypothesis, and then we would calculate the *t*-scores between the model and the data.

Data

Table 4: Prediction, Measured Period with Error, *t*-scores, and Interpretation

Angle (degrees)	Prediction (s)	Measured (s)	t-score	Interpretation
5	1.02	0.90 +/- 0.05	1.43	Inconclusive
8	1.04	0.90 +/- 0.05	0.14	Indistinguishable
10	1.05	1.12 +/- 0.01	4.44	Distinguishable
20	1.11	1.13 +/- 0.01	5.24	Distinguishable
30	1.17	1.14 +/- 0.01	6.23	Distinguishable

Figure 4



Analysis

This is interesting, because Figure 4 makes it appear as if the linear model proposed in our hypothesis is a slight improvement over the simple model for the period, whereas the *t*-scores in Table 4 indicate that this model is very incorrect. Perhaps if we were using residuals to measure the error, we would see something more in favor of our hypothesis, but whereas we are restricted to using *t*-scores, we have to conclude that our hypothesis is not correct.

Now again, we go through the questions posed on Canvas. (1) Where does the model fail and what might be the reason? This model fails most clearly in the large amplitude regime, that being amplitudes greater than or equal to 10 degrees. The reason for this is that the measurements for the periods are much more tightly distributed for these amplitudes, so the *t*-score is able to distinguish them from the model much more easily. This is why, although Figure 4 makes it seem as if the model is more accurate for these amplitudes, the *t*-scores are in the distinguishable range. (2) What parameter would you change and why? Well for one thing, it would be good practice to get more data to compare the model against, instead of using the data the model was fitted on. In that vein, we would also change the mass and length of the string, as was suggested in part 1; this would allow for a more complete exploration of the parameter space; we would have done that in this part, but because we were instructed to use the training data to validate the model, we didn't have the opportunity this time. (3) Limitations of your hypothesis. This hypothesis is actually very limited, as the coefficients were fitted to a specific setting of l and g. This means that we would expect very poor predictions for different values of l or g. (4) Does your data and conclusion match with the data from last week? Why or why not? Well, as before, we were testing a different hypothesis, so the conclusion is not the same; in fact, this point applies much more here than in part 1, because we were testing a different model.

Regarding sample calculations, we again used Jupyter Notebooks to carry these out for us. The code is attached in Figures 5 and 6 for consideration.

Figure 5: Code for Predictions

```
amplitudes = [5,8,10,20,30]
```

```
def prediction(data):
    c1 = 0.8576847192
    c2 = 0.0053992651
    T_orig = 1.15065019
    out = []
    for theta in data:
        value = T_orig*(c1+c2*theta)
        out.append(value)
    return(out)
```

```
data_md = prediction(amplitudes)
print(data_md)
```

```
[1.0179584121734533, 1.0365964084129795, 1.0490217392393304, 1.111148393371
084, 1.1732750475028375]
```

Figure 6: Analysis Code

```
data_sets = [data_05, data_08, data_10, data_20, data_30]
angles = ["05", "08", "10", "20", "30"]

print("PART 2")
print("Angle (deg), Mean Period (s), StDev of Mean Period (s)")
for i in range(5):
    st = angles[i] + "degrees, "
    st += str( np.mean(data_sets[i]) ) + ", "
    st += str( stdev_of_mean(data_sets[i]) )
    print(st)
```

```
PART 2
Angle (deg), Mean Period (s), StDev of Mean Period (s)
05degrees, 0.898, 0.1255945858705701
08degrees, 1.098, 0.1441665703275208
10degrees, 1.122, 0.0019999999999957
20degrees, 1.13, 0.0
30degrees, 1.14, 0.0
```

```
# Calc t-scores between model and data
print("t-scores between data and model")
print("amplitude, t-score")
for i in range(5):
    st = angles[i] + "degrees, "
    st += str(t_score(data_sets[i], data_md, 0.01, model=1))
    print(st)
```

```
t-scores between data and model
amplitude, t-score
05degrees, 1.4299979485184282
08degrees, 0.14150298376189482
10degrees, 4.439999986006327
20degrees, 5.239999986006305
30degrees, 6.239999986006306
```

Note that the arrays data_05, data_08, and so on are just 5 element arrays containing the trial data from Table 1. **This should count as the sample calculation for part 2, because we have provided the inputs and the machinery that carried out the calculation.** If we needed to calculate these values by hand, we could do so, but this is just so much more convenient.

Part 2: Conclusion

From our analysis, we determined that a linear model does not predict the pendulum's period very well. For future experiments, we recommend a quadratic model or a power model. This is in line with our hypothesis of our new derived model formula. A noteworthy query was a methodological quirk we noticed; it would be better for the accuracy of the experiment to collect a denser sample of amplitudes. We suggest in the future to collect this larger sample in reiterations of this experiment. However during our experiment, we were instructed not to. In conclusion, we found that for amplitudes over 10 degrees, the model is inaccurate. For amplitudes under 10 degrees, the model is more accurate, but still not completely accurate. As seen from the data and *t*-scores, the period measurements are significantly less spread out for these amplitudes.