Readings from Sipser

- Section 5.1
  - Theorem 5.1 (Halting Problem)

Halting Problem

- We have proven that A_{TM} is undecidable, though recognizable
- Halting Problem – Given a TM M and input w, can we determine if M(w) will even terminate?

$$HALT_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$$

- Question: Is $HALT_{TM}$ decidable, or even recognizable?

Relationship between A_{TM} and $HALT_{TM}$

Possible

Not Possible!

???

???

On input <M, w>, if…	UTM… (recognizer for A_{TM})	Decider for A_{TM} must…	Recognizer for $HALT_{TM}$ must…	Decider for $HALT_{TM}$ must…
M accepts w	Accepts	Accepts	Accepts	Accepts
M rejects w	Rejects	Rejects	Accepts	Accepts
M loops forever on w	Loops forever	Rejects	Rejects or loops forever	Rejects

$HALT_{TM}$ is Recognizable

- Main idea: The UTM is almost a recognizer for $HALT_{TM}$, we just need to accept in those cases where the UTM rejects
- The following TM R_{HALT} is a recognizer for $HALT_{TM}$:

1. On input <M, w>, run the UTM on the same input <M, w>:
   a) If the UTM accepts or rejects, simply <u>accept</u> (since it halted in either case)
   b) If the UTM happens to loop forever, then R_{HALT} will necessarily loop forever (which is okay, since it doesn't have to reject in these cases)

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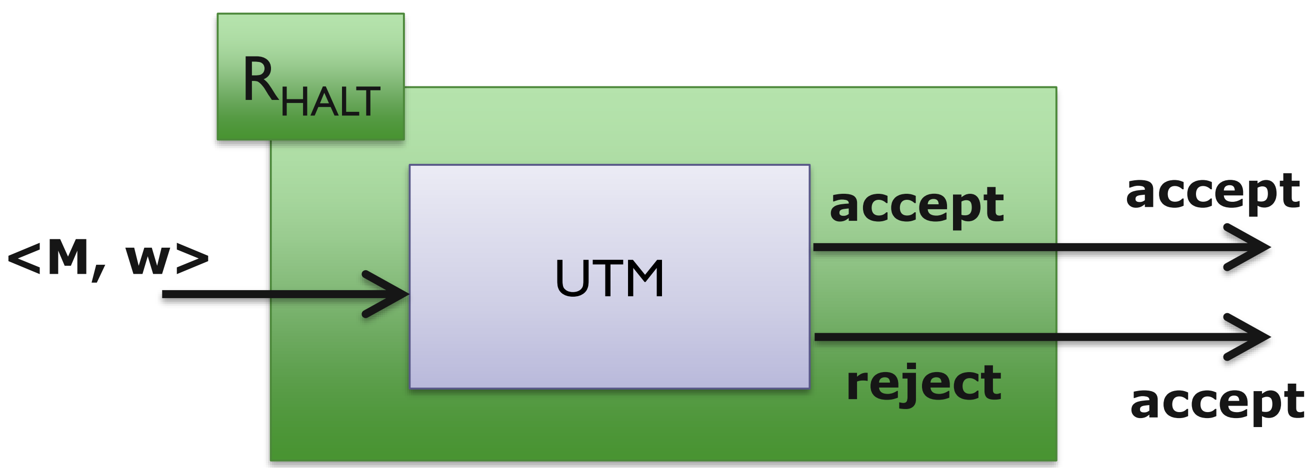

Relationship between A_{TM} and $HALT_{TM}$

	Possible	Not Possible!	Possible	???
On input <M, w>, if…	**UTM… (recognizer for A_{TM})**	**Decider for A_{TM} must…**	**R_{HALT}… (recognizer for $HALT_{TM}$)**	**Decider for $HALT_{TM}$ must…**
M accepts w	Accepts	Accepts	Accepts	Accepts
M rejects w	Rejects	Rejects	Accepts	Accepts
M loops forever on w	Loops forever	Rejects	Loops forever	Rejects

Theorem: $HALT_{TM}$ is Undecidable

- Proof (by contradiction): Suppose R is a TM that acts as a decider for the language $HALT_{TM}$. With R, we can construct the following TM which decides A_{TM}:

1. On input <M, w>, run R on the same input <M, w>
    a) If R rejects <M, w>, then <u>reject</u> (since M will loop forever on w, and never accept it)
    b) If R accepts <M, w>, then simulate M on w (we know this will terminate)
        i. if M accepts w, <u>accept</u>
        ii. if M rejects w, <u>reject</u>

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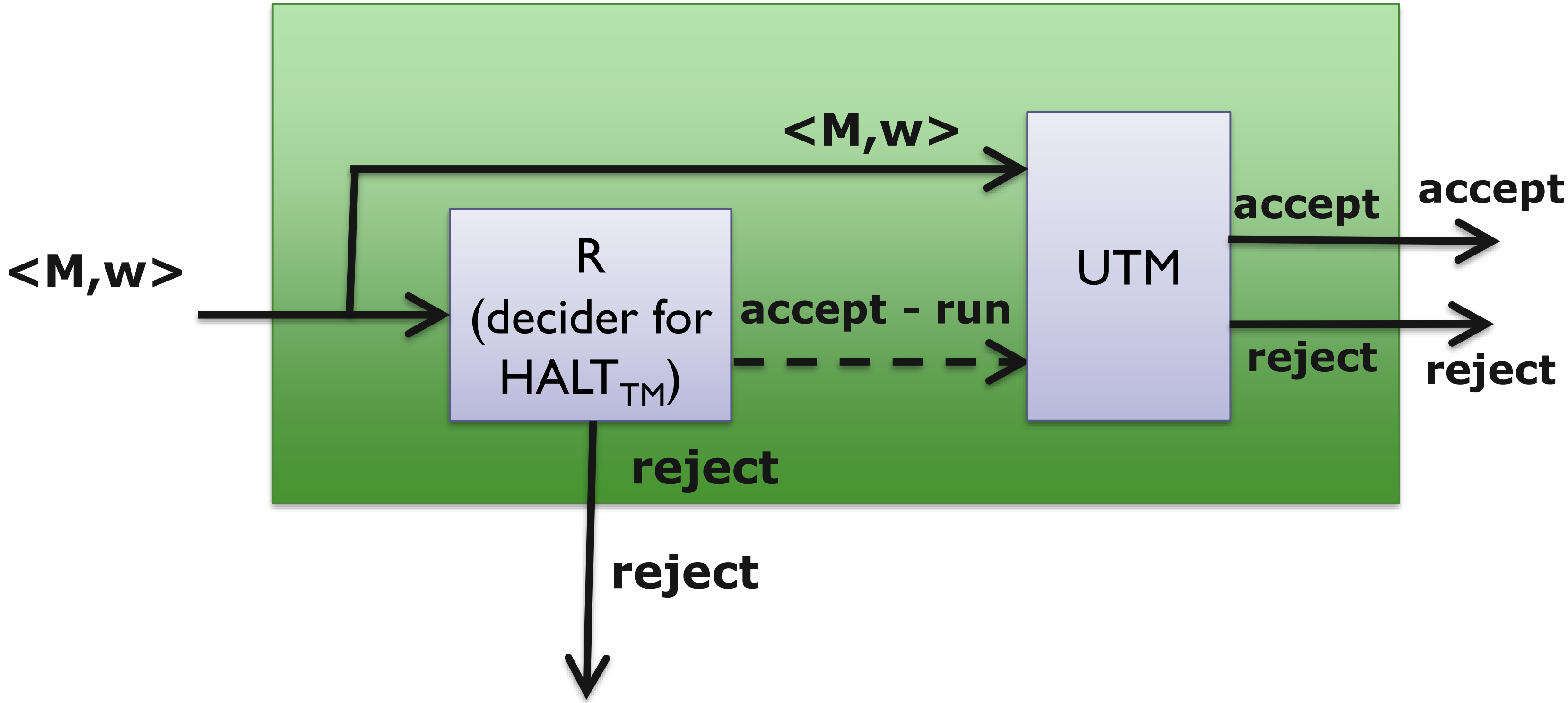

Theorem: $HALT_{TM}$ is Undecidable

- Proof (by contradiction): Suppose R is a TM that acts as a decider for the language $HALT_{TM}$. With R, we can construct the following TM which decides A_{TM}:

However, since we have already proven that so such decider for A_{TM} can exist, no such decider R for $HALT_{TM}$ can exist either!

Relationship between A_{TM} and $HALT_{TM}$

	Possible	Not Possible!	Possible	Not Possible!
On input <M, w>, if…	**UTM… (recognizer for A_{TM})**	**Decider for A_{TM} must…**	**R_{HALT}… (recognizer for $HALT_{TM}$)**	**Decider for $HALT_{TM}$ must…**
M accepts w	Accepts	Accepts	Accepts	Accepts
M rejects w	Rejects	Rejects	Accepts	Accepts
M loops forever on w	Loops forever	Rejects	Loops forever	Rejects

Kinds of Languages

- Decidable (includes Context-Free and Regular languages)
- Undecidable, but still recognizable (A_{TM} and $HALT_{TM}$)
- Unrecognizable ($\overline{A_{TM}}$ and $\overline{HALT_{TM}}$)

Exercises – Which Language Class?

- Binary-Palindromes
- $\overline{F}$ (where F is some finite language)
- A_{aaa} = { <M> | M is s Turing Machine that accepts the string aaa }
- E_{TM} = { <M> | M is s Turing Machine that accepts nothing }
- $\overline{E_{TM}}$
- EQ_{TM} = {<M_1, M_2> | M_1 and M_2 are TMs that accept exactly the same strings }
- $\overline{EQ_{TM}}$