Notto - Category for the Working Mathematician

tkunn

August 2020

This is solutions for problems on the second edition of *Category for the Working Mathematician* by S. Mac Lane.

1

1.1

$$\langle \wedge \_ \wedge \rangle$$

1.2

$$\langle Q \perp Q \rangle$$

1.3

1.3.1

TODO

1.3.2

TODO

1.3.3

TODO

1.3.4

TODO

1.3.5

TODO

1.4

1.4.1

TODO

1.4.2

TODO

1.4.3

TODO

1.4.4

TODO

1.4.5

TODO

1.4.6

TODO

1.5

1.5.1

TODO

1.5.2

TODO

1.5.3

TODO

1.5.4

TODO

1.5.5

TODO

1.5.6

TODO

1.5.7

TODO

1.5.8

TODO

1.5.9

TODO

1.6

Let U be a set that satisfies following conditions:

(i) $x \in u \in U \Rightarrow x \in U$

(ii) $(u \in U \land v \in U) \Rightarrow (\{u,v\}, \langle u,v \rangle, u \times v \in U)$

(iii) (1) $x \in U \Rightarrow \mathcal{P}x \in U$

     (2) $x \in U \Rightarrow \bigcup x \in U$

(iv) $\omega \in U$, where $\omega = \{0,1,2,\dots\}$ is a set of all finite ordinal numbers.

(v) If there exists a surjection $f : a \to b$ and $a \in U$ and $b \subset U$, then $b \in U$.

1.6.1

For all $q \in \prod_i f_i$, we can construct a bijection $r : I \to q$. $q \subset U$ because $\forall w \in q$, $\exists j \in I$, $w \in f_j \in U$. Since $I \in U$ and $q \subset U$, we can say $q \in U$. Therefore $\prod_i f_i \subset U$.

Let $|f_k| \geq |f_i|$ for all i. Then, we can construct a surjection $g : f_k^I \to \prod_i f_i$. Also, we can construct a surjection $h : X \to f_k^I$, with X is either $\mathcal{P}f_k$ or $\mathcal{P}I$. As $X \in U$, $g \circ h : X \to \prod_i f_i$ and $\prod_i f_i \subset U$, we can say $\prod_i f_i \in U$.

1.6.2

$$\langle Q \smile Q \rangle$$

(a) We can construct a bijection $g : I \to \{f_i \mid i \in I\}$, $g(i) = f_i$. As $I \in U$ and $f_i \in U$ for all i, $\{f_i \mid i \in I\} \in U$. Therefore $\bigcup_i f_i = \bigcup \{f_i \mid i \in I\} \in U$.

(b) Because $x \in U$, we have $y \in U$ for all $y \in x$. Therefore we can apply $f : x \to x, f(i) = i$ to (a) to get $\bigcup x \in U$.

Because $a \in U$ and $b \subset U$, we can apply f to (a) to get $\bigcup_i f_i \in U$. f is surjective, therefore $b = \bigcup_i f_i$. Hence $b \in U$.
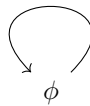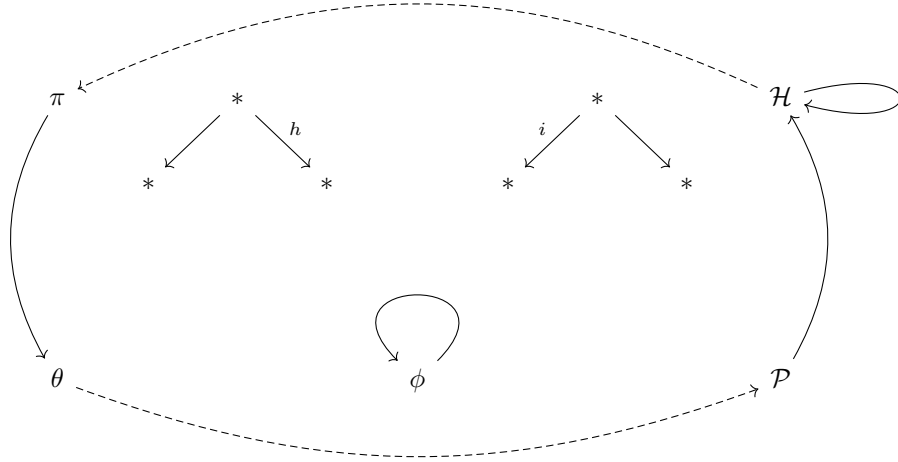
1.7

$$
\begin{array}{ccccccc}
& * & & & & * & \\
\swarrow & & \searrow^{h} & & {}^{i}\swarrow & & \searrow \\
* & & * & & * & & *
\end{array}
$$

$$\phi \longrightarrow \theta$$

1.8

$$
\begin{array}{ccccccc}
& * & & & & * & \\
\swarrow & & \searrow^{h} & & {}^{i}\swarrow & & \searrow \\
* & & * & & * & & *
\end{array}
$$

$$\phi \longrightarrow \theta$$

2

2.1

$$
\begin{array}{ccccccc}
& * & & & & * & \\
\swarrow & & \searrow^{h} & & {}^{i}\swarrow & & \searrow \\
* & & * & & * & & *
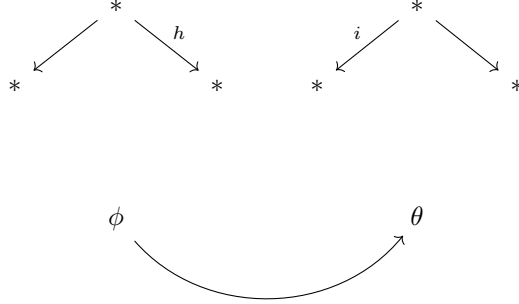\end{array}
$$

$$\phi$$

2.2

2.3

2.3.1

In this section, $\times_S$ is a product operation for sets, $\times_C$ is for categories, $\times_G$ is for groups and $\times_M$ is for monoids.

Monoids. Let M, N be a monoid with object m, n respectively. The only object in $M \times_C N$ is $\langle m, n \rangle$. ... (TODO)

Groups. Let G, H be a group with object a, b respectively. $G \times_C H$ ... (TODO)

Sets. Let A, B be discrete categories. The set of all objects in $A \times_C B$ is $X = A \times_S B$. The set of all arrows in $A \times_C B$ is $\{\langle f, g \rangle \mid a \in A, \ b \in B, \ f \in A(a, a) \ \wedge \ g \in B(b, b)\} = \{\langle \mathrm{id}_A(c_0), \mathrm{id}_B(c_1) \rangle \mid c \in X\}$. Therefore $A \times_C B$ is a discrete category of $X = A \times_S B$.

2.3.2

$$\langle \wedge \_ \wedge \rangle$$

Let P, Q be preorders. $\forall a \forall b \, |\hom(a,b)| \leq 1$ for both P and Q. Therefore, for all $p_1 \in P$, $p_2 \in P$, $q_1 \in Q$, $q_2 \in Q$, $|\hom_{P \times Q}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle)| = |\hom_P(p_1, p_2) \times \hom_Q(q_1, q_2)| \leq 1$. Hence $P \times Q$ is preorder.

2.3.3

$$\langle Q \perp Q \rangle$$

Let $\mathrm{id}_X(a) : a \to a$ be identity in category X.

Let the set of all object in C be the product set $\prod_i C_i$ and $C(a,b) = \prod_i C_i(a_i, b_i)$. Let $\mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i)$ for all $c \in C$. Now, we prove C has a universal property:

1. For every i there is a functor $P_i : C \to C_i$.

2. For every category B such that a functor $G_i : B \to C_i$ presents for every C_i, there is a functor $F : B \to C$, which makes the following diagram commute.

$$
\begin{array}{ccc}
B & & \\
\big\downarrow{\scriptstyle F} & \searrow^{G_i} & \\
C & \xrightarrow{P_i} & C_i
\end{array}
$$

First, we prove $P_i : C \to C_i$ exists. Let the object function be $P_i(x) = x_i$. Let the arrow function be $P_i(f) = f_i$. For all object $c \in C$, $P_i(\mathrm{id}_C(c)) = \mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i) = \mathrm{id}_{C_i}(P_i(c))$. For all arrow f, g in C, $P_i(g \circ f) = (g \circ f)_i = g_i \circ f_i = P_i(g) \circ P_i(f)$. Therefore P_i is a functor.
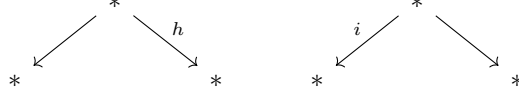
Second, we prove $F : B \to C$ exists. Let the object function be $F(x)_i = G_i(x)$. Let the arrow function be $F(f)_i = G_i(f)$. For all object $b \in B$, $F(\mathrm{id}_B(b))_i = G_i(\mathrm{id}_B(b)) = \mathrm{id}_{C_i}(G_i(b)) = \mathrm{id}_{C_i}(F(b)_i)$. Thus $F(\mathrm{id}_B(b)) = \mathrm{id}_C(F(b))$. For all arrow f, g in B, $F(f \circ g)_i = G_i(f \circ g) = G_i(f) \circ G_i(g)$. Thus $F(f \circ g) = F(f) \circ F(g)$. Therefore F is a functor.

2.3.4

$$\langle Q \smile Q \rangle$$

In $\mathbf{Matr}_K$, the object set is all positive integers $\{1, 2, 3, ...\} = \omega \setminus \{0\}$. $\mathbf{Matr}_K(n, m)$ is all rectangular matrix on K of shape $m \times n$. Therefore $\mathbf{Matr}_K^{\mathrm{op}}$ has the same objects $\omega \setminus \{0\}$ and $\mathbf{Matr}_K^{\mathrm{op}}(n, m)$ is all rectangular matrix on K of shape $n \times m$.

2.3.5

$$
\begin{array}{ccc}
& * & \\
\swarrow & \ {}^{h} & \searrow \\
* & & *
\end{array}
\qquad
\begin{array}{ccc}
& * & \\
{}^{i}\ \swarrow & & \searrow \\
* & & *
\end{array}
$$

$$\phi \longrightarrow \theta$$

Let $R_T \subseteq (T \to \mathbb{R})$ be a ring whose elements are continuous functions from a topological space T to real number. We construct R_T as follows:

1. Additive identity. $0_{R_T} : x \mapsto 0$.

2. Multiplicative identity. $1_{R_T} : x \mapsto 1$.

3. Addition. $f + g : x \mapsto f(x) + g(x)$.

4. Multiplication. $f \times g : x \mapsto f(x) \times g(x)$.

Let X and Y be any topological spaces. If we have a continuous function $f : Y \to X$, we can construct a ring homomorphism $H(f) = h : R_X \to R_Y$. We define $h(r) = r \circ f$. Then $h(0_{R_X}) = 0_{R_Y}$, $h(1_{R_X}) = 1_{R_Y}$, $(h(s + t))(x) = (s+t)(f(x)) = s(f(x))+t(f(x))$, $(h(s\times t))(x) = (s\times t)(f(x)) = s(f(x))\times t(f(x))$. Therefore $H(f)$ is a ring homomorphism.

Now we construct a functor $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Rng}$. Let the object function be $F(A) = R_A$ and the arrow function be $F(g) = H(g^{\mathrm{op}})$. For all arrow a, b in $\mathbf{Top}^{\mathrm{op}}$, $F(b \circ a) = H((b \circ a)^{\mathrm{op}}) = H(a^{\mathrm{op}} \circ b^{\mathrm{op}}) = H(b^{\mathrm{op}}) \circ H(a^{\mathrm{op}}) = F(b) \circ F(a)$. For all topological space $T \in \mathbf{Top}^{\mathrm{op}}$, $F(\mathrm{id}(T)) = H(\mathrm{id}(T)) = \mathrm{id}(R_T)$. Therefore F is a functor and $\overline{F}$ is a contravariant functor on $\mathbf{Top}$ to $\mathbf{Rng}$.

2.4

2.4.1

TODO

2.4.2

For any functor $T : X \to B$, if its object function is $T(a) = b$, its arrow function maps $\mathrm{id}(a)$ to $\mathrm{id}(b)$. Such a functor T is an object of B^X.

Let $R, S : X \to B$ be functors and τ be a map on an object of X to an arrow in B. $(\tau : R \dotto S) \Leftrightarrow (\forall x \in X, \ \tau_x(R(x)) = S(x))$. Therefore $\hom(R, S) = \{\tau \mid \forall x \in X, \ \tau_x(R(x)) = S(x)\}$. Therefore an arrow on R to S exists iff $(\forall x, y \in X, \ e_R(x,y) \to e_S(x,y)) \ \wedge \ (\forall x \in X, \ \hom(R(x), S(x)) \neq \emptyset)$, where $e_T(x,y) \Leftrightarrow (\exists a \in B, \{x,y\} \subseteq \{w \mid a = T(w)\})$.

2.4.3
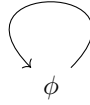
An object of $\mathbf{Ab^N}$ is a map on $\mathbb{N}$ to $\mathbf{Ab}$. Same as above, $\hom(R, S) = \{\tau \mid \forall n \in N, \ \tau_n(R(n)) = S(n)\}$. In other words, a map $\tau : \mathbb{N} \to (\mathbf{Ab} \to \mathbf{Ab})$ is an arrow iff, for every $n \in \mathbb{N}$, there is a corresponding group homomorphism τ_n on $R(n)$ to $S(n)$, and, for every $m \in \mathbb{N}$ such that $R(n) = R(m)$, $S(n) = S(m)$.

2.4.4
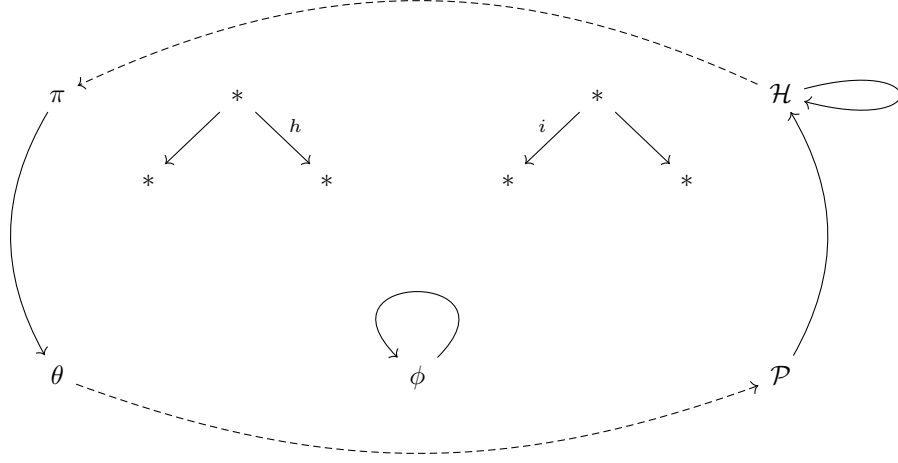
Let $R, S : P \to Q$. Then R and S are objects of Q^P. Let τ be a natural transform $\tau : R \dot{\to} S$. τ is an arrow on R to S in Q^P. Since τ is natural and P is preorder, following diagram is commute for every pair of objects $p, p' \in P$. $a = f(p, p')$, where $f(p, p')$ is the only arrow on p to p'. Since Q is preorder, $\tau p = g(Rp, Sp)$, where $g(Rp, Sp)$ is the only arrow on Rp to Sp.

$$
\begin{array}{ccc}
Rp & \xrightarrow{\ \tau p\ } & Sp \\
{\scriptstyle Ra}\big\downarrow & & \big\downarrow{\scriptstyle Sa} \\
Rp' & \xrightarrow{\ \tau p'\ } & Sp'
\end{array}
$$

From the two downward arrows in the diagram, we can say that $\mathrm{Im}(R)$ and $\mathrm{Im}(S)$ contain the preorder structure of P. There are two functors $P \to \mathrm{Im}(R)$ and $P \to \mathrm{Im}(S)$, where $\mathrm{Im}(T)$ is a category from the image of the object function of T and all arrows between any two pairs in the image.

As explained above, $\sigma p = g(Rp, Sp)$ for all $\sigma : R \dot{\to} S$. Thus $|\hom(R, S)| \leq |\{\sigma \mid \forall p \in P,\ \sigma p = g(Rp, Sp)\}| = 1$. Therefore Q^P is preorder.

2.4.5
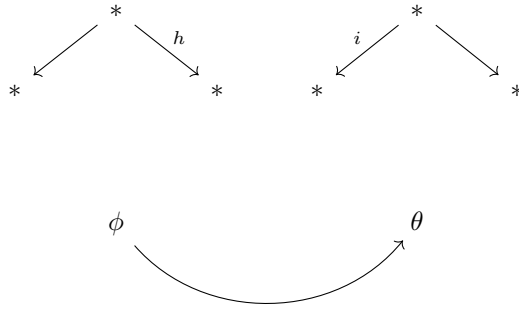
Let **Fin** be a category of all finite sets. The object is every finite set and the arrow is every mapping between every pair of finite sets.

Let G be a finite group. G is a category of only one object. Every arrow a in G has its inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}$.

Fin^G is a category that have any functor on G to **Fin** as objects and any natural transform between two objects as arrows. The group G has only one object $x \in G$, thus any functor $T \in \mathbf{Fin}^G$ map x to a finite set $T(x) \in \mathbf{Fin}$, and endomorphisms of x to endomorphisms of $T(x)$. For any arrow a, b in G, $T(b \circ a) = T(b) \circ T(a)$. Also, $\mathrm{id} = T(\mathrm{id}) = T(a \circ a^{-1}) = T(a) \circ T(a^{-1})$. Thus any element in the image of the arrow function of T is invertible. Therefore T is a permutation representation of G.

Now, let $\tau : R \overset{\cdot}{\to} S$ be an arrow in **Fin**^{G}. Then, the following diagram commutes for any arrow a in G:

$$
\begin{array}{ccc}
Rx & \xrightarrow{\ \tau x\ } & Sx \\
{\scriptstyle Ra}\downarrow & & \downarrow{\scriptstyle Sa} \\
Rx & \xrightarrow{\ \tau x\ } & Sx
\end{array}
$$

Therefore, $\hom(R, S) = \{\tau \mid \forall a,\ \tau x \circ Ra = Sa \circ \tau x\}$... what does it mean????
TODO

2.4.6

TODO

2.4.7

TODO

2.4.8

TODO

2.5

2.5.1

$$\langle \wedge \_ \wedge \rangle$$

symbols: A B C F S T

We prove there is a bijection $F : \mathbf{Cat}(A \times B, C) \to \mathbf{Cat}(A, C^B)$.

Let $FT = S$, where $T : A \times B \to C$ and $S : A \to C^B$. First, we make the object function of S. Given $a \in A$, we make a subcategory $a \subseteq A$, a category of an object a and an arrow $\mathrm{id}(a)$. Trivially, there is a functor $f_a : B \to a \times B$ ($f_a(b) = \langle a, b \rangle$ for objects, $f_a(b) = \langle \mathrm{id}(a), b \rangle$ for arrows). As $a \times B \subseteq A \times B$, now we define $Sa : B \to C$, $Sa = T \circ f_a$. Second, we make the arrow function of S. Let $Sa = \tau$, where $a : a_1 \to a_2$ is an arrow in A, $\tau : Sa_1 \overset{\cdot}{\to} Sa_2$,

$\tau b = T\langle a, \mathrm{id}(b)\rangle$. Let $g : b_1 \to b_2$ is an arrow in B. Then $Sa_2 g(\tau b_1(Sa_1 b_1)) = Sa_2 b_2 = \tau b_2(Sa_1 g(Sa_1 b_1))$. Thus τ is natural.

Next, we prove for all $S : A \to C^B$, there exists $T : A \times B \to C$, such that $FT = S$. First, we prove for the object function of T. We define $T\langle a, b\rangle = Sab$. $FTab = T\langle a, b\rangle$, thus $FT = S$. Second, we prove for the arrow function. Let $T\langle a, b\rangle = Sa(\mathrm{dom}(b))$. Then $FTab = Sab$ for any arrow a in A, object $b \in B$.

Finally, we prove that for all T_1, $T_2 : A \times B \to C$, if $FT_1 = FT_2$, then $T_1 = T_2$. Let $FT_1 = FT_2$ and $n \in \{1,\ 2\}$. We write $T_=\langle x, y\rangle$ when $T_1\langle x, y\rangle = T_2\langle x, y\rangle$. First, for all $a \in A$, $b \in B$, $FT_n ab = T_n\langle a, b\rangle$. Thus $T_=\langle a, b\rangle$. Second, for every object $a \in A$, arrow b in B, $FT_n ab = T_n\langle \mathrm{id}(a), b\rangle$. For every arrow a in A, object $b \in B$, $FT_n ab = T_n\langle a, \mathrm{id}(b)\rangle$. Now, let $a : a_1 \to a_2$ is any arrow in A, $b : b_1 \to b_2$ is any arrow in B. Then $T_=\langle \mathrm{id}(a_1), b\rangle$, $T_=\langle a, \mathrm{id}(b_2)\rangle$ and $T_=(\langle \mathrm{id}(a_1), b\rangle \circ \langle a, \mathrm{id}(b_2)\rangle)$. Thus $T_=\langle a, b\rangle$.

2.5.2

$$\langle Q \perp Q \rangle$$

For any $x \in (A \times B)^C$, there are two functions $G(c) = x(c)_0$ and $H(c) = x(c)_1$. Let the object function be $F(x) = \langle G, H\rangle$, where $F : (A \times B)^C \to A^C \times B^C$.

For any natural transform $\tau : (A \times B) \xrightarrow{\cdot} C$, there are two natural transforms $\tau_0 : A \xrightarrow{\cdot} C$ and $\tau_1 : B \xrightarrow{\cdot} C$. Let $\tau_0(x) =$ and $\tau_1(x) =.$

Thus we can construct the arrow function $F(\tau) = \langle \tau_0, \tau_1\rangle$.