Notto - Category for the Working Mathematician

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August 2020

This is solutions for problems on the second edition of *Category for the Working Mathematician* by S. Mac Lane. Problems are placed in the first line of each sections. They may be modified from the original text for simplicity, clarity and some laziness.

1

1.1

$$\langle \wedge \_ \wedge \rangle$$

1.2

$$\langle Q \perp Q \rangle$$

1.3

1.3.1

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1.6

Let U be a set that satisfies following conditions:

(i) $x \in u \in U \Rightarrow x \in U$

(ii) $(u \in U \wedge v \in U) \Rightarrow (\{u, v\}, \langle u, v\rangle, u \times v \in U)$

(iii) (1) $x \in U \Rightarrow \mathcal{P}x \in U$

(2) $x \in U \Rightarrow \bigcup x \in U$

(iv) $\omega \in U$, where $\omega = \{0, 1, 2, \dots\}$ is a set of all finite ordinal numbers.

(v) If there exists a surjection $f : a \to b$ and $a \in U$ and $b \subset U$, then $b \in U$.

1.6.1

Let $I \in U$, $f : I \to b$ and $f_i \in U$ for all $i \in I$. Prove that $\prod_i f_i \in U$.

For all $q \in \prod_i f_i$, we can construct a bijection $r : I \to q$. $q \subset U$ because $\forall w \in q,\ \exists j \in I,\ w \in f_j \in U$. Since $I \in U$ and $q \subset U$, we can say $q \in U$. Therefore $\prod_i f_i \subset U$.

Let $|f_k| \geq |f_i|$ for all i. Then, we can construct a surjection $g : f_k^I \to \prod_i f_i$. Also, we can construct a surjection $h : X \to f_k^I$, with X is either $\mathcal{P}f_k$ or $\mathcal{P}I$. As $X \in U$, $g \circ h : X \to \prod_i f_i$ and $\prod_i f_i \subset U$, we can say $\prod_i f_i \in U$.

1.6.2

(a) Let $I \in U$, $f : I \to b$ and $f_i \in U$ for all $i \in I$. Prove $\bigcup_i f_i \in U$.
(b) Prove that (a) implies following if (i), (ii), (iii)(1), (iv) and (v) holds true:
(iii)(2) $x \in U \Rightarrow \bigcup x \in U$.
(v) If $f : a \to b$ is surjective and $a \in U$ and $b \subset U$, then $b \in U$.

(a) We can construct a bijection $g : I \to \{f_i \mid i \in I\}$, $g(i) = f_i$. As $I \in U$ and $f_i \in U$ for all i, $\{f_i \mid i \in I\} \in U$. Therefore $\bigcup_i f_i = \bigcup\{f_i \mid i \in I\} \in U$.

(b) Because $x \in U$, we have $y \in U$ for all $y \in x$. Therefore we can apply $f : x \to x, f(i) = i$ to (a) to get $\bigcup x \in U$.

Because $a \in U$ and $b \subset U$, we can apply f to (a) to get $\bigcup_i f_i \in U$. f is surjective, therefore $b = \bigcup_i f_i$. Hence $b \in U$.

1.7

$$\langle Q \,\dot{\smile}\, Q \rangle$$

1.8

$$\begin{tikzcd}
& * \arrow[dl] \arrow[dr, "h"] & & & * \arrow[dl, "i"'] \arrow[dr] & \\
* & & * & * & & *
\end{tikzcd}$$

$$\phi \longrightarrow \theta$$

2

2.1

$$\begin{tikzcd}
& * \arrow[dl] \arrow[dr, "h"] & & & * \arrow[dl, "i"'] \arrow[dr] & \\
* & & * & * & & *
\end{tikzcd}$$

$$\begin{tikzcd}
\phi \arrow[r, bend right] & \theta
\end{tikzcd}$$

2.2

$$
\begin{tikzcd}
 & * \arrow[dl] \arrow[dr, "h"] & \\
* & & *
\end{tikzcd}
\qquad
\begin{tikzcd}
 & * \arrow[dl, "i"'] \arrow[dr] & \\
* & & *
\end{tikzcd}
$$

$$
\begin{tikzcd}
\phi \arrow[loop above]
\end{tikzcd}
$$

2.3

2.3.1

Show product of categories includes product of monoids, product of groups and product of sets.

In this section, $\times_S$ is a product operation for sets, $\times_C$ is for categories, $\times_G$ is for groups and $\times_M$ is for monoids.

Monoids. Let M, N be a monoid with object m, n respectively. The only object in $M \times_C N$ is $\langle m, n \rangle$. ... (TODO)

Groups. Let G, H be a group with object a, b respectively. $G \times_C H$ ... (TODO)

Sets. Let A, B be discrete categories. The set of all objects in $A \times_C B$ is $X = A \times_S B$. The set of all arrows in $A \times_C B$ is $\{\langle f, g \rangle \mid a \in A,\ b \in B,\ f \in A(a, a)\ \wedge\ g \in B(b, b)\} = \{\langle \mathrm{id}_A(c_0), \mathrm{id}_B(c_1) \rangle \mid c \in X\}$. Therefore $A \times_C B$ is a discrete category of $X = A \times_S B$.

2.3.2

Prove that product of two preorders is preorder.

Let P, Q be preorders. $\forall a \forall b\ |\hom(a, b)| \leq 1$ for both P and Q. Therefore, for all $p_1 \in P,\ p_2 \in P,\ q_1 \in Q,\ q_2 \in Q,\ |\hom_{P \times Q}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle)| = |\hom_P(p_1, p_2) \times \hom_Q(q_1, q_2)| \leq 1$. Hence $P \times Q$ is preorder.

2.3.3

Let $\{C_i \mid i \in I\}$ be a family of categories indexed by a set I. Show product $C = \prod_i C_i$, its projections $P_i : C \to C_i$ and universal property of these projections.

Let $\mathrm{id}_X(a) : a \to a$ be identity in category X.

Let the set of all object in C be the product set $\prod_i C_i$ and $C(a, b) = \prod_i C_i(a_i, b_i)$. Let $\mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i)$ for all $c \in C$. Now, we prove C has a universal property:

1. For every i there is a functor $P_i : C \to C_i$.

2. For every category B such that a functor $G_i : B \to C_i$ presents for every C_i, there is a functor $F : B \to C$, which makes the following diagram commute.

$$
\begin{tikzcd}
B \arrow[d, dashed, "F"'] \arrow[dr, "G_i"] & \\
C \arrow[r, "P_i"] & C_i
\end{tikzcd}
$$

First, we prove $P_i : C \to C_i$ exists. Let the object function be $P_i(x) = x_i$. Let the arrow function be $P_i(f) = f_i$. For all object $c \in C$, $P_i(\mathrm{id}_C(c)) = \mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i) = \mathrm{id}_{C_i}(P_i(c))$. For all arrow f, g in C, $P_i(g \circ f) = (g \circ f)_i = g_i \circ f_i = P_i(g) \circ P_i(f)$. Therefore P_i is a functor.

Second, we prove $F : B \to C$ exists. Let the object function be $F(x)_i = G_i(x)$. Let the arrow function be $F(f)_i = G_i(f)$. For all object $b \in B$, $F(\mathrm{id}_B(b))_i = G_i(\mathrm{id}_B(b)) = \mathrm{id}_{C_i}(G_i(b)) = \mathrm{id}_{C_i}(F(b)_i)$. Thus $F(\mathrm{id}_B(b)) = \mathrm{id}_C(F(b))$. For all arrow f, g in B, $F(f \circ g)_i = G_i(f \circ g) = G_i(f) \circ G_i(g)$. Thus $F(f \circ g) = F(f) \circ F(g)$. Therefore F is a functor.

2.3.4

Show opposite of $\mathbf{Matr}_K$.

In $\mathbf{Matr}_K$, the object set is all positive integers $\{1, 2, 3, ...\} = \omega \setminus \{0\}$. $\mathbf{Matr}_K(n, m)$ is all rectangular matrix on K of shape $m \times n$. Therefore $\mathbf{Matr}_K^{\mathrm{op}}$ has the same objects $\omega \setminus \{0\}$ and $\mathbf{Matr}_K^{\mathrm{op}}(n, m)$ is all rectangular matrix on K of shape $n \times m$.

2.3.5

Show that the mapping between a topological space and the ring of real continuous functions on it is the object function of a contravariant functor on Top to Rng.

Let $R_T \subseteq (T \to \mathbb{R})$ be a ring whose elements are continuous functions from a topological space T to real number. We construct R_T as follows:

1. Additive identity. $0_{R_T} : x \mapsto 0$.
2. Multiplicative identity. $1_{R_T} : x \mapsto 1$.
3. Addition. $f + g : x \mapsto f(x) + g(x)$.
4. Multiplication. $f \times g : x \mapsto f(x) \times g(x)$.

Let X and Y be any topological spaces. If we have a continuous function $f : Y \to X$, we can construct a ring homomorphism $H(f) = h : R_X \to R_Y$. We define $h(r) = r \circ f$. Then $h(0_{R_X}) = 0_{R_Y}$, $h(1_{R_X}) = 1_{R_Y}$, $(h(s + t))(x) =$

$(s+t)(f(x)) = s(f(x))+t(f(x))$, $(h(s\times t))(x) = (s\times t)(f(x)) = s(f(x))\times t(f(x))$. Therefore $H(f)$ is a ring homomorphism.

Now we construct a functor $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Rng}$. Let the object function be $F(A) = R_A$ and the arrow function be $F(g) = H(g^{\mathrm{op}})$. For all arrow a, b in $\mathbf{Top}^{\mathrm{op}}$, $F(b \circ a) = H((b \circ a)^{\mathrm{op}}) = H(a^{\mathrm{op}} \circ b^{\mathrm{op}}) = H(b^{\mathrm{op}}) \circ H(a^{\mathrm{op}}) = F(b) \circ F(a)$. For all topological space $T \in \mathbf{Top}^{\mathrm{op}}$, $F(\mathrm{id}(T)) = H(\mathrm{id}(T)) = \mathrm{id}(R_T)$. Therefore F is a functor and $\overline{F}$ is a contravariant functor on $\mathbf{Top}$ to $\mathbf{Rng}$.

2.4

2.4.1

Show that for any ring R, R-Mod is a full subcategory of $\mathbf{Ab}^R$.

TODO

2.4.2

For a finite discrete category X, describe B^X.

For any functor $T : X \to B$, if its object function is $T(a) = b$, its arrow function maps $\mathrm{id}(a)$ to $\mathrm{id}(b)$. Such a functor T is an object of B^X.

Let $R, S : X \to B$ be functors and τ be a map on an object of X to an arrow in B. $(\tau : R \dot\to S) \Leftrightarrow (\forall x \in X,\ \tau_x(R(x)) = S(x))$. Therefore $\hom(R, S) = \{\tau \mid \forall x \in X,\ \tau_x(R(x)) = S(x)\}$. Therefore an arrow on R to S exists iff $(\forall x, y \in X,\ e_R(x, y) \to e_S(x, y)) \ \wedge\ (\forall x \in X,\ \hom(R(x), S(x)) \neq \emptyset)$, where $e_T(x, y) \Leftrightarrow (\exists a \in B, \{x, y\} \subseteq \{w \mid a = T(w)\})$.

2.4.3

Let N be a discrete category of natural numbers. Describe $\mathbf{Ab}^{\mathbf{N}}$.

An object of $\mathbf{Ab}^{\mathbf{N}}$ is a map on $\mathbb{N}$ to $\mathbf{Ab}$. Same as above, $\hom(R, S) = \{\tau \mid \forall n \in N,\ \tau_n(R(n)) = S(n)\}$. In other words, a map $\tau : \mathbb{N} \to (\mathbf{Ab} \to \mathbf{Ab})$ is an arrow iff, for every $n \in \mathbb{N}$, there is a corresponding group homomorphism τ_n on $R(n)$ to $S(n)$, and, for every $m \in \mathbb{N}$ such that $R(n) = R(m)$, $S(n) = S(m)$.

2.4.4

Let P and Q be preorders. Describe Q^P and show it is a preorder.

Let $R, S : P \to Q$. Then R and S are objects of Q^P. Let τ be a natural transform $\tau : R \dot\to S$. τ is an arrow on R to S in Q^P. Since τ is natural and P is preorder, following diagram is commute for every pair of objects $p, p' \in P$. $a = f(p, p')$, where $f(p, p')$ is the only arrow on p to p'. Since Q is preorder, $\tau p = g(Rp, Sp)$, where $g(Rp, Sp)$ is the only arrow on Rp to Sp.

$$\begin{tikzcd} Rp \arrow[r, "\tau p"] \arrow[d, "Ra"'] & Sp \arrow[d, "Sa"] \\ Rp' \arrow[r, "\tau p'"] & Sp' \end{tikzcd}$$

From the two downward arrows in the diagram, we can say that $\mathrm{Im}(R)$ and $\mathrm{Im}(S)$ contain the preorder structure of P. There are two functors $P \to \mathrm{Im}(R)$ and $P \to \mathrm{Im}(S)$, where $\mathrm{Im}(T)$ is a category from the image of the object function of T and all arrows between any two pairs in the image.

As explained above, $\sigma p = g(Rp, Sp)$ for all $\sigma : R \dot{\to} S$. Thus $|\hom(R, S)| \leq |\{\sigma \mid \forall p \in P,\ \sigma p = g(Rp, Sp)\}| = 1$. Therefore Q^P is preorder.

2.4.5

Let Fin be a category of all finite sets and G be a finite group. Describe $\mathbf{Fin}^G$.

Let **Fin** be a category of all finite sets. The object is every finite set and the arrow is every mapping between every pair of finite sets.

Let G be a finite group. G is a category of only one object. Every arrow a in G has its inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}$.

$\mathbf{Fin}^G$ is a category that have any functor on G to **Fin** as objects and any natural transform between two objects as arrows. The group G has only one object $x \in G$, thus any functor $T \in \mathbf{Fin}^G$ map x to a finite set $T(x) \in \mathbf{Fin}$, and endomorphisms of x to endomorphisms of $T(x)$. For any arrow a, b in G, $T(b \circ a) = T(b) \circ T(a)$. Also, $\mathrm{id} = T(\mathrm{id}) = T(a \circ a^{-1}) = T(a) \circ T(a^{-1})$. Thus any element in the image of the arrow function of T is invertible. Therefore T is a permutation representation of G.

Now, let $\tau : R \dot{\to} S$ be an arrow in $\mathbf{Fin}^G$. Then, the following diagram commutes for any arrow a in G:

$$\begin{tikzcd} Rx \arrow[r, "\tau x"] \arrow[d, "Ra"'] & Sx \arrow[d, "Sa"] \\ Rx \arrow[r, "\tau x"] & Sx \end{tikzcd}$$

Therefore, $\hom(R, S) = \{\tau \mid \forall a,\ \tau x \circ Ra = Sa \circ \tau x\}$... what does it mean???? TODO

2.4.6

TODO

2.4.7

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2.4.8

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2.5

2.5.1

Let A, B and C are small categories. Prove that $\mathbf{Cat}(A \times B, C) \cong \mathbf{Cat}(A, C^B)$ and that they are natural.

First, we prove there is a bijection $f : \mathbf{Cat}(A \times B, C) \to \mathbf{Cat}(A, C^B)$. Let $f(F) = G$, where $F : A \times B \to C$ and $G : A \to C^B$.

We define the object function of G. Let $G(a) = H$, where $H : B \to C$ and $a \in A$. Let $H(b) = F(\langle a, b \rangle)$ for objects and $H(b) = F(\langle \mathrm{id}(a), b \rangle)$ for arrows. Then for any $b \in B$, $H(\mathrm{id}(b)) = F(\langle \mathrm{id}(a), \mathrm{id}(b) \rangle) = \mathrm{id}(H(b))$. For any arrow x, y in A, $H(x \circ y) = F(\langle \mathrm{id}(a), x \circ y \rangle) = F(\langle \mathrm{id}(a), x \rangle) \circ F(\langle \mathrm{id}(a), y \rangle) = H(x) \circ H(y)$. Therefore H is a functor.

We define the arrow function of G. Let $G(a) = \tau$, where a is an arrow in A and $\tau : B \dot{\to} C$. Let $\tau(b) = F(\langle a, \mathrm{id}(b) \rangle)$.

TODO

2.5.2

Let A, B and C are categories. Prove that $(A \times B)^C \cong A^C \times B^C$ and $C^{A \times B} \cong (C^B)^A$.

For any $x \in (A \times B)^C$, there are two functions $G(c) = x(c)_0$ and $H(c) = x(c)_1$. Let the object function be $F(x) = \langle G, H \rangle$, where $F : (A \times B)^C \to A^C \times B^C$.

For any natural transform $\tau : (A \times B) \dot{\to} C$, there are two natural transforms $\tau_0 : A \dot{\to} C$ and $\tau_1 : B \dot{\to} C$. Let $\tau_0(x) =$ and $\tau_1(x) =$.

Thus we can construct the arrow function $F(\tau) = \langle \tau_0, \tau_1 \rangle$.

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