# Notto - Category for the Working Mathematician

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This is solutions for problems on the second edition of *Category for the Working Mathematician* by S. Mac Lane. Questions are placed in the first line of each sections. Questions may be modified from the original text for simplicity and clarity.

# 1

## 1.1

$$\langle \wedge \_ \wedge \rangle$$

## 1.2

$$\langle Q \perp Q \rangle$$

## 1.3

### 1.3.1

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### 1.3.2

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### 1.3.3

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### 1.3.4

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### 1.3.5

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## 1.4

### 1.4.1

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### 1.4.2

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### 1.4.3

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### 1.4.5

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### 1.4.6

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## 1.5

### 1.5.1

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### 1.5.2

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### 1.5.9

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## 1.6

Let $U$ be a set that satisfies following conditions:

(i) $x \in u \in U \Rightarrow x \in U$

(ii) $(u \in U \wedge v \in U) \Rightarrow (\{u, v\}, \langle u, v \rangle, u \times v \in U)$

(iii) (1) $x \in U \Rightarrow \mathcal{P}x \in U$

(2) $x \in U \Rightarrow \bigcup x \in U$

(iv) $\omega \in U$, where $\omega = \{0, 1, 2, \dots\}$ is a set of all finite ordinal numbers.

(v) If there exists a surjection $f : a \to b$ and $a \in U$ and $b \subset U$, then $b \in U$.

### 1.6.1

**Let $I \in U$, $f : I \to b$ and $f_i \in U$ for all $i \in I$. Proof $\prod_i f_i \in U$.**

For all $q \in \prod_i f_i$, we can construct a bijection $r : I \to q$. $q \subset U$ because $\forall w \in q,\ \exists j \in I,\ w \in f_j \in U$. Since $I \in U$ and $q \subset U$, we can say $q \in U$. Therefore $\prod_i f_i \subset U$.

Let $|f_k| \geq |f_i|$ for all $i$. Then, we can construct a surjection $g : f_k^I \to \prod_i f_i$. Also, we can construct a surjection $h : X \to f_k^I$, with X is either $\mathcal{P}f_k$ or $\mathcal{P}I$. As $X \in U$, $g \circ h : X \to \prod_i f_i$ and $\prod_i f_i \subset U$, we can say $\prod_i f_i \in U$.

### 1.6.2

**(a) Let $I \in U$, $f : I \to b$ and $f_i \in U$ for all $i \in I$. Proof $\bigcup_i f_i \in U$.**
**(b) Proof that (a) implies following if (i), (ii), (iii)(1), (iv) and (v) holds true:**

 **(iii)(2) $x \in U \Rightarrow \bigcup x \in U$.**
 **(v) If $f : a \to b$ is surjective and $a \in U$ and $b \subset U$, then $b \in U$.**

(a) We can construct a bijection $g : I \to \{f_i \mid i \in I\}$, $g(i) = f_i$. As $I \in U$ and $f_i \in U$ for all $i$, $\{f_i \mid i \in I\} \in U$. Therefore $\bigcup_i f_i = \bigcup\{f_i \mid i \in I\} \in U$.

(b) Because $x \in U$, we have $y \in U$ for all $y \in x$. Therefore we can apply $f : x \to x, f(i) = i$ to (a) to get $\bigcup x \in U$.

Because $a \in U$ and $b \subset U$, we can apply $f$ to (a) to get $\bigcup_i f_i \in U$. $f$ is surjective, therefore $b = \bigcup_i f_i$. Hence $b \in U$.

## 1.7

$$\langle Q \mathbin{\dot\smile} Q \rangle$$

## 1.8

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr, "h"] & & & * \arrow[dl, "i"'] \arrow[dr] & \\ * & & * & * & & * \end{tikzcd}$$

$$\begin{tikzcd} \phi \arrow[r] & \theta \end{tikzcd}$$

# 2

## 2.1

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr, "h"] & & & * \arrow[dl, "i"'] \arrow[dr] & \\ * & & * & * & & * \end{tikzcd}$$

$$\begin{tikzcd} \phi \arrow[r, bend right] & \theta \end{tikzcd}$$

## 2.2

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr, "h"] & \\ * & & * \end{tikzcd} \qquad \begin{tikzcd} & * \arrow[dl, "i"'] \arrow[dr] & \\ * & & * \end{tikzcd}$$

$$\begin{tikzcd} \phi \arrow[loop above] \end{tikzcd}$$

## 2.3

### 2.3.1

**Show product of categories includes product of monoids, product of groups and product of sets.**

In this section, $\times_S$ is a product operation for sets, $\times_C$ is for categories, $\times_G$ is for groups and $\times_M$ is for monoids.

*Monoids.* Let $M, N$ be a monoid with object $m, n$ respectively. The only object in $M \times_C N$ is $\langle m, n \rangle$. ... (TODO)

*Groups.* Let $G, H$ be a group with object $a, b$ respectively. $G \times_C H$ ... (TODO)

*Sets.* Let $A, B$ be discrete categories. The set of all objects in $A \times_C B$ is $X = A \times_S B$. The set of all arrows in $A \times_C B$ is $\{\langle f, g \rangle \mid a \in A,\ b \in B,\ f \in \hom_A(a, a) \wedge g \in \hom_B(b, b)\} = \{\langle \mathrm{id}_A(c_0), \mathrm{id}_B(c_1) \rangle \mid c \in X\}$. Therefore $A \times_C B$ is a discrete category of $X = A \times_S B$.

### 2.3.2

**Proof that product of two preorders is preorder.**

Let P, Q be preorders. $\forall a \forall b\ |\hom(a, b)| \leq 1$ for both P and Q. Therefore, for all $p_1 \in P,\ p_2 \in P,\ q_1 \in Q,\ q_2 \in Q,\ |\hom_{P \times Q}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle)| = |\hom_P(p_1, p_2) \times \hom_Q(q_1, q_2)| \leq 1$. Hence $P \times Q$ is preorder.

### 2.3.3

**Let $\{C_i \mid i \in I\}$ be a family of categories indexed by a set $I$. Show product $C = \prod_i C_i$, its projections $P_i : C \to C_i$ and universal property of these projections.**

Let $\mathrm{id}_X(a) : a \to a$ be identity in category $X$.

Let the set of all object in $C$ be the product set $\prod_i C_i$ and $\hom_C(a, b) = \prod_i \hom_{C_i}(a_i, b_i)$. Let $\mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i)$ for all $c \in C$. Now, we proof $C$ has a universal property:

1. For every $i$ there is a functor $P_i : C \to C_i$.

2. For every category $B$ such that a functor $G_i : B \to C_i$ presents for every $C_i$, there is a functor $F : B \to C$, which makes the following diagram commute.

$$\begin{tikzcd} B \arrow[d, dashed, "F"'] \arrow[dr, "G_i"] & \\ C \arrow[r, "P_i"] & C_i \end{tikzcd}$$

First, we proof $P_i : C \to C_i$ exists. Let the object function be $P_i(x) = x_i$. Let the arrow function be $P_i(f) = f_i$. For all object $c \in C$, $P_i(\mathrm{id}_C(c)) = \mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i) = \mathrm{id}_{C_i}(P_i(c))$. For all arrow $f$, $g$ in C, $P_i(g \circ f) = (g \circ f)_i = g_i \circ f_i = P_i(g) \circ P_i(f)$. Therefore $P_i$ is a functor.

Second, we proof $F : B \to C$ exists. Let the object function be $F(x)_i = G_i(x)$. Let the arrow function be $F(f)_i = G_i(f)$. For all object $b \in B$, $F(\mathrm{id}_B(b))_i = G_i(\mathrm{id}_B(b)) = \mathrm{id}_{C_i}(G_i(b)) = \mathrm{id}_{C_i}(F(b)_i)$. Thus $F(\mathrm{id}_B(b)) = \mathrm{id}_C(F(b))$. For all arrow $f$, $g$ in B, $F(f \circ g)_i = G_i(f \circ g) = G_i(f) \circ G_i(g)$. Thus $F(f \circ g) = F(f) \circ F(g)$. Therefore $F$ is a functor.

### 2.3.4

**Show opposite of $\mathbf{Matr}_K$.**

In $\mathbf{Matr}_K$, the object set is all positive integers $\{1, 2, 3, ...\} = \omega \setminus \{0\}$. $\hom_{\mathbf{Matr}_K}(n, m)$ is all rectangular matrix on $K$ with shape $m \times n$. Therefore $\mathbf{Matr}_K^{\mathrm{op}}$ has the same objects $\omega \setminus \{0\}$ and $\hom_{\mathbf{Matr}_K^{\mathrm{op}}}(n, m)$ is all rectangular matrix on $K$ with shape $n \times m$.

### 2.3.5

**Show that the mapping between a topological space and the ring of real continuous functions on it is the object function of a contravariant functor on Top to Rng.**

Let $R_T \subseteq (T \to \mathbb{R})$ be a ring whose elements are continuous functions from a topological space $T$ to real number. We construct $R_T$ as follows:

1. Additive identity. $0_{R_T} : x \mapsto 0$.
2. Multiplicative identity. $1_{R_T} : x \mapsto 1$.
3. Addition. $f + g : x \mapsto f(x) + g(x)$.
4. Multiplication. $f \times g : x \mapsto f(x) \times g(x)$.

Let $X$ and $Y$ be any topological spaces. If we have a continuous function $f : Y \to X$, we can construct a ring homomorphism $H(f) = h : R_X \to R_Y$. We define $h(r) = r \circ f$. Then $h(0_{R_X}) = 0_{R_Y}$, $h(1_{R_X}) = 1_{R_Y}$, $(h(s + t))(x) =$

$(s+t)(f(x)) = s(f(x))+t(f(x))$, $(h(s\times t))(x) = (s\times t)(f(x)) = s(f(x))\times t(f(x))$. Therefore $H(f)$ is a ring homomorphism.

Now we construct a functor $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Rng}$. Let the object function be $F(A) = R_A$ and the arrow function be $F(g) = H(g^{\mathrm{op}})$. For all arrow $a, b$ in $\mathbf{Top}^{\mathrm{op}}$, $F(b \circ a) = H((b\circ a)^{\mathrm{op}}) = H(a^{\mathrm{op}} \circ b^{\mathrm{op}}) = H(b^{\mathrm{op}}) \circ H(a^{\mathrm{op}}) = F(b) \circ F(a)$. For all topological space $T \in \mathbf{Top}^{\mathrm{op}}$, $F(\mathrm{id}(T)) = H(\mathrm{id}(T)) = \mathrm{id}(R_T)$. Therefore $F$ is a functor and $\overline{F}$ is a contravariant functor on $\mathbf{Top}$ to $\mathbf{Rng}$.

## 2.4

### 2.4.1

**Show that for any ring $R$, $R$-Mod is a full subcategory of $\mathbf{Ab}^R$.**

TODO

### 2.4.2

**For a finite discrete category $X$, describe $B^X$.**

For any functor $T : X \to B$, if its object function is $T(a) = b$, its arrow function maps $\mathrm{id}(a)$ to $\mathrm{id}(b)$. Such a functor $T$ is an object of $B^X$.

Let $R, S : X \to B$ be functors and $\tau$ be a map on an object of X to an arrow in $B$. $(\tau : R \dot{\to} S) \Leftrightarrow (\forall x \in X,\ \tau_x(R(x)) = S(x))$. Therefore $\hom(R, S) = \{\tau \mid \forall x \in X,\ \tau_x(R(x)) = S(x)\}$. Therefore an arrow on $R$ to $S$ exists iff $(\forall x, y \in X,\ e_R(x, y) \to e_S(x, y)) \ \wedge\ (\forall x \in X,\ \hom(R(x), S(x)) \neq \emptyset)$, where $e_T(x, y) \Leftrightarrow (\exists a \in B, \{x, y\} \subseteq \{w \mid a = T(w)\})$.

### 2.4.3

**Let N be a discrete category of natural numbers. Describe $\mathbf{Ab}^{\mathbf{N}}$.**

An object of $\mathbf{Ab}^{\mathbf{N}}$ is a map on $\mathbb{N}$ to $\mathbf{Ab}$. Same as above, $\hom(R, S) = \{\tau \mid \forall n \in N,\ \tau_n(R(n)) = S(n)\}$. In other words, a map $\tau : \mathbb{N} \to (\mathbf{Ab} \to \mathbf{Ab})$ is an arrow iff, for every $n \in \mathbb{N}$, there is a corresponding group homomorphism $\tau_n$ on $R(n)$ to $S(n)$, and, for every $m \in \mathbb{N}$ such that $R(n) = R(m)$, $S(n) = S(m)$.

### 2.4.4

**Let $P$ and $Q$ be preorders. Describe $Q^P$ and show it is a preorder.**

Let $R, S : P \to Q$. Then $R$ and $S$ are objects of $Q^P$. Let $\tau$ be a natural transform $\tau : R \dot{\to} S$. $\tau$ is an arrow on $R$ to $S$ in $Q^P$. Since $\tau$ is natural and $P$ is preorder, following diagram is commute for every pair of objects $p, p' \in P$. $a = f(p, p')$, where $f(p, p')$ is the only arrow on $p$ to $p'$. Since $Q$ is preorder, $\tau p = g(Rp, Sp)$, where $g(Rp, Sp)$ is the only arrow on $Rp$ to $Sp$.

$$\begin{tikzcd} Rp \arrow[r, "\tau p"] \arrow[d, "Ra"'] & Sp \arrow[d, "Sa"] \\ Rp' \arrow[r, "\tau p'"] & Sp' \end{tikzcd}$$

From the two downward arrows in the diagram, we can say that $\mathrm{Im}(R)$ and $\mathrm{Im}(S)$ contain the preorder structure of $P$. There are two functors $P \to \mathrm{Im}(R)$ and $P \to \mathrm{Im}(S)$, where $\mathrm{Im}(T)$ is a category from the image of the object function of $T$ and all arrows between any two pairs in the image.

As explained above, $\sigma p = g(Rp, Sp)$ for all $\sigma : R \dot{\to} S$. Thus $|\hom(R,S)| \leq |\{\sigma \mid \forall p \in P,\ \sigma p = g(Rp, Sp)\}| = 1$. Therefore $Q^P$ is preorder.

### 2.4.5

**Let Fin be a category of all finite sets and $G$ be a finite group. Describe $\mathbf{Fin}^G$.**

Let **Fin** be a category of all finite sets. The object is every finite set and the arrow is every mapping between every pair of finite sets.

Let $G$ be a finite group. $G$ is a category of only one object. Every arrow $a$ in $G$ has its inverse $a^{-1}$ such that $a \circ a^{-1} = a^{-1} \circ a = \mathrm{id}$.

$\mathbf{Fin}^G$ is a category that have any functors on $G$ to **Fin** as objects and any natural transform between two objects as arrows. The group $G$ has only one object $x \in G$, thus any functor $T \in \mathbf{Fin}^G$ maps a object $x$ to a finite set $T(x) \in$ **Fin**, and endomorphisms of $x$ to endomorphisms of $T(x)$. For any arrows $a$, $b$ in $G$, $T(b \circ a) = T(b) \circ T(a)$. Also, $\mathrm{id} = T(\mathrm{id}) = T(a \circ a^{-1}) = T(a) \circ T(a^{-1})$. Thus any elements in the image of the arrow function of $T$ is invertible. Therefore $T$ is a permutation representation of $G$.

Now, let $\tau : R \dot{\to} S$ be an arrow in $\mathbf{Fin}^G$. Then, the following diagram commutes for any arrow $a$ in $G$:

$$\begin{tikzcd}
Rx \arrow[r, "\tau x"] \arrow[d, "Ra"'] & Sx \arrow[d, "Sa"] \\
Rx \arrow[r, "\tau x"] & Sx
\end{tikzcd}$$

Therefore, $\hom(R,S) = \{\tau \mid \forall a,\ \tau x \circ Ra = Sa \circ \tau x\}$.

### 2.4.6

TODO

### 2.4.7

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### 2.4.8

TODO

## 2.5

### 2.5.1

**Let A, B and C are small categories. Proof that $C^{(A\times B)} \cong (C^B)^A$ and proof that they are natural. Proof that**

### 2.5.2

### 2.5.3

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