

# Notto - Category for the Working Mathematician

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This is solutions for problems on the second edition of *Category for the Working Mathematician* by S. Mac Lane.

## 1

### 1.1

$$\langle \wedge \_ \wedge \rangle$$

### 1.2

$$\langle Q \perp Q \rangle$$

### 1.3

#### 1.3.1

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#### 1.3.2

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#### 1.3.3

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#### 1.3.4

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#### 1.3.5

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## **1.4**

### **1.4.1**

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### **1.4.6**

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## **1.5**

### **1.5.1**

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### **1.5.2**

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### 1.5.9

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## 1.6

Let  $U$  be a set that satisfies following conditions:

- (i)  $x \in u \in U \Rightarrow x \in U$
- (ii)  $(u \in U \wedge v \in U) \Rightarrow (\{u, v\}, \langle u, v \rangle, u \times v \in U)$
- (iii) (1)  $x \in U \Rightarrow \mathcal{P}x \in U$   
 (2)  $x \in U \Rightarrow \bigcup x \in U$
- (iv)  $\omega \in U$ , where  $\omega = \{0, 1, 2, \dots\}$  is a set of all finite ordinal numbers.
- (v) If there exists a surjection  $f : a \rightarrow b$  and  $a \in U$  and  $b \subset U$ , then  $b \in U$ .

### 1.6.1

For all  $q \in \prod_i f_i$ , we can construct a bijection  $r : I \rightarrow q$ .  $q \subset U$  because  $\forall w \in q, \exists j \in I, w \in f_j \in U$ . Since  $I \in U$  and  $q \subset U$ , we can say  $q \in U$ . Therefore  $\prod_i f_i \subset U$ .

Let  $|f_k| \geq |f_i|$  for all  $i$ . Then, we can construct a surjection  $g : f_k^I \rightarrow \prod_i f_i$ . Also, we can construct a surjection  $h : X \rightarrow f_k^I$ , with  $X$  is either  $\mathcal{P}f_k$  or  $\mathcal{P}I$ . As  $X \in U$ ,  $g \circ h : X \rightarrow \prod_i f_i$  and  $\prod_i f_i \subset U$ , we can say  $\prod_i f_i \in U$ .

### 1.6.2

$$\langle Q \circ Q \rangle$$

- (a) We can construct a bijection  $g : I \rightarrow \{f_i \mid i \in I\}$ ,  $g(i) = f_i$ . As  $I \in U$  and  $f_i \in U$  for all  $i$ ,  $\{f_i \mid i \in I\} \in U$ . Therefore  $\bigcup_i f_i = \bigcup \{f_i \mid i \in I\} \in U$ .

(b) Because  $x \in U$ , we have  $y \in U$  for all  $y \in x$ . Therefore we can apply  $f : x \rightarrow x, f(i) = i$  to (a) to get  $\bigcup x \in U$ .

Because  $a \in U$  and  $b \subset U$ , we can apply  $f$  to (a) to get  $\bigcup_i f_i \in U$ .  $f$  is surjective, therefore  $b = \bigcup_i f_i$ . Hence  $b \in U$ .

1.7



$$\phi \longrightarrow \theta$$

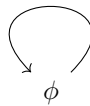
1.8



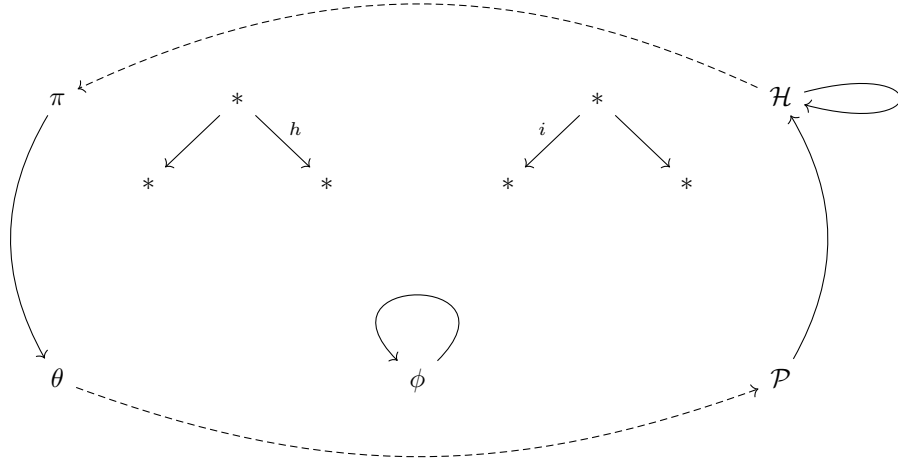
$$\phi \curvearrowright \theta$$

2

2.1

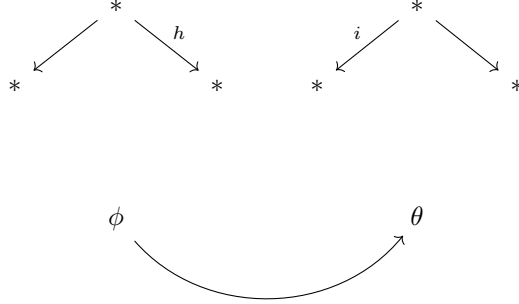


## 2.2



## 2.3

### 2.3.1



In this section,  $\times_S$  is a product operation for sets,  $\times_C$  is for categories,  $\times_G$  is for groups and  $\times_M$  is for monoids.

*Monoids.* Let  $M, N$  be a monoid with object  $m, n$  respectively. The only object in  $M \times_C N$  is  $\langle m, n \rangle$ . ... (TODO)

*Groups.* Let  $G, H$  be a group with object  $a, b$  respectively.  $G \times_C H$  ... (TODO)

*Sets.* Let  $A, B$  be discrete categories. The set of all objects in  $A \times_C B$  is  $X = A \times_S B$ . The set of all arrows in  $A \times_C B$  is  $\{\langle f, g \rangle \mid a \in A, b \in B, f \in A(a, a) \wedge g \in B(b, b)\} = \{\langle \text{id}_A(c_0), \text{id}_B(c_1) \rangle \mid c \in X\}$ . Therefore  $A \times_C B$  is a discrete category of  $X = A \times_S B$ .

### 2.3.2

$$\langle \wedge \_ \wedge \rangle$$

Let  $P, Q$  be preorders.  $\forall a \forall b \ | \text{hom}(a, b)| \leq 1$  for both  $P$  and  $Q$ . Therefore, for all  $p_1 \in P, p_2 \in P, q_1 \in Q, q_2 \in Q, |\text{hom}_{P \times Q}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle)| = |\text{hom}_P(p_1, p_2) \times \text{hom}_Q(q_1, q_2)| \leq 1$ . Hence  $P \times Q$  is preorder.

### 2.3.3

$$\langle Q \perp Q \rangle$$

Let  $\text{id}_X(a) : a \rightarrow a$  be identity in category  $X$ .

Let the set of all object in  $C$  be the product set  $\prod_i C_i$  and  $C(a, b) = \prod_i C_i(a_i, b_i)$ . Let  $\text{id}_C(c)_i = \text{id}_{C_i}(c_i)$  for all  $c \in C$ . Now, we prove  $C$  has a universal property:

1. For every  $i$  there is a functor  $P_i : C \rightarrow C_i$ .
2. For every category  $B$  such that a functor  $G_i : B \rightarrow C_i$  presents for every  $C_i$ , there is a functor  $F : B \rightarrow C$ , which makes the following diagram commute.

$$\begin{array}{ccc} B & & \\ \downarrow F & \searrow G_i & \\ C & \xrightarrow{P_i} & C_i \end{array}$$

First, we prove  $P_i : C \rightarrow C_i$  exists. Let the object function be  $P_i(x) = x_i$ . Let the arrow function be  $P_i(f) = f_i$ . For all object  $c \in C$ ,  $P_i(\text{id}_C(c)) = \text{id}_C(c)_i = \text{id}_{C_i}(c_i) = \text{id}_{C_i}(P_i(c))$ . For all arrow  $f, g$  in  $C$ ,  $P_i(g \circ f) = (g \circ f)_i = g_i \circ f_i = P_i(g) \circ P_i(f)$ . Therefore  $P_i$  is a functor.

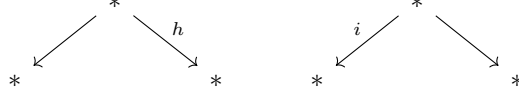
Second, we prove  $F : B \rightarrow C$  exists. Let the object function be  $F(x)_i = G_i(x)$ . Let the arrow function be  $F(f)_i = G_i(f)$ . For all object  $b \in B$ ,  $F(\text{id}_B(b))_i = G_i(\text{id}_B(b)) = \text{id}_{C_i}(G_i(b)) = \text{id}_{C_i}(F(b)_i)$ . Thus  $F(\text{id}_B(b)) = \text{id}_C(F(b))$ . For all arrow  $f, g$  in  $B$ ,  $F(f \circ g)_i = G_i(f \circ g) = G_i(f) \circ G_i(g)$ . Thus  $F(f \circ g) = F(f) \circ F(g)$ . Therefore  $F$  is a functor.

### 2.3.4

$$\langle Q \circ Q \rangle$$

In  $\mathbf{Matr}_K$ , the object set is all positive integers  $\{1, 2, 3, \dots\} = \omega \setminus \{0\}$ .  $\mathbf{Matr}_K(n, m)$  is all rectangular matrix on  $K$  of shape  $m \times n$ . Therefore  $\mathbf{Matr}_K^{\text{op}}$  has the same objects  $\omega \setminus \{0\}$  and  $\mathbf{Matr}_K^{\text{op}}(n, m)$  is all rectangular matrix on  $K$  of shape  $n \times m$ .

### 2.3.5



$$\phi \longrightarrow \theta$$

Let  $R_T \subseteq (T \rightarrow \mathbb{R})$  be a ring whose elements are continuous functions from a topological space  $T$  to real number. We construct  $R_T$  as follows:

1. Additive identity.  $0_{R_T} : x \mapsto 0$ .
2. Multiplicative identity.  $1_{R_T} : x \mapsto 1$ .
3. Addition.  $f + g : x \mapsto f(x) + g(x)$ .
4. Multiplication.  $f \times g : x \mapsto f(x) \times g(x)$ .

Let  $X$  and  $Y$  be any topological spaces. If we have a continuous function  $f : Y \rightarrow X$ , we can construct a ring homomorphism  $H(f) = h : R_X \rightarrow R_Y$ . We define  $h(r) = r \circ f$ . Then  $h(0_{R_X}) = 0_{R_Y}$ ,  $h(1_{R_X}) = 1_{R_Y}$ ,  $(h(s + t))(x) = (s + t)(f(x)) = s(f(x)) + t(f(x))$ ,  $(h(s \times t))(x) = (s \times t)(f(x)) = s(f(x)) \times t(f(x))$ . Therefore  $H(f)$  is a ring homomorphism.

Now we construct a functor  $F : \mathbf{Top}^{\text{op}} \rightarrow \mathbf{Rng}$ . Let the object function be  $F(A) = R_A$  and the arrow function be  $F(g) = H(g^{\text{op}})$ . For all arrow  $a, b$  in  $\mathbf{Top}^{\text{op}}$ ,  $F(b \circ a) = H((b \circ a)^{\text{op}}) = H(a^{\text{op}} \circ b^{\text{op}}) = H(b^{\text{op}}) \circ H(a^{\text{op}}) = F(b) \circ F(a)$ . For all topological space  $T \in \mathbf{Top}^{\text{op}}$ ,  $F(\text{id}(T)) = H(\text{id}(T)) = \text{id}(R_T)$ . Therefore  $F$  is a functor and  $\bar{F}$  is a contravariant functor on  $\mathbf{Top}$  to  $\mathbf{Rng}$ .

## 2.4

### 2.4.1

TODO

### 2.4.2

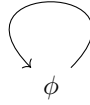


$$\phi \longrightarrow \theta$$

For any functor  $T : X \rightarrow B$ , if its object function is  $T(a) = b$ , its arrow function maps  $\text{id}(a)$  to  $\text{id}(b)$ . Such a functor  $T$  is an object of  $B^X$ .

Let  $R, S : X \rightarrow B$  be functors and  $\tau$  be a map on an object of  $X$  to an arrow in  $B$ .  $(\tau : R \rightarrow S) \Leftrightarrow (\forall x \in X, \tau_x(R(x)) = S(x))$ . Therefore  $\text{hom}(R, S) = \{\tau \mid \forall x \in X, \tau_x(R(x)) = S(x)\}$ . Therefore an arrow on  $R$  to  $S$  exists iff  $(\forall x, y \in X, e_R(x, y) \rightarrow e_S(x, y)) \wedge (\forall x \in X, \text{hom}(R(x), S(x)) \neq \emptyset)$ , where  $e_T(x, y) \Leftrightarrow (\exists a \in B, \{x, y\} \subseteq \{w \mid a = T(w)\})$ .

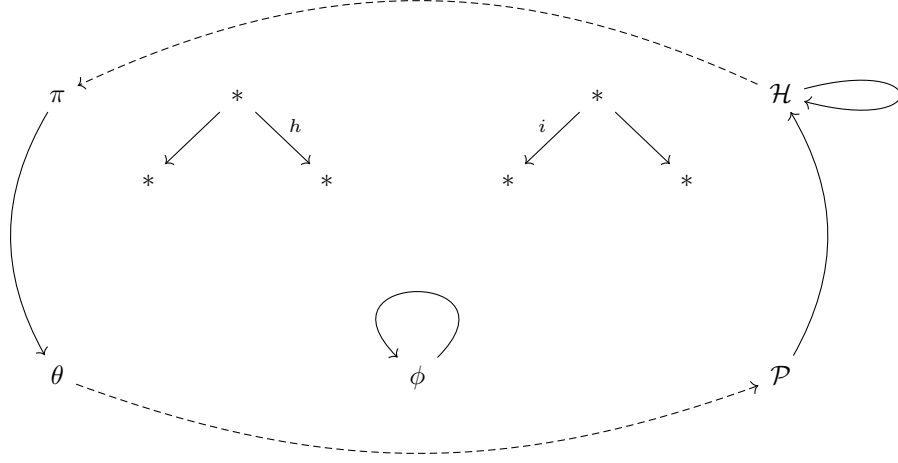
### 2.4.3



An object of  $\mathbf{Ab}^{\mathbb{N}}$  is a map on  $\mathbb{N}$  to  $\mathbf{Ab}$ . Same as above,  $\text{hom}(R, S) = \{\tau \mid \forall n \in \mathbb{N}, \tau_n(R(n)) = S(n)\}$ . In other words, a map  $\tau : \mathbb{N} \rightarrow (\mathbf{Ab} \rightarrow \mathbf{Ab})$  is an arrow iff, for every  $n \in \mathbb{N}$ , there is a corresponding group homomorphism  $\tau_n$  on  $R(n)$  to  $S(n)$ , and, for every  $m \in \mathbb{N}$  such that  $R(n) = R(m)$ ,  $S(n) = S(m)$ .

### 2.4.4





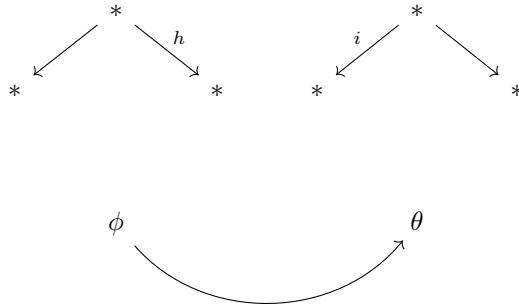
Let  $R, S : P \rightarrow Q$ . Then  $R$  and  $S$  are objects of  $Q^P$ . Let  $\tau$  be a natural transform  $\tau : R \rightarrow S$ .  $\tau$  is an arrow on  $R$  to  $S$  in  $Q^P$ . Since  $\tau$  is natural and  $P$  is preorder, following diagram is commute for every pair of objects  $p, p' \in P$ .  $a = f(p, p')$ , where  $f(p, p')$  is the only arrow on  $p$  to  $p'$ . Since  $Q$  is preorder,  $\tau p = g(Rp, Sp)$ , where  $g(Rp, Sp)$  is the only arrow on  $Rp$  to  $Sp$ .

$$\begin{array}{ccc} Rp & \xrightarrow{\tau p} & Sp \\ Ra \downarrow & & \downarrow Sa \\ Rp' & \xrightarrow{\tau p'} & Sp' \end{array}$$

From the two downward arrows in the diagram, we can say that  $\text{Im}(R)$  and  $\text{Im}(S)$  contain the preorder structure of  $P$ . There are two functors  $P \rightarrow \text{Im}(R)$  and  $P \rightarrow \text{Im}(S)$ , where  $\text{Im}(T)$  is a category from the image of the object function of  $T$  and all arrows between any two pairs in the image.

As explained above,  $\sigma p = g(Rp, Sp)$  for all  $\sigma : R \rightarrow S$ . Thus  $|\text{hom}(R, S)| \leq |\{\sigma \mid \forall p \in P, \sigma p = g(Rp, Sp)\}| = 1$ . Therefore  $Q^P$  is preorder.

#### 2.4.5



Let **Fin** be a category of all finite sets. The object is every finite set and the arrow is every mapping between every pair of finite sets.

Let  $G$  be a finite group.  $G$  is a category of only one object. Every arrow  $a$  in  $G$  has its inverse  $a^{-1}$  such that  $a \circ a^{-1} = a^{-1} \circ a = \text{id}$ .

$\mathbf{Fin}^G$  is a category that have any functor on  $G$  to **Fin** as objects and any natural transform between two objects as arrows. The group  $G$  has only one object  $x \in G$ , thus any functor  $T \in \mathbf{Fin}^G$  map  $x$  to a finite set  $T(x) \in \mathbf{Fin}$ , and endomorphisms of  $x$  to endomorphisms of  $T(x)$ . For any arrow  $a, b$  in  $G$ ,  $T(b \circ a) = T(b) \circ T(a)$ . Also,  $\text{id} = T(\text{id}) = T(a \circ a^{-1}) = T(a) \circ T(a^{-1})$ . Thus any element in the image of the arrow function of  $T$  is invertible. Therefore  $T$  is a permutation representation of  $G$ .

Now, let  $\tau : R \rightarrow S$  be an arrow in  $\mathbf{Fin}^G$ . Then, the following diagram commutes for any arrow  $a$  in  $G$ :

$$\begin{array}{ccc} Rx & \xrightarrow{\tau x} & Sx \\ Ra \downarrow & & \downarrow Sa \\ Rx & \xrightarrow{\tau x} & Sx \end{array}$$

Therefore,  $\text{hom}(R, S) = \{\tau \mid \forall a, \tau x \circ Ra = Sa \circ \tau x\} \dots$  what does it mean???

#### 2.4.6

TODO

#### 2.4.7

TODO

#### 2.4.8

TODO

### 2.5

#### 2.5.1

$$\langle \wedge \_ \wedge \rangle$$

First, we prove there is a bijection  $f : \mathbf{Cat}(A \times B, C) \rightarrow \mathbf{Cat}(A, C^B)$ . Let  $f(F) = G$ , where  $F : A \times B \rightarrow C$  and  $G : A \rightarrow C^B$ .

We define the object function of  $G$ . Let  $G(a) = H$ , where  $H : B \rightarrow C$  and  $a \in A$ . Let  $H(b) = F(\langle a, b \rangle)$  for objects and  $H(b) = F(\langle \text{id}(a), b \rangle)$  for arrows. Then for any  $b \in B$ ,  $H(\text{id}(b)) = F(\langle \text{id}(a), \text{id}(b) \rangle) = \text{id}(H(b))$ . For any arrow  $x, y$  in  $A$ ,  $H(x \circ y) = F(\langle \text{id}(a), x \circ y \rangle) = F(\langle \text{id}(a), x \rangle) \circ F(\langle \text{id}(a), y \rangle) = H(x) \circ H(y)$ . Therefore  $H$  is a functor.

We define the arrow function of  $G$ . Let  $G(a) = \tau$ , where  $a$  is an arrow in  $A$  and  $\tau : B \rightarrow C$ . Let  $\tau(b) = F(\langle a, \text{id}(b) \rangle)$ .

TODO

### 2.5.2

$$\langle Q \perp Q \rangle$$

For any  $x \in (A \times B)^C$ , there are two functions  $G(c) = x(c)_0$  and  $H(c) = x(c)_1$ . Let the object function be  $F(x) = \langle G, H \rangle$ , where  $F : (A \times B)^C \rightarrow A^C \times B^C$ .

For any natural transform  $\tau : (A \times B) \rightarrow C$ , there are two natural transforms  $\tau_0 : A \rightarrow C$  and  $\tau_1 : B \rightarrow C$ . Let  $\tau_0(x) =$  and  $\tau_1(x) =$ .

Thus we can construct the arrow function  $F(\tau) = \langle \tau_0, \tau_1 \rangle$ .

**2.5.3**

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