Not to - Categories for the Working Mathematician

tkunn

August 2020

This is notes for problems on $\it Categories$ for the Working Mathematician by S. Mac Lane.

1 1.1 $\langle \land _ \land \rangle$ 1.2 $\langle Q \perp Q \rangle$ 1.3 1.3.1 TODO
1.3.2 TODO
1.3.3 TODO

1.3.4 TODO

1.3.5

TODO

1.4

1.4.1

TODO

1.4.2

TODO

1.4.3

TODO

1.4.4

TODO

1.4.5

TODO

1.4.6

TODO

1.5

1.5.1

TODO

1.5.2

TODO

1.5.3

TODO

1.5.4

TODO

1.5.5

TODO

1.5.6

TODO

1.5.7

TODO

1.5.8

TODO

1.5.9

TODO

1.6

Let U be a set that satisfies following conditions:

- (i) $x \in u \in U \Rightarrow x \in U$
- (ii) $(u \in U \land v \in U) \Rightarrow (\{u, v\}, \langle u, v \rangle, u \times v \in U)$
- (iii) (1) $x \in U \Rightarrow \mathcal{P}x \in U$
 - (2) $x \in U \Rightarrow \bigcup x \in U$
- (iv) $\omega \in U$, where $\omega = \{0, 1, 2, \dots\}$ is a set of all finite ordinal numbers.
- (v) If there exists a surjection $f: a \to b$ and $a \in U$ and $b \subset U$, then $b \in U$.

1.6.1

Let $I \in U$, $f: I \to b$ and $f_i \in U$ for all $i \in I$. Proof $\prod_i f_i \in U$.

For all $q \in \prod_i f_i$, we can construct a bijection $r: I \to q$. $\forall w \in q, \exists j \in I, w \in f_j \in U$, hence $q \subset U$. As $I \in U$ and $q \subset U$, we can say $q \in U$. Therefore $\prod_i f_i \subset U$.

Let $|f_k| \geq |f_i|$ for all i. Then, we can construct a surjection $g: f_k^I \to \prod_i f_i$. Also, we can construct a surjection $h: X \to f_k^I$, with X is either $\mathcal{P}f_k$ or $\mathcal{P}I$. As $X \in U$, $g \circ h: X \to \prod_i f_i$ and $\prod_i f_i \subset U$, we can say $\prod_i f_i \in U$. 1.6.2

(a) Let $I \in U$, $f: I \to b$ and $f_i \in U$ for all $i \in I$. Proof $\bigcup_i f_i \in U$.

(b) Proof that (a) implies following if (i), (ii), (iii), (iv) and (v) holds true:

(iii)(2) $x \in U \Rightarrow \bigcup x \in U$.

(v) If $f: a \to b$ is surjective and $a \in U$ and $b \subset U$, then $b \in U$.

(a) We can construct a bijection $g: I \to \{f_i \mid i \in I\}, g(i) = f_i$. As $I \in U$ and $f_i \in U$ for all $i, \{f_i \mid i \in I\} \in U$. Therefore $\bigcup_i f_i = \bigcup \{f_i \mid i \in I\} \in U$.

(b) Because $x \in U$, we have $y \in U$ for all $y \in x$. Therefore we can apply $f: x \to x, f(i) = i$ to (a) to get $\bigcup x \in U$.

Because $a \in U$ and $b \subset U$, we can apply f to (a) to get $\bigcup_i f_i \in U$. f is surjective, therefore $b = \bigcup_i f_i$. Hence $b \in U$.

1.7

 $\langle Q \ \dot{\ } \ Q
angle$

1.8



 $\phi \longrightarrow \epsilon$

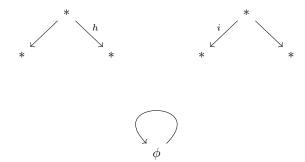
 $\mathbf{2}$

2.1



 $\phi \longrightarrow \theta$

2.2



2.3

2.3.1

Show product of categories includes product of monoids, product of groups and product of sets.

In this section, \times_S is a product operation for sets, \times_C is for categories, \times_G is for groups and \times_M is for monoids.

Monoids. Let M,N be a monoid with object m,n respectively. The only object in $M \times_C N$ is $\langle m,n \rangle$ (TODO)

Groups. Let G, H be a group with object a, b respectively. $G \times_C H$... (TODO)

Sets. Let A, B be discrete categories. The set of all objects in $A \times_C B$ is $X = A \times_S B$. The set of all arrows in $A \times_C B$ is $\{\langle f, g \rangle \mid a \in A, b \in B, f \in \text{hom}_A(a,a) \land g \in \text{hom}_B(b,b)\} = \{\langle \text{id}_A(c_0), \text{id}_B(c_1) \rangle \mid c \in X\}$. Therefore $A \times_C B$ is a discrete category of $X = A \times_S B$.

2.3.2

Proof that product of two preorders is preorder.

Let P, Q be preorders. $\forall a \forall b \mid \text{hom}(a,b) \mid \leq 1$ for both P and Q. Therefore, for all $p_1 \in P$, $p_2 \in P$, $q_1 \in Q$, $q_2 \in Q$, $|\text{hom}_{P \times Q}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle)| = |\text{hom}_P(p_1, p_2) \times \text{hom}_Q(q_1, q_2)| \leq 1$. Hence $P \times Q$ is preorder.

2.3.3

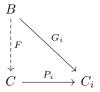
Let $\{C_i \mid i \in I\}$ be a family of categories indexed by a set I. Show product $C = \prod_i C_i$, its projections $P_i : C \to C_i$ and universal property of these projections.

Let $id_X(a): a \to a$ be identity in category X.

Let the set of all object in C be the product set $\prod_i C_i$ and $\hom_C(a,b) = \prod_i \hom_{C_i}(a_i,b_i)$. Let $\mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i)$ for all $c \in C$. Now, we proof C has a universal property:

1. For every i there is a functor $P_i: C \to C_i$.

2. For every category B such that a functor $G_i: B \to C_i$ presents for every C_i , there is a functor $F: B \to C$, which makes the following diagram commute.



First, we proof $P_i: C \to C_i$ exists. Let the object function be $P_i(x) = x_i$. Let the arrow function be $P_i(f) = f_i$. For all object $c \in C$, $P_i(\mathrm{id}_C(c)) = \mathrm{id}_C(c)_i = \mathrm{id}_{C_i}(c_i) = \mathrm{id}_{C_i}(P_i(c))$. For all arrow f, g in $C, P_i(g \circ f) = (g \circ f)_i = g_i \circ f_i = P_i(g) \circ P_i(f)$. Therefore P_i is a functor.

Second, we proof $F: B \to C$ exists. Let the object function be $F(x)_i = G_i(x)$. Let the arrow function be $F(f)_i = G_i(f)$. For all object $b \in B$, $F(\mathrm{id}_B(b))_i = G_i(\mathrm{id}_B(b)) = \mathrm{id}_{C_i}(G_i(b)) = \mathrm{id}_{C_i}(F(b)_i)$. Thus $F(\mathrm{id}_B(b)) = \mathrm{id}_{C}(F(b))$. For all arrow f, g in B, $F(f \circ g)_i = G_i(f \circ g) = G_i(f) \circ G_i(g)$. Thus $F(f \circ g) = F(f) \circ F(g)$. Therefore F is a functor.

2.3.4

Show opposite of $Matr_K$.

In \mathbf{Matr}_K , the object set is all positive integers $\{1,2,3,...\} = \omega \setminus \{0\}$. hom $_{\mathbf{Matr}_K}(n,m)$ is all rectangular matrix on K with shape $m \times n$. Therefore $\mathbf{Matr}_K^{\mathrm{op}}$ has the same objects $\omega \setminus \{0\}$ and $\mathrm{hom}_{\mathbf{Matr}_K^{\mathrm{op}}}(n,m)$ is all rectangular matrix on K with shape $n \times m$.

2.3.5

Show that the ring of real continuous functions on a topological space is the object function of a contravariant functor from Top to Rng.

Let $R_T \subseteq (T \to \mathbb{R})$ be a ring whose elements are continuous functions from a topological space T to real number. We construct R_T as follows:

- 1. Additive identity: $0_{R_T} = x \mapsto 0$.
- 2. Multiplicative identity: $1_{R_T} = x \mapsto 1$.
- 3. Addition: $f + g = x \mapsto f(x) + g(x)$.
- 4. Multiplication: $f \times g = x \mapsto f(x) \times g(x)$.

Let X and Y be any topological spaces. If we have a continuous function $f: Y \to X$, we can construct a ring homomorphism $H(f) = h: R_X \to R_Y$. We define $h(r) = r \circ f$. Then $h(0_{R_X}) = (x \mapsto 0) \circ f = 0_{R_Y}$, $h(1_{R_X}) = (x \mapsto 0) \circ f = 0_{R_Y}$.

1) $\circ f = 1_{R_Y}$, (h(s+t))(x) = (s+t)(f(x)) = s(f(x)) + t(f(x)), $(h(s \times t))(x) = (s \times t)(f(x)) = s(f(x)) \times t(f(x))$. Therefore H(f) is a ring homomorphism.

Now we construct a functor $F: \mathbf{Top^{op}} \to \mathbf{Rng}$. Let the object function be $F(A) = R_A$ and the arrow function be $F(g) = H(g^{op})$, except for any $T \in \mathbf{Top^{op}}$, $F(\mathrm{id}(T)) = \mathrm{id}(R_T)$. For all arrow a, b in $\mathbf{Top^{op}}$ such that $a \neq \mathrm{id} \land b \neq \mathrm{id}$, $F(b \circ a) = H((b \circ a)^{op}) = H(a^{op} \circ b^{op}) = H(b^{op}) \circ H(a^{op}) = F(b) \circ F(a)$. For all arrow a in $\mathbf{Top^{op}}$, $F(\mathrm{id} \circ a) = F(a \circ \mathrm{id}) = F(\mathrm{id}) \circ F(a) = F(a) \circ F(\mathrm{id}) = F(a)$. Therefore F is a functor and \overline{F} is a contravariant functor from \mathbf{Top} to \mathbf{Rng} .

2.4

2.4.1

Show that for any ring R, R-Mod is a full subcategory of Ab^R .

2.4.2

For a finite discrete category X, describe B^X .

2.4.3

Let N be a discrete category of natural numbers. Describe Ab^{N} .

2.4.4

Let P and Q be preorders. Describe Q^P and show it is a preorder.

2.4.5

Let Fin be a category of all finite sets and G be a finite group. Describe Fin^G .