

Notto - Category for the Working Mathematician

tkunn

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This is solutions for problems on the second edition of *Category for the Working Mathematician* by S. Mac Lane. Problems are placed in the first line of each sections. They may be modified from the original text for simplicity, clarity and some laziness.

1

1.1

$$\langle \wedge _ \wedge \rangle$$

1.2

$$\langle Q \perp Q \rangle$$

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1.6

Let U be a set that satisfies following conditions:

- (i) $x \in u \in U \Rightarrow x \in U$
- (ii) $(u \in U \wedge v \in U) \Rightarrow (\{u, v\}, \langle u, v \rangle, u \times v \in U)$
- (iii) (1) $x \in U \Rightarrow \mathcal{P}x \in U$
 (2) $x \in U \Rightarrow \bigcup x \in U$
- (iv) $\omega \in U$, where $\omega = \{0, 1, 2, \dots\}$ is a set of all finite ordinal numbers.
- (v) If there exists a surjection $f : a \rightarrow b$ and $a \in U$ and $b \subset U$, then $b \in U$.

1.6.1

Let $I \in U$, $f : I \rightarrow b$ and $f_i \in U$ for all $i \in I$. Prove that $\prod_i f_i \in U$.

For all $q \in \prod_i f_i$, we can construct a bijection $r : I \rightarrow q$. $q \subset U$ because $\forall w \in q, \exists j \in I, w \in f_j \in U$. Since $I \in U$ and $q \subset U$, we can say $q \in U$. Therefore $\prod_i f_i \subset U$.

Let $|f_k| \geq |f_i|$ for all i . Then, we can construct a surjection $g : f_k^I \rightarrow \prod_i f_i$. Also, we can construct a surjection $h : X \rightarrow f_k^I$, with X is either $\mathcal{P}f_k$ or $\mathcal{P}I$. As $X \in U$, $g \circ h : X \rightarrow \prod_i f_i$ and $\prod_i f_i \subset U$, we can say $\prod_i f_i \in U$.

1.6.2

(a) Let $I \in U$, $f : I \rightarrow b$ and $f_i \in U$ for all $i \in I$. Prove $\bigcup_i f_i \in U$.

(b) Prove that (a) implies following if (i), (ii), (iii)(1), (iv) and (v) holds true:

(iii)(2) $x \in U \Rightarrow \bigcup x \in U$.

(v) If $f : a \rightarrow b$ is surjective and $a \in U$ and $b \subset U$, then $b \in U$.

(a) We can construct a bijection $g : I \rightarrow \{f_i \mid i \in I\}$, $g(i) = f_i$. As $I \in U$ and $f_i \in U$ for all i , $\{f_i \mid i \in I\} \in U$. Therefore $\bigcup_i f_i = \bigcup \{f_i \mid i \in I\} \in U$.

(b) Because $x \in U$, we have $y \in U$ for all $y \in x$. Therefore we can apply $f : x \rightarrow x, f(i) = i$ to (a) to get $\bigcup x \in U$.

Because $a \in U$ and $b \subset U$, we can apply f to (a) to get $\bigcup_i f_i \in U$. f is surjective, therefore $b = \bigcup_i f_i$. Hence $b \in U$.

1.7

$$\langle Q \smile Q \rangle$$

1.8



$$\phi \longrightarrow \theta$$

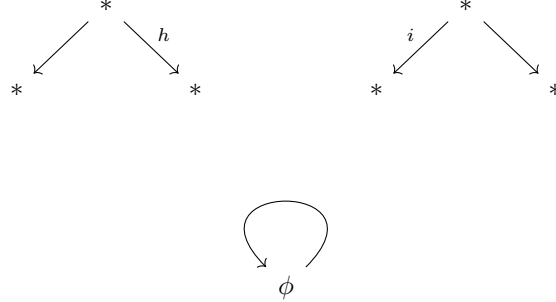
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2.1



$$\phi \curvearrowright \theta$$

2.2



2.3

2.3.1

Show product of categories includes product of monoids, product of groups and product of sets.

In this section, \times_S is a product operation for sets, \times_C is for categories, \times_G is for groups and \times_M is for monoids.

Monoids. Let M, N be a monoid with object m, n respectively. The only object in $M \times_C N$ is $\langle m, n \rangle$ (TODO)

Groups. Let G, H be a group with object a, b respectively. $G \times_C H$... (TODO)

Sets. Let A, B be discrete categories. The set of all objects in $A \times_C B$ is $X = A \times_S B$. The set of all arrows in $A \times_C B$ is $\{\langle f, g \rangle \mid a \in A, b \in B, f \in A(a, a) \wedge g \in B(b, b)\} = \{\langle \text{id}_A(c_0), \text{id}_B(c_1) \rangle \mid c \in X\}$. Therefore $A \times_C B$ is a discrete category of $X = A \times_S B$.

2.3.2

Prove that product of two preorders is preorder.

Let P, Q be preorders. $\forall a \forall b \mid \text{hom}(a, b) \mid \leq 1$ for both P and Q . Therefore, for all $p_1 \in P, p_2 \in P, q_1 \in Q, q_2 \in Q$, $\mid \text{hom}_{P \times Q}(\langle p_1, q_1 \rangle, \langle p_2, q_2 \rangle) \mid = \mid \text{hom}_P(p_1, p_2) \times \text{hom}_Q(q_1, q_2) \mid \leq 1$. Hence $P \times Q$ is preorder.

2.3.3

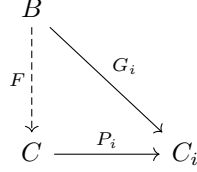
Let $\{C_i \mid i \in I\}$ be a family of categories indexed by a set I . Show product $C = \prod_i C_i$, its projections $P_i : C \rightarrow C_i$ and universal property of these projections.

Let $\text{id}_X(a) : a \rightarrow a$ be identity in category X .

Let the set of all object in C be the product set $\prod_i C_i$ and $C(a, b) = \prod_i C_i(a_i, b_i)$. Let $\text{id}_C(c)_i = \text{id}_{C_i}(c_i)$ for all $c \in C$. Now, we prove C has a universal property:

1. For every i there is a functor $P_i : C \rightarrow C_i$.

2. For every category B such that a functor $G_i : B \rightarrow C_i$ presents for every C_i , there is a functor $F : B \rightarrow C$, which makes the following diagram commute.



First, we prove $P_i : C \rightarrow C_i$ exists. Let the object function be $P_i(x) = x_i$. Let the arrow function be $P_i(f) = f_i$. For all object $c \in C$, $P_i(\text{id}_C(c)) = \text{id}_C(c)_i = \text{id}_{C_i}(c_i) = \text{id}_{C_i}(P_i(c))$. For all arrow f, g in C , $P_i(g \circ f) = (g \circ f)_i = g_i \circ f_i = P_i(g) \circ P_i(f)$. Therefore P_i is a functor.

Second, we prove $F : B \rightarrow C$ exists. Let the object function be $F(x)_i = G_i(x)$. Let the arrow function be $F(f)_i = G_i(f)$. For all object $b \in B$, $F(\text{id}_B(b))_i = G_i(\text{id}_B(b)) = \text{id}_{C_i}(G_i(b)) = \text{id}_{C_i}(F(b)_i)$. Thus $F(\text{id}_B(b)) = \text{id}_C(F(b))$. For all arrow f, g in B , $F(f \circ g)_i = G_i(f \circ g) = G_i(f) \circ G_i(g)$. Thus $F(f \circ g) = F(f) \circ F(g)$. Therefore F is a functor.

2.3.4

Show opposite of \mathbf{Matr}_K .

In \mathbf{Matr}_K , the object set is all positive integers $\{1, 2, 3, \dots\} = \omega \setminus \{0\}$. $\mathbf{Matr}_K(n, m)$ is all rectangular matrix on K of shape $m \times n$. Therefore $\mathbf{Matr}_K^{\text{op}}$ has the same objects $\omega \setminus \{0\}$ and $\mathbf{Matr}_K^{\text{op}}(n, m)$ is all rectangular matrix on K of shape $n \times m$.

2.3.5

Show that the mapping between a topological space and the ring of real continuous functions on it is the object function of a contravariant functor on \mathbf{Top} to \mathbf{Rng} .

Let $R_T \subseteq (T \rightarrow \mathbb{R})$ be a ring whose elements are continuous functions from a topological space T to real number. We construct R_T as follows:

1. Additive identity. $0_{R_T} : x \mapsto 0$.
2. Multiplicative identity. $1_{R_T} : x \mapsto 1$.
3. Addition. $f + g : x \mapsto f(x) + g(x)$.
4. Multiplication. $f \times g : x \mapsto f(x) \times g(x)$.

Let X and Y be any topological spaces. If we have a continuous function $f : Y \rightarrow X$, we can construct a ring homomorphism $H(f) = h : R_X \rightarrow R_Y$. We define $h(r) = r \circ f$. Then $h(0_{R_X}) = 0_{R_Y}$, $h(1_{R_X}) = 1_{R_Y}$, $(h(s + t))(x) =$

$(s+t)(f(x)) = s(f(x)) + t(f(x))$, $(h(s \times t))(x) = (s \times t)(f(x)) = s(f(x)) \times t(f(x))$. Therefore $H(f)$ is a ring homomorphism.

Now we construct a functor $F : \mathbf{Top}^{\text{op}} \rightarrow \mathbf{Rng}$. Let the object function be $F(A) = R_A$ and the arrow function be $F(g) = H(g^{\text{op}})$. For all arrow a, b in \mathbf{Top}^{op} , $F(b \circ a) = H((b \circ a)^{\text{op}}) = H(a^{\text{op}} \circ b^{\text{op}}) = H(b^{\text{op}}) \circ H(a^{\text{op}}) = F(b) \circ F(a)$. For all topological space $T \in \mathbf{Top}^{\text{op}}$, $F(\text{id}(T)) = H(\text{id}(T)) = \text{id}(R_T)$. Therefore F is a functor and \bar{F} is a contravariant functor on \mathbf{Top} to \mathbf{Rng} .

2.4

2.4.1

Show that for any ring R , $R\text{-Mod}$ is a full subcategory of \mathbf{Ab}^R .

TODO

2.4.2

For a finite discrete category X , describe B^X .

For any functor $T : X \rightarrow B$, if its object function is $T(a) = b$, its arrow function maps $\text{id}(a)$ to $\text{id}(b)$. Such a functor T is an object of B^X .

Let $R, S : X \rightarrow B$ be functors and τ be a map on an object of X to an arrow in B . $(\tau : R \rightarrow S) \Leftrightarrow (\forall x \in X, \tau_x(R(x)) = S(x))$. Therefore $\text{hom}(R, S) = \{\tau \mid \forall x \in X, \tau_x(R(x)) = S(x)\}$. Therefore an arrow on R to S exists iff $(\forall x, y \in X, e_R(x, y) \rightarrow e_S(x, y)) \wedge (\forall x \in X, \text{hom}(R(x), S(x)) \neq \emptyset)$, where $e_T(x, y) \Leftrightarrow (\exists a \in B, \{x, y\} \subseteq \{w \mid a = T(w)\})$.

2.4.3

Let \mathbb{N} be a discrete category of natural numbers. Describe $\mathbf{Ab}^{\mathbb{N}}$.

An object of $\mathbf{Ab}^{\mathbb{N}}$ is a map on \mathbb{N} to \mathbf{Ab} . Same as above, $\text{hom}(R, S) = \{\tau \mid \forall n \in \mathbb{N}, \tau_n(R(n)) = S(n)\}$. In other words, a map $\tau : \mathbb{N} \rightarrow (\mathbf{Ab} \rightarrow \mathbf{Ab})$ is an arrow iff, for every $n \in \mathbb{N}$, there is a corresponding group homomorphism τ_n on $R(n)$ to $S(n)$, and, for every $m \in \mathbb{N}$ such that $R(n) = R(m)$, $S(n) = S(m)$.

2.4.4

Let P and Q be preorders. Describe Q^P and show it is a preorder.

Let $R, S : P \rightarrow Q$. Then R and S are objects of Q^P . Let τ be a natural transform $\tau : R \rightarrow S$. τ is an arrow on R to S in Q^P . Since τ is natural and P is preorder, following diagram is commute for every pair of objects $p, p' \in P$. $a = f(p, p')$, where $f(p, p')$ is the only arrow on p to p' . Since Q is preorder, $\tau p = g(Rp, Sp)$, where $g(Rp, Sp)$ is the only arrow on Rp to Sp .

$$\begin{array}{ccc} Rp & \xrightarrow{\tau p} & Sp \\ Ra \downarrow & & \downarrow Sa \\ Rp' & \xrightarrow{\tau p'} & Sp' \end{array}$$

From the two downward arrows in the diagram, we can say that $\text{Im}(R)$ and $\text{Im}(S)$ contain the preorder structure of P . There are two functors $P \rightarrow \text{Im}(R)$ and $P \rightarrow \text{Im}(S)$, where $\text{Im}(T)$ is a category from the image of the object function of T and all arrows between any two pairs in the image.

As explained above, $\sigma p = g(Rp, Sp)$ for all $\sigma : R \rightarrowtail S$. Thus $|\text{hom}(R, S)| \leq |\{\sigma \mid \forall p \in P, \sigma p = g(Rp, Sp)\}| = 1$. Therefore Q^P is preorder.

2.4.5

Let \mathbf{Fin} be a category of all finite sets and G be a finite group. Describe \mathbf{Fin}^G .

Let \mathbf{Fin} be a category of all finite sets. The object is every finite set and the arrow is every mapping between every pair of finite sets.

Let G be a finite group. G is a category of only one object. Every arrow a in G has its inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = \text{id}$.

\mathbf{Fin}^G is a category that have any functor on G to \mathbf{Fin} as objects and any natural transform between two objects as arrows. The group G has only one object $x \in G$, thus any functor $T \in \mathbf{Fin}^G$ map x to a finite set $T(x) \in \mathbf{Fin}$, and endomorphisms of x to endomorphisms of $T(x)$. For any arrow a, b in G , $T(b \circ a) = T(b) \circ T(a)$. Also, $\text{id} = T(\text{id}) = T(a \circ a^{-1}) = T(a) \circ T(a^{-1})$. Thus any element in the image of the arrow function of T is invertible. Therefore T is a permutation representation of G .

Now, let $\tau : R \rightarrowtail S$ be an arrow in \mathbf{Fin}^G . Then, the following diagram commutes for any arrow a in G :

$$\begin{array}{ccc} Rx & \xrightarrow{\tau x} & Sx \\ Ra \downarrow & & \downarrow Sa \\ Rx & \xrightarrow{\tau x} & Sx \end{array}$$

Therefore, $\text{hom}(R, S) = \{\tau \mid \forall a, \tau x \circ Ra = Sa \circ \tau x\} \dots$ what does it mean????
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2.4.6

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2.5

2.5.1

Let A , B and C be small categories. Prove that $\mathbf{Cat}(A \times B, C) \cong \mathbf{Cat}(A, C^B)$ and that they are natural.

First, we prove there are two bijections $f : \mathbf{Cat}(A \times B, C) \rightarrow \mathbf{Cat}(A, C^B)$ and $g : \mathbf{Cat}(A, C^B) \rightarrow \mathbf{Cat}(A \times B, C)$ such that $f \circ g = g \circ f = \text{id}$. For any $x : A \times B \rightarrow C$, let $f(x) =$ and $g(x) =$.

Second, we prove f and g are natural.

TODO

2.5.2

Let A , B and C be categories. Prove that $(A \times B)^C \cong A^C \times B^C$ and $C^{A \times B} \cong (C^B)^A$.

For any $x \in (A \times B)^C$, there are two functions $G(c) = x(c)_0$ and $H(c) = x(c)_1$. Let the object function be $F(x) = \langle G, H \rangle$, where $F : (A \times B)^C \rightarrow A^C \times B^C$.

For any natural transform $\tau : (A \times B) \rightarrow C$, there are two natural transforms $\tau_0 : A \rightarrow C$ and $\tau_1 : B \rightarrow C$. Let $\tau_0(x) =$ and $\tau_1(x) =$.

Thus we can construct the arrow function $F(\tau) = \langle \tau_0, \tau_1 \rangle$.

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