Cahn-Hilliard with Deal.II

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For our purposes, the Cahn-Hilliard equation is described by the following partial differential equation

$$\frac{\partial \phi}{\partial t} = \nabla \cdot ((1 - \phi^2)\nabla \eta) \tag{1}$$
$$\eta(\phi) = F'(\phi) - \epsilon^2 \nabla^2 \phi \tag{2}$$
$$\tag{3}$$

where

$$F(\phi) = \frac{1}{4}(1 - \phi^2)^2 \tag{4}$$

so that

$$F'(\phi) = -\phi(1 - \phi^2) \tag{5}$$

We'll begin by solving this in the simplest way possible. In particular, a first order forward Euler method.

$$\frac{c^{n+1} - c^n}{\Delta t} = \nabla \cdot ((1 - (c^n)^2)\nabla \eta^n)) \tag{6}$$
$$\eta^n = -c^n(1 - (c^n)^2) - \epsilon^2 \nabla^2 c^n \tag{7}$$

Multiplying through by the test function, we have

$$(\phi, c^{n+1})_\Omega = (\phi, c^n)_\Omega + \Delta t(\nabla \phi, (1 - (c^n)^2)\nabla(\eta^n))_\Omega \tag{8}$$
$$(\phi, \eta^n)_\Omega = -(\phi, c^n(1 - (c^n)^2))_\Omega - \epsilon^2(\nabla \phi, \nabla c^n)_\Omega \tag{9}$$

We will impose periodic conditions on both boundaries. As a consequence, there are no boundaries to integrate over. Since we are time stepping explicitly, we can solve for η^n independent of c^{n+1} and use the previous solution to construct the necessary values, after some work, we have:

$$MC^{n+1} = MC^n + \Delta t(\nabla \phi_i, F(c^n, \eta^n)) \tag{10}$$
$$M\eta^n = -\epsilon^2 AC^n - (\phi, G(c^n)) \tag{11}$$

We could implicitly step the laplace term, however because ϵ^2 is small, this should generally not be super stiff. It would also couple the system which may make it more difficult to solve for the next timestep.