Cahn-Hilliard with Deal.II

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For our purposes, the Cahn-Hilliard equation is described by the following partial differential equation

$$\frac{\partial \phi}{\partial t} = \nabla \cdot (M(\phi)\nabla \eta) \tag{1}$$
$$\eta(\phi) = F'(\phi) - \epsilon^2 \nabla^2 \phi \tag{2}$$
$$\tag{3}$$

where

$$F(\phi) = \frac{1}{4}(\phi^2 - 1)^2, \qquad M(\phi) = 1 - \phi^2 \tag{4}$$

so that

$$\frac{\partial \phi}{\partial t} = \nabla \cdot ((1 - \phi^2)\nabla \eta) \tag{5}$$
$$\eta(\phi) = \phi(\phi^2 - 1) - \epsilon^2 \nabla^2 \phi \tag{6}$$

We'll begin by solving this in the simplest way possible. In particular, a first order forward Euler method. Before this, we will replace the ϕ's in the system with c since we frequently use ϕ to represent functions in the trial space. As a consequence, our system takes the form:

$$\frac{c^{n+1} - c^n}{\Delta t} = \nabla \cdot ((1 - (c^n)^2)\nabla \eta^n)) \tag{7}$$
$$\eta^n = c^n((c^n)^2 - 1) - \epsilon^2 \nabla^2 c^n \tag{8}$$

Notice that we can solve for η^n before finding c^{n+1} and provides a reasonable method for updating the system. Projecting into the test space, we have

$$(\phi, c^{n+1})_\Omega = (\phi, c^n)_\Omega + \Delta t(\phi,\, \nabla \cdot (1 - (c^n)^2)\nabla(\eta^n))_\Omega, \tag{9}$$
$$(\phi, \eta^n)_\Omega = (\phi, c((c^n)^2 - 1)))_\Omega - \epsilon^2(\phi, \nabla^2 c^n). \tag{10}$$

After integrating by parts, and assuming periodic boundary conditions

$$(\phi, c^{n+1})_\Omega = (\phi, c^n)_\Omega - \Delta t(\nabla\phi,\, (1 - (c^n)^2)\nabla(\eta^n))_\Omega, \tag{11}$$
$$(\phi, \eta^n)_\Omega = (\phi, c((c^n)^2 - 1)))_\Omega + \epsilon^2(\nabla\phi, \nabla c^n). \tag{12}$$

If we assume that the solution lives in the trial space, and take some finite basis of the function space we can replace (ϕ, c^n) with a mass matrix, $(\nabla\phi, \nabla c^n)$ with a laplace matrix and perform similar substitutions for η. If M and A represent the two matrices, the system is approximately governed by

$$MC^{n+1} = MC^n - \Delta t(\nabla\phi_i, F(c^n, \eta^n)) \tag{13}$$
$$M\eta^n = \epsilon^2 AC^n + (\phi, G(c^n)) \tag{14}$$