Cahn-Hilliard with Deal.II

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We use the linearly stable splitting scheme described in Kim

$$\phi^n = \phi^{n-1} + k\left[\nabla^2\eta + \nabla^2\left(f'(\phi^{n-1}) - 2\phi^{n-1}\right)\right] \tag{1}$$
$$\eta = 2\phi^n - \epsilon^2\nabla^2\phi^n \tag{2}$$

where $f'(\phi^{n-1}) = (\phi^{n-1})^3 - \phi^{n-1}$. After projecting into the test space, we obtain

$$\langle\varphi, \phi^n\rangle = \left\langle\varphi, \phi^{n-1}\right\rangle - k\left\langle\nabla\varphi, \nabla\eta\right\rangle - k\left\langle\nabla\varphi, \nabla f'(\phi^{n-1})\right\rangle + 2k\left\langle\nabla\varphi, \nabla\phi^{n-1}\right\rangle \tag{3}$$
$$\langle\varphi, \eta\rangle = 2\left\langle\varphi, \phi^n\right\rangle + \epsilon^2\left\langle\nabla\varphi_i, \nabla\phi^n\right\rangle \tag{4}$$

If φ is assumed to be from a finite dimensional test space, and ϕ^n, η are also in the test space then we obtain a finite dimensional system of equations

$$\langle\varphi_i, \varphi_j\rangle\phi_j^n = \left\langle\varphi_i, \phi^{n-1}\right\rangle - k\left\langle\nabla\varphi_i, \nabla\varphi_j\right\rangle\eta_j - k\left\langle\nabla\varphi_i, \nabla f'(\phi^{n-1})\right\rangle + 2k\left\langle\nabla\varphi_i, \nabla\phi^{n-1}\right\rangle \tag{5}$$
$$\langle\varphi_i, \varphi_j\rangle\eta_j = 2\left\langle\varphi_i, \varphi_j\right\rangle\phi_j^n + \epsilon^2\left\langle\nabla\varphi_i, \nabla\varphi_j\right\rangle\phi_j^n \tag{6}$$

Noting that $M = \langle\varphi_i, \varphi_j\rangle$ is the mass matrix and $A = \langle\nabla\varphi_i, \nabla\varphi_j\rangle$ is the stiffness or Laplace matrix we obtain

$$M\phi^n = \left\langle\varphi_i, \phi^{n-1}\right\rangle - kA\eta - k\left\langle\nabla\varphi_i, \nabla f'(\phi^{n-1})\right\rangle + 2k\left\langle\nabla\varphi_i, \nabla\phi^{n-1}\right\rangle \tag{7}$$
$$M\eta_j = 2M\phi^n + \epsilon^2 A\phi^n \tag{8}$$

which can be compactly represented with the following linear system

$$\begin{bmatrix} M & kA \\ -(2M + \epsilon^2 A) & M \end{bmatrix}\begin{bmatrix}\phi^n \\ \eta\end{bmatrix} = \begin{bmatrix}F(\phi^{n-1}) \\ 0\end{bmatrix} \tag{9}$$

where

$$F_i(\phi^{n-1}) = \left\langle\varphi_i, \phi^{n-1}\right\rangle - k\left\langle\nabla\varphi_i, \nabla(\phi^{n-1})^3\right\rangle + k\left\langle\nabla\varphi_i, \nabla\phi^{n-1}\right\rangle + 2k\left\langle\nabla\varphi_i, \nabla\phi^{n-1}\right\rangle \tag{10}$$
$$= \left\langle\varphi_i, \phi^{n-1}\right\rangle - k\left\langle\nabla\varphi_i, 3(\phi^{n-1})^2\nabla\phi^{n-1}\right\rangle + 3k\left\langle\nabla\varphi_i, \nabla\phi^{n-1}\right\rangle \tag{11}$$