My grades for Homework 2

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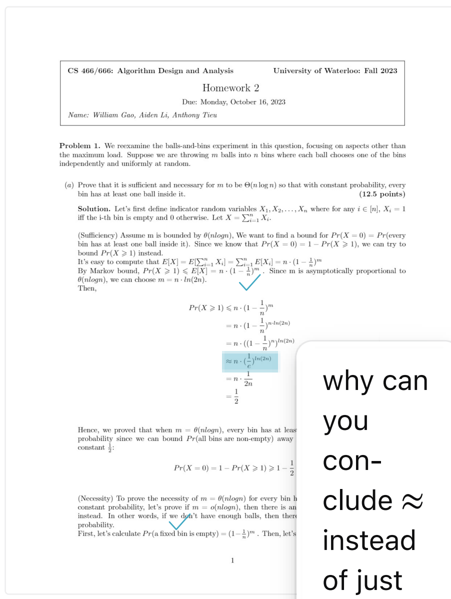
Refer to the assignment PDF.



CS 466/666: Algorithm Design and Analysis University of Waterloo: Fall 2023

Homework 2

Due: Monday, October 16, 2023

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Problem 1. We reexamine the balls-and-bins experiment in this question, focusing on aspects other than the maximum load. Suppose we are throwing m balls into n bins where each ball chooses one of the bins independently and uniformly at random.

(a) Prove that it is sufficient and necessary for m to be $\Theta(n \log n)$ so that with constant probability, every bin has at least one ball inside it. **(12.5 points)**

Solution. Let's first define indicator random variables $X_1, X_2, \ldots, X_n$ where for any $i \in [n]$, $X_i = 1$ iff the i-th bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$.

(Sufficiency) Assume m is bounded by $\theta(nlogn)$, We want to find a bound for $Pr(X = 0) = Pr($every bin has at least one ball inside it). Since we know that $Pr(X = 0) = 1 - Pr(X \geqslant 1)$, we can try to bound $Pr(X \geqslant 1)$ instead.
It's easy to compute that $E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = n \cdot (1 - \frac{1}{n})^m$
By Markov bound, $Pr(X \geqslant 1) \leqslant E[X] = n \cdot (1 - \frac{1}{n})^m$. Since m is asymptotically proportional to $\theta(nlogn)$, we can choose $m = n \cdot ln(2n)$.
Then,

$$\begin{aligned} Pr(X \geqslant 1) &\leqslant n \cdot (1 - \frac{1}{n})^m \\ &= n \cdot (1 - \frac{1}{n})^{n \cdot ln(2n)} \\ &= n \cdot ((1 - \frac{1}{n})^n)^{ln(2n)} \\ &\approx n \cdot (\frac{1}{e})^{ln(2n)} \\ &= n \cdot \frac{1}{2n} \\ &= \frac{1}{2} \end{aligned}$$

Hence, we proved that when $m = \theta(nlogn)$, every bin has at leas
probability since we can bound $Pr($all bins are non-empty) away
constant $\frac{1}{2}$:

$$Pr(X = 0) = 1 - Pr(X \geqslant 1) \geqslant 1 - \frac{1}{2}$$

(Necessity) To prove the necessity of $m = \theta(nlogn)$ for every bin h
constant probability, let's prove if $m = o(nlogn)$, then there is an
instead. In other words, if we don't have enough balls, then there
probability.
First, let's calculate $Pr($a fixed bin is empty$) = (1 - \frac{1}{n})^m$. Then, let's

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why can you con-clude ≈ instead of just ≤?

that there is an empty bin:

$$
\begin{aligned}
Pr(\text{there is an empty bin}) &= Pr(X_i = 1 \cup \cdots \cup X_n = 1) \\
&= (-1)^2 \cdot \binom{n}{1} \cdot (1 - \frac{1}{n})^m + \cdots + (-1)^n \cdot \binom{n}{n-1} \cdot (1 - \frac{n-1}{n})^m \\
&\text{(by the principle of inclusion-exclusion)} \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} \cdot \binom{n}{k} \cdot (1 - \frac{k}{n})^m
\end{aligned}
$$

To be proved: we need to lower bound $Pr(X \geqslant 1)$ to show that it t
slower than nlogn

good try

(b) Find the sufficient and necessary asymptotic value for m to be, so that with constant probability, at least one bin has two or more balls inside it. **(12.5 points)**

Solution. This question is similar to the birthday paradox problem. Since Pr(at least one bin has two or more balls inside it) = 1 - Pr(m balls end up in m different bins), we can interpret the question as find a tight asymptotic bound for m in terms of n such that m balls end up in m different bins with a constant probability. Let's choose m = $\theta(\sqrt{n})$

$$
\begin{aligned}
Pr(\text{m balls end up in m different bins}) &= 1 \cdot (1 - \frac{1}{n}) \cdot (1 - \frac{2}{n}) \cdot \cdots \cdot (1 - \frac{m-1}{n}) \\
&\leqslant e^{-\frac{1}{n}} \cdot e^{-\frac{2}{n}} \cdot \cdots \cdot e^{-\frac{m-1}{n}}
\end{aligned}
$$

usually when dealing with asymptotics, there is no need to solve for such a precise expression.

Let's choose $m = \sqrt{n}$, then it's c
that Pr(at least one bin has two

Therefore, if $m = \theta(\sqrt{n})$, at lea
(Sufficiency proved)

more precisely you need to show that if $m = o(\sqrt{n})$ then this probability is $o(1)$.

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Q2

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Refer to the assignment PDF.

Problem 2. For any $\varepsilon \in (0, 1/4)$, a $(1 \pm \varepsilon)$-**cut sparsifier** of an undirected unweighted graph $G = (V, E)$ is a *weighted* spanning subgraph $H = (V, E_H)$ of G with weights $w : E_H \to \mathbb{R}^+$ such that for *every* cut $S \subseteq V$,

$$(1-\varepsilon) \cdot |\delta_G(S)| \leqslant w(\delta_H(S)) \leqslant (1+\varepsilon) \cdot |\delta_G(S)|;$$

here, $\delta_G(S)$ is the set of edges in the cut S in G and $w(\delta_H(S))$ denotes the *weight* of the cut S in H. In other words, the weight of every cut in H is a $(1 \pm \varepsilon)$-approximation of the size of the same cut in G.

We are generally interested in constructing cut sparsifiers that are *sparse*, i.e., has few edges (otherwise, we could have taken $H = G$ with weight 1 everywhere). In this question, we see a simple (but not that efficient) way of constructing a cut sparsifier using random sampling.

Let λ denote the minimum cut value of G. Suppose we sample each edge in G with probability

$$p := \frac{100 \log n}{\varepsilon^2 \cdot \lambda},$$

to obtain the graph H and set the weights of all sampled edges to be $1/p$. Prove that with high probability, H is a $(1 \pm \varepsilon)$-cut sparsifier of G with

$$O(\frac{m \log n}{\varepsilon^2 \cdot \lambda})$$

many edges. Note that for "large" values of λ, this approach indeed sparsifies the graph. **(25 points)**

Note: There is a fundamental graph structural result that is crucial for solving this problem: in any graph $G = (V, E)$ with minimum cut λ, and for any $\alpha \geqslant 1$, the number of cuts with size at most $\alpha \cdot \lambda$ is $O(n^{2\alpha})$. This is a generalization of the bound we proved earlier in the course that the number of minimum cuts is $O(n^2)$ (the aforementioned result extends this from exact minimum cuts to approximate ones). You can prove this generalization along the same lines of the proof for exact minimum cuts, but for this question, you can just directly use this fact without a proof.

Solution. To show that the sampled graph has $O\left(m\frac{\log(n)}{\varepsilon^2\lambda}\right)$, with high probability, we use Chernoff bounds. Define m indicator variables $I_1, \ldots I_m$ where

$$I_i = \begin{cases} 1 & \text{if edge } i \text{ in } G \text{ is sampled} \\ 0 & \text{otherwise} \end{cases}$$

And let $I = \sum I_i$. Clearly, $E[I] = m\frac{\log(n)}{\varepsilon^2\lambda}$. Chernoff says the probability that the expectation deviates by a factor of 2 is:

$$\begin{aligned} P(|I - E[I]| \geq 2E[I]) &\leq 2 \cdot \exp\left(-\frac{2E[I]}{3}\right) \\ &= 2 \cdot \exp\left(-\frac{2m\log(n)}{\varepsilon^2\lambda}\right) \\ &= \frac{1}{poly(n)} \end{aligned}$$

big O

So the sampled graph has $O\left(m\frac{\log(n)}{\varepsilon^2\lambda}\right)$ edges wit

Now, we wish to show that the sampled graph is obability. We do this problem in 2 steps. First, we will use Chernoff b a single cut is not a $(1 \pm \varepsilon)$-cut sparsifier with probability $\frac{1}{poly(n)}$.

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Then, we perform a union bound over all cuts to bound the probability that our sampled graph is not a $(1 \pm \varepsilon)$-cut sparsifier. However, there are exponentially many cuts so just naively union bounding over all cuts will not yield the desired result. We use the hint to union bound in a more careful way (details below).

First, take a cut S and let $k = |\delta(S)|$. Define k indicator variables $X_1, \ldots, X_k$ where

$$X_i = \begin{cases} 1 & \text{if edge } i \text{ in } S \text{ is sampled} \\ 0 & \text{otherwise} \end{cases}$$

and define $X = \sum X_i$

Clearly,

$$\begin{aligned} E[X] &= \sum_{i=1}^{k} P(X_i = 1) \\ &= k \cdot p \end{aligned}$$

To satisfy a $(1 \pm \varepsilon)$-cut sparsifier for the cut S, we need that

$$(1-\varepsilon) \cdot |\delta_G(S)| \leq w(\delta_H(S)) \leq (1+\varepsilon) \cdot |\delta_G(S)|$$

Since we assign a weight of $1/p$ to each edge, $w(\delta_H(S)) = \frac{1}{p} \cdot X$

So we want,

$$\begin{aligned} &(1-\varepsilon) \cdot |\delta_G(S)| \leq w(\delta_H(S)) \leq (1+\varepsilon) \cdot |\delta_G(S)| \\ \Rightarrow &(1-\varepsilon) \cdot k \leq \frac{1}{p} \cdot X \leq (1+\varepsilon) \cdot k \\ \Rightarrow &(1-\varepsilon) \cdot pk \leq X \leq (1+\varepsilon) \cdot pk \\ \Rightarrow &-\varepsilon pk \leq X - pk \leq \varepsilon pk \\ \Rightarrow &|X - pk| \leq \varepsilon pk \\ \Rightarrow &|X - E[X]| \leq \varepsilon E[X] & \text{(Since } E[X] = pk\text{)} \end{aligned}$$

Using Chernoff, we see that

$$Pr(|X - E[X]| \leq \varepsilon E[X]) \leq 2\exp\left(-\frac{\varepsilon^2 pk}{3}\right)$$

Now, we rewrite the size of our cut S as a multiple of the size of the smallest cut: $k = \alpha\lambda$. So,

$$\begin{aligned} Pr\left(|X - E[X]| \leq \varepsilon E[X]\right) &\leq 2\exp\left(-\frac{\varepsilon^2 pk}{3}\right) \\ &= 2\exp\left(-\frac{\varepsilon^2 p(\alpha\lambda)}{3}\right) \\ &= 2\exp\left(-\frac{100\alpha}{3} \cdot \log n\right) \\ &= \frac{2}{n^{100\alpha/3}} \end{aligned}$$

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So for 1 cut, the probability that it does not satisfy the cut sparsifier property is small (less than $\frac{2}{n^{100\alpha/3}}$). We wish to union bound this for all cuts. For this, we use the fact that the number of cuts with size at most $\alpha\lambda$ is $O(n^{2\alpha})$. So we "group" cuts by their size and union bound by integrating over α:

$$
\begin{aligned}
Pr(\text{fail}) &\leq \int_1^\infty O(n^{2\alpha}) \cdot \frac{2}{n^{100\alpha/3}} d\alpha \\
&\leq \int_1^\infty cn^{2\alpha} \cdot \frac{2}{n^{100\alpha/3}} d\alpha && \text{(for some } c > 0) \\
&= 2c \int_1^\infty n^{-\frac{94\alpha}{3}} d\alpha \\
&= 2c \int_1^\infty n^{-\frac{94\alpha}{3}} d\alpha \\
&= 2c \left(-\frac{3}{94 n^{94\alpha/3} \ln(n)} \Bigg|_1^\infty \right) \\
&= \frac{6c}{94 n^{94/3} \ln(n)} \\
&\leq \frac{c}{n^{94/3}} \\
&\leq \frac{c}{n^{10}}
\end{aligned}
$$

So the probability of success is $\geq 1 - \frac{c}{n^{10}}$ which is $1 - \frac{1}{poly(n)}$ as desired. ■

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Q3

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Refer to the assignment PDF.

Problem 3. In this question, we prove two other simple results from *random graph theory* (for a slightly different family of random graphs than the ones studied in the lecture). Let $G = (V, E)$ be a graph on n vertices obtained by adding an edge between each pair of vertices independently and with probability

$$p := \frac{100 \log n}{n}.$$

(*a*) Prove that with high probability G is connected. **(10 points)**

Solution. Solution.

To prove that G is connected with high probability, we seek to prove that G is disconnected with low probability.

- By definition, a graph is disconnected if V can be divided into two sets of vertices, $A, B \neq \emptyset$ such that no edge exists between A and B.
- We union bound over all possible sets A, B.
- Note that since we are only interested in the cut between A and B (and not the vertices in A and B), we iterate over all possible sizes of A (and all possible sets of those sizes) from 1 to $\frac{n}{2}$. Anything more than that, and the cut of the configuration $\{A, B\}$ where $|A| \geqslant \frac{n}{2}$ would be equal to the cut of some configuration $\{B, A\}$ where $|B| \geqslant \frac{n}{2}$ (some cut that we have already counted).

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$$\begin{aligned}
\Pr(G \text{ is disconnected}) &= \sum_{k=1}^{n/2} \binom{n}{k} \left(1 - \frac{100 \log n}{n}\right)^{k(n-k)} \\
&\leqslant \sum_{k=1}^{n/2} \left(\frac{en}{k}\right)^k \left(1 - \frac{100 \log n}{n}\right)^{k(n-k)} && \text{Lecture 4 Fact 5} \\
&= \sum_{k=1}^{n/2} \left(\frac{en}{k}\right)^k e^{(k)(n-k)\log\left(1 - \frac{100 \log n}{n}\right)} \\
&= \sum_{k=1}^{n/2} \left(\frac{en}{k}\right)^k e^{(k)(n-k)\left(-\frac{100 \log n}{n}\right)} \\
&= \sum_{k=1}^{n/2} \left(\frac{e^{1 - 100 \log n + \left(\frac{100 \log n k}{n}\right)} n}{k}\right)^k \\
&\leqslant \sum_{k=1}^{n/2} \left(\frac{e^{1 - 100 \log n + \left(\frac{100 \log n \frac{n}{2}}{n}\right)} n}{k}\right)^k \\
&= \sum_{k=1}^{n/2} \left(\frac{e^{1 - 100 \log n + 50 \log n} n}{k}\right)^k \\
&\leqslant \sum_{k=1}^{n/2} \left(\frac{e^{\log n - 100 \log n + 50 \log n} n}{k}\right)^k \\
&= \sum_{k=1}^{n/2} \left(\frac{n}{e^{49 \log n} k}\right)^k \\
&= \sum_{k=1}^{n/2} \left(\frac{n}{n^{49} k}\right)^k \\
&\leqslant \sum_{k=1}^{n/2} \left(\frac{1}{n^{48}}\right) && \text{Since } \frac{1}{n^{\cdots}} \\
&= \frac{n}{2}\left(\frac{1}{n^{48}}\right) \\
&\leqslant \frac{1}{n^{47}}
\end{aligned}$$

The probability that G is connected is $1 - \Pr(G \text{ is disconnected}) \geqslant$
with high probability, as desired. ■

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not "equal"; and also the overall expression can differ a lot even if the log term can be well approximated

(think
$n \cdot 1/n$)

(b) Prove that with high probability the *chromatic number* of G is $\Omega($ … number is the minimum number of colors we can assign to the vert… same color on both its endpoints.

Hint: Try to upper bound the size of the largest *independent set* in … relate that to the chromatic number.

Solution. Solution.

Let S be an independent set of G (with $|S| = k$ for some arbitrary integer $k > 0$). By definition, there does not exist an edge between any two vertices in S, as such, by Handshaking lemma on k vertices, each with possible degree $k-1$, all $\frac{k(k-1)}{2}$ possible edges of S were not selected.

$$\begin{aligned}
\Pr(\exists \text{ set } S \text{ of size } k) &= \sum_{\substack{S\subseteq V\\|S|=k}} \left(1-\frac{100\log n}{n}\right)^{\frac{k(k-1)}{2}} && \text{Union bound}\\
&= \binom{n}{k}\left(1-\frac{100\log n}{n}\right)^{\frac{k(k-1)}{2}}\\
&\leqslant \left(\frac{en}{k}\right)^k\left(1-\frac{100\log n}{n}\right)^{\frac{k(k-1)}{2}} && \text{By Lecture 4 Fact 5}\\
&= \left(\frac{en}{k}\right)^k e^{\frac{k(k-1)}{2}\log\left(1-\frac{100\log n}{n}\right)}\\
&= \left(\frac{en}{k}\right)^k e^{\frac{k(k-1)}{2}\left(-\frac{100\log n}{n}\right)} && \text{As } n\to\infty\\
&= \left(\frac{e^{\frac{-50k\log n+n+50\log n}{n}}n}{k}\right)^k\\
&= \left(\frac{e^{\frac{-50k\log n+2n}{n}}n}{k}\right)^k && \text{By Claim}^1
\end{aligned}$$

We wish to show that the largest independent set of G is bounded by $O(\frac{n\log\log n}{\log n})$ with high probability. Let $k_0 = \frac{n\log\log n}{\log n}$, we will show that a set of size $k \geqslant k_0$ occurs with low probability. Observe that even with the smallest possible value of $k = k_0$, we have that $\frac{e^{\frac{-50k\log n+2n}{n}}n}{k} < 1$ as $n\to\infty$:

$$\begin{aligned}
\frac{e^{\frac{-50k_0\log n+2n}{n}}n}{k_0} &= \frac{e^{\frac{-50\frac{n\log\log n}{\log n}\log n+2n}{n}}n}{\frac{n\log\log n}{\log n}}\\
&= \frac{e^{\frac{-50n\log\log n+2n}{n}}\log n}{\log\log n}\\
&\leqslant \frac{e^{\frac{-50n\log\log n+2n\log\log n}{n}}\log n}{\log\log n}\\
&\leqslant \frac{\log n}{e^{\log\log n}\log\log n}\\
&= \frac{\log n}{\log n\log\log n}\\
&= \frac{1}{\log\log n}
\end{aligned}$$

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As such, for $k \geqslant k_0$:

$$\frac{e^{\frac{-50k\log n+2n}{n}}n}{k} \leqslant \frac{e^{\frac{-50k_0\log n+2n}{n}}n}{k_0} \leqslant \frac{1}{\log\log n}$$

Hence:

$$\begin{aligned}
\Pr(\exists \text{ set } S \text{ of size } k \text{ with } k\geqslant k_0) &\leqslant \left(\frac{1}{\log\log n}\right)^k\\
&\leqslant \left(\frac{1}{\log\log n}\right)^{k_0}\\
&= \left(\frac{1}{\log\log n}\right)^{\frac{n\log\log n}{\log n}}\\
&\leqslant \frac{1}{2^{\frac{n}{\log n}}} && \text{As } n\to\infty\\
&\leqslant \frac{1}{2^{\frac{n}{n^{0.5}}}} && \text{By Claim}^1\\
&\leqslant \frac{1}{2^{\sqrt{n}}}
\end{aligned}$$

Observe that $\frac{1}{2^{\sqrt{n}}} \leqslant \frac{1}{n^2}$ as $n\to\infty$:

$$\begin{aligned}
\lim_{n\to\infty}\frac{2^{\sqrt{n}}}{n^2} &= \lim_{n\to\infty} 2 \ldots\\
&= \lim_{n\to\infty} 2 \ldots\\
&= \lim_{n\to\infty} 2 \ldots\\
&= \lim_{n\to\infty} 2 \ldots\\
&= +\infty
\end{aligned}$$

perhaps easier to see by taking logs on both sides

Thus:

$\Pr(\exists$ s…

[1] Claim: $O(\log n) \in O(n^c)$ for any constant $c >$ …

- Proof. Let $a, b, c > 0$ be constants:

$$\begin{aligned}
\lim_{n\to\infty}\frac{an^c}{b\ln n} &= \lim_{n\to\infty} \ldots\\
&= \lim_{n\to\infty}\frac{acn^c}{b}\\
&= +\infty
\end{aligned}$$

- As such, $50\log n \leqslant n$.
- As such, $\log n \leqslant n^{0.5}$

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We union bound over all $n - k_0 + 1$ choices of k in $[k_0 : n]$ and obtain:

$$\begin{aligned}
\Pr\left(\exists \text{ a set } S \text{ of size } \geqslant \frac{n \ln\ln n}{\ln n}\right) &\leqslant \left(n - \frac{n \ln\ln n}{\ln n} + 1\right)\left(\frac{1}{n^2}\right) \\
&= n\left(1 - \frac{\ln\ln n}{\ln n} + \frac{1}{n}\right)\left(\frac{1}{n^2}\right) \\
&\leqslant n\left(\frac{1}{n^2}\right) && \text{Claim}^2 \\
&= \frac{1}{n^1}
\end{aligned}$$

Thus:

$$\Pr(\text{largest independent set of } G \text{ is } O\left(\frac{n \log\log n}{\log n}\right)) \geqslant 1 - \frac{1}{n}$$

Chromatic Colouring:

- We have shown that the largest independent set of G is upper bounded by $\frac{n \log\log n}{\log n}$ with high probability.
- Observe that the minimum number of independent sets in G (with high probability) must then be given by:
$$\frac{n}{\frac{n \log\log n}{\log n}} = \frac{\log n}{\log\log n}$$
- By definition of chromatic coloring, if any two vertices in G receive the same color, then they are not adjacent, that is, they are in the same independent set. If any two vertices are adjacent, they must be in different independent sets.
- As such, the minimum chromatic number (with high probability) is the same as the minimum number of independent sets (with high probability) which is given by $\frac{\log n}{\log\log n}$

Therefore, the chromatic number of G is $\Omega(\frac{\log n}{\log\log n})$ with high probability, as desired. ■

very good

[2] Claim:

- Observe that both $\frac{\ln\ln n}{\ln n}$ and $\frac{1}{n}$ are asymptotically between $(0,1)$, exclusively ($\ln\ln n$ grows slower than $\ln n$ slower than n).
- Additionally $\frac{1}{n}$ is asymptotically less than $\frac{\ln\ln n}{\ln n}$ given that the denominator n is asymptotically greater th the numerator 1 is asymptotically less than $\ln\ln n$.
- Thus, $1 - \frac{\ln\ln n}{\ln n} + \frac{1}{n}$ is asymptotically between $(0,1)$, exclusively.
- Therefore, $n\left(1 - \frac{\ln\ln n}{\ln n} + \frac{1}{n}\right)\left(\frac{1}{n^2}\right) \leqslant n\left(\frac{1}{n^2}\right)$

Q4

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Refer to the assignment PDF.

Problem 4. In this question, we design another simple algorithm for MSTs with runtime better than the classical algorithms (although not as good as the advanced ones we studied). Recall the following two facts:

- Each round of Boruvka's algorithm takes $O(m)$ time and reduces the number of vertices by at least a half.
- Prim's algorithm can be implemented in $O(m + n \log n)$ time using Fibonacci heaps.

Combine these two algorithms in a careful way to obtain an $O(m \log \log n)$ time algorithm for MSTs.

(25 points)

Solution. This solution uses $\log(n)$ to mean $\log_2(n)$.

We claim that we can implement a $O(m \log \log n)$ time algorithm for MSTs by running $\log \log n$ rounds of Boruvka's then run Prim's on the resulting graph.

- $\log \log n$ rounds of Boruvka's takes $O(m \log \log n)$ time resulting in a graph that has at most $\frac{n}{2^{\log \log n}} = \frac{n}{\log n}$ vertices and m edges.
- Running Prim's on the resulting then takes

$$O\left(m + \frac{n}{\log n}\left(\log\left(\frac{n}{\log n}\right)\right)\right) = O(m + n - \log n)$$
$$\in O(m + n)$$

The runtime is dominated by running the $\log \log n$ rounds of Boruvka's so the resulting runtime is $O(m \log \log n)$.

The correctness simply follows from the correctness of Boruvka's and Prim's.

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Q5

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Refer to the assignment PDF.

Problem 5 (Extra Credit). Design a *deterministic* $O(m)$ time algorithm for finding MST of a given *planar* graph. A planar graph is a graph that can be drawn on a surface so that no two edges cross each other.

(+10 points)

Hint: This is a pretty simple question. Think about (or search) how many edges are in a planar graph? And, what happens if you contract an edge in a planar graph?

Solution. Claim 1: Planar graphs have $O(n)$ edges. This follows simply from Euler's Formula on graphs taught in MATH 239, a prerequisite to this course.

Claim 2: Contracting an edge in a planar graph leaves the graph planar. One simple way to see this is using Kuratowski's Theorem which states that a graph is planar if and only if you can obtain $K_{3,3}$ or K_5 by a sequence of contracting edges, removing vertices, and removing edges. This theorem was also taught in MATH 239.

Consider simply running Boruvka's algorithm on the planar graph:

- Each round takes $O(n)$
- The number of components (vertices) halves each round
- The graph remains planar (the resulting graph has $O(n')$ edges where n' is the number of remaining components)

So the runtime is:

$$\begin{aligned} T(n) &\leq c \cdot n + c \cdot \frac{n}{2} + c \cdot \frac{n}{4} + ... \\ &\in O(n) \\ &\in O(m) \end{aligned}$$

As desired.

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