**CS 466/666: Algorithm Design and Analysis** | **University of Waterloo: Fall 2023**

# Lecture 9

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*Instructor: Sepehr Assadi*



## Topics of this Lecture



# 1 A Motivating Example for Duality

In the last lecture, we introduced linear programming and three of its main applications: (1) solving "day-to-day life" (continuous optimization) problems, (2) designing algorithms for combinatorial optimization problems via *rounding*, and (3) proving different structural results. However, we only saw examples of the first two types of applications.

In this lecture, we will see an example of a type (3) application and then switch back to type (2) ones. To do so, we need to introduce one of the most important notions in LPs: *duality*. We use this section to present a motivating application of this concept and then formalize it in the next section.

Consider the following LP:

$$\begin{aligned}
\max_{x_1,x_2} \quad & 2x_1 + 3x_2 \\
\text{subject to} \quad & 4x_1 + 8x_2 \leqslant 12 \\
& 2x_1 + x_2 \leqslant 3 \\
& 3x_1 + 2x_2 \leqslant 4 \\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

While the optimal solution to this LP may not be immediately clear, it is easy to see that it is at most 12 just given the first constraint:

$$\underbrace{2x_1 + 3x_2}_{\text{objective function}} \leqslant \underbrace{4x_1 + 8x_2 \leqslant 12}_{\text{first constraint}}.$$

In fact, we can divide both sides of the same constraint by two and obtain an even better upper bound of 6 on the objective function:

$$\underbrace{2x_1 + 3x_2}_{\text{objective function}} \leqslant 2x_1 + 4x_2 = \underbrace{\frac{1}{2} \cdot (4x_1 + 8x_2) \leqslant 6}_{\text{first constraint}}.$$

Yet another way is to sum up the first two constraints and then divide by three to obtain an upper bound of 5 as follows:

$$\underbrace{2x_1 + 3x_2}_{\text{objective function}} = \frac{1}{3} \cdot (6x_1 + 9x_2) = \frac{1}{3} \cdot (\underbrace{4x_1 + 8x_2}_{\text{LHS of first constraint}} + \underbrace{2x_1 + x_3}_{\text{LHS of second constraint}}) \leqslant \frac{1}{3} \cdot (12 + 3) = 5.$$

We can continue playing this game to find better and better upper bound. But let us instead formulate this game entirely as follows:

1. Multiply *new* variables $y_1, y_2, y_3 \geqslant 0$ separately on the three inequalities, which leads to

$$\begin{aligned}
&y_1 \cdot (4x_1 + 8x_2) \leqslant 12 \cdot y_1 \\
&y_2 \cdot (2x_1 + x_2) \quad \leqslant 3 \cdot y_2 \\
&y_3 \cdot (3x_1 + 2x_2) \leqslant 4 \cdot y_3 \\
&x_1, x_2 \geqslant 0 \\
&y_1, y_2, y_3 \geqslant 0.
\end{aligned}$$

   (We ensures the order of inequalities do not change by making $y_1, y_2, y_3$ non-negative).

2. Sum up the above inequality constraints to obtain the following linear combination of the constraints:

$$(4y_1 + 2y_2 + 3y_3) \cdot x_1 + (8y_1 + y_2 + 2y_3) \cdot x_2 \leqslant 12y_1 + 3y_2 + 4y_3. \tag{1}$$

3. Ensure that the coefficient of $x_1$ is at least 2 and the coefficient of $x_2$ is at least 3 in the above equation:

$$\begin{aligned}
4y_1 + 2y_2 + 3y_3 &\geqslant 2 \\
8y_1 + y_2 + 2y_3 \quad &\geqslant 3.
\end{aligned}$$

4. With these constraints, we now have that RHS of Eq (1) will be an *upper bound* on the objective value of the original program; thus, to obtain the best upper bound, we should minimize this expression as much as possible. Putting all these together gives us the following formulation:

$$\begin{aligned}
\min_{y_1, y_2, y_3} \quad & 12y_1 + 3y_2 + 4y_3 \\
\text{subject to} \quad & 4y_1 + 2y_2 + 3y_3 \ \geqslant 2 \\
& 8y_1 + y_2 + 2y_3 \quad \geqslant 3 \\
& y_1, y_2, y_3 \geqslant 0.
\end{aligned}$$

   This is another LP itself that is called the **dual** of the original LP, which is called the **primal**. It is worth taking the dual of this LP again and see that we get back to the primal LP.

An immediate consequence of our dual construction is that *any* feasible solution to the dual LP already gives us a valid upper bound on the optimal objective of the primal LP (this is often referred to as *"weak duality"*). In other words, finding an upper bound for the primal LP reduces to finding a feasible solution in the dual. But how tight is this approach, i.e., how well can we upper bound the objective of the primal LP if we manage to minimize the dual? The answer is *completely tight*! Minimum value of the dual LP coincides with the maximum value of the primal LP (this is often referred to as *"strong duality"*).

In the rest of this lecture, we formalize these notions andl then see some simple applications of duality.

# 2 Duality in Linear Programming

## 2.1 Primal and Dual LPs

Consider a primal LP as follows:

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & A \cdot x \leqslant b \\ & x \geqslant 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, i.e., we have $m$ constraints (beside non-negativity constraints). Recall that any LP can be written in this form.

1. Let $y = [y_1, \ldots, y_m]$ be the vector of non-negative coefficients in the four step approach of previous section for writing the dual. Let $a_i$ denote the $i$-th row of matrix $A$. We can thus write:

$$\begin{aligned} y_1 \cdot \langle a_1 \, , \, x \rangle &\leqslant b_1 \cdot y_1 \\ y_2 \cdot \langle a_2 \, , \, x \rangle &\leqslant b_2 \cdot y_2 \\ &\vdots \\ y_m \cdot \langle a_m \, , \, x \rangle &\leqslant b_m \cdot y_m. \end{aligned}$$

   In the matrix form, this can be written as:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \odot \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leqslant \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \odot \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},$$

   where $\odot$ stands for the element-wise multiplication.

2. Now, by adding all these constraints together and factoring out $x_1, \ldots, x_n$, we obtain that:

$$\langle y \, , \, a_1 \rangle \cdot x_1 + \langle y \, , \, a_2 \rangle \cdot x_2 + \ldots + \langle y \, , \, a_m \rangle \cdot x_m \leqslant \langle b \, , \, y \rangle.$$

3. In order for the LHS to form an upper bound on the objective value of the primal, we need to have,

$$\begin{aligned} \langle y \, , \, a_1 \rangle &\geqslant c_1 \\ \langle y \, , \, a_2 \rangle &\geqslant c_2 \\ &\vdots \\ \langle y \, , \, a_m \rangle &\geqslant c_m. \end{aligned}$$

   In the matrix form, this is simply

$$A^\top y \geqslant c.$$

4. Finally, we would like to minimize the RHS of the equation in step (2), which gives us the following:

$$\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & b^\top y \\ \text{subject to} \quad & A^\top \cdot y \geqslant c \\ & y \geqslant 0. \end{aligned}$$

This way, we get:

**Definition 1.** For any primal LP (P) as below, the dual LP (D) is defined as:

**Primal LP:**

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & A \cdot x \leqslant b \\ & x \geqslant 0, \end{aligned}$$

**Dual LP:**

$$\begin{aligned} \min_{y \in \mathbb{R}^m} \quad & b^\top y \\ \text{subject to} \quad & A^\top \cdot y \geqslant c \\ & y \geqslant 0. \end{aligned}$$

An immediate observation given the above formula is that:

**Observation 2** (“Dual of the dual is primal”). *Given an LP (P) and its dual (D), the dual of (D) is (P).*

## 2.2 Weak and Strong Duality Theorems

**Weak duality.** The weak duality theorem states that for any maximization primal LP (P), any feasible solution to its dual LP (D) gives an upper bound on the objective value of (P). Formally,

**Theorem 3** (“Weak Duality”). *Let (P) be any maximization LP and (D) be its dual in Definition 1. For any feasible point $x$ in (P) and any feasible point $y$ in (D):*

$$c^T \cdot x \leqslant b^T \cdot y.$$

*Proof.* Proof of this theorem is straightforward basically by our construction of the dual (and the whole motivation behind it). For any $i \in [m]$ and $j \in [n]$, let $a_i$ and $a^j$ denote the $i$-th row and the $j$-th column of the matrix $A \in \mathbb{R}^{m \times n}$, respectively. This way, the primal and dual constraints will be

$$\forall i \in [m] \quad \langle a_i \, , \, x \rangle \leqslant b_i \qquad \text{and} \qquad \forall j \in [\cancel{m}] \quad \langle a^j \, , \, y \rangle \geqslant c_j. \tag{2}$$

We now have

$$\begin{aligned} c^\top \cdot x = \sum_{j=1}^{n} c_j \cdot x_j &\leqslant \sum_{j=1}^{n} \langle a^j \, , \, y \rangle \cdot x_j && \text{(by the right inequality of Eq (2))} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} a_{i,j} \cdot y_i \cdot x_j = \sum_{i=1}^{m} \langle a_i \, , \, x \rangle \cdot y_i \\ &\leqslant \sum_{i=1}^{m} b_i \cdot y_i = b^\top \cdot y. && \text{(by the left inequality of Eq (2))} \end{aligned}$$

□

We have the following immediate corollary of Theorem 3.

**Observation 4.** *Let (P) be any maximization LP and (D) be its dual in the form of Definition 1. Then,*

1. *If the objective value of (P) can go to $+\infty$ (is unbounded from above), then (D) has no feasible solution.*
2. *If the objective value of (D) can go to $-\infty$ (is unbounded from below), then (P) has to feasible solution.*

*Proof.* For the first one, suppose (D) has a feasible solution $y$. Then, by weak duality, we need $b^\top \cdot y$ to be larger than any feasible objective value of (P), but the latter is going to $+\infty$, a contradiction. The second part holds via the same argument. $\square$

We also have a simple example that shows it is possible for both primal and dual LP to be infeasible.

**Observation 5.** *The following is an example of a primal (P) and dual (D) pair which are both infeasible*[1]*:*

(P)

$$\begin{aligned} \max_{x_1,x_2\in\mathbb{R}^2} \quad & x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = -2 \end{aligned}$$

(D)

$$\begin{aligned} \min_{y_1,y_2\in\mathbb{R}^2} \quad & y_1 - 2y_2 \\ \text{subject to} \quad & y_1 + 2y_2 = 1 \\ & y_1 + 2y_2 = -1. \end{aligned}$$

Just using weak duality and the above observations, we can infer the following possibilities between the primal and dual LPs:

| (P)/(D) | Unbounded | Infeasible | Feasible |
|---|---|---|---|
| Unbounded | no | yes | no |
| Infeasible | yes | yes | ??? |
| Feasible | no | ??? | ??? |

Table 1: Here, for the primal (P), being unbounded means having objective value going to $+\infty$ (first row) and for the dual (D), means having objective value going to $-\infty$ (first column).

What happens in the remaining cases? What can we say about primal and dual when they are both feasible and bounded? Strong duality handles these cases (and more!).

**Strong duality.** Strong duality states that whenever at least one of the primal or dual is feasible and bounded, then so is the either one and more importantly, their objective values are equal. Formally,

**Theorem 6** ("Strong Duality")**.** *Let (P) be any maximization LP and (D) be its dual in Definition 1. Suppose at least one of these LPs is both feasible and bounded. Then, both LPs are feasible and bounded and have optimal solutions $x^*$ for (P) and $y^*$ for (D) such that*

$$c^\top \cdot x^* = b^\top \cdot y^*.$$

The proof of this theorem is beyond the scope of our course Theorem 6 and is thus omitted. But we can now use it to complete Table 2 as follows:

| (P)/(D) | Unbounded | Infeasible | Feasible |
|---|---|---|---|
| Unbounded | no | yes | no |
| Infeasible | yes | yes | no |
| Feasible | no | no | yes$^\dagger$ |

Table 2: $^\dagger$: In this case, both optimal objective value of (P) and (D) is the same.

# 3 LP Duality Applications

Let us now visit some basic applications of LP duality.

[1]You should first convince yourself that these two LPs are indeed primal and dual of each other.

## 3.1 Optimizing vs Feasibility Checking LPs

We can use strong duality to say that checking feasibility of LPs in general is as hard as solving general LPs. In other words, if we have an algorithm that given a polyhedron can decide if it is feasible or not, then we can use the algorithm to also find optimal solutions of any given LP.

Concretely, suppose we are given a LP of the following form

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{subject to} \quad & A \cdot x \leqslant b \\ & x \geqslant 0, \end{aligned}$$

where $A$ is in $\mathbb{R}^{m \times n}$. We build a $(m+n)$-dimensional polyhedron as below by combining the LP and its dual and add one more constraint that requires the objective function of the LP and its dual LP to be equal.

Polyhedron (P1):

$$\begin{aligned} A \cdot x &\leqslant b \\ A^\top \cdot y &\geqslant c \\ c^\top \cdot x &= b^\top \cdot y \\ x, y &\geqslant 0. \end{aligned}$$

Then we give this polyhedron to the black box algorithm as input. If the algorithms decides that the polyhedron has a feasible point $(x^*, y^*)$, then we know the original LP has an optimal solution $x^*$ by the strong duality theorem (Theorem 6), and if this polyhedron is infeasible, then the original LP should also be either unbounded or infeasible.

## 3.2 $\mathsf{LP} \in \mathsf{NP} \cap \mathsf{coNP}$

Consider the decision version of the LP problem as the set

$$\mathsf{LP} := \{(P, q) \mid \text{(P) is an LP of the form in Definition 1 with objective value at least } q\}.$$

For simplicity, we further assume that in this problem we are promised that (P) is bounded and feasible (this assumption is not necessary). What complexity class this problem belongs to?

An immediate consequence of strong duality is that $\mathsf{LP} \in \mathsf{NP} \cap \mathsf{coNP}$, meaning that both the “yes” instances and “no” instances of the $\mathsf{LP}$ problem can be efficiently *verified* in polynomial time:

- $\mathsf{LP} \in \mathsf{NP}$: The certificate for verification is a feasible solution $x'$ for (P) such that $c^\top x' \geqslant q$. The algorithm simply verifies that $x'$ is a feasible solution of (P) by checking each constraint respectively and verify that $c^\top x' \geqslant q$. Then, we can conclude the optimal value of (P) is at least $q$ and the algorithm only takes polynomial time[2].
- $\mathsf{LP} \in \mathsf{coNP}$: To show this, we need to be able to prove that the objective value of (P) is less than the given parameter $q$. To do this, we take the dual (D) and the certificate is a feasible solution $y'$ for (D) such that $b^\top y' < q$. By strong duality (Theorem 6), such a certificate must always exist and by weak duality (Theorem 3), given $y'$, we have the proof that $c^\top x \leqslant b^\top y' < q$ for all feasible $x$ in (P). We can verify $y'$ is a feasible solution in (D) exactly as in the first part.

[2] We are a bit *cheating* here because we are ignoring the *bit-complexity* of $x'$, namely, how large the proof itself needs to be. But it is true that the bit-complexity of the optimal solution of the LP is polynomial in the input size

**Remark.** Recall that we have $\mathsf{LP} \in \mathsf{P} \subseteq \mathsf{NP} \cap \mathsf{coNP}$, i.e., there are polynomial time algorithms for optimizing any linear program, in particular, Ellipsoid algorithms and Interior-point methods.

### 3.3 Bipartite Matching and Bipartite Vertex Covers

We defined the bipartite matching and bipartite vertex cover problems in the previous lecture: in the former problem we want to pick the maximum number of edges that do not share any vertices, and in the latter the goal is to pick the minimum number of vertices so that every edge has at least one chosen endpoint. We prove that for both LPs, the optimal solution can be rounded to a matching and a vertex cover of the same size. I.e., the optimal value of each LP is equal to the value of the original graph problem. These results however did not say anything about the connections between these two problems. We use strong duality to address this.

Recall that the LP for the fractional matching in a bipartite graph $G = (L \sqcup R, E)$ is as follows:

$$\begin{aligned}
\max_{x \in \mathbb{R}^E} \quad & \sum x_e \\
\text{subject to} \quad & \sum_{e \ni v} x_e \leqslant 1 \qquad \forall\, v \in L \sqcup R, \\
& x_e \geqslant 0 \qquad\quad\;\; \forall\, v \in L \sqcup R.
\end{aligned}$$

Let (P) denote this LP. We can obtain the dual (D) of (P) using the same approach as before to get:

$$\begin{aligned}
\min_{y \in \mathbb{R}^V} \quad & \sum y_v \\
\text{subject to} \quad & y_u + y_v \geqslant 1 \qquad \forall\, (u, v) \in E, \\
& y_v \geqslant 0 \qquad\qquad \forall\, v \in L \sqcup R.
\end{aligned}$$

As before, *if* assume the values in this dual (D) are integral, then we get a program for minimum vertex cover, because $y_v = 1$ can be interpreted as picking the vertex $v$ in the solution, and the $y_u + y_v \geqslant 1$ constraints ensures that from every edge we are picking at least one vertex. So:

*Dual of the maximum bipartite matching LP (P) is the minimum bipartite vertex cover LP (D).*

By the results we have proven for fractional matching and vertex cover, plus strong duality (Theorem 6), we have that for any *bipartite* graph:

size of maximum matching = optimal value of (P) = optimal value of (D) = size of minimum vertex cover;

here, the first equality is by Proposition 4 of Lecture 8, the second one is by strong duality, and the third one is by Proposition 5 of Lecture 8. This result is also known as *Kőnig's theorem* that was proved by Kőnig in 1931 for the first time (combinatorially, without using any connection to LPs). This is a great example of type (3) applications of LP: proving a structural result without any algorithmic aspects.

## 4 Integrality Gaps, Rounding, and Approximation Algorithms

Let us now go back again to type (2) applications of LPs for designing algorithms for combinatorial optimization problems. So far, we examined bipartite matching/vertex cover and in both case it turned out that the optimal fractional solution can be *rounded* to an integral solution (a maximum matching or a minimum vertex cover, respectively) without changing the value at all.

Nevertheless, in many scenarios, we cannot expect to have such a "loss-less" rounding. For instance, consider the general (non-bipartite) vertex cover problem wherein the input graph can be arbitrary. You might already have a strong suspicion that we should not be able to round the value of LP without any loss to an integral vertex cover. After all, we can solve LPs in polynomial time while (general) vertex cover is an NP-hard problem; so, if optimal value of vertex cover LP is always equal to the minimum vertex cover, then we get a polynomial time algorithm for an NP-hard problem. While, technically speaking, this is not known to be impossible[3], it is still something to be extremely suspicious of. We will examine this in more details in the rest of this lecture.

## 4.1 Minimum Vertex Cover and Its LP

Consider the following LP for the minimum (non-bipartite) vertex cover problem on a graph $G = (V, E)$:

$$\begin{aligned} \min_{y \in \mathbb{R}^V} \quad & \sum_{v \in V} y_v \\ \text{subject to} \quad & y_u + y_v \geqslant 1 \qquad \forall\ (u, v) \in E, \\ & y_v \geqslant 0 \qquad\qquad\quad \forall\ v \in V. \end{aligned}$$

We define the **integrality gap** of this LP as the largest ratio between the minimum vertex cover and the optimal value of the LP (roughly speaking, the maximum value of how "costly" this relaxation is). Formally, the integrality gap of the vertex cover LP is:

$$\sup_G \quad \frac{\text{size of minimum vertex cover of } G}{\text{optimal value of vertex cover LP on } G}.$$

So our question from before is what is the integrality gap of the minimum vertex cover problem? We claim it is almost 2, i.e., there are graphs where size of the integral minimum vertex cover is (almost) twice as large as the optimum solution of the LP. A very simple example is a complete graph on $n$ vertices:

- The solution $y_v = 1/2$ for all $v \in V$ is a feasible fractional to the LP: for every edge $(u, v) \in E$, we have $y_u + y_v = 1/2 + 1/2 = 1$ and thus the edge is covered. Hence, the optimal LP value is $\leqslant n/2$ (we did not prove the optimality of this solution, only its feasibility, hence the '$\leqslant$' sign and not '$=$' although the latter is also true; try proving it yourself).
- On the other hand, the size of minimum vertex cover is $n - 1$; if we do not pick any two vertices $u$ and $v$ in the vertex cover, then the edge between them is not covered.

This implies that the integrality gap on $n$-vertex graph is

$$\frac{n-1}{n/2} = 2 - \frac{1}{n}.$$

Given the supremum definition of integrality gap, for $n \to \infty$, we have that the integrality gap is at least 2.

But what about an upper bound? That is the topic of **approximation algorithms**, i.e., finding a way of rounding the fractional LP so that we do not lose "too much" in the value (in general, approximation algorithms try to find a solution which is "approximately" optimal – in the formal sense, this means that we can bound the ratio between the value of the returned solution and the optimal solution to the problem).

Let us show that integrality gap of vertex cover LP is at most 2 also (which combined with the previous part implies it is precisely 2).

**Proposition 7.** *Given any feasible solution $y \in \mathbb{R}^V$ to the vertex cover LP, there is a polynomial time algorithm that returns a vertex cover of size at most $2 \sum_{v \in V}$.*

*Proof.* Define the set $U \subseteq V$ as:

$$U := \{v \in V \mid y_v \geqslant 1/2\},$$

[3] Of course, we do not know whether P = NP or not yet ...

i.e., the set of vertices with $y$-value at least $1/2$.

Firstly, we argue that $U$ is a valid vertex cover of $G$. This is because for any edge $e = (u, v) \in E$, we have $y_u + y_v \geqslant 1$ (by the feasibility of LP) which means at least one of $u$ or $v$ has $y$-value at least $1/2$ and is thus added to the set $U$. So, every edge has at least one endpoint in $U$ and thus $U$ is a vertex cover.

Secondly, we can bound the size of $U$ as follows:

$$\begin{aligned}
|U| &= \sum_{v \in V} \mathbb{I}(v \in U) && \text{(where } \mathbb{I}(v \in U) = 1 \text{ if } v \in U \text{ and 0 otherwise)} \\
&= \sum_{v \in V} \mathbb{I}(y_v \geqslant 1/2) && \text{(where } \mathbb{I}(\cdot) \text{ in a similar way as above)} \\
&\leqslant \sum_{v \in V} 2 \cdot y_v, \\
&&& \text{(because either } y_v < 1/2 \text{ and thus } \mathbb{I}(y_v \geqslant 1/2) = 0 \text{ or } y_v \geqslant 1/2 \text{ and thus } \mathbb{I}(v \geqslant 1/2) = 1 \leqslant 2y_v)
\end{aligned}$$

concluding the proof. □

Proposition 7 now implies that integrality gap of the vertex cover LP is at most 2 because we can round any optimal solution to get a vertex cover of size at most twice as large. But this also tells us something very useful algorithmically: we have a polynomial time algorithm for 2-approximation of minimum vertex cover. Firstly, solve the LP optimally in polynomial time to get the optimal solution $y^*$ and then run Proposition 7 to turn it into a vertex cover of size at most $2 \sum_v y_v^*$. But recall that optimal value of LP is always a lower bound on the minimum vertex cover (as we relaxed the integer linear program to an LP and thus can only improve the optimal solution). Thus, this algorithm also outputs a vertex cover of size at most twice that of minimum vertex cover.

**Remark.** There are much easier ways of getting a 2-approximation algorithm to minimum vertex cover. We will see those later in the course. That being said, the above approach has its positives, for instance, it is quite robust and versatile; you can extend it as it is to the *weighted* version of the problem where the vertices have weights and we want to pick the cover with minimum weight. Just apply the same exact strategy to the following weighted vertex cover LP instead:

$$\begin{aligned}
\min_{y \in \mathbb{R}^V} \quad & \sum_{v \in V} w(v) \cdot y_v \\
\text{subject to} \quad & y_u + y_v \geqslant 1 \qquad \forall\ (u, v) \in E, \\
& y_v \geqslant 0 \qquad \forall\ v \in V,
\end{aligned}$$

where $w(v)$ in the objective is the given weight of the vertex $v \in V$.