

Lecture 24

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1 A Quick Recap of Random Walks

In the last lecture, we looked at Markov chains and stationary distributions. A Markov chain consists of n states and a probability matrix—called the *transition matrix*— $P \in \mathbb{R}^{n \times n}$, where P_{ij} denotes the probability that the next state of the random walk is j assuming the current state is i . Recall the following standard definitions:

- For any states i and j , the *hitting time* $h(i, j)$ is the *expected* length of the random walk starting from i and ending in j . We refer to $h_{i,i}$ as the *return time*.
- The *stationary distribution* is a distribution π over the states which remains unchanged after taking another step of the random walk, i.e., $\pi = \pi \cdot P$.
- An *irreducible* Markov chain is a one where the underlying directed graph of transition matrix (having any edge (i, j) whenever $P_{ij} > 0$), is strongly connected.

We will be using the following special case of the fundamental theorem of Markov Chains in this lecture.

Theorem 1. In any irreducible Markov chain, for any state i , $h(i, i) = 1/\pi_i$; namely, the return time is equal to inverse stationary probability.

We now use this theorem to design a brilliant and yet extremely simple algorithm for finding perfect matching in *regular* bipartite graphs, due to Goel, Kapralov, and Khanna [GKK13].

2 Perfect Matching in Regular Bipartite Graphs

In this section, we will use a random walk to construct an algorithm which finds a perfect matching in d -regular bipartite graphs in $O(n \log n)$ time. Notice that since the input is of size $\Theta(nd)$, this is even faster than reading the input once! Let us first show that regular bipartite graphs *always* have a perfect matching to begin with.

Proposition 2. For any integers $n, d \geq 1$, any d -regular bipartite graph $G = (L, R, E)$ with $n := |L| = |R|$ has a perfect matching.

Proof. This can be proven easily using Hall's theorem or alternatively via fractional matchings.

Proof via Hall's theorem. Recall that Hall's (Marriage) Theorem states:

Let $G = (L \cup R, E)$ be a bipartite graph. G has a matching that matches all vertices in L iff for every subset $A \subseteq L$, we have $|A| \leq |N(A)|$ in G .

Now consider our d -regular graph G and fix any set $A \subseteq L$. Let $m(A, N(A))$ denote the number of edges between A and $N(A)$. We have,

$$|A| \cdot d = m(A, N(A)) \leq |N(A)| \cdot d;$$

the left equality holds because G is d -regular and all edges in A go to $N(A)$, and the right inequality holds because subset of edges of $N(A)$ are going back to A . This implies $|A| \leq |N(A)|$. Thus G has a matching that matches all of L by Hall's theorem. But since $|L| = |R|$, this means G has a perfect matching.

Proof of via fractional matchings. Recall that a fractional matching is an assignment $x \in \mathbb{R}^E$ satisfying

$$\forall v \in V : \sum_{e \ni v} x_e \leq 1 \quad \text{and} \quad \forall e \in E : x_e \geq 0.$$

We proved earlier in the course that any fractional matching in bipartite graphs can be turned into an integral (standard) matching with $\sum_{e \in E} x_e$ many edges. In other words, integrality gap of the linear program for bipartite matching is 1.

Now consider our d -regular graph G and assign $x_e = 1/d$ to every edge. This is clearly a fractional matching as

$$\forall v \in V : \sum_{e \ni v} x_e = \deg(v) \cdot \frac{1}{d} = 1,$$

and we have

$$\sum_{e \in E} x_e = n \cdot d \cdot \frac{1}{d} = n;$$

hence, there is an integral matching of size n also in G , namely a perfect matching. \square

Now that we established that there is a perfect matching, let us see how we can find it.

Theorem 3 ([GKK13]). There is a randomized algorithm that for every integer $n, d \geq 1$, given a bipartite d -regular graph $G = (L \cup R, E)$ with $n = |L| = |R| = n$, outputs a perfect matching of G in $O(n \log n)$ expected time.

The algorithm proceeds by finding augmenting paths in the graph using random walks. First, we establish what augmenting paths are and their role in increasing the size of a matching. Then, we will see how we can construct these augmenting paths.

2.1 Role of Augmenting Paths

We define augmenting paths formally.

Definition 4. Given a graph $G = (V, E)$ and a matching $M \subseteq E$, a path $P = (u_1, u_2, \dots, u_{2k})$ in G is said to be an **augmenting path** for M if,

- Edge (u_{2i-1}, u_{2i}) is not present in M for all $i \in [k]$.
- Edge (u_{2i}, u_{2i+1}) is present in M for all $i \in [k-1]$.

That is, path P starts and ends at unmatched vertices, and alternates between edges from M and $E \setminus M$.

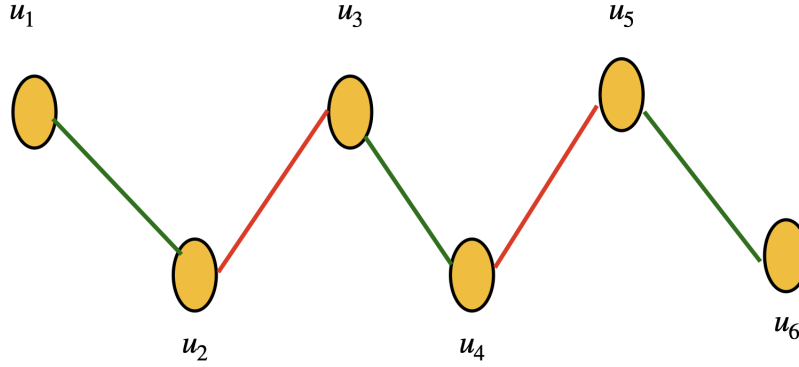


Figure 1: An augmenting path of length 5. The red edges are matching edges, and green edges are non-matching edges. Vertices u_1, u_6 are unmatched. Replacing red edges with green edges in M gives us a larger matching M' .

It is easy to see that whenever we find an augmenting path for a matching M , we can find another matching M' , the size of which is larger than that of M by one.

Claim 5. Given a matching M of graph $G = (L \cup R, E)$ and an augmenting path P of M in graph G , there is a matching M' of G such that $|M'| = |M| + 1$.

Proof. Let path P be (u_1, u_2, \dots, u_k) for some even k . Let $E(P)$ be the set of all edges which are present in path P . Then we construct M' as,

$$M' = (M \setminus E(P)) \cup (E(P) \setminus M).$$

In other words, we replace all the edges in path P which are in M by all the edges in path P which are not in M . Refer to Figure 1 for an illustration.

To see that M' remains a matching, note that no edge which is not present in path P is affected, so the status of all vertices which are not in P is unchanged. For vertices in P , both u_1, u_k are unmatched, and we add one edge incident to each of them to M' . For all the other vertices u_i for $1 < i < k$, we remove edge (u_i, u_{i+1}) and add edge (u_{i-1}, u_i) or vice-versa. M' never has two edges incident to one vertex.

Finally, the size of M' is larger than that of M by one, because we remove $(k/2) - 1$ edges from M , and add $(k/2)$ edges. \square

The algorithm in [Theorem 3](#) works by repeatedly find augmenting paths to increase the size of a maintained matching. The following lemma states that we can quickly find such paths, for any matching M which is not perfect.

Lemma 6. *For any $k, n, d \geq 1$, given a bipartite graph $G = (L \cup R, E)$ which is d -regular with $|L| = |R| = n$, and a matching M which leaves $2k$ vertices unmatched from $L \cup R$, there is a randomized algorithm which finds an augmenting path for M in expected $O(n/k)$ time.*

We will prove this lemma in the next subsection. Let us show that finding these augmenting paths proves [Theorem 3](#).

Proof of Theorem 3. We start with matching $M = \emptyset$. Then we can find an augmenting path in $O(1)$ expected time by [Lemma 6](#), and increase the size of the matching to 1 by [Claim 5](#). We continue to do this until M becomes a perfect matching.

The size of M slowly increases from 0 to n . When the total number of unmatched vertices is $2k$, we know from [Lemma 6](#) that we can find an augmenting path in $O(n/k)$ expected time. Then we use [Claim 5](#) to reduce the number of unmatched vertices by 2. The number of unmatched vertices decreases from $2n$ to 0, so the total expected running time is,

$$2n \cdot \left(\frac{1}{2n} + \frac{1}{2n-2} + \dots + \frac{1}{4} + \frac{1}{2} \right) \leq 2n \cdot \sum_{i=1}^n \frac{1}{i} = O(n \log n),$$

where in the last step we used that the harmonic sum up to n terms is bounded by $O(\log n)$. □

2.2 Finding Augmenting Paths

In this section, we will see how to find augmenting paths by using a random walk.

Notation. We use $V(M) \subseteq L \cup R$ to denote the set of all vertices which is matched by M . For any matching M , and vertex v which is matched by M , we use $M(v)$ to denote the vertex M matches v to. That is, $(v, M(v)) \in M$.

Informally, when we try to find an augmenting path, we want to alternate between matching edges and the non-matching edges. We pick all the non-matching edges at random out of the current vertex. The matching edges are picked deterministically; there is only one choice for the edge in the matching.

Algorithm 1. Algorithm Aug-Walk for finding a single augmenting path for a matching M .

- (i) Pick an unmatched vertex $u_1 \in L$ at random. Start walk W from u_1 , and repeat the following.
- (ii) Let $u \in L$ be the current vertex for walk W . Pick an edge (u, v) with $v \in R$ at random out of vertex u , and add it to W . Stop if v is unmatched.
- (iii) If v is matched, add edge $(v, M(v))$ to W , and move to vertex $M(v) \in L$. Continue from step (ii).

To prove [Lemma 6](#), we need to show that algorithm **Aug-Walk** terminates in $O(n/k)$ time when $2k$ vertices are left unmatched by M . The walk W is not an *undirected random walk* on the underlying graph, as all the matching edges are picked deterministically. We will construct a Markov Chain from graph G and matching M that captures this walk.

Definition 7. Given a bipartite d -regular graph $G = (L \cup R, E)$ and a matching M which leaves $2k$ vertices unmatched, we construct the **matching Markov chain** as follows.

- (i) Create a state for each vertex $v \in L \cup R$, and add all edges by orienting them from L to R .
- (ii) For each matching edge (u, v) with $u \in L, v \in R$, combine both states u, v into one state. This state has all the edges incoming to v as well as all the edges outgoing from u .
- (iii) Create a new state \star which denotes the start and final states. Add edges from \star to all unmatched vertices in L , and add edges from all unmatched vertices in R to \star .
- (iv) For each state, make all outgoing edges equiprobable.

Refer to Figure 2 for an illustration of Definition 7.

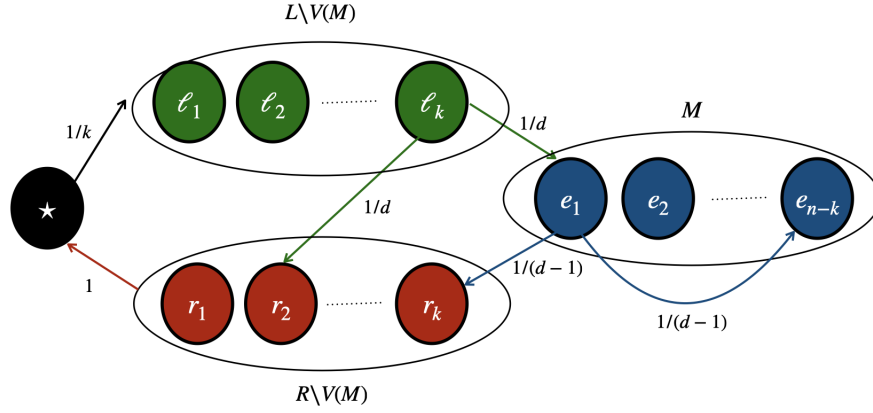


Figure 2: The black state \star has k outgoing equiprobable edges to $L \setminus V(M)$. The green states $L \setminus V(M)$ have d outgoing edges each corresponding to edges in $E \setminus M$. The brown states $R \setminus V(M)$ have one outgoing edge each to \star . The blue states M have $d - 1$ outgoing edges corresponding to edges in $E \setminus M$.

The key point is that when we want to pick a matching edge from $v \in R$, there is only one choice for such an edge, and the state of the Markov chain does not change. We have *fused* all matching edges into a single state. Let us formally argue why **Aug-Walk** is a random walk on our matching Markov chain.

Claim 8. *Algorithm Aug-Walk, for graph G and matching M performs a random walk on the Markov chain from Definition 7.*

Proof. We start by picking an unmatched vertex at random, and moving out of state \star picks such a vertex $u_1 \in L$, by step (iii) of the construction. We pick an edge (u_1, u_2) out of u_1 at random, and all edges out of u are equiprobable, by step (v). If $u_2 \in R$ is unmatched, we move out of u_2 into final state \star by step (iii).

If u_2 is matched, u_2 is fused in state $u_2 M(u_2)$ by step (ii). We move out of $M(u_2) \in L$ at random into another vertex from R in the graph G , which may be a matching edge state in Markov chain, or a state for an unmatched vertex in R . We repeat till we reach an unmatched vertex in R , and in one more timestep we reach final state \star . The walk starts from \star , and ends when we reach \star again. \square

Remark. Note that **Aug-Walk** returns a *walk* on Markov chain and not a path. However, it can be converted into a path by removing cycles.

Given **Claim 8**, bounding the runtime of **Aug-Walk** is the same (up to an $O(1)$ factor) as bounding the return time of the state marked by \star , i.e., $h(\star, \star)$ in the matching Markov chain. To do so, we only need to compute the stationary distribution of our Markov chain and we can then apply **Theorem 1** to conclude the proof.

We will look at the transition matrix of our Markov chain to find the stationary distribution. Let P_{ij} denote the probability of entering state j from state i . There are $n + k + 1$ states in total, divided into four types (see **Figure 2**). We order them as follows:

1. State \star corresponding to the start and final states. There are k edges outgoing from \star to $L \setminus V(M)$, each with probability $1/k$. There are k edges incoming to \star from $R \setminus V(M)$, each with probability 1 .
2. State $\ell_i \in L \setminus V(M)$ for each unmatched vertex $\ell_i \in L$ for $i \in [k]$. Each such state has d outgoing edges, each with probability $1/d$, to either states corresponding to matching edges or unmatched vertices in R . There is one edge incoming to each ℓ_i from \star with probability $1/k$.
3. State $r_i \in R \setminus V(M)$ for each unmatched vertex $r_i \in R$ for $i \in [k]$. They have only one outgoing edge to state \star . There are d edges incoming to each r_i , based on edges $(u, r_i) \in E \setminus M$. They may have probability $1/(d-1)$ (if they are from matching edge states), or $1/d$ (if they are from $L \setminus V(M)$).
4. State $e_i = vM(v)$ corresponding to edge $e_i = (v, M(v)) \in M$ incident on matched vertex $v \in R$ for $i \in [n-k]$. Each state has $d-1$ outgoing edges, each with probability $1/(d-1)$, to other matching edge states or unmatched vertices in R . There are $d-1$ incoming edges, based on edges incident to $v \in R$. Again, they have probability $1/(d-1)$ from matching edge states, or $1/d$ from $L \setminus V(M)$.

Let us show what the stationary distribution is.

Claim 9. Let $t := (n-k)(d-1) + 3kd$. The stationary distribution of matching Markov chain is π where:

$$\pi_\star = \frac{kd}{t}, \quad \pi_{\ell_i} = \pi_{r_j} = \frac{d}{t} \text{ for } i, j \in [k], \quad \pi_{e_i} = \frac{d-1}{t} \text{ for } i \in [n-k].$$

Proof. First, let us show that π is a probability distribution over the states:

$$\begin{aligned} \sum_{\text{State } s} \pi_s &= \pi_\star + \sum_{\ell \in L \setminus V(M)} \pi_\ell + \sum_{r \in R \setminus V(M)} \pi_r + \sum_{e \in M} \pi_e \\ &= \frac{kd}{t} + k \cdot \frac{d}{t} + k \cdot \frac{d}{t} + (n-k) \cdot \frac{(d-1)}{t} \\ &= \frac{1}{t} \cdot (3kd + (n-k)(d-1)) \\ &= 1. \end{aligned}$$

We will check what the probability is for transitioning into every state starting from distribution π . It is easy to check for states \star and any $\ell \in L \setminus V(M)$.

$$\Pr(\text{Entering } \star \text{ starting with } \pi) = \sum_{r \in R \setminus V(M)} \pi_r \cdot p_{r, \star} = k \cdot \frac{d}{t} = \frac{kd}{t}.$$

$$\Pr(\text{Entering any } \ell \in L \setminus V(M) \text{ starting with } \pi) = \pi_\star \cdot p_{\star, \ell} = \frac{kd}{t} \cdot \frac{1}{k} = \frac{d}{t}.$$

For any state $r \in R \setminus V(M)$, we have an incoming edge from all $\ell \in L \setminus V(M)$ with $(\ell, r) \in E$, and all edge states $e = (u, v) \in M$ with $(v, r) \in E$. There are totally d incoming edges.

$$\begin{aligned}
\Pr(\text{Entering } r \in R \setminus V(M) \text{ starting with } \pi) &= \sum_{e \in M} \pi_e \cdot p_{e,r} + \sum_{\ell \in L \setminus V(M)} \pi_\ell \cdot p_{\ell,r} \\
&= \sum_{(u,v) \in M; (v,r) \in E} \frac{d-1}{t} \cdot \frac{1}{d-1} + \sum_{\ell \in L \setminus V(M); (\ell,r) \in E} \frac{d}{t} \cdot \frac{1}{d} \\
&= \sum_{(u,v) \in M; (v,r) \in E} \frac{1}{t} + \sum_{\ell \in L \setminus V(M); (\ell,r) \in E} \frac{1}{t} \\
&= \frac{d}{t}.
\end{aligned}$$

Similarly, for any edge $e = (u, v) \in M$, with $d-1$ incoming edges,

$$\begin{aligned}
\Pr(\text{Entering } e = (u, v) \in M \text{ starting with } \pi) &= \sum_{e' \in M} \pi_{e'} \cdot p_{e',e} + \sum_{\ell \in L \setminus V(M)} \pi_\ell \cdot p_{\ell,e} \\
&= \sum_{(u',v') \in M; (v',u) \in E} \frac{d-1}{t} \cdot \frac{1}{d-1} + \sum_{\ell \in L \setminus V(M); (\ell,u) \in E} \frac{d}{t} \cdot \frac{1}{d} \\
&= \sum_{(u',v') \in M; (v',u) \in E} \frac{1}{t} + \sum_{\ell \in L \setminus V(M); (\ell,u) \in E} \frac{1}{t} \\
&= \frac{d-1}{t}.
\end{aligned}$$

This proves that π is indeed the stationary distribution of the matching Markov chain. \square

Remark. We could have alternatively changed the matching Markov chain slightly so that it becomes the transition matrix of a random walk on a *directed* graph wherein in-degree and out-degree of all vertices is the same. One can generally show that random walk on any directed graph with this property—often called a balanced directed graph—has stationary distribution wherein probability of each vertex is its out-degree divided by the number of edges (namely, a direct analogue of the stationary distribution in undirected graphs).

We are ready to prove [Lemma 6](#), as we know what the stationary distribution is now.

Proof of Lemma 6. We use algorithm [Aug-Walk](#) to find an augmenting path in graph G for matching M . By [Claim 8](#), we know that this is a random walk on the matching Markov chain from [Definition 7](#).

We know that any random walk on the matching Markov chain which starts at state \star reaches \star again in expected number of steps $\frac{1}{\pi_\star}$ by [Theorem 1](#). We know what π_\star is from [Claim 9](#). Therefore, the expected number of steps of the random walk is,

$$\frac{t}{kd} = \frac{(n-k)(d-1) + 3kd}{kd} = \frac{nd + 2kd - (n-k)}{kd} \leq \frac{2n}{k}.$$

\square

References

[GKK13] Ashish Goel, Michael Kapralov, and Sanjeev Khanna. Perfect matchings in $O(n \log n)$ time in regular bipartite graphs. *SIAM Journal on Computing*, 42(3):1392–1404, 2013. [1](#), [2](#)