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Lecture 22

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Topics of this Lecture

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1 Proof of Lovász Local Lemma

In the previous two lectures, we worked with Lovász Local Lemma (LLL) and its algorithmic version. We now prove a slightly weaker form of LLL to provide more intuition about it (notice that we effectively replace $1/e$ in the original statement with a weaker constant of $1/4$).

Theorem 1 (Symmetric LLL – "Weak" Form). *Suppose $B_1, \ldots, B_n$ are a collection of events. If:*

1. $\Pr(B_i) \leqslant p$ *for every* $i \in [n]$*, for some* $p \in (0, 1)$*;*
2. *and, the events admits a dependency graph with maximum degree* $d \geqslant 1$*,*

Then, as long as

$$p \cdot d \leqslant 1/4$$

the probability that ***none of*** $B_1, \ldots, B_n$ *happens is strictly more than zero.*

Proof. Define A_i to be the complement of the event B_i. Moreover, for any $i \in [n]$, define $A_{<i}$ as the event that $A_1, \ldots, A_{i-1}$ happens. Our goal is to prove that

$$0 < \Pr\left(\wedge_{i=1}^{n} \bar{B}_i\right) = \Pr\left(\wedge_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \Pr\left(A_i \mid \wedge_{j=1}^{i-1} A_j\right) = \prod_{i=1}^{n} \Pr\left(A_i \mid A_{<i}\right).$$

Thus, we have to prove that

$$\Pr\left(A_i \mid A_{<i}\right) > 0 \iff \Pr\left(B_i \mid A_{<i}\right) < 1$$

for all $i \in [n]$. We actually prove a stronger statement inductively:

> **Induction hypothesis**: *For any $i \in [n]$ and any set $S \subseteq [n] \setminus \{i\}$,*
>
> $$\Pr\left(B_i \mid A_S\right) \leqslant 2p,$$
>
> *where A_S is defined as $\wedge_{j \in S} A_j$.*

The base case for each $i \in [n]$ and $S = \emptyset$ follows immediately because $\Pr(B_i) \leqslant p$ by the theorem statement. We now prove the induction step.

Step 1. We know that B_i only depends on d other events in $N(i)$ so we should find a way to "get rid of" the remaining terms in A_S. To do so, we write,

$$\begin{aligned}
\Pr\left(B_i \mid A_S\right) &= \Pr\left(B_i \mid A_{S\cap N(i)}\ A_{S\setminus N(i)}\right) \\
&= \frac{\Pr\left(B_i \wedge A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right)}{\Pr\left(A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right)} && \text{(by the definition of conditional probability)} \\
&\leqslant \frac{\Pr\left(B_i \mid A_{S\setminus N(i)}\right)}{\Pr\left(A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right)} && \text{(as } \Pr(C \wedge D) \leqslant \Pr(C) \text{ for any events } C, D\text{)} \\
&= \frac{\Pr\left(B_i\right)}{\Pr\left(A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right)} && \text{(because } B_i \text{ is independent of events outside } N(i)\text{)} \\
&\leqslant \frac{p}{\Pr\left(A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right)}. && \text{(as } \Pr\left(B_i\right) \leqslant p \text{ in the theorem statement)}
\end{aligned}$$

The only "real" inequality above is the step that we are dropping $\wedge A_{S\setminus N(i)}$ (the other inequality might as well be tight also because we have no control over the gap between $\Pr(B_i)$ and p in the theorem statement). As we shall see, the "math is going to work out" even when taking this inequality but it is good to see why we should intuitively make such a step. This is because, $A_{S\cap N(i)}$ only contains d terms and in the next step we are going to prove that these terms actually happen with a "large enough" probability (some constant more than zero); As a result, we are *not* "dropping" a very low probability event that can make the inequality quite loose.

Step 2. We know need to lower bound the denominator of the RHS above. But now, this term only depends on d events in total and we can try to simply use a *union bound* to get a loose bound here. Specifically, we can write

$$\Pr\left(A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right) = 1 - \Pr\left(\vee_{j\in S\cap N(i)} B_j \mid A_{S\setminus N(i)}\right) \geqslant 1 - \sum_{j\in S\cap N(i)} \Pr\left(B_j \mid A_{S\setminus N(i)}\right).$$

Given that $N(i)$ has at least one element (as otherwise B_i is independent of all other events and trivially satisfies the induction hypothesis), we have that $|S \setminus N(i)| < |S|$. Thus, we can apply our induction hypothesis and obtain that for every $j \in S \cap N(i)$,

$$\Pr\left(B_j \mid A_{S\setminus N(i)}\right) \leqslant 2p.$$

Plugging this bound above gives us

$$\Pr\left(A_{S\cap N(i)} \mid A_{S\setminus N(i)}\right) \geqslant 1 - \sum_{j\in S\cap N(i)} 2p \geqslant 1 - 2p \cdot d \geqslant 1/2,$$

where the last inequality is by the assumption in the theorem statement that $p \cdot d \leqslant 1/2$.

Plugging in the bounds of step 2 on the last equation of step 1 give us

$$\Pr\left(B_i \mid A_S\right) \leqslant \frac{p}{1/2} \leqslant 2p,$$

as desired. This concludes the proof. □

It is worth mentioning that the statement of Theorem 1 is actually the original version of LLL proven by Erdős and Lovász in [EL75].

2 Asymmetric Lovász Local Lemma

A (slightly) careful view of the proof of Theorem 1 reveals that we actually really do not need a "symmetric" bound on the probabilities and degrees of all bad events. Instead, we can even prove the following variant:

Theorem 2 (Asymmetric LLL – "Weak" Form). *Suppose $B_1, \ldots, B_n$ are a collection of events. If:*

1. $\Pr(B_i) \leqslant p$ *for every* $i \in [n]$, *for some* $p \in (0,1)$;
2. *and, the events admits a dependency graph wherein event* B_i *depends only on the events* B_j *for* $j \in N(i)$,

Then, as long as for every $i \in [n]$,

$$\Pr(B_i) + \sum_{j \in N(i)} \Pr(B_j) \leqslant 1/4,$$

the probability that ***none of*** $B_1, \ldots, B_n$ *happens is strictly more than zero.*

We leave the proof of this version as an exercise (*Hint:* Change the induction hypothesis to prove that $\Pr(B_i \mid A_S) \leqslant 2 \cdot \Pr(B_i)$; the rest should follow exactly as before). However, let us see how this stronger version can be helpful in the following simple application.

2.1 Frugal Coloring

Consider the problem of vertex coloring a graph $G = (V, E)$ with maximum degree Δ. We already discussed that we can always color the graph with $\Delta + 1$ color without creating any monochromatic edges. We are now going to throw in another side constraint. We say that a coloring is **β-frugal** for some integer $\beta \geqslant 1$ iff in the neighborhood of every vertex $v \in V$, any single color c is used at most β times. In other words, no color can appear more than β times in any $N(v)$ for $v \in V$.

It is easy to see that every graph has an $O(\Delta^2)$ coloring which is 1-frugal: For each $v \in V$, connect all vertices $N(v)$ to each other to turn the "old" neighborhood of v into a clique. Then, properly color this graph which can be done with its maximum degree plus one color, which is $\Delta^2 + 1$. The following result of Hind, Molloy, and Reed [HMR97] shows a relaxing the value of β allows for much fewer number of colors.

Proposition 3 ([HMR97]). *For every integers $\beta \geqslant 1$ and $\Delta > \beta^\beta$, every graph $G = (V, E)$ with maximum degree Δ admits a proper $k := 100 \cdot \Delta^{1+1/\beta}$ coloring which is β-frugal.*

Proof. Suppose for every vertex $v \in V$, we pick a color $c(v)$ uniformly at random from $[k]$. Is this a proper k-coloring which is also β-frugal with non-zero probability?

We have the following two types of "bad" events:

- For each edge $(u, v) \in E$, B_{uv}: the color of both vertex u and v is the same, i.e., we have $c(u) = c(v)$ (this violates the proper coloring condition);
- For each vertex $v \in V$ and $S \subseteq N(v)$ with $\beta + 1$ vertices, B_{vS}: the color of all vertices in S is the same (this violates the β-frugal coloring condition).

We have,

$$\Pr(B_{uv}) = \frac{1}{k} \qquad \text{and} \qquad \Pr(B_{v,S}) = \frac{1}{k^\beta}.$$

Dependency for B_{uv}. Each event B_{uv} depends on the following:

- B_{wz} if $\{u, v\} \cap \{w, z\}$ is non-empty. This means there are most 2Δ such events (one for each of 2Δ edges incident on u or v).
- B_{wT} if u or v belongs to T. This means there are at most $2\Delta \cdot \binom{\Delta}{\beta}$ such events (one $\binom{\Delta}{\beta}$ for each of the Δ neighbors of u, and similarly for v).

This means that

$$\Pr\left(B_{uv}\right) + \sum_{B_{wz} \in N(B_{uv})} \frac{1}{k} + \sum_{B_{wT} \in N(B_{uv})} \frac{1}{k^\beta} \leqslant \frac{1}{k} + \frac{2\Delta}{k} + \frac{2\Delta \cdot \Delta^\beta}{k^\beta} \leqslant \frac{1}{\Delta^{1/\beta}} + \frac{2}{100^\beta} < 1/4.$$

Dependency for B_{vS}. Each event B_{vS} depends on the following:

- B_{wz} if $\{w, z\} \cap S$ is non-empty. This means there are most $(\beta + 1) \cdot \Delta$ such events (one for each Δ neighbor of each of the $\beta + 1$ vertices in S).
- B_{wT} if $S \cap T$ is non-empty. This means there are at most $(\beta + 1) \cdot \Delta \cdot \binom{\Delta}{\beta}$ such events ($\beta + 1$ choice for picking a vertex from S to intersect with T, Δ choice for picking w, and $\binom{\Delta}{\beta}$ for the rest of T).

This means that

$$\Pr\left(B_{vS}\right) + \sum_{B_{wz} \in N(B_{vS})} \frac{1}{k} + \sum_{B_{wT} \in N(B_{vS})} \frac{1}{k^\beta} \leqslant \frac{1}{k} + \frac{(\beta+1) \cdot \Delta}{k} + \frac{(\beta+1) \cdot \Delta \cdot \binom{\Delta}{\beta}}{k^\beta} \leqslant \frac{\beta+1}{\Delta^{1/\beta}} + \frac{2e^\beta \cdot (\beta+1)}{(100\beta)^\beta} < 1/4.$$

Consequently, the total probability in the neighborhood of each vertex is at most $1/4$. This allows us to apply Theorem 2 and have that there is an assignment of colors wherein none of the bad events happens. In this case, we obtain a coloring as desired, concluding the proof. □

We conclude this part by mentioning that there are even more general versions of the LLL that one may want to use depending on the application in mind. In some relatively rare cases also, one should instead use the Entropy Compression method of last lecture to prove LLL-type results that do not exactly fit the framework of LLL.

References

[EL75] Paul Erdős and László Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. *Infinite and finite sets*, 10(2):609–627, 1975. 2

[HMR97] Hugh Hind, Michael Molloy, and Bruce Reed. Colouring a graph frugally. *Combinatorica*, 17(4):469–482, 1997. 3